MGMT 237M2 **Statistical Arbitrage**

Lecture 07: Portfolio Optimization

Professor Olivier Ledoit

University of California Los Angeles
Anderson School of Management
Master of Financial Engineering

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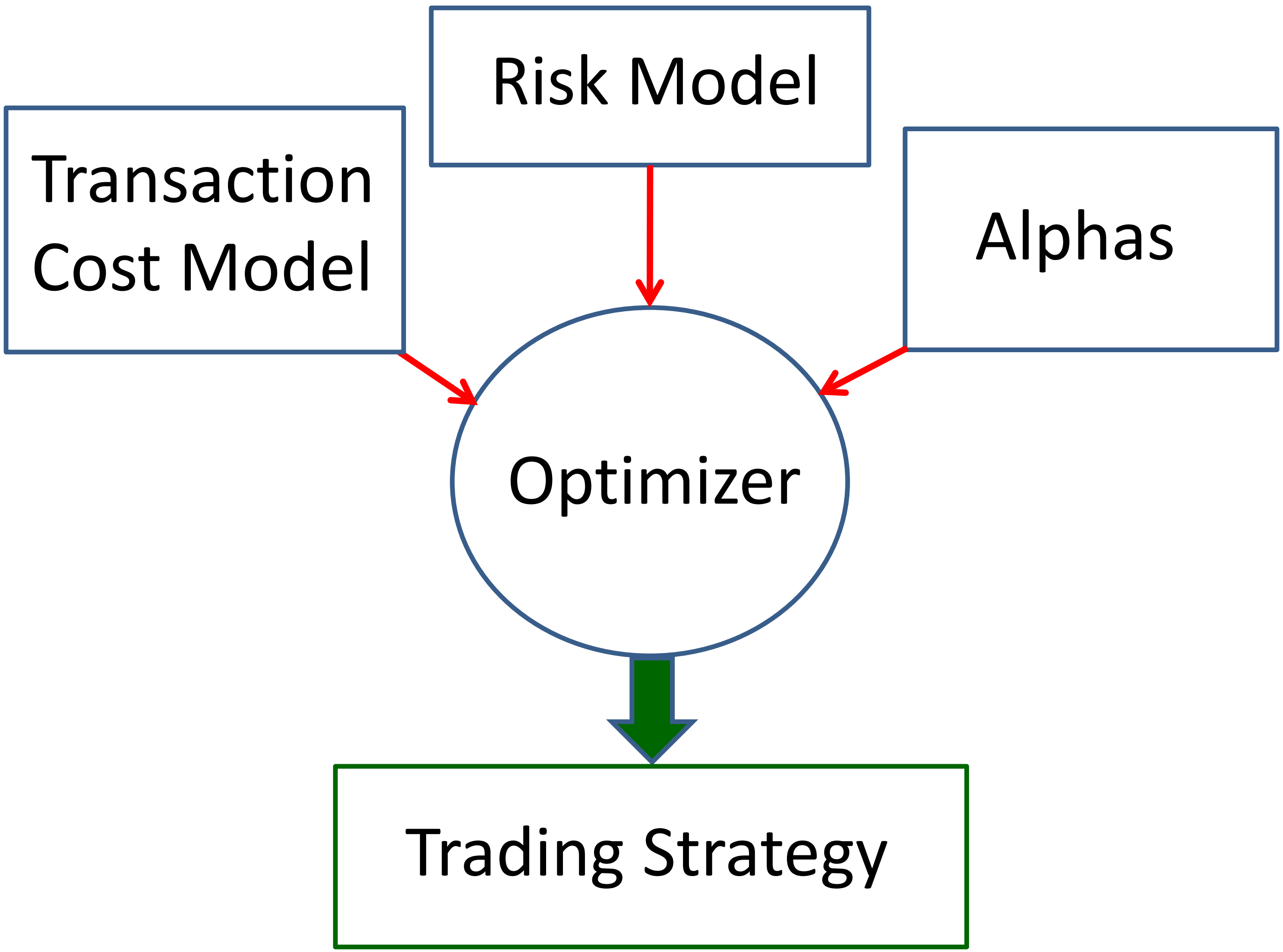
Overall Structure



What We've Seen So Far

- Risk Model: shrinkage estimator of the covariance matrix of stock returns
- Transaction Cost Model: 1bp + ½ bid-ask spread
- Alpha: Weighted blend of various standardized, windsorized alphas

Optimizer

- Inputs:
  - position as of close of business on day *t-1*
  - alphas using data observed up to day *t-1*
  - t-costs using data observed up to day *t-1*
  - risk model using data observed up to day *t-1*
  - constraints using data observed up to day *t-1*
    country exposure, beta exposure, industry exposure
- Output: trade to be executed on day *t*

final position(*t+1*) = final position(*t*) + trade (*t*)

