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Assignment 2

Rajpreet

21104081

① $L\{f(t)\} = \frac{e^{-1/s}}{s}$

using change of scale property

$L\{f(3t)\} = \frac{1}{3}\left[\frac{e^{-3/s}}{s/3}\right]$

$= \frac{e^{-3/s}}{s}$

② a) $L\{\sin^3 t\}$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\sin^3 A = \frac{3}{4}\sin A - \frac{\sin 3A}{4}$

$\Rightarrow \sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$

Using Linearity after taking laplace,

$L\{\sin^3 t\} = \frac{3}{4}L\{\sin t\} - \frac{1}{4}L\{\sin 3t\}$

$= \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{1}{4}\left(\frac{3}{s^2+9}\right)$

$L\{e^{2t}\sin^3 t\}$ using first shifting prop.

$$L\{e^{2t}\sin^3 t\} = \frac{3}{4}\left[\frac{1}{(s-2)^2+1}\right] - \frac{3}{4}\left[\frac{1}{(s-2)^2+9}\right]$$

$$= \frac{3}{4}\left[\frac{1}{(s-2)^2+1} - \frac{1}{(s-2)^2+9}\right]$$

(3) $L\left\{\cos\left(t-\frac{2\pi}{3}\right)\right\} = e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}$

using second shifting property, and

$\left[\cancel{\cos} L\{\cos t\} = \frac{s}{s^2+1}\right]$ $\cancel{\left[L\{e\right.}$

(4) $L\left[2\sqrt{\frac{t}{\pi}}\right] = \frac{1}{s^{3/2}}$

by linearity $= L\left\{\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{2s^{3/2}}$

$$L\left\{\frac{1}{t}\sqrt{\frac{t}{\pi}}\right\} = \int_s^{\infty} \frac{1}{2s^{3/2}}\,ds$$

(division property)

$$= \frac{1}{2}\left[\frac{s^{-1/2}}{-1/2}\right]_s^{\infty} = \frac{1}{s^{1/2}}$$

$$\Rightarrow \boxed{L\left\{\frac{1}{\sqrt{t\pi}}\right\} = \frac{1}{s^{1/2}}}$$

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(5) $\int_0^\infty e^{-t} \frac{\sin t}{3} dt$

$L\{\sin t\} = \frac{1}{s^2+1}$ $(a=1)$

$= \frac{\pi}{2} - \tan^{-1}(s) = \cot^{-1}(s) = \tan^{-1}\left(\frac{1}{s}\right)$

by defination $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

$L\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-st} \frac{\sin t}{t} dt = \tan^{-1}\frac{1}{s}$

as $s = 1$

$L\int_0^\infty e^{-st} \frac{\sin t}{t} dt = \pi/4$

(6) $\int_0^t e^t \frac{\sin t}{t} dt$

$L[\sin t] = \frac{1}{s^2+1}$ $(a=1)$

$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1} ds = [\tan^{-1}(s)]_0^\infty = \frac{\pi}{2} - \tan^{-1}(s)$

$= \cot^{-1}(s)$

$L\left[\frac{e^t \sin t}{t}\right] = \cot^{-1}(s-1)$ { Property of integral shifting }

Now, $\int_0^\infty e^t \frac{\sin t}{t} dt = \cot^{-1}(s-1)$

$$\Rightarrow \int_0^\infty e^{t} \frac{\sin t}{t} \, dt = \cot^{-1}(s-1)$$

$$\Rightarrow \int_0^t e^{t} \frac{\sin t}{t} \, dt = \cot^{-1}\left[\frac{s-1}{s}\right]$$

