Portfolio Optimization

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Outline

Single asset investment

Portfolio investment

Portfolio optimization

Return of an asset over one period

- *asset* can be stock, bond, real estate, commodity, ...
- invest in a single asset over period (quarter, week, day, ...)
- buy q shares at price p (at beginning of investment period)
- $h = pq$ is dollar value of holdings
- sell q shares at new price p^+ (at end of period)
- profit is $qp^+ - qp = q(p^+ - p) = \frac{p^+ - p}{p} h$
- define **return** $r = (p^+ - p)/p$
- return $= \frac{\text{profit}}{\text{investment}}$
- profit $= rh$
- example: invest $h = \$1000$ over period, $r = +0.03$: profit $= \$30$

Short positions

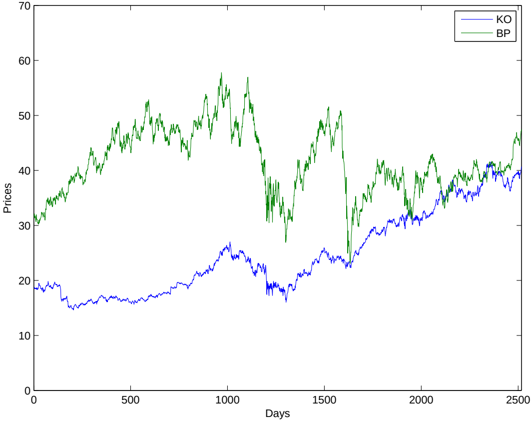
- basic idea: holdings h and share quantities q are **negative**
- called *shorting* or *taking a short position on* the asset
  (h or q positive is called a *long position*)
- how it works:
  - you borrow q shares at the beginning of the period and sell them at price p
  - at the end of the period, you have to buy q shares at price p^+ to return them to the lender
- all formulas still hold, *e.g.*, profit $= rh$
- example: invest $h = -\$1000$, $r = -0.05$: profit $= +\$50$
- no limit to how much you can lose when you short assets
- normal people (and mutual funds) don't do this; hedge funds do

Return of an asset over multiple periods

- invest over periods $t = 1, 2, \ldots, T$
  (quarters, trading days, minutes, seconds ...)
- p_t is price at the beginning of period t
- return over period t is $r_t = \frac{p_{t+1} - p_t}{p_t}$
- invest dollar amount h_t in period t, or share number $q_t = h_t / p_t$
- profit over period t is $r_t h_t$
- total profit is $\sum_{t=1}^{T} r_t h_t$
- per period profit is $\frac{1}{T} \sum_{t=1}^{T} r_t h_t$

