Financial Risk Management

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Credit VaR – Normal Copula Factor Model

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THINK IN THE NEXT

Credit VaR Agenda

- VaR for Independent Loans
- Simulating Dependent Loans
- Copula Factor Model
  - Closed-form solution
  - Extensions through simulation
  - Application to capital requirements

Credit VaR for Uncorrelated Loans

- A bank has 100 loans of $15 million each. The probability of default (PD) of each loan is 2.5%. In case of default there is no recovery. Loan defaults are independent of each other.
- What is the Expected Loss (EL) on the portfolio?
  - EL = 100*15*2.5% = $37.5

Credit VaR for Uncorrelated Loans (Example – Cont.)

- What is the VaR 96% of the portfolio?
  - The probability of K loans defaulting is given by the Binomial Distribution
  - P(K loans default) = BINOMDIST(K, 100,2.5%,0)
  - P(# defaults $\leq K$) = BINOMDIST(K, 100,2.5%,1)
- Excel gives us the following values:
- There is 96% that 5 loans or less will default.

Num Defaults	Cumulative Prob
0	0.08
1	0.28
2	0.54
3	0.76
4	0.89
5	**0.96**
6	0.99

Credit VaR for Uncorrelated Loans (Example – Cont.)

- VaR-96% is therefore: 5*15 = $75M, or 5% of the portfolio value.
- 5% is called Worst Case Default Rate (WCDR)
- The Unexpected Loss VaR 96% is equal to 75-37.5 = 37.5 or 2.5% of the portfolio value.

Binomial to Normal

- The Central Limit Theorem tells us that at the limit the Binomial will tend to Normal.
- Assume a portfolio of *N* equal size loans, with total value of *V*.

$$Loss = \sum_{i=1}^{N} \frac{V}{N} D_i, \qquad D_i = \begin{cases} 0 & 1-p \\ 1 & p \end{cases}$$

$$\mu = E\left[Loss\right] = \frac{V}{N} \cdot Np = Vp$$

$$\sigma^2 = Var\left[Loss\right] = \left(\frac{V}{N}\right)^2 Np\left(1-p\right) = \frac{V^2}{N} p\left(1-p\right)$$

$$\sigma = SD\left[Loss\right] = \frac{V}{\sqrt{N}} \sqrt{p\left(1-p\right)} = \mu\sqrt{\frac{1-p}{Np}}$$

Binomial to Normal - Example

- In our example:

$$\mu = Vp = 1500 \cdot 0.025 = 37.5$$

$$\sigma = \frac{V}{\sqrt{N}} \sqrt{p(1-p)} = \frac{1500}{10} \cdot \sqrt{0.025 \cdot (1-0.025)} = 23.4$$

- What is the 96 percentile?

$$Loss_{96\%} = \mu + \sigma \cdot \Phi^{-1}(0.96) = 78.5$$

- UL $VaR_{96\%}$ is 78.5 – 37.5 = 41

Binomial to Normal - Example

- What would happen if we had the same size portfolio with 1000 loans?

$$\mu = Vp = 1500 \cdot 0.025 = 37.5$$

$$\sigma = \frac{V}{\sqrt{N}}\sqrt{p(1-p)} = \frac{1500}{\sqrt{100}} \cdot \sqrt{0.025 \cdot (1-0.025)} = 7.4$$

- What is the 96 percentile?

$$Loss_{96\%} = \mu + \sigma \cdot \Phi^{-1}(0.96) = 50.5$$

- UL $VaR_{96\%}$ is 50.5 – 37.5 = 13

Limit of Independent Case

$$\mu = E[Loss] = \frac{V}{N} \cdot Np = Vp$$

$$Var[Loss] = \left(\frac{V}{N}\right)^2 Np(1-p) = \frac{V^2}{N} p(1-p)$$

$$VaR_{96\%} = \mu \cdot \left[1 + \Phi^{-1}(0.96) \cdot \sqrt{\frac{1-p}{Np}}\right]$$

- As N increases, the variance of the loss decreases, and eventually we are almost guaranteed Loss=*Vp*.
- The VaR tends to the mean, and the Unexpected Loss VaR goes to zero.

Simulating Defaults

- First, we look at one loan:
- To simulate a loss on one loan with probability of default = PD, we can sample from a uniform, $V_i \sim U[0,1]$, and count as default if $V_i < PD$
- We can alternatively sample from a Normal distribution, $U_i \sim N(0,1)$ and count as default if $U_i < N^{-1}(PD)$

