Financial Risk Management

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Market VaR I

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THINK IN THE NEXT

Modeling Market VaR

- Parametric Models
  - Delta-Gamma
  - Monte Carlo Simulation
- Non-Parametric Simulation
  - Historical VaR
  - Semi-parametric: time and volatility weighting
- Simulations: Allocation and Confidence Intervals
- We look at 1-day VaR, unless stated otherwise

Model-Building / Parametric Approach

Model-based or Parametric VaR

1. Select a set of market variables or factors that underlie the prices and values of the portfolio
   – E.g. stock indices, interest rates, principal components
2. Assume returns of factors follow certain stochastic processes, i.e. changes in their value in the next day have certain probability distributions
   – E.g. daily stock returns are Normal
3. Estimate parameters for the underlying processes
   – E.g. use GARCH(1,1) to estimate exchange rate volatility
4. Figure out the distribution of daily changes of the portfolio based on the distribution of underlying factors
   – Closed-form or by simulation
5. Find the appropriate percentile of the distribution

Linear-Normal Model Assumptions

- Daily change in the value of a portfolio is linearly related to the daily returns of market variables or factors
- Returns on factors are normally distributed, with mean zero, and a covariance matrix
  - Each factor return, *i*, has variance σ_i
  - Every 2 factor returns, *i* and *j*, have covariance cov_{ij}
- Under these assumptions, returns on the portfolio are also Normal with mean zero.
- To find VaR, we need only find the portfolio Variance

Linear Model / Delta Method

- Define returns on market variables: $\Delta x_i = \dfrac{\Delta S_i}{S_i}$
- And deltas of the portfolio with respect to asset *i*:

$$\delta_i = \frac{\partial P}{\partial S_i}$$

- Then changes in portfolio value are approximated by:

$$\Delta P = \sum_i S_i \delta_i \, \Delta x_i$$

Variance of Portfolio Value

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i$$

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} S_i \delta_i S_j \delta_j \sigma_i \sigma_j$$

$$\sigma_P^2 = \sum_{i=1}^{n} (S_i \delta_i)^2 \sigma_i^2 + 2 \sum_{i<j} \rho_{ij} S_i \delta_i S_j \delta_j \sigma_i \sigma_j$$

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{cov}_{ij}\, S_i \delta_{ii} S_j \delta_j$$

How should we apply this method to:

1. Portfolio of Options
2. Portfolio of Bonds

Delta Method – Example w/options

1. Consider an investment in options on Microsoft and AT&T. Suppose the stock prices are $120 and $30 respectively and the deltas of the portfolio with respect to the two stock prices are 1 and 20 respectively.

- Approximate the change in portfolio value as function of Δx_1 and Δx_2, the returns on the two stocks:

$$\Delta P = 120 \cdot 1 \cdot \Delta x_1 + 30 \cdot 20 \cdot \Delta x_2$$

Delta Method Example - Cont

2. Assume daily return volatility for Microsoft is 2% and that of AT&T is 1%, correlation between the two is 0.3, what is the 5-day 95% VaR?

- The variance of the portfolio is:

