Topic 1: Introduction to Fundamental of Investments

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Learning objectives

- Introduce some basic ideas about investments
- Give some terminology
- Types of investments and investment philosophies
- Types of securities

What is an investment ?

An investment is the current commitment of resources for a period of time in the expectation of receiving future resources greater than the current outlay.

- Real Assets
  - Assets used to produce goods and services (machinery, buildings)
    - Projects
    - Companies
- Financial Assets
  - Claims on real assets
    - Bonds
    - Stocks
    - Convertible Stocks
    - (Options, Futures)

Required Rate of Return

- In order to part with their money, investors require compensation for:

  - the time resources are committed

  - the expected rate of inflation

  - the uncertainty of the future payments

Compensation for time

- The real risk-free rate of interest is the exchange rate between future consumption and present consumption.
- This rate of interest can be thought of as the “pure” rental rate on money in the absence of inflation and risk.
- Borrowers are willing to pay to be able to spend more than their current resources allow.
- Savers need compensation in order to give up the right to consume today.

Compensation for Inflation

- Inflation is the rate at which prices as a whole are increasing
- If the future payment is worth less (measured in consumption) because of inflation, then investors will demand a higher interest rate so that their expected purchasing power will actually increase from the investment
- Nominal interest rates reflect expected inflation and compensation for time

Real returns and inflation

- Consider a $1,000 bond maturing in one year with a $100 interest payment. If there is no risk of default the investor is guaranteed to get $1,100 next year (a 10% *nominal* rate of return). However, while the *nominal* cash flow is certain, the *real* cash flow is not because of the possibility of inflation.

Inflation

- Relation between real and nominal rates

$$1 + r_n = (1 + r_r)(1 + i_f)$$

$$r_n = r_r + i_f + i_f \times r_r$$

$$r_n = 0.02 + 0.03 + 0.02 \times 0.03 = 0.0506 \Rightarrow 5.06\%$$

To convert a nominal rate of return into a real rate of return

$$(1 + r_r) = \frac{1 + r_n}{(1 + i_f)}$$

The formula above is exact. Here's an approximation:

$$\begin{aligned} &\text{real interest rate} \\ &\approx \text{nominal interest rate - inflation rate} \end{aligned}$$

$$r_n = r_r + i_f + i_f \times r_r$$

$$r_n \approx r_r + i_f$$

$$r_r \approx r_n - i_f$$

Inflation

*Example*

If the nominal interest rate on one year govt. bonds is 5.9% and the inflation rate is 3.3%, what is the real interest rate?

Inflation

Example

If the nominal interest rate on one year govt. bonds is 5.9% and the inflation rate is 3.3%, what is the real interest rate?

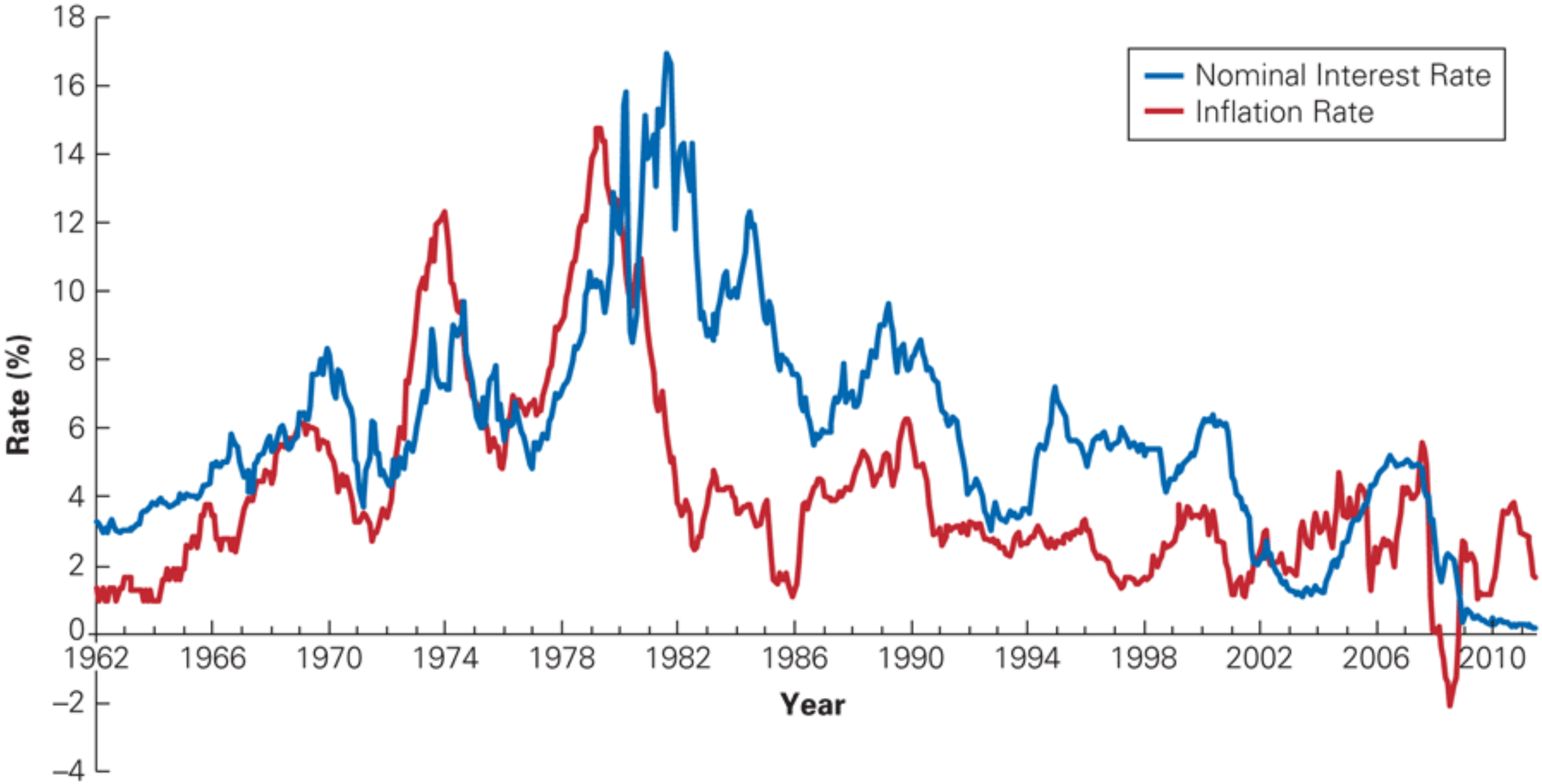
$$1 + \text{real interest rate} = \frac{1+.059}{1+.033}$$

$$= 1.025$$

$$==> \text{real interest rate} = .025 \text{ or } 2.5\%$$

$$\text{Approx.} \approx .059 - .033 = .026 \text{ or } 2.6\%$$

U.S. Interest Rates and Inflation Rates,1960–2012



Compensation for the time value of money

- The nominal risk-free rate of interest adjusts the real risk-free rate to reflect expected inflation over the life of the investment.
- Taking into account these two factors (time and expected inflation) compensates investors for the "time value" of their money.

$$r_n = r_f \approx r_r + i_f$$

Compensation for Risk-bearing

- Investors tend to be risk-averse, meaning that they need sufficient expected additional compensation in order to bear additional risk.
- If the future payment from an investment is uncertain, investors will demand an interest rate that exceeds the nominal risk-free rate of interest to provide a risk premium.
- The riskier the investment the higher is the risk premium required.

The Required Rate of Return

- The sum of the nominal risk-free interest rate and the risk premium on an investment gives that investment's required rate of return:

$$r = r_f + risk\ premium$$

- The required rate of return is the discount rate used in present value calculations. It is also sometimes called the expected rate of return or the opportunity cost of capital.

Depending on the context

- Required rate of return
- Expected rate of return
- Opportunity cost of capital
- Discount rate

= nominal risk free rate + risk premium

Risk premium=risk of investment x price of risk

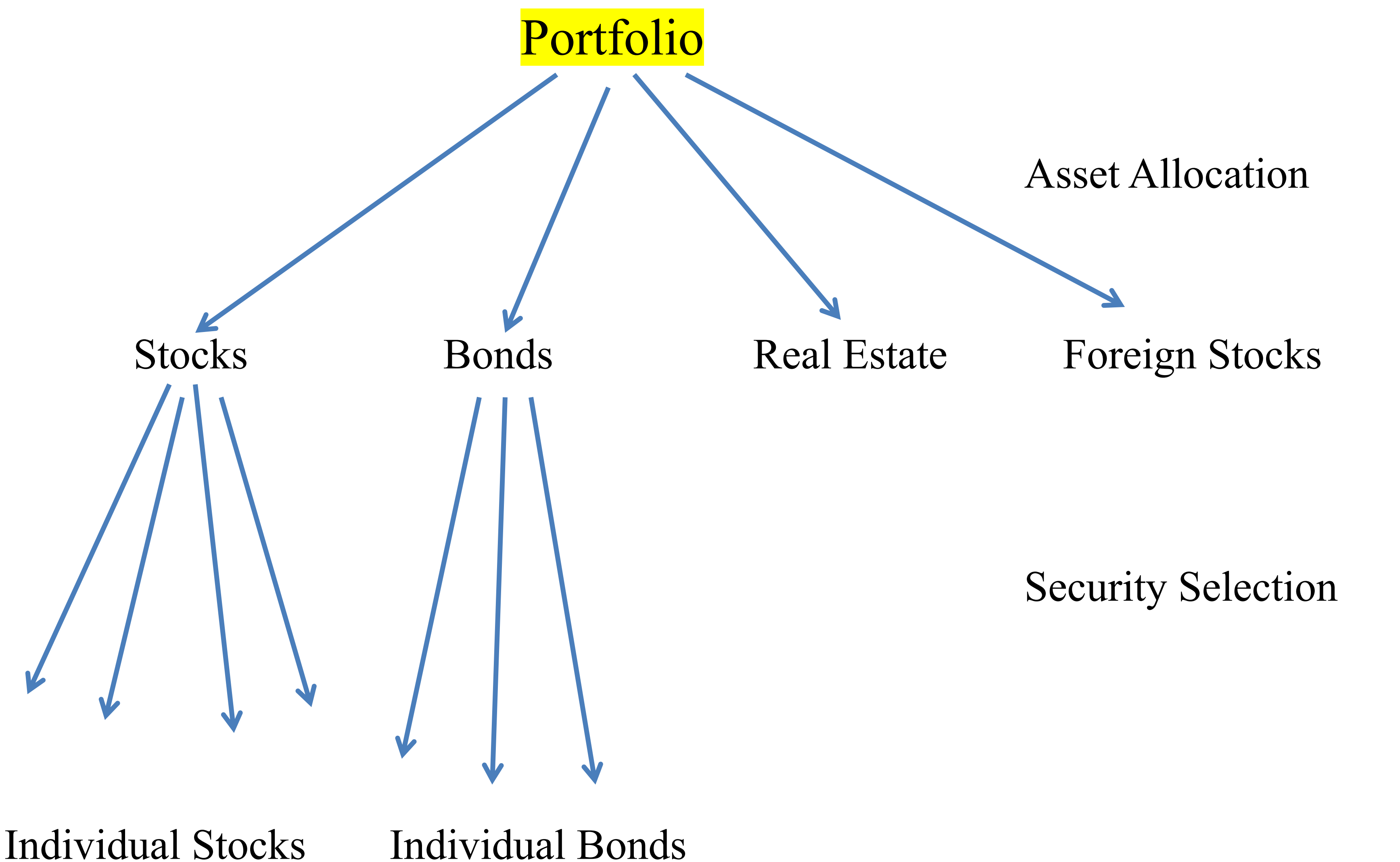
What types of investments can one make?

- Real assets vs. Financial assets
  - Tangible assets vs. Claims on assets
- Direct vs. Indirect financial investments
  - Individual securities vs. “pools” of assets
- Equities vs. Debt (capital structure of firms)
- Derivatives (risk management)
  - Futures, forwards, options, swaps

Basic Investment Philosophies

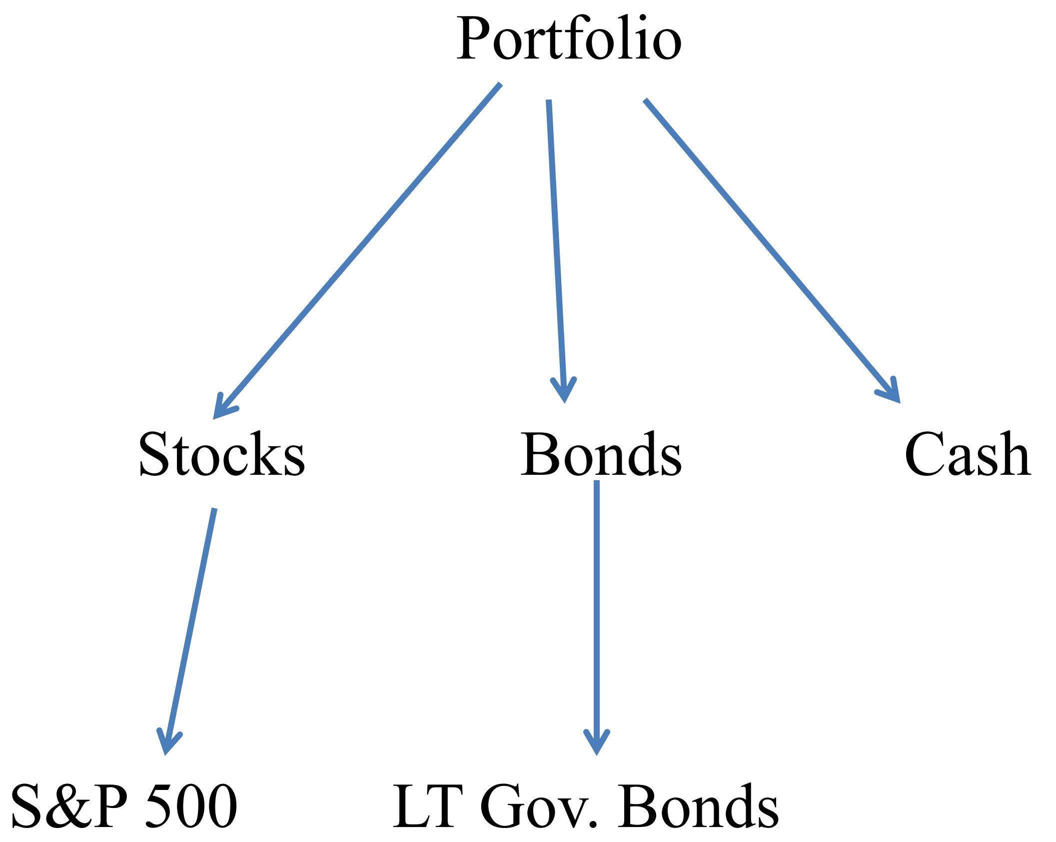
In forming an investment portfolio, several questions are paramount:

- In what types of securities should I invest?
  - Asset Allocation
- Within each security type, how do I select which assets to purchase?
  - Security Selection
- Finally, how active should I manage my portfolio?
  - Should I be an active or passive investor?


Portfolio
Asset Allocation
Stocks
Bonds
Real Estate
Foreign Stocks
Security Selection
Individual Stocks
Individual Bonds

Summarizing Basic Strategies

	Asset Allocation	Security Selection
Active	Market timing	Stock picking
Passive	Maintain pre-determined allocation(s)	Try to track a well-known market index


Portfolio
Stocks
Bonds
Cash
S&P 500
LT Gov. Bonds

Security Types

What are the different types of securities that are routinely bought and sold in financial markets around the world?

Classifying Securities

Basic Types	**Major Subtypes**
Interest-bearing	Money market instruments Fixed-income securities
Equities	Common stock Preferred stock
Derivatives	Options, Forwards Futures, Swaps

Interest-Bearing Assets

Money market instruments

Short-term debt obligations of large corporations and governments that mature in a year or less.

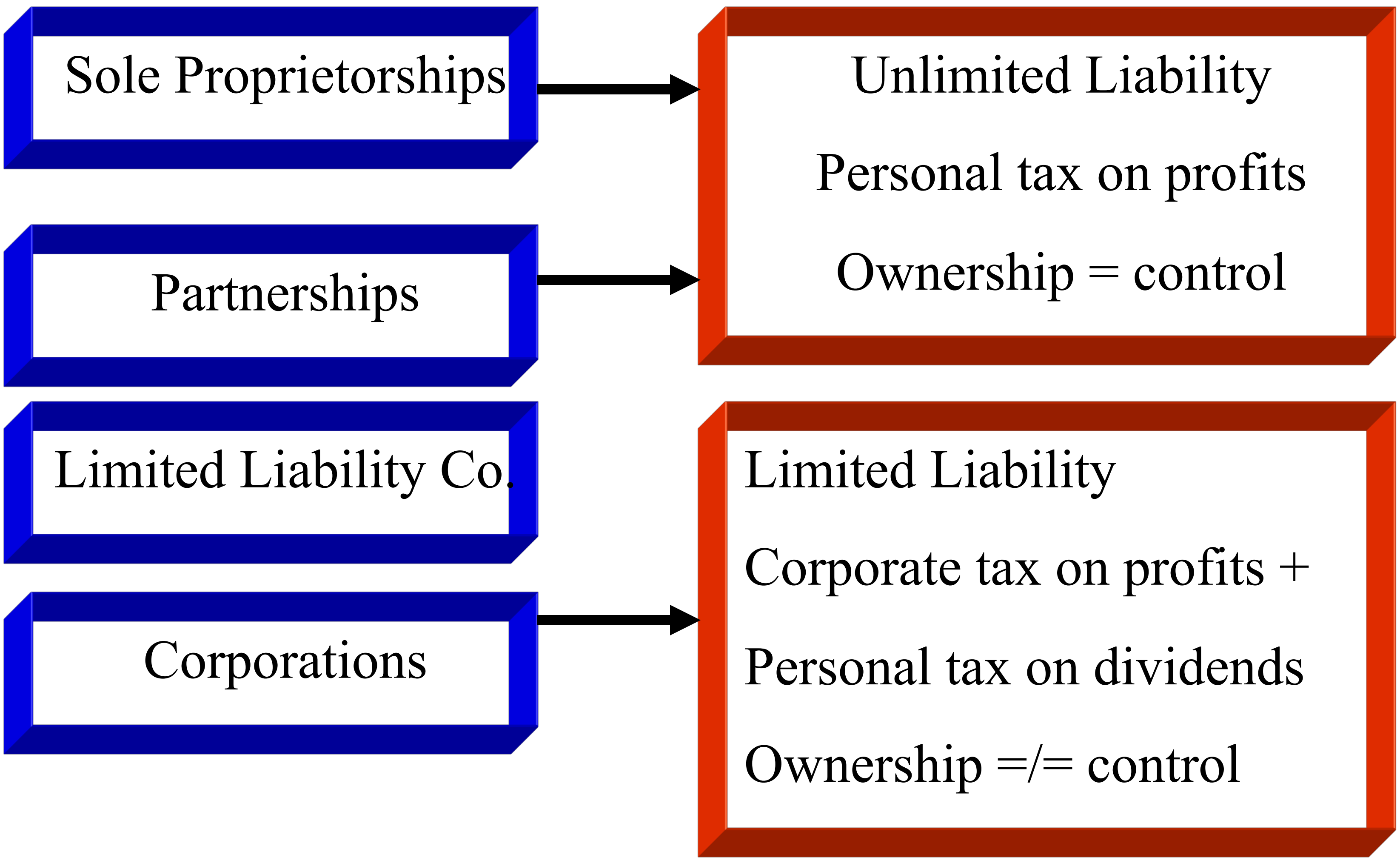
- *Examples:* U.S. Treasury bills (T-bills), bank certificates of deposit (CDs), corporate and municipal money market instruments.
- *Potential gains/losses:* Fixed future payment, except when the borrower defaults.

Fixed-income securities

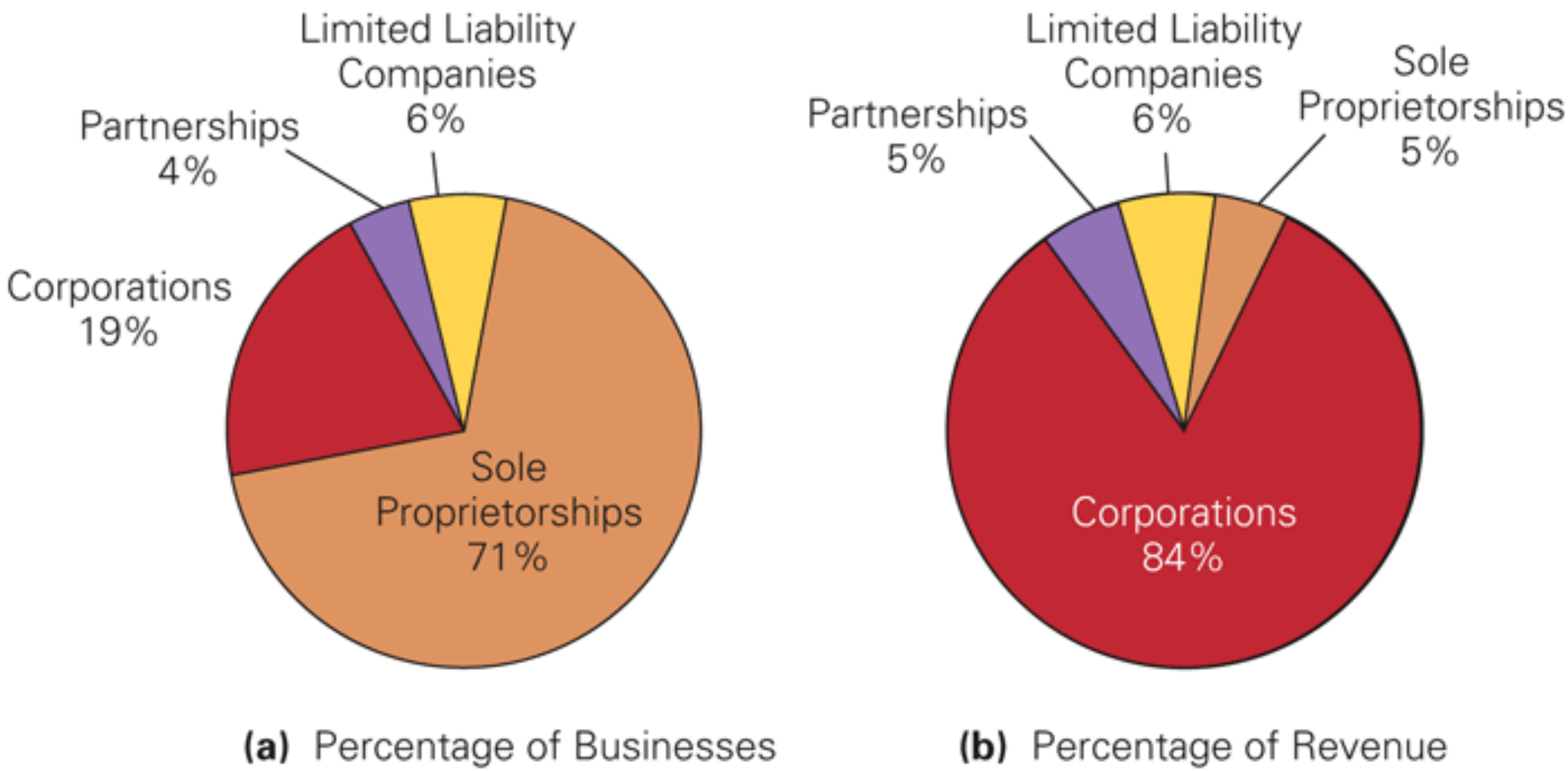
Longer-term debt obligations, often of corporations or governments, that promise to make fixed payments according to a preset schedule.

- *Examples:* U.S. Treasury notes, corporate bonds, car loans, student loans, mortgages.
- *Potential gains/losses:*
  - Fixed coupon payments and final payment at maturity, except when the borrower defaults.
  - Possibility of gain/loss from fall/rise in interest rates.
  - Can be quite illiquid and quite risky (sub-prime mortgages)

Corporate Structure



Types of U.S. Firms



Source: www.bizstats.com

Equities

Common stock

Represents ownership in a corporation. A part owner receives a pro rata share of dividends and whatever is left over after all obligations have been met in the event of a liquidation. Elect directors.

Preferred stock

The dividend is usually fixed and must be paid before any dividends for the common shareholders. In the event of a liquidation, preferred shares have a particular face value.

Common Stock

- *Examples:* IBM shares, Microsoft shares, etc.
- *Potential gains/losses:*
  - Many companies pay cash dividends to their shareholders. However, neither the timing nor the amount of any dividend is guaranteed.
  - The stock value may rise or fall depending on the prospects for the company and market-wide circumstances.

Preferred Stock

- *Example:* Citigroup preferred stock.
- *Potential gains/losses:*
  - Dividends are "promised." However, there is no legal requirement that the dividends be paid, as long as no common dividends are distributed.
  - The stock value may rise or fall depending on the prospects for the company and market-wide circumstances.

Derivatives

Primary asset

Security originally sold by a business or government to raise money.

Derivative asset

A financial asset that is derived from an existing traded asset rather than issued by a business or government to raise capital. Used to hedge risks or to speculate.

Derivatives

Futures contract

An agreement made today regarding the terms of a trade that will take place later.

Option contract

An agreement that gives the owner the right, but not the obligation, to buy or sell a specific asset at a specified price for a set period of time.

Buyers and Sellers: Zero net supply. Zero Sum Game.

Futures Contracts

- *Examples:* financial futures, commodity futures.
- *Potential gains/losses:*
  - At maturity, you gain if your contracted futures price is lower than the market price of the underlying asset.
  - If you sell your contract before its maturity, you may gain or lose depending on the market price for the contract.
  - Note that enormous gains/losses are possible.

Option Contracts

- A *call option* gives the owner the right, but not the obligation, to *buy* an asset, while a *put option* gives the owner the right, but not the obligation, to *sell* an asset.
- The price you pay to buy an option is called the *option premium*.
- The specified price at which the underlying asset can be bought or sold is called the *strike price*, or *exercise price*.

Option Contracts

- An *American option* can be exercised anytime up to and including the expiration date, while a *European option* can be exercised only on the expiration date.
- Options differ from futures in two main ways:
  ① There is no obligation to buy/sell the underlying asset.
  ② There is a premium associated with the contract.

Option Contracts

- *Potential gains/losses:*
  - Buyers gain if the exercise price is lower (call) or higher (put) than the market price of the underlying asset, and if the difference is greater than the option premium. In the worst case, buyers lose the entire premium.
  - Sellers gain the premium if the market price is lower (call) or higher (put) than exercise price.

Managed Investments

Investment companies sell shares in themselves and use the proceeds to invest in other investment instruments.

- Closed-end investment companies: offer a fixed number of shares.
- Open-end investment companies (Mutual Funds): offer fluctuating number of shares based on purchases/sales of fund shares.
  - Stock funds, Bond funds, Money market funds, Mixed funds
- Exchange Traded Funds (ETF): trades like a stock

Managed Investments

- Hedge Funds: typically act as a partnership where one partner manages funds for all other partners according to some investment strategy. Great flexibility in strategies. Largely unregulated. “Accredited Investors” only.
- Venture Capital pools: Similar to hedge funds, these partnerships obtain an equity interest in promising start-up or privately held firms.
- Real Estate Investment Trusts (REITs): provides investors with an indirect means of investing in real estate.

Topic 2: Present Values

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Learning objectives

- Our goal is to develop a framework for valuing streams of (potentially uncertain) future cash flows
  - value bonds
  - value stocks
  - value investment projects – capital budgeting
  - value companies
- Must account for time value of money and uncertainty

Interest on Risk Free Loans

- Let's start with risk free loans.
- Suppose you make a $100 deposit into a bank that pays, say, 5% per year.

0 yr		1 yr
\$100 →	$100 \times (1+0.05) =$	\$105

- This is the time value of money (nominal)

- Similarly, in 2 years:

<u>0 yr</u> <u>2 yr</u>

$100 \rightarrow 100\times1.05\times1.05 = 100\times1.05^2 = 110.25$

Compound value; you get interest on the interest

- If you put the \$100 for 10 years at 5%, $100\times1.05^{10} = 162.89$
- If you put \$PV for n-years and the interest is i, what will the future value (FV) be?

Future Value

$$FV_n = PV_0 \times (1+i)^n$$

FV_n : Future value at period n.

PV_0 : Present value (at period 0).

i : Interest rate rate per period.

n : Number of periods.

Can be easily obtained in Excel, Tables, Calculators.

Future Value: Example

- How much will be in an account if $1,500 is deposited and allowed to compound for 20 years at 8% (compounded annually)?

$$FV_{20} = 1,500 \times (1.08)^{20}$$
$$= 1,500 \times 4.661 = 6,991.44$$

- In 2035, the account will have $6,991.44.

- Note: The interest rate is assumed fixed over the period.

Future Value vs. Present Value

- Suppose someone wants to sell you an account that will have $6,991.44 in 2035.
- Q: How much will you pay for this account today if the interest is 8%?

- PV: what is the value today of some future amount?

Present Value (PV)

Recall that

$$FV_n = PV_0 \times (1+i)^n$$

$$PV_0 = FV_n \times \left[\frac{1}{(1+i)^n} \right]$$

We call $\left[\frac{1}{(1+i)^n} \right]$ the n-period discount factor.

Present Value: Example

- President Obama wants to borrow from you. He promises to give you \$10,000 three years from today (3 year Note).
- When alternative investment opportunities are available at 5% per year, how much are you willing to lend him today?

$$PV_0 = \$10,000 \times \left[\frac{1}{1.05^3} \right] = \$8,638.38$$

Compounding More Than One Time Per Year

- Interest may be paid semi-annually, quarterly, monthly, weekly, or daily.
- The Future Value Formula and the Present Value Formula need to be adjusted to handle interest being paid more than once a year.

Compounding More Than One Time Per Year

- Example
  - Suppose that you invest $1 million in an investment that promises to pay an annual interest rate of 6.4% for 6 years.
  - What is the future value of this investment for different compounding periods?

If interest is paid annually:

$$FV6 = \$1,000,000 \times (1.064)^{6} = \$1,450,941$$

If interest is paid semi-annually:

$$FV6 = \$1,000,000 \times (1.032)^{12} = \$1,459,340$$

If interest is paid quarterly:

$$FV6 = \$1,000,000 \times (1.016)^{24} = \$1,463,690$$

If interest is paid continuously:

$$FV6 = \$1,000,000 \times e^{.064 \times 6} = \$1,468,145$$

Future Value with Non-annual Compounding

$$FV_n = PV_0 \times \left(1 + \frac{i}{m}\right)^{m \times n}$$

FV_n : Future value at period n.

PV_0 : Present value (at period 0).

i : Interest rate rate per period.

n : Number of years.

m : Number of times compounded per year.

APR (Annual Percentage Rate)

- A credit card loan (or automobile loan, home improvement loan, etc.) that charges a monthly interest rate 1% has an APR of 12% (1% × 12 months)

  - **1+ effective annual rate**
    = (1+monthly rate)^{12}
    = (1.01)^{12} = 1.126825

  - In effect, 12% APR rate (1% per month) is equivalent to 12.6825%
    (The "Effective Annual Rate")

Effective Annual Rate Example

Compounding Interval	Equation	Effective Annual Rate
Annual (m=1)	EAR = $(1.12)^1 - 1$	12.00%
Semiannual (m=2)	EAR = $(1.06)^2 - 1$	12.36
Quarterly (m=4)	EAR = $(1.03)^4 - 1$	12.55
Monthly (m=12)	EAR = $(1.01)^{12} - 1$	12.68
Weekly (m=52)	EAR = $(1.0023)^{52} - 1$	12.73
Daily (m=365)	EAR = $(1.0003288)^{365} - 1$	12.7475
Continuously ($m=\infty$)	EAR = $e^{0.12} - 1$	12.7497

Effective Annual Rate

$$\text{Effective Interest Rate} = \left(1+\frac{i}{m}\right)^m - 1$$

- The effective annual rate increases with m.
- What happens to the effective annual rate when m $\rightarrow \infty$?

$$\lim_{m\to\infty}\left(1+\frac{i}{m}\right)^m = e^i$$

Technical Note: Definition of e

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e \quad = 2.7182818$$

Base of natural logarithms

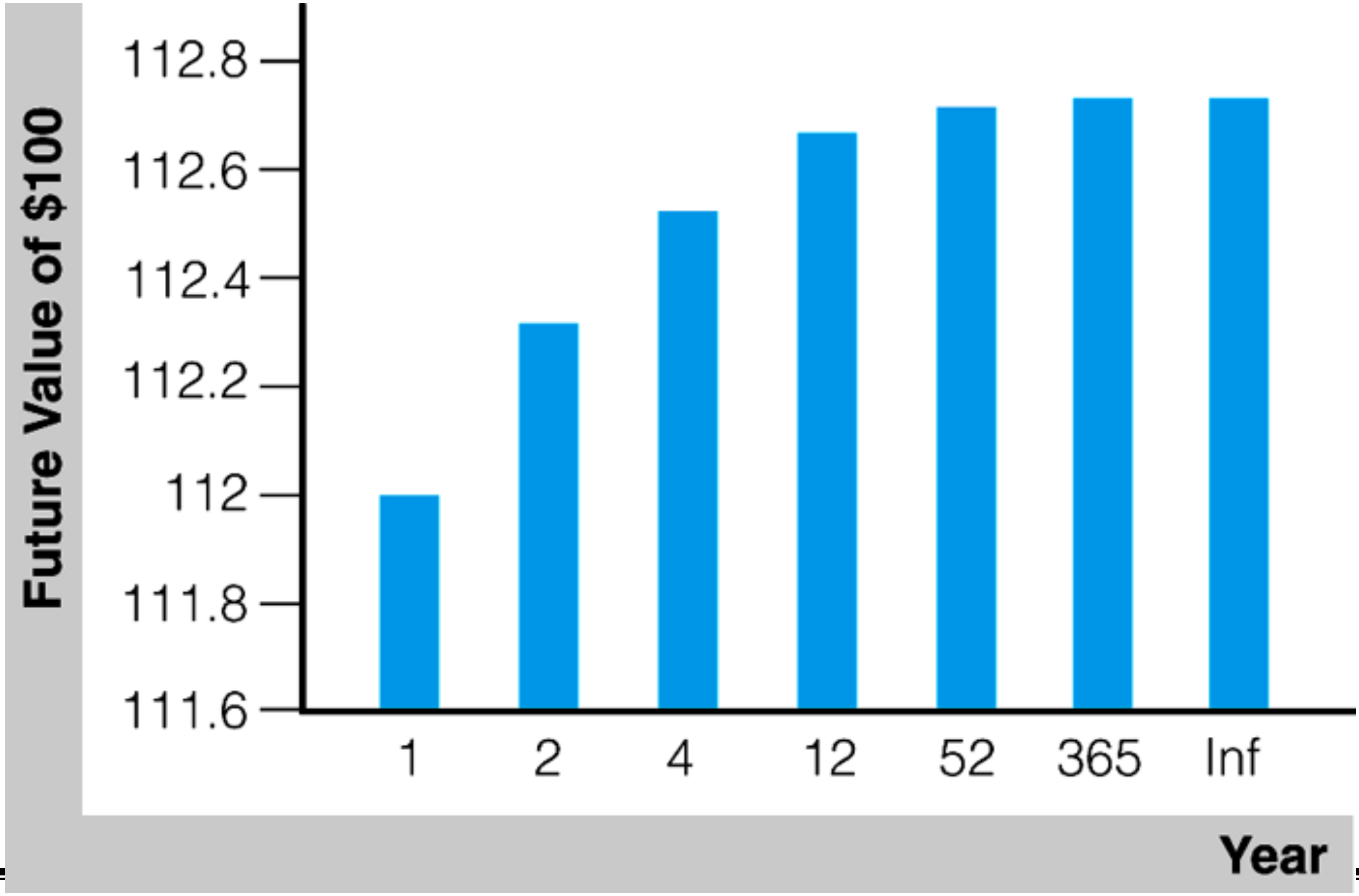
$$\lim_{m\to\infty}\left(1+\frac{i}{m}\right)^{m} = \lim_{m\to\infty}\left(1+\frac{1}{\frac{m}{i}}\right)^{\frac{m}{i}i} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{xi} = e^{i}$$

Continuous Compounding

- In the limit as we compound more and more frequently we obtain continuously compounded interest rates
- \$100 grows to **\$100e*^in^*** when invested at a continuously compounded rate *i* for time *n*
- \$100 received at time *n* discounts to **\$100e*^-in^*** at time zero when the continuously compounded discount rate is *i*

$$FV_n = PV_0 e^{ni} \qquad PV_0 = FV_n e^{-ni}$$

Future Value and Compounding Frequencies



Present Value with Non-annual Compounding

Recall that, when $m > 1$,

$$FV_n = PV_0 \times \left(1 + \frac{i}{m}\right)^{m \times n}$$

$$PV_0 = \frac{FV_n}{\left(1 + \frac{i}{m}\right)^{m \times n}}$$

Perpetuity

- A Perpetuity is an investment which makes constant payments forever ($C_1=C_2=C_3=\ldots=C$).
  - To finance Napoleonic wars, the British government borrowed money by issuing British Consol Bonds.
  - Instead of repaying their loans, the British government pays (to this day) a fixed annual payment in perpetuity (forever).

Valuing Perpetuity

- If a British Consol Bond pays £4 a year forever and the interest rate is 5.2% per year, what is the value of the bond?

```
0          1      2          3…          forever...
|--------|-------|-------|---------
                                                      (i = 5.2%)
          £4      £4        £4...        forever…
```

First payment is at the end of the first year

$$\text{Present value of } PMT_1: \ 4 \times \frac{1}{1.052}$$

$$\text{Present value of } PMT_2: \ 4 \times \frac{1}{1.052^2}$$

$$\text{Present value of } PMT_2: \ 4 \times \frac{1}{1.052^3}$$

$$\vdots$$

$$PV_0 = \frac{4}{1.052} + \frac{4}{1.052^2} + \frac{4}{1.052^3} + \ldots = \sum_{j=1}^{\infty} \frac{4}{1.052^j}$$

Present Value of a Perpetuity

$$PV_0 = \frac{4}{1.052} + \frac{4}{1.052^2} + \frac{4}{1.052^3} + \ldots$$

Let us multiply both sides by $\frac{1}{1.052}$

$$\frac{1}{1.052} PV_0 = \frac{4}{1.052^2} + \frac{4}{1.052^3} + \ldots$$

Let us subtract the second equation from the first equation:

$$PV_0 - \frac{1}{1.052} PV_0 = \frac{4}{1.052} \Rightarrow \left(1.052 - 1\right) PV_0 = 4$$

$$PV_0 = \frac{4}{0.052} = 76.9231$$

Present Value of a Perpetuity

- In general, the present value of a perpetuity that pays PMT forever is

$$PV_{perpetuity} = \frac{PMT}{i}$$

For example, when $i = 5\%$, a perpetuity that pays \$100 every year forever has a present value of

$$PV_{perpetuity} = \frac{100}{0.05} = \$2,000$$

Growing Perpetuity

- Pays \$PMT in the next period and grows at rate of g (< i) thereafter forever.

0 1 2 3… forever...

|--------|-------|-------|---------

\$PMT \$PMT(1+g) \$PMT$(1+g)^2$ **... forever…**

- Convenient assumption for the valuation of many projects.

$$PV_{perpetuity} = \frac{PMT}{1+i} + \frac{PMT(1+g)}{(1+i)^2} + \frac{PMT(1+g)^2}{(1+i)^3} + \bullet\bullet\bullet$$

Present Value of a Growing Perpetuity

$$PV_{perpetuity} = \frac{PMT}{i-g}$$

For example, when $i = 5\%$, a perpetuity that pays \$100 in the first year and grows 3% per year thereafter forever has a present value of

$$PV_{perpetuity} = \frac{100}{0.05-0.03} = \frac{100}{0.02} = \$5,000$$

Example: Gordon's Constant Growth Model

- A common stock is expected to pay a dividend of $10 next year and dividends are expected to grow at a constant rate of g=10% thereafter. If the opportunity cost of investing in this common stock is i=15%, what is the current value of the common stock?

Example: Gordon's Constant Growth Model

- D=$10
- g=10%
- i=15%

$$PV = \frac{10}{0.15 - 0.10} = \frac{10}{0.05} = \$200$$

- Without the growth in dividends, the value of the stock is only $66.67.

$$PV = \frac{10}{0.15} = \frac{10}{0.15} = \$66.67$$

Annuity

- Annuity
  - A stream of constant cash flows that lasts for a fixed number of periods (as opposed to a perpetuity that lasts for ever)
    - Ordinary Annuity: end of period
    - Annuity Due: beginning of period

Ordinary Annuity

- Stream of constant cash flows (PMT) for fixed number of periods.

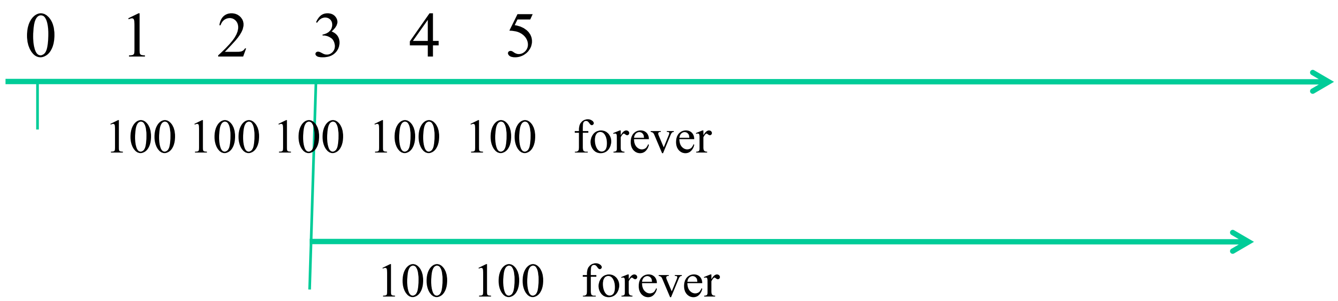
0 1 2 3 years

|--------|-------|-------| (i = 10%)

$100 $100 $100

$$PV = \frac{100}{1+0.1} + \frac{100}{(1+0.1)^2} + \frac{100}{(1+0.1)^3}$$

- You can think of this as the difference of two perpetuities: one starting at time 0 and the other at time 3.



$$PV = \frac{PMT}{i} \qquad PV = \frac{PMT}{i}$$

$$PV_{annuity} = \frac{PMT}{i} - \frac{1}{(1+i)^n}\frac{PMT}{i} = \frac{100}{0.10} - \frac{1}{(1+0.10)^3}\frac{100}{0.10}$$

PV of an Ordinary Annuity

$$PV_{annuity} = PMT \times \underbrace{\frac{1 - \frac{1}{(1+i)^n}}{i}}_{PVIFA_{i,n}}$$

when $PMT = \$100,\ i = 10\%,$ n=3 years,

$$PV_{annuity} = \$100 \times \underbrace{\frac{1 - \frac{1}{(1.1)^3}}{0.1}}_{PVIFA_{10\%,3}} = \$249$$

Example: Winning Big at a Slot Machine

- In May 1992, a 60 year old nurse "invested" \$12 in a Reno casino and walked away with \$9.3 million. The sum was to be paid out in 20 annual installments of \$465,000 each (20x\$465,000 = \$9.3 million).
- What is the "present value" of the prize?

