Financial Risk Management

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Value at Risk

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THINK IN THE NEXT

Agenda

- What is Value at Risk (VaR)?
- Applications of VaR
- Simple Examples
- Expected Shortfall
- Coherent Risk Measures
- Aggregation Through Time
- Choice of VaR Parameters

A Concise Single Measure

- Summarizes multiple aspects of risk in a single number.
  - As opposed to measures like the Greeks and Duration that look at particular sensitivities.
- It is easy to understand and communicate.
- It asks the simple question: “How bad can things get in a given period of time?”

The VaR Statement

"We are α percent certain that we will not lose more than L dollars in time T."

L is the VaR

T is the Time Horizon

α is the Confidence Level

Formal Definition of VaR

- The *VaR* at confidence level, α, is the smallest number *L*, such that the probability that losses exceed *L* is no larger than 1-α.

Or,

$$VaR_{\alpha} = \min\left\{L : P\left(Loss > L\right) \le 1 - \alpha\right\}$$

$$VaR_{\alpha} = \min\left\{L : P\left(Loss \le L\right) \ge \alpha\right\}$$

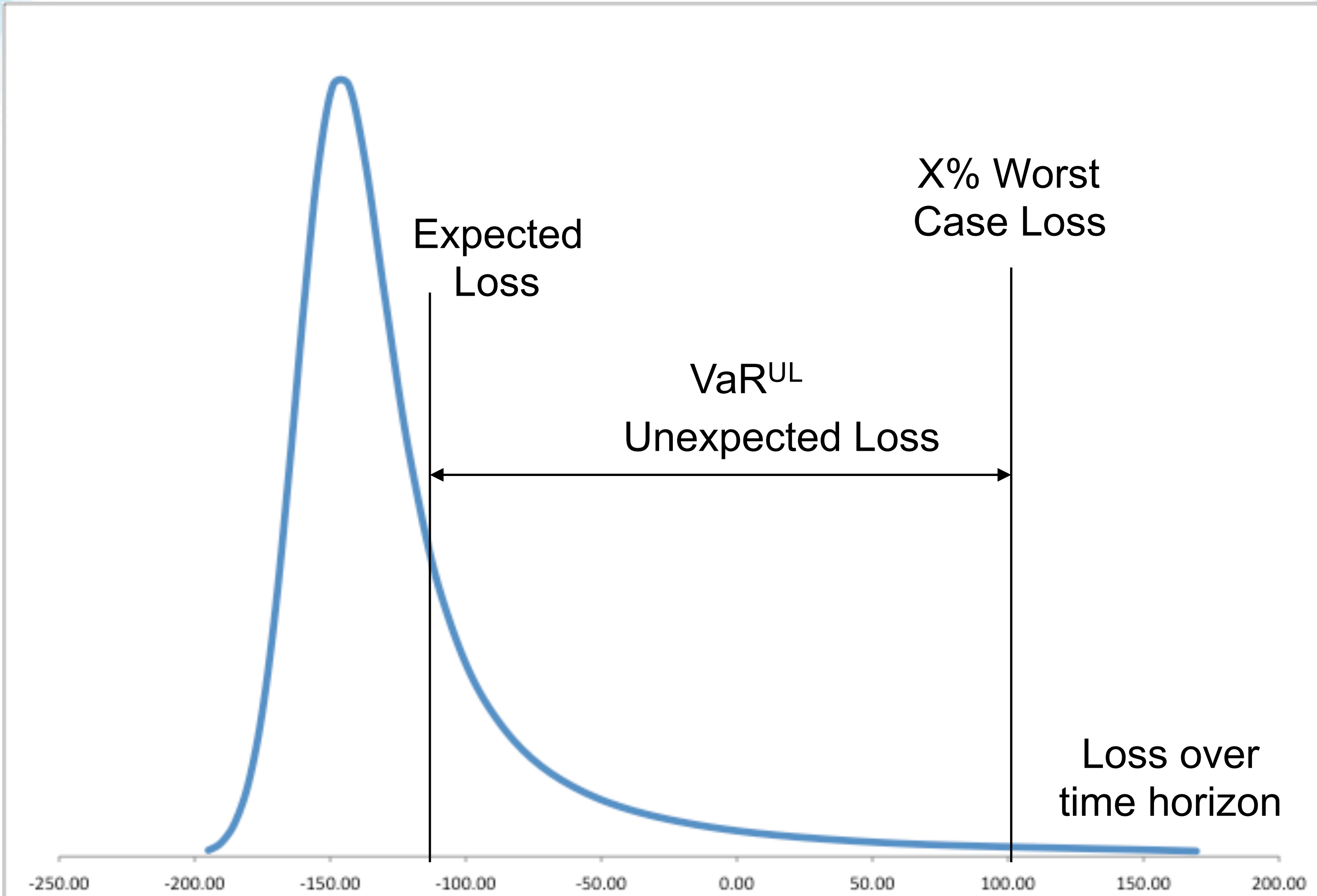
Applications of VaR

- VaR is the industry standard for reporting, both internally and externally.
- It is part of most regulatory frameworks.
- It is used for calculating Capital requirements – How big a cushion do we need to cover losses at a certain probability?
  - Calculations for internal purposes are usually called **Economic Capital**
  - Calculations based on regulatory specifications for reporting purposes are called **Regulatory Capital**.