$\sigma_P^2 = (120*0.02)^2 + (600*0.01)^2 + 2*120*0.02*600*0.01*0.3 = 50.40$

- The five-day 95% VaR is

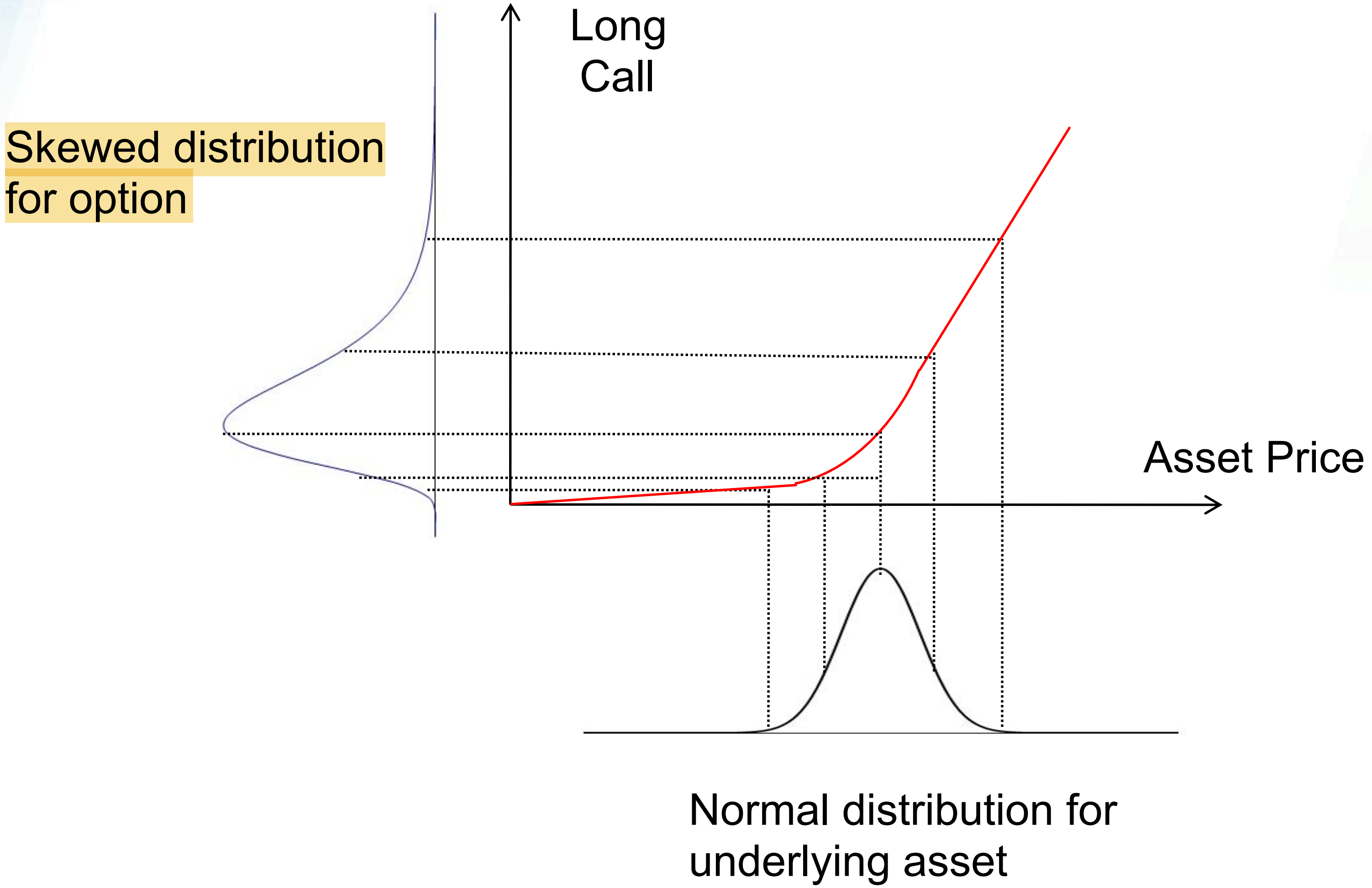
$$VaR = \Phi^{-1}(0.95) \times \sqrt{T} \times \sigma_P = 1.65 \times \sqrt{5} \times \sqrt{50.4} = 26,193$$