- The interest rate at the time was 8%.

$$PV = \$465,000 \times \underbrace{\frac{1 - \frac{1}{(1.08)^{20}}}{0.08}}_{PVIFA_{8\%,20}}$$

$$= \$465,000 \times 9.818 = \$4.565MM$$

- From the perpetuity formula, we would get $5.813MM

Example: Fixed Rate, Level Payment Mortgage

- What is the monthly payment for a 30 year, 12% Annual Percentage Rate (APR), $100,000 mortgage?

Example: Fixed Rate, Level Payment Mortgage

- What is the monthly payment for a 30 year, 12% compounded monthly, $100,000 mortgage?

12% APR means 1% per month.

30 years = 360 months.

$$100,000 = PMT \times \underbrace{\frac{1 - \frac{1}{(1.01)^{360}}}{0.01}}_{PVIFA_{1\%,360}} = PMT \times 97.218331$$

$$PMT = \frac{100,000}{97.218331} = \$1,028.61$$

- This amount includes principal and interest.
- Principal at time 0: $100,000.00
- First month payment: $1,028.61
- Interest (1% per month): $1,000.00
- Principal payment $ 28.61
- Principal at time 1: $99,971.39
- At the beginning payment is mostly interest.
- Interest is tax deductable, principal is not.
- If prepayment is possible, you need to know principal outstanding.

FV of an Ordinary Annuity

$$FV_{annuity} = PMT \times \underbrace{\frac{(1+i)^n - 1}{i}}_{FVIFA_{i,n}}$$

when $PMT = \$100,\ i = 10\%,\ \text{n=3 years},$

$$FV_{annuity} = \$100 \times \underbrace{\frac{(1.1)^3 - 1}{0.1}}_{FVIFA_{10\%,3}} = \$331$$

Example: Retirement Planning

- You will retire in 30 more years.
- You would like to have $1,000,000 by your retirement date to support your modest lifestyle.
- How much must you save each year between now and then to meet this goal, assuming a 3% interest rate?
- How about 10% interest rate?

$$1{,}000{,}000 = PMT \times \frac{(1.03)^{30} - 1}{0.03} = PMT \times 47.56$$

$$PMT = \frac{1{,}000{,}000}{47.56} = 21{,}019$$

$$1{,}000{,}000 = PMT \times \frac{(1.10)^{30} - 1}{0.10} = PMT \times 164.49$$

$$PMT = \frac{1{,}000{,}000}{164.49} = 6{,}079$$

Prices of different types of Bonds

1. *Coupon Bonds*

$$P = \frac{C}{(1+r)} + \frac{C}{(1+r)^2} + \bullet\bullet \quad + \frac{C+F}{(1+r)^n} = \sum_{t=1}^{n} \frac{C}{(1+r)^t} + \frac{F}{(1+r)^n}$$

2. *Consol Bonds*

$$P = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

3. *Discount Bond*

$$P = \frac{F}{(1+r)^n}$$

4. *Amortizing Bond* : *mortgage*

$$P = \sum_{t=1}^{n} \frac{A}{(1+r)^t}$$

5. *Floating Rate Bonds*

To summarize PV

- To compute PV of an asset we need:
  - Discount rate (r)
  - Future cash flows
- So far we have assumed that we know them with certainty.
- But, for most of the assets we will be interested in valuing both of those quantities are more or less unknown or uncertain.
- Herein lies the valuation problem.

Valuation of uncertain cash flows

- Discount the future cash flows by the opportunity cost of capital (the expected rate of return that can be earned on securities of comparable risk)
- Things get more difficult here because:
  - If future cash flows are *uncertain* there is a probability distribution of future cash flows?
  - How do you measure risk and determine the opportunity cost of capital?

Valuation of uncertain cash flows

- An uncertain future cash flow *D* is a random variable with an expected value *E[D]*
- We discount *E[D]* to the present using as discount rate the expected return on an investment with similar risk

  r = risk-free rate + risk premium

- Present value

$$\frac{E[D]}{1+r}$$

Gross and Net Return

$$P_0 : price\ today \quad 100$$

$$P_1 : price\ next\ period \quad 110$$

$$D_1 : dividend\ next\ period \quad 5$$

$$Gross\ return \quad R_1 = \frac{P_1 + D_1}{P_0} = \frac{110+5}{100} = 1.15$$

$$Net\ return \quad r_1 = \frac{P_1 + D_1}{P_0} - 1 = .15 = 15\%$$

$$r_1 = \frac{D_1}{P_0} + \frac{P_1 - P_0}{P_0} = dividend\ yield + capital\ gain\ or\ loss$$

The present value formula

- In general for any period t, definition of returns

$$r_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1$$

  - Realized returns are very different from expected returns!!!
- You want asset to give a certain *expected return*

$$\mathrm{E}[r_{t+1}] = r$$

- Where *r* compensates you for
  - Time
  - Inflation
  - Risk of asset's cash flows

The present value formula

- Then

$$1+r=\frac{E[D_{t+1}+P_{t+1}]}{P_t}$$

- Or

$$P_t=\frac{E[D_{t+1}+P_{t+1}]}{1+r}$$

- But we also have

$$P_{t+1}=\frac{E[D_{t+2}+P_{t+2}]}{1+r}$$

- Substituting

$$P_t=\frac{E[D_{t+1}]}{1+r}+\frac{E[D_{t+2}]}{(1+r)^2}+\frac{E[P_{t+2}]}{(1+r)^2}$$

The present value formula

- And so on

$$P_t = \frac{E[D_{t+1}]}{1+r} + \frac{E[D_{t+2}]}{(1+r)^2} + \frac{E[D_{t+3}]}{(1+r)^3} + \ldots$$

- The value of an asset is the *sum of discounted expected future cash flows*
- The expected return *r* is also called the *discount rate* or the *required rate of return*
- What is the right discount rate?
  - Need an asset pricing model such as the CAPM to tell us how to measure risk and what is the compensation for risk

Uses of present value formula

- Present values are the keystone to finance
- Some uses:
  - Determining fair prices of securities
  - Making capital budgeting decisions
- Present value problems involve streams of *cash flows*
  - Exchange present dollars for future (uncertain) dollars, or vice versa

Implications of present value formula

- Prices should (only) move when
  - Expected future cash flows change
  - Discount rate changes
    - Changes in riskless interest rate (time value of money)
      - Expected inflation
      - Investor impatience
    - Changes in risk premium (higher risk or increase in risk aversion

- Understanding the effect of economic reports

Bond Prices

Bond with maturity n:

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \ldots + \frac{C + FV}{(1+r)^n}$$

News about inflation make bond prices fall:
Higher (nominal) discount rate and same expected cash flows

Stock Prices

$$P_t = \frac{E[D_{t+1}]}{1+r} + \frac{E[D_{t+2}]}{(1+r)^2} + \frac{E[D_{t+3}]}{(1+r)^3} + \ldots$$

News about recession make stocks fall:
Expected cash flows drop

Topic 3: Risk and Return

Eduardo Schwartz
UCLA Anderson School

Definition

- Gross return

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t}$$

- Net return

$$r_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1$$

where

P_t : price today

P_{t+1} : price tomorrow

D_{t+1} : dividend tomorrow

Net return

$$r_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1$$

$$= \frac{D_{t+1}}{P_t} + \frac{P_{t+1} - P_t}{P_t}$$

$$= \text{income yield} + \text{capital gain/loss}$$

- Income yield : cash payout
- Capital gain/loss : change in security price

Expected versus Realized return

- At the start of the period, some variables are not known, so we can calculate only expected return

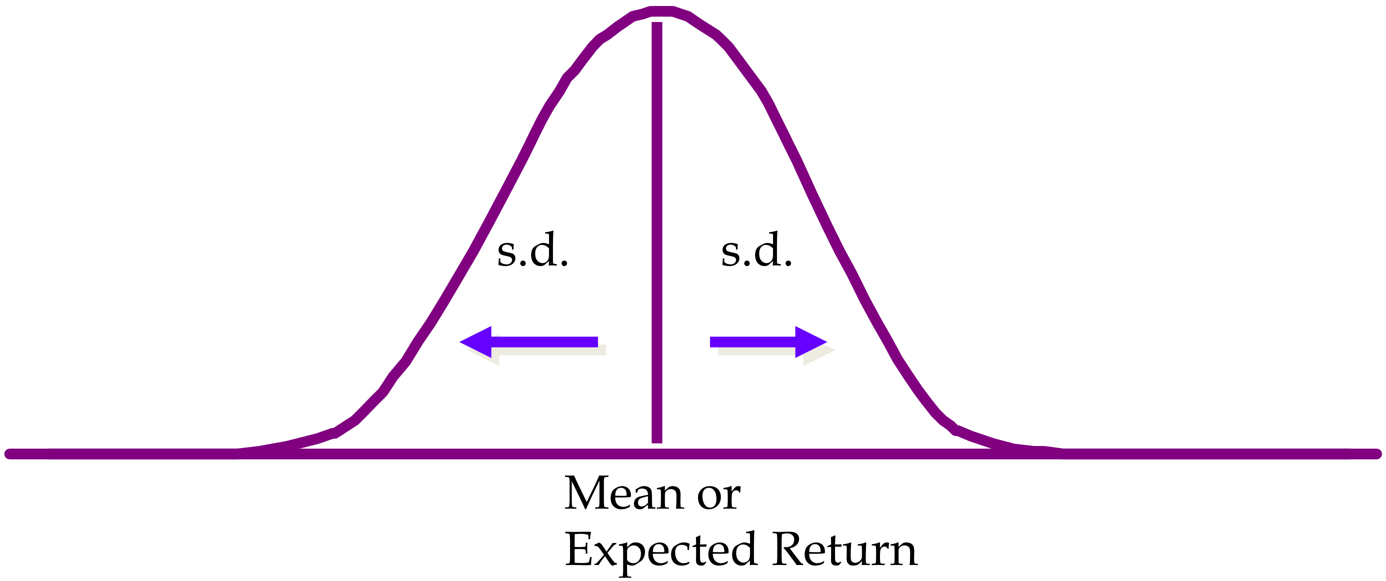
$$\text{Expected Return; } \mathrm{E}_t\left[r_{t+1}\right] = \frac{\mathrm{E}_t\left[P_{t+1}\right] + \mathrm{E}_t\left[D_{t+1}\right]}{P_t} - 1$$

$$\text{Realized Return; } \quad r_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} - 1$$

- ‘Expected’ and ‘Realized’ returns are very different!!!

Descriptive statistics

- Mean : Expected return
- Variance (standard deviation) : dispersion
- Symmetric Distribution



The Normal Distribution

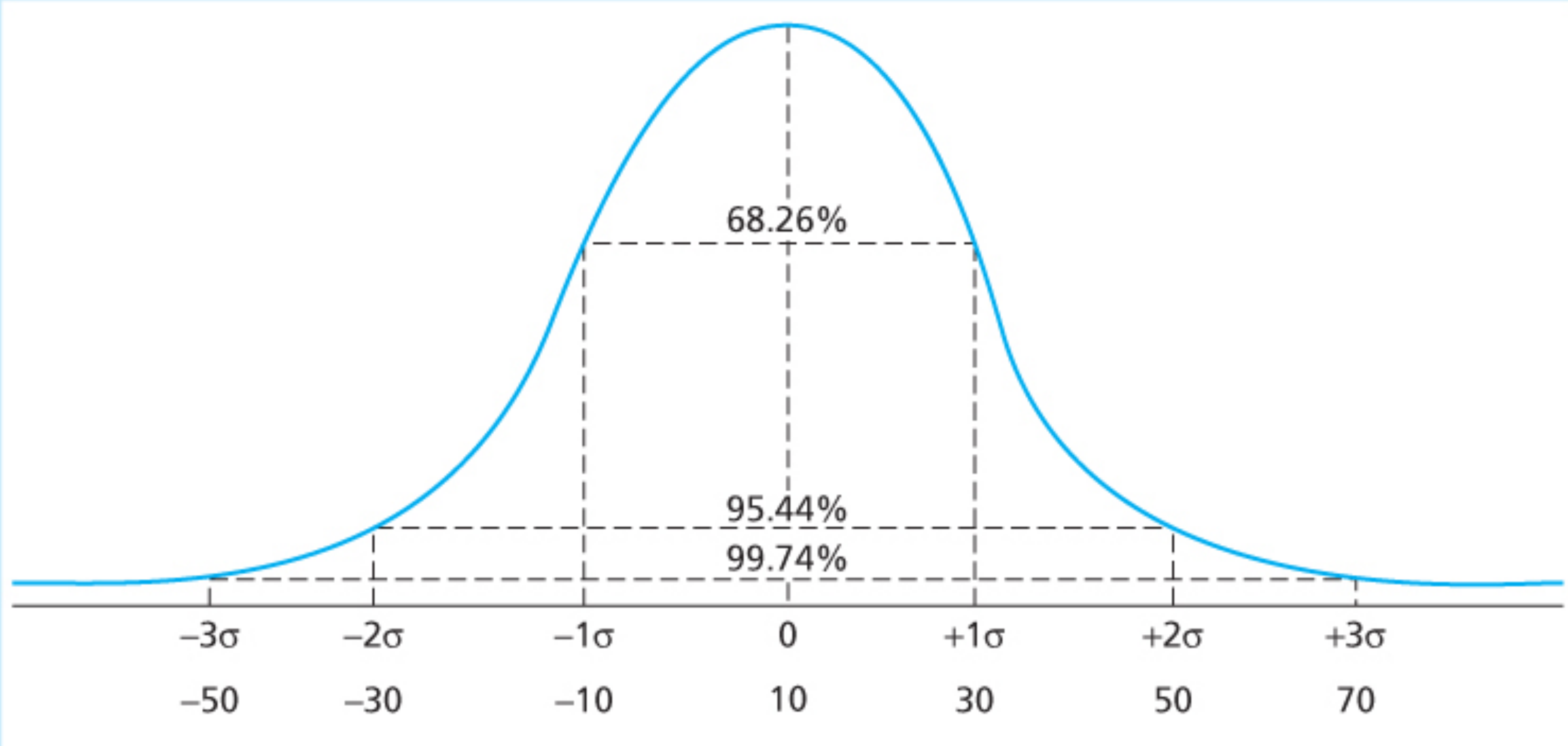
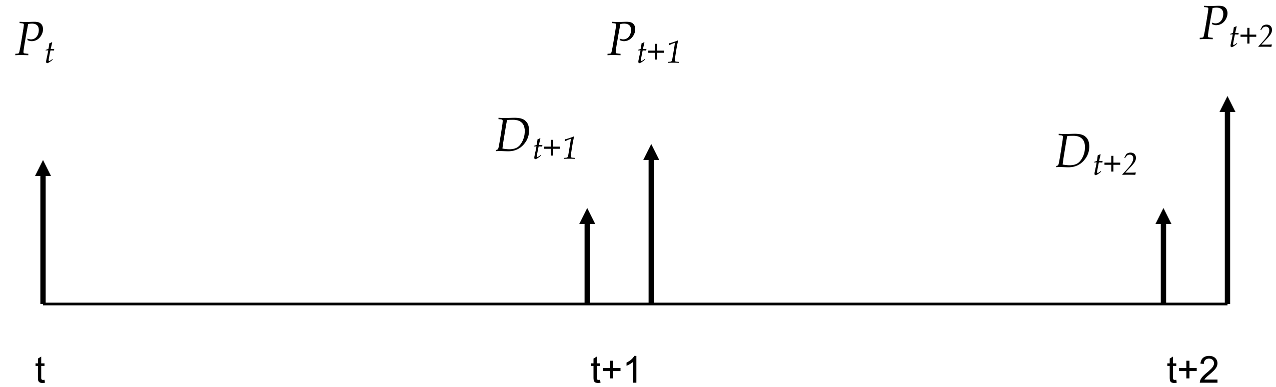


Figure 5.4 The normal distribution with mean 10% and standard deviation 20%.

Multi-period returns



- r_{t+1} : return from time t to t+1 $= \frac{P_{t+1} + D_{t+1}}{P_t} - 1$
- r_{t+2} : return from time t+1 to t+2 $= \frac{P_{t+2} + D_{t+2}}{P_{t+1}} - 1$
- $r_{t+2}(2)$: return from time t to t+2

$$(1 + r_{t+2}(2)) = (1 + r_{t+1})*(1 + r_{t+2})$$

Compounding returns

- Example: you earn 10% in year 1 and 20% in year 2; your 2-year return is:

$$(1+0.1)(1+0.2)-1=1.32-1=0.32=32\%$$

  - Calculation assumes that we immediately reinvest the dividends
  - The 2-year return is *not the sum of* annual returns (30%)
- What is the *average annual* return? What is the return per annum that would give us the same amount in 2 years?

Compounding returns

- We want a number such that

$(1+ r)^* (1+ r) = (1 + 0.1)^*(1 + 0.2) = 1.32$

$(1+ r)^* (1+ r) = (1 + r_{t+1})^*(1 + r_{t+2}) = 1.32$

$r = \sqrt{[(1 + r_{t+1})^*(1 + r_{t+2})]} - 1$

$r = \sqrt{1.1^*1.2} - 1 = 0.149 = 14.9\%$

  – Note $.32 = (1+.149)^2 - 1$

  – It is *not the arithmetic average* annual return [15%=(10%+20%)/2]

  – It is the *geometric average* of 10% and 20%

Arithmetic vs. Geometric Mean Return

Year	Re*turn*
1	0.30
2	−0.20
3	0.20
4	0.50

Arithmetic mean return

$$r_a = \frac{1}{T}\sum_{t=1}^{T} r_t = \frac{1}{4}0.80 = 0.20 \Rightarrow 20\%$$

Geometric mean return

$$(1+r_g)^T = (1+r_1)(1+r_2)\bullet\bullet\bullet\bullet(1+r_T) = (1.30)(0.80)(1.20)(1.50) = 1.87$$

$$1+r_g = [(1+r_1)(1+r_2)\bullet\bullet\bullet\bullet(1+r_T)]^{\frac{1}{T}}$$

$$1+r_g = 1.87^{0.25} \Rightarrow r_g = 0.17 \Rightarrow 17\%$$

Arithmetic vs. Geometric Mean Return

S_0	S_1	S_2
100	200	100

r_1=100% r_2=-50%

$r_a = (1.00\text{-}.50)/2 = .25 = 25\%$

$1+r_g = [(1+1.0)(1-.50)]^{0.5} = [(2)(.50)]^{0.5} = 1$

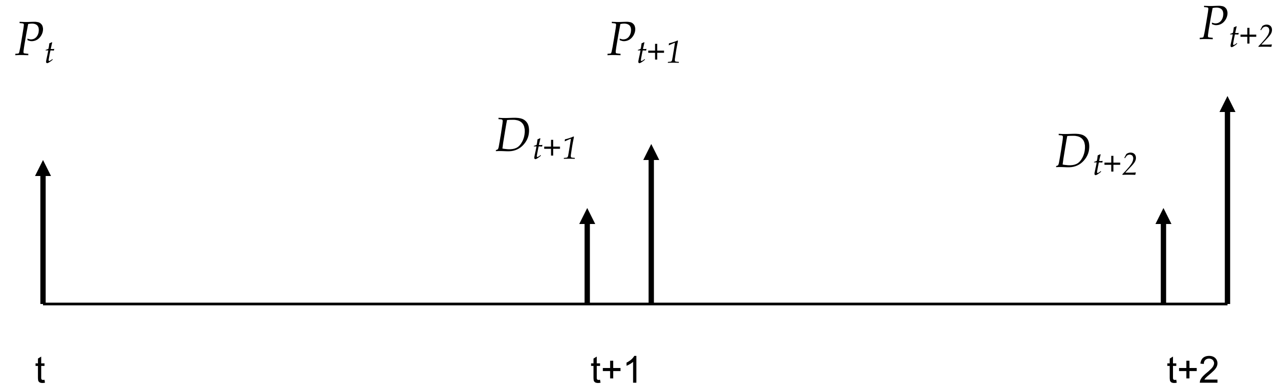
$r_g = 0\%$

Which one makes more sense?

Arithmetic vs. Geometric average

- Which one makes more sense?
  - Geometric average is an excellent measure of *past* performance
  - Arithmetic average is an unbiased estimate of the *expected future* return in one period
- Geometric average is always lower than the arithmetic average
  - Difference depends on the horizon and the variability of returns

Continuously compounded returns



$$e^{r_{t+1}^c} = \frac{P_{t+1} + D_{t+1}}{P_t} = 1 + r_{t+1} \qquad e^{r_{t+2}^c} = \frac{P_{t+2} + D_{t+2}}{P_{t+1}} = 1 + r_{t+2}$$

$$r_{t+1}^c = \ln\left(\frac{P_{t+1} + D_{t+1}}{P_t}\right) \qquad r_{t+2}^c = \ln\left(\frac{P_{t+2} + D_{t+2}}{P_{t+1}}\right)$$

Continuous compounded returns = Log returns

Continuously compounded returns

Data Annual return CCR

$$\frac{P_{t+1}+D_{t+1}}{P_t} = (1+r_{t+1}) = e^{r_{t+1}^c}$$

$$\frac{P_{t+1}+D_{t+1}}{P_t} = 1.01$$

$$r_{t+1}^c = \ln\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right) = \ln(1.01) = 0.00995$$

$$\frac{P_{t+1}+D_{t+1}}{P_t} = 1.10$$

$$r_{t+1}^c = \ln\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right) = \ln(1.10) = 0.09531$$

Continuous compounding

- Continuously compounded returns make our lives easier
  - Two-period returns are just the sum of one period returns

$$e^{r_{t+2}^c(2)} = e^{r_{t+1}^c} e^{r_{t+2}^c} = e^{r_{t+1}^c + r_{t+2}^c}$$

$$r_{t+2}^c(2) = r_{t+1}^c + r_{t+2}^c$$

- Average return

$$e^{r^c \times 2} = e^{r_{t+1}^c} e^{r_{t+2}^c} = e^{r_{t+1}^c + r_{t+2}^c}$$

$$r^c = \frac{r_{t+1}^c + r_{t+2}^c}{2}$$

Past and future

- We want to find the best possible portfolio to invest in.
- For that, we need to forecast returns in the future and evaluate the risk of the investments.
- The best guide for the future performance of assets is their past performance
  - But we need to be very careful extrapolating past performance to the future!!!

Historical averages

- We look at the historical returns of 5 portfolios
  - Treasury Bills: as safe as an investment we can get
  - Long-term government bonds: interest risk
  - Long-term corporate bonds: interest risk + default risk
  - S&P Composite Index (S&P 500): 500 stocks of over 7000, but they account for over 70% of market
  - Small Stocks: smallest 20% of NYSE

Historical averages

- Nominal and Real Averages
- We take averages over long periods of time (85 y) because returns are very volatile, especially for stocks
- Returns on these portfolios coincide with our intuitive risk ranking

Historical Averages

AVERAGE RATES OF RETURN (1928 - 2013 : 85 YEARS OF DATA*)

PORTFOLIO	AVERAGE ANNUAL RATE OF RETURN NOMINAL	REAL	AVERAGE RISK PREMIUM (EXTRA RETURN VS. TBILLS)
Treasury bills	3.6%	0.5%	0.0%
Long-Term government bonds (10 years)	5.2%	2.1%	1.6%
Long-Term corporate bonds	6.2%	3.1%	2.6%
Large company stocks	11.5%	8.4%	7.9%
Small company stocks	16.4%	13.3%	12.8%

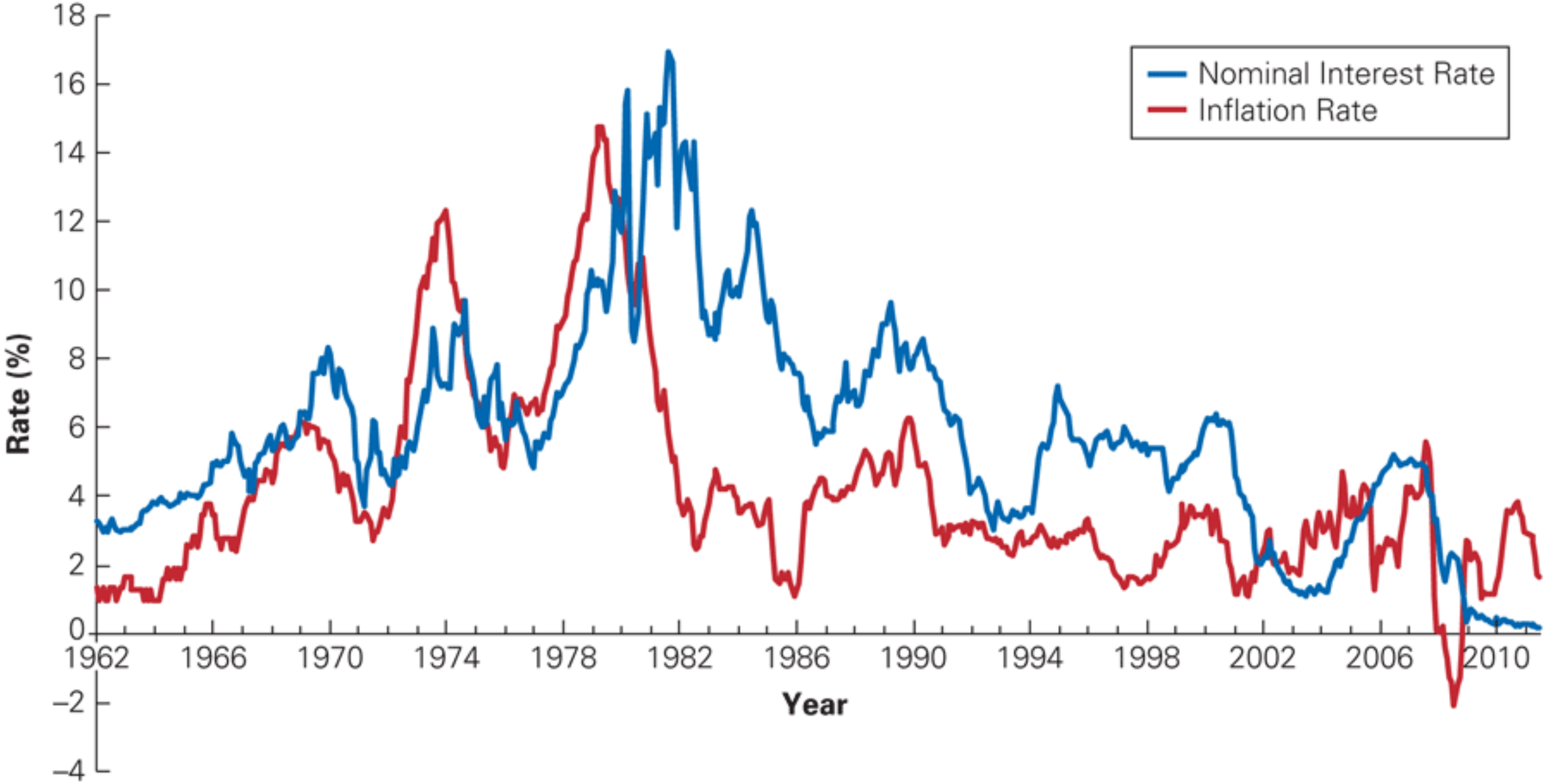
Equity risk premium

Inflation = 3.1%

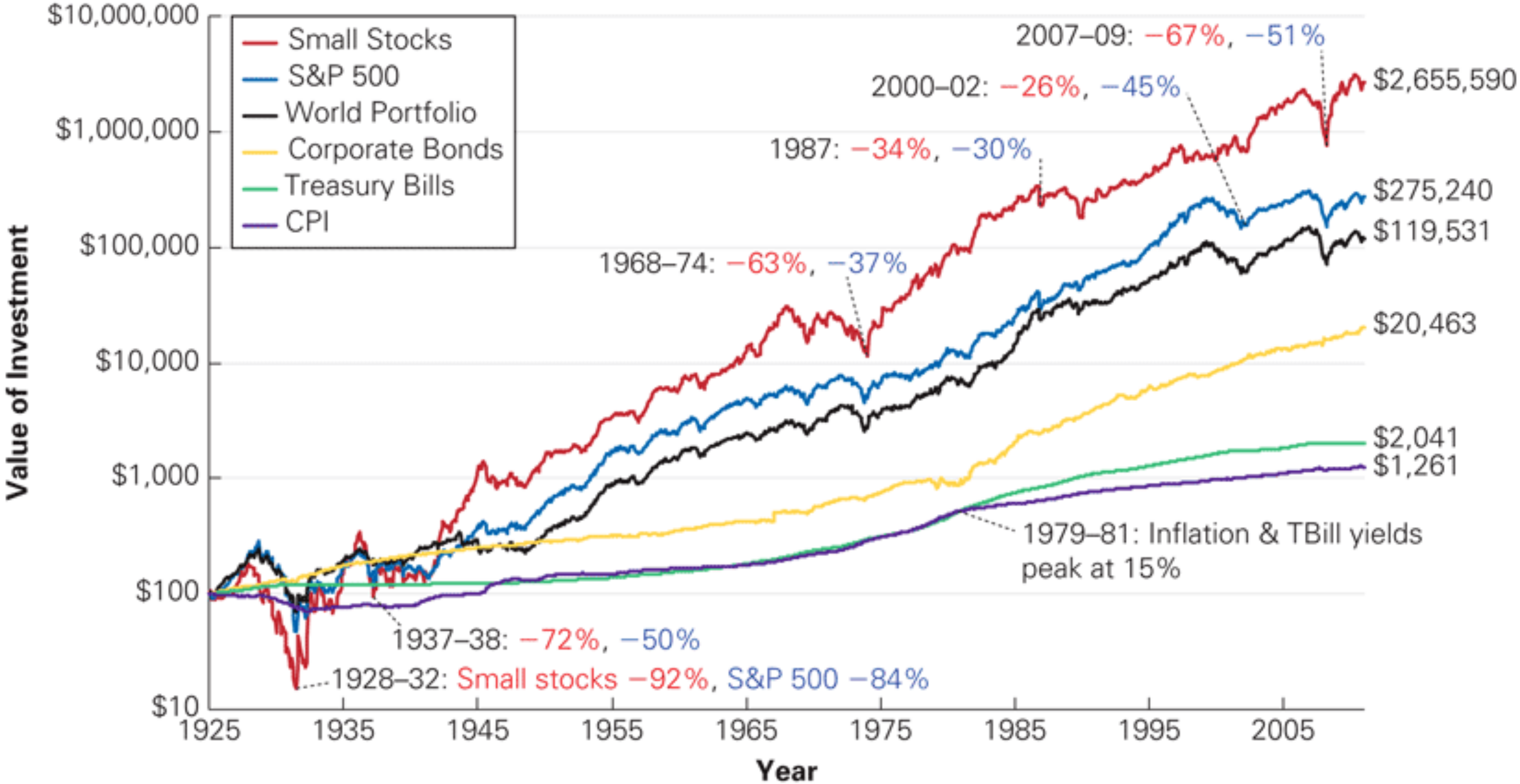
$r = r_f$ + risk premium

*Some of the data is for a slightly different period

U.S. Interest Rates and Inflation Rates,1960–2012



Value of $100 Invested at the End of 1925

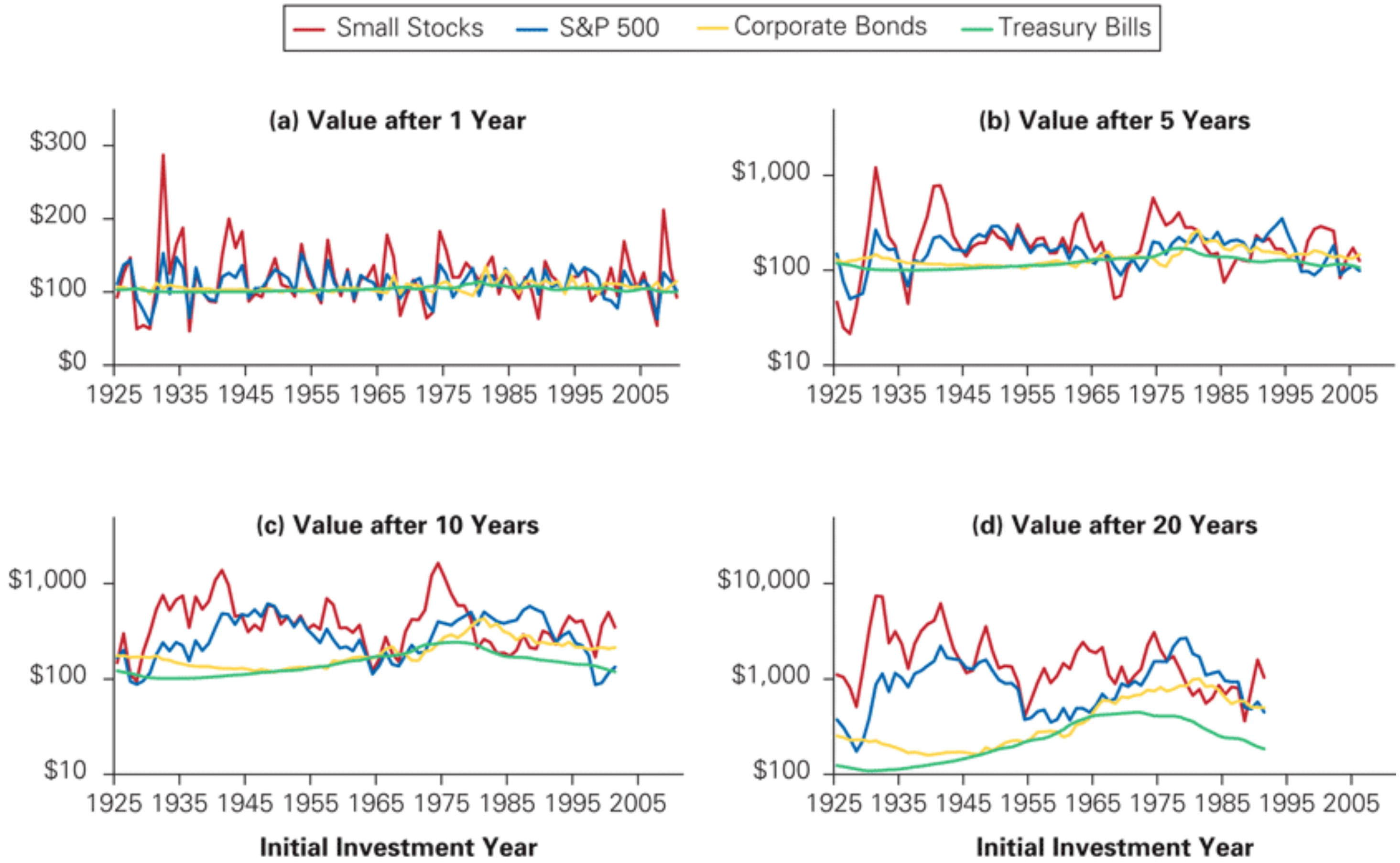


Source: Chicago Center for Research in Security Prices, Standard and Poor's, MSCI, and Global Financial Data.

Risk and Return: Insights from 85 Years of Investor History

- Few people ever make an investment for 85 years.
- More realistic investment horizons and different initial investment dates can greatly influence each investment's risk and return.

Value of $100 Invested for Alternative Investment Horizons

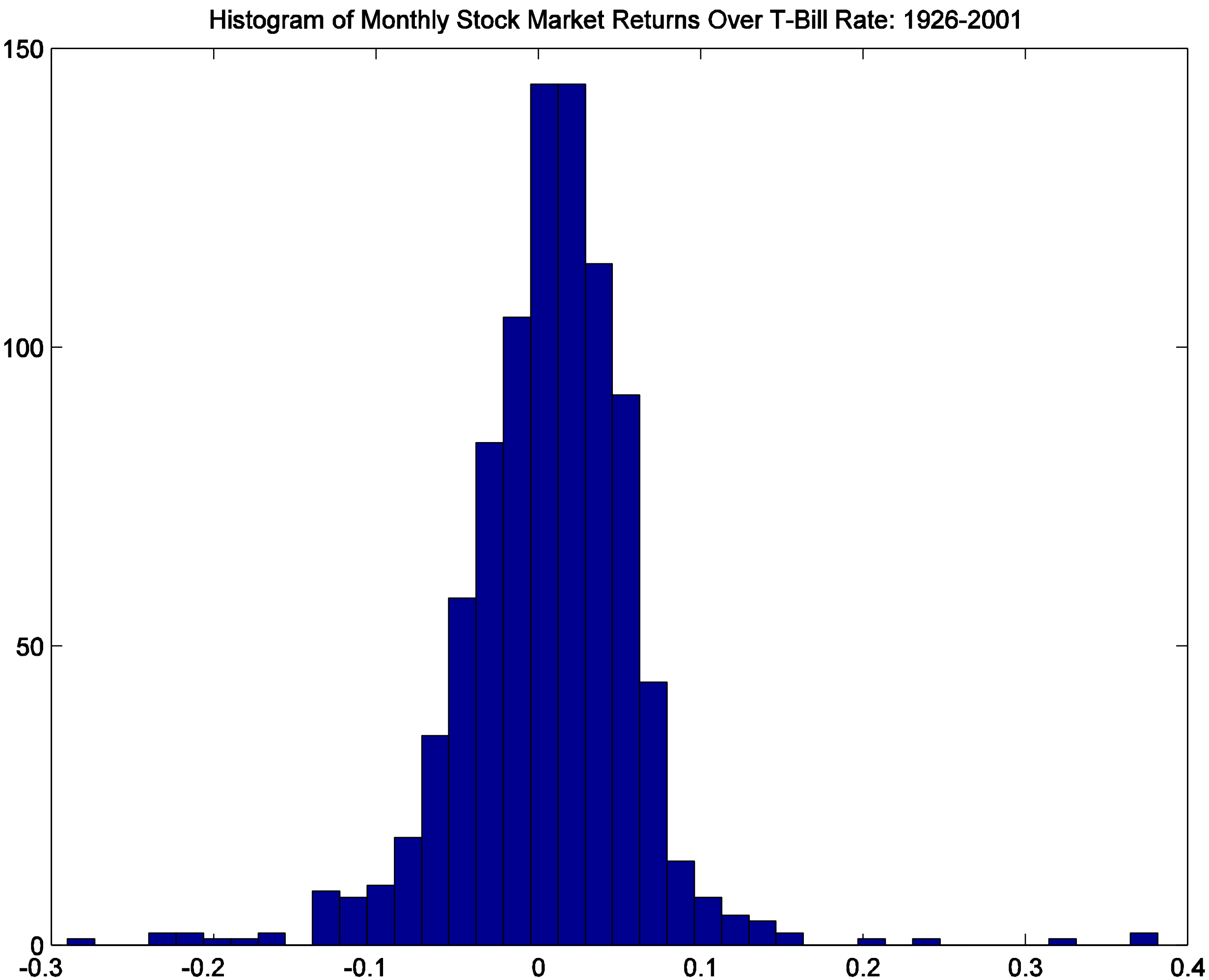


Source: Chicago Center for Research in Security Prices, Standard and Poor's, MSCI, and Global Financial Data.

Equity premium

- Excess of stock returns over risk-free (T-bills) return
  - 7.9% over 1928-2013
  - Note that there is no need to adjust excess returns for inflation
- Given the superior performance of stocks over such long period, why does anyone hold T-bills or bonds?

But Stocks are Riskier!



Risk

- Risk is uncertainty about the future
  - Probability Distributions
  - While stocks do better on average, investors know that in any one year, stocks may do much worse
- Summarize risk through standard deviation, σ, a measure of dispersion
  - Using historical data
  - Frequency Distributions (Histograms)

Standard Deviations

AVERAGE RATES OF RETURN (1928 - 2013 : 85 YEARS OF DATA*)

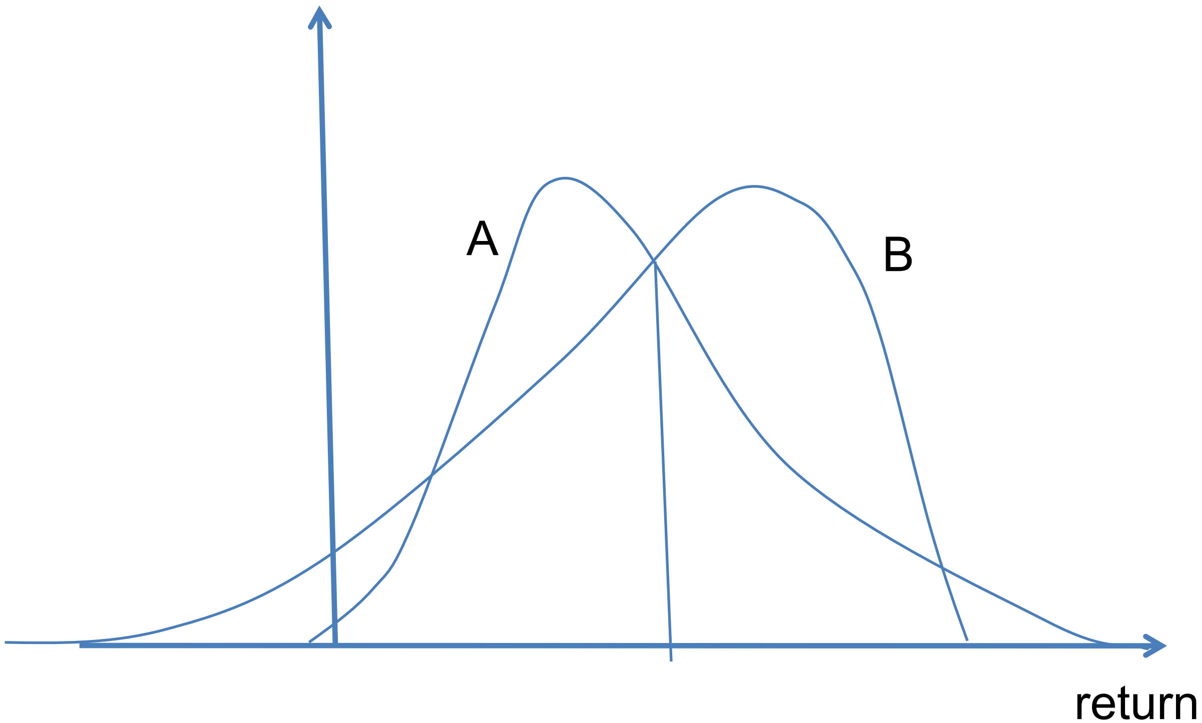
PORTFOLIO	AVERAGE ANNUAL RATE OF RETURN		Standard Deviation
	Arithmetic mean	Geometric mean	
Treasury bills	3.6%	3.5%	3.1%
Long-Term government bond	5.2%	4.9%	9.4%
Long-Term corporate bonds	6.2%	5.9%	8.4%
Large company stocks	11.5%	9.6%	20.6%
Small company stocks	16.4%	11.7%	33.0%

*Some of the data is for a slightly different period

LTGB vs. LTCB?