(2) $f(t) = \begin{cases} t^2, & 0 \le t \le 2 \\ 4t, & t > 2 \end{cases}$

$f(t) = [\, t^2 u(t-a) - t^2 u(t-2)\,]$

$\cancel{+4t u(t-2)} + 4t\, u(t-2)$

$f(t) = t^2 u(t-0) + (4t - t^2)\, u(t-2)$

Taking Laplace Both sides

$L\{f(t) . u(t-a)\} = e^{-as} L[f(t+a)]$

$\Rightarrow e^{0} L[(t+0)^2] + e^{-2s} L[4(t+2) - (t+2)^2]$

$\Rightarrow L[t^2] + e^{-2s} [4 - t^2]$

$\Rightarrow \frac{2}{s^3} + e^{-2s} \left[\frac{4}{s} - \frac{2}{s^3}\right]$

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(8) $L\{u(t-1)\} = \frac{e^{-s}}{s}$

using Multiplication of t property

$$t^2 u(t-1) = \frac{d^2}{ds^2}\left(\frac{e^{-s}}{s}\right)$$

$$= \frac{(s^3 - 3s^2 - s)e^{-s}}{s}$$

$$L\{f(t-1)\} = e^{-s}$$

$$= \frac{(s^3 - 3s^2 - s)e^{-s}}{s} + e^{-s}$$

$$= (s^2 - 3s)\, e^{-s}$$

(9) $$L^{-1}\left\{\frac{s^{\cancel{4}}}{s^4 + 4a^4}\right\}$$

$$\frac{s}{(s^2 + 2a^2)^2 - (2as)^2}$$

$$= \frac{s}{(s^2 + 2a^2 + 2as)(s^2 + 2a^2 - 2as)}$$

$$= \frac{1}{4a} \cdot \frac{1}{(s^2 + 2a^2 - 2as)} - \frac{1}{4a}\frac{1}{(s^2 + 2a^2 + 2as)}$$

taking inverse Laplace both side

$$L^{-1}\left\{\frac{s}{s^4+4a^4}\right\} = \frac{1}{4a} L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$

$$- \frac{1}{4a} L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{4a} \frac{\sin at}{a} e^{at} - \frac{1}{4a} \frac{\sin at}{a} e^{-at}$$

$$= \frac{1}{4a} \sin at \left(e^{at} - e^{-at}\right)$$

$$= \frac{\sin at \, \sinh at}{2a}$$

(ii) $L^{-1}\left\{\frac{s+8}{s^2+4s+5}\right\} + L^{-1}\left\{\frac{6}{(s+2)^2+1}\right\}$

$$= e^{-2t}\cos t + 6e^{-2t}\sin t$$

(iii) $L^{-1}\left\{\frac{s e^{-2\pi s/3}}{s^2+9}\right\}$

$$= \cos 3\left(t - \frac{2\pi}{3}\right)$$

$$= \cos 3t - 2\pi$$

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(10) $L^{-1}\left\{\cot^{-1}\left(\frac{s}{2}\right)\right\}$

$\frac{d}{ds}\cos^{-1}\left(\frac{s}{2}\right) = \frac{-2}{s^2+4}$

taking inverse Laplace both sides

$L^{-1}\left\{\frac{d}{ds}\cot^{-1}\left(\frac{s}{2}\right)\right\} = L^{-1}\left\{\frac{2}{s^2+4}\right\}$

$= -t f(t) = -2 \times \frac{1}{2} \sin 2t$

$\Rightarrow f(t) = \frac{\sin 2t}{t}$

(ii) $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$

$\frac{d}{ds}\left[\frac{\log(s+1)}{s-1}\right] = \frac{1}{s+1} - \frac{1}{s-1}$

taking inverse laplace both side

$L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} = -L^{-1}\left\{\frac{1}{s-1}\right\}$

$-d f(t) = e^{-t} - e^{t}$

$f(t) = \frac{e^{t} - e^{-t}}{t} = \frac{2\sinh t}{t}$

(iii)

$$L^{-1}\left\{\frac{1}{(s^2+a^2)s^3}\right\}$$

~~$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$~~

$$L^{-1}\left[\frac{F(s)}{s^n}\right] = \int_0^t\int_0^t \cdots \int_0^t f(x)\,dx\,dx \,(n \text{ times})$$