Delta – Gamma

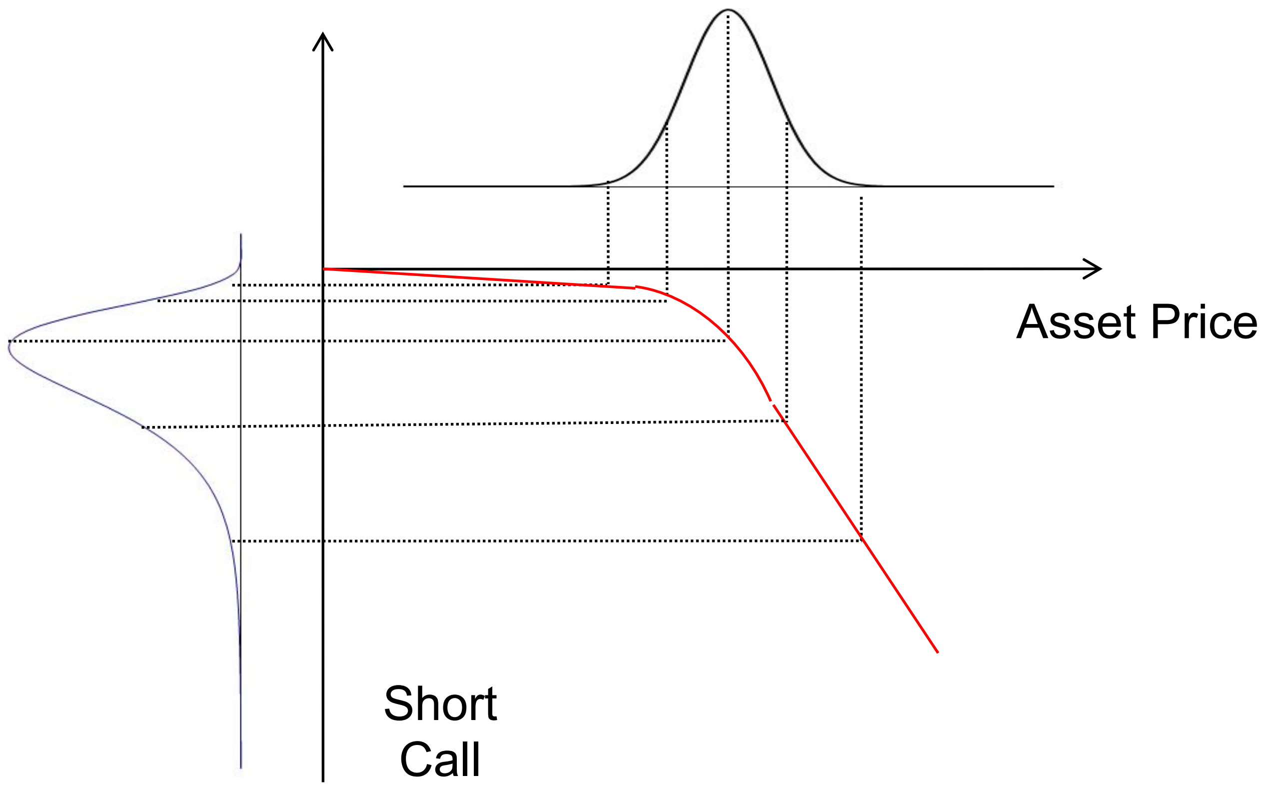
- The linear model will not be accurate because option prices are linear only for small changes in the underlying.
- We can improve our estimation by using gamma as well:

$$\Delta P \approx \delta \cdot \Delta S + \frac{1}{2}\gamma \cdot (\Delta S)^2$$

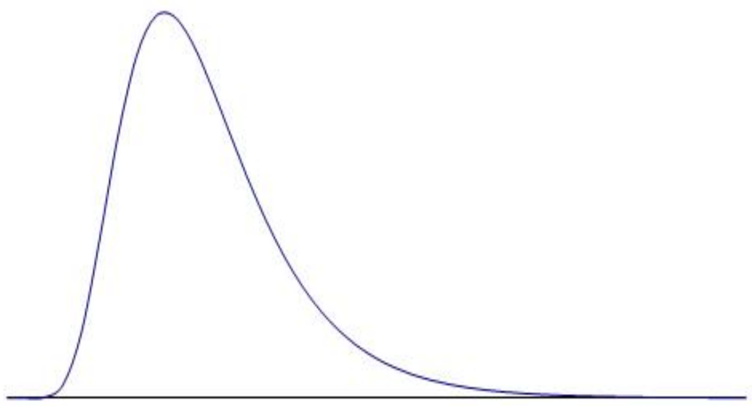
Translation of Asset Price Changes to Price Changes for Long Call



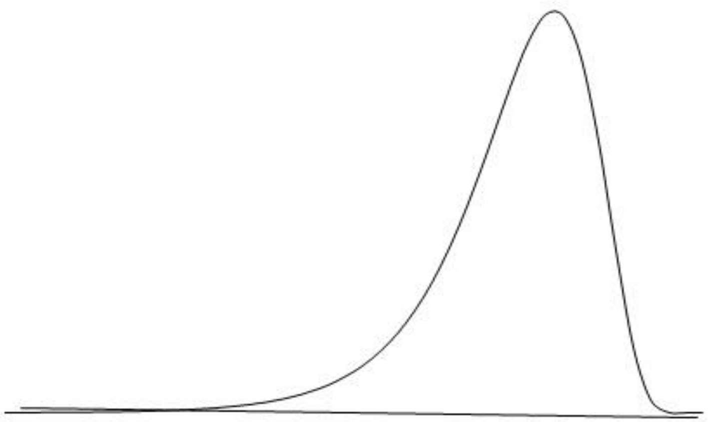
Translation of Asset Price Change to Price Change for Short Call