- LTGB may have longer duration
- Some LTCB may be callable
- Liquidity
- Distribution of LTCB is skewed to the left: mean and variance are not enough! Skewness preference.
- Maybe LTCB were good buys?

If A and B have the same mean and standard deviation.
Which one do you prefer?



Skewness Preference: most people prefer right skeness

Normal and Skewed Distributions

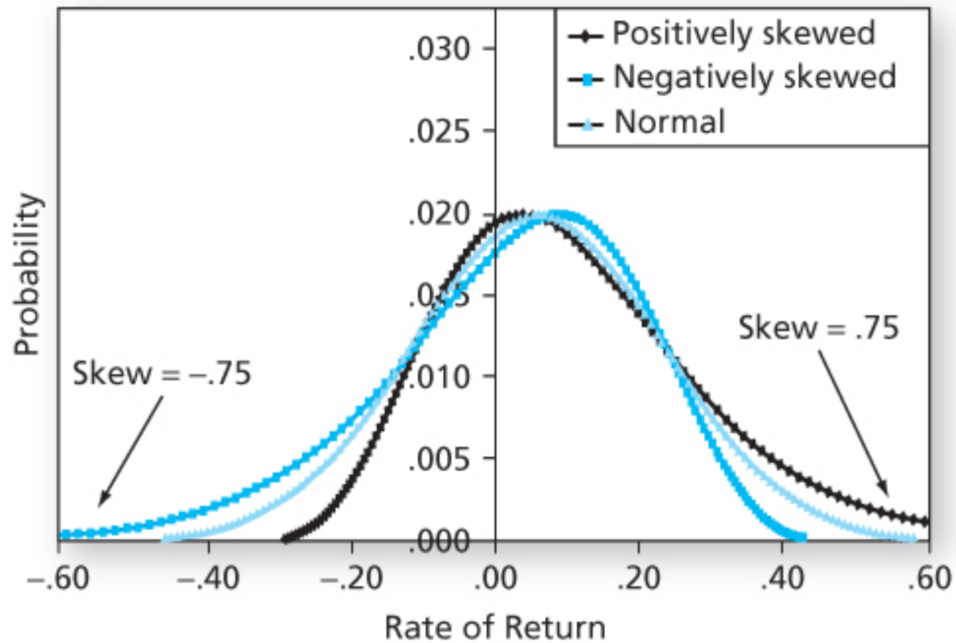


Figure 5.5A Normal and skewed distributions (mean = 6%, SD = 17%)

Normal and Fat-Tailed Distributions

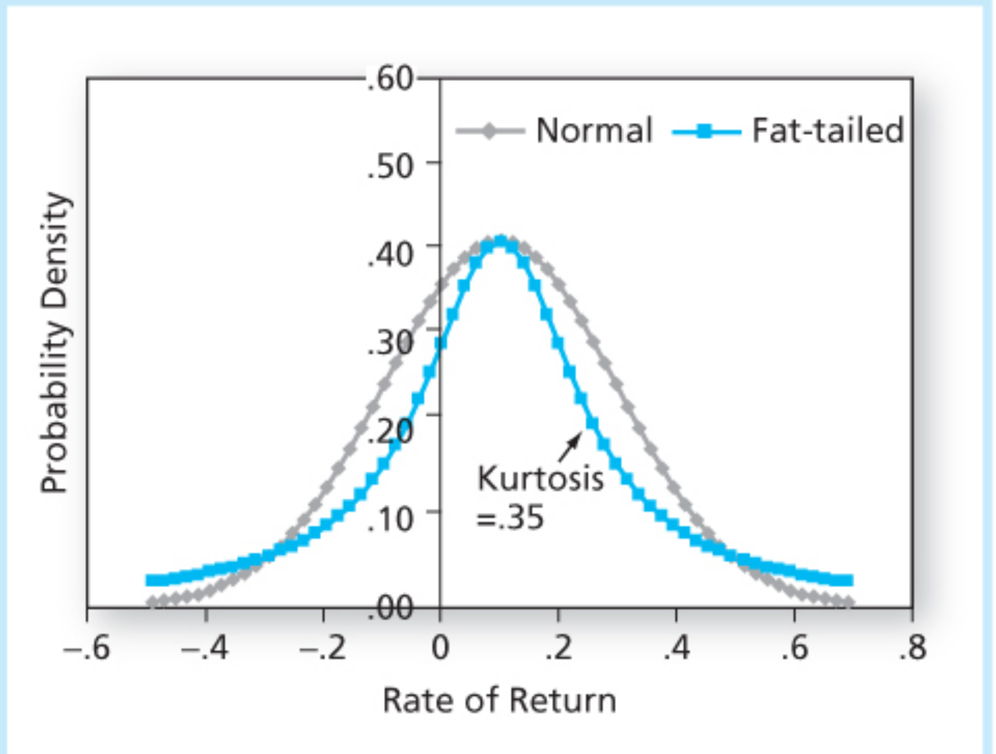


Figure 5.5B Normal and fat-tailed distributions (mean = .1, SD = .2)

Standard deviation

- If r_1, r_2, r_3, …, r_T, are yearly returns, first compute the sample mean of the returns

$$\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

  – Using the sample mean, the sample variance is

$$\mathrm{V}[\tilde{r}] = \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2$$

  – Sample standard deviation is then

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$$

  – Functions *average*(), *var*(), and *stdev*() in Excel

Computing historic standard deviation

Year	Return	
1	.10	$\bar{r} = .08$
2	.30	$\sigma^2 = .0370$
3	-.20	$\sigma = .192$
4	.0	
5	.20	

$$\sigma^2 = [(.10-.08)^2+(.30-.08)^2+(-.20-.08)^2+(.0-.08)^2+(.20-.08)^2]/4$$

$$= [.0004+.0484+.0784+.0064+.0144]/4$$

$$= [.1480]/4$$

In general it is better to do the calculation in decimals (.10) instead of in percentages (10%)

St. dev. in the same units as returns (not variance)

Estimating means and variances

- We are implicitly assuming that the returns came from the same probability distribution in each year of the sample
- The estimated mean and variance are themselves random variables since there is estimation error that depends on the particular sample of data used (sampling error)
  - We can calculate the standard error of our estimates and figure out a confidence interval for them
  - This contrasts with the true (but unknown) mean and variance which are fixed numbers, not random variables

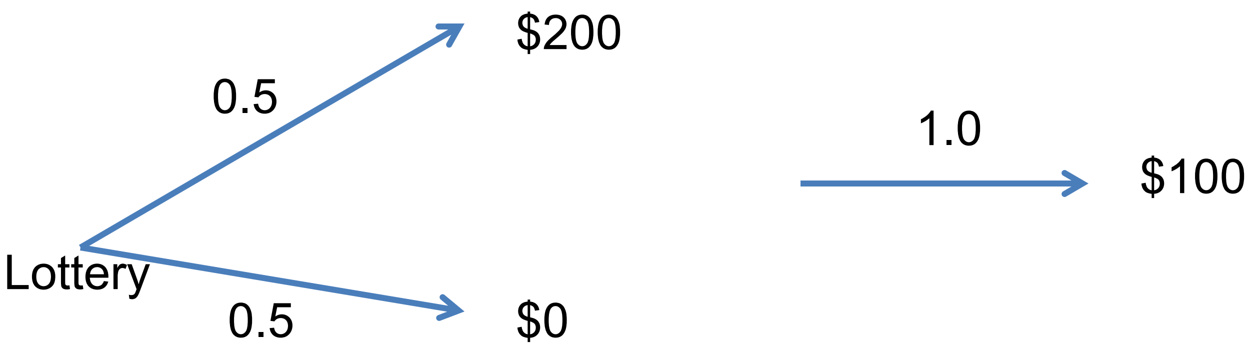
Risk and standard deviation

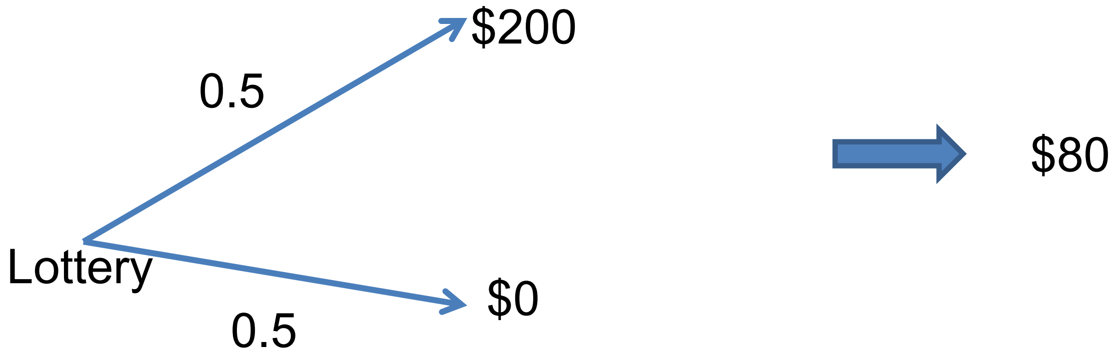
- We use historic estimates as approximations for true *ex ante* means and variances, since we cannot see into the future
  - Is using past data to forecast the future the best method?
  - For variances is not bad, but for means…..
- Is risk just standard deviation?
  - What about skewness, fat tails?

Risk aversion

- People are *risk averse*
  - They prefer a *sure* outcome of $100 versus a *lottery* of $200 (heads) and $0 (tails)
  - Demand compensation for risk
- Stocks offer higher average returns than bonds because:
  - To be willing to hold risky security, investors must receive higher expected return as reward
- However, studies have shown that we would need unreasonably high risk aversion to justify an equity premium of 6% to 8%, given the risk of stocks

Certainty Equivalent





Equity premium puzzle

- The equity premium is a very important number
  - At what age can you retire? How much money will you have?
  - How much should you save now?
- The consequences of an equity premium of 6% to 8% are huge!
  - Borrow and invest in equities
  - Only invest in projects with high expected returns (the opportunity cost is high)
- However, historical (realized) returns are not necessarily an indication of future (expected) returns

Some pause for consideration

- We are very uncertain about the size of the premium
  - Standard error on mean estimate is 2.2%
  - Two-standard error confidence interval is [3.5%,12.3%]
- Survivorship bias
  - Only examined U.S. market, which is one of the best performing and surviving market over the last century
  - When examining the global market, including emerging equities, the premium is smaller
- Other risks besides historical volatility
  - Catastrophe risk

Nominal and Real Equity Returns Around the World, 1900-2000

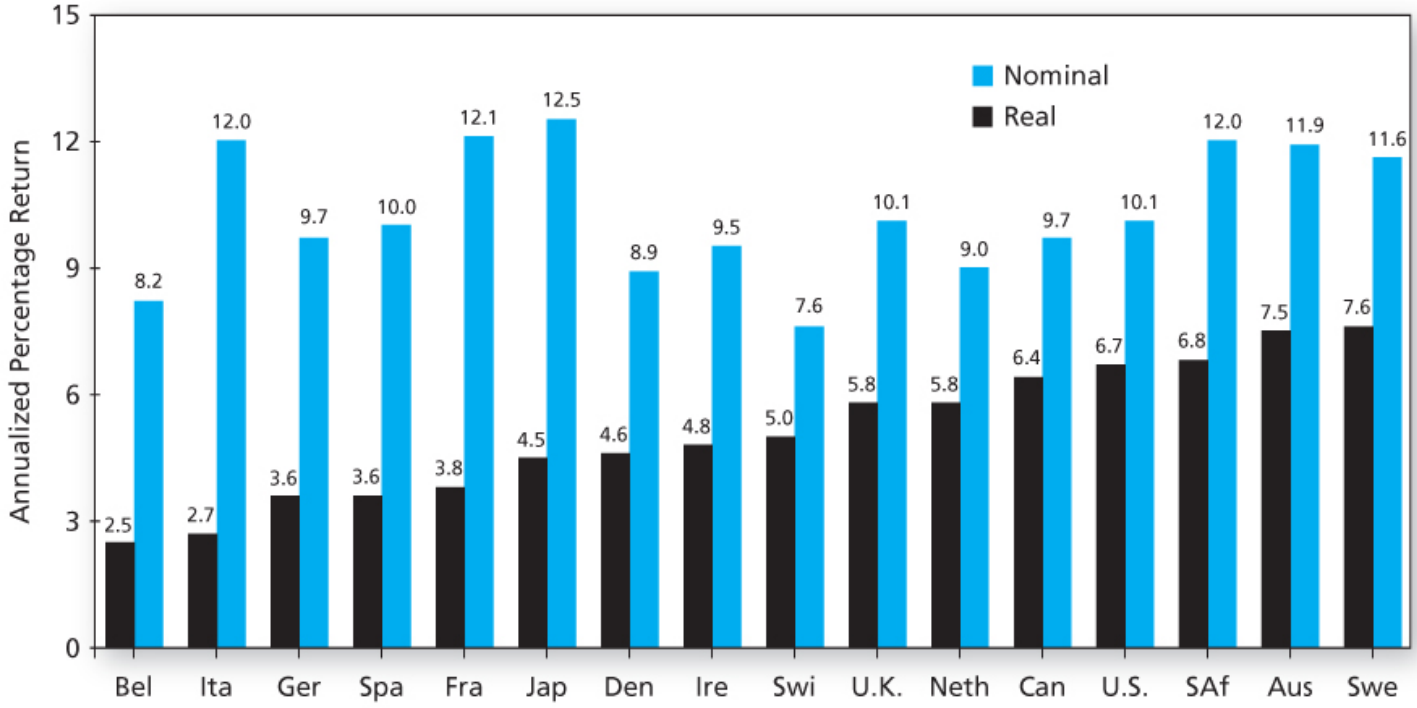


Figure 5.7 Nominal and real equity returns around the world, 1900–2000

Source: Elroy Dimson, Paul Marsh, and Mike Staunton, *Triumph of the Optimists: 101 Years of Global Investment Returns* (Princeton: Princeton University Press, 2002), p. 50. Reprinted by permission of the Princeton University Press.

Standard Deviations of Real Equity and Bond Returns Around the World, 1900-2000

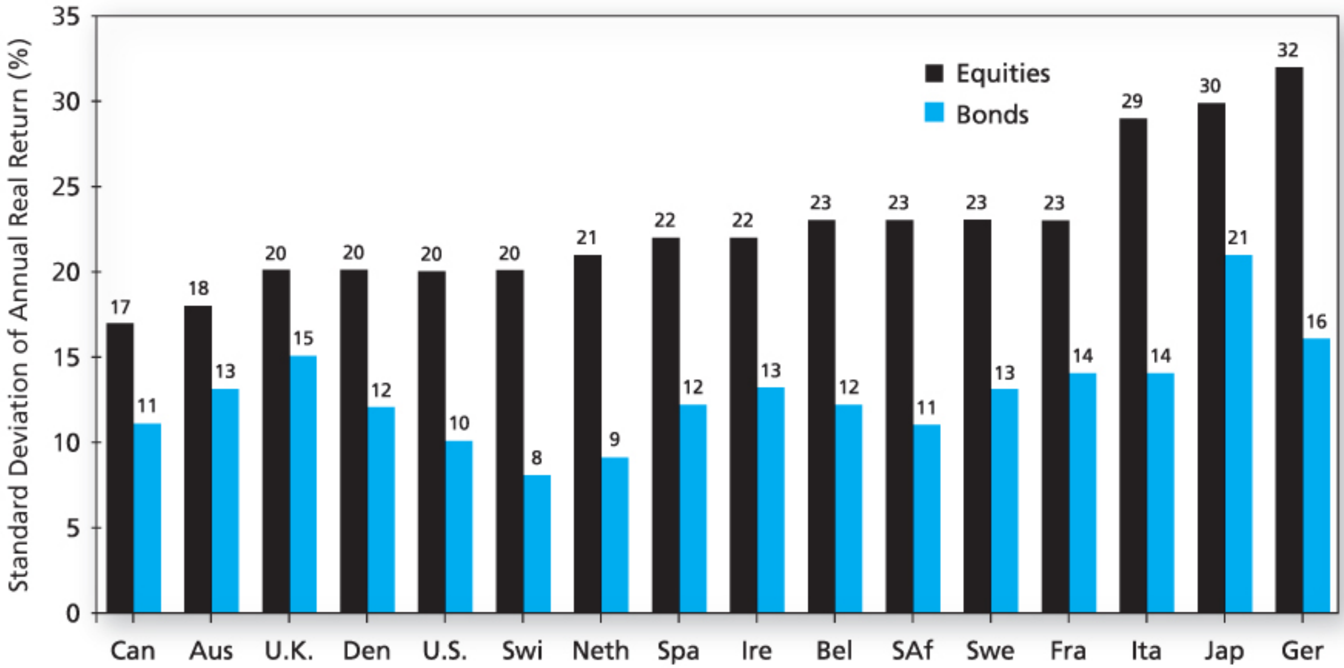


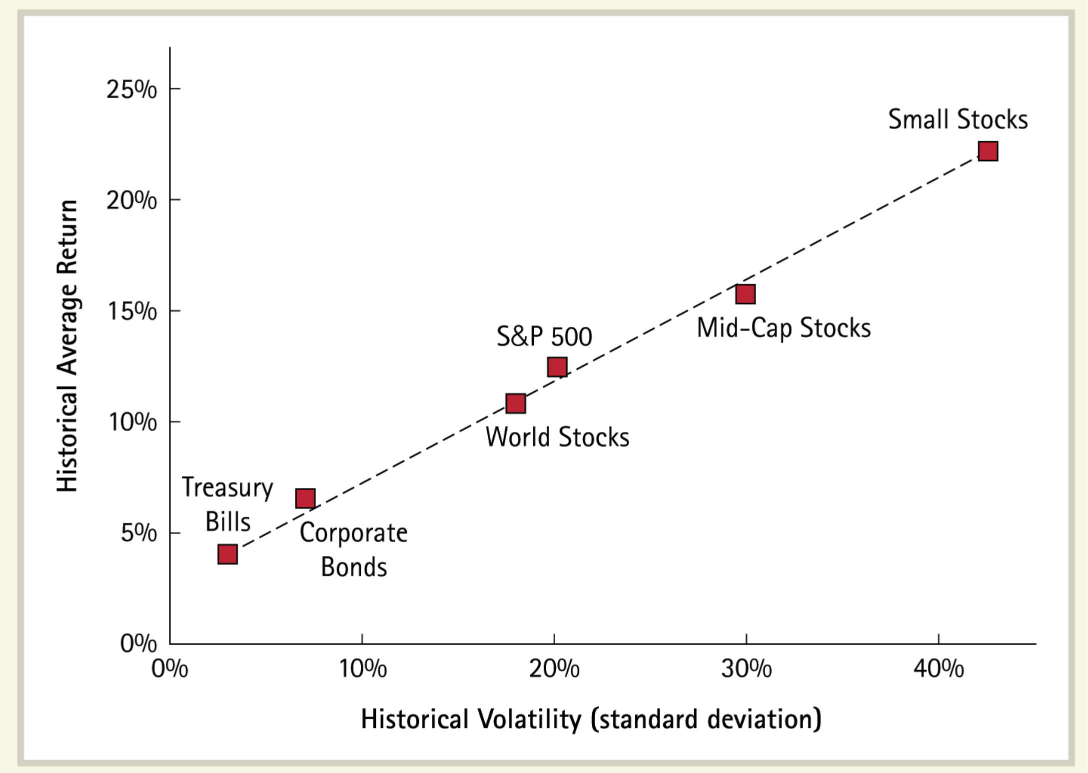
Figure 5.8 Standard deviations of real equity and bond returns around the world, 1900–2000

Source: Elroy Dimson, Paul Marsh, and Mike Staunton, *Triumph of the Optimists: 101 Years of Global Investment Returns* (Princeton: Princeton University Press, 2002), p. 61. Reprinted by permission of the Princeton University Press.

Points to remember

- How to compound returns?
- Difference between
  - Income yield and capital gain/loss
  - Expected vs. realized returns
  - Geometric vs. arithmetic averages
- Historical behavior of returns
  - The equity premium puzzle
- Risk is uncertainty about future
  - Usually measured by standard deviation

The Historical Tradeoff Between Risk and Return in Large Portfolios, 1926–2004



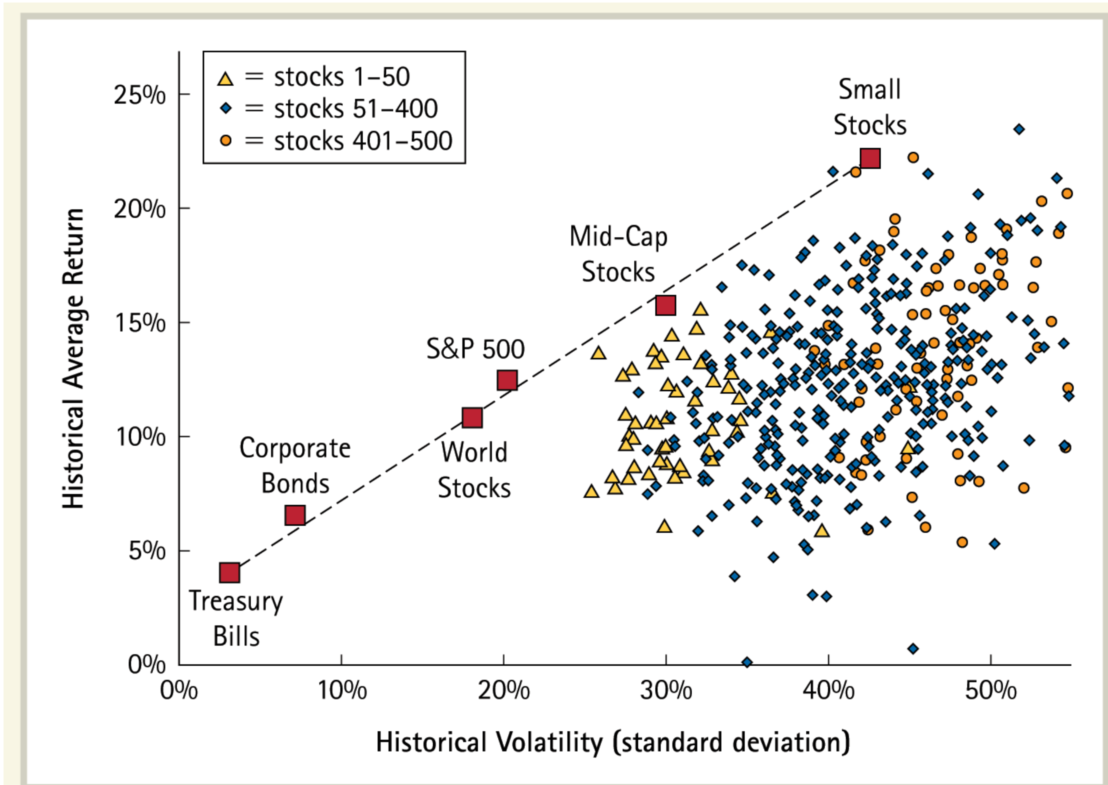
Also included are a mid-cap portfolio composed of the 10% of U.S. stocks whose size is just below the median of all U.S. stocks, and a world portfolio of large stocks from North America, Europe, and Asia. Note the general increasing relationship between historical volatility and average return for these large portfolios.

Source: CRSP, Morgan Stanley Capital International and Global Financial Data.

Portfolios

- There seems to be linear relation between expected return and volatility
- Should expected return rise proportionally with volatility?
- Approximately true for portfolios
- Is it true in general?
- Does it apply to individual stocks?

Historical Volatility and Return for 500 Individual Stocks, by Size, Updated Quarterly, 1926–2004



Unlike the case for large portfolios, there is no precise relationship between volatility and average return for individual stocks. Individual stocks have higher volatility and lower average returns than the relationship shown for large portfolios.

Individual Stocks

- No clear relation between volatility and expected return
- There seems to be a relation between risk and size: larger stocks have lower volatility
- Even larger stocks are typically more volatile than the S&P500
- Smaller stocks have a slightly higher return
- All stocks have higher volatility and lower return than would be predicted by the line

Topic 4: Expected Utility and Risk Aversion

Eduardo Schwartz
UCLA Anderson School

St. Petersburg Paradox

- How much are you willing to pay to play this game?
- Coin flipping game: toss a coin and continue to do so until it lands "heads"
- Pays Probabilities

\$1 if "heads" the first time (H)	½
\$2 if "heads" the second time (TH)	¼
\$4 if "heads" the third time (TTH)	1/8
$\$2^{i-1}$ if "heads" the i^{th} time (TT…TH)	$(½)^i$

St. Petersburg Paradox

- Expected payoff:

$$\bar{x} = \sum_{i=1}^{\infty} p_i x_i = \frac{1}{2}1 + \frac{1}{4}2 + \frac{1}{8}4 + \cdots + \left(\frac{1}{2}\right)^i 2^{i-1} + \cdots$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} + \cdots = \frac{1}{2}(1+1+1+\cdots+1+\cdots) = \infty$$

The "paradox" is that the expected value is infinity, but most individuals would pay only a moderate amount to play this game

Expected Utility

- Bernoulli (1738) provided and explanation of the paradox: individuals do not maximize expected value, but **expected utility:**

$$V \equiv E[u(\widetilde{x})] = \sum_{i=1}^{n} p_i u(x_i)$$

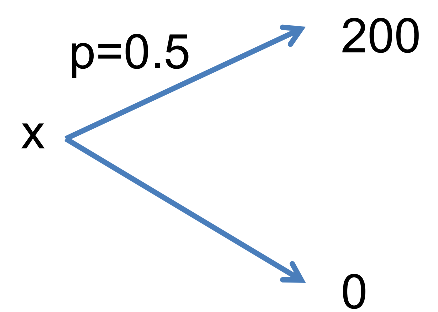
- The maximum they would pay is an amount that has the same utility as the expected utility of this lottery (Certainty Equivalent).
- The first complete axiomatic development of expected utility is due to von Neumann and Morgenstern (1944).

Axioms of Rational Behavior

1. Completeness: there exists a preference ordering over all risky prospects which is complete. i.e. for any two gambles x and y

$$x \succ y \quad or \quad x \prec y \quad or \quad x \approx y$$

y=80



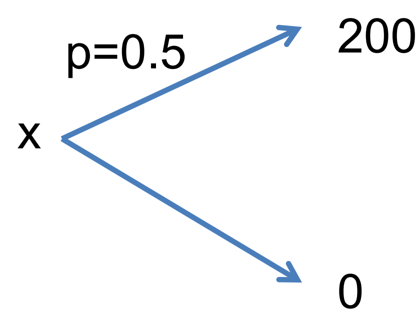
Axioms of Rational Behavior

2. Transitivity: the preference ordering is transitive

$$If \quad x \succ y \quad and \quad y \succ z \quad \Rightarrow \quad x \succ z$$

$$If \quad x \approx y \quad and \quad y \approx z \quad \Rightarrow \quad x \approx z$$

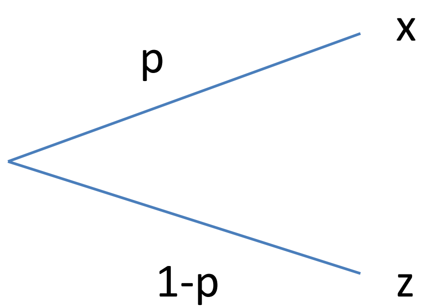
y=80
z=60



Axioms of Rational Behavior

3. Independence: Let *g(x,z;p)* represent a gamble with probability *p* of paying *x* and *(1-p)* of paying *z*

g(x,z;p)



$$\text{If} \qquad x \approx y$$

$$\text{Then} \quad g(x,z;p) \approx g(y,z;p)$$

Axioms of Rational Behavior

4. Continuity: $\quad$ *If* $\quad x \succ\approx y \succ\approx z$

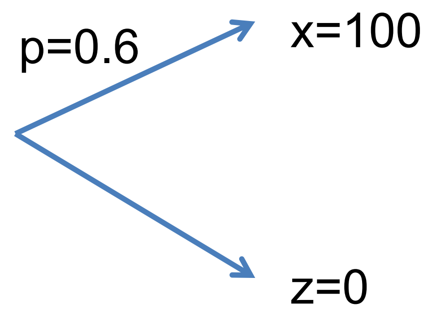
then there exist a probability p

$(\textit{unique unless } x \approx y \approx z)$

such that

$y \approx g(x, z; p)$

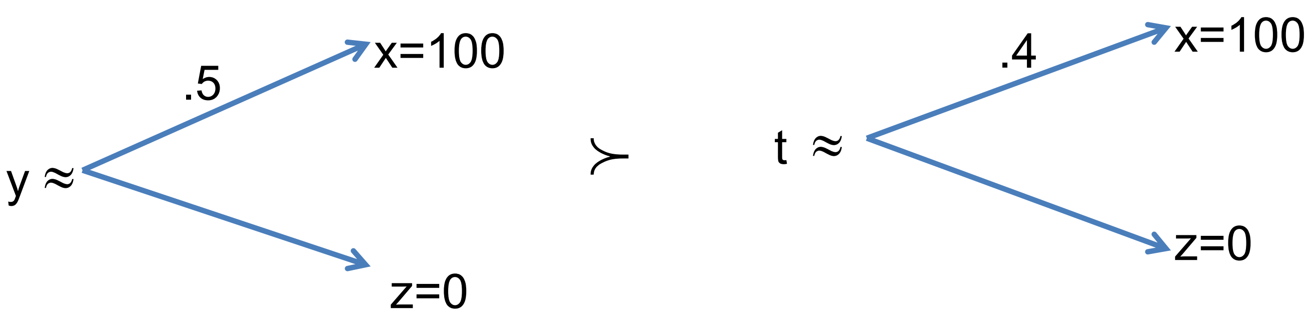
y=50 ≈



Axioms of Rational Behavior

5. Dominance:

$$\begin{aligned}
&If \quad x \succ y \succ z \quad , \quad x \succ t \succ z \\
&and \quad y \approx g(x,z;p_1) \quad , \quad t \approx g(x,z;p_2) \\
&Then \quad p_1 > p_2 \quad \Rightarrow \quad y \succ t \\
&and \quad p_1 = p_2 \quad \Rightarrow \quad y \approx t
\end{aligned}$$



Expected Utility Theorem

Rational Preferences under Uncertainty

- ***Existence***: A preference ordering on risky prospects which satisfy this rationality requirements can be represented by a utility function *u* defined over the set of outcomes in the sense that

$$E[u(x)] \geq E[u(y)]$$
$$iff \quad x \succ\approx y$$

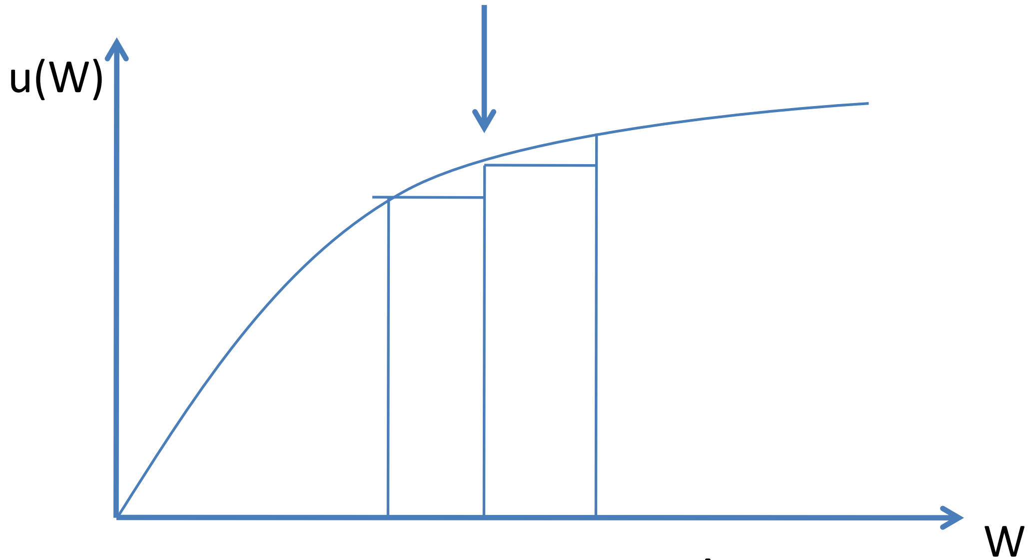
Expected Utility Theorem
Rational Preferences under Uncertainty

- ***Uniqueness***: *u(x)* is unique up to a positive linear transformation.
- If *u(x)* is a utility function
- So is v(x)=*au(x)+b* with *a>0*

$$E[v(x)] \geq E[v(y)]$$
$$E[au(x)+b] \geq E[au(y)+b]$$
$$aE[u(x)]+b \geq aE[u(y)]+b$$
$$E[u(x)] \geq E[u(y)]$$

Risk Aversion

- Concave utility function for wealth: $u'' < 0$

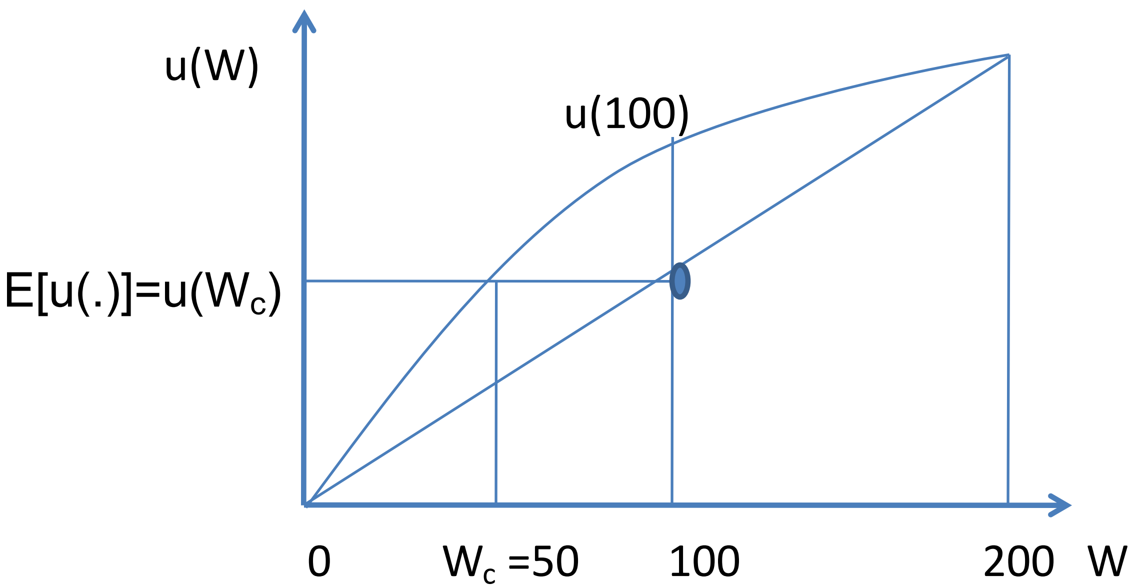


- Feel more "pain" to loose \$100 than "pleasure" to win \$100. Decreasing marginal utility of money
- ($u' > 0$ since more is preferred to less!)

Risk Aversion

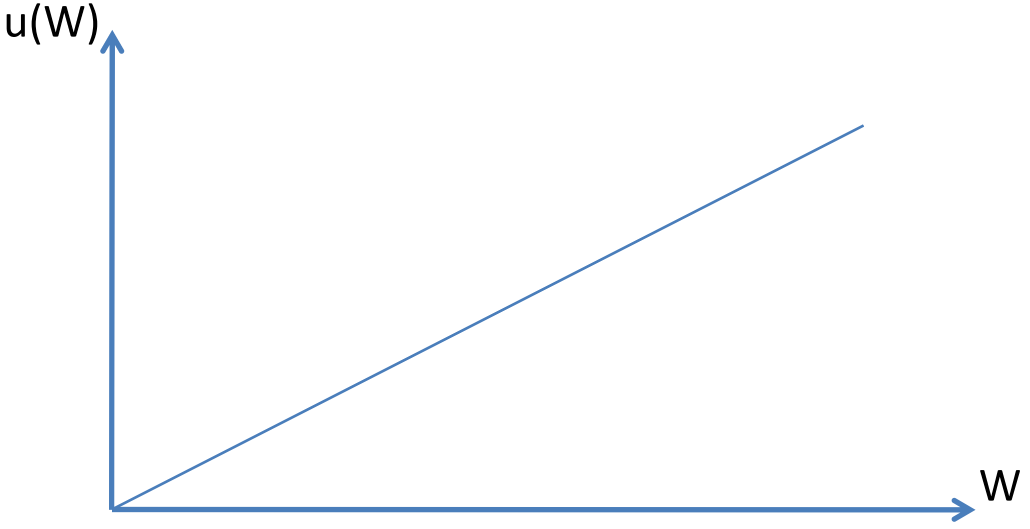
- Investor will not trade \$100 for sure for 50/50 chance of getting \$200 or nothing.
- With concave utility:

$$U(100) > 0.5\ U(200) + 0.5\ U(0)$$



Risk Neutral

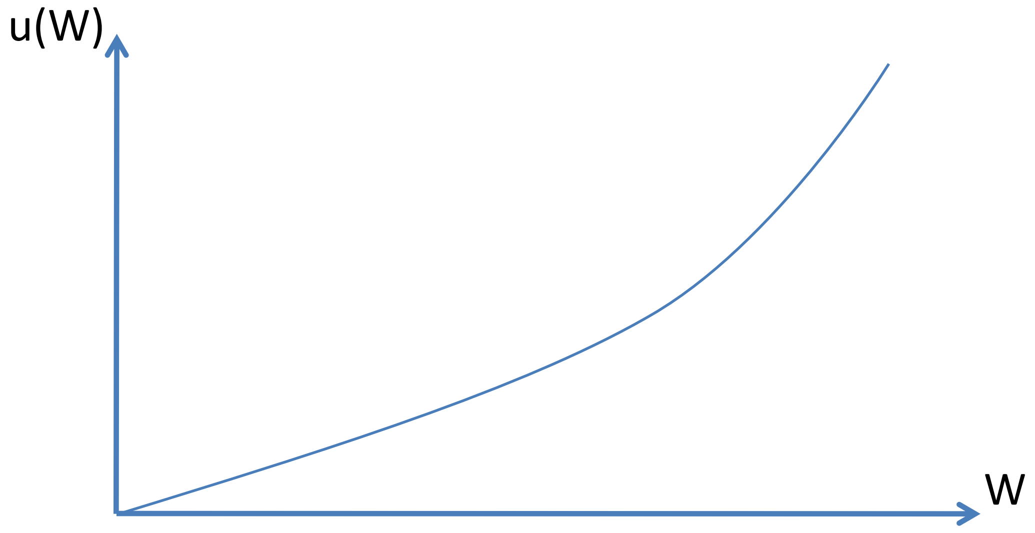
- Linear utility function for wealth: u’’=0



Investor maximizes expected wealth

Risk Seeker

- Convex utility function for wealth: $u'' > 0$



Gambling?

Certainty Equivalent for Complex Gambles (W_c)

$$W_c \approx g(M, m; H)$$

$$u(W_c) = Eu(complex\ gamble)$$

Amount you would accept for certain instead of the gamble.

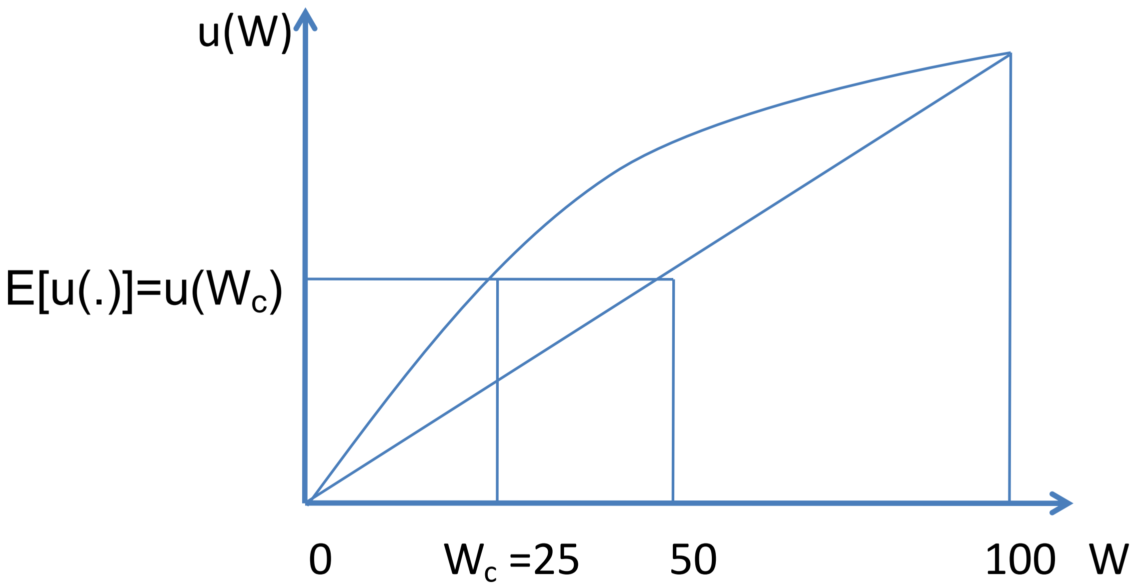
$$If\ \overline{W}\ is\ the\ \mathrm{exp}ected\ outcome$$

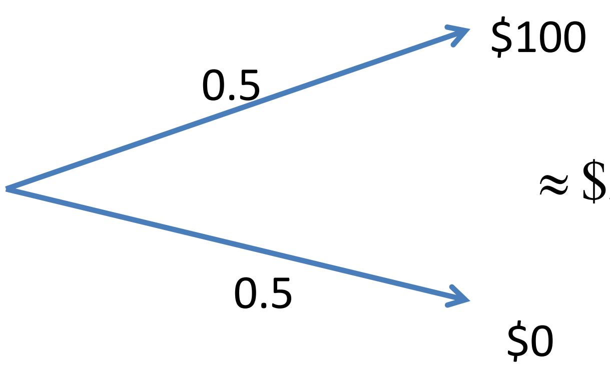
$$risk\ averse \quad if \quad W_c < \overline{W} \quad \Leftarrow \quad u'' < 0$$

$$risk\ neutral \quad if \quad W_c = \overline{W} \quad \Leftarrow \quad u'' = 0$$

$$risk\ see\mathrm{ker} \quad if \quad W_c > \overline{W} \quad \Leftarrow \quad u'' > 0$$

Risk Averse: Certainty Equivalent





$$\approx \$25 = W_c < \overline{W} = \$50$$

Measures of Risk Aversion

- Size of u''(W)<0 is not meaningful (unique up to a positive linear transformation), u'(W)>0 more is preferred to less.

$$\text{Absolute Risk Aversion} \quad A(W) = -\frac{u''(W)}{u'(W)}$$

$$\text{Relative Risk Aversion} \quad R(W) = -\frac{Wu''(W)}{u'(W)} = WA(W)$$

- Independent of positive linear transformation of the ut. fn. *v(W)=au(W)+b*, with *a>0*
- Allows comparisons between individuals.