$n=2$, $f(t) = \frac{\sin at}{a}$

$$\int_0^t\int_0^t\int_0^t \frac{\sin at}{a}\,.dt = \int_0^t\int_0^t \left[-\frac{\cos at}{a^2}\right]_0^t$$

$$\int_0^t\int_0^t \frac{1-\cos at}{a^2} = \int_0^t \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]_0^t$$

$$\Rightarrow \int_0^t \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right) = \frac{1}{a^2}\left[\frac{t^2}{2} + \frac{\cos at}{a^2}\right]_0^t$$

$$\Rightarrow \frac{1}{a^2}\left[\frac{t^2}{2} + \frac{\cos 2at}{a^2} - \frac{1}{a^2}\right] \text{ Ans}$$

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(11) $y'' + 2ty' \cancel{z} - y\ 2t$

Taking LT on both sides.

$$s^2F(s) - sf(0) - f'(0) + 2L[ty'] - F(s) = 1/s^2$$

$$s^2(F(s)) - 1 - 2\frac{d}{ds}(sF(s) - f(0)) - F(s) = \frac{1}{s^2}$$

$$s^2F(s) - 1 - 2\frac{d}{ds}(sF(s) - f(0)) - F(s) = \frac{1}{s^2}$$

$$\Rightarrow s^2F(s) - 1 - 2(F(s) + F'(s)) - F(s) = 1/s^2$$

or $$F'(s) + \left(\frac{3}{2s} - \frac{s}{2}\right)F(s) = \frac{1}{2s^3} - \frac{1}{2s}$$

IF $= e^{\int P.dx}$ using $\left\{\frac{dy}{dx} + Py = Q\right\}$

$$\therefore = e^{\int\left(\frac{3}{2s} - \frac{s}{2}\right)ds} = e^{\frac{3}{2}\ln s - \frac{s^2}{4}} = e^{-s^2/4} \cdot e^{3/2}$$

Now, $F(s) \cancel{z}\ e^{-s^2/4}\cancel{3}\ s^{3/2}$

$$= \int \frac{-(s^2+1)}{2s^3} e^{-s^2/4} e^{3/2} ds$$

$$F(s)e^{-s^2/4} \cdot s^{3/2} = \int -\frac{(s^2+1)e^{-s^2/4}}{2s^{3/2}} ds$$

using by parts,

$$I = \frac{-1}{2}\left[e^{-s^2/4}\left(\frac{2}{3}s^{3/2} - 2s^{-1/2}\right) \cancel{+\frac{z}{2}}\right.$$

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$$+\frac{1}{2}\int \frac{s}{2}\left(\frac{2}{3}s^{3/2}-2s^{-1/2}\right)e^{-s^2/4}\,ds\Big]$$

$$\left\{\frac{d}{ds}\left(e^{-s^2/4}s^{3/2}\right)=s^{1/2}e^{-s^2/4}\left(-\frac{s^2}{4}+\frac{3}{2}\right)\right\}$$

$$=\frac{-1}{2}\left[2s^{-1/2}e^{-s^2/4}\left(\frac{s^2-3}{8}\right)+\frac{1}{2}\int\frac{s^{1/2}}{3}(s^2-3)e^{-s^2/4}ds\right.$$

$$=\frac{-1}{2}\left[2s^{-1/2}e^{-s^2/4}\left(\frac{s^2-3}{3}\right)-\int\frac{s^{1/2}}{3}\left(\frac{-s^2+3}{2}\right)e^{-s^2/4}ds\right]$$

$$=-\frac{1}{2}\left[2s^{-1/2}e^{-s^2/4}\left(\frac{s^2-3}{3}\right)-\frac{1}{3}e^{-s^2/4}\cdot s^{3/2}\right]$$