Timeline

- Day *t-1*:   most recent available data

- Day *t*:   trade gets executed

- Day *t+1*: returns start to be earnt

Backtest Code

- Load all necessary data into memory
- Create the alphas
- Start from portfolio with zero dollar invested
- Loop over all days in backtest period
  - Every day: call optimizer to find optimal rebalancing trade given initial position
  - End-of-day position becomes initial position of next day
- Compute P&L

Notation

- x: (n × 1) vector of desired portfolio weights
- w: (n × 1) vector of initial portfolio weights
- Σ: (n × n) covariance matrix of stock returns
- α: (n × 1) vector of aggregate alphas
- β: (n × 1) vector of historical betas
- τ: (n × 1) vector of transaction costs

Objectives and Constraints

- Minimize risk: $x' \Sigma x$
- Maximize exposure to alpha: $\alpha' x$
- Neutralize exposure to beta: $\beta' x = 0$
- Minimize transaction costs: $\tau' |x-w|$
- Other constraints:
  - maximum trade size
  - maximum position size
  - maximum industry and country exposure

Optimization Problem

$$\max_x \ \alpha' x - \lambda \cdot \tau' |x-w| - \mu \cdot x'\Sigma x$$

<u>subject to:</u> $\beta' x = 0$

and other constraints:

- maximum trade size
- maximum position size
- maximum industry exposure
- maximum country exposure

Objective Function

$$\max_x \underbrace{\alpha' x}_{\text{maximize alpha exposure}} - \overbrace{\lambda}^{\text{controls turnover}} \cdot \underbrace{\tau'|x-w|}_{\text{minimize transaction cost}} - \overbrace{\mu}^{\text{controls book size}} \cdot \underbrace{x'\Sigma x}_{\text{minimize portfolio variance}}$$

Maximum Trade Size

- 1% of Average Daily Volume (ADV)
  Can go to 2% if necessary (big book)

- Capped so liquid stocks do not dominate
- Example: cap at $150K
  Can go higher if necessary (big book)

Maximum Position Size

- Multiple of maximum trade size
- I want to be able to liquidate every position in how many days?
- 10 days $\Rightarrow$ 10 × max trade size
- Can be big relative to book size
- Keep balance between liquid and illiquid stocks

min(10 × max trade size, 2.5% of long side of book)

Merge the 2 Constraints

- Max trade size for i^{th} stock: θ_i

$$\Rightarrow \quad w_i - \theta_i \le x_i \le w_i + \theta_i$$

- Max position size for i^{th} stock: π_i

$$\Rightarrow \quad -\pi_i \le x_i \le \pi_i$$

- Enforce both constraints simultaneously:

$$\underbrace{\max(w_i - \theta_i, -\pi_i)}_{\gamma_i} \le x_i \le \underbrace{\min(w_i + \theta_i, \pi_i)}_{\delta_i}$$

$$\gamma_i \le x_i \le \delta_i$$

Industry Constraints

- Sectors are a factor of risk
- Difficult to time sector performance
- Constrain industry exposure
- But not to zero (too much transaction cost)
- For $50 × 50M book size: r* = $300,000 limit

Industry Dummy

- ρ industries
- Boolean matrix R of dimension $(n \times \rho)$
- $R(i,j) = 1$ if i^{th} stock belongs to j^{th} industry
- $R(i,j) = 0$ if i^{th} stock is outside j^{th} industry
- Every row of matrix R has exactly one entry equal to 1; all other entries are equal to 0
- Constraint: $-r^* \cdot \mathbf{1} \leq R'x \leq r^* \cdot \mathbf{1}$
  where **1** = vector of ones of the right dimension

Country Constraints

- Countries are a factor of risk
- Difficult to time country performance
- Constrain country exposure
- But not to zero (too much transaction cost)
- For $50 × 50M book size: f* = $100,000 limit
- Tighter than industry exposure

Country Dummy

- φ countries
- Boolean matrix F of dimension $(n \times \varphi)$
- $F(i,j) = 1$ if i^{th} stock belongs to j^{th} country
- $F(i,j) = 0$ if i^{th} stock does not belong to j^{th} country
- Every row of matrix F has exactly one entry equal to 1; all other entries are equal to 0
- <u>Constraint:</u> $-f^* \cdot \mathbf{1} \leq F'x \leq f^* \cdot \mathbf{1}$

Overall Problem

$$\max_x \ \alpha' x - \lambda \cdot \tau' |x - w| - \mu \cdot x' \Sigma x$$

<u>Subject to:</u>

- beta neutrality: $\beta' x = 0$
- max trade and position: $\gamma \leq x \leq \delta$
- industry constraint: $-r^* \cdot \mathbf{1} \leq R' x \leq r^* \cdot \mathbf{1}$
- country constraint: $-f^* \cdot \mathbf{1} \leq F' x \leq f^* \cdot \mathbf{1}$

Is this standard Quadratic Programming?