Measures of Risk Aversion

$$v(W) = au(W) + b, \quad a > 0$$

$$v'(W) = au'(W) > 0$$

$$v''(W) = au''(W) < 0$$

$$A(W) = -\frac{u''(W)}{u'(W)} = -\frac{v''(W)}{v'(W)} > 0$$

$$R(W) = -\frac{Wu''(W)}{u'(W)} = -\frac{Wv''(W)}{v'(W)} > 0$$

Some useful utility functions

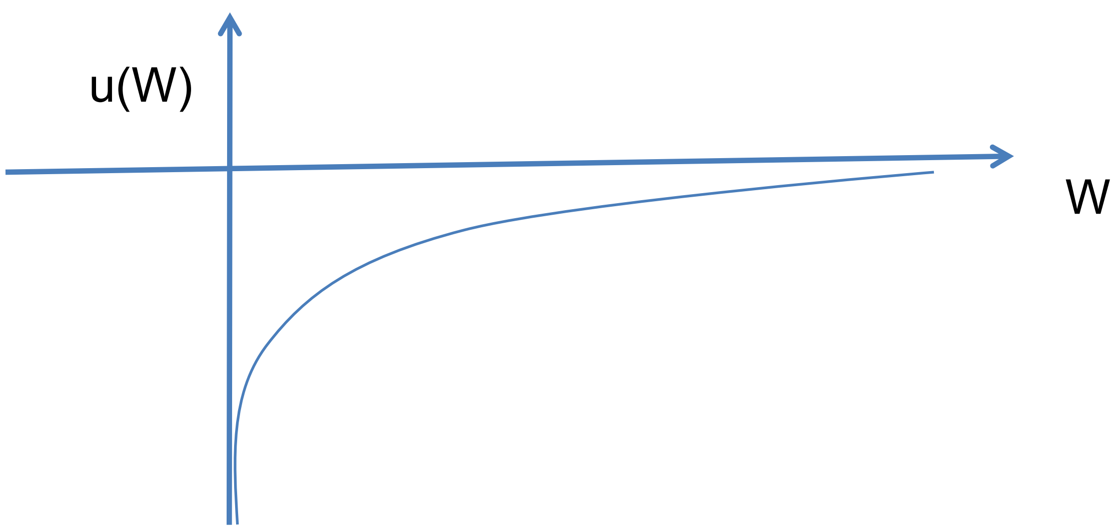
- Exponential: constant absolute risk aversion

$$u(W) = -\frac{1}{a} e^{-aW} \quad with \quad a > 0$$

$$u'(W) = e^{-aW} > 0$$

$$u''(W) = -ae^{-aW} < 0$$

$$A(W) = a$$



Some useful utility functions

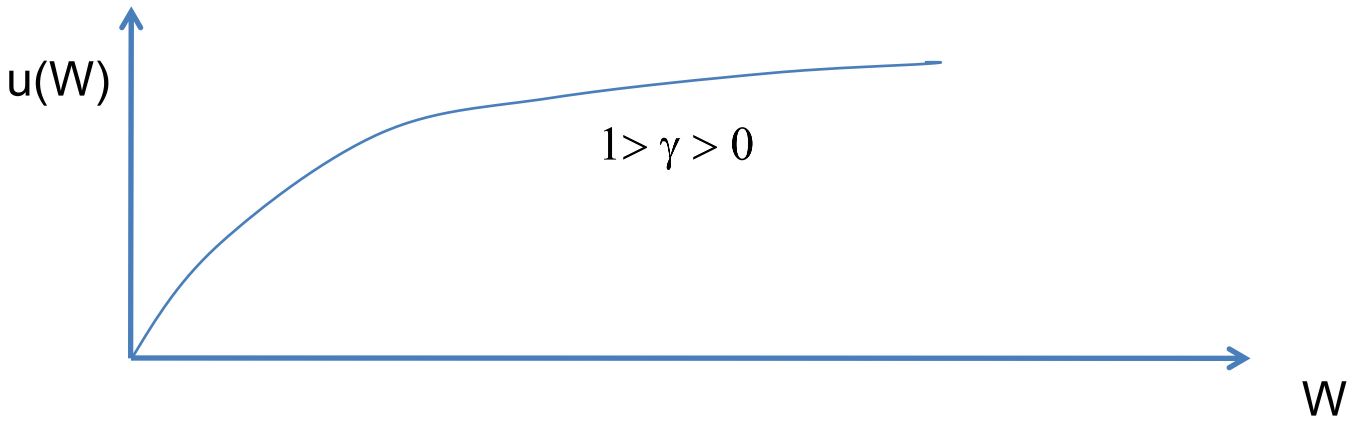
- Power: constant relative risk aversion

$$u(W) = \frac{W^{\gamma}}{\gamma} \quad for \quad \gamma < 1, \quad \gamma \neq 0$$

$$u'(W) = W^{\gamma-1} > 0$$

$$u''(W) = (\gamma - 1)W^{\gamma-2} < 0$$

$$R(W) = 1 - \gamma = c > 0, \quad \gamma < 1 \qquad u(W) = \frac{W^{1-c}}{1-c}$$



Some useful utility functions

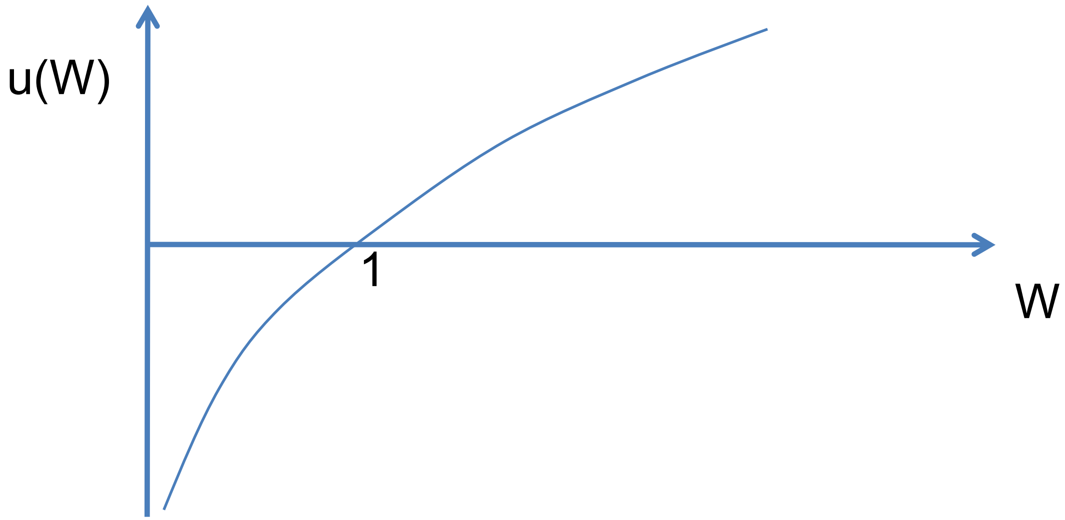
- Log: also constant relative risk aversion

$$u(W) = \ln W \quad (\gamma = 0)$$

$$u'(W) = \frac{1}{W} > 0$$

$$u''(W) = -\frac{1}{W^2} < 0$$

$$R(W) = 1$$



Some useful utility functions

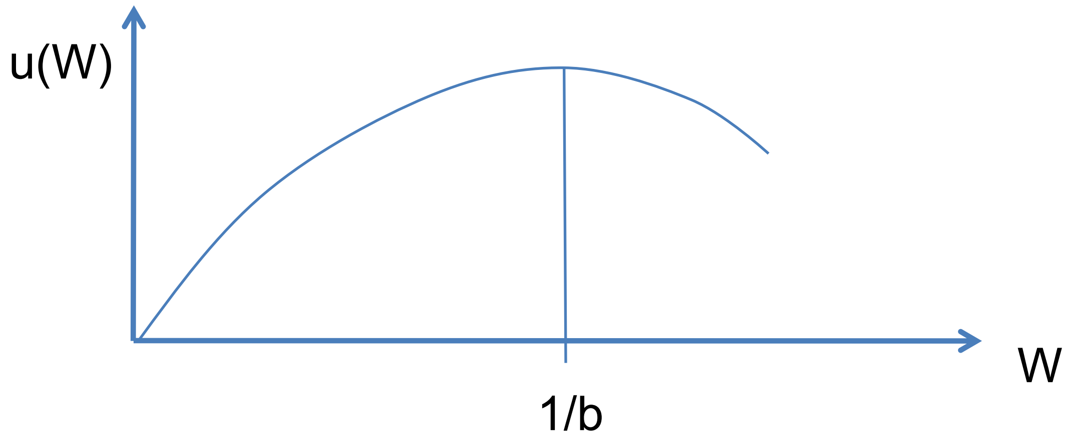
- Quadratic

$$u(W) = W - \frac{1}{2}bW^2 \quad for \quad 0 \le W \le \frac{1}{b}$$

$$u'(W) = 1 - bW > 0 \quad for \quad 0 \le W \le \frac{1}{b}$$

$$u''(W) = -b < 0$$

$$A(W) = \frac{b}{1 - bW} \quad increasing \ with \ W \quad (not\ good)$$



Expected utility max. and mean variance analysis are equivalent when:

- Utility functions are quadratic

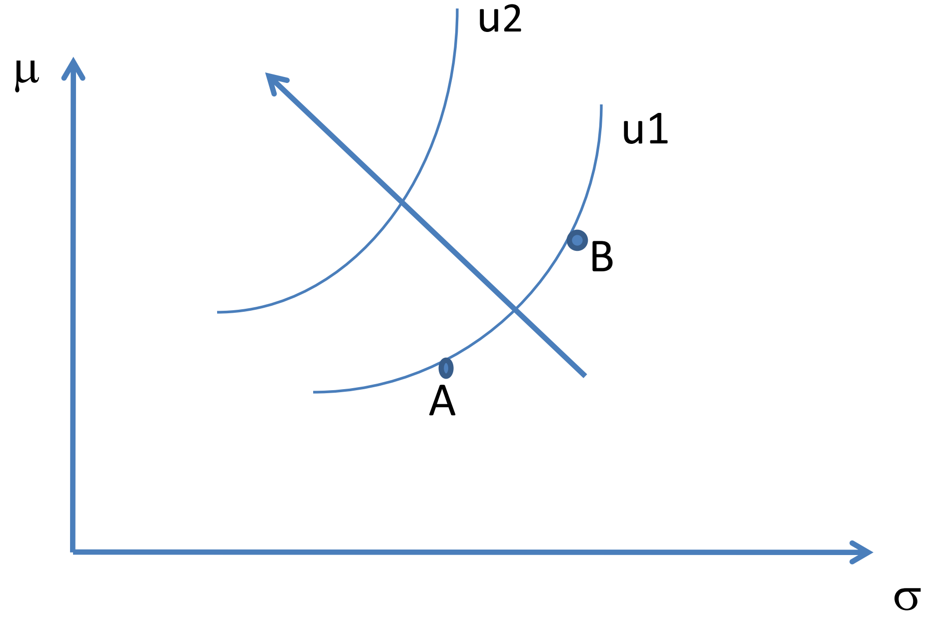
$$u(W) = W - \frac{1}{2} b W^2$$

$$E[u(W)] = \overline{W} - \frac{1}{2} b [\overline{W}^2 + Var(W)]$$

- Distributions of returns are joint normal
  - Normal is completely defined by mean and variance
  - Additive
  - Indifference curves (constant Eu) can be written in terms of mean and s.d. only
- Be careful when distributions are not normal (options)

Indifference Curves (for risk averse)

A and B have the same expected utility



Expected Utility - Example

- Your current wealth is \$ 100,000.
- You want to purchase a home owner's insurance policy to cover potential loss due to fire.
- You estimate that the probability of fire is 10 %.
- If your house is damaged by fire your loss would be \$ 60,000.
- You have determined that your utility of wealth is U(W) = $\sqrt{W}$.
- What is the maximum amount you would be willing to pay to fully insure against possible losses due to fire?

Expected Utility - Solution

- Let us assume that you will at the most be willing to pay \$ D
- If you buy insurance then your wealth is \$(100,000 – D) for sure.
- If you do not buy insurance then there is a 10% chance that you will end with a wealth of \$40,000 and a 90% chance that you will end with a wealth of \$100,000

Expected Utility - Solution

Buy insurance → 100,000 – D

Don't by insurance

.9 → 100,000

.1 → 40,000

Expected loss = 100,000 – 94,000 = 6,000

Expected Utility - Solution

- We defined certainty equivalent, W_c as that amount of wealth which makes you indifferent between this wealth and a gamble
- That is $U(W_c) = E[U]$
- $U(100{,}000 – D) = 0.1*U(40{,}000) + 0.9*U(100{,}000)$
- $\sqrt{100000-D} = 0.1 \sqrt{40000} + 0.9 \sqrt{100000}$
- $D = 7{,}215.80$

Topic 5a: Introduction to Portfolio Choice

Eduardo Schwartz
UCLA Anderson School

Learning objectives

- How to calculate the expected return and standard deviation of portfolios
- How to combine risky and risk-free assets
- How to find the most efficient combination of stocks into a portfolio?

Motivation

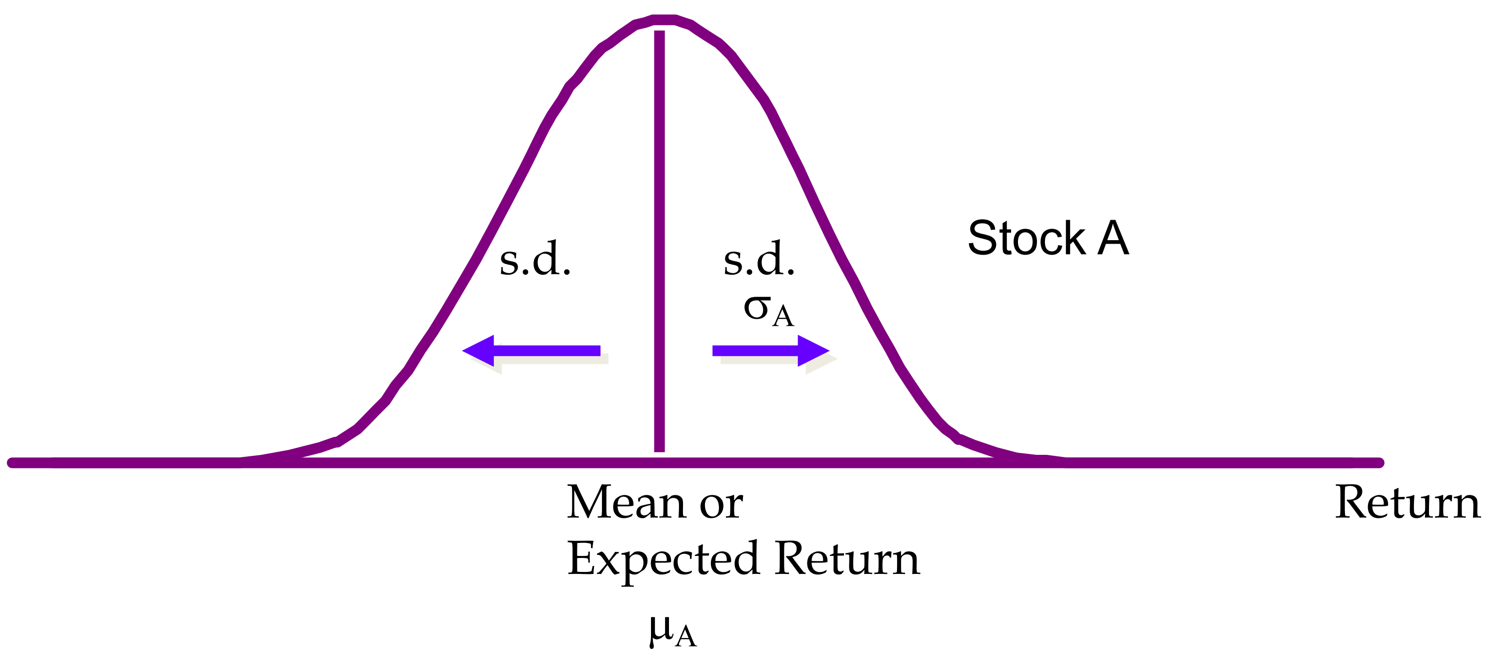
- Mean-variance portfolio analysis
  - Developed by Harry Markowitz in the early 1960's (1990 Nobel Prize in Economics)
  - Foundation of modern finance
- Used by all pension plans, endowments, wealthy individuals, banks, insurance companies, ...
- There is an industry of advisors and software makers that implement what we will learn in these couple of classes

Portfolio Theory

- Markowitz showed exactly how an investor can reduce the s.d. of portfolio return by diversification
- If we look a histogram of daily returns of most stocks we observe that they are (close to) normally distributed
- The normal distribution can completely be described by two numbers:
  - Average, mean or “expected return”
  - Standard deviation of return

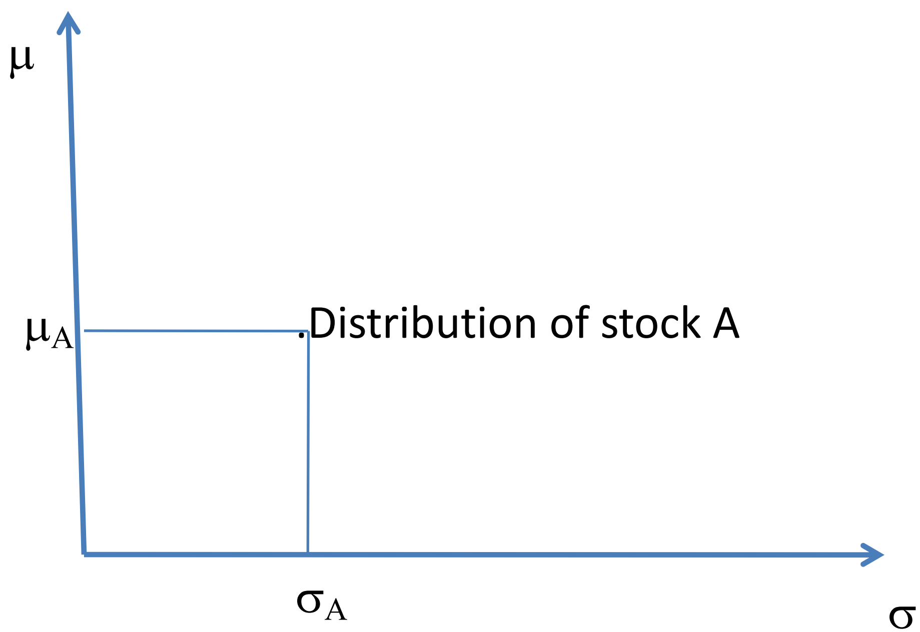
Probability Distribution of Returns

- Normal Distribution is completely described by
  - Mean : Expected return
  - Variance (standard deviation) : dispersion



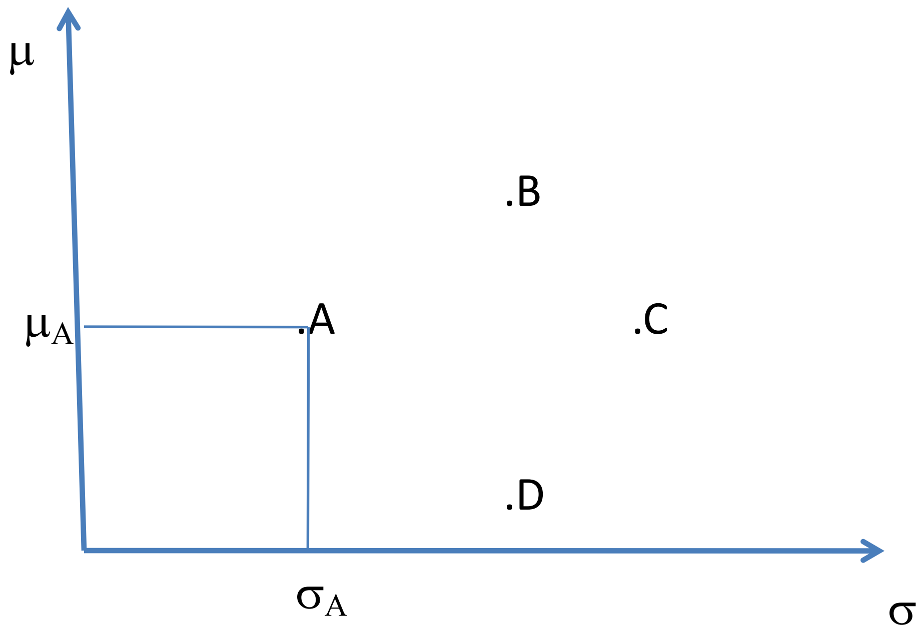
Distribution can be represented by one point in the graph

- Expected return (μ) vs. standard deviation (σ)



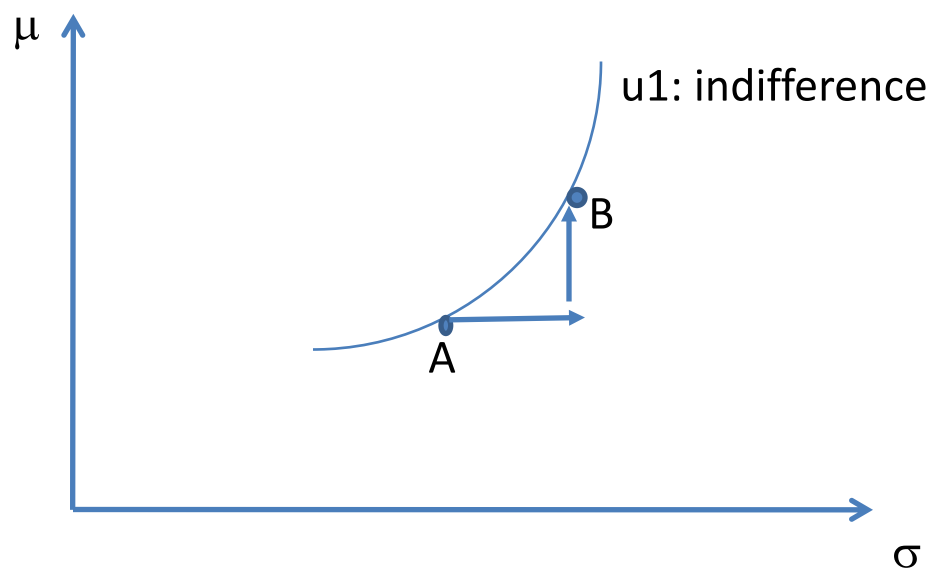
If you can only choose one stock?

- Most investors like expected return and dislike uncertainty (risk aversion)



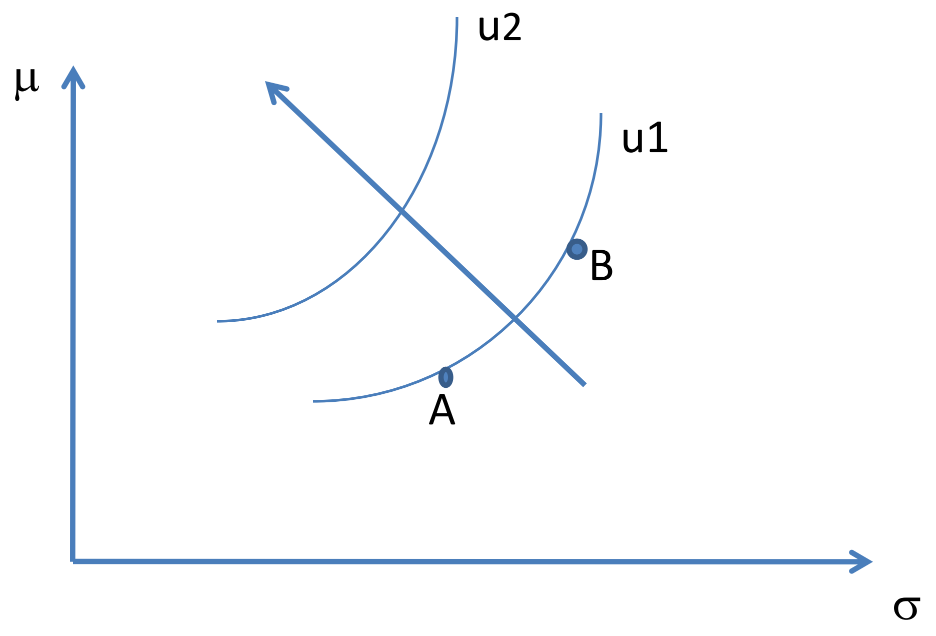
Indifference Curves (for risk averse)

The choice between A and B depends on your preferences
Utility function



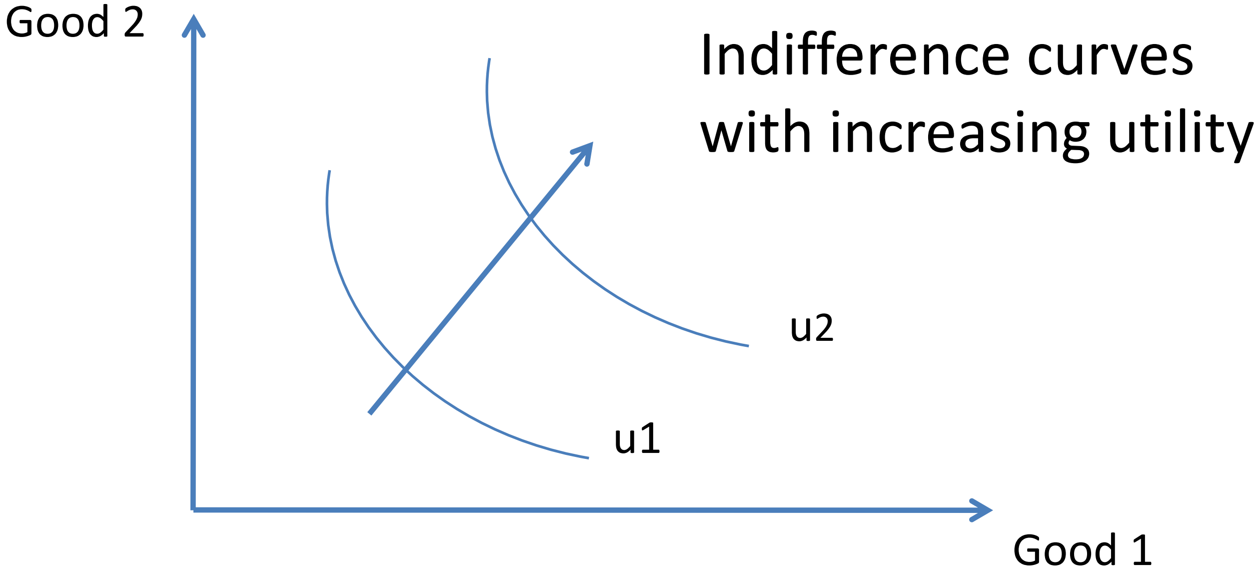
Indifference Curves (for risk averse)

Indifference curves with increasing utility

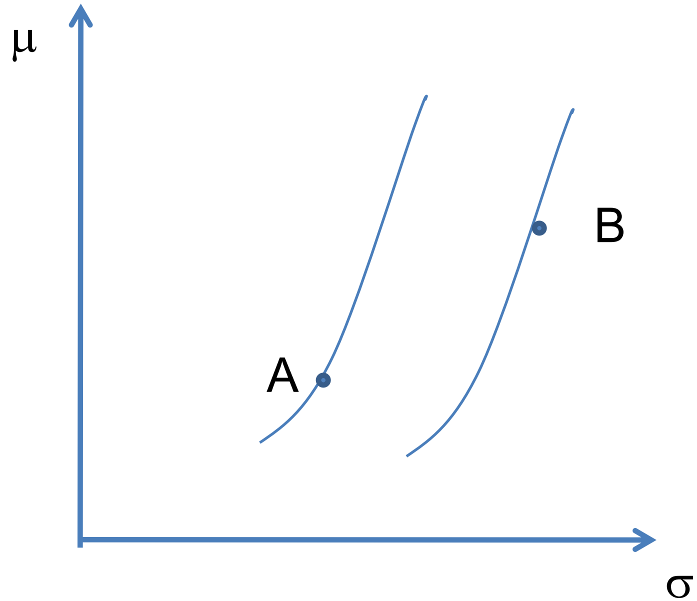


Indifference Curves in Economics

In this case both goods are desirable

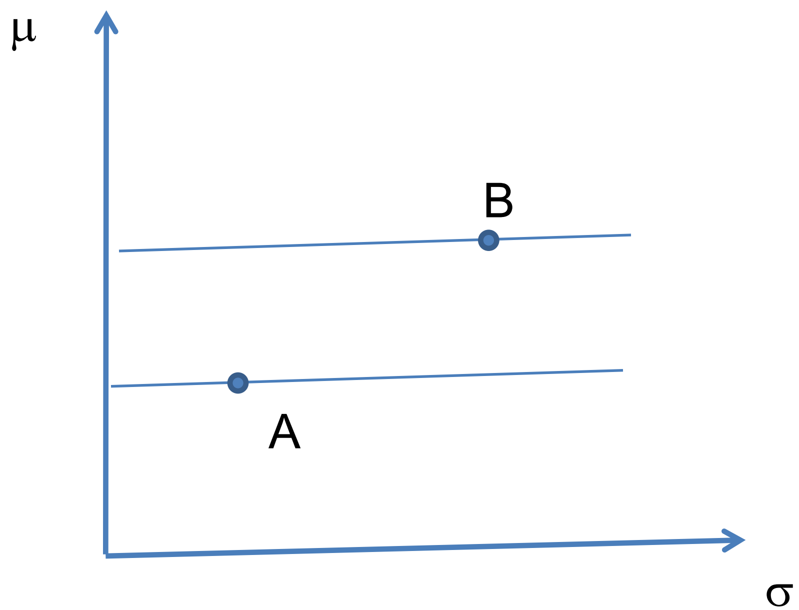


Indifference Curves



Very risk averse (steep)

Indifference Curves



Risk neutral

But, there is no reason to restrict your portfolio to holding one security, you could buy a combination of both (portfolio)

Portfolio weights

- Fraction of wealth invested in different assets
  - Fractions
  - Add up to 1.0
  - Usually denoted by ‘w’
- Example
  - \$100 in savings account at bank, \$200 in IBM
    - Total investment: \$100+\$200=\$300
  - Portfolio weights
    - Savings account: \$100/\$300 = 1/3
    - IBM: \$200/\$300 = 2/3
- Can we have negative portfolio weights?
  - \$600 in IBM, Borrow \$300 (weights: 2, -1)

Notation

- Portfolio weights

  w_x and w_y (with $w_x + w_y = 1.0$)

- Expected (mean) returns

  $\mu_x = E(r_x)$ and $\mu_y = E(r_y)$

- Variances of returns

  $\sigma^2_x = Var(r_x)$ and $\sigma^2_y = Var(r_y)$

- Covariance of returns

  $\sigma_{xy} = Cov(r_x, r_y) = \rho_{xy} \sigma_x \sigma_y$

Portfolio return and expected return

- Portfolio return (random)
  - Dollar value at end of period (plus cash flows) divided by dollar value at beginning of period
  - Can be computed as average of returns on individual securities weighted by their portfolio weights

$$r_p = w_x r_x + w_y r_y$$

- Then the expected return on the portfolio is

$$\mu_p = w_x \mu_x + w_y \mu_y$$

Remember from stats that E(aX+bY)=aE(X)+bE(Y)

Expected Return

- $\mu_x=0.10$
- $\mu_y=0.20$
- $w_x=0.5$
- $w_y=0.5$
- $\mu_p=0.5 \times 0.1 + 0.5 \times 0.2 = 0.15$

Portfolio expected return and variance

- The expected return is a weighted average of the expected return on the assets in the portfolio.
- But, the variance is not a weighted average. It depends on how the returns on the assets in the portfolio *covary (correlation, covariance)*.
- Diversification can reduce the variance of a portfolio (extreme example: life insurance company).

The Volatility of a Two-Stock Portfolio

- Combining Risks

Returns for Three Stocks, and Portfolios of Pairs of Stocks

	Stock Returns			Portfolio Returns	
Year	North Air	West Air	Tex Oil	$1/2R_N + 1/2R_W$	$1/2R_W + 1/2R_T$
2007	21%	9%	−2%	15.0%	3.5%
2008	30%	21%	−5%	25.5%	8.0%
2009	7%	7%	9%	7.0%	8.0%
2010	−5%	−2%	21%	−3.5%	9.5%
2011	−2%	−5%	30%	−3.5%	12.5%
2012	9%	30%	7%	19.5%	18.5%
Average Return	10.0%	10.0%	10.0%	10.0%	10.0%
Volatility	13.4%	13.4%	13.4%	12.1%	5.1%

The Volatility of a Two-Stock Portfolio

- Combining Risks
  - While the three stocks have the same volatility and average return, the pattern of their returns differs.
  - For example, when the airline stocks performed well, the oil stock tended to do poorly, and when the airlines did poorly, the oil stock tended to do well.

The Volatility of a Two-Stock Portfolio

- Combining Risks
  - Consider the portfolio which consists of equal investments in West Air and Tex Oil. The average return of the portfolio is equal to the average return of the two stocks
  - However, the volatility of 5.1% is much less than the volatility of the two individual stocks.

The Volatility of a Two-Stock Portfolio

- Combining Risks
  - By combining stocks into a portfolio, we reduce risk through diversification.
  - The amount of risk that is eliminated in a portfolio depends on the degree to which the stocks face common risks and their prices move together.

Covariance and Correlation

Covariance

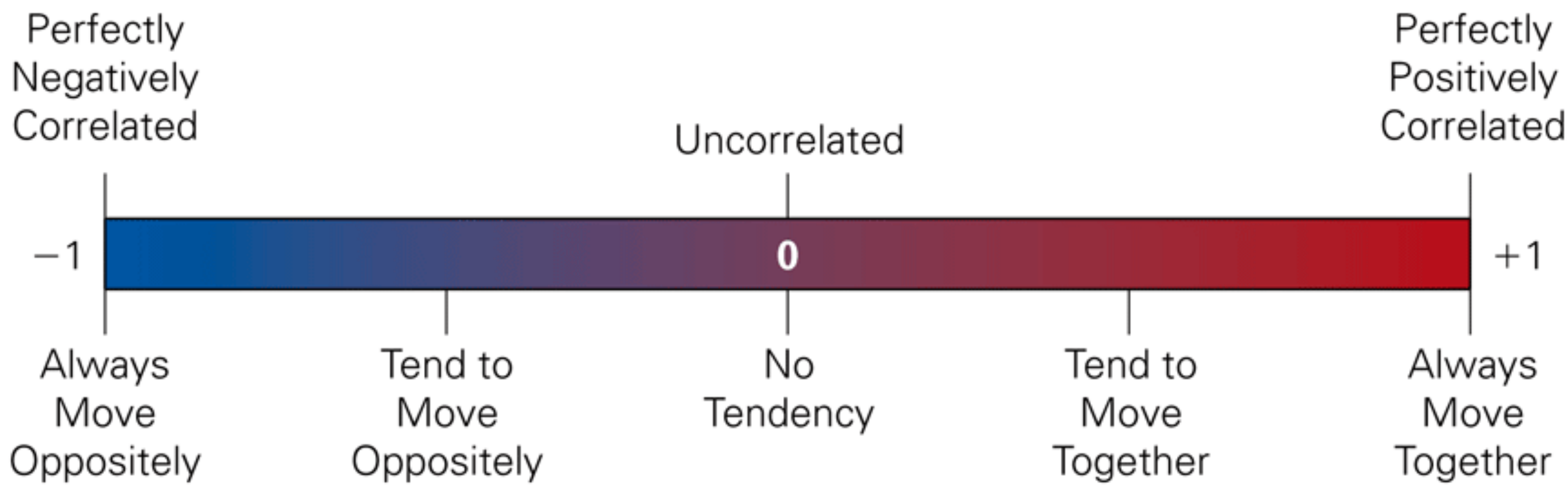
$$\sigma_{xy} = \rho_{xy}\,\sigma_x\,\sigma_y$$

Correlation

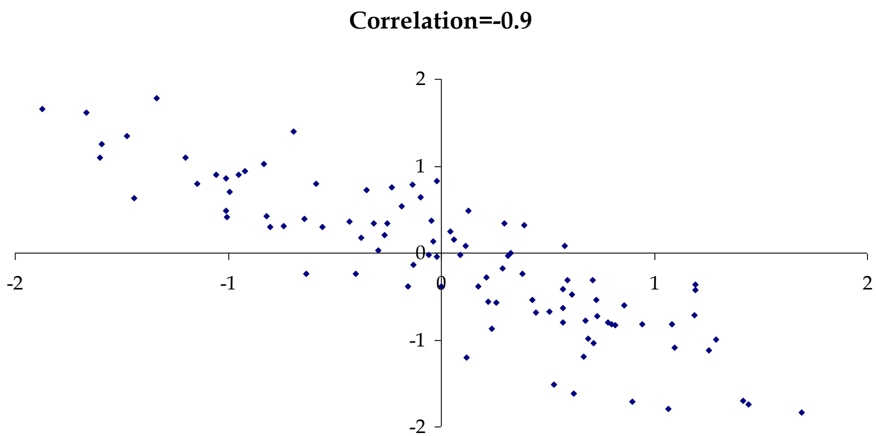
$$\rho_{xy} \begin{cases} 1 & \text{perfectly correlated} \\ 0 & \text{uncorrelated} \\ -1 & \text{perfectly negatively correlated} \end{cases}$$

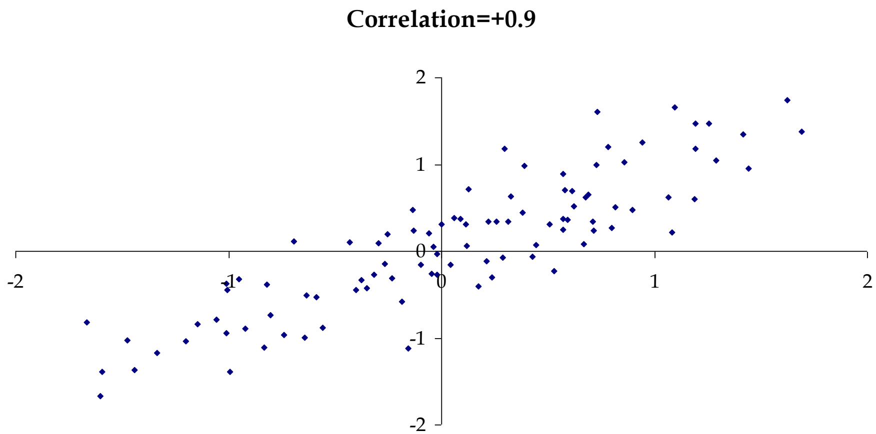
Covariance with itself = Variance= $\sigma_{xx} = 1\sigma_x\,\sigma_x = \sigma_x^2$

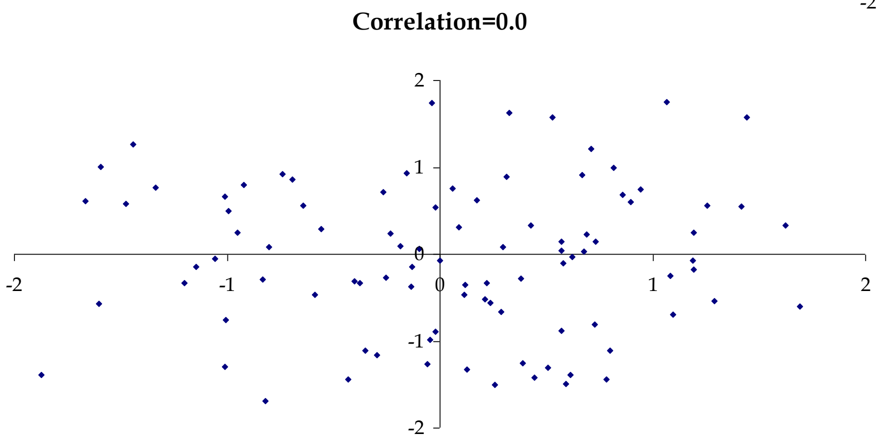
Correlation



Correlations







Determining Covariance and Correlation

- Covariance between Returns r_i and r_j
  - The expected product of the deviations of two returns from their means

$$\sigma_{ij} = E[(r_i - \bar{r}_i)(r_j - \bar{r}_j)]$$

  - Covariance Estimate of the Covariance from Historical Data

$$\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)$$

    - If the covariance is positive, the two returns tend to move together.
    - If the covariance is negative, the two returns tend to move in opposite directions.

Determining Covariance and Correlation

- Correlation
  - A measure of the common risk shared by stocks that does not depend on their volatility

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

    - The correlation between two stocks will always be between –1 and +1.