- It is used for setting limits, predominately on trading floors and portfolios.

Dell, Inc. 10-K

"Based on our foreign currency cash flow hedge instruments …, **we estimate a maximum potential one-day loss in fair value** of … $65 million, … using a Value-at-Risk ("VaR") model. By using market implied rates and incorporating volatility and correlation among the currencies of a portfolio, **the VaR model simulates 3,000 randomly generated market prices and calculates the difference between the fifth percentile and the average as the Value-at-Risk.**"

Expected and Unexpected Loss



Normal Distribution Example

- The daily <u>gains</u> on a portfolio of stocks are distributed normally with mean=0, and standard deviation =$5M
- What is the 95% VaR?

Normsinv(0.05)*σ=-1.645×5 = -8.22mil

$VaR_{95\%}$ = $8.22mil

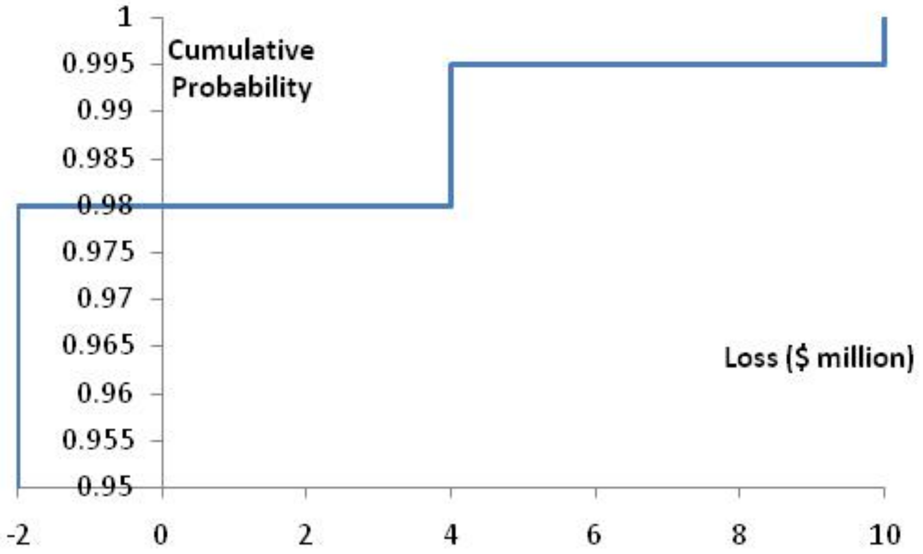
Normal Distribution Example (2)

- The profit from a portfolio over six months is normally distributed with mean \$2 million and standard deviation \$10 million
- The 1% point of the distribution of gains is $2-2.33\times10$ or − \$21.3 million
- The VaR for the portfolio with a six month time horizon and a 99% confidence level is \$21.3M million.
- If we look at Unexpected Losses:
  - Expected Loss = -\$2M
  - Unexpected Loss VaR = \$21.3M – (-\$2M) = \$23.3M

VaR Discrete Example

- A one-year project has a 98% chance of leading to a gain of $2 million, a 1.5% chance of a loss of $4 million, and a 0.5% chance of a loss of $10 million
- What is the VaR at a 99% confidence level?
  - $4 million
- What if the confidence level is 99.9%?
  - $10 million
- What if it is 99.5%?