Impact of Gamma



Positive Gamma



Negative Gamma

Quadratic / Delta-Gamma Model

- For a portfolio dependent on a single asset price it is approximately true that

$$\Delta P = \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

- so that

$$\Delta P = S\delta\,\Delta x + \frac{1}{2}S^2\gamma\,(\Delta x)^2$$

- Recall $\Delta x \sim N(0,\sigma^2)$, hence:

$$\mu_P = E(\Delta P) = 0.5S^2\gamma\sigma^2$$

$$E(\Delta P^2) = S^2\delta^2\sigma^2 + 0.75S^4\gamma^2\sigma^4$$

$$Var(\Delta P) = E(\Delta P^2) - E(\Delta P)^2 = \delta^2 S^2\sigma^2 + \frac{1}{2}\gamma^2(S^2\sigma^2)^2 = \delta^2 Var(\Delta S) + \frac{1}{2}\gamma^2[Var(\Delta S)]^2$$

$$E(\Delta P^3) = 4.5S^4\delta^2\gamma\sigma^4 + 1.875S^6\gamma^3\sigma^6$$

Quadratic Model

- When there are a small number of underlying market variable moments can be calculated analytically from the delta/gamma approximation
- The Cornish – Fisher expansion can then be used to convert moments to quantiles

Quadratic Model – Estimating Quantiles

- Use Moments to find skewness:

$$\xi_P = \frac{1}{\sigma_P^3} E[(\Delta P - \mu_P)^3] = \frac{E[(\Delta P)^3] - 3E[(\Delta P)^2]\mu_P + 2\mu_P^3}{\sigma_P^3}$$

- Cornish – Fisher: The *q* percentile is $\mu_p + w_q \sigma_p$

where $$w_q = z_q + \frac{1}{6}\left(z_q^2 - 1\right)\xi_p$$

Z_q is the relevant quantile of the standard normal. For example, if we're looking for $VaR_{95\%}$ then it will be $N^{-1}(0.05) = -1.645$.

Delta-Gamma Example

- The current value of a stock index is \$1,500, and its daily volatility is 2%
- A portfolio of options on the index has delta=0.5, gamma=0
- Use the delta-gamma method to estimate the mean, variance and skewness of dollar returns of the portfolio.
- What is 1-day VaR-95%?

Delta-Gamma Example

- Mean=0
- Variance=$\delta^2 S^2 \sigma^2$=(0.5)^2*1500^2*(0.02)^2=225
- Skewness=0
- The 5%-ile is: $0-1.645\cdot\sqrt{225}=-25$
- VaR-95%=|-25|=25

Effect of Gamma on Portfolio and VaR

What if Portfolio Gamma = 0.07? Gamma = -0.07?

Gamma	0.07	0	-0.07
E(ΔP)	31.5	0	-31.5
var(ΔP)	2,210	225	2,210
skewness	2.817	0.000	-2.817
w_q	-0.844	-1.645	-2.446
5%-ile	-8	-25	-146

Modeling Bonds in Linear Model

- Duration Approach: Linear relation between ΔP and Δy (allows parallel shifts only)
- Zero Coupon Bonds: Underlying variables are zero-coupon bond returns with many different maturities
- Principal Components Approach: 2 or 3 independent shifts with their own volatilities, capture most of the variance in term-structure moves

Duration Approach

- Recall $\frac{\Delta P}{P} \sim -Duration \cdot \Delta y$
- Therefore: $\Delta P \approx -P \cdot Duration \cdot \Delta y = -Dollar\, Duration \cdot \Delta y$
- Assume yield follows: $\Delta y \sim N\left(\mu_y, \sigma_y^2\right)$
- Then the price will follow: $\Delta P \sim N\left(\mu_P, \sigma_P^2\right)$

s.t. $\mu_P = -DD \cdot \mu_y \qquad \sigma_P = DD \cdot \sigma_y$

$$\boxed{VaR_{95\%} = -\left(\mu_P - 1.645 \times \sigma_P\right) = DD \cdot \left(\mu_y + 1.645 \times \sigma_y\right)}$$