$$=\cancel{-\frac{1}{2}\left[e^{-s/4}s^{3/2}\right.}$$

$$=-\frac{1}{2}\left[\frac{e^{-s^2/4}s^{3/2}}{3}-2s^{-1/2}e^{-s^2/4}\right]$$

$$\Rightarrow F(s)=\frac{1}{s^2}-\frac{1}{6}$$

Taking Laplace inverse

$$\boxed{f(t)=t-\frac{1}{6}\delta(t)}$$

This also satisfies the equation

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(12) $\int_0^\infty \frac{\cos x\omega + \omega \sin x\omega}{1+\omega^2} d\omega = \begin{cases} 0, & \text{if } x < 0 \\ \pi/2, & \text{if } x = 0 \\ \pi e^{-x}, & \text{if } x > 0 \end{cases}$

$f(x) = \int_0^\infty [A(\omega)\cos \omega x + B(\omega)\sin \omega x] \, d\omega$ — ①

from LHS

$\int_0^\infty \frac{1}{1+\omega^2} \cos x\omega + \frac{\omega}{1+\omega^2} \sin x\omega . d\omega$ — ②

from ① & ②

$A(\omega) = \frac{1}{1+\omega^2}$, $B(\omega) = \frac{\omega}{1+\omega^2}$

$A(\omega) = \frac{1}{\pi} \left[\int_{-\infty}^{0} 0 \cos \omega x . dx + \int_0^\infty \pi e^{-x} \cos \omega x . dx \right.$

$A(\omega) = \frac{1}{\pi} \left[0 + \pi \int_0^\infty e^{-x} \cos \omega x . dx \right]$

$A(\omega) = \frac{\pi}{\pi} \left[\frac{e^{-x}}{1+\omega^2} \left\{ -1 \cos \omega x + \omega \sin \omega x \right\} \right]_0^\infty$

$A(\omega) = 1 . \left[0 - \frac{1}{1+\omega^2} (-1+0) \right]$

$$\boxed{A(\omega) = \frac{1}{1+\omega^2}}$$

$\Rightarrow B(\omega) = \frac{1}{\pi} \left[\int_{-\infty}^{0} 0 \sin \omega x \, dx + \int_0^\pi \pi e^{-x} \sin \omega x \, dx \right)$

$$B(\omega) = \frac{1}{\pi}\left[0 + \pi \int_0^{\pi} e^{-x} \sin \omega x \, dx\right]$$

$$B(\omega) = \frac{\pi}{\pi}\left[\frac{e^{-x}}{1+\omega^2}(-\sin\omega x - \omega\cos\omega x)\right]_0^{\infty}$$

$$B(\omega) = \left[\frac{-e^{-x}}{1+\omega^2}[\sin\omega x + \omega\cos\omega x]\right]_0^{\infty}$$

$$B(\omega) = \frac{-1}{1+\omega^2}[0-(0+\omega)]$$

$$\Rightarrow \boxed{B(\omega) = \frac{\omega}{1+\omega^2}}$$

at $x=0$,

$$\int_0^{\infty} \frac{\cos 0 + \omega \sin 0}{1+\omega^2} \, d\omega = \int \frac{1}{1+\omega^2} \, d\omega = [\tan^{-1}(\omega)]_0^{\infty}$$

$$= \pi/2$$

(13) $$F(e^{-x^2}) = \int_{-\infty}^{\infty} e^{-x^2} e^{-i\omega x} \, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-(x^2+i\omega x)} \, dx = \int_{-\infty}^{\infty} e^{-\left(x+\frac{i\omega}{2}\right)^2 + \left(\frac{i\omega}{2}\right)^2} \, dx$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\left(x+\frac{i\omega}{2}\right)^2 - \frac{\omega^2}{4}} \, dx$$

$$\Rightarrow e^{-\omega^2/4} \int_{-\infty}^{\infty} e^{-(x+i\omega/2)^2} \cdot dx$$

$$\left\{ \begin{array}{l} x + \frac{i\omega}{2} = t \\ dx = dt \end{array} \right.$$