Simulating VaR for Independent Loans

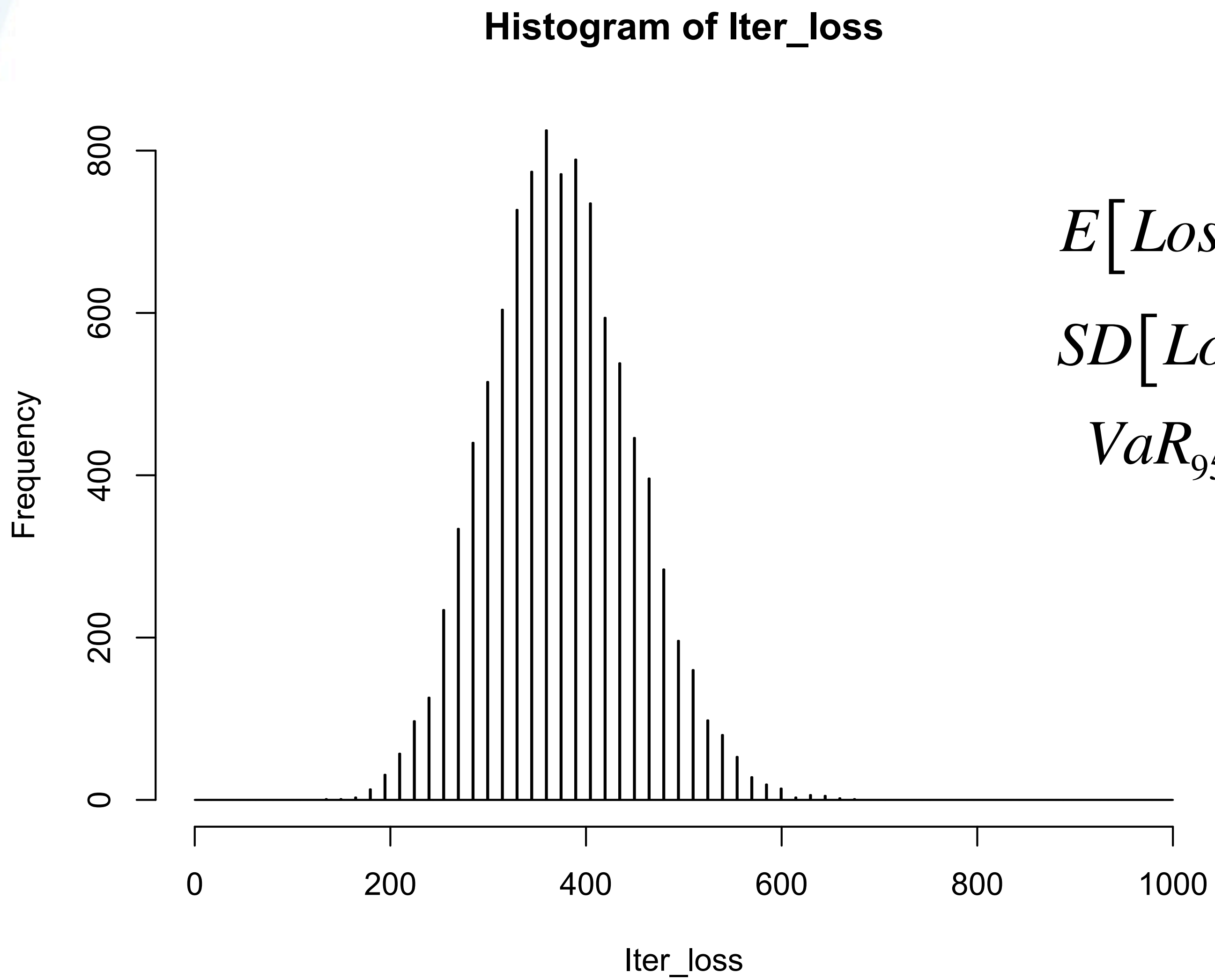
```
Num=1000 #Number of loans
Size=15 #Dollar size of each loan
PD=0.025 #PD for one loan
alpha=0.95 #VaR alpha

N=10000 #Number of iterations
Iter_loss = array (0, dim=c(N)) #Distribution of losses per iteration

for (iter in 1:N) {
        U = matrix(rnorm(Num,mean=0,sd=1), 1, Num) #Generate U_i
        Default = (U<qnorm(PD)) #Every loan, every iteration, did it default
        loan_loss = Default*Size #Total loss on each loan for this iteration (assuming LGD is 
100%)
        Iter_loss[iter] = sum(loan_loss) #Total loss for this iteration
}

hist(Iter_loss) #Histogram of losses over iterations
EL = mean(Iter_loss) #Expected loss
VaR = quantile(Iter_loss, alpha) #VaR
```

Independent Loans



$$E[Loss] = 375$$

$$SD[Loss] = 74$$

$$VaR_{95\%} = 497$$

Normal Copula Factor Model

- We can generate a set of *N* correlated variables with standard normal distribution using a Factor Model.
- We generate *N+1* independent standard normal variables: the common factor F, and *N* idiosyncratic components Z_i
- Generate new variables, U_i as:

$$U_i = a_i F + \sqrt{1 - a_i^2}\, Z_i$$

- They have standard normal distributions and correlation between U_i and U_j is $a_i a_j$ – show this.

Normal Copula Factor Model

- We consider a case where all $a_i = \sqrt{\rho}$

$$U_i = \sqrt{\rho}F + \sqrt{1-\rho}Z_i$$

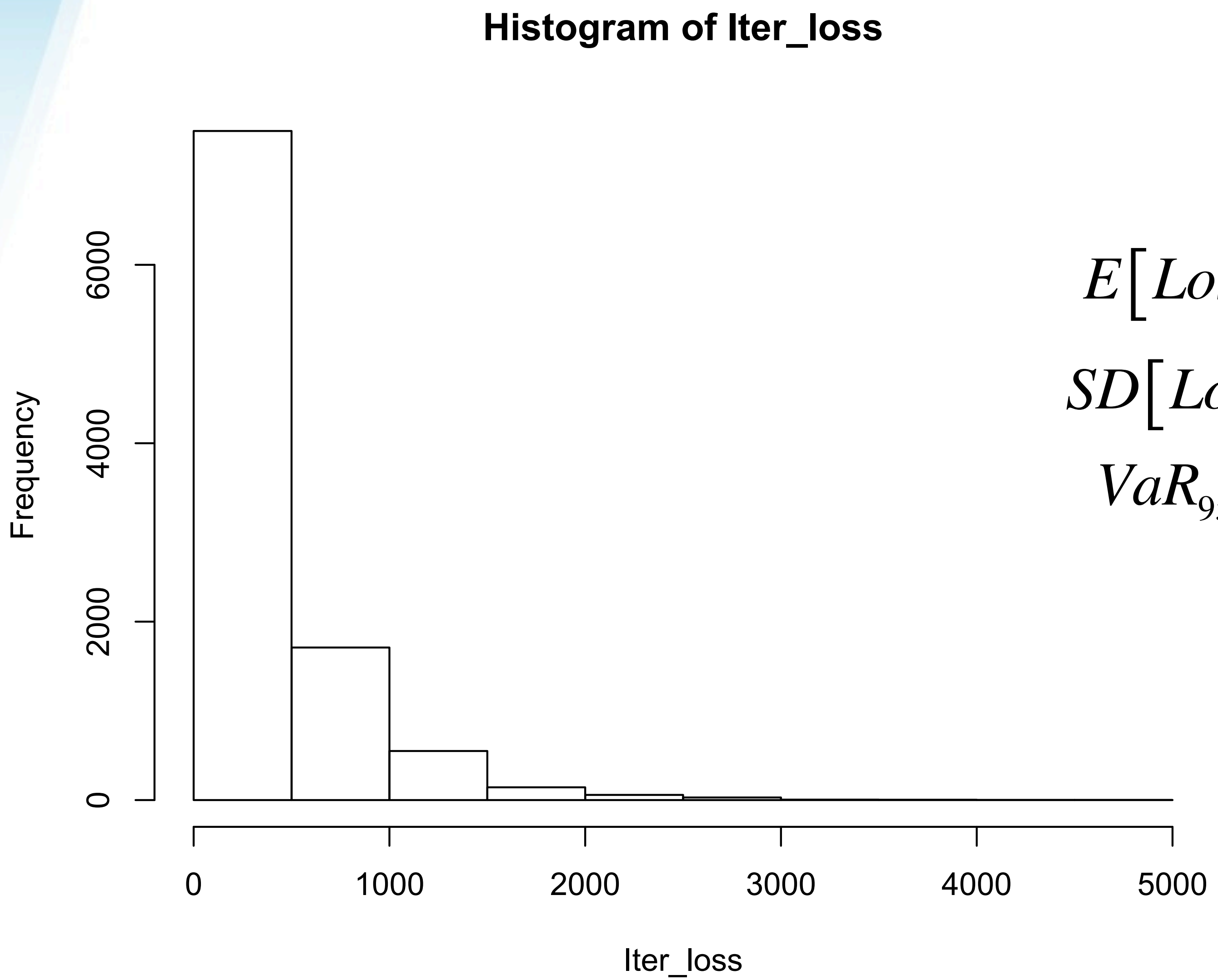
- U_i is also distributed normally: $U_i \sim N(0,1)$
- The correlation between every two latent variables (U_i) is ρ
- Count as default if