Duration Approach

- Suppose that the volatility of daily changes in interest rates is 0.1% with mean=0
- Our Portfolio is worth \$820 and has duration of 5
- Using the normal-linear approach find the 1-day $VaR_{95\%}$:
  - The dollar duration is 820*5=4,100
    - Portfolio value will drop by \$41 for 1% rise in yield.
  - $VaR_{95\%} = 4100*[0+1.645*0.001] = \$6.74$

Duration Approach Caveats

- The duration approximation is for small changes in yield, VaR might involve large changes, where the approximation fails
  - Include convexity and use delta-gamma.
- The portfolio might be affected by non-parallel shifts in the yield curve.

Zero-coupon Bond Returns as Underlying Variables

- We can choose as market variables zero-coupon bond price changes with standard maturities (for example: 1m, 3m, 6m, 1yr, 2yr, 5yr, 7yr, 10yr, 30yr)
- We need to estimate the covariance matrix of all these bond price returns.
- We need to map the portfolio to each of the maturities.
- Suppose we have *n* maturities:

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}_{ij}\,P_iP_j$$

Bond Portfolio Example

- Consider a portfolio invested $37,397 in 3mm, $331,382 in 6mm and $678,074 in 1-yr bonds
- Rates, vols and correlations for bond prices:

	3-Month	6-Month	1-Year
Zero rate (% with ann. comp.)	5.50	6.00	7.00
Bond price vol (% per day)	0.06	0.10	0.20

	Correlation between daily returns		
	3-Month Bond	6-Month Bond	1-Year Bond
3-month bond	1.0	0.9	0.6
6-month bond	0.9	1.0	0.7
1-year bond	0.6	0.7	1.0

Example – cont.

Portfolio Variance =

$37{,}397^2*(0.06\%)^2+331{,}382^2*(0.10\%)^2+678{,}074^2*(0.20\%)^2+2*37{,}397*331{,}382*(0.06\%)*(0.10\%)*0.9+2*331{,}382*678{,}074*(0.10\%)*(0.20\%)*0.7+2*37{,}397*678{,}074*(0.06\%)*(0.20\%)*0.6 = 2{,}628{,}518 = 1{,}621.3^2$

10-day VaR-99% =

$1621.3*2.33*\text{sqrt}(10)=\$11{,}946$

Zero-Coupon Bond Return Disadvantages

- Requires many underlying variables
- We need to map bond cash flows that arrive at times different than our underlying bonds.

Using PCA to Calculate VaR

- We can use 2 or 3 PCAs as underlying factors.
- It requires:
  - Portfolio sensitivities to those factors
  - Volatilities of the factors
- We estimate less deltas and don't need covariance matrix as PCs are orthogonal

Table 8.8 Standard Deviation of Factor Scores

PC1	PC2	PC3	PC4	PC5	PC6	PC7	PC8
17.55	4.77	2.08	1.29	0.91	0.73	0.56	0.53

PCA Example

Suppose a portfolio has the following sensitivities to 1-basis-point rate moves, in $ millions:

3-Year Rate	4-Year Rate	5-Year Rate	7-Year Rate	10-Year Rate
+10	+4	−8	−7	+2

What are the portfolio sensitivities to PC1 and PC2?

PC1: $10 \times 0.372 + 4 \times 0.392 - 8 \times 0.404 - 7 \times 0.394 + 2 \times 0.376 = -0.05$

PC2: $10 \times (-0.267) + 4 \times (-0.110) - 8 \times 0.019 - 7 \times 0.194 + 2 \times 0.371 = -3.87$

PCA Example – Cont.

- We get:

$$\Delta P = -0.05 f_1 - 3.87 f_2$$

where f_1 is the first factor and f_2 is the second factor

- If the SD of the factor scores are 17.55 and 4.77 the SD of ΔP is

$$\sqrt{0.05^2 \times 17.55^2 + 3.87^2 \times 4.77^2} = 18.48$$

Thanks