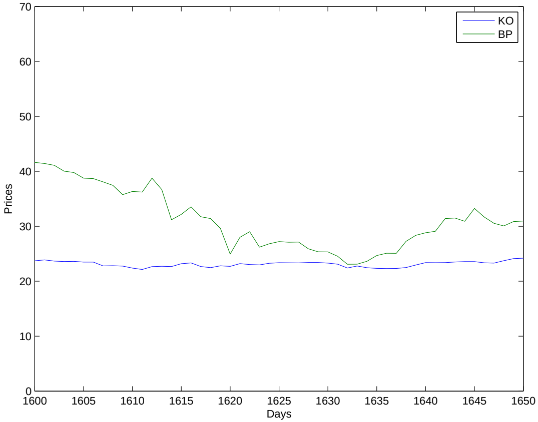
Examples

stock prices of BP (BP) and Coca-Cola (KO) for last 10 years



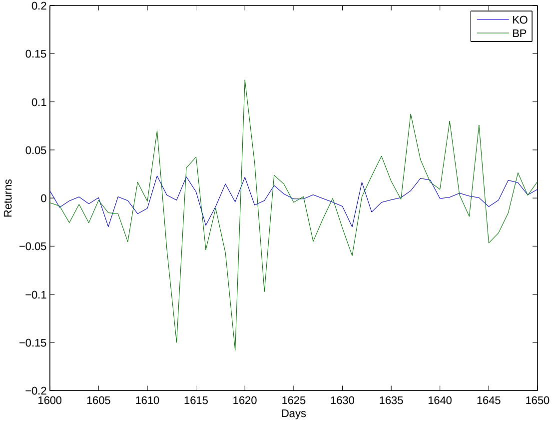
Examples

price changes over a few weeks



Examples

returns of the assets over the same period



Buy and hold

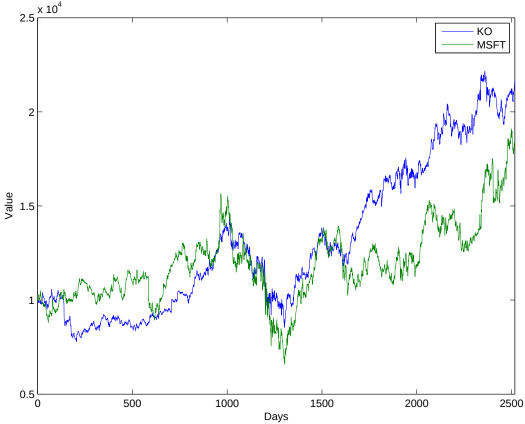
- a very simple choice of h_t
- $q_t = q$ for all $t = 1, 2, \ldots, T$
  - buy q shares at the beginning of period 1
  - sell q shares at the end of period T
- hence $h_t = p_t q$
- profit is

$$\sum_{t=1}^{T} r_t h_t = \sum_{t=1}^{T} \left(\frac{p_{t+1} - p_t}{p_t} \right) (p_t q) = q(p_{T+1} - p_1)$$

- same as combining periods $1, \ldots, T$ into a single period

Cumulative value plot

plot of $h_t = p_t q$ versus t ($h_1 = \$10{,}000$ by tradition)



Constant value

- another simple choice of h_t: $h_t = h$, $t = 1, \ldots, T$
- number of shares $q_t = h/p_t$ (which varies with t)
- requires buying or selling shares every period to keep value constant
- profit is $\sum_{t=1}^{T} r_t h$
- **per period profit** is $(1/T) \sum_{t=1}^{T} r_t h = \mathbf{avg}(r_t) h$ (in \$)
- **mean return** is $\mathbf{avg}(r_t)$ (fractional; often expressed in %)
- **profit standard deviation** is $\mathbf{std}(r_t h) = \mathbf{std}(r_t) h$ (in \$)
- **risk** is $\mathbf{std}(r_t)$ (fractional)
- want per period profit high, risk low

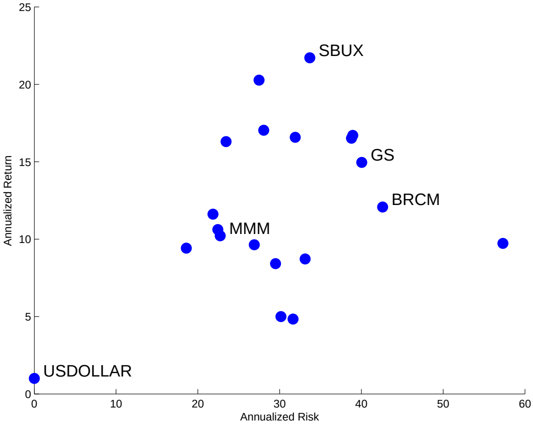
Annualizing return and risk

- mean return and risk are often expressed in **annualized form** (*i.e.*, per year)
- if there are P trading periods per year
  - annualized return $= P\,\mathbf{avg}(r_t)$
  - annualized risk $= \sqrt{P}\,\mathbf{std}(r_t)$

  (the squareroot in risk annualization comes from the assumption that the fluctuations in return around the mean are independent)
- if t denotes trading days, with 250 trading days in a year
  - annualized return $= 250\,\mathbf{avg}(r_t)$
  - annualized risk $= \sqrt{250}\,\mathbf{std}(r_t)$

Risk-return plot

annualized risk versus annualized return of various assets
up (high return) and left (low risk) is good



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Portfolio of assets

- n assets
- n-vector p_t is prices of assets in period t, $t = 1, 2, \ldots, T$
- n-vector h_t is dollar value holdings of the assets
- total portfolio value: $V_t = \mathbf{1}^T h_t$
- n-vector q_t is the number of shares: $(q_t)_i = (h_t)_i / (p_t)_i$
- $w_t = (1/\mathbf{1}^T h_t) h_t$ gives **portfolio weights** or **allocation** (fraction of portfolio, defined only for $\mathbf{1}^T h_t > 0$)