Quadratic Programming

- Quadratic programming (QP) is fast, efficient and guaranteed to converge
- Excellent off-the-shelf software
- Matlab optimization toolbox

- <u>Problem: </u>the absolute value in the transaction cost term is not *standard* quadratic programming: $\tau'|x-w|$

Split Variables

- Classic solution: split each variable into 2
- Drawback: twice as many variables
- Advantage: no need to use nonlinear programming
- Define:
  - $y = \max(x-w,0)$
  - $z = \max(w-x,0)$
- Then $y \geq 0$, $z \geq 0$, $x = w + y - z$ and $|x-w| = y+z$

Indeterminacy?

- Initial problem strictly convex
  $\Rightarrow$ unique solution in x
- Twice as many variables:
  solution still unique in y and z?
- Replace y by y+1 and z by z+1
  $\Rightarrow$ x = w + y – z remains unchanged!

- Still OK because |x-w| = y+z penalized

New Formulation

$\max_{y,z}\ \alpha'(w+y-z) - \lambda \cdot \tau'(y+z) - \mu \cdot (w+y-z)'\Sigma(w+y-z)$

<u>Subject to:</u>

- beta neutrality: $\beta'(w+y-z) = 0$
- max trade and position: $\gamma \le w+y-z \le \delta$
- industry constraint: $-r^* \cdot \mathbf{1} \le R'(w+y-z) \le r^* \cdot \mathbf{1}$
- country constraint: $-f^* \cdot \mathbf{1} \le F'(w+y-z) \le f^* \cdot \mathbf{1}$

Very close to standard Quadratic Programming

Standard Quadratic Programming

$$\min_u \; 0.5\, u' H u + g' u$$

Subject to:

- $A u \leq b$
- $C u = d$
- $LB \leq u \leq UB$

Rewrite Optimization Problem

$$\min_{y,z} -\alpha'(y-z) + \lambda \cdot \tau'(y+z) + 2\mu \cdot w'\Sigma(y-z)$$
$$+ \mu \cdot (y-z)'\Sigma(y-z) + \text{constant}$$

<u>Subject to:</u>

- beta neutrality: $\beta'(y-z) = -\beta' w$
- max trade and position: $\gamma - w \le y-z \le \delta - w$
- industries: $-r^* \cdot \mathbf{1} - R'w \le R'(y-z) \le r^* \cdot \mathbf{1} - R'w$
- countries: $-f^* \cdot \mathbf{1} - F'w \le F'(y-z) \le f^* \cdot \mathbf{1} - F'w$

Maps into standard Quadratic Programming

Mapping Objective Function

- u = $\begin{bmatrix} y \\ z \end{bmatrix}$

- H = 2 μ $\begin{pmatrix} \Sigma & -\Sigma \\ -\Sigma & \Sigma \end{pmatrix}$

- g = $\begin{pmatrix} 2\mu\,\Sigma w - \alpha + \lambda\,\tau \\ -\,2\mu\,\Sigma w + \alpha + \lambda\,\tau \end{pmatrix}$

Mapping Inequality Constraints

- $A = \begin{pmatrix} R' & -R' \\ -R' & R' \\ F' & -F' \\ -F' & F' \end{pmatrix}$
- $b = \begin{pmatrix} r^* \cdot \mathbf{1} - R'w \\ r^* \cdot \mathbf{1} + R'w \\ f^* \cdot \mathbf{1} - F'w \\ f^* \cdot \mathbf{1} + F'w \end{pmatrix}$

Mapping Equality Constraints

- $C = \left(\beta' \quad -\beta' \right)$
- $d = -\beta' w$

Bounds on Optimization Variables

- Lower bound:
  LB = vector of zeros of dimension (2n × 1)

- Upper bound:
  $$\text{UB} = \begin{pmatrix} \max(0,\min(\theta,\pi-w)) \\ \max(0,\min(\theta,\pi+w)) \end{pmatrix}$$

Matlab Quadratic Optimizer

- quadprog.m
- No starting point needed
- options = optimset('Algorithm','interior-point-convex')
- options = optimset(options,'Display','iter')

- [u,fval,exitflag,output] = quadprog(H,g,A,b,C,d,LB,UB,[],options)

Other Good Optimizers

- IBM: CPLEX
- FICO: Xpress
- Sunset: XA
- Stanford: QPOPT, SQOPT and MINOS
- Roger Fletcher: BQPD
- KNITRO

- Not cheap!

Required Readings for Next Lecture

1. Cristi A. Gleason and Charles M. C. Lee. Analyst forecast revisions and market price discovery. The Accounting Review, 78(1):pp. 193–225, 2003.
2. Narasimhan Jegadeesh, Joonghyuk Kim, Susan D. Krische, and Charles M. C. Lee. Analyzing the analysts: When do recommendations add value? The Journal of Finance, 59(3):1083–1124, 2004.