Variance and Covariance

- Use historical data to estimate average returns, variance of returns and covariance:

$$\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \bar{r})^2 = \frac{1}{T-1}\sum_{t=1}^{T} r_t^{\,2} - \bar{r}^{\,2}$$

$$\hat{\sigma}_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)$$

Computing the Covariance and Correlation Between Pairs of Stocks

- First compute the mean and standard deviation

	Stock Returns		
Year	North Air	West Air	Tex Oil
2007	21%	9%	−2%
2008	30%	21%	−5%
2009	7%	7%	9%
2010	−5%	−2%	21%
2011	−2%	−5%	30%
2012	9%	30%	7%
Average Return	10.0%	10.0%	10.0%
Volatility	13.4%	13.4%	13.4%

Computing the Covariance and Correlation Between Pairs of Stocks

	Deviation from Mean			North Air and West Air	West Air and Tex Oil
Year	$(R_N - \bar{R}_N)$	$(R_W - \bar{R}_W)$	$(R_T - \bar{R}_T)$	$(R_N - \bar{R}_N)(R_W - \bar{R}_W)$	$(R_W - \bar{R}_W)(R_T - \bar{R}_T)$
2007	11%	−1%	−12%	−0.0011	0.0012
2008	20%	11%	−15%	0.0220	−0.0165
2009	−3%	−3%	−1%	0.0009	0.0003
2010	−15%	−12%	11%	0.0180	−0.0132
2011	−12%	−15%	20%	0.0180	−0.0300
2012	−1%	20%	−3%	−0.0020	−0.0060
		$\text{Sum} = \sum_t (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j) =$		0.0558	−0.0642
Covariance:		$Cov(R_i, R_j) = \frac{1}{T-1}\text{Sum} =$		0.0112	−0.0128
Correlation:		$Corr(R_i, R_j) = \frac{Cov(R_i, R_j)}{SD(R_i)SD(R_j)} =$		0.624	−0.713

Historical Annual Volatilities and Correlations for Selected Stocks

	Microsoft	Dell	Alaska Air	Southwest Airlines	Ford Motor	General Motors	General Mills
Volatility (Standard Deviation)	37%	50%	38%	31%	42%	41%	18%
Correlation with							
Microsoft	1.00	0.62	0.25	0.23	0.26	0.23	0.10
Dell	0.62	1.00	0.19	0.21	0.31	0.28	0.07
Alaska Air	0.25	0.19	1.00	0.30	0.16	0.13	0.11
Southwest Airlines	0.23	0.21	0.30	1.00	0.25	0.22	0.20
Ford Motor	0.26	0.31	0.16	0.25	1.00	0.62	0.07
General Motors	0.23	0.28	0.13	0.22	0.62	1.00	0.02
General Mills	0.10	0.07	0.11	0.20	0.07	0.02	1.00

Portfolio variance

- The variance of a portfolio is

$$\sigma^2_p = w_x^2\,\sigma^2_x + w_y^2\,\sigma^2_y + 2w_x\,w_y\,\sigma_{xy}$$

Rem $Var(aX+bY)=a^2Var(X)+b^2Var(Y)+2abCov(X,Y)$

Covariance matrix

Portfolio Weights	w_x	w_y
w_x	$cov(r_x,r_x)$	$cov(r_x,r_y)$
w_y	$cov(r_x,r_y)$	$cov(r_y,r_y)$

Border-multiplied covariance matrix

Portfolio Weights	w_x	w_y
w_x	$w_x w_x cov(r_x,r_x)$	$w_x w_y cov(r_x,r_y)$
w_y	$w_y w_x cov(r_y,r_x)$	$w_y w_y cov(r_y,r_y)$

Variance

- $\sigma^2_p = w_x^2 \sigma^2_x + w_y^2 \sigma^2_y + 2w_x w_y \sigma_{xy}$
- σ_x=0.10
- σ_y=0.30
- ρ = 0.6
- σ_{xy}= 0.1 x 0.3 x 0.6

- $\sigma^2_p = 0.5^2\, 0.1^2 + 0.5^2\, 0.3^2 + 2(0.5)(0.5)\ 0.1 \times 0.3 \times 0.6$

In Matrix Notation

$$\Omega : 2\times 2\, Variance\ Co\mathrm{var}iance\ matrix\ of\ returns = \begin{vmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{vmatrix}$$

$$\mu : 2\ column\ vector\ of\ \mathrm{exp}ected\ returns = \begin{vmatrix} \mu_x \\ \mu_y \end{vmatrix}$$

$$w : 2\ column\ vector\ of\ weights = \begin{vmatrix} w_x \\ w_y \end{vmatrix}$$

$$\mu_P = w_x\mu_x + w_y\mu_y = w^{Tr}\mu \quad scalar\ product$$

$$\sigma_P^2 = w_x^2\sigma_x^2 + w_x w_y \sigma_{xy} + w_y w_x \sigma_{yx} + w_y^2\sigma_y^2 = w^{Tr}\Omega w$$

$$\sigma_{xy} = \rho_{xy}\sigma_x\sigma_y$$

One risky asset and one risk-free asset

- Individual assets
  - Portfolio weights:

    w in risky asset (stocks)

    (**1-w**) in risk-free asset (T-bill)
  - Expected (mean) returns :

    μ=7.5% and r_f =1.5%
  - Standard deviation of returns :

    σ =20% and 0
  - Covariance of returns : 0

One risky asset and one risk-free asset

- Portfolio expected return

$$\mu_p = w\,\mu + (1\text{-}w)\, r_f = r_f + w(\mu\text{-}\, r_f)$$

$$= 0.015 + 0.06w$$

- Portfolio variance

$$\sigma_p{}^2 = w^2\,\sigma^2$$

$$= 0.04w^2$$

Standard deviation

$$\sigma_p = w\sigma = 0.2w$$

One risky asset and one risk-free asset

$$w_x = w \quad , \quad w_y = 1 - w$$

$$\mu_P = w\mu + (1 - w)r_f$$

$$\mu_P = r_f + w(\mu - r_f)$$

$$\mu_P = 0.015 + w(0.075 - 0.015) = 0.015 + 0.06w$$

$$\sigma_P^2 = w^2\sigma^2 + (1 - w)^2 0 + 2w(1 - w)0 = w^2\sigma^2$$

$$\sigma_P = w\sigma = 0.2w$$

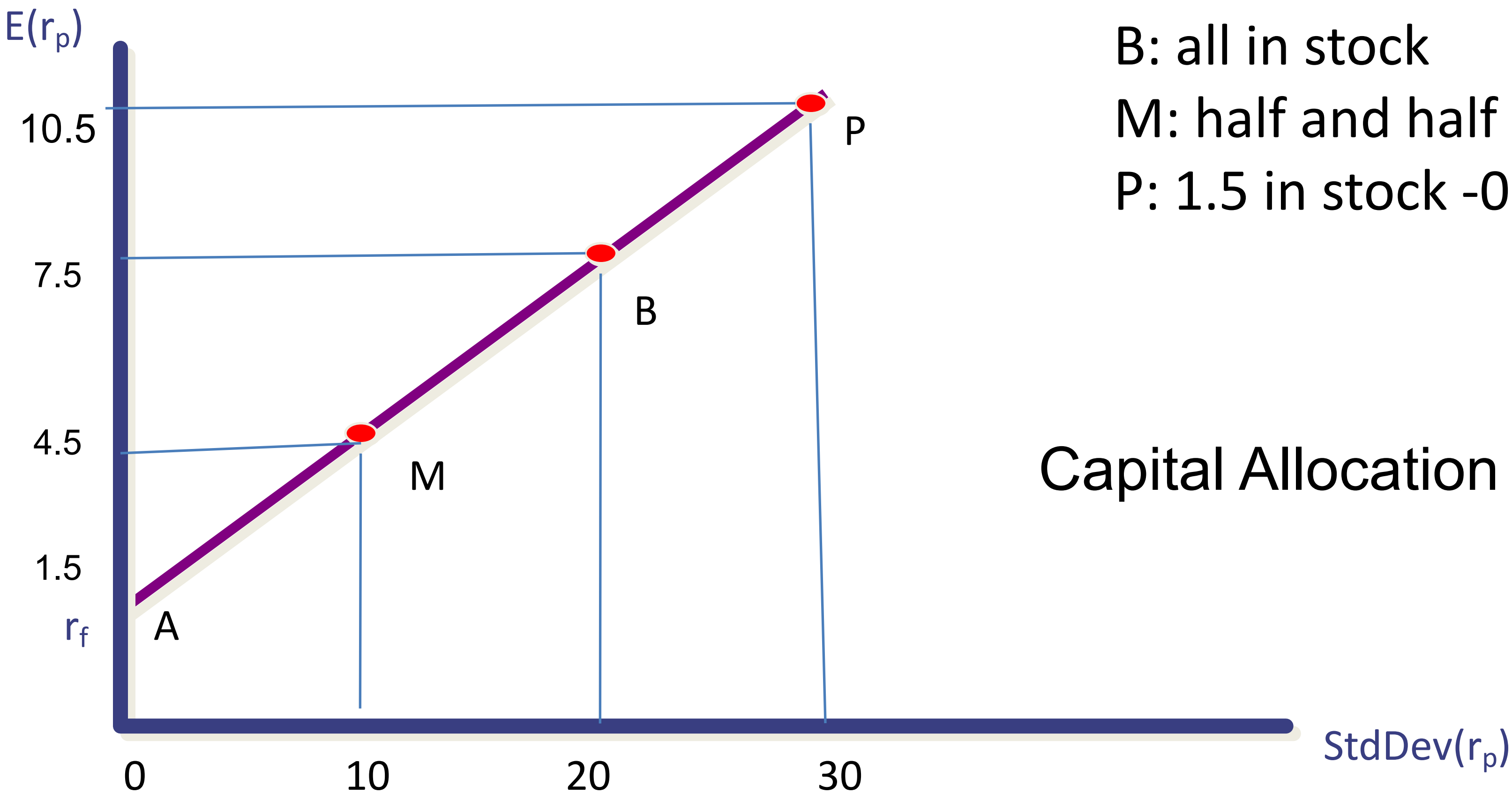
Different weights in the stock and the risk free asset

Portfolio	Stock	Risk free A	Exp. Ret. μ_p	S.D. σ_p
A	0	1	1.5%	0
B	1	0	7.5%	20%
M	0.5	0.5	4.5%	10%
P	1.5	-0.5	10.5%	30%

$$\mu_p = w\,\mu + (1-w)\,r_f = 0.015 + 0.06w$$

$$\sigma_p = w\sigma = 0.2w$$

One risky asset and one risk-free asset



A: all in risk free asset

B: all in stock

M: half and half

P: 1.5 in stock -0.5 in r_f

Capital Allocation Line

One risky asset and one risk-free asset

- Lending
  - Invest in the risk free asset (long position)
  - Take a positive position in the risk free asset
- Borrowing
  - Short the risk free asset
  - Take a negative position in the risk free asset
- Short a stock
  - Borrow a stock and sell it
  - Take a negative position in the stock

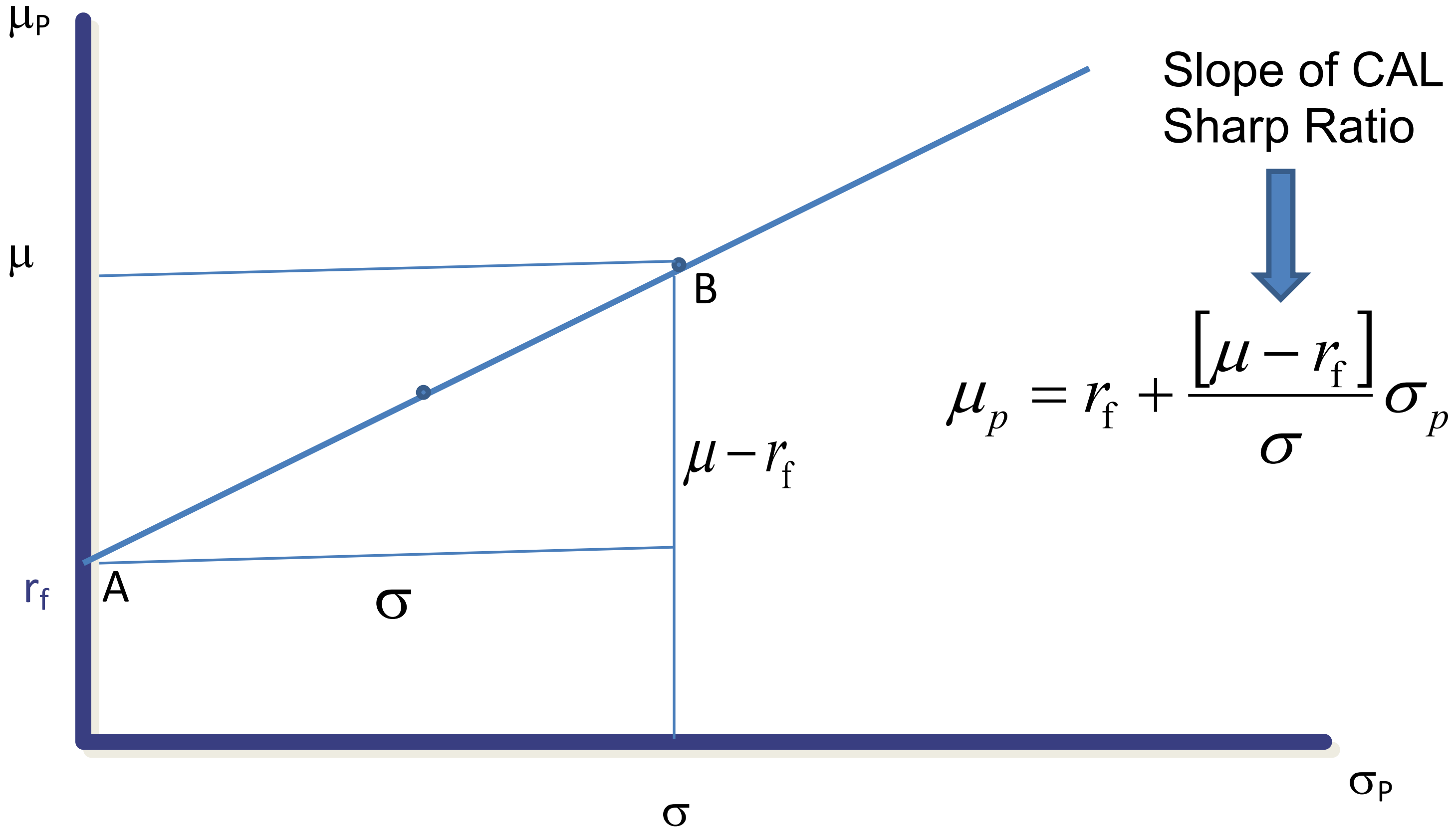
Short Selling

- Short selling involves selling securities you do not own
- Your broker borrows the securities from another client and sells them in the market in the usual way
- Having a negative position in the security (long: positive position)

Short Selling

- At some stage you must buy the securities back so they can be replaced in the account of the client
- You must pay dividends and other benefits the owner of the securities receives
- You benefit when the security price goes down

Capital Allocation Line



$$\mu_p = r_f + \frac{[\mu - r_f]}{\sigma}\sigma_p$$

Equation of a Line

$$y = \alpha + \beta x$$

$$\alpha : \text{int}\, ercept \qquad \beta : slope$$

$$\mu_p = r_{\mathrm{f}} + \frac{[\mu - r_{\mathrm{f}}]}{\sigma} \sigma_p$$

Capital Allocation Line

- Feasible combinations of mean and standard deviation – Capital Allocation Line

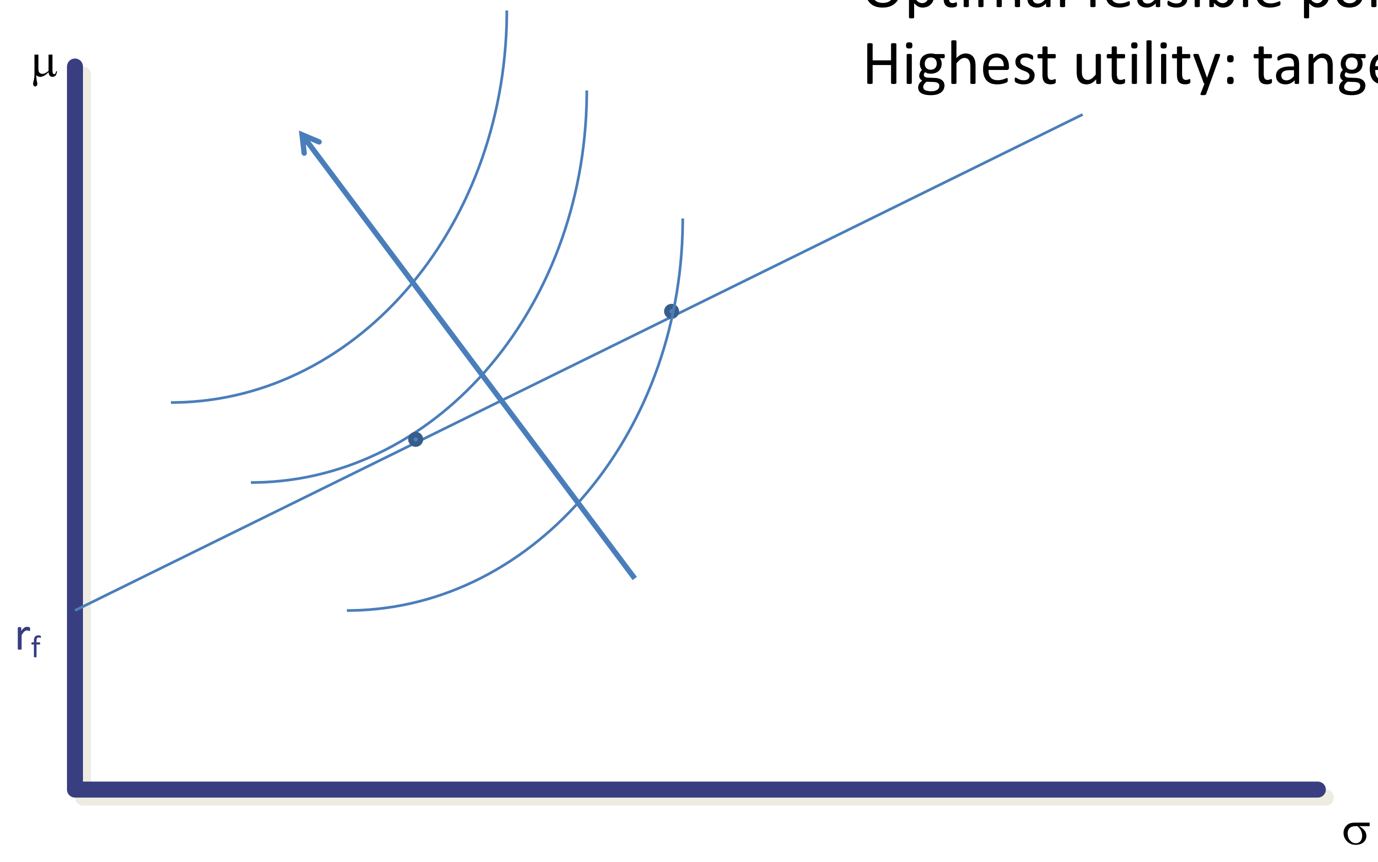
$$\mu_p = r_{\mathrm{f}} + \sigma_p \frac{\left[\mu - r_{\mathrm{f}}\right]}{\sigma} = 0.015 + 0.3\sigma_p$$

- Slope of CAL is **Sharpe ratio**: excess return per unit of risk

$$S = \frac{\mu - r_{\mathrm{f}}}{\sigma} = \frac{0.06}{0.2} = 0.3$$

Optimal Portfolio on the CAL

Optimal feasible portfolio
Highest utility: tangency point



Mean-variance utility functions

- Utility function (mean-variance)

  $U(r_p) = E(r_p) - \gamma Var(r_p)/2 = \mu_p - \gamma \sigma_p^2/2$

  Where γ is coefficient of risk aversion: e.g. $\gamma=4$
- We saw that there are other utility functions that are often used in practice
  - Not as easy to use, but with better properties

Mean-variance utility functions
(unique up to a positive linear transformation)

$$u(W) = W - \frac{b}{2}W^2 \approx k_1 W - k_2 (W - \overline{W})^2$$

$$\approx k_1 W_0 (r_P + 1) - k_2 {W_0}^2 (r_P - \mu_P)^2$$

$$E[u(W)] \approx E[u(r_P)] = U(r_P) = \mu_P - \frac{\gamma}{2}\sigma_P^2$$

Portfolio of risky and risk-free assets

- To find optimal portfolio choice

  $\max_w U(r_p) = w\,\mu + (1-w)\,r_f - \gamma\, w^2\,\sigma^2/2$

- From first-order conditions

$$w = \frac{\mu - r_{\mathrm{f}}}{\gamma\sigma^2} = \frac{0.06}{4\times 0.2^2} = \frac{0.06}{0.16} = 0.375$$

- Invest 37.5% of portfolio in stock and 62.5% in T-bills
- What if the risk aversion coefficient was 2?

Optimal Portfolio

$$U(r_P) = \mu_P - \frac{\gamma}{2}\sigma_P^2$$

$$Max_w U(r_P) = w\mu + (1-w)r_f - \frac{\gamma}{2}w^2\sigma^2$$

$$\frac{dU(r_P)}{dw} = \mu - r_f - \gamma w\sigma^2 = 0$$

$$w^* = \frac{\mu - r_f}{\gamma\sigma^2}$$

Utility as a Function of Allocation to the Risky Asset

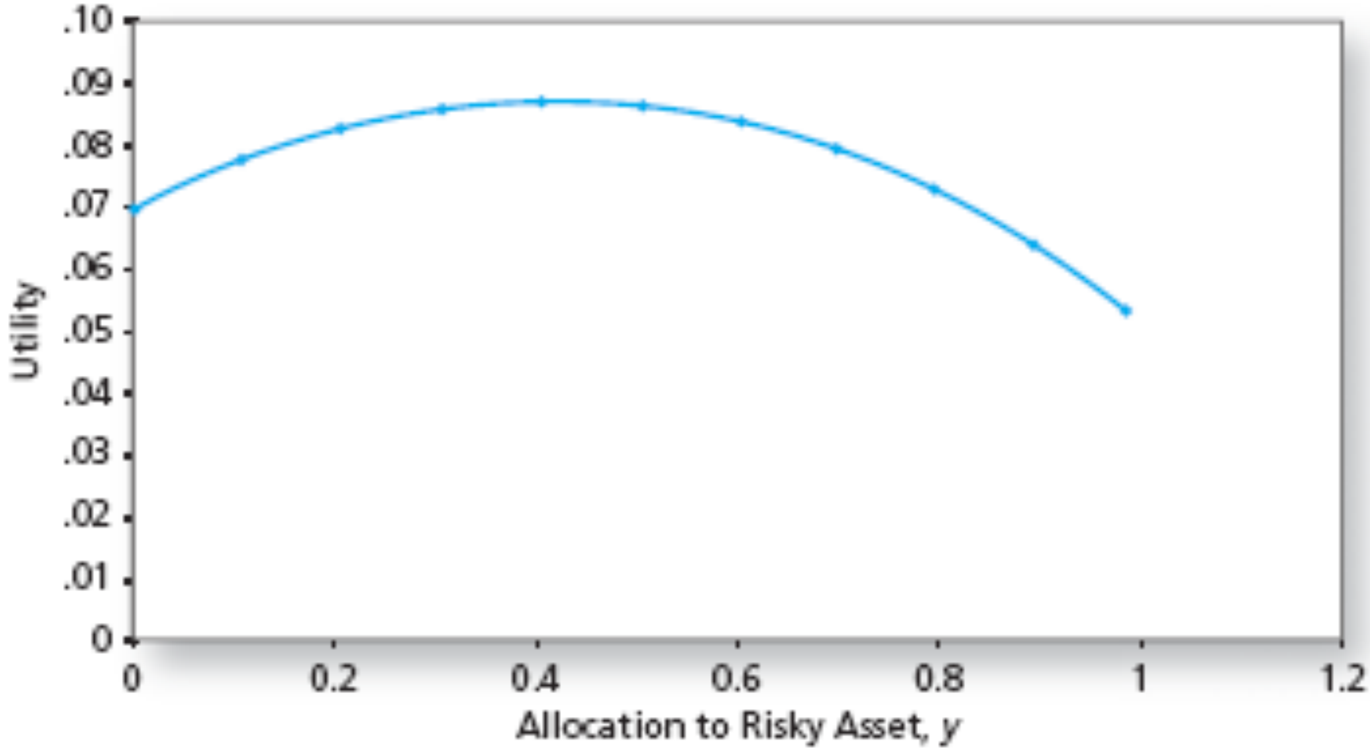
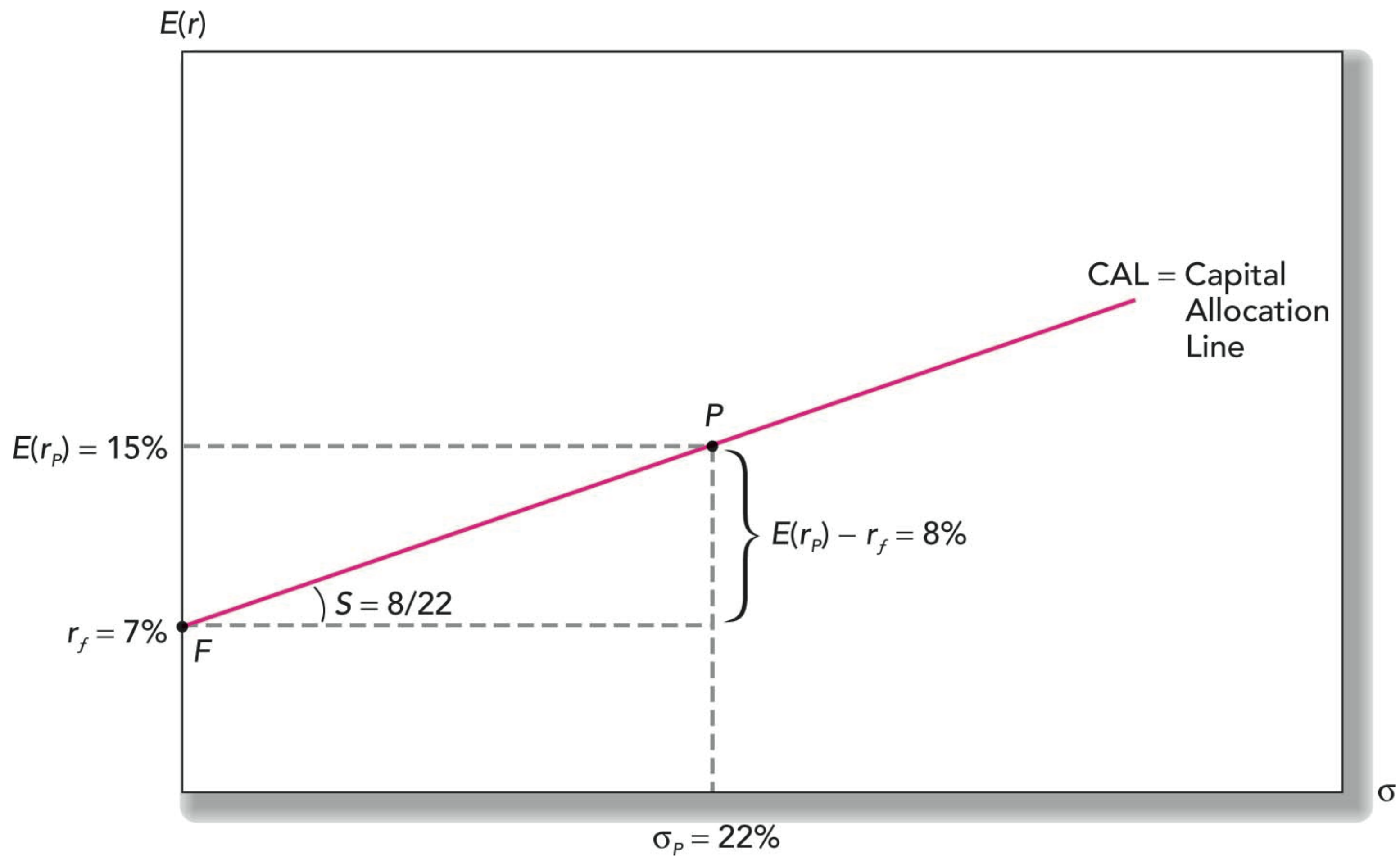


Figure 6.6 Utility as a function of allocation to the risky asset, *y*

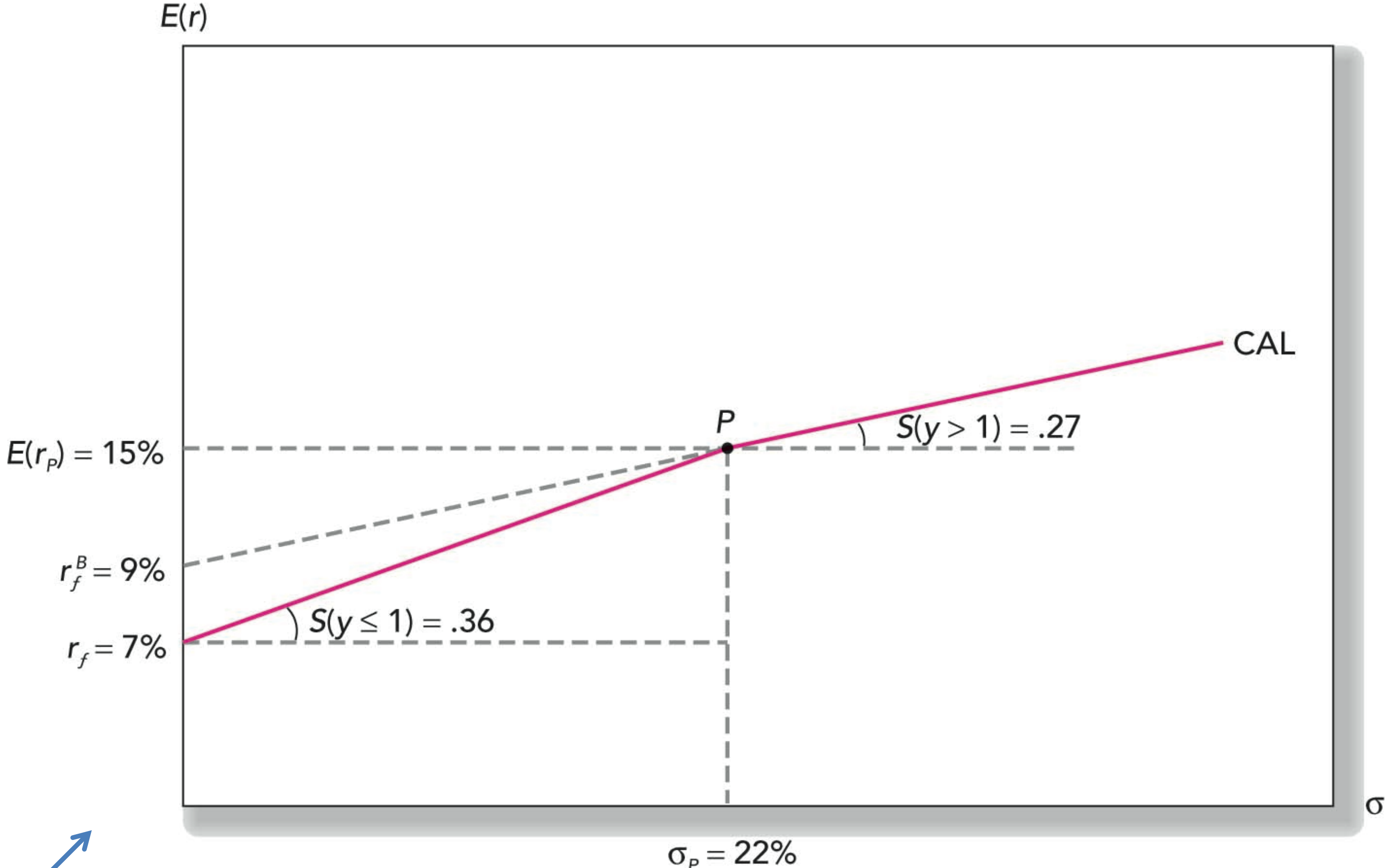
The Investment Opportunity Set with a Risky Asset and a Risk-free Asset in the Expected Return-Standard Deviation Plane



Different Borrowing and Lending Rates

- What happens when you lend (invest in the risk-free asset) at 7%, but your borrowing (short in the risk free asset) is 9%?
- You cannot prolong the line in the CAL!
- In practice borrowing and lending rates are not the same.
- But, for simplicity we often abstract from this in portfolio theory.

The Opportunity Set with Differential Borrowing and Lending Rates



y=w

Topic 5b: Portfolio Choice with Two Risky Assets

Eduardo Schwartz
UCLA Anderson School

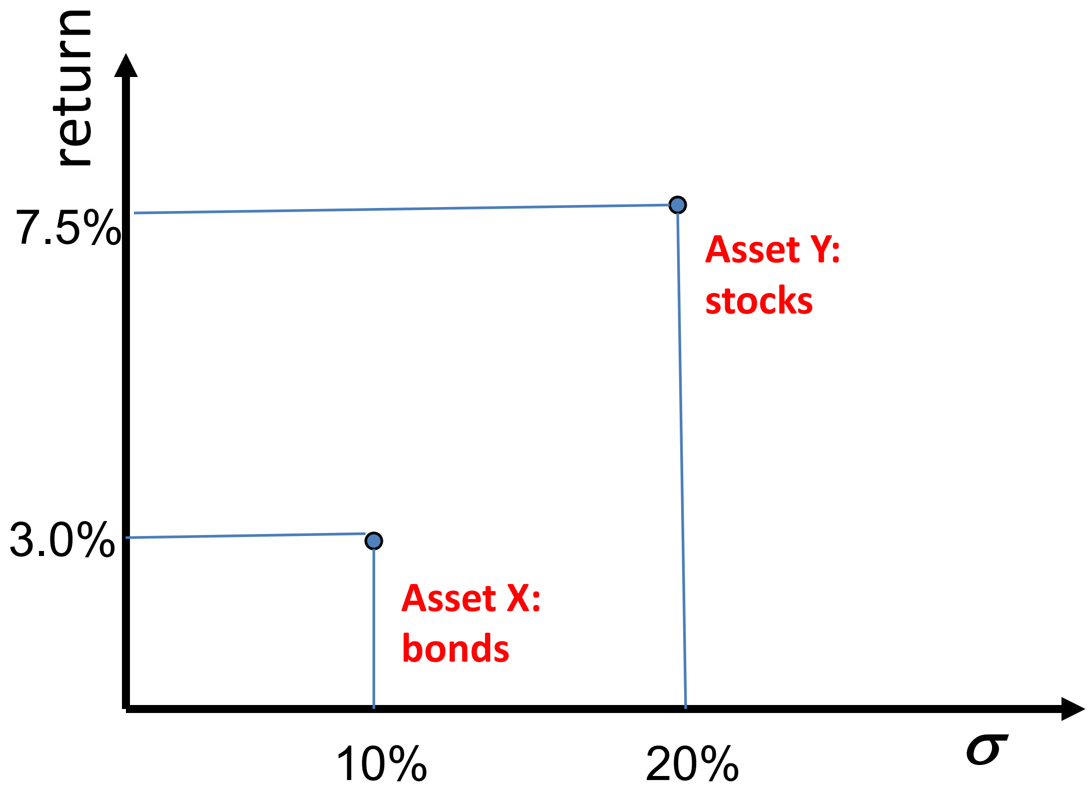
Two risky assets

- We now know how to optimally combine one risky asset with a risk-free asset
- What if we have several risky assets?
  - First combine the risky assets into an optimal risky portfolio
  - Then treat the optimal risky portfolio as if it was a single asset and combine it with the risk-free asset
- Let’s start with two risky assets

Two risky assets

- Two risky assets – bond and stock portfolios (it could also be two stocks)
  - Means of 3.0% and 7.5%
  - Standard deviations of 10% and 20%
  - Correlation of 0.2
- Consider a portfolio with 1/3 of funds invested in bond and 2/3 of funds invested in stock portfolios

Two risky assets



Two risky assets with a correlation of 0.2

Two risky assets

- Portfolio expected return

  $\mu_p = 1/3 \times 0.03 + 2/3 \times 0.075$

  $= 0.06 = 6.0\%$

- Portfolio variance= $\boxed{w_x^2 \sigma^2_x + w_y^2 \sigma^2_y + 2 w_x w_y \sigma_{xy}}$ $\boxed{\sigma_{xy} = \rho \sigma_x \sigma_y}$

  $\sigma_p^2 = (1/3)^2 (0.1)^2 + (2/3)^2 (0.2)^2 + 2\,(1/3)(2/3)(0.2 \times 0.1 \times 0.2)$

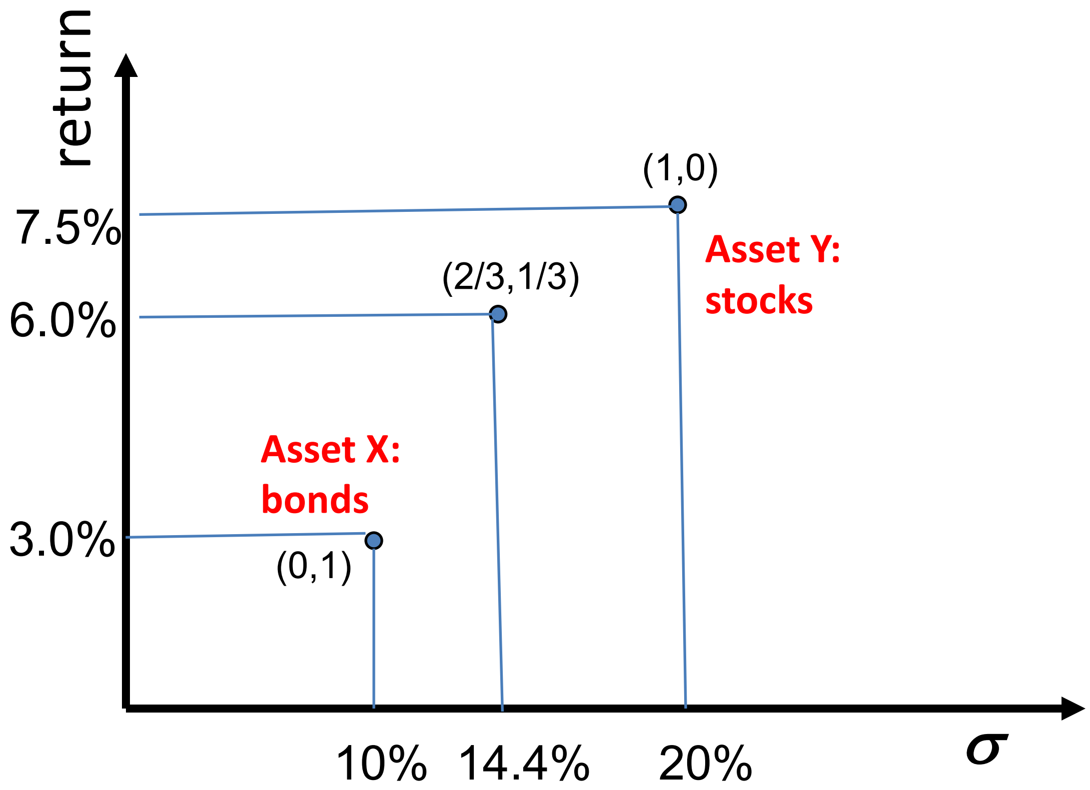
  $= 0.0207$

- Portfolio standard deviation

  $\sigma_p = \sqrt{0.0207}$

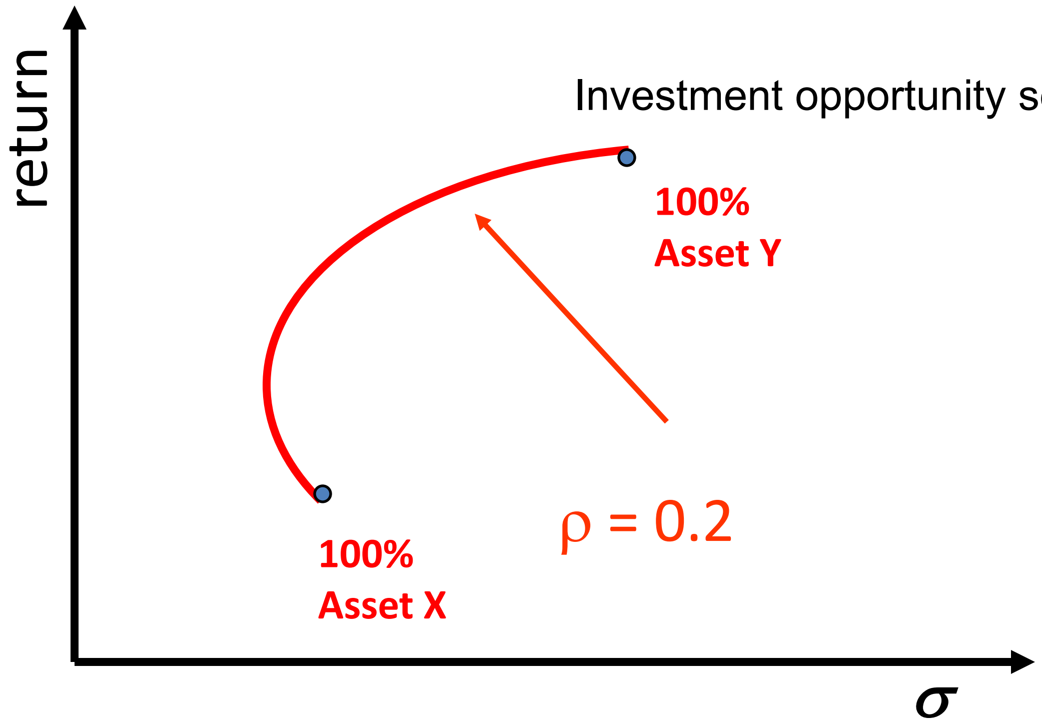
  $= 0.1438 = 14.38\%$

Two risky assets



What happens if the proportions change?