Examples

- $(h_3)_5 = -1000$ means you short asset 5 in investment period 3 by \$1,000
- $(w_2)_4 = 0.20$ means 20% of total portfolio value in period 2 is invested in asset 4
- $w_t = (1/n, \ldots, 1/n)$, $t = 1, \ldots, T$ means total portfolio value is equally allocated across assets in all investment periods
- $\mathbf{1}^T h_t = 0$ means total short positions = total long positions

Buy and hold portfolio

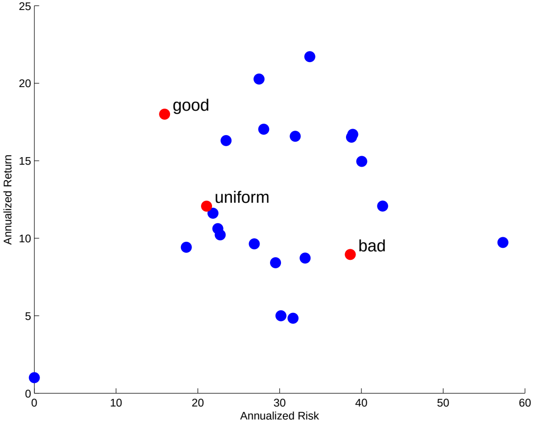
- same idea as single asset case
- (n-vector) $q_t = q$ for all $t = 1, \ldots, T$
- holdings given by $(h_t)_i = (p_t)_i q_i$, $i = 1, \ldots, n$
- profit is

$$\sum_{t=1}^{T} r_t^T h_t = \sum_{t=1}^{T} \sum_{i=1}^{n} (r_t)_i \left(\frac{(p_{t+1})_i - (p_t)_i}{(p_t)_i} \right) ((p_t)_i q_i) = q^T (p_{T+1} - p_1)$$

Constant value portfolio

- simple choice of h_t: $h_t = h$, $t = 1, \ldots, T$
- requires *rebalancing* (buying and selling) shares to maintain $(h_t)_i = h_i$ every period
- profit is $\sum_{t=1}^{T} r_t^T h$
- **per period profit** is $(1/T) \sum_{t=1}^{T} r_t^T h$ (in \$)
- **mean return** is $\mathbf{avg}(r_t^T h)/\mathbf{1}^T h$ (fractional; often expressed in %)
- **profit standard deviation** is $\mathbf{std}(r_t^T h)$ (in \$)
- **risk** is $\mathbf{std}(r_t^T h)/\mathbf{1}^T h$ (fractional)

Risk-return plot for constant value portfolio

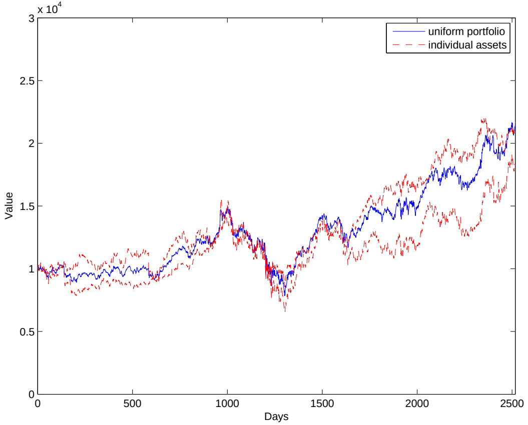


Constant weight portfolio with re-investment

- fix weight vector w
- given initial total investment V_1, set $h_1 = V_1 w$
- $V_2 = V_1 + r_1^T h_1$
- set $h_2 = V_2 w$, *i.e.*, re-invest total portfolio value using allocation w
- and so on . . .
- $V_T = V_1(1 + r_1^T w)(1 + r_2^T w) \cdots (1 + r_T^T w)$
- $V_t \leq 0$ (or some small value like $0.1 V_1$) called **going bust** or **ruin**