$$U_i = \sqrt{\rho}F + \sqrt{1-\rho}Z_i < N^{-1}(PD)$$

Simulating VaR for Correlated Loans

```
Num=1000 #Number of loans
Size=15 #Dollar size of each loan
PD=0.025 #PD for one loan
rho = 0.15 #Correlation between latent variables
alpha=0.95 #VaR alpha
N=10000 #Number of iterations
Iter_loss = array (0, dim=c(N)) #Distribution of losses per iteration
for (iter in 1:N) {
        F = matrix(rnorm(1),1,Num) #One F Factor value per iteration
        Z = matrix(rnorm(Num,mean=0,sd=1), 1, Num) #idiosyncratic errors for all loans
        U = sqrt(rho)*F + sqrt(1-rho)*Z #Generate the U_i for all loans
        Default = (U<qnorm(PD)) #Every loan, every iteration, did it default
        loan_loss = Default*Size #Total loss on each loan for this iteration (assuming LGD is 100%)
        Iter_loss[iter] = sum(loan_loss) #Total loss for this iteration
}
hist(Iter_loss) #Histogram of losses over iterations
EL = mean(Iter_loss) #Expected loss
VaR = quantile(Iter_loss, alpha) #VaR
```

Correlated Loans



$$E\left[Loss\right] = 375$$

$$SD\left[Loss\right] = 394$$

$$VaR_{95\%} = 1140$$

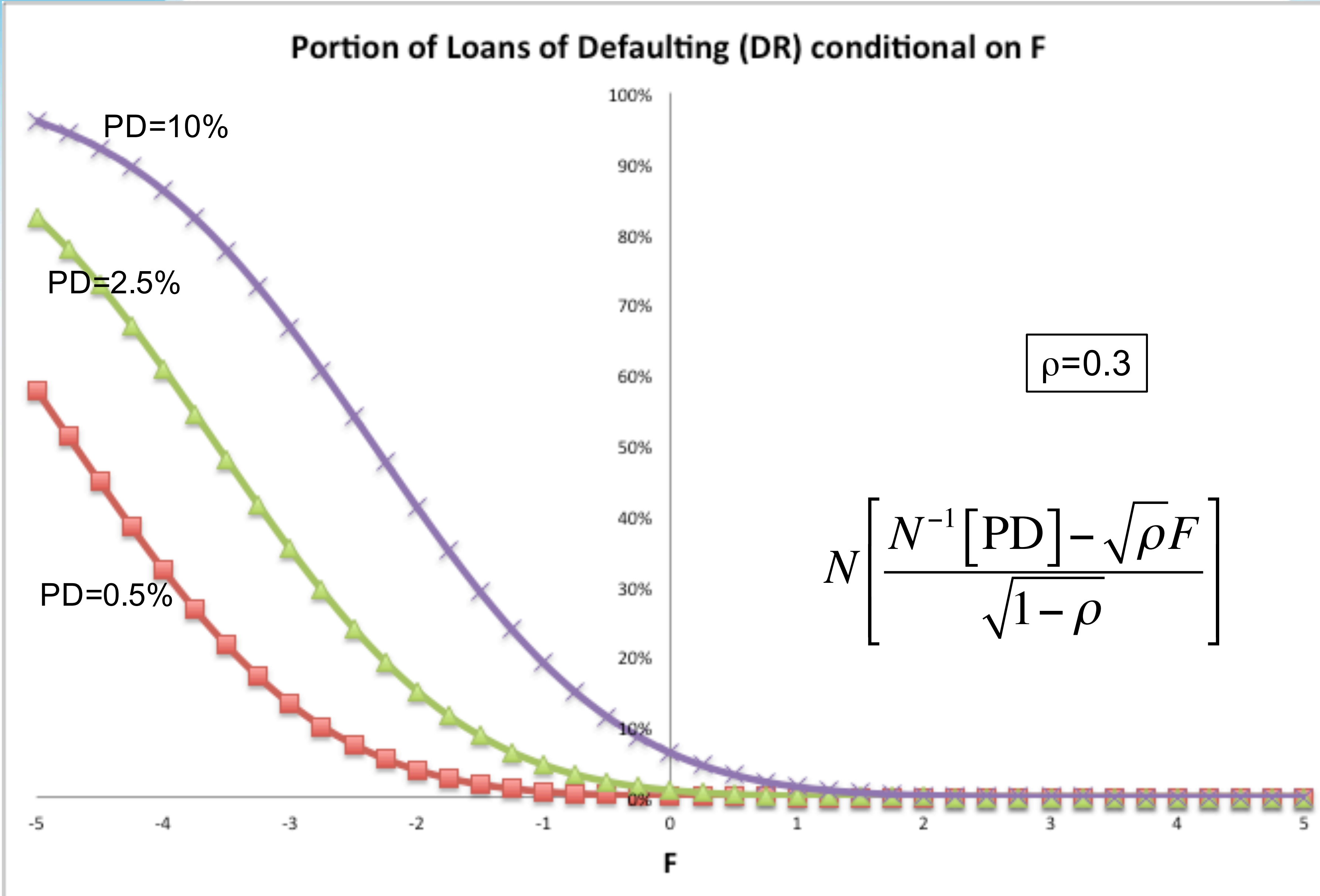
Closed Form Solution

- If we assume all loans are of same size, L, with same PD and same LGD we can reach a closed form solution
- For a given F, rewrite the condition of default as:

$$Z_i < \frac{N^{-1}(PD) - \sqrt{\rho}F}{\sqrt{1-\rho}}$$

- Given that Z_i is standard normal, the portion of loans that default conditional on F is:

$$N\left[\frac{N^{-1}\left[\mathrm{PD}\right] - \sqrt{\rho}F}{\sqrt{1-\rho}}\right]$$

