SOP: MATH F266

# GRAPH COLOURING AND ITS VARIANTS


| | |
|---|---|
| **Name** | Rajvi Sampat |
| **ID No.** | 2018B4A70820G |
| **Professor** | Tarkeshwar Singh |
| **Date** | December 7, 2020 |

# Contents

# 1. Star Colouring of Some Graphs

A *k - star colouring* of a graph G is a k-colouring $V_1, V_2, V_3 ... V_k$ such that every component of the graph G$[V_i U V_j]$ is a star for $1 \leq i, j \leq k$. The minimum *k* for which a graph G has a *k-star colouring* is known as the *star chromatic number* $\chi_S(G)$ of the graph G.

A star colouring is a colouring without a bicoloured $P_4$.

## 1.1 Tree

$\chi_S = min\{r(G) + 1, 3\}$ where r(G) is the radius of the tree.

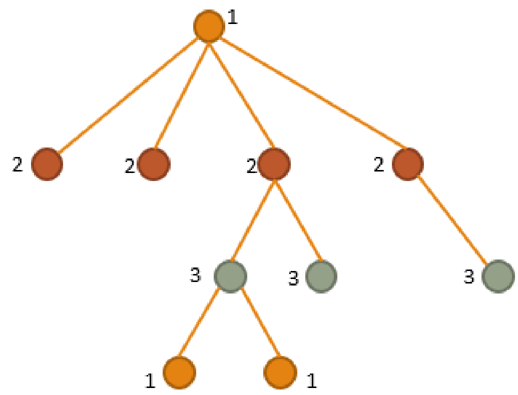



Figure 1.1: Tree of radius 3

**Proof:**

If the radius $r(G)$ = 0, then the graph G consists of a single isolated vertex (since a tree is a connected graph there exists only 1 vertex). Hence $\chi_S(G) = 1 = r(G) + 1$.

If the radius $r(G)$ = 1, then the graph G is a star with a single internal node and atleast 1 leaf node. The internal node is assigned colour 1 and the leaf nodes are assigned colour 2, which gives a star colouring of 2 colours. Hence $\chi_S(G) = 2 = r(G) + 1$.

Now, let $r(G)$ > 1. Let $V_1, V_2, V_3..V_k$ be a vertex partition of the vertex set V(G). Consider the tree to be a rooted tree, with root node $x$.
Let $x \in V_1$. Consider all the neighbours $y$ of $x$ and let all $y \in V_2$. Since $x$ is the root node and $r(G)$ > 1, $\exists\, y_1, y_2 \in V_2$.
Consider a neighbour $z \neq x$ of $y_1$. $xz \neq E(G)$ since x$y_1 \in E(G)$, $y_1 z \in E(G)$ and G is acyclic.
$\therefore$ $x$ and $z$ are independent. If $z \in V_1$, then $y_1 x y_2 z$ is a bicoloured path of length 3 ($P_4$) and hence does not permit a star colouring.
$\Rightarrow z \notin V_1$. Thus $z \in V_2$.

Consider a neighbor $w$ of $z$ , $w \neq y_1$.
$wx, wy_i \notin E(G)$ since G is acyclic. w and x are independent. If $w \in V_1, \exists$ no bicoloured $P_4$ through $x$ and $w$.
$\Rightarrow w \in V_1$.
Thus, a vertex v at a distance 'a' from the root node $x$ can be assigned to the colour set $V_i$, where i = a mod 3. Since there exists no bicoloured $P_4$ in such a colouring, it is a star colouring. For any i,j $\in$ (1,2,3) , G( $V_i U V_j$) has each component as a star.
Thus, $\exists$ a 3 - star colouring of a tree. Since r(G) > 1, there exists an induced subgraph $P_4$ , and since $P_4$ cannot be bicoloured, $\chi_S(G) \geq 3$.
$\therefore \chi_S(G) = 3$.

An example of a 3-star coloured tree is shown in Fig. 1.1

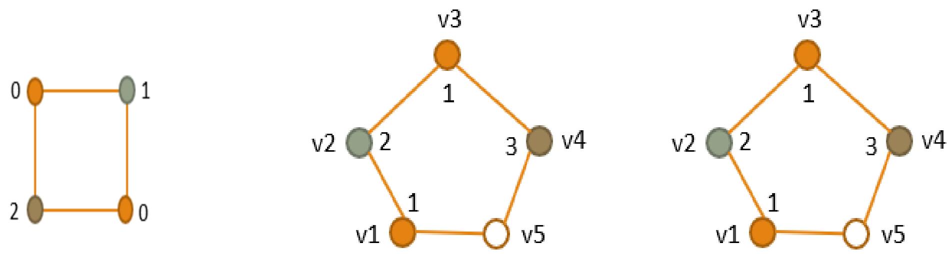
## 1.2 Cycle

$$\chi_S(C_n) = 3;\ n \geq 3;\ n \neq 5;$$
$$\chi_S(C_n) = 4;\ n = 5;$$

$C_3$ : Trivially $\chi_S(C_3)$ = 3.