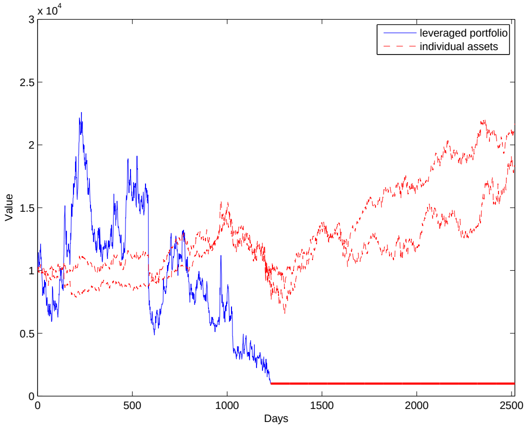
Cumulative value plot

uniform portfolio between two assets, with $h_1 = \$10{,}000$ (by tradition)



Cumulative value plot

portfolio with large short positions (heavily leveraged) going **bust** (dropping to 10% of starting value)



Comparison: Re-investment or not

- constant value portfolio (without re-investment) gives total profit $\sum_{t=1}^{T} r_t^T h$
- constant weight portfolio (with re-investment) gives total profit

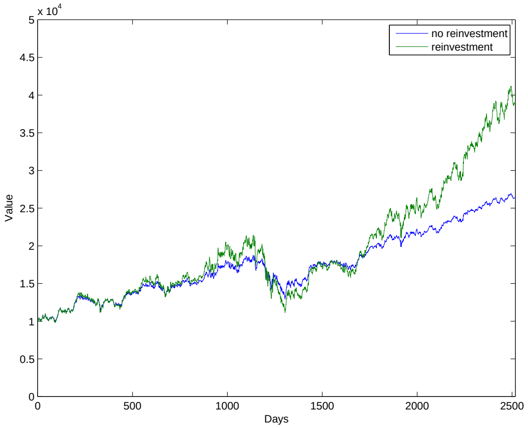
$$V_T - V_1 = ((1 + r_T^T w) \cdots (1 + r_1^T w) - 1)V_1$$

- for $|r_t^T w|$ all small (say, ≤ 0.01)

$$(1 + r_T^T w) \cdots (1 + r_1^T w) \approx 1 + \sum_{t=1}^{T} r_t^T w$$

  so $V_T - V_1 \approx \sum_{t=1}^{T} r_t^T w V_1$
- profit with constant value $h \approx$ profit with constant weight $w = (1/\mathbf{1}^T h)h$ and initial investment h

Comparison: Re-investment or not



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- how should we choose a portfolio value vector h, or a portfolio weight vector w, over some investment period?
- more generally, these vectors could change with time, as our information or goals changes
- when we choose h or w, we know past returns ('realized returns') but (of course) not future ones
- in all cases, we want high (mean) return, low risk

Returns matrix

- define **returns matrix**

$$R = \begin{bmatrix} r_1^T \\ \vdots \\ r_T^T \end{bmatrix}$$

- jth column is asset j return time series
- Rh is profit time series
- $\mathbf{1}^T Rh$ is total profit
- $\mathbf{avg}(Rh) = (1/T)\mathbf{1}^T Rh$ is per period profit
- $\mathbf{std}(Rh)$ is per period risk
- goal: choose h that makes $\mathbf{avg}(Rh)$ high, $\mathbf{std}(Rh)$ low

Portfolio optimization via least squares on past returns

$$\begin{array}{ll} \text{minimize} & \mathbf{std}(Rh)^2 = (1/T)\|Rh - \rho B \mathbf{1}\|^2 \\ \text{subject to} & \mathbf{1}^T h = B, \quad \mathbf{avg}(Rh) = \rho B \end{array}$$

- h is holdings vector to be found
- R is the returns matrix for **past returns**
- Rh is the (past) profit time series
- require mean (past) profit ρB
- minimize the standard deviation of (past) profit

- we are really asking what **would have been** the best constant allocation, had we known future returns

Constant weight portfolio optimization

$$\begin{array}{ll} \text{minimize} & \mathbf{std}(Rw)^2 = (1/T)\|Rw - \rho\mathbf{1}\|^2 \\ \text{subject to} & \mathbf{1}^T w = 1, \quad \mathbf{avg}(Rw) = \rho \end{array}$$

