More About Energy Transfer in Turbomachines

Equation (1.7) can be better explained by demonstrating a steady flow through a container having uniform angular velocity ω as shown in Fig.1.3b. The centrifugal force on an infinitesimal body of a fluid of mass dm at radius r gives rise to a pressure differential dp across the thickness dr of the body in a manner that a differential force of dpdA acts on the body radially inward. This force, in fact, is the centripetal force responsible for the rotation of the fluid element and thus becomes equal to the centrifugal force under equilibrium conditions in the radial direction. Therefore, we can write

$$dp.dA = dm\,\omega^2\, r$$

with dm = dA dr ρ where ρ is the density of the fluid, it becomes

$$dp/\rho = \omega^2\, r\; d\,r$$

For a reversible flow (flow without friction) between two points, say, 1 and 2, the work done per unit mass of the fluid (i.e., the flow work) can be written as

$$\int_1^2 \frac{dp}{\rho} = \int_1^2 \omega^2\, r\; dr = \frac{\omega^2 r_2^2 - \omega^2 r_1^2}{2} = \frac{U_2^2 - U_1^2}{2}$$

The work is, therefore, done on or by the fluid element due to its displacement from radius r_1 to radius r_2 and hence becomes equal to the energy held or lost by it. Since the centrifugal force field is responsible for this energy transfer, the corresponding head (energy per unit weight) $U^2/2g$ is termed as centrifugal head. The transfer of energy due to a change in centrifugal head $[(U_2^2 - U_1^2)/2g]$ causes a change in the static head of the fluid.

The third term represents a change in the static head due to a change in fluid velocity relative to the rotor. This is similar to what happens in case of a flow through a fixed duct of variable cross-sectional area. Regarding the effect of flow area on fluid velocity V_r relative to the rotor, a converging passage in the direction of flow through the rotor increases the relative velocity $(V_{r2} > V_{r1})$ and hence decreases the static pressure. This usually happens in case of turbines. Similarly, a diverging passage in the direction of flow through the rotor decreases the relative velocity $(V_{r2} < V_{r1})$ and increases the static pressure as occurs in case of pumps and compressors.

The fact that the second and third terms of Eq. (1.7) correspond to a change in static head can be demonstrated analytically by deriving Bernoulli's equation in the frame of the rotor.

In a rotating frame, the momentum equation for the flow of a fluid, assumed "inviscid" can be written as

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} + 2\vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r})\right] = -\nabla p$$

where $\vec{v}$ is the fluid velocity relative to the coordinate frame rotating with an angular velocity $\vec{\omega}$.

We assume that the flow is steady in the rotating frame so that $\frac{\partial \vec{v}}{\partial t} = 0$. We choose a cylindrical coordinate system (r, θ, z) with z-axis along the axis of rotation. Then the momentum equation reduces to

$$\vec{v}\cdot\nabla\vec{v} + 2\omega\vec{i}_z\times\vec{v} - \omega^2 r\,\vec{i}_r = -\frac{1}{\rho}\nabla p$$

where $\vec{i}_z$ and $\vec{i}_r$ are the unit vectors along z and r direction respectively. Let $\vec{i}_s$ be a unit vector in the direction of $\vec{v}$ and s be a coordinate along the stream line. Then we can write

$$v\frac{\partial v}{\partial s}\vec{i}_s + v^2\frac{\partial \vec{i}_s}{\partial s} + 2\omega v\,\vec{i}_z\times\vec{i}_s - \omega^2 r\, i_r = -\frac{1}{\rho}\nabla p$$