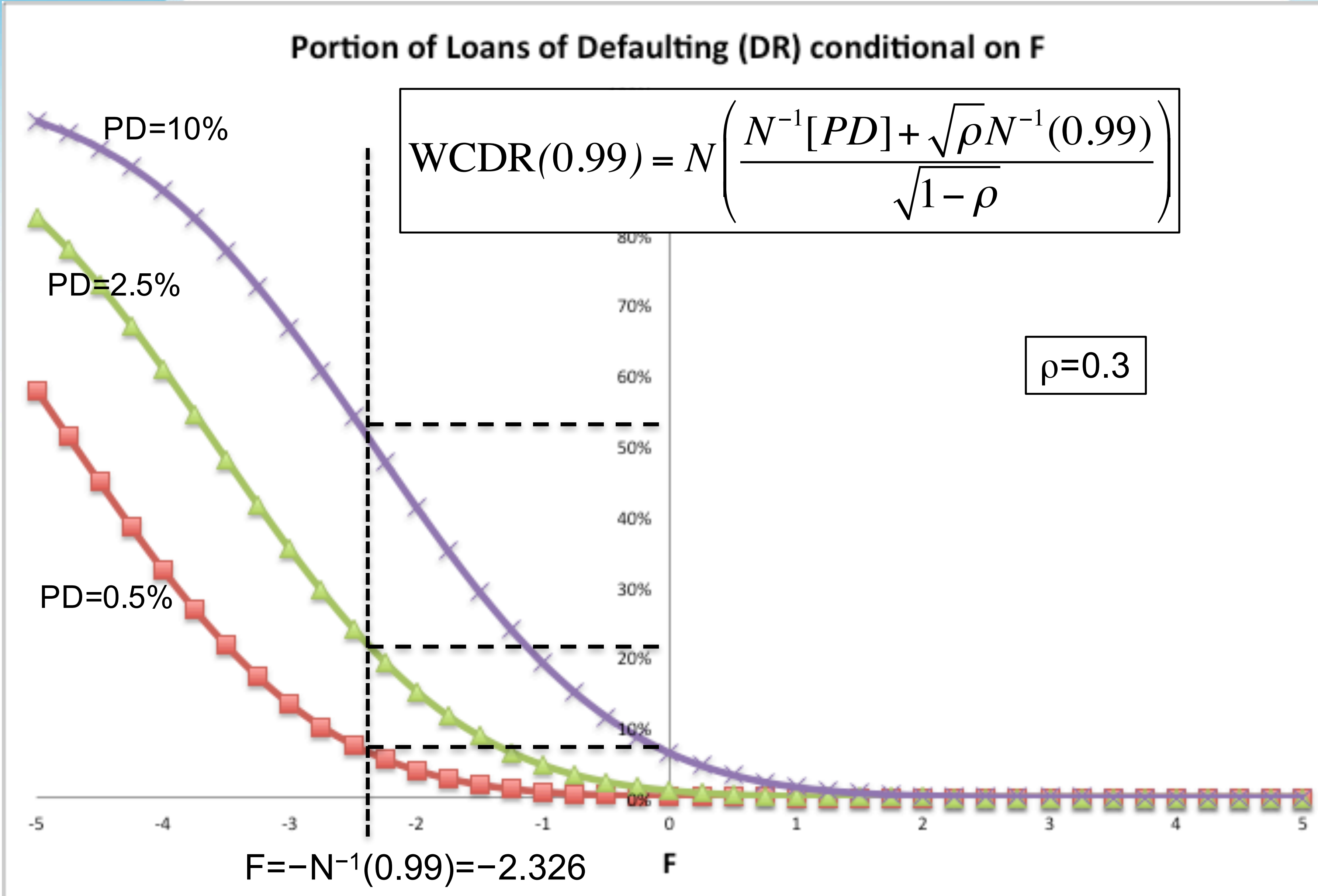


$$N\left[\frac{N^{-1}\left[\mathrm{PD}\right]-\sqrt{\rho}F}{\sqrt{1-\rho}}\right]$$

Closed Form Solution cont.

- F is the state variable (e.g. condition of the economy), when it's high the probability of default is low.
- The *X*% worst case is when *F* is $N^{-1}(1-X)=-N^{-1}(X)$
- For example, 99% worst case is when F=$N^{-1}(0.01)=-N^{-1}(0.99)$
- The worst case default rate with a confidence level of *X* is therefore:

$$\mathrm{WCDR}(X) = N\left(\frac{N^{-1}[PD]+\sqrt{\rho}N^{-1}(X)}{\sqrt{1-\rho}}\right)$$



Portion of Loans of Defaulting (DR) conditional on F
$$\mathrm{WCDR}(0.99) = N\left(\frac{N^{-1}[PD] + \sqrt{\rho}N^{-1}(0.99)}{\sqrt{1-\rho}}\right)$$
PD=10%
PD=2.5%
PD=0.5%
ρ=0.3
80%
70%
60%
50%
40%
30%
20%
10%
0%
-5
-4
-3
-2
-1
0
1
2
3
4
5
F
F=−N⁻¹(0.99)=−2.326

Credit VaR Formula

- The Unexpected Loss (UL) Credit VaR in Dollars is:

$$Credit\mathrm{VaR}(X) = L \times \mathrm{LGD} \times \left[\mathrm{WCDR}(X) - PD\right]$$

where *L* is loan principal and LGD is loss given default, *WCDR* based on previous formula.

Credit VaR Example

- A bank has a total of $100 million of loans, each exposure is small in relation to the total portfolio. The one-year probability of default (PD) for each loan is 2% and the loss given default (LGD) for each loan is 40%. The copula correlation parameter ρ is 0.1.
  - What is Worst Case Default Rate (WCDR) at 99.9%?
  - What is Unexpected Loss $VaR_{99\%}$?