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$$\Rightarrow e^{-\omega^2/4} \int_{-\infty}^{\infty} e^{-t^2} \cdot dt$$

$$\Rightarrow \boxed{e^{-\omega^2/4} \cdot \sqrt{\pi}}$$

(i) $$F(e^{-ax^2}) = \int_{-\infty}^{\infty} e^{-ax^2} e^{-i\omega x} \cdot dx$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-a\left(x^2 + \frac{i\omega x}{2}\right)} \cdot dx = \int_{-\infty}^{\infty} e^{-a\left[\left(x + \frac{i\omega}{2a}\right)^2 - \left(\frac{i\omega}{2a}\right)^2\right]} dx$$

$$\Rightarrow e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-\left[\sqrt{a}\left(x + \frac{i\omega}{2a}\right)\right]^2} dx$$

$$\Rightarrow e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-t^2} \cdot \frac{dt}{\sqrt{a}}$$

$$\Rightarrow \frac{e^{-\omega^2/4a}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-t^2} \cdot dt = \boxed{\frac{e^{-\omega^2/4a}}{\sqrt{a}} \cdot \sqrt{\pi}}$$

(ii) $F(e^{-x^2/2}) \quad (a = 1/2)$

So, $$\frac{e^{-x^2/2}}{\sqrt{1/2}} \sqrt{\pi} = \sqrt{2\pi}\, e^{-\omega^2/2}$$

(iii) $F(x) = e^{-4(x-3)^2}$

by shifting property, $\Rightarrow \frac{\sqrt{\pi}}{2} e^{\left(3is - \frac{s^2}{16}\right)}$

(iv) $F(x) = e^{-x^2} \cos 2x$

by modulation theorem,

$$\frac{1}{2} \times \sqrt{2\pi} \left[e^{-\frac{(s+2)^2}{2}} + e^{-(s-2)^2/2} \right]$$

$$\Rightarrow \boxed{\sqrt{\frac{\pi}{2}} \left[e^{-(s+2)^2/2} + e^{-(s+2)^2/2} \right]}$$

(14) $\int_0^\infty F(x)\cdot\cos px\, dx = \begin{cases} 1-P, & 0 \le P \le 1 \\ 0, & P > 1 \end{cases}$

Let, $\int_0^\infty F(x)\cdot\cos px\cdot dx = F_C(P)$

Now,

$$F(x) = \frac{2}{\pi}\int_0^\infty \cancel{f}\, f_c(P)\cos px\cdot dp$$

$$f(x) = \frac{2}{\pi}\left[\int_0^1 f_c(P)\cos px\cdot dp + \int_1^\infty f_c(P)\cos px\cdot dx\right]$$

$$f(x) = \frac{2}{\pi}\int_0^1 (1-P)\cos px\cdot dp$$

$$f(x) = \frac{2}{\pi}\left[(1-P)\left(\frac{\sin px}{x}\right) - \int(-1)\frac{\sin px}{x}\right]_0^1$$

$$= \frac{2}{\pi}\left[(1-P)\frac{\sin px}{x} - \frac{\cos px}{x^2}\right]_0^1$$

$$\Rightarrow \frac{2}{\pi}\left[-\frac{\cos x}{x^2} - \left(0 - \frac{1}{x^2}\right)\right] = \frac{2(1-\cos x)}{\pi x^2}$$

Now,

$$f_c(p) = \int_0^\infty F(x)\cdot\cos px\cdot dx = \int_0^\infty \frac{2(1-\cos x)}{\pi x^2}\cos px\cdot dx \quad \text{—(1)}$$

Also, $f_c(p) = \begin{cases} 1-p, & 0 \le p \le 1 \\ 0, & p > 1 \end{cases}$ —(2)

from ① & ②

$$\frac{2}{\pi}\int_{1}^{\infty}\left[\frac{1-\cos x}{x^2}\right]\cos px \cdot dx = \begin{cases} 1-p, & 0\le p\le 1 \\ 0, & p>1 \end{cases}$$