Two risky assets when proportions change



Efficient Portfolios: upper part of the curve. Highest expected return for a given standard deviation

Perfect Correlation?

$$\sigma_P^2 = w_x^2\sigma_x^2 + w_y^2\sigma_y^2 + 2w_x w_y \rho\sigma_x\sigma_y$$

$$\rho = 1$$

$$\sigma_P^2 = (w_x\sigma_x + w_y\sigma_y)^2$$

$$\sigma_P = w_x\sigma_x + w_y\sigma_y$$

$$\rho = -1$$

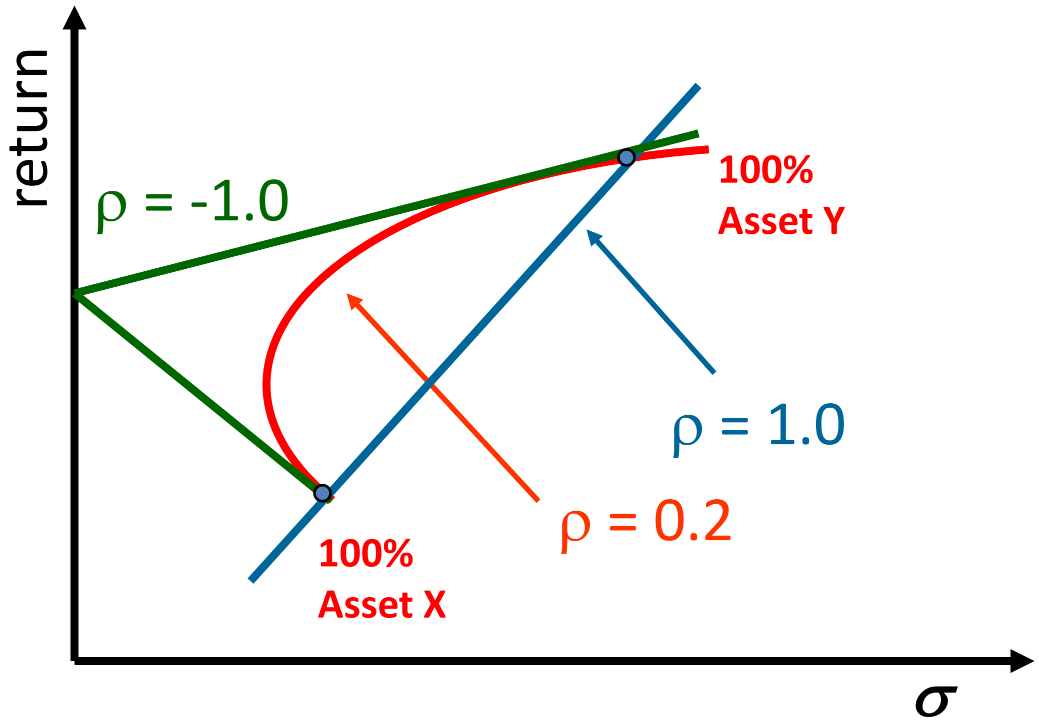
$$\sigma_P^2 = (w_x\sigma_x - w_y\sigma_y)^2$$

$$\sigma_P = w_x\sigma_x - w_y\sigma_y > 0$$

$$\sigma_P = w_y\sigma_y - w_x\sigma_x > 0$$

$$\sigma_P = 0 \ \ when \quad \frac{w_x}{w_y} = \frac{\sigma_y}{\sigma_x}$$

Two-Assets: Different Correlations



Efficient Portfolios: highest expected return for a given σ

Perfect Negative Correlation

$$\sigma_P = 0 \ \ when \quad \frac{w_x}{w_y} = \frac{\sigma_y}{\sigma_x}$$

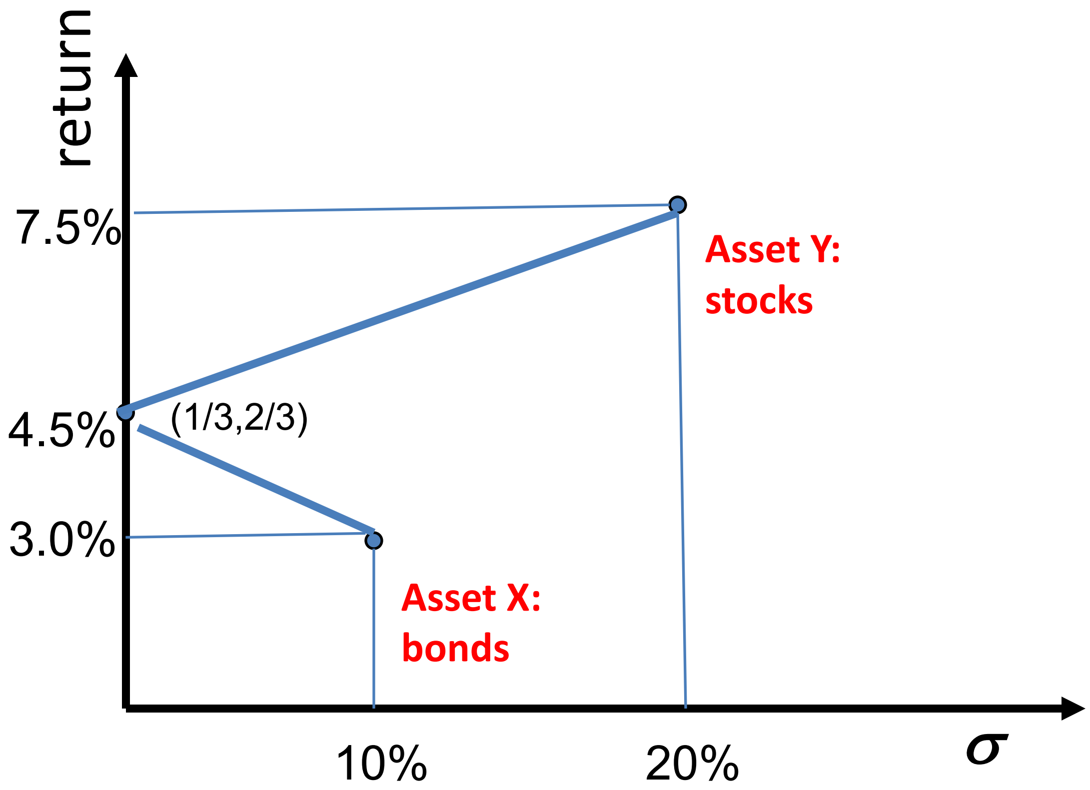
$$\sigma_x = 0.10 \quad \sigma_y = 0.20$$

$$\frac{w_x}{w_y} = \frac{.20}{.10} \quad and \quad w_x + w_y = 1$$

$$w_x = 2/3 \quad w_y = 1/3$$

$$\mu_p = w_x\mu_x + w_y\mu_y = 0.67(0.03) + 0.33(0.075) = 0.045$$

Two risky assets



Two risky assets with a correlation of -1

Could the risk free rate be 1.5%?

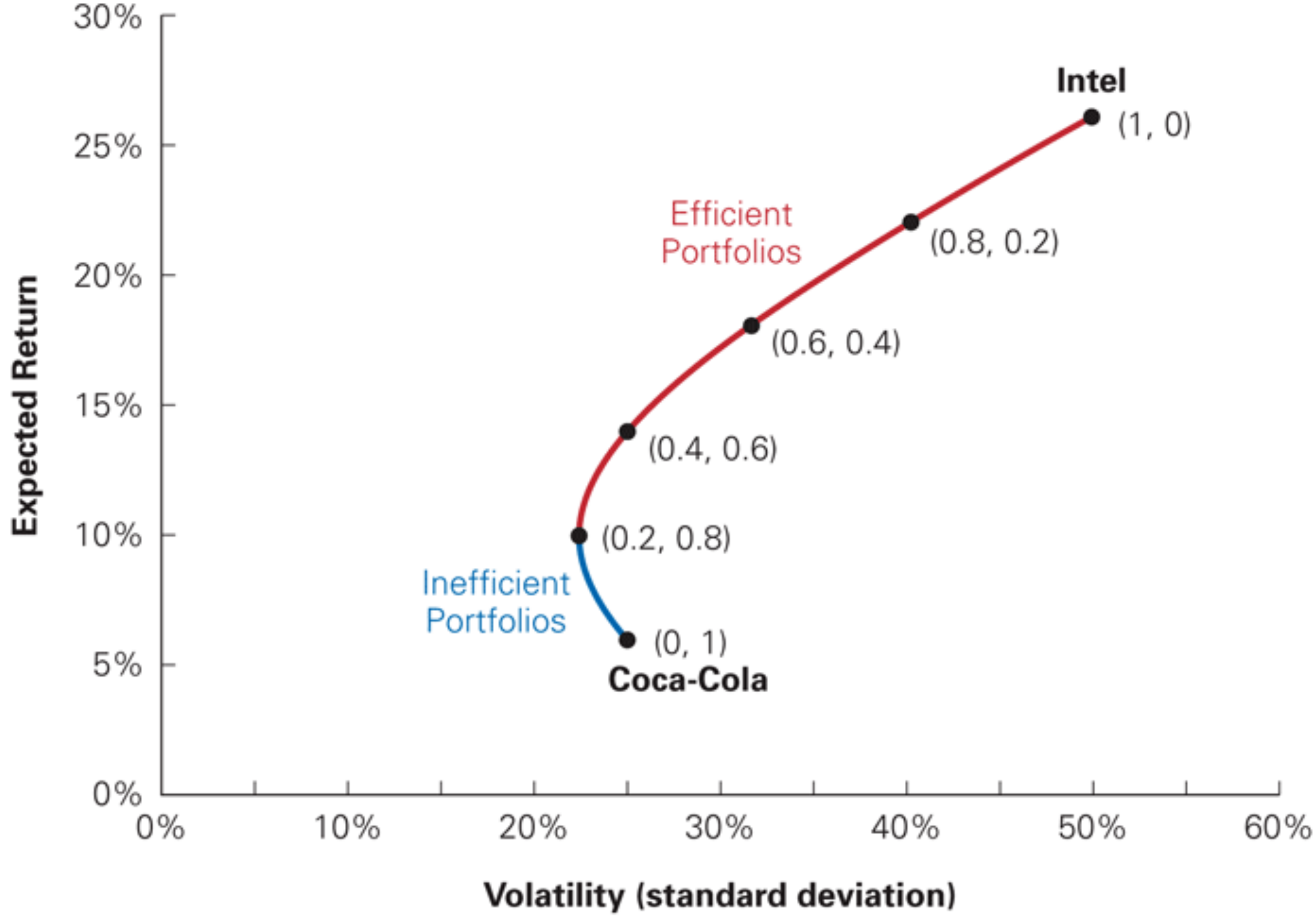
Volatility Versus Expected Return for Portfolios of Intel and Coca-Cola Stock

Expected Returns and Volatility for Different Portfolios of Two Stocks

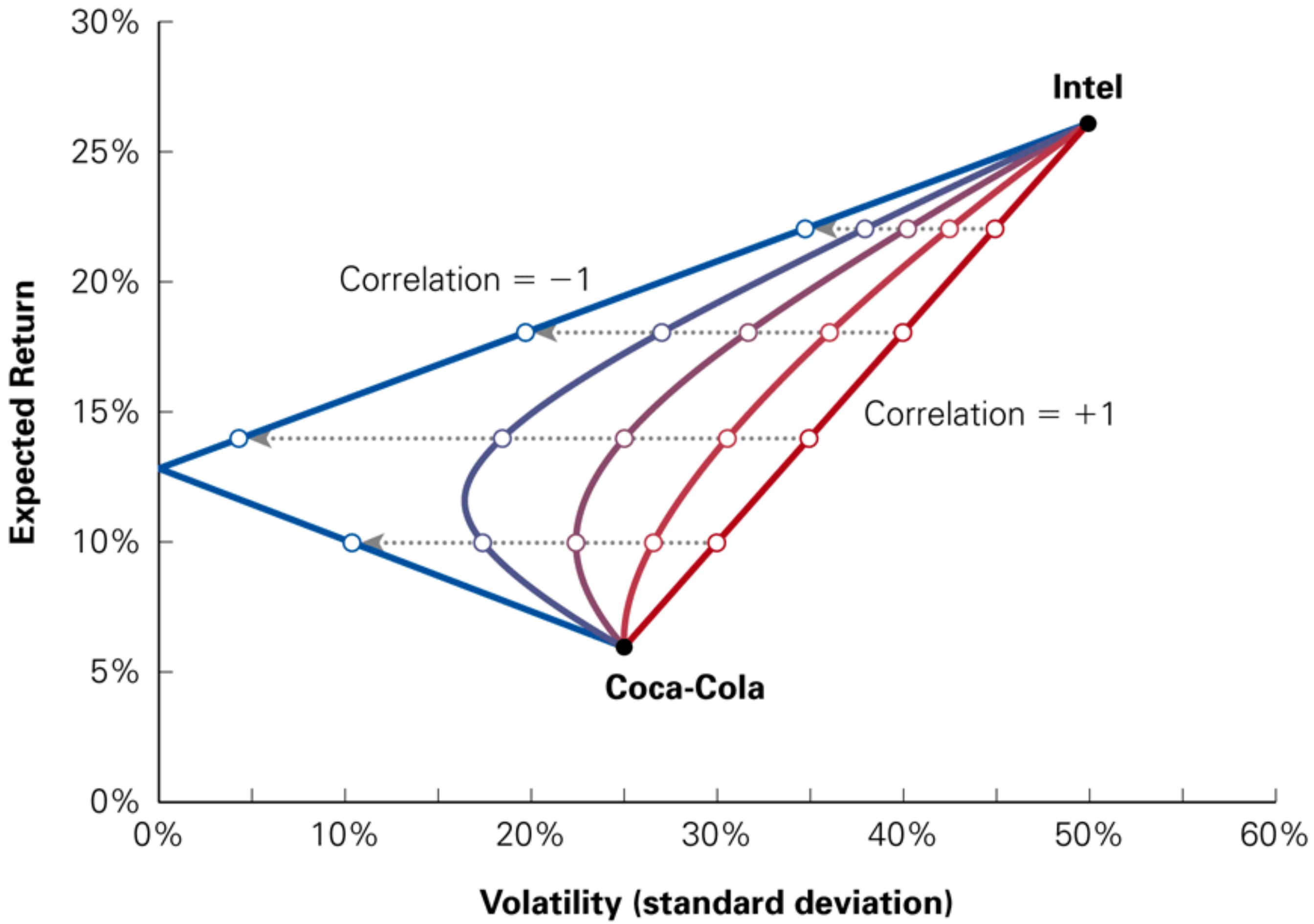
Portfolio Weights		Expected Return (%)	Volatility (%)
x_I	x_C	$E[R_P]$	$SD[R_P]$
1.00	0.00	26.0	50.0
0.80	0.20	22.0	40.3
0.60	0.40	18.0	31.6
0.40	0.60	14.0	25.0
0.20	0.80	10.0	22.4
0.00	1.00	6.0	25.0

Assume that the returns on these stocks are uncorrelated.

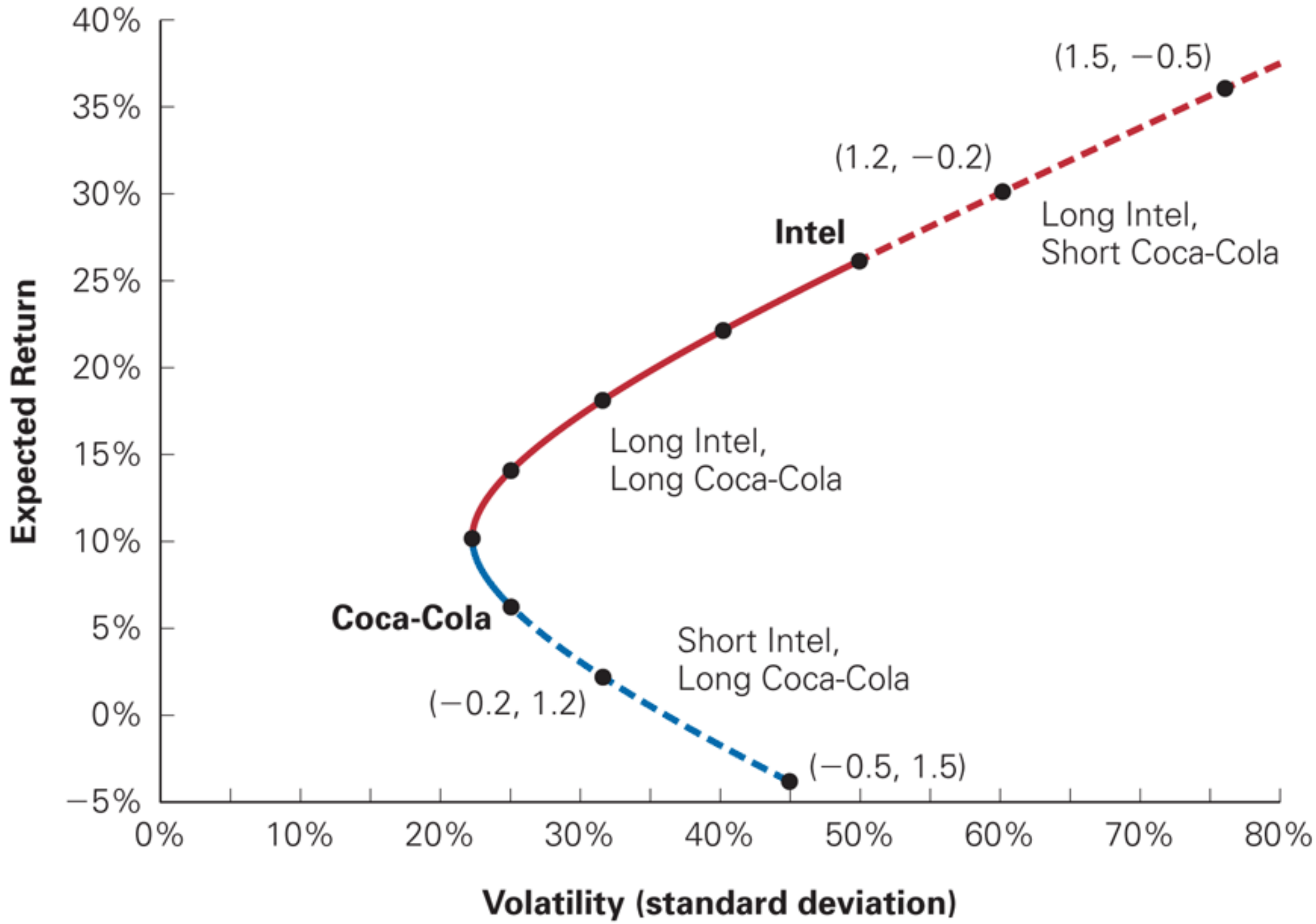
Volatility Versus Expected Return for Portfolios of Intel and Coca-Cola Stock



Effect on Volatility and Expected Return of Changing the Correlation between Intel and Coca-Cola Stock



Portfolios of Intel and Coca-Cola Allowing for Short Sales



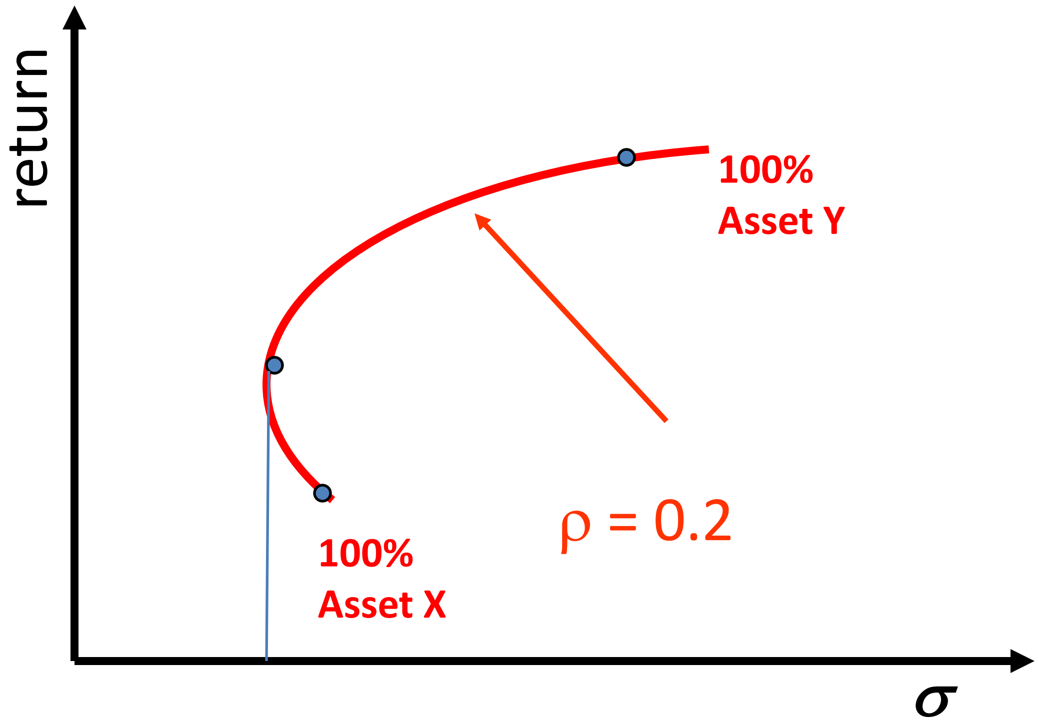
Two Assets: Correlation Effects

- Relationship depends on correlation coefficient
- $-1.0 \leq \rho \leq +1.0$
- The smaller the correlation, the greater the risk reduction potential
- If $\rho = +1.0$, no risk reduction is possible

Correlation and Diversification

- The various combinations of risk and return available all fall on a smooth curve.
- This curve is called an *investment opportunity set* because it shows the possible combinations of risk and return available from portfolios of these two assets.
- A portfolio that offers the highest return for its level of risk is said to be an *efficient portfolio*.
- The undesirable portfolios are said to be *dominated* or *inefficient*.

Minimum Variance Portfolio



Efficient Portfolios: upper part of the curve

Minimum Variance Portfolio

$$\boxed{\sigma_p^{\,2} = w_x^{\,2}\,\sigma^2_x + w_y^{\,2}\,\sigma^2_y + 2w_x\,w_y\,\sigma_{xy}}$$

$w_x + w_y = 1.0$

$$\sigma_P^2 = w^2\sigma_x^2 + (1-w)^2\sigma_y^2 + 2w(1-w)\sigma_{xy}$$

$$\frac{d\sigma_P^2}{dw} = 2w\sigma_x^2 - 2(1-w)\sigma_y^2 + 2\sigma_{xy} - 4w\sigma_{xy} = 0$$

$$w^* = \frac{\sigma_y^2 - \sigma_{xy}}{\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}} = w_x \quad , \quad w_y = 1 - w^*$$

Minimum Variance Portfolio

$$\sigma_x = 0.10 \quad bonds$$

$$\sigma_y = 0.20 \quad stocks$$

$$\rho = 0.2$$

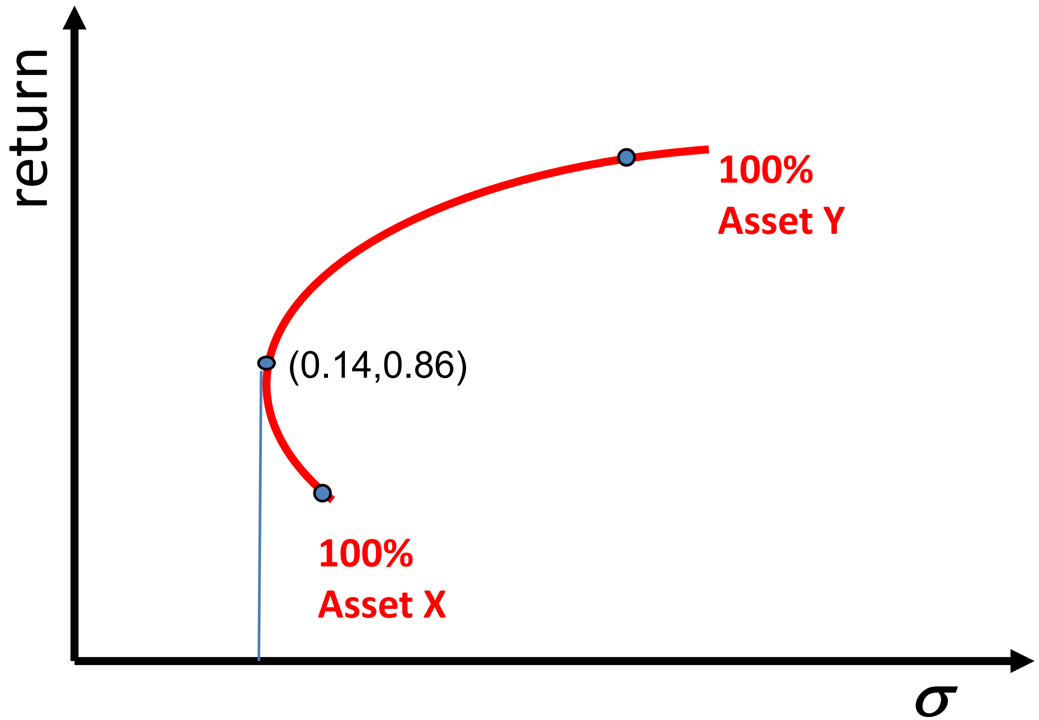
$$\sigma_{xy} = 0.2(0.10)(0.20) = 0.004$$

$$w^* = \frac{\sigma_y^2 - \sigma_{xy}}{\sigma_x^2 + \sigma_y^2 - 2\sigma_{xy}} = w_x \quad , \quad w_y = 1 - w^*$$

$$w_x = \frac{0.04 - 0.004}{0.01 + 0.04 - 0.008} = \frac{0.036}{0.042} = 0.86$$

$$w_y = 0.14$$

Minimum Variance Portfolio

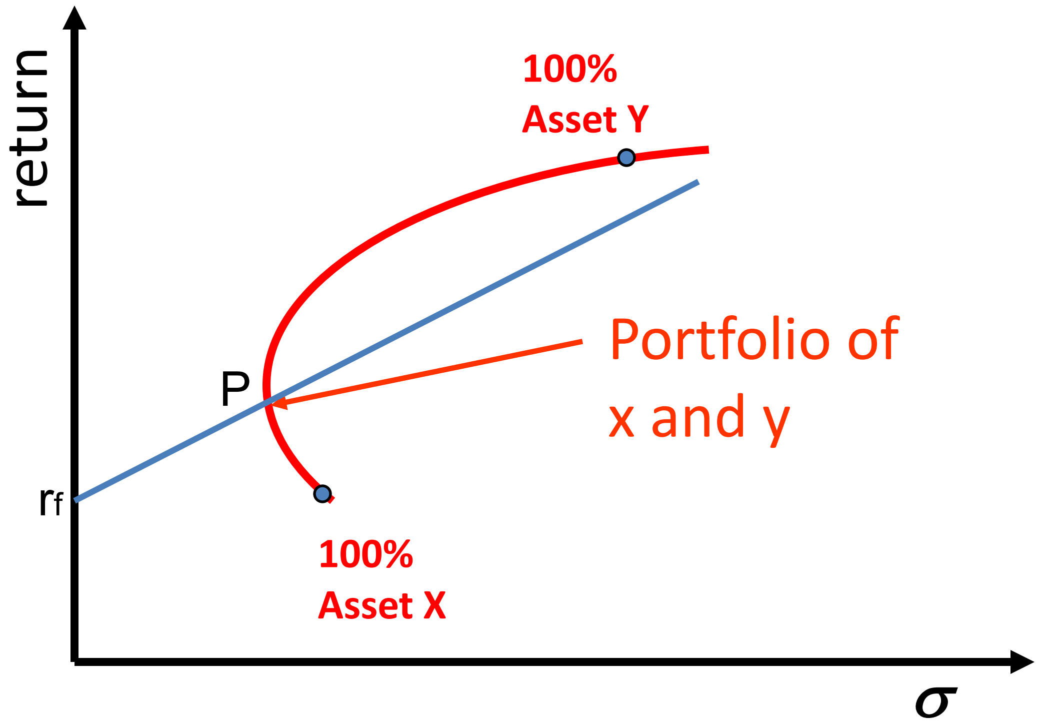


How do we find the expected return and standard deviation of the minimum variance portfolio?

Two risky and one risk-free asset

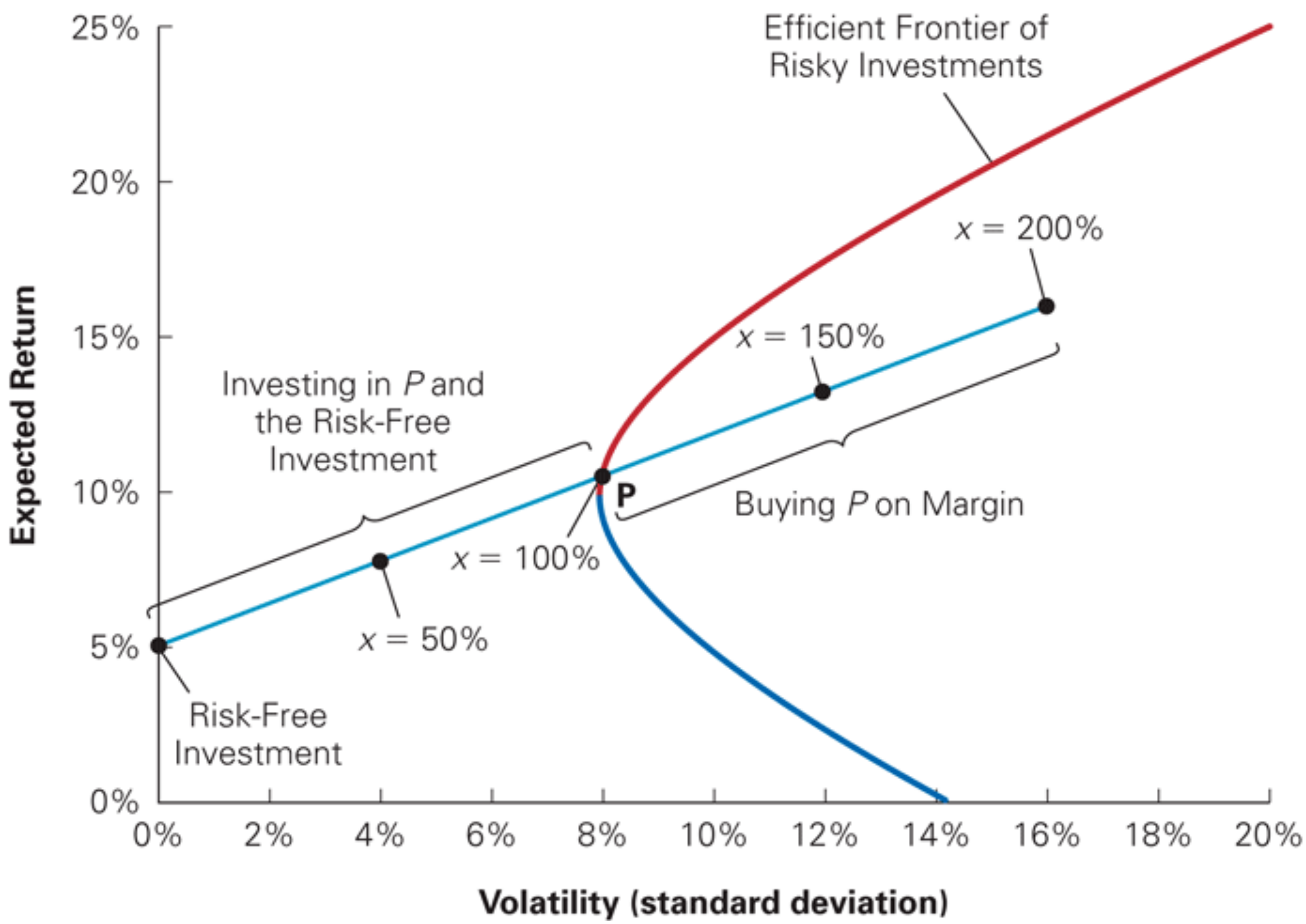
- We now find the portfolio of the two risky assets that can be optimally combined with the risk-free asset
  - This is the *optimal risky portfolio*
- For each risky portfolio, find out the corresponding CAL
- What is the CAL that would give the investor maximum utility?

Two risky assets and one risk-free asset

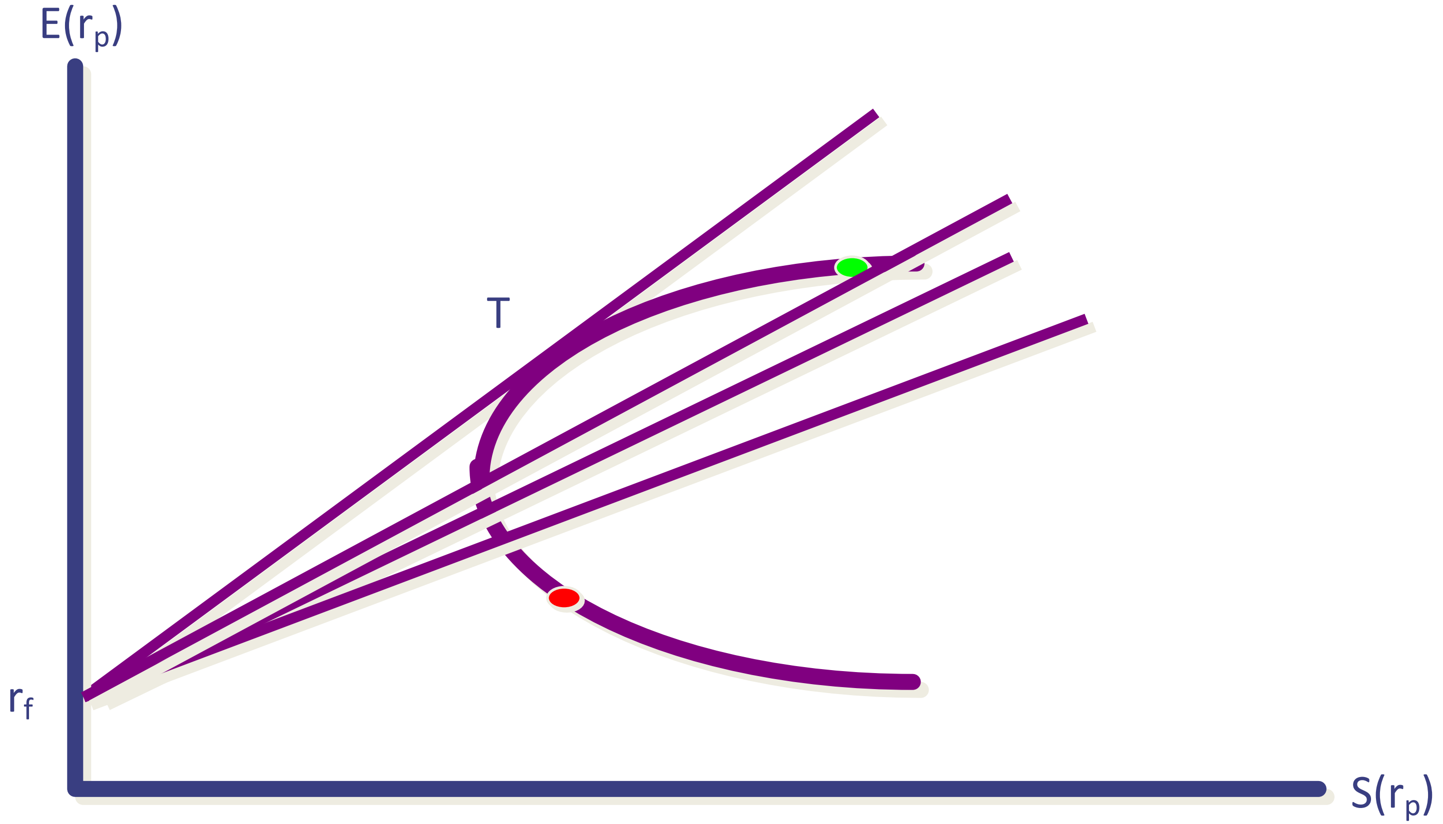


CAL for any portfolio of x and y

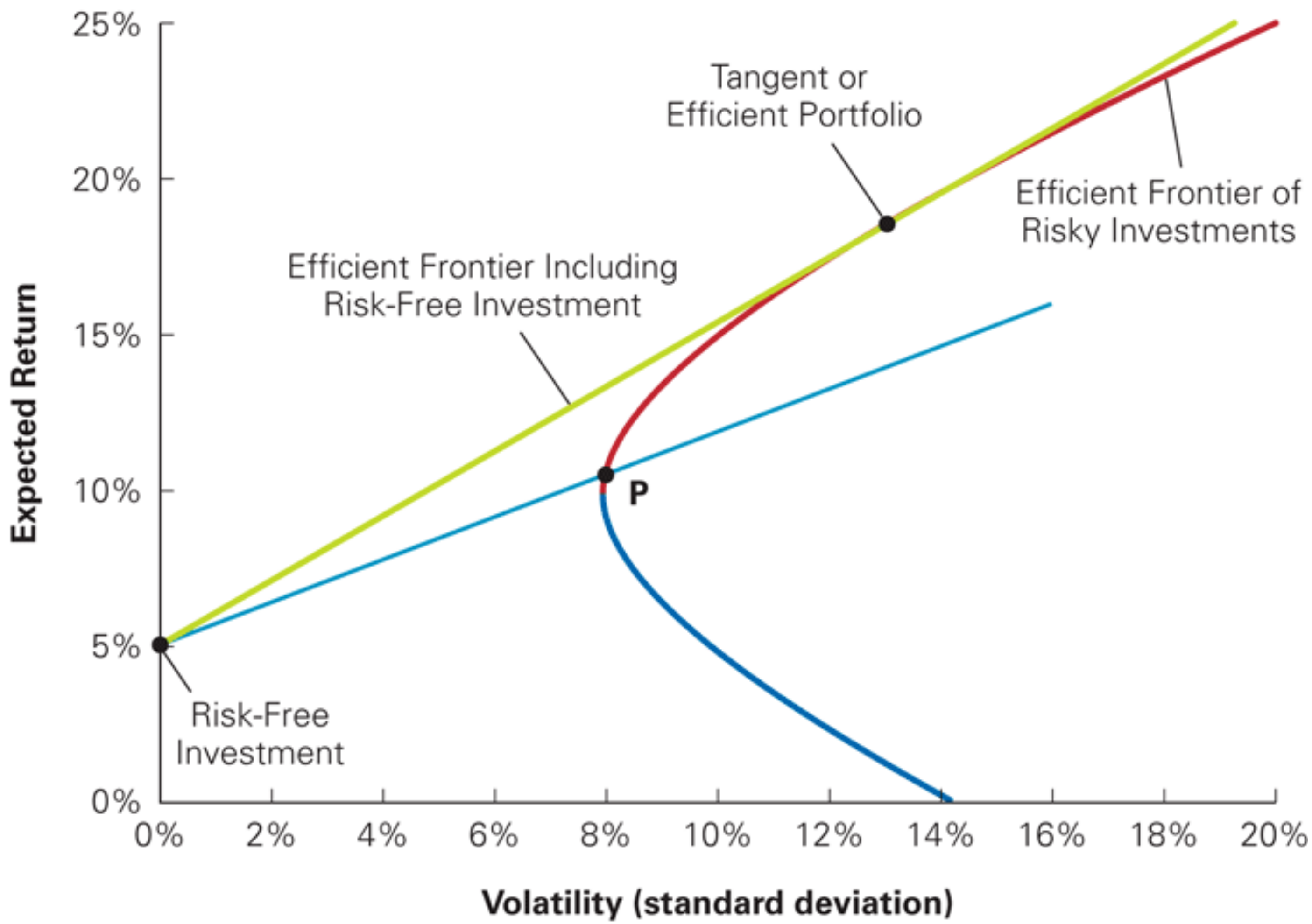
The Risk–Return Combinations from Combining a Risk-Free Investment and a Risky Portfolio (of two assets)



Two risky and one risk-free asset

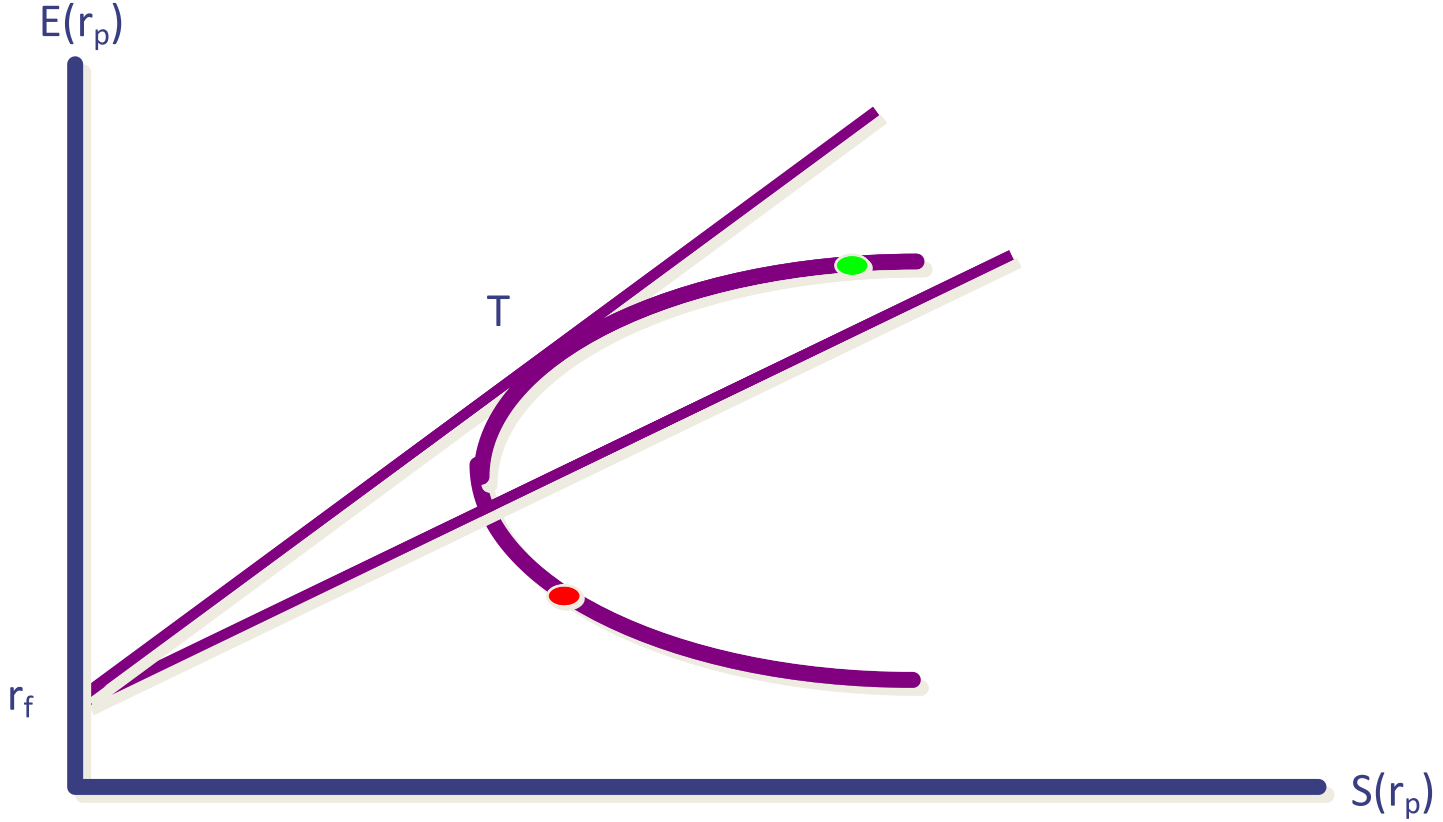


The Tangent or Efficient Portfolio



Efficient frontier and tangency portfolio

- Tangency – maximum Sharpe ratio portfolio

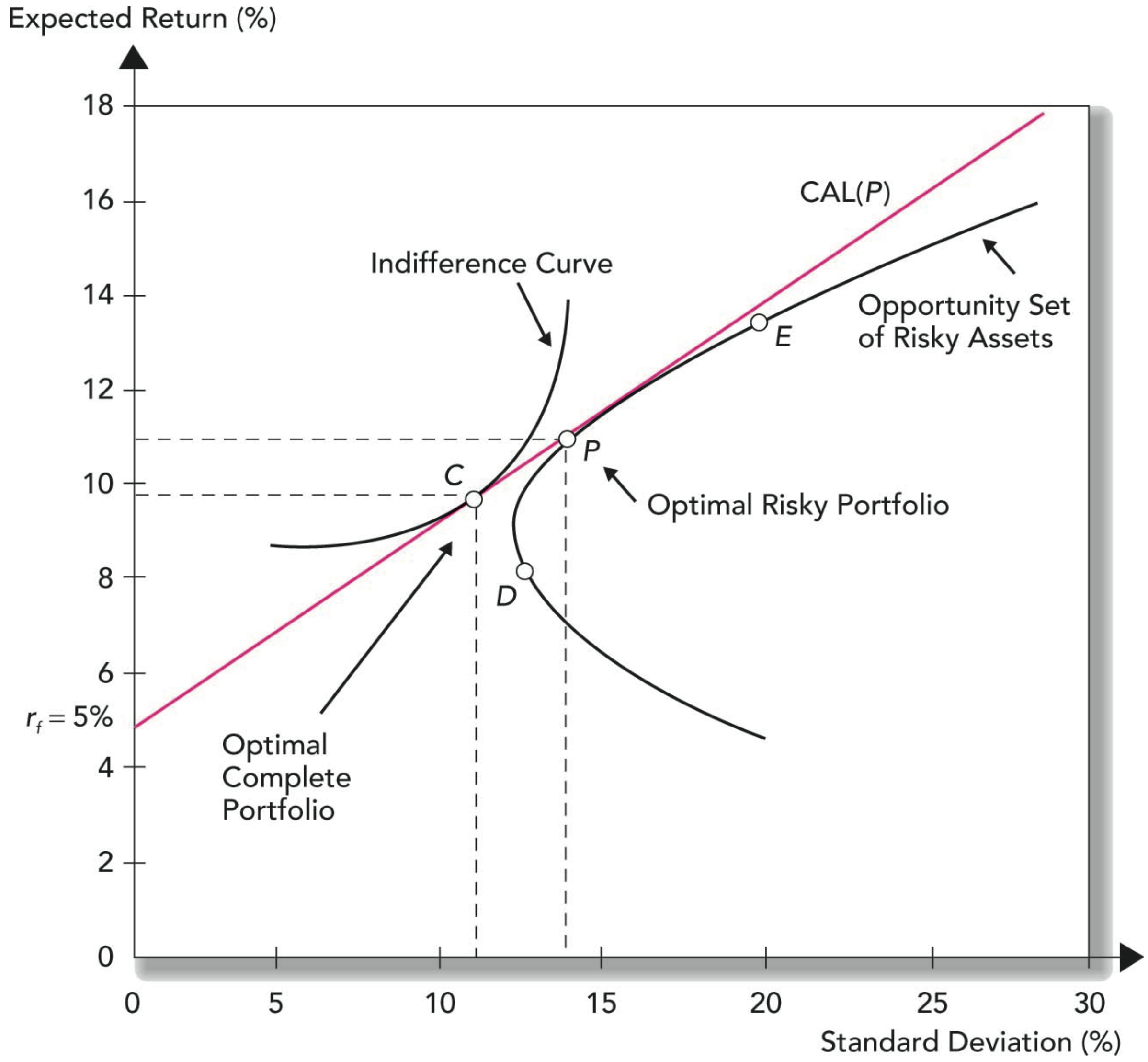


Tangency portfolio: optimal combination of risky securities.
Efficient frontier becomes linear: CAL for T