Credit VaR Example

$$\mathrm{WCDR}(0.999) = N\left(\frac{N^{-1}\left(0.02\right) + \sqrt{0.1}N^{-1}(0.999)}{\sqrt{1-0.1}}\right) = 0.128$$

$$\mathit{Credit}\mathrm{VaR}(0.999) = 100 \times 0.4 \times \left[12.8\% - 2\%\right] = 4.32$$

Gordy's Result

- In a large portfolio of *M* loans where each loan is small in relation to the size of the portfolio it is approximately true that

$$Credit\text{VaR}(X) = \sum_{i=1}^{M} L_i \times \text{LGD}_i \times \left[\text{WCDR}_i(X) - PD_i \right]$$

- Note that: loan size, probability of default and loss given default can vary between loans.

RBS Asset Protection Scheme

In 2009, Royal Bank of Scotland (RBS) was bailed out by the UK government using an Asset Protection Scheme (APS).

$325B of the Bank's assets (i.e. loans and bonds) were placed in the scheme. RBS would be liable for the first $19.5B of losses on the portfolio, and the government would be liable for the rest. Assume every asset is a small part of the portfolio.

Suppose the Probability of Default (PD) of each asset is 1% and the Loss Given Default (LGD) is 100%. The copula correlation is 0.4. What is the probability that the government will have to pay anything?

RBS Asset Protection Scheme

Call *x* the probability of losing \$19.5B or less on the portfolio. We are looking for *1-x*:

$$19.5 = L * LGD * N\left[\frac{N^{-1}(PD) + \sqrt{\rho}N^{-1}(x)}{\sqrt{1-\rho}}\right]$$

$$19.5 = 325 * 100\% * N\left[\frac{N^{-1}(1\%) + \sqrt{0.4}N^{-1}(x)}{\sqrt{0.6}}\right]$$

$$x = N\left[\frac{\sqrt{0.6}N^{-1}(0.06) - N^{-1}(0.01)}{\sqrt{0.4}}\right] = 96.2\%$$

$$1 - x = 3.8\%$$

RBS Asset Protection Scheme

What if the loan defaults were independent of each other, and there were 1000 loans?

$$Loss \sim N\left[Vp, \frac{V^2}{N} p(1-p)\right]$$

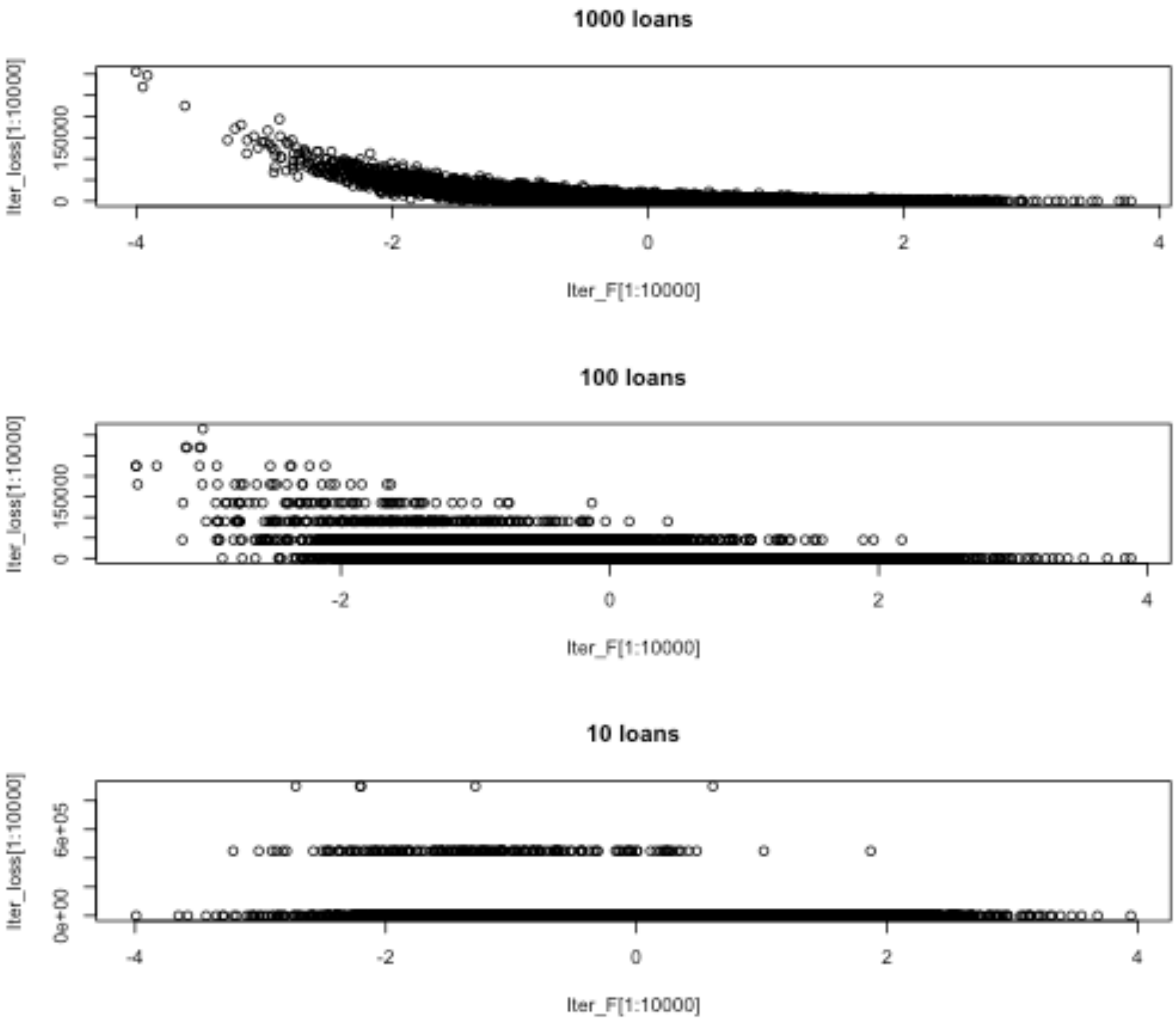
$$\mu = 3.25, \sigma^2 = 1.04$$

$$P[Loss > 19.5] = P\left[z > \frac{19.5 - 3.25}{\sqrt{1.04}}\right] = 0$$

If the loans were independent there would be almost no chance of the government paying out!