- very similar to constant weight optimization
  (in fact the two solutions are the same, except for scaling)
- w is weight allocation vector to be found
- Rw is the (past) return time series
- require mean (past) return ρ
- minimize the standard deviation of (past) return

Examples

- optimal w for annual return 1% (last asset is risk-less with 1% return)

$$w = (0.0000, 0.0000, 0.0000, \ldots, 0.0000, 0.0000, 1.0000)$$

- optimal w for annual return 13%

$$w = (0.0250, -0.0715, -0.0454, \ldots, -0.0351, 0.0633, 0.5595)$$

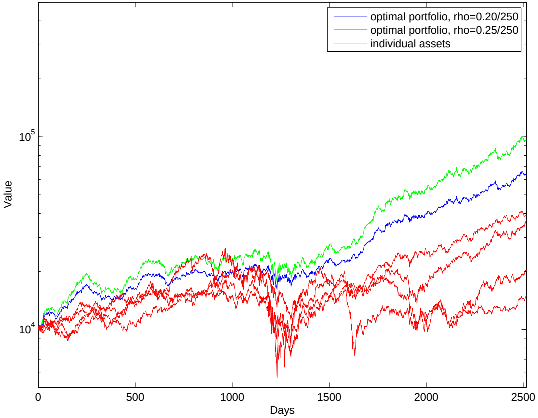
- optimal w for annual return 25%

$$w = (0.0500, -0.1430, -0.0907, \ldots, -0.0703, 0.1265, 0.1191)$$

- asking for higher annual return yields
  - more invested in risky, but high return assets
  - larger short positions ('leveraging')

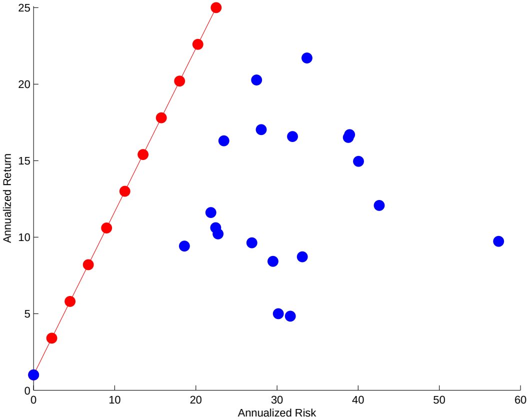
Cumulative value plots for optimal portfolios

cumulative value plot for optimal portfolios and some individual assets



Optimal risk-return curve

red curve obtained by solving problem for various values of ρ



Optimal portfolios

- perform significantly better than individual assets
- risk-return curve forms a straight line
  - one end of the line is the risk-free asset
- *two-fund theorem:* optimal portfolio w is an affine function in ρ

$$\left[\begin{array}{c} w \\ \nu_1 \\ \nu_2 \end{array}\right] = \left[\begin{array}{ccc} R^T R & \mathbf{1} & R^T \mathbf{1} \\ \mathbf{1}^T & 0 & 0 \\ \mathbf{1}^T R & 0 & 0 \end{array}\right]^{-1} \left[\begin{array}{c} R^T \mathbf{1} \\ 1 \\ \rho T \end{array}\right]$$

The big assumption

- now we make the big assumption (BA):

  future returns will look something like past ones

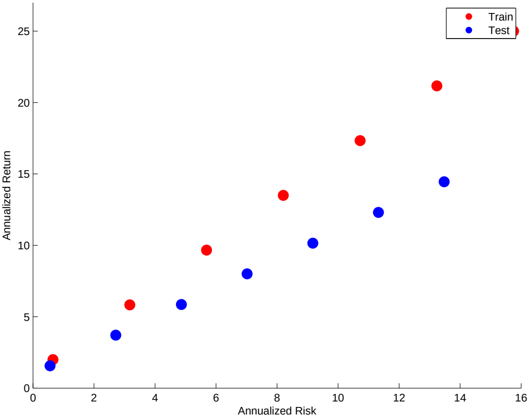
  - you are warned this is false, every time you invest
  - it is often reasonably true
  - in periods of 'market shift' it's much less true