Put $p=0$,

$$\frac{2}{\pi}\int_{0}^{\infty}\left[\frac{1-\cos x}{x^2}\right]\cdot dx = 1$$

$$\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin^2 x/2}{x^2}\,dx = 1$$

$$\int_{0}^{\infty}\frac{\sin^2 x/2}{x^2}\cdot dx = \pi/4$$

Now, $x = 2t$

$dx = 2dt$

$$\int_{0}^{\infty}\frac{\sin^2 t}{4t^2}\cdot 2\cdot dt = \pi/4$$

$$\Rightarrow \int_{0}^{\infty}\frac{\sin^2 t}{t}\cdot dt = \pi/2$$

(15) $$\int_{0}^{\infty}\frac{t^2}{(4+t^2)(9+t^2)}\cdot dt$$

$f(t) = e^{-2t}$, $g(t) = e^{-2t}$

$$F_c[f(t)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty}e^{-2t}\sin \omega t = \sqrt{\frac{2}{\pi}}\left[\frac{\omega}{\omega^2+4}\right] - ①$$

$$F_c[g(t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-3t} \sin \omega t = \sqrt{\frac{2}{\pi}} \left[\frac{\omega}{\omega^2+9}\right] \quad (2)$$

$$\int_0^\infty f(t) \cdot g(t) \cdot dt = \int_0^\infty F_c[f(t)] \cdot F_c[g(t)] \, d\omega$$

from (1) & (2)

$$\int_0^\infty e^{-2t} \cdot e^{-3t} \cdot dt = \frac{2}{\pi} \int_0^\infty \frac{\omega^2}{(\omega^2+4)(\omega^2+9)} \, d\omega$$

$$\Rightarrow \frac{\pi}{2} \int_0^\infty e^{-5t} \cdot dt = \int_0^\infty \frac{\omega^2}{(\omega^2+4)(\omega^2+9)} \, d\omega$$

$$\Rightarrow \frac{\pi}{2} \left[\frac{e^{-5t}}{-5}\right]_0^\infty = \int_0^\infty \frac{\omega^2}{(\omega^2+4)(\omega^2+9)} \, d\omega$$

$$-\frac{\pi}{10} [0 - 1] = \int_0^\infty \frac{\omega^2}{(\omega^2+4)(\omega^2+9)} \, d\omega$$

Replace ω by t

$$\int_0^\infty \frac{t^2}{(t^2+4)(t^2+9)} \, dt = \boxed{\pi/10}$$

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(16) $z[{}^nC_p] = \sum_{p=0}^{n} ({}^nC_p \, z^{-p})$

$\Rightarrow +1 + {}^nC_1 z^{-1} + {}^nC_2 z^{-2} \dots + {}^nC_n z^{-n}$

$\Rightarrow 1 + {}^nC_1\left(\frac{1}{z}\right) + {}^nC_2\left(\frac{1}{z}\right)^2 \dots {}^nC_n\left(\frac{1}{z}\right)^n$

(BINOMIAL Exp^n) $= \left(1+\frac{1}{z}\right)^n = \boxed{(1+z^{-1})^n}$

(17) $z[{}^{n+p}C_p] = \sum_{p=0}^{n} {}^{n+p}C_p \, z^{-p}$

$\Rightarrow 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + \dots + {}^{2n}C_n z^{-n}$

$= \boxed{(1-z^{-1})^{n-1}}$ (BINOMIAL)

(17) $z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} z^{-n}\left(1/n\right)$ $(n \neq 0)$

$\Rightarrow z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} \dots \infty$

i.e $\log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \infty\right)$

Replace x by $1/z$

$\log\left(1-\frac{1}{z}\right) = -\left[z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \dots \infty\right]$