What does simulation get us?

- Using simulation we can address specific features of the portfolio
  - Size Concentration
  - Industry/Sector Concentration
- Mark to Market VaR vs. Default VaR
- LGD Simulation
- Expected Shortfall


1000 loans
Iter_loss[1:10000]
0
150000
-4
-2
0
2
4
Iter_F[1:10000]
100 loans
Iter_loss[1:10000]
0
150000
-2
0
2
4
Iter_F[1:10000]
10 loans
Iter_loss[1:10000]
0e+00
6e+05
-4
-2
0
2
4
Iter_F[1:10000]

Industry/Sector

- Use multiple factors instead of one
- For example, 2 factors will lead to every loan variable being generated as:

$$U_i = w_1 F_1 + w_2 F_2 + w_z Z_i$$

- Common choice for factors are equity indices
  - Credit Metrics uses MSCI indexes
- F_1 and F_2 are correlated Normal variables, with mean zero. Z_i are independent of Fs and other Z_i.

Sector Concentration - Example

- Estimate the correlation matrix for the factors.

Index	**Volatility**	**Correlations**		
		U.S. Chemicals	Germany Insurance	Germany Banking
U.S.: Chemicals	2.03%	1.00	0.16	0.08
Germany: Insurance	2.09%	0.16	1.00	0.34
Germany: Banking	1.25%	0.08	0.34	1.00

- Regress equity returns on the factors to get betas

$$R_{XYZ} = \alpha + \beta_1 R_{F_1} + \beta_2 R_{F_2} + \varepsilon_{XYZ}$$

- σ^2_{XYZ} is the total variance, R^2_{XYZ} out of it is due to the factors, while $1\text{-}R^2_{XYZ}$ is due to idiosyncratic risk.

Sector Concentration - Example

- Simulate factors as correlated mean-zero Normals and simulate idiosyncratic as uncorrelated

$$U_{XYZ} = \frac{\beta_1}{\sigma_{XYZ}} F_1 + \frac{\beta_2}{\sigma_{XYZ}} F_2 + \sqrt{1 - R_{XYZ}^2} \cdot Z_{XYZ}$$

- F_1, F_2 for example are indexes: *Germany:Banking* and *Germany:Insurance*
- Coefficients are set so variance due to Factors is R^2_{XYZ}. Leading to U_{XYZ} being standard normal.
- As before, count as default if $U_i < N^{-1}(PD)$

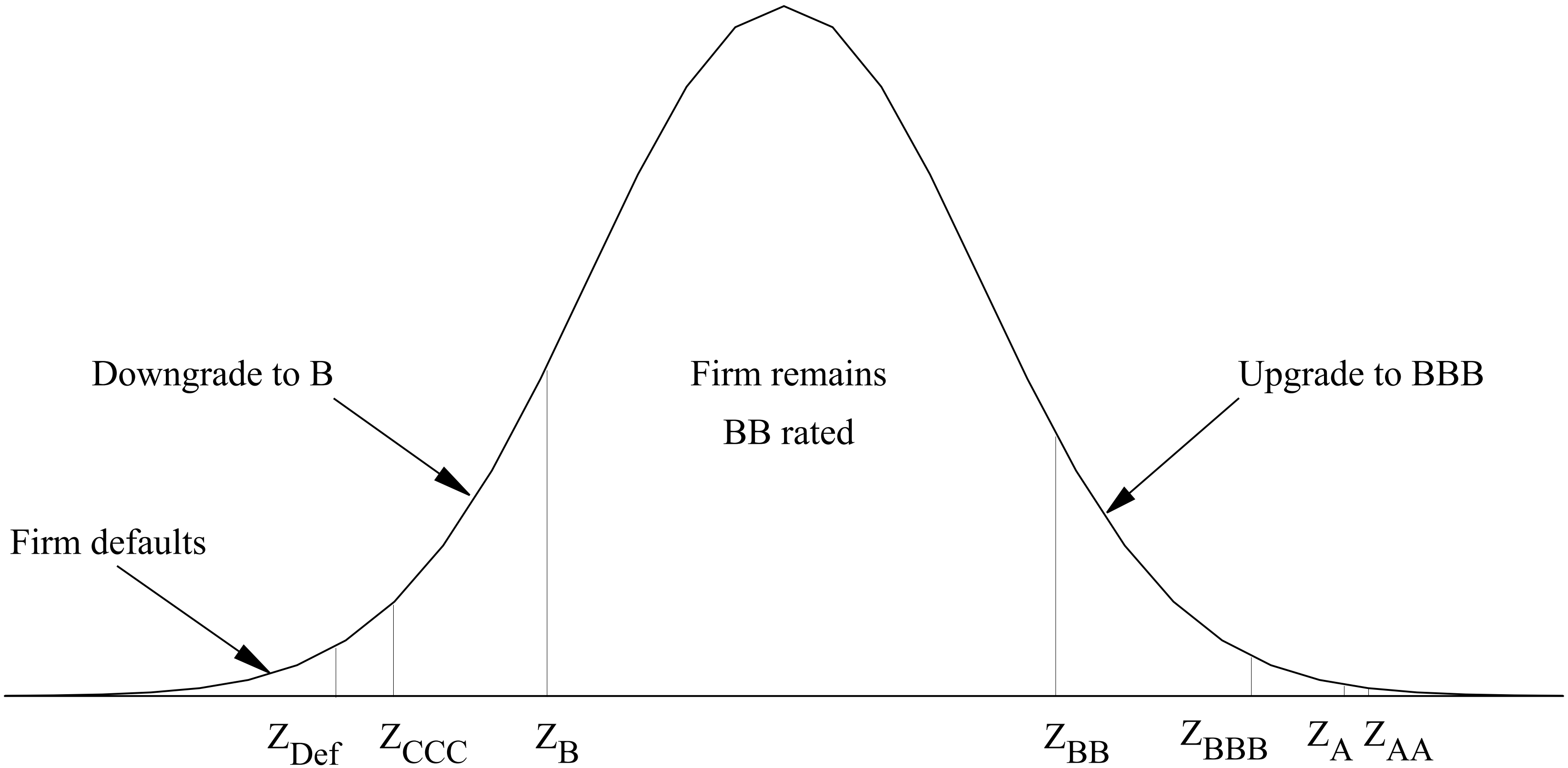
Mark-to-Market VaR

- So far, we modeled losses due to default only.
- At the end of the period some credits may be downgraded or upgraded. This will affect their price.
- Bondholders may be concerned with the MTM of the bond at the end of the period, rather than loss to default.
- In fact, loan holders too are interested in changes in the ratings, because the model is quantifying one period losses, whereas the loan maturities are longer.