- if BA holds (even approximately), then a good weight vector for past (realized) returns should be good for future (unknown) returns

- for example:
  - choose w based on last 2 years of returns
  - then use w for next 6 months

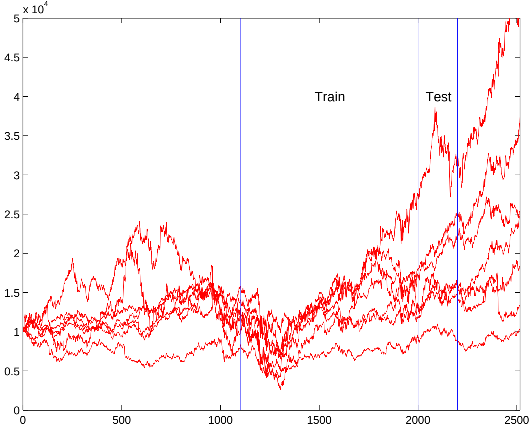
Optimal risk-return curve

- trained on 900 days (red), tested on the next 200 days (blue)
- here BA held reasonably well



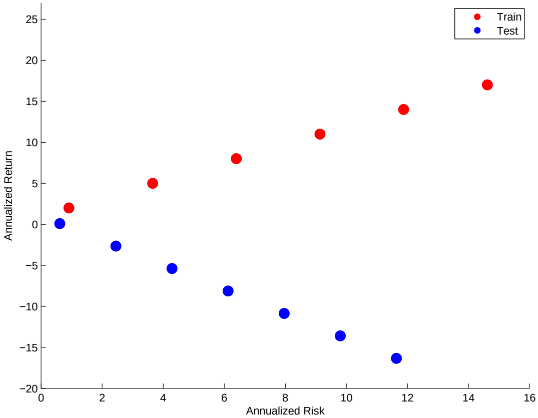
Optimal risk-return curve

- corresponding train and test periods



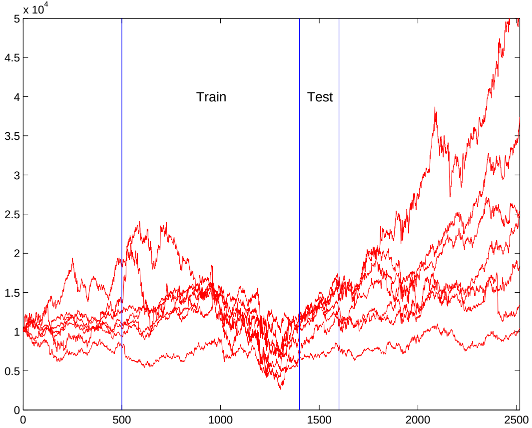
Optimal risk-return curve

- and here BA didn't hold so well
- (can you guess when this was?)



Optimal risk-return curve

- corresponding train and test periods



Rolling portfolio optimization

for each period t, find weight w_t using L past returns

$$r_{t-1}, \ldots, r_{t-L}$$

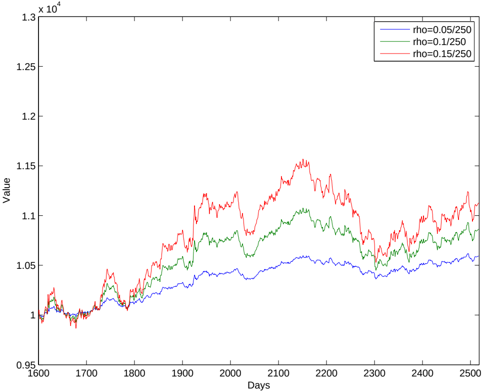
variations:

- update w every K periods (say, monthly or quarterly)
- add cost term $\kappa \|w_t - w_{t-1}\|^2$ to objective to discourage turnover, reduce transaction cost
- add logic to detect when the future is likely to not look like the past
- add 'signals' that predict future returns of assets

(...and pretty soon you have a quantitative hedge fund)

Rolling portfolio optimization example

- cumulative value plot for different target returns
- update w daily, using $L = 400$ past returns



Rolling portfolio optimization example

- same as previous example, but update w every quarter (60 periods)