Optimal Risky Portfolio

- Optimal combination of risky securities
- Tangency Portfolio (the same for all!)
- We find the CAL that maximizes the Sharp Ratio
- The efficient frontier becomes linear
- You can separate the determination of the optimal risky portfolio from preferences (Separation Property)

Determination of the Optimal Overall Portfolio



Portfolio weights for tangency portfolio

- We can find the portfolio weights of the tangency portfolio (Max. Sharpe ratio)

$$\max_{w_x, w_y} \frac{\mu_p - r_{\mathrm{f}}}{\sigma_{\mathrm{p}}} = \max_{w_x, w_y} \frac{w_x \mu_x + w_y \mu_y - r_{\mathrm{f}}}{\sqrt{w_x^2 \sigma_x^2 + w_y^2 \sigma_y^2 + 2 w_x w_y \sigma_{xy}}}$$

$$\text{s.t.} \quad w_x + w_y = 1$$

$$w_x = \frac{(\mu_x - r_{\mathrm{f}})\sigma_y^2 - (\mu_y - r_{\mathrm{f}})\sigma_{xy}}{(\mu_x - r_{\mathrm{f}})\sigma_y^2 + (\mu_y - r_{\mathrm{f}})\sigma_x^2 - (\mu_x - r_{\mathrm{f}} + \mu_y - r_{\mathrm{f}})\sigma_{xy}}$$

$$w_y = 1 - w_x$$

Back to Example

- Two risky assets – bond and stock
  - Means of 3.0% and 7.5%
  - Standard deviations of 10% and 20%
  - Correlation of 0.2
- Assume r_f=1.5%. Find for the Tangency Portfolio:
  - Stock weight
  - Bond weight
  - Expected return
  - Standard deviation
  - Sharp ratio

Example

$$w_x = \frac{\left(\mu_x - r_\mathrm{f}\right)\sigma_y^2 - \left(\mu_y - r_\mathrm{f}\right)\sigma_{xy}}{\left(\mu_x - r_\mathrm{f}\right)\sigma_y^2 + \left(\mu_y - r_\mathrm{f}\right)\sigma_x^2 - \left(\mu_x - r_\mathrm{f} + \mu_y - r_\mathrm{f}\right)\sigma_{xy}}$$

$$w_x = \frac{\left(0.03 - 0.015\right)0.04 - (0.075 - 0.015)0.004}{\left(0.03 - 0.015\right)0.04 + (0.075 - 0.015)0.01 - (0.03 - 0.015 + 0.075 - 0.015)0.004}$$

$$w_x = 0.4$$

$$w_y = 1 - w_x = 0.6$$

Example

- Portfolio expected return

  μ_p $= 0.4 \times 0.03 + 0.6 \times 0.075$

  $= 0.057 = 5.7\%$

- Portfolio variance= $\boxed{w_x^2 \sigma^2_x + w_y^2 \sigma^2_y + 2w_x w_y \sigma_{xy}}$ $\boxed{\sigma_{xy} = \rho\sigma_x\sigma_y}$

  $\sigma_p^2 = (0.4)^2 (0.1)^2 + (0.6)^2 (0.2)^2 + 2\ (0.4)(0.6)(0.2 \times 0.1 \times 0.2)$

  $= 0.0180$

- Portfolio standard deviation

  $\sigma_p = \sqrt{0.0180}$

  $= 0.134 = 13.4\%$

  S = (5.7 – 1.5)/13.4 = 0.31

$$S = \frac{\mu_p - r_{\mathrm{f}}}{\sigma_{\mathrm{p}}}$$

The Tangency Portfolio

- In this case (for r_f=1.5%)
  - Stock weight: 0.6
  - Bond weight: 0.4
  - Expected return: 5.7%
  - Std.dev.: 13.4%,
  - Sharp ratio: 0.31

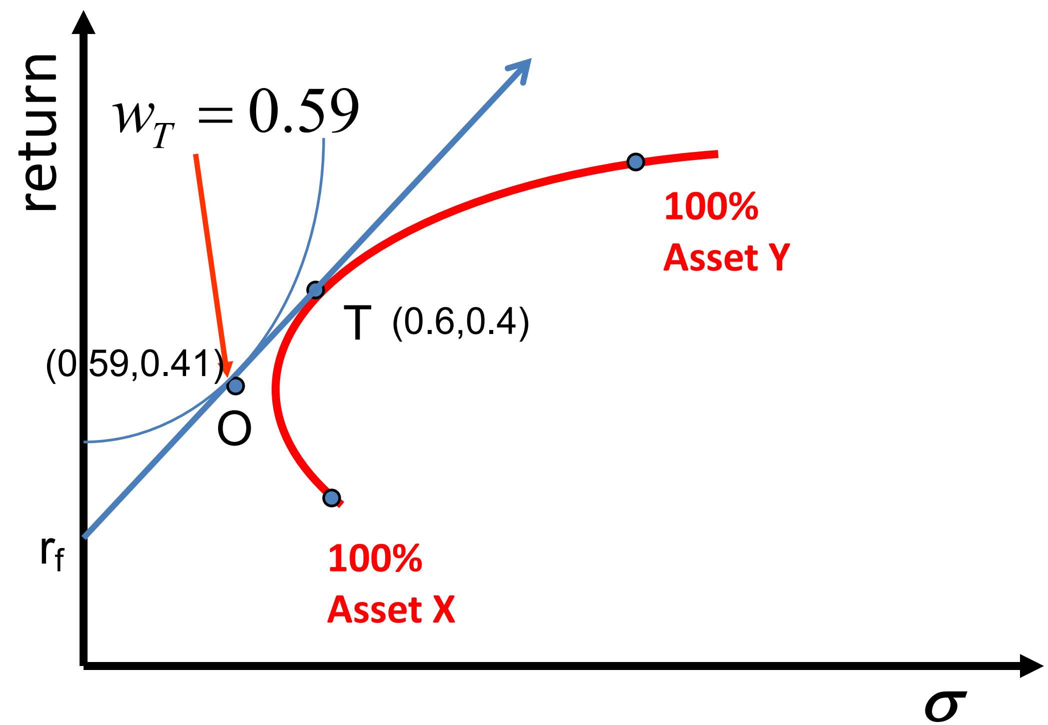
Optimal portfolio of tangency and risk-free asset

- We can figure out the optimal combination of tangency and risk-free asset for an investor with risk aversion $\gamma = 4$
- The optimal weight on the tangency portfolio

$$w_T = \frac{\mu_T - r_{\mathrm{f}}}{\gamma \sigma_T^2} = \frac{0.057 - 0.015}{4 \times 0.1339^2} = 0.59$$

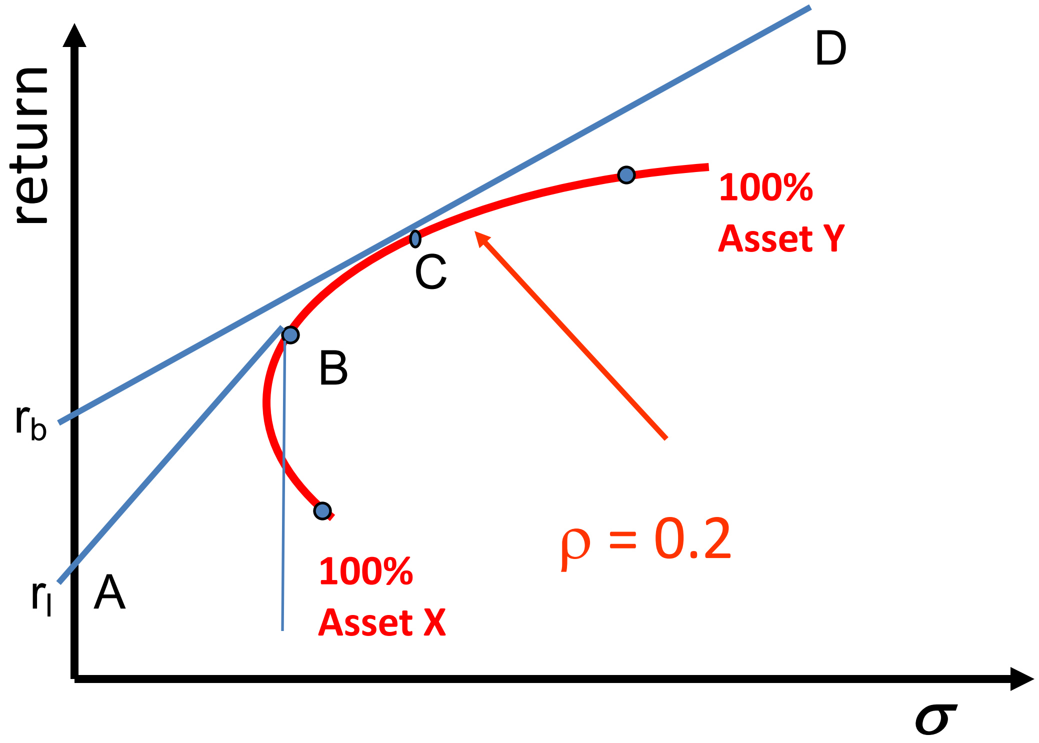
- So the weight on the risk-free Tbill is 0.41
- To find the weights on stocks and bonds, multiply the weight on T by the weights that stocks and bonds have in T
  - Weight on stock 0.59 × 0.6 = 0.35 (rounding)
  - Weight on bond 0.59 × 0.4 = 0.24 (rounding)

Optimal portfolio of tangency and risk-free asset



The overall optimal portfolio (O) has 59% in T and 41% in the risk free asset

Opportunity Set when Borrowing and Lending rates are different



Efficient Portfolios: A-B-C-D

Lessons

- Same risky portfolio (tangency portfolio) chosen by all investors regardless of their risk aversion (separation property).
- Depending on risk aversion, investors choose more or less of the tangency portfolio and put the rest in the risk-free asset.
- How do we handle many risky assets?

Annualizing monthly continuously compounded expected returns and variances

$$E[ax+by] = a\mu_x + b\mu_y$$

$$Var[ax+by] = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\rho\sigma_x\sigma_y$$

If a=b=1, the correlation is 0, and the monthly returns are cc and iid:

$$r_a = r_1 + r_2 + r_3 + r_2 + \ldots\ldots\ldots\ldots r_{12}$$

$$\mu_a = \mu_1 + \mu_2 + \mu_3 + \mu_4 + \ldots\ldots\ldots\ldots \mu_{12} = 12\mu_m$$

$$\sigma_a^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \ldots\ldots\ldots\ldots \sigma_{12}^2 = 12\sigma_m^2$$

Topic 5c: Portfolio Choice with Many Assets

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Agenda

- Portfolios of many risky assets
- The efficient frontier
- The tangency portfolio with many assets
- The minimum variance portfolio with many assets
- Estimating the inputs of the problem
- Benefits of diversification

Arbitrary Number of Assets

- Consider a portfolio of an arbitrary number *N* of assets with weight w_i in Asset *i.*
- Portfolio mean return and variance of return

$$\mu_P = \sum_{i=1}^{N} w_i \mu_i$$

$$\sum_{i=1}^{N} w_i = 1$$

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

Three Assets

$$\mu_P = w_1\mu_1 + w_2\mu_2 + w_3\mu_3$$

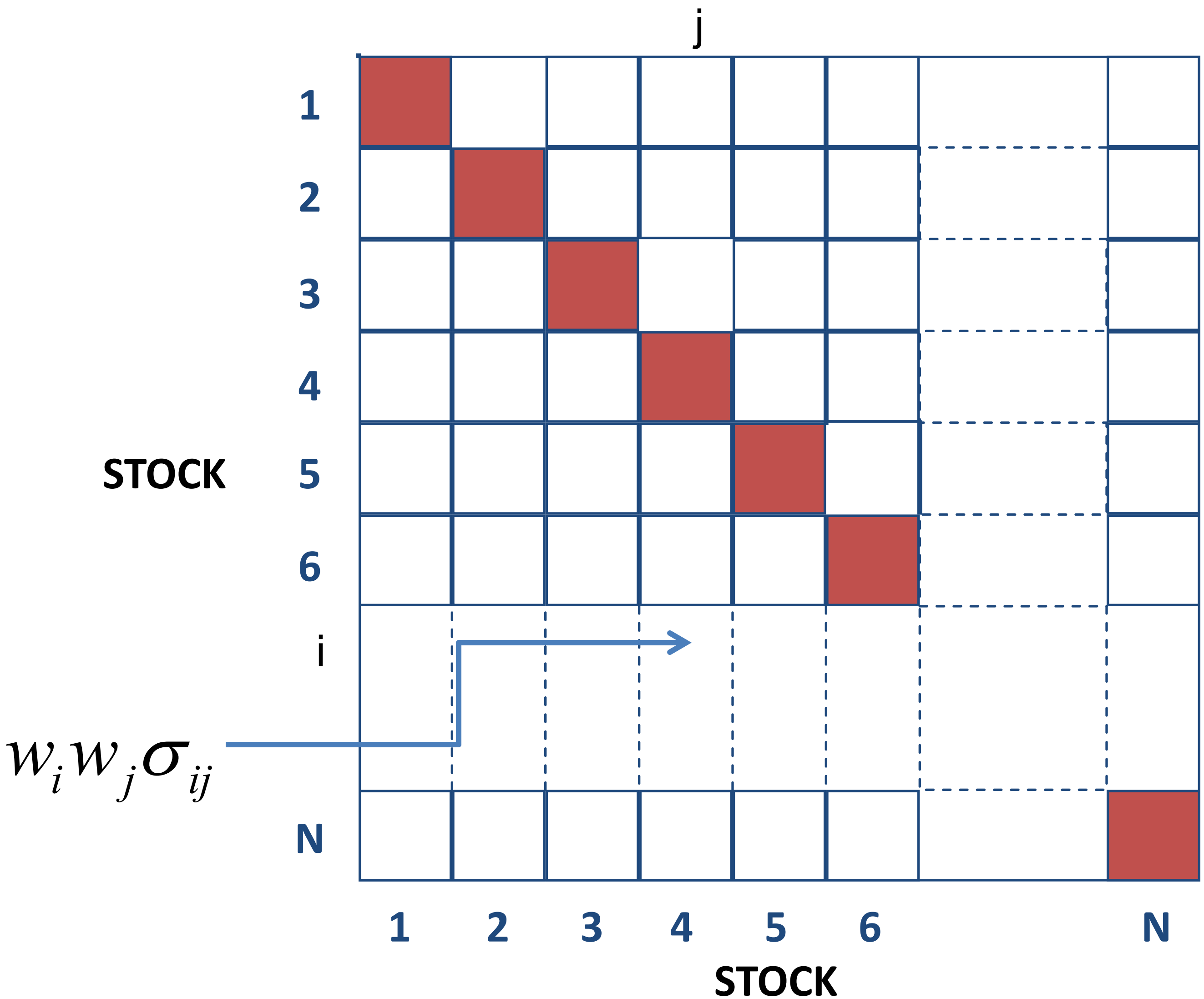
$$w_1 + w_2 + w_3 = 1$$

$$\sigma_P^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} w_i w_j \sigma_{ij}$$

$$\sigma_P^2 = w_1 w_1 \sigma_{11} + w_2 w_2 \sigma_{22} + w_3 w_3 \sigma_{33} + 2w_1 w_2 \sigma_{12} + 2w_1 w_3 \sigma_{13} + 2w_2 w_3 \sigma_{23}$$

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1 w_2 \sigma_{12} + 2w_1 w_3 \sigma_{13} + 2w_2 w_3 \sigma_{23}$$

Variance and Covariance



The shaded boxes contain variance terms; the remainder contain covariance terms.

Variance of a Portfolio

- If there are N stocks (say 100)
- The number of boxes will be N^2 (10,000)
- The number of variances will be N (100)
- Therefore, the total number of covariances will be: $N^2 – N$. (9,900)
- But, $\sigma_{i,k} = \sigma_{k,i}$ The number of different covariaces is then $(N^2 – N)/2$ (4,950)

In Matrix Notation

$\Omega : N \times N\ Variance\ Co\mathrm{var}iance\ matrix\ of\ returns$

$\vec{\mu} : N\ column\ vector\ of\ \exp ected\ returns$

$\vec{w} : N\ column\ vector\ of\ weights$

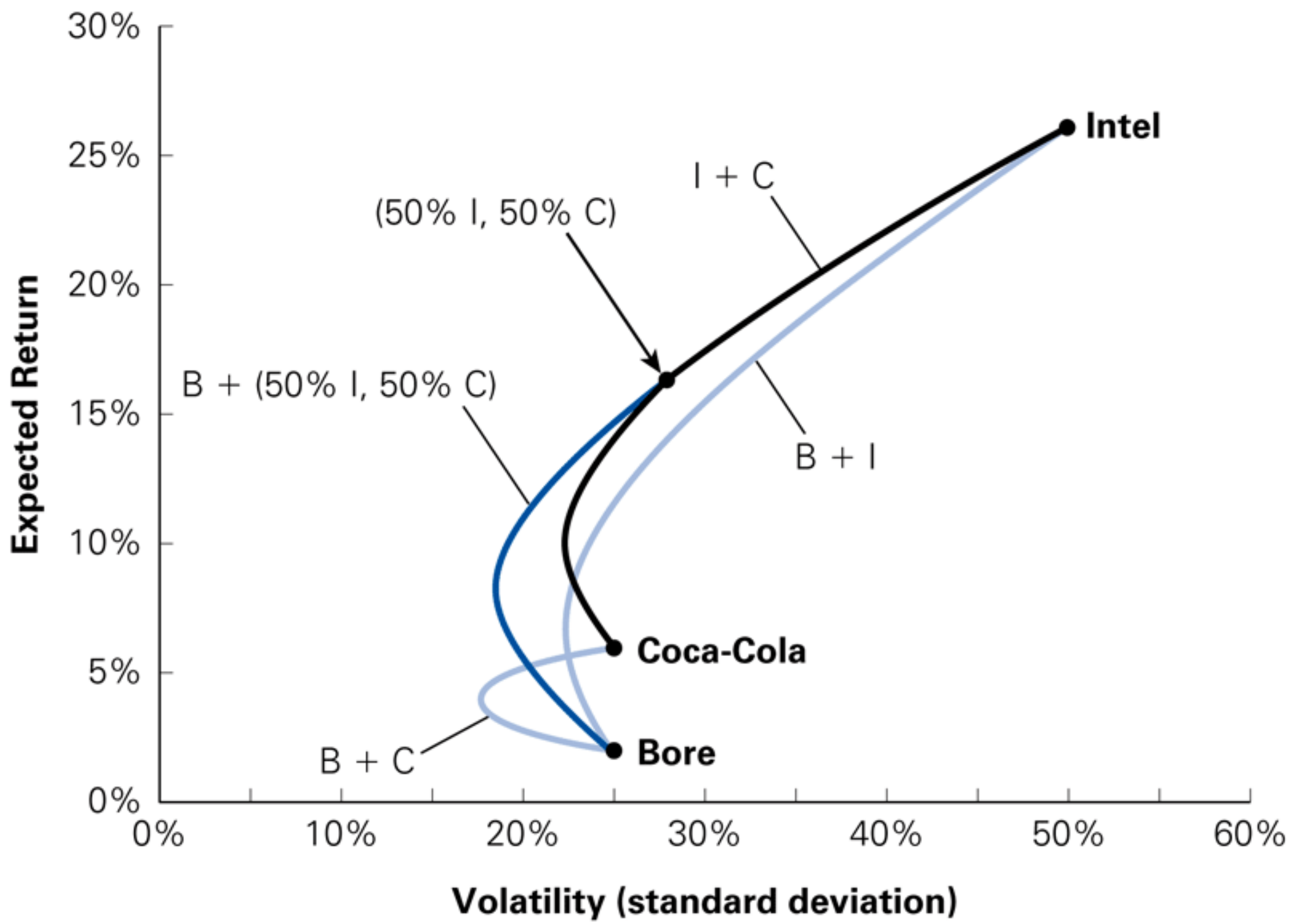
$\vec{1} : N\ column\ vector\ of\ ones$

$\vec{w}^{Tr}\vec{1} = 1 \quad scalar\ product$

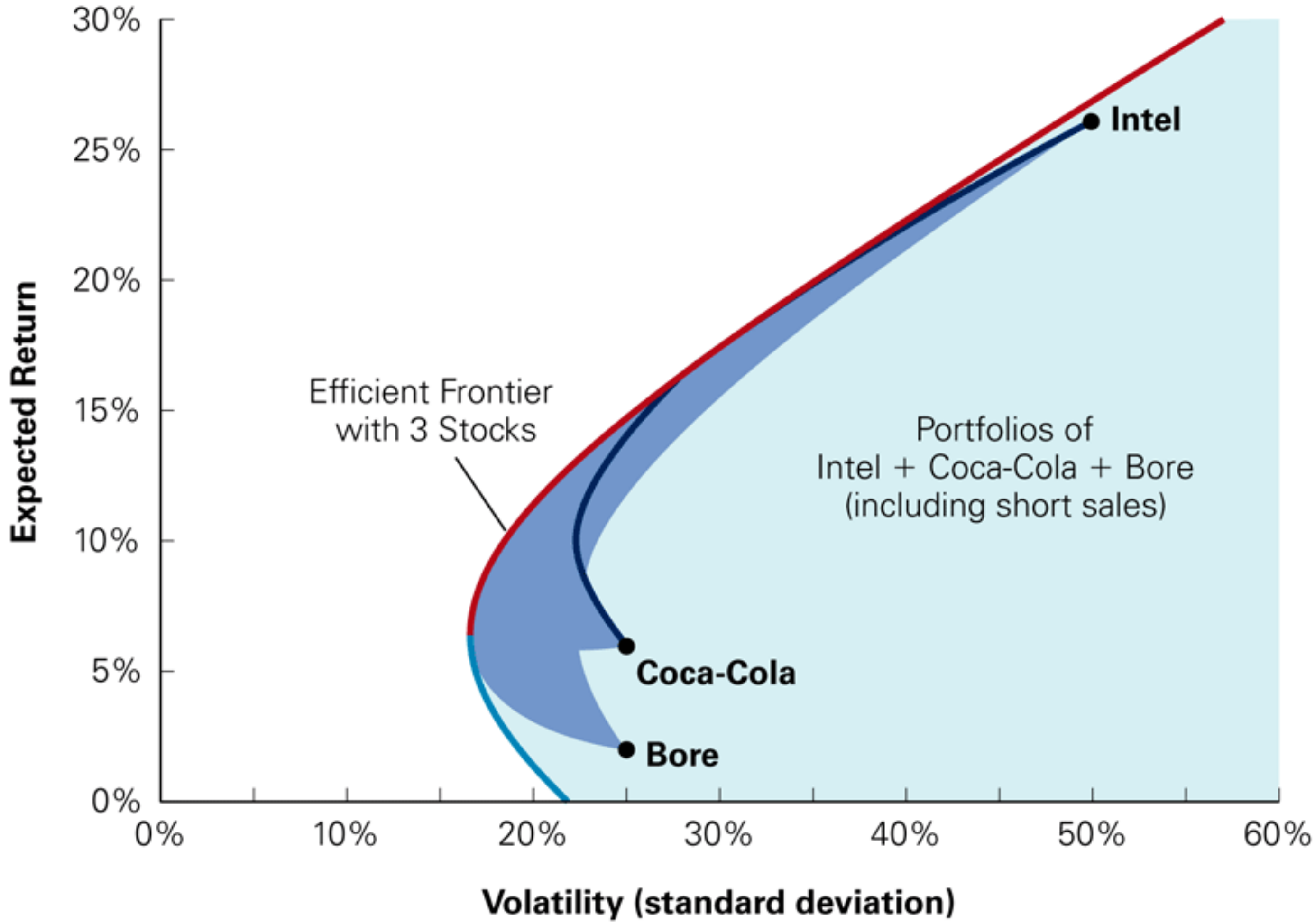
$\mu_P = \vec{w}^{Tr}\vec{\mu} \quad scalar\ product$

$\sigma_P^2 = \vec{w}^{Tr}\Omega\vec{w}$

Expected Return and Volatility for Selected Portfolios of Intel, Coca-Cola, and Bore Industries Stocks



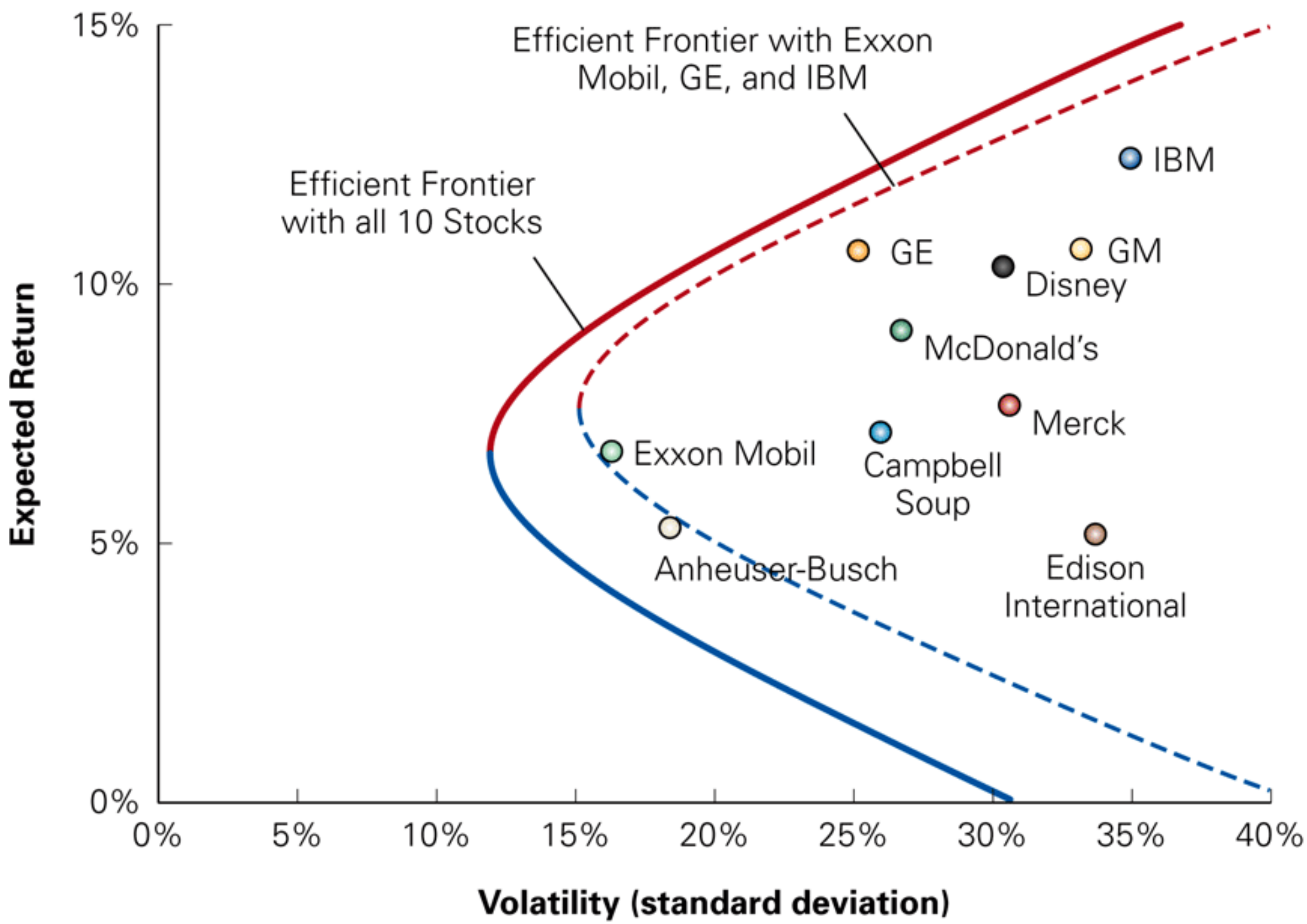
The Volatility and Expected Return for All Portfolios of Intel, Coca-Cola, and Bore Stock



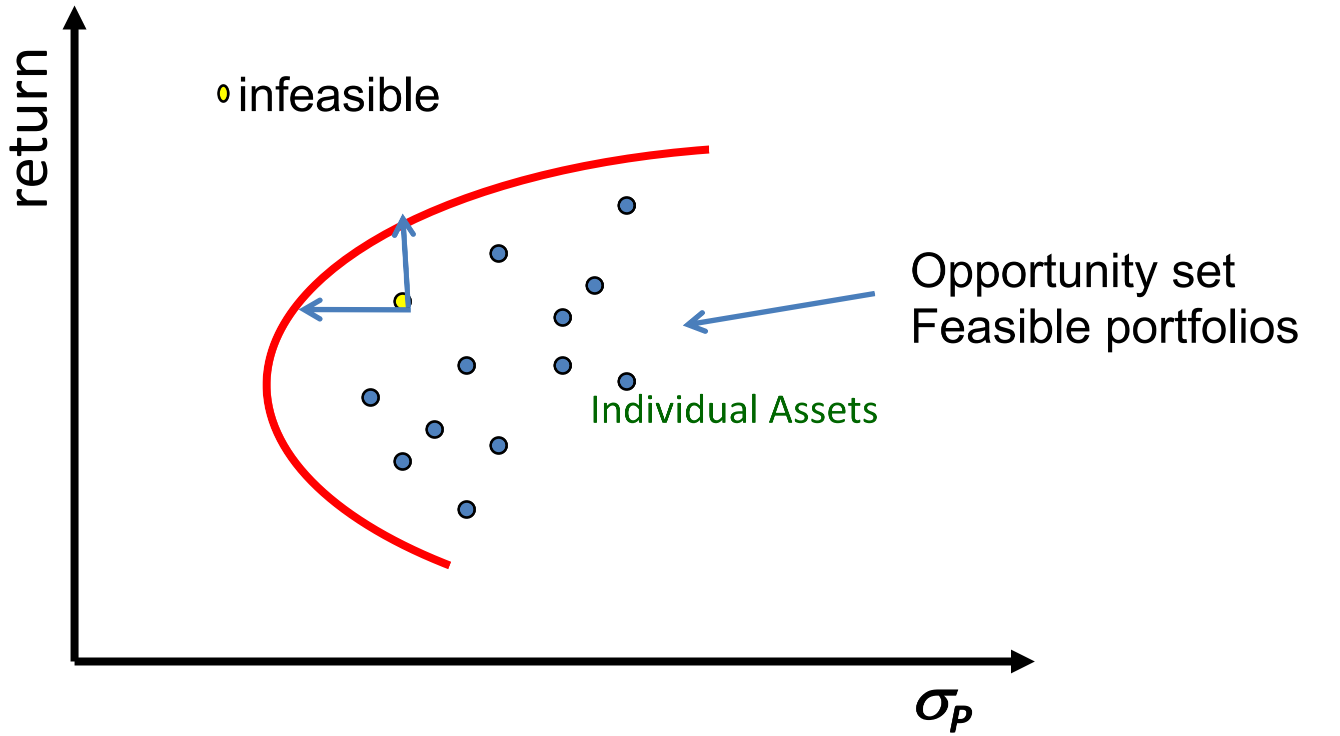
Risk Versus Return: Many Stocks

- The efficient portfolios, those offering the highest possible expected return for a given level of volatility, are those on the northwest edge of the shaded region, which is called the **efficient frontier** for these three stocks.
  - In this case none of the stocks, on its own, is on the efficient frontier, so it would not be efficient to put all our money in a single stock.

Efficient Frontier with Ten Stocks Versus Three Stocks

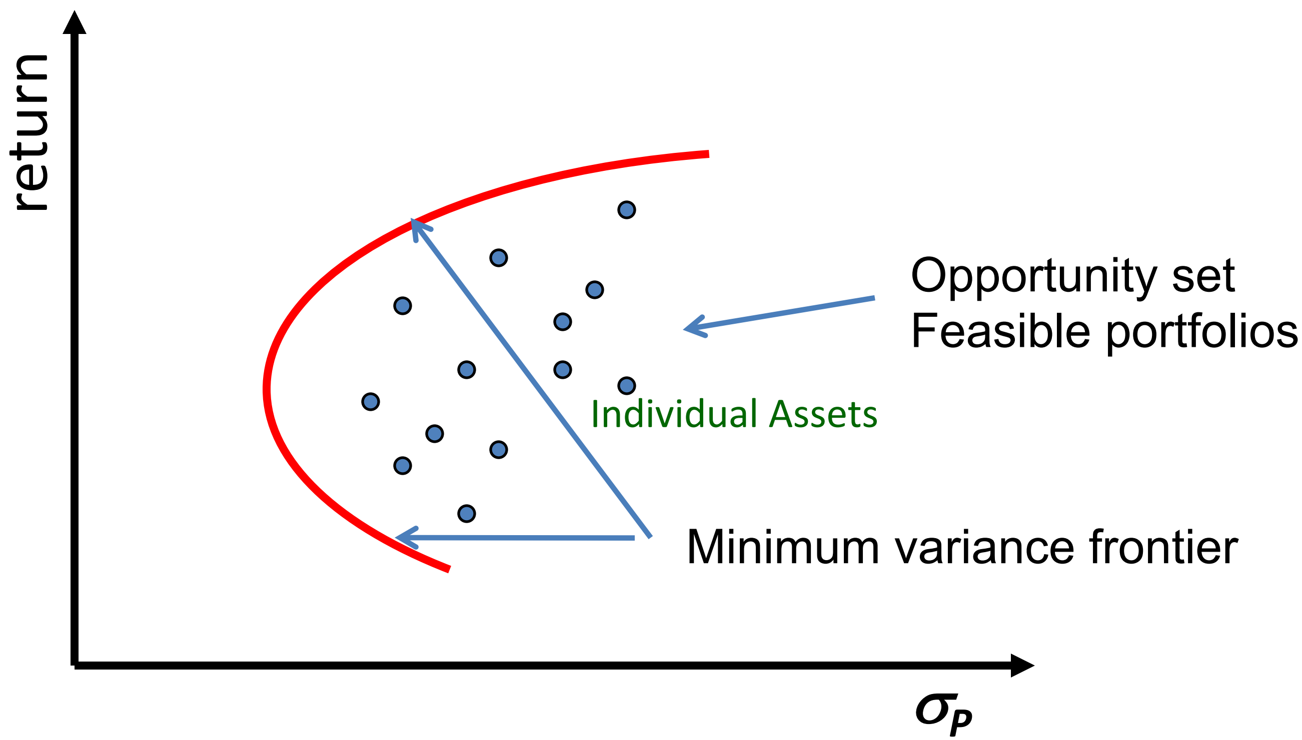


Opportunity Set for Many Securities



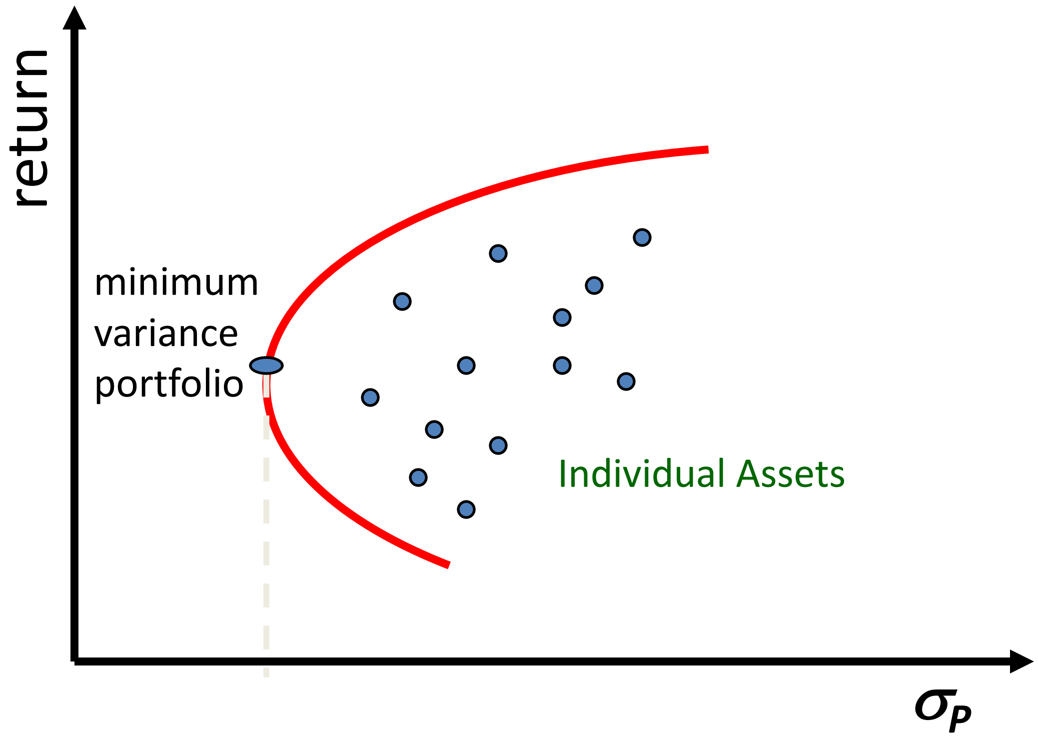
Efficient portfolios are those with the highest expected return for a given standard deviation.

Opportunity Set for Many Securities



We can still identify the *opportunity set* of risk-return combinations of various portfolios, and the envelope of this set has the same shape as before.

Minimum Variance Portfolio

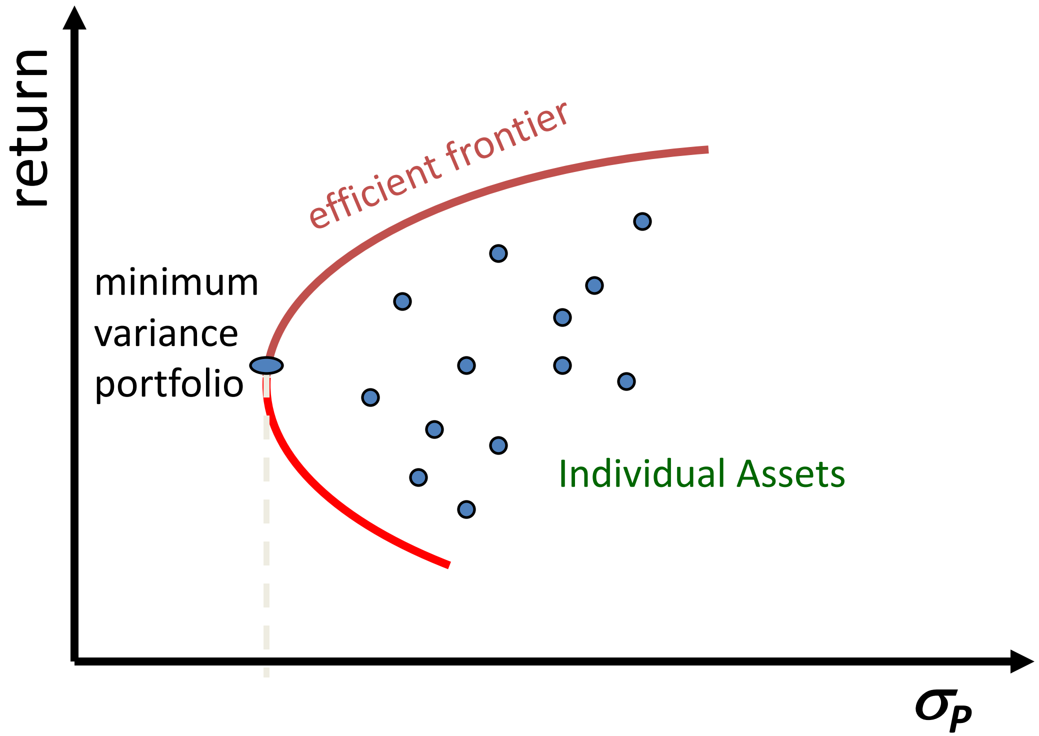


Given the *opportunity set* we can also identify the **minimum variance portfolio**. Now it has many securities.