One-Year Rating Transition Matrix (% probability, Moody's 1970-2010)

Initial	Rating at year end								
Rating	Aaa	Aa	A	Baa	Ba	B	Caa	Ca-C	Default
Aaa	90.42	8.92	0.62	0.01	0.03	0.00	0.00	0.00	0.00
Aa	1.02	90.12	8.38	0.38	0.05	0.02	0.01	0.00	0.02
A	0.06	2.82	90.88	5.52	0.51	0.11	0.03	0.01	0.06
Baa	0.05	0.19	4.79	89.41	4.35	0.82	0.18	0.02	0.19
Ba	0.01	0.06	0.41	6.22	83.43	7.97	0.59	0.09	1.22
B	0.01	0.04	0.14	0.38	5.32	82.19	6.45	0.74	4.73
Caa	0.00	0.02	0.02	0.16	0.53	9.41	68.43	4.67	16.76
Ca-C	0.00	0.00	0.00	0.00	0.39	2.85	10.66	43.54	42.56
Default	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Simulating Transitions



CreditMetrics Technical Document 1997

Credit Metrics MTM VaR

- Simulate for every loan: $U_i = \sqrt{\rho}F + \sqrt{1-\rho}Z_i$
- Determine the loan's new ratings based on the standard normal cut-off points
- Re-price the loan with the credit spreads for the new rating
  - Forwards for interest rates and credit spreads have to be used
- Sum up all loan values for one possible portfolio MTM at the end of period.

MTM Example for One Bond

- Consider a 5-year, A-rated bond, paying 6% coupon.
- Simulate U_i a standard Normal, and determine the new ratings based on the transition matrix:

End of Year Rating	Probability (%)	Cum Prob (%)	lower bound	upper bound
Aaa	0.06	100	3.239	∞
Aa	2.82	99.94	1.899	3.239
A	90.88	97.12	-1.535	1.899
Baa	5.52	6.24	-2.447	-1.535
Ba	0.51	0.72	-2.863	-2.447
B	0.11	0.21	-3.090	-2.863
Caa	0.03	0.1	-3.195	-3.090
Ca-C	0.01	0.07	-3.239	-3.195
D	0.06	0.06	-∞	-3.239

MTM Example Cont.

- Use the forward curves for each rating to revalue the bond:

Example one-year forward zero curves by credit rating category (%)

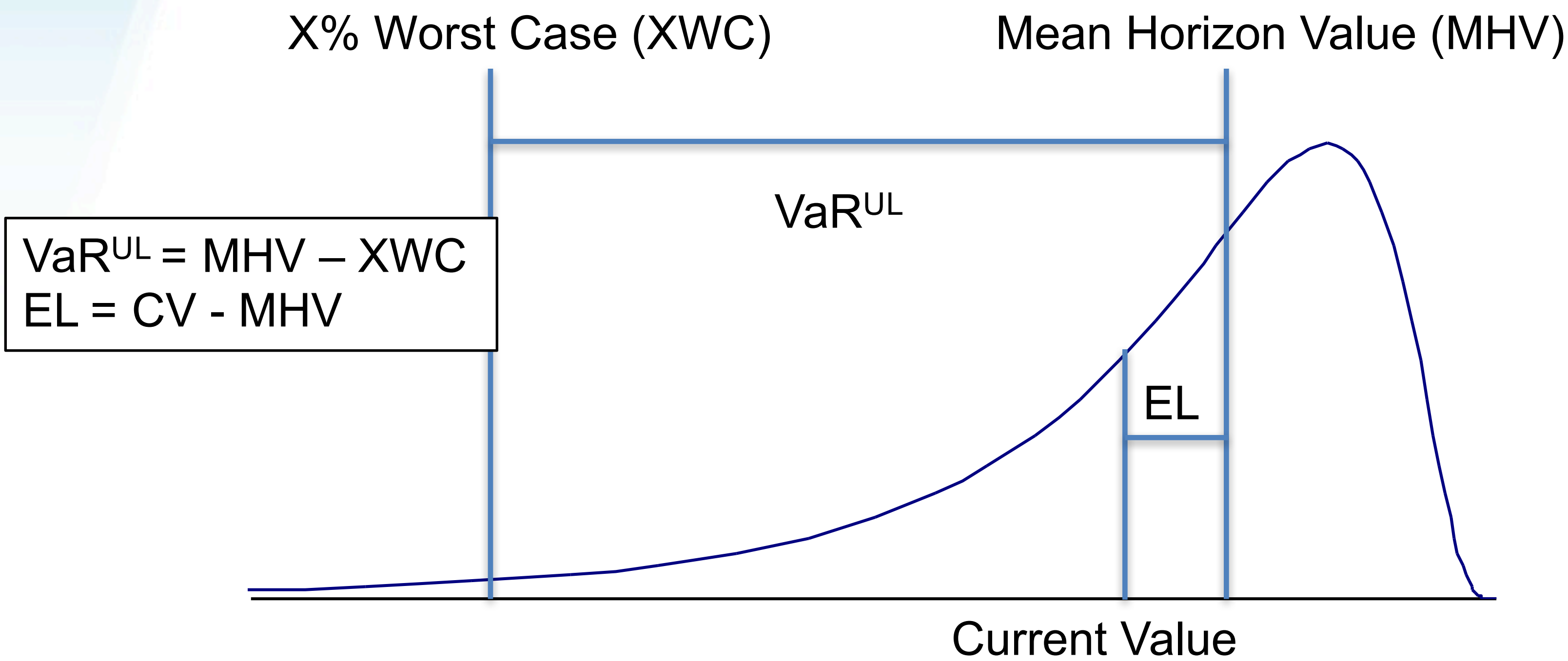
Category	Year 1	Year 2	Year 3	Year 4
AAA	3.60	4.17	4.73	5.12
AA	3.65	4.22	4.78	5.17
A	3.72	4.32	4.93	5.32
BBB	4.10	4.67	5.25	5.63
BB	5.55	6.02	6.78	7.27
B	6.05	7.02	8.03	8.52
CCC	15.05	15.02	14.03	13.52

- For example, if it stays A

$$Value = 6 + \frac{6}{1+3.72\%} + \frac{6}{(1+4.32\%)^2} + \frac{6}{(1+4.93\%)^3} \frac{106}{(1+5.32\%)^4}$$

Year-end rating	Value ($)
AAA	109.37
AA	109.19
A	108.66
BBB	107.55
BB	102.02
B	98.10
CCC	83.64
Default	51.13

MTM VaR (cont)



Sum all bond values to create an end of period distribution for portfolio value.