Cumulative Loss Distribution for Example



$$VaR_{\alpha} = \min\left\{L : P\left(Loss \le L\right) \ge \alpha\right\}$$

L	Prob.	Cum. Prob	P(Loss>L)
-2	98%	98%	2%
4	1.5%	99.5%	0.5%
10	0.5%	100%	0%

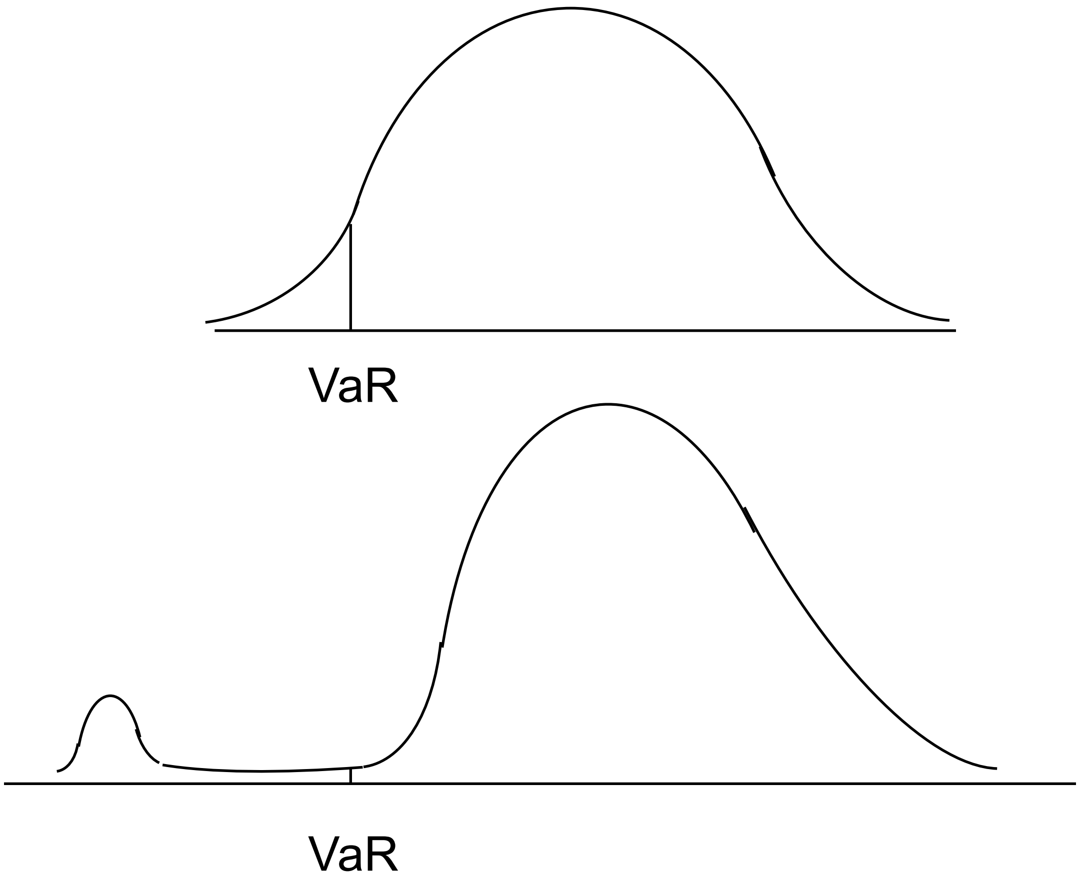
Expected Shortfall

- VaR is the loss level that will not be exceeded with a specified probability, α.
- Expected Shortfall is the average loss over the *1*-α worst cases.
- For continuous distributions: It is the mean loss given that the loss is greater or equal to the VaR level:

$$ES_\alpha = E[L \mid L \geq VaR_\alpha]$$

- Also called: CVaR (Conditional VaR) or Tail Loss
- Two portfolios with the same VaR can have very different expected shortfalls
- Basel has recently moved to using ES rather than VaR