Efficient Frontier

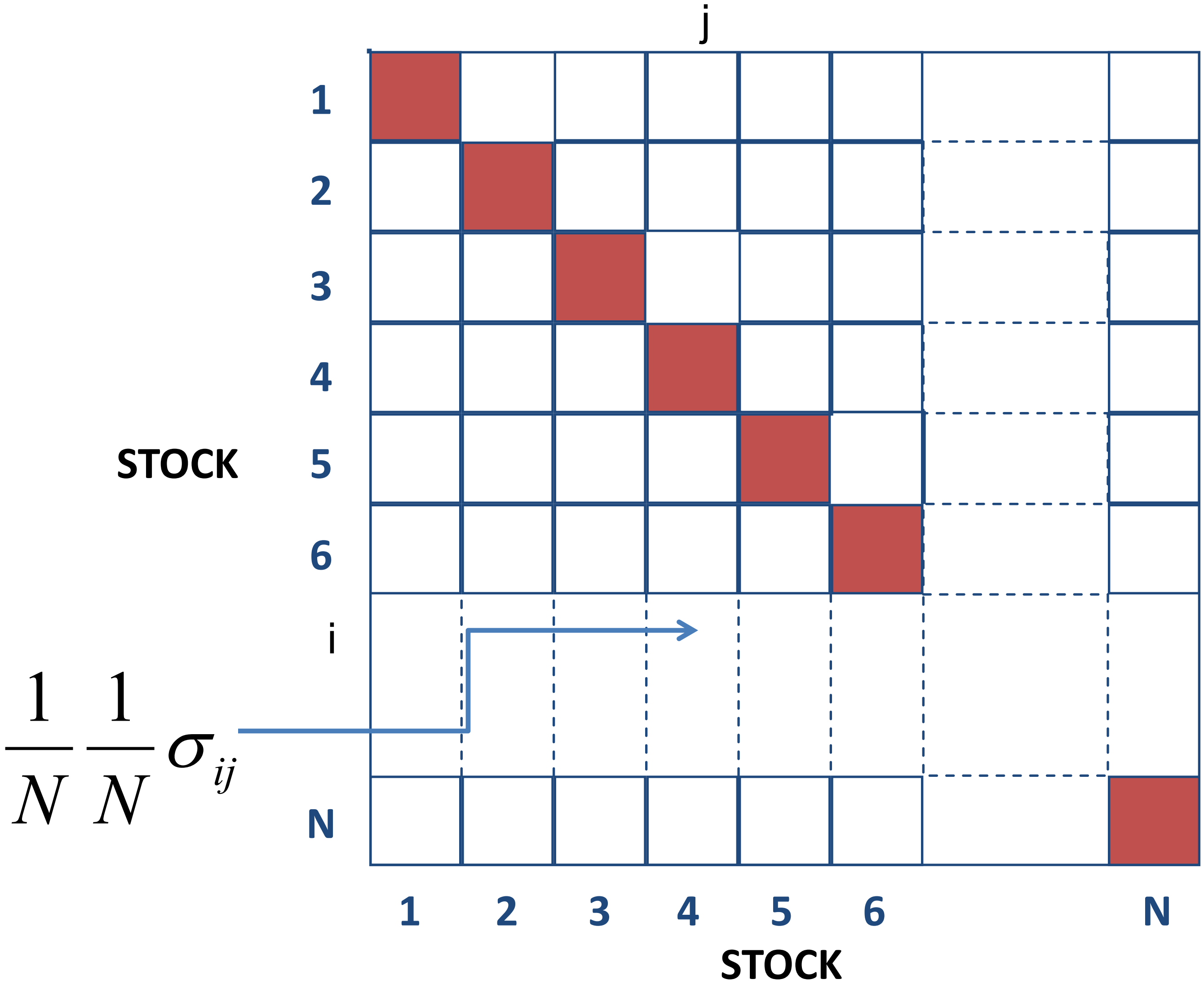
- With more than two risky assets, the investment opportunity set becomes an *area* (rather than a line)
- Efficient Portfolios:
  - Those with the highest expected return for a given variance
- Only the north-west edge of the feasible area will be the *efficient frontier*

Efficient Set for Many Securities



The section of the opportunity set above the minimum variance portfolio is the efficient frontier.

Limits to diversification: equally weighted portfolio



The shaded boxes contain variance terms; the remainder contain covariance terms.

Limits to diversification

- Consider equal-weighted portfolio w_i=1/N

$$\sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} = \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N^2} \sigma_{ij}$$

$$\sigma_P^2 = \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2 + \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \sigma_{ij} = \frac{1}{N^2} N \overline{\sigma}_i^2 + \frac{1}{N^2} (N^2 - N) \overline{\sigma}_{ij}$$

$$\overline{\sigma}_i^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 \Rightarrow \sum_{i=1}^{N} \sigma_i^2 = N \overline{\sigma}_i^2$$

$$\overline{\sigma}_{ij} = \frac{\sum_{i} \sum_{j \neq i} \sigma_{ij}}{N^2 - N} \Rightarrow \sum_{i} \sum_{j \neq i} \sigma_{ij} = (N^2 - N) \overline{\sigma}_{ij}$$

Limits to diversification

$$\sigma_p^2 = \frac{1}{N}\sum_i \frac{1}{N}\sigma_i^2 + \sum_i \sum_{j \neq i} \frac{1}{N^2}\sigma_{ij}$$

$$= \frac{1}{N}\bar{\sigma}_i^2 + \frac{N-1}{N}\bar{\sigma}_{ij}$$

$$\approx \bar{\sigma}_{ij} \text{ as N becomes very large}$$

- There is a certain amount of risk that cannot be diversified away corresponding to the average covariance. The var. of a well diversified portfolio is equal to the average cov.

Diversification and Portfolio Risk

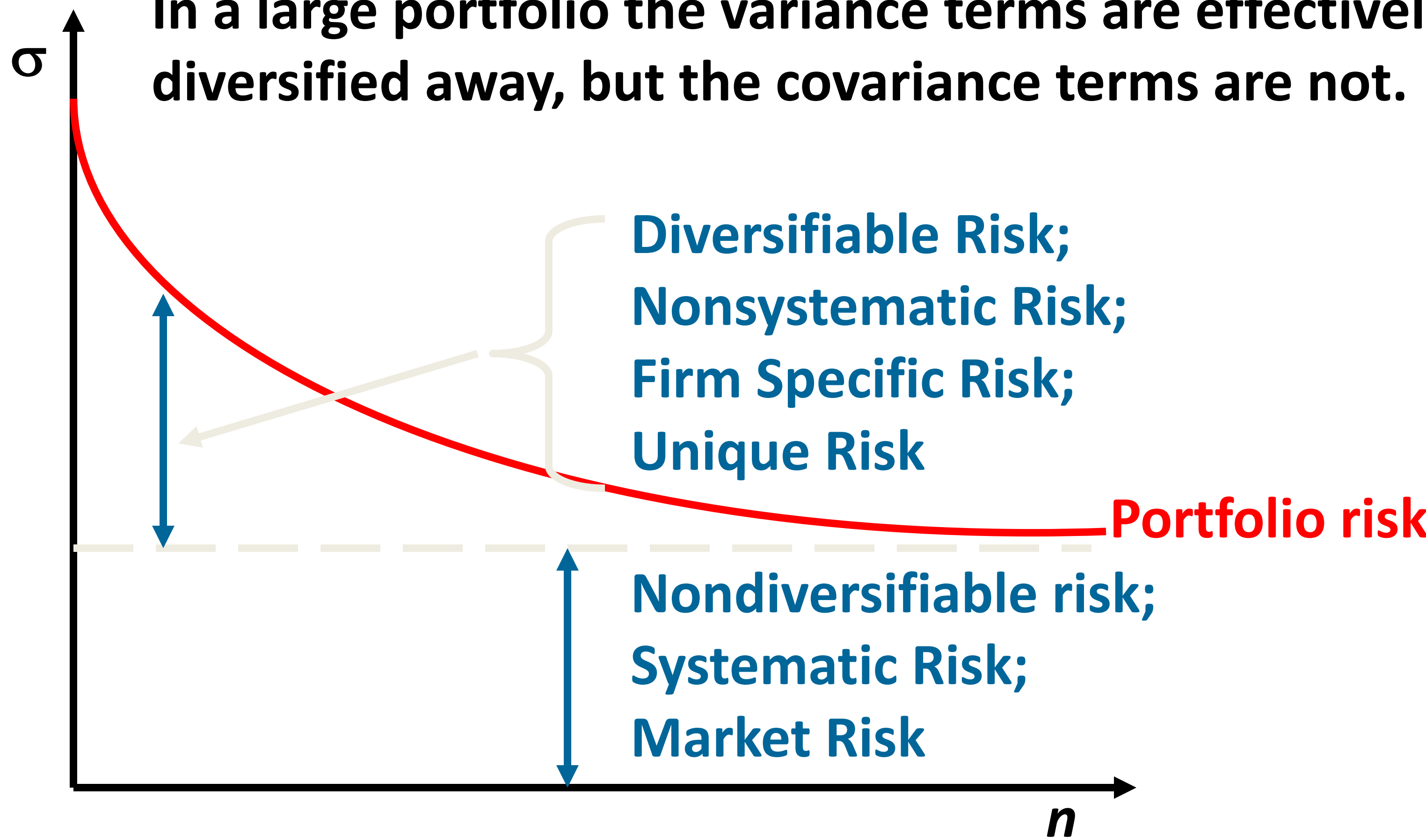
Portfolio Standard Deviations

(1) Number of Stocks in Portfolio	(2) Average Standard Deviation of Annual Portfolio Returns	(3) Ratio of Portfolio Standard Deviation to Standard Deviation of a Single Stock
1	49.24%	1.00
2	37.36	.76
4	29.69	.60
6	26.64	.54
8	24.98	.51
10	23.93	.49
20	21.68	.44
30	20.87	.42
40	20.46	.42
50	20.20	.41
100	19.69	.40
200	19.42	.39
300	19.34	.39
400	19.29	.39
500	19.27	.39
1,000	19.21	.39

Source: These figures are from Table 1 in Meir Statman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* 22 (September 1987), pp. 353–64. They were derived from E. J. Elton and M. J. Gruber, "Risk Reduction and Portfolio Size: An Analytic Solution," *Journal of Business* 50 (October 1977), pp. 415–37.

Portfolio Risk

In a large portfolio the variance terms are effectively diversified away, but the covariance terms are not.



Thus diversification can eliminate some, but not all of the risk of individual securities.

Life Insurance Case

- Diversification and risk in the life insurance industry
- Death and mortality tables
- Deaths are uncorrelated: average covariance is zero
- All risk can be diversified away
- But, what if all the policyholders live in the San Andres

Diversification in Stock Portfolios

- Firm-Specific Versus Systematic Risk
  - Firm Specific News
    - Good or bad news about an individual company (strikes, new discoveries, death of CEO)
  - Market-Wide News
    - News that affects all stocks, such as news about the economy (inflation, oil prices, interest rates)

Firm-Specific Versus Systematic Risk

- Independent Risks (Due to firm-specific news)
  - Firm-Specific Risk
  - Idiosyncratic Risk
  - Unique Risk
  - Unsystematic Risk
  - Diversifiable Risk
- Common Risks (Due to market-wide news)
  - Systematic Risk
  - Undiversifiable Risk
  - Market Risk

Firm-Specific Versus Systematic Risk

- When many stocks are combined in a large portfolio, the firm-specific risks for each stock will average out and be diversified.
- The systematic risk, however, will affect all firms and will not be diversified.
- Actual firms are affected by both. Only the unsystematic risk will be diversified when many firm’s stocks are combined into a portfolio. The volatility will therefore decline until only the systematic risk remains.

No Arbitrage and the Risk Premium

- *The risk premium for diversifiable risk is zero, so investors are not compensated for holding firm-specific risk.*
  - If the diversifiable risk of stocks were compensated with an additional risk premium, then investors could buy the stocks, earn the additional premium, and simultaneously diversify and eliminate the risk.

No Arbitrage and the Risk Premium

– By doing so, investors could earn an additional premium without taking on additional risk. This opportunity to earn something for nothing would quickly be exploited and eliminated. Because investors can eliminate firm-specific risk “for free” by diversifying their portfolios, they will not require or earn a reward or risk premium for holding it.

No Arbitrage and the Risk Premium

- *The risk premium of a security is determined by its systematic risk and does not depend on its diversifiable risk.*
  - This implies that a stock's volatility, which is a measure of total risk (that is, systematic risk plus diversifiable risk), is not especially useful in determining the risk premium that investors will earn.

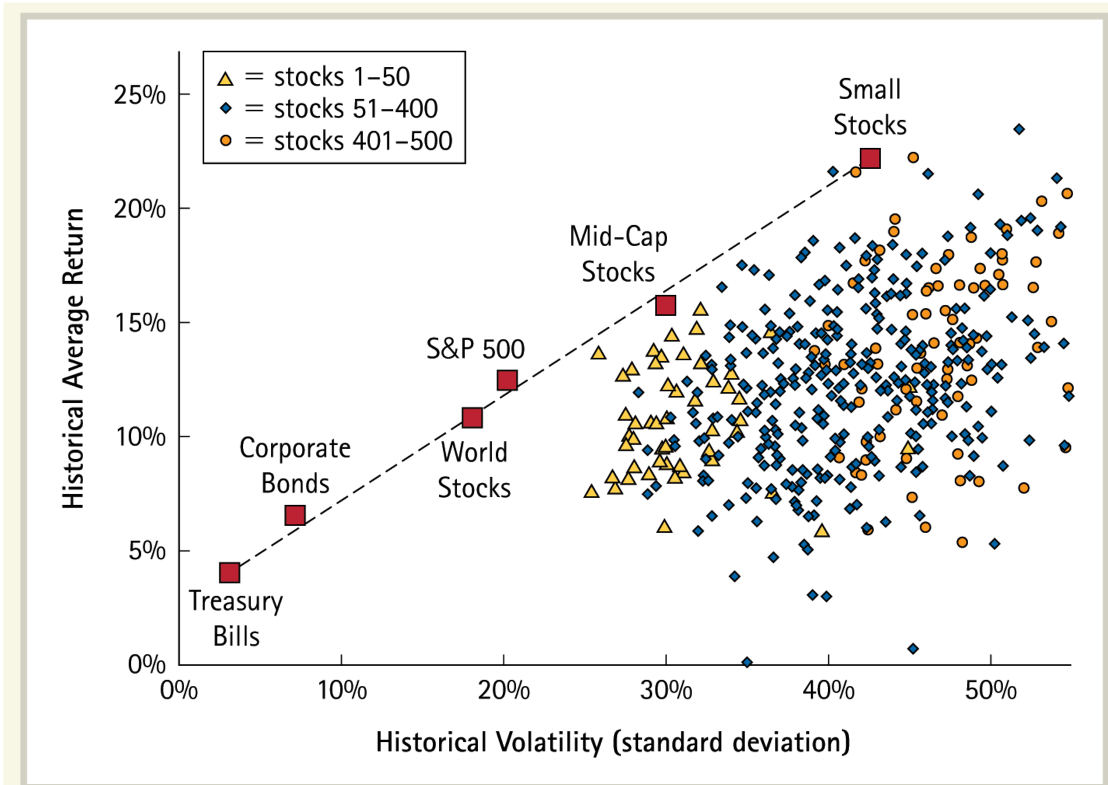
No Arbitrage and the Risk Premium

- Standard deviation is not an appropriate measure of risk for an individual security. There should be no clear relationship between volatility and average returns for individual securities. Consequently, to estimate a security’s expected return, we need to find a measure of a security’s systematic risk.

Diversification and Risk

- In a large portfolio, some stocks will go up in value because of positive company-specific events, while others will go down in value because of negative company-specific events.
- Unsystematic risk is essentially eliminated by diversification, so a portfolio with many assets has almost no unsystematic risk.
- Unsystematic risk is also called *diversifiable* risk, while systematic risk is also called *nondiversifiable* risk.

Historical Volatility and Return for 500 Individual Stocks, by Size, Updated Quarterly, 1926–2004



Unlike the case for large portfolios, there is no precise relationship between volatility and average return for individual stocks. Individual stocks have higher volatility and lower average returns than the relationship shown for large portfolios.

Systematic Risk

- The risk of a well diversified portfolio depends on the market risk of the securities included in the portfolio
- The reward for bearing risk depends only on the systematic risk of an investment, not on the risk that can be diversified away.
- So, no matter how much total risk an asset has, only the systematic portion is relevant in determining the expected return (and the risk premium) on that asset.

Market Risk is Measured by Beta

- If you want to measure the contribution of an individual security to the risk of a well-diversified portfolio, it is no good thinking about how risky that security is in isolation, you need to measure its market risk, and that boils down to measuring how sensitive it is to market movements. This sensitivity is called **beta (β)**.
- Stocks with $\beta>1$ amplify movements in the market
- Stocks with $0<\beta<1$ tend to move in the same direction as the market, but not so far.
- β of the market is 1 (portfolio of all stocks, "average")

Measuring Beta

- Beta measures the responsiveness of a security to movements in the market portfolio.

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

- Standardized measure of covariance with market
- A high-beta security is simply one that is relatively sensitive to overall market movements, whereas a low-beta security is one that is relatively insensitive

Where Do Betas Come From?

- A security's beta depends on
  - how closely correlated the security's return is with the overall market's return, and
  - how volatile the security is relative to the market.
- A security's beta is equal to the correlation multiplied by the ratio of the standard deviations.

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\rho_{im}\sigma_i\sigma_m}{\sigma_m^2} = \rho_{im}\frac{\sigma_i}{\sigma_m}$$

Betas with Respect to the S&P 500 for Individual Stocks (based on monthly data for 2004–2008)

Company	Ticker	Industry	Equity Beta
Family Dollar Stores	FDO	Retail	0.10
Abbott Laboratories	ABT	Pharmaceuticals	0.18
Consolidated Edison	ED	Utilities	0.19
Hershey	HSY	Food Processing	0.19
Piedmont Natural Gas	PNY	Gas Utilities	0.24
General Mills	GIS	Food Processing	0.25
Wal-Mart Stores	WMT	Superstore	0.31
Altria Group	MO	Tobacco	0.31
Kellogg	K	Food Processing	0.44
Amgen	AMGN	Biotechnology	0.45
DeVry	DV	Education Services	0.49
Exxon Mobil	XOM	Oil and Gas	0.56
Procter & Gamble	PG	Household Products	0.57
The Coca-Cola Company	KO	Soft Drinks	0.60
Newmont Mining	NEM	Gold	0.65
McDonald's	MCD	Restaurants	0.79
United Parcel Service	UPS	Air Freight and Logistics	0.79
Southwest Airlines	LUV	Airline	0.83
Costco Wholesale	COST	Superstore	0.85
Walt Disney	DIS	Movies and Entertainment	0.96
Microsoft	MSFT	Systems Software	0.98
Starbucks	SBUX	Restaurants	1.04
Target	TGT	Retail	1.07
General Electric	GE	Conglomerates	1.12
Cisco Systems	CSCO	Communications Equipment	1.27
Marriott International	MAR	Hotels and Resorts	1.29
Intel	INTC	Semiconductors	1.35
Dell	DELL	Computer Hardware	1.36
Sears	SHLD	Department Stores	1.36
Google	GOOG	Internet Services	1.45
Tiffany & Co.	TIF	Specialty Stores	1.64
Coach	COH	Apparel and Luxury Goods	1.65
Apple	AAPL	Computer Hardware	1.89
Amazon.com	AMZN	Internet Retail	1.89
eBay	EBAY	Internet Services	1.93
Sotheby's	BID	Auction Services	2.07
Autodesk	ADSK	Application Software	2.31
Salesforce.com	CRM	Application Software	2.39

Source: CapitalIQ

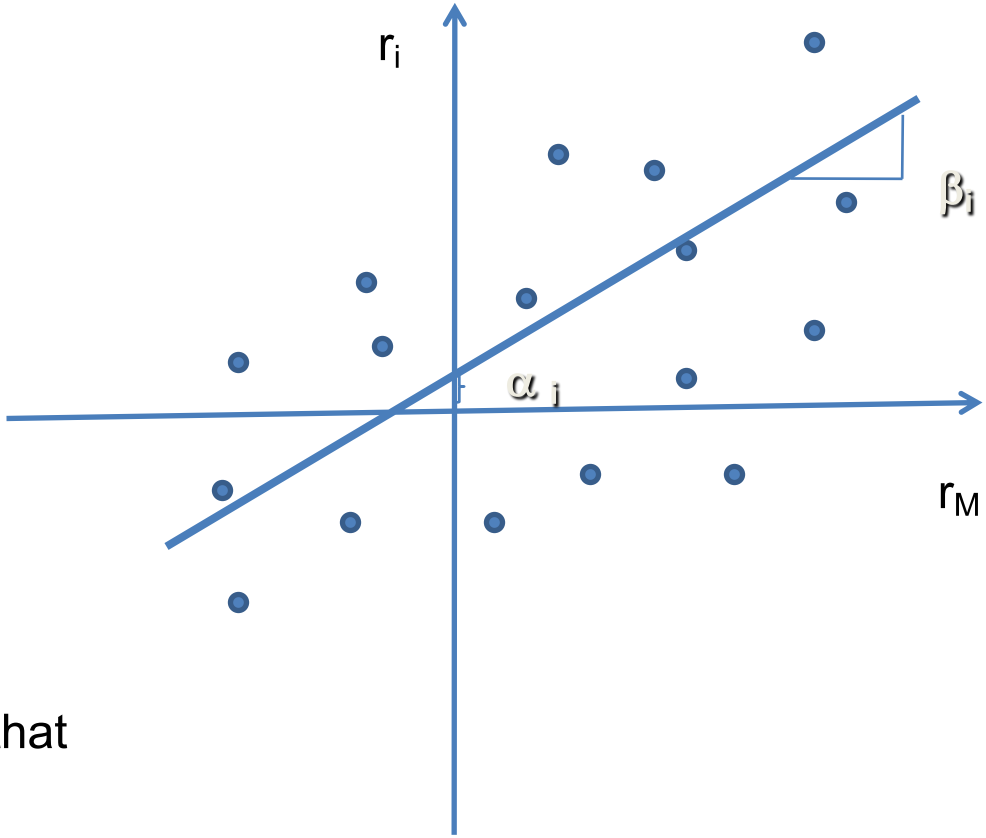
Measuring Beta

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$$

- The easiest way to measure historical beta is through running a regression of the individual asset return on the market return.
- Example: 5 years of monthly data on the market return and the return on stock i.
- The slope of that regression (beta) is precisely what we are looking for.

Linear Regression

Date	r_i	r_M
1	r_{i1}	r_{M1}
2	r_{i2}	r_{M2}
t	r_{it}	r_{Mt}
T	r_{iT}	r_{MT}



We want to find the straight line that best fits the data:

-average deviation equals zero

-minimizes the squared deviations form the line

Regression

- Regression Equation

$$r_{it} = \alpha_i + \beta_i r_{Mt} + \varepsilon_{it} \qquad E[\varepsilon_i] = 0$$

- Line Equation

$$r_{it} = \alpha_i + \beta_i r_{Mt}$$

- Estimate of beta

$$\hat{\beta}_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

- Both alpha and beta are estimated with error

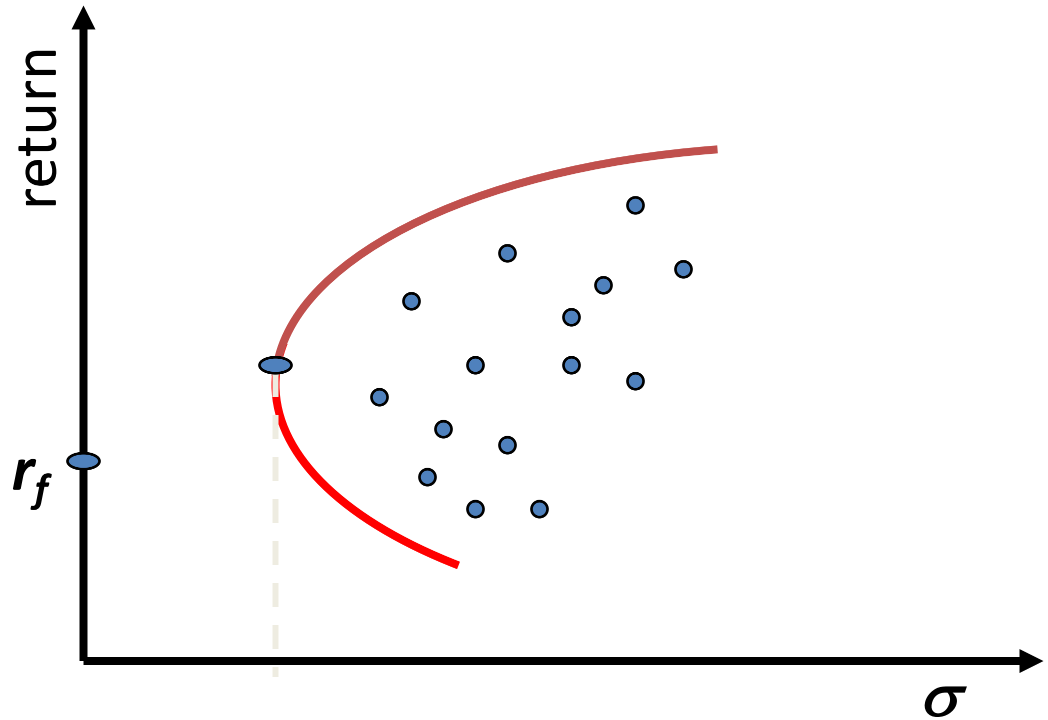
The Market Portfolio

- Because it contains all risky assets, it is a completely diversified portfolio, which means that all the unique risk of individual assets (unsystematic risk) is diversified away
- Only systematic risk remains in the market portfolio
- The risk that matters is the contribution of the risk of an asset to the risk of this perfectly diversified market portfolio, and this is measured by covariance of the asset with the market portfolio, or by Beta
  - Beta is a measure of *Systematic Risk* of the asset

Intuition

- High beta stocks are risky, and must therefore offer a higher return on average to compensate for the risk
- Why are high beta stocks risky?
  - Because they pay up just when you need the money least – when the overall market is doing well
  - If anyone is to hold this security, it must offer a high expected return
- More about this later when we discuss the CAPM

Optimal Risky Portfolio with a Risk-Free Asset



In addition to risky securities, consider a world that also has risk-free securities like T-bills

Optimal Risky Portfolio with a Risk-Free Asset

- Recall that a portfolio with weight *w* in the risky asset and a weight *(1-w)* in the risk-free asset has:

Mean return

$\mu_p = w\,\mu + (1-w)\,r_f = r_f + w(\mu - r_f)$

Portfolio variance $\sigma_p^2 = w^2\,\sigma^2$

Standard deviation $\sigma_p = w\sigma$

Optimal Risky Portfolio with a Risk-Free Asset

$$\mu_p = r_f + w(\mu - r_f)$$

$$\sigma_P = w\sigma$$

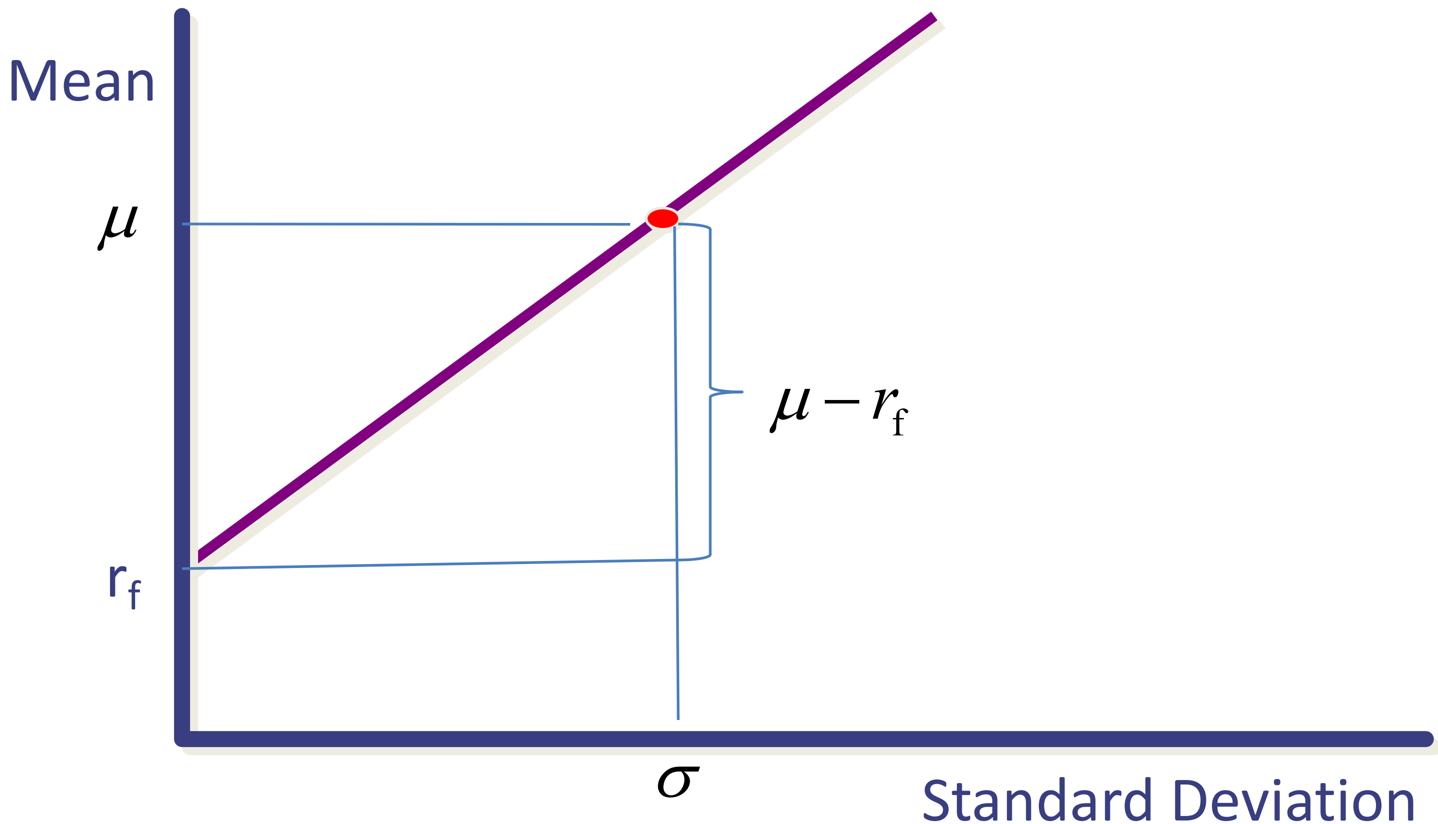
- Both expected return and risk increase linearly with *w*
- *Plot of portfolio return on portfolio risk is a straight line with an intercept of r_f and a slope equal to:*

$$S = \frac{\mu - r_{\mathrm{f}}}{\sigma}$$

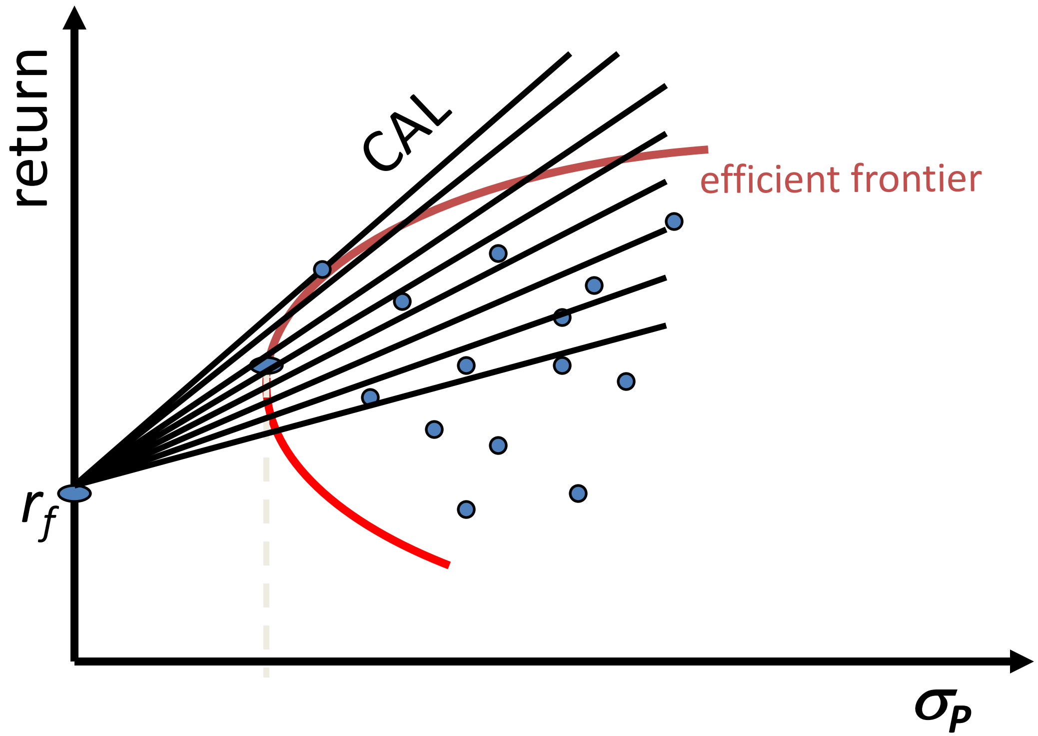
- *Sharpe Ratio, Reward to Variability Ratio or Excess Return per unit of Risk.*

Optimal Risky Portfolio with a Risk-Free Asset

Slope of the Investment Opportunity Set: $$S = \frac{\mu - r_f}{\sigma}$$

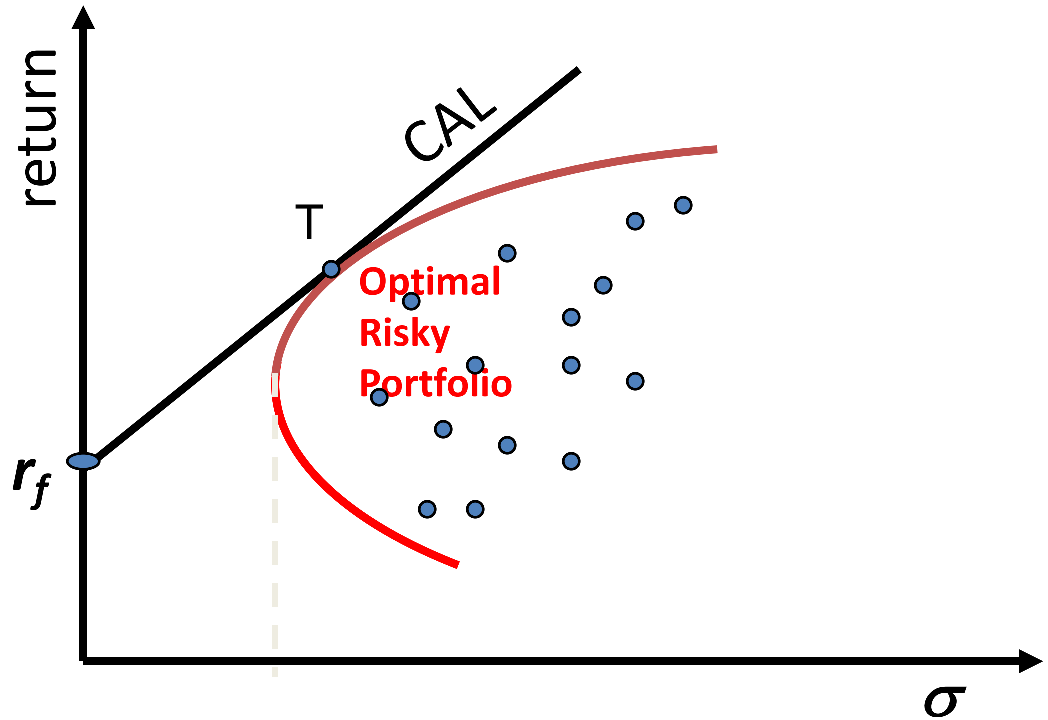


Riskless Borrowing and Lending



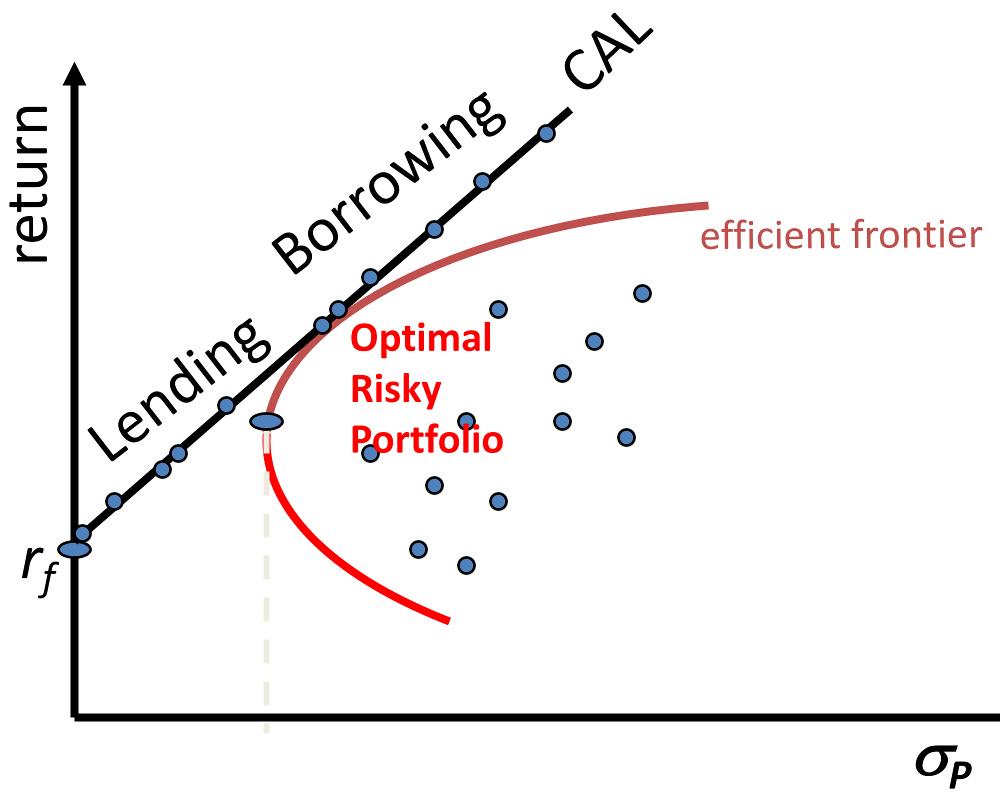
With a risk-free asset available and the efficient frontier identified, we choose the capital allocation line with the steepest slope because that line always has a portfolio with a higher return for the same risk

Riskless Borrowing and Lending



Now investors allocate their money across the T-bills and an optimal portfolio of risky securities: the tangency portfolio for a straight line starting from the risk-free rate point.

The Separation Property

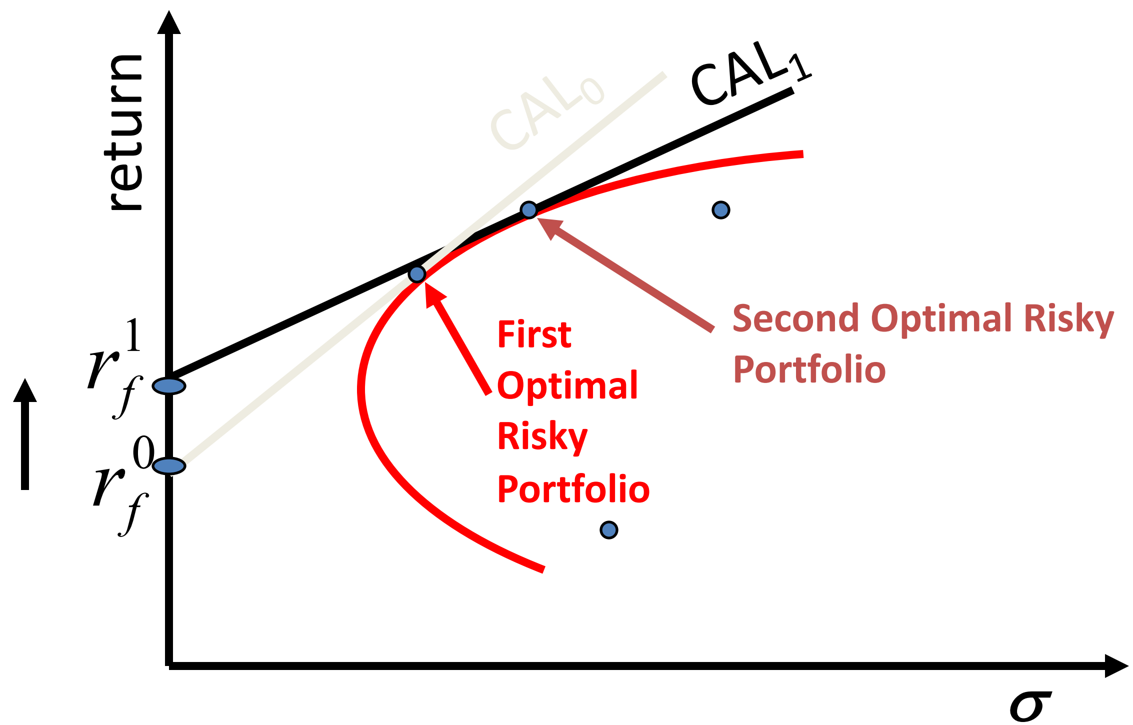


Investor risk aversion is revealed in their choice of where to stay along the capital allocation line: not in their choice of different risky portfolios, or different capital allocation lines

The Separation Property

- Just where the investor chooses to be along the CAL depends on her risk tolerance: she will lend or borrow as needed.
- The separation property states that portfolio choice can be separated into two tasks:
  - (1) determine the optimal risky portfolio, and
  - (2) selecting a point on the CAL

Optimal Risky Portfolio with a Risk-Free Asset



The optimal risky portfolio depends on the risk-free rate as well as the risky assets.

Topic 5d: Portfolio Choice and the Tangency Portfolio

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Properties of Covariance:
Covariance of a stock with a portfolio

$$Cov(r_1, w_2 r_2 + w_3 r_3) = w_2 Cov(r_1, r_2) + w_3 Cov(r_1, r_3)$$
$$= w_2 \sigma_{12} + w_3 \sigma_{13}$$