Credit Metrics Incorporating LGD

Recovery statistics by seniority class
Par (face value) is $100.00.

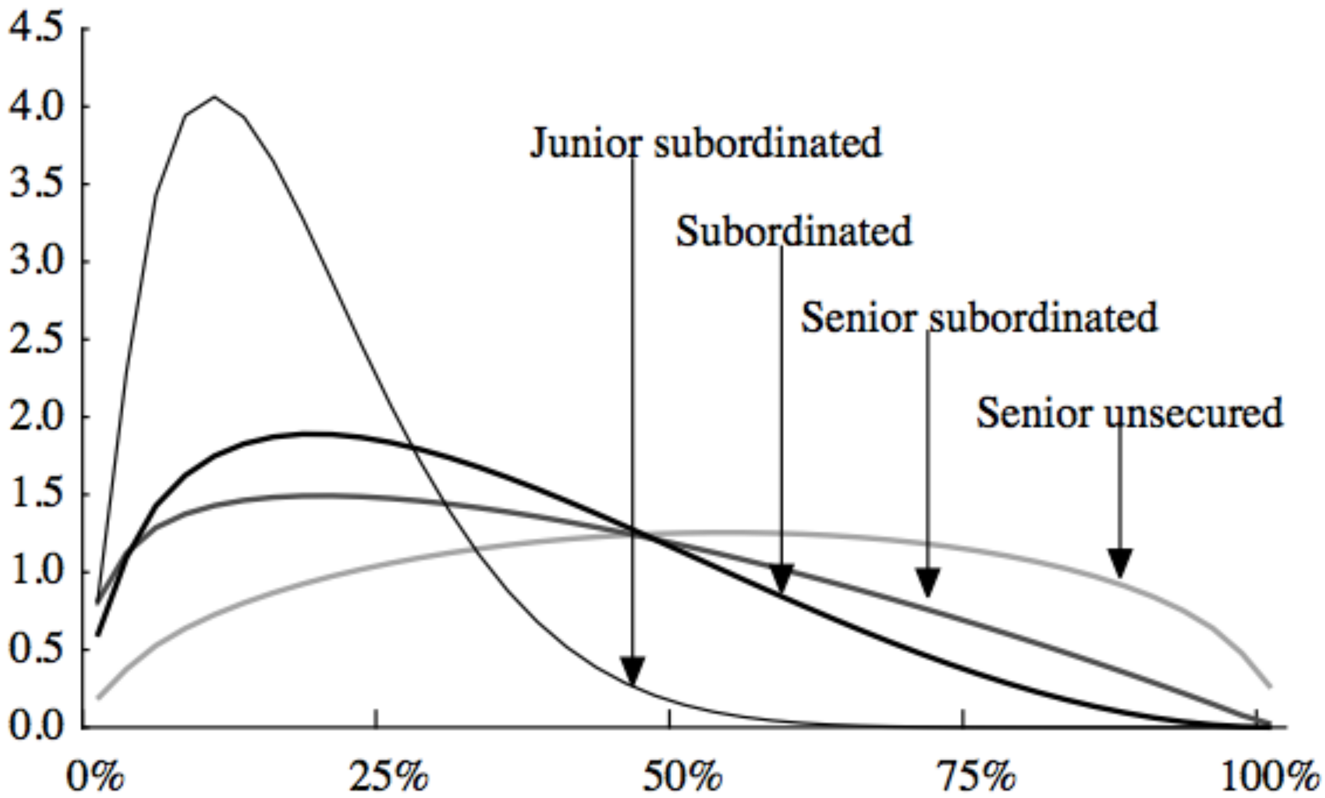
	Carty & Lieberman [96a]			Altman & Kishore [96]		
Seniority Class	**Number**	**Average**	**Std. Dev.**	**Number**	**Average**	**Std. Dev.**
Senior Secured	115	$53.80	$26.86	85	$57.89	$22.99
Senior Unsecured	278	$51.13	$25.45	221	$47.65	$26.71
Senior Subordinated	196	$38.52	$23.81	177	$34.38	$25.08
Subordinated	226	$32.74	$20.18	214	$31.34	$22.42
Junior Subordinated	9	$17.09	$10.90	—	—	—

- Recovery rates have wide variation
- Low recovery rates are correlated with high PD over time and industry

CreditMetrics Technical Document 1997

Incorporating LGD cont.

- Credit Metrics samples Recovery from Beta distribution
  - bounded between 0 and 1.
  - fitted based on empirical studies.
- More complex simulations can correlate Recovery to PDs



Basel II – Internal Rating Based Approach

Capital requirement is based on 99.9% worst case default rate using Normal copula:

$$WCDR = N\left[\frac{N^{-1}(PD) + \sqrt{\rho} \times N^{-1}(0.999)}{\sqrt{1-\rho}}\right]$$

- ✕– correlation between two exposures
- ✕- depends on PD and the type of exposure (corporate, SMB, retail, mortgage)

Capital Requirements

$$\text{Capital} = \text{EAD} \times \text{LGD} \times (\text{WCDR} - \text{PD}) \times \text{MA}$$

$$\text{where MA} = \frac{1 + (\text{M} - 2.5) \times b}{1 - 1.5 \times b}$$

M is the effective maturity and

$$b = [0.11852 - 0.05478 \times \ln(PD)]^2$$

The risk - weighted assets are 12.5 times the Capital so that Capital = 8% of RWA

Capital Requirements

- Requirements are calculated exposure by exposure and summed up
- *MA* formula was approximated by comparing a MTM simulation to Default only case
- Portfolio features like size concentration or industry concentration are NOT taken into account
  - Banks use internal Credit VaR models to approximate the additional capital required for these features
  - This process is called ICAAP

Thanks