Distributions with the Same VaR but Different Expected Shortfalls



Example: Same VaR, Different ES

- Suppose there are two possible states of the world:
  - A good state with probability = 0.8: Portfolio gains 100
  - A bad state with probability = 0.2: Portfolio loss is randomly drawn from U[50,100].
- What is 90% VaR?
  - Let *x* be the VaR. It must be between 50 and 100.
  - The probability of loss being less than or equal to *x* is 0.9:

$$\Pr\left[Loss \le x\right] = 0.8 + 0.2 \times \frac{x - 50}{100 - 50} = 0.9$$

$$x = 75$$

- What is 90% Expected Shortfall?

$$E\left[Loss \mid Loss \ge 75\right] = \frac{75 + 100}{2} = 87.5$$

Example: Same VaR, Different ES (2)

- Suppose the two states are now:
  - A good state with probability = 0.8: Portfolio gains 100
  - A bad state with probability = 0.2: Portfolio loss is randomly drawn from U[0,150].
- What is 90% VaR?
  - Let *x* be the VaR. It must be between 0 and 150.
  - The probability of loss being less than or equal to *x* is 0.9:

$$\Pr\left[Loss \le x\right] = 0.8 + 0.2 \times \frac{x - 0}{150 - 0} = 0.9$$

$$x = 75$$

- What is 90% Expected Shortfall?

$$E\left[Loss \mid Loss \ge 75\right] = \frac{75 + 150}{2} = 112.5$$

Expected Shortfall - Normal Distribution

- The daily losses on a portfolio of stocks are distributed normally with mean=0, and standard deviation of 5 million
- What is the 95% VaR?
  - VaR = 1.645×5 = 8.22 million
- What is the 95% Expected Shortfall?

$$E[Loss \mid Loss \geq VaR] = \frac{\int_{VaR}^{\infty} L \times \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{L^2}{2\sigma^2}} dL}{P[Loss \geq VaR]} =$$

$$= \frac{\varphi\left(VaR/\sigma\right)}{1-p}\sigma = \frac{\varphi(1.645)}{0.05} \bullet 5 = 10.31$$

Expected Shortfall - Normal Distribution

- For the **Mean Zero Normal Distribution**, Expected Shortfall at a certain confidence level, is like VaR at some higher level
- In our case:

$$\frac{\varphi(1.645)}{0.05} = 2.06 = NORMSINV(0.98)$$

- 95% Expected Shortfall = 98% VaR

Coherent Risk Measures

- **Monotonicity:** If one portfolio always produces a worse outcome than another, its risk measure should be greater
- **Translation Invariance:** If we add an amount of riskless asset paying K to a portfolio its risk measure should go down by K
- **Homogeneity:** Multiplying the size of a portfolio by λ should result in the risk measure being multiplied by λ
- **Sub-additivity:** The risk measures for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged = benefit to diversification

VaR vs Expected Shortfall

- VaR satisfies the first three conditions but not the fourth one
  - VaR is not coherent
- Expected shortfall satisfies all four conditions.