$-\log\left[\frac{z-1}{z}\right] = z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} \dots \infty$

$\Rightarrow \log\left(\left[\frac{z}{z-1}\right]\right) = z(1/n)$

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(18) $\dfrac{z^2}{(z-1)(z-3)}$

Let $f(z) = \dfrac{z}{z-1}$ & $G(z) = \dfrac{z}{z-3}$

$z^{-1}[F(z)] = 1^n = 1$ $\quad z^{-1}[G(z)] = 3^n$

$z^{-1}[F(z) \cdot G(z)] = f_n * g_n = 1^n * 3^n$

$= \sum_{k=0}^{n} (1)^k (3)^{n-k}$

$\Rightarrow 3^n + 3^{n-1} + 3^{n-2} + \ldots + 1$

$r = \dfrac{3^{n-1}}{3^n} = \dfrac{1}{3}$

$\therefore \dfrac{3^n\left(1-\left(\frac{1}{3}\right)^{n+1}\right)}{1-1/3} = \dfrac{3^n[3^{n+1}-1]}{3^{n+1}[2/3]}$

$= \boxed{\dfrac{3^{n+1}-1}{2}}$

(19) $F(z) = \dfrac{z^2+3z}{(z-1)^2(z^2+1)}$

$\dfrac{F(z)}{z} = \dfrac{z+3}{(z-1)^2(z^2+1)}$

$\dfrac{z+3}{(z-1)^2(z^2+1)} = \dfrac{A}{z-1} + \dfrac{B}{(z-1)^2} + \dfrac{Cz+D}{z^2+1}$

By comparing LHS & RHS

$A = \frac{-3}{2}$, $B = 2$, $C = \frac{3}{2}$, $D = -1/2$

$$\frac{F(z)}{z} = -\frac{3}{2}\left(\frac{1}{z-1}\right) + \frac{2}{(z-1)^2} + \frac{3z-1}{2(z^2+1)}$$

$$\frac{F(z)}{z} = \frac{-3}{2}\left(\frac{1}{z-1}\right) + \frac{2}{(z-1)^2} + \frac{3}{2}\left(\frac{z}{z^2+1}\right) - \frac{1}{2}\cdot\frac{1}{(z^2+1)}$$

$$F(z) = -\frac{3}{2}\left(\frac{z}{z-1}\right) + \frac{2z}{(z-1)^2} + \frac{3}{2}\left(\frac{z^2}{z^2+1}\right) - \frac{1}{2}\cdot\left(\frac{z}{z^2+1}\right)$$

(20) $y_n + \frac{1}{4} y_{n-1} = u_n + \frac{1}{3}(u_{n-1})$

Taking z- transform on both sides,

$$Y(z) + \frac{1}{4} z^{-1} Y(z) = 1 + \frac{1}{3} z^{-1}$$

$$Y(z) = \left(1 + \frac{1}{3} z^{-1}\right) / \left(1 + \frac{1}{4} z^{-1}\right)$$

$$= \frac{(z+1/3)}{(z+1/4)}$$

There being only 1 simple pole at $z = -1/4$

consider the contour $|z| > 1/4$

$$\therefore \text{Res}\left[Y(z)\, z^{n-1}\right]_{z = -\frac{1}{4}}$$

$$\lim_{z \to -1/4} \left(z+\frac{1}{4}\right)\left(z+\frac{1}{3}\right) \cdot z^{n-1} \Big/ z+1/4$$

$$\Rightarrow \lim_{z \to -1/4} \left(z+\frac{1}{3}\right) z^{n-1} = \left(-\frac{1}{4}+\frac{1}{3}\right)\left(-\frac{1}{4}\right)^{n-1}$$

$$= \frac{1}{12}\left(-\frac{1}{4}\right)^{n-1}$$

Hence by inversion integral method, we have,

$$\boxed{y_n = \frac{1}{12}\left(\frac{-1}{4}\right)^{n-1}}$$

Rajpreet