Since all $C_n$ induce a path of length 3 for n $\geq$ 4, and a bicoloured $P_4$ does not permit a star colouring, so $\chi_S(C_n) \geq 3$ for n $\geq$ 4.

$C_4$ : $C_4$ has a 3-star colouring as shown in Fig. 1.4a $\chi_S(C_4)$ = 3.




(a) $C_4$ (b) $C_5$ Case a (c) $C_5$ Case b

Figure 1.2: Star colouring of some cycles

$C_5$ : Since any $P_4$ has atleast 3 colours, colour vertices 1 to 4 using a 3- colouring.
Case a) $v_5$ is adjacent to 2 different colours as shown in Fig. 1.4b. Let $v_1, v_3 \in V_1$ , $v_2 \in V_2$ , $v_4 \in V_3$, where $V_i$ is a vertex partition of V(G). $\Rightarrow v_5 \notin V_1$ , $v_5 \notin V_3$ since $v_5$ is adjacent to $v_1$ and $v_4$. If $v_5 \in V_2$, then $\exists$ a bicoloured $P4$ : $v_5v_1v_2v_3$.
$\Rightarrow v_5 \notin V_2$. $v_5 \in V_4$.

Case b) $v_5$ is adjacent to 2 same colours as shown in Fig. 1.2c. Let $v_1, v_4 \in V_1$ , $v_2 \in V_2$ , $v_3 \in V_3$, where $V_i$ is a vertex partition of V(G). $\Rightarrow v_5 \notin V_1$since $v_5$ is adjacent to $v_1$ and $v_4$. If $v_5 \in V_2$,or $v_5 \in V_3$ then $\exists$ a bicoloured $P4$ : $v_4v_5v_1v_2$ or $v_3v_4v_5v_1$.
$\Rightarrow v_5 \notin V_2, V_3$. $v_5 \in V_4$.

$\Rightarrow \chi_S(C_5)$ = 4

$C_n$ , n > 5

Label vertices from 0 to n-1 , let $v_i \in V_{imod3}$.

Case a) n mod 3 = 0 $\Rightarrow$ (n-1) mod 3 = 2
$\Rightarrow v_{n-1} \in V_2, v_0 \in V_0$
$\therefore$ any path $P_3$ $v_{xmod3}v_{(x+1)mod3}v_{(x+2)mod3}v_{(x+3)mod3}$ is 3- coloured.
Since $\chi_S \geq 2$ and a 3-star colouring exists, $\chi_S(C_n)$= 3 for n mod 3 = 0, n > 5.
Case b) n mod 3 = 1 ,$\Rightarrow$ (n-1) mod 3 = 0
$v_0, v_{n-1} \in V_0 \Rightarrow notacolouring$
Let $v_{n-1} \in V_2, v_{n-2} \in V_0$ (swap colours of $v_{n-1}$ and $v_{n-2}$).
For and $P_4$, if $v_{n-1}, v_{n-2} \notin P_4$, it is 3-coloured ( same as case a).
If only one of $v_{n-1}$ and $v_{n-2}$ is present in $P_4$, i.e. the paths $v_2v_1v_0v_{n-1}$ or $v_{n-2}v_{n-3}v_{n-4}v_{n-5}$ it is still 3-coloured as n $\geq$ 5 and the other 3 vertices are of different colours and no 2 adjacent vertices are of the same colour.
If both $v_{n-1}, v_{n-2} \in P_4$, then no two adjacent vertices are of the same colour, and either $v_1$ or $v_{n-3} \in P_4$ .Since $v_1$ and $v_{n-3} \in V_1$, the path is 3 - coloured.
$\Rightarrow$ any path of length 3 has an admissible 3-colouring. Thus the given colouring is a 3- star colouring of the $C_n$ where n mod 3 = 1.
Since $\chi_S \geq 2$ and a 3-star colouring exists, $\chi_S(C_n)$ = 3 for n mod 3 = 1, n > 5.
Case c) n mod 3 = 2 $\Rightarrow$ (n-1) mod 3 = 1
$v_{n-2}, v_{n-1}, v_0, v_1$ belong to the sets $V_0, v_1, V_0, V_1$ respectively, hence this is a bicoloured path of length 3, and not a star colouring.
Let $v_{n-2} \in V_2, v_{n-3} \in V_0$ (swap colours of $v_{n-2}$ and $v_{n-3}$).
Consider any $P_4$. If $v_{n-2}, v_{n-3} \notin P_4$, it is 3-coloured ( same as case a).
If $v_{n-2} \in P_4$ and $v_{n-3} \notin P_4$, then $P_4$ is the path $v_{n-2}v_{n-1}v_0v_1$ which are in the colour sets $V_2, V_1, V_0, V_1$ respectively, so it is 3-coloured.
If $v_{n-2} \notin P_4$ and $v_{n-3} \in P_4$, then $P_4$ is the path $v_{n-6}v_{n-5}v_{n-4}v_{n-3}$. If n= 5, then this will be the path $v_{n-1}v_0v_1v_2$, the vertices of which are in the colour sets $V_1, V_0, V_1, V_0$ respectively, so it is a bicoloured path, hence does not permit a star colouring. As shown earlier, $C_5$ does not have a 3-star colouring. If n $\neq$ 5, then the vertices of this path belong to the sets $V_2, V_0, V_1, V_0$ respectively, so it is a 3-coloured path, with no adjacent vertices of the same colour, hence acceptable.
If both $v_{n-2}, v_{n-3} \in P_4$, then no two adjacent vertices are of the same colour, and either $v_{n-1}$ or $v_{n-5} \in P_4$ .Since $v_{n-1}$ and $v_{n-5} \in V_1$, when n > 5, the path is 3 - coloured.
$\Rightarrow$ any path of length 3 has an admissible 3-colouring. Thus the given colouring is a 3- star colouring of the $C_n$ where n mod 3 = 2, n > 5.
Since $\chi_S \geq 2$ and a 3-star colouring exists, $\chi_S(C_n)$ = 3 for n mod 3 = 1, n > 5.

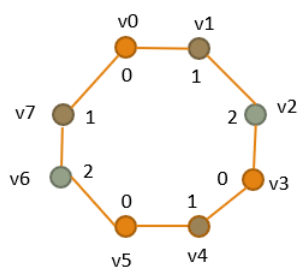



Figure 1.3: Star colouring of $C_8$

## 1.3 Peterson Graph

$$\chi_S(G) = 5$$


(a) Initial colouring

(b) An optimal star colouring

Figure 1.4: Star colouring of Peterson Graphs

A peterson graph contains $C_5$ as an induced subgraph, so $\chi_S(G) \geq 4$.
Consider the outer $C_5$. Any 2 vertices have the same colour and are at distance 2,the rest have distinct colours.Label the outer $C_5$ as shwon in **??**
Consider the inner $C_5$. If it can be coloured with 4 colours, any 2 vertices must have the same colour, rest have different colours.
Case 1: Suppose the common colour of the inner $C_5$ is the same as that of the outer $C_5$, i.e. 2 in this case. Since e, b can't be coloured 2, c,d will be coloured 2. So colour of a will be 3 or 4.However, either of these assignments will lead to a bicoloured $P_4$. Hence this is not possible.
Case 2: Suppose the common colour is 1.Vertices e, a, b can't be coloured 1 as a is adjacent to a vertex of colour 1 and if e or b is taken, a bicoloured $P_4$ is induced. If the vertices c and d are coloured 1, any assignment for a(2,3 or 4) will induce a bicoloured $P_4$. Hence this is not possible.
Case 3: Common colour is 3 or 4. w.l.o.g, let common colour be 3. Again, any assignment induces a bicoloured $P_4$. If a,e = 3. Then c = 1 generates a bicoloured $P_4$ of colours 1 and 3, and c = 4 induces a bicoloured $P_4$ of colours 3 and 4. If e,d = 3.Then c = 1 generates a bicoloured $P_4$ of colours 1 and 2, and c = 4 induces a bicoloured $P_4$ of colours 3 and 4.
Hence, a 4- star colouring is not possible.$\chi_S(G)$> 4.
A 5 star colouring exists as shown in **??**. Hence, $\chi_S(G) \leq 5$.
$\chi_S(G) = 5$.

# 2. CD Colouring of Some Graphs

A k-coloring $V_1, V_2, ..., V_k$ of a graph G is called a *k-CD coloring* if for every $V_i$ , 1 i k, there exists a vertex $x_i$ such that $V_i \subseteq N[x_i]$.

## 2.1 Peterson Graph

In a CD colouring, all vertices in a colour set must be at a distance 2. In a Peterson graph, all vertices are mutually at a distance 1 or 2.
Consider Fig. 2.1
$S_1$ := { x | d(a,x) = 2 } = c,d,f,h,i,j.
$S_1$ := { x | d(a,x) = 1 } = b,e,j.
$\therefore$w$_s$=2, $\Rightarrow \chi_S(G) \geq 2$.
For each vertex, N(v) = 3, and all x in N(v) are mutually at a distance 2.
For all the vertices x in a colour set $V_i$ , there should be a y $\in Gsuchthatd(x, y) = 1.Sinceallx \in N[v]areatadistance1fromv, N[v]canbeacoloursetdominatedbyv.SinceN[v] = 3, themaximumsizeofacoloursetis3.$
$Sincethenumberofverticesare10 = 3(3) + 1.\therefore \chi_S(G)$ > 3.$\Rightarrow \chi_S(G)$ = 4.
We have a 4-colouring.
$\Rightarrow \chi_S(G)$ = 4.

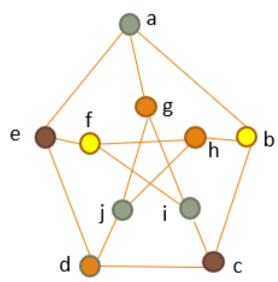



Figure 2.1: CD colouring of Peterson Graph

# 3. References


Sandhya T.P. (2017) Graph Colouring and its Variants

L.Jethruth Emelda Mary and Dr. K. Ameenal Bibi A. Lydia Mary Julietterayan(2017), A Study on Star Chromatic Number of Some Special Classes of Graphs,*Global Journal of Pure and Applied Mathematics*.13.9