Sub-Additivity –Example

- Each of two independent projects has a probability 0.98 of a loss of $1 million and 0.02 probability of a loss of $10 million
- What is the 97.5% VaR for each project? $1M
- What is the 97.5% expected shortfall for each project?
  - Within the 2.5% worst results for the project:
    - Conditional Probability of $1M = (98-97.5)/2.5=0.5/2.5
      Conditional Probability of 10mil = (100-98)/2.5=2/2.5
  - ES=(0.5/2.5)*1 + (2/2.5)*10 = 8.2

Sub-Additivity –Example (2)

- What is the 97.5% VaR for the portfolio? 11

Loss	Probability	Cum. Probability
1+1=2	0.98*0.98=96.04%	96.04%
10+1=11	2*0.98*0.02=3.92%	99.96%
10+10=20	0.02*0.02=0.04%	100%

- What is the 97.5% expected shortfall for the portfolio?
  - Conditional Prob. of 11 = (99.96-97.5)/2.5, Conditional Prob. of 20 = (100-99.96)/2.5
  - (2.46/2.5)*11 + (0.04/2.5)*20 = 11.144

Sub-Additivity –Example (3)

- This is an example of VaR not satisfying sub-additivity:
  - VaR for 1 project = 1, VaR for 2 projects = 11
  - VaR (2 projects)>2*VaR(1 project)
- Expected Shortfall satisfies sub-additivity
  - ES for 1 project = 8.2, ES for 2 projects = 11.144
  - ES(2 projects)<2*ES(1 project)
  - ES always satisfies sub-additivity

Spectral Risk Measures

- A spectral risk measure assigns weights to quantiles of the loss distribution
- VaR assigns all weight to Xth quantile of the loss distribution
- Expected shortfall assigns equal weight to all quantiles greater than the Xth quantile
- For a coherent risk measure weights must be a non-decreasing function of the quantiles

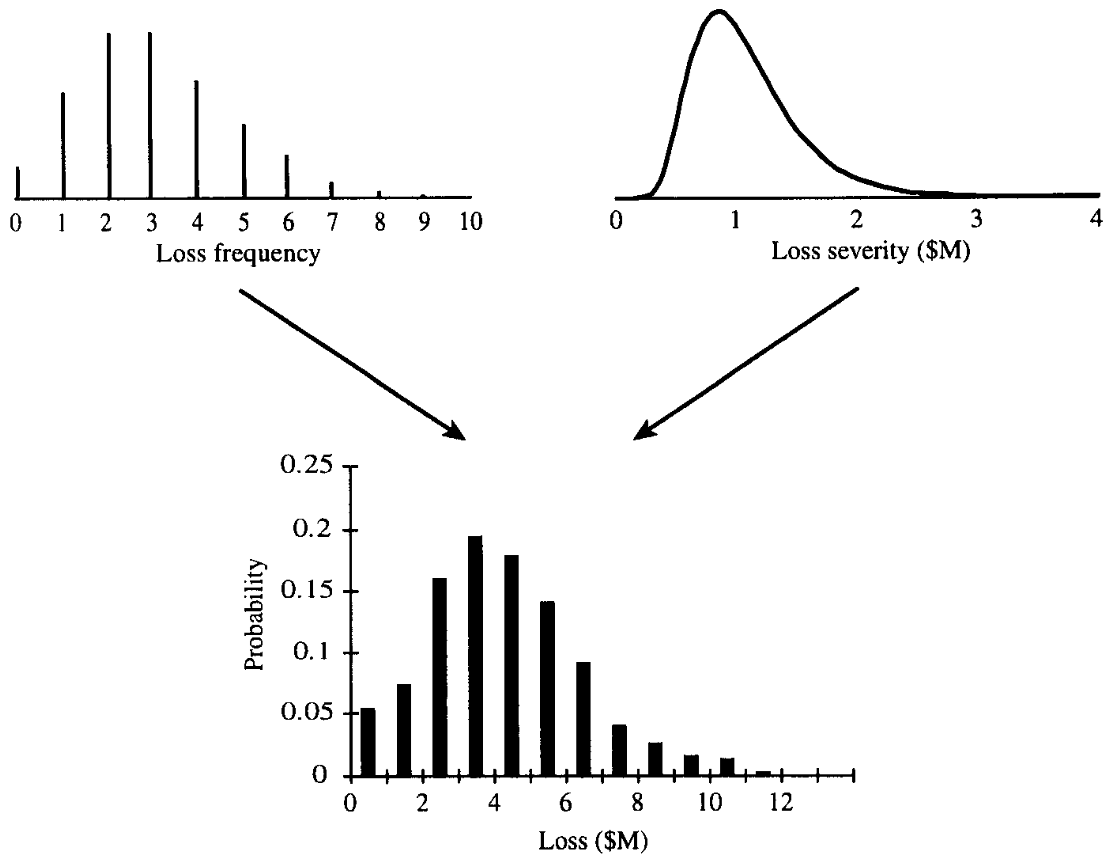
Example: Simulation for Operational Risk VaR

- Assume a bank has figured out that the probability of k events of fraud happening in a year is distributed according to the Poisson distribution:

$$\Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

- The severity of the fraud is distributed lognormally.
- What is the VaR-95% for annual loss due to fraud?

Using Monte Carlo to combine the Distributions



Monte Carlo Simulation

- Create distribution of annual losses, by simulating many trials:
  - Sample from Poisson distribution to determine the number of loss events (=k)
  - Sample k times from the loss severity distribution to determine the loss severity for each loss event
  - Sum loss severities to determine total loss
- Find the 95 percentile of the distribution

Credit VaR for Uncorrelated Loans

- A bank has 100 loans of $15 million each. The probability of default (PD) of each loan is 2.5%. In case of default there is no recovery. Loan defaults are independent of each other.
- What is the Expected Loss (EL) on the portfolio?
  - EL = 100*15*2.5% = $37.5

Credit VaR for Uncorrelated Loans (Example – Cont.)

- What is the $VaR_{96\%}$ of the portfolio?
  - The probability of K loans defaulting is given by the Binomial Distribution
  - P(K loans default) = BINOMDIST(K, 100, 2.5%, 0)
  - P(# defaults $\leq K$) = BINOMDIST(K, 100, 2.5%, 1)
- The cumulative Probability is given by:
- There is 96% that 5 loans or less will default.

Num Defaults	Cumulative Prob
0	0.08
1	0.28
2	0.54
3	0.76
4	0.89
5	**0.96**
6	0.99

Credit VaR for Uncorrelated Loans (Example – Cont.)

- $VaR_{96\%}$ is therefore: 5*15 = $75M
- The Unexpected Loss $VaR_{96\%}$ is equal to 75-37.5 = $37.5M
- What if there was 60% recovery on each loan?
  - LGD = 40%
  - VaR-96% = 5*15*0.4 = $30M
- But loan defaults in a portfolio are generally not independent … later in the course.

Aggregating VaR Over Time

- The simplest assumption is that daily gains/losses are normally distributed and independent with mean zero
- The T-day VaR equals $\sqrt{T}$ times the one-day VaR
- If there is positive autocorrelation the T-day VaR will be greater

Independence Assumption in VaR Calculations

- When there is autocorrelation equal to ρ, instead of T-days variance being T times daily variance, the multiplier is:

$$T + 2(T-1)\rho + 2(T-2)\rho^2 + 2(T-3)\rho^3 + \ldots 2\rho^{T-1}$$

Impact of Autocorrelation: Ratio of *T*-day VaR to 1-day VaR

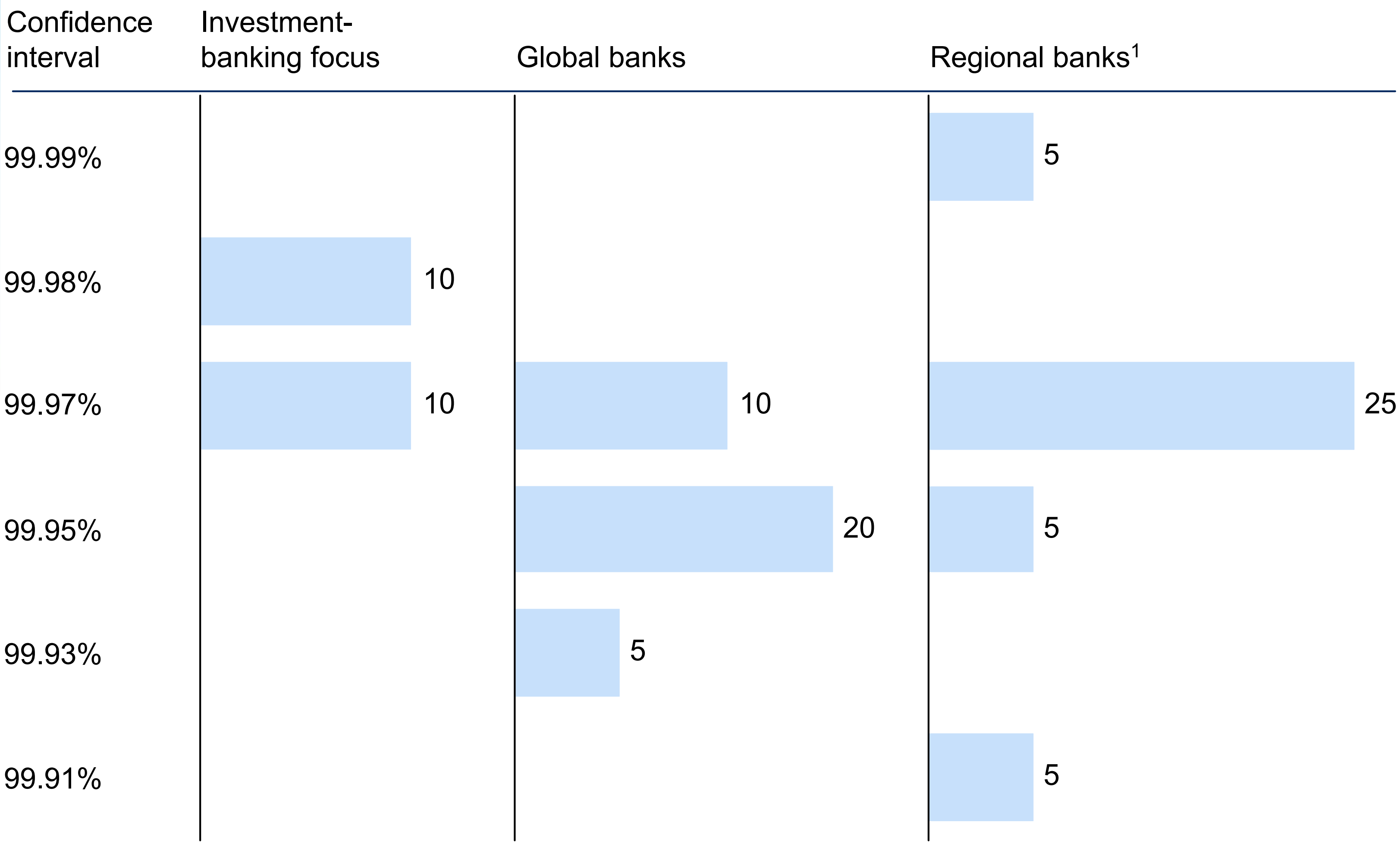
	T=1	T=2	T=5	T=10	T=50	T=250
ρ=0	1.0	1.41	2.24	3.16	7.07	15.81
ρ=0.05	1.0	1.45	2.33	3.31	7.43	16.62
ρ=0.1	1.0	1.48	2.42	3.46	7.80	17.47
ρ=0.2	1.0	1.55	2.62	3.79	8.62	19.35

Choice of VaR Parameters

- Time horizon:
  - Should depend on how quickly portfolio can be unwound.
  - Bank regulators use 1-day for market risk scaled by the square root of time to 10-days, and 1-year for credit/operational risk.
- Confidence level:
  - Depends on objectives. Regulators use 99% for market risk and 99.9% for credit/operational risk.
  - A bank aiming to maintain a AA credit rating might use confidence levels as high as 99.98% for internal Economic Capital calculations.

Exhibit 3 Banks use a range of confidence intervals for economic capital models.

Economic capital-modeling practices at 17 financial institutions, %



1 Some public information of additional regional banks has been included for comparison.
Note: Numbers may not add up to 100 due to rounding.

Source: McKinsey Market Risk Survey and Benchmarking 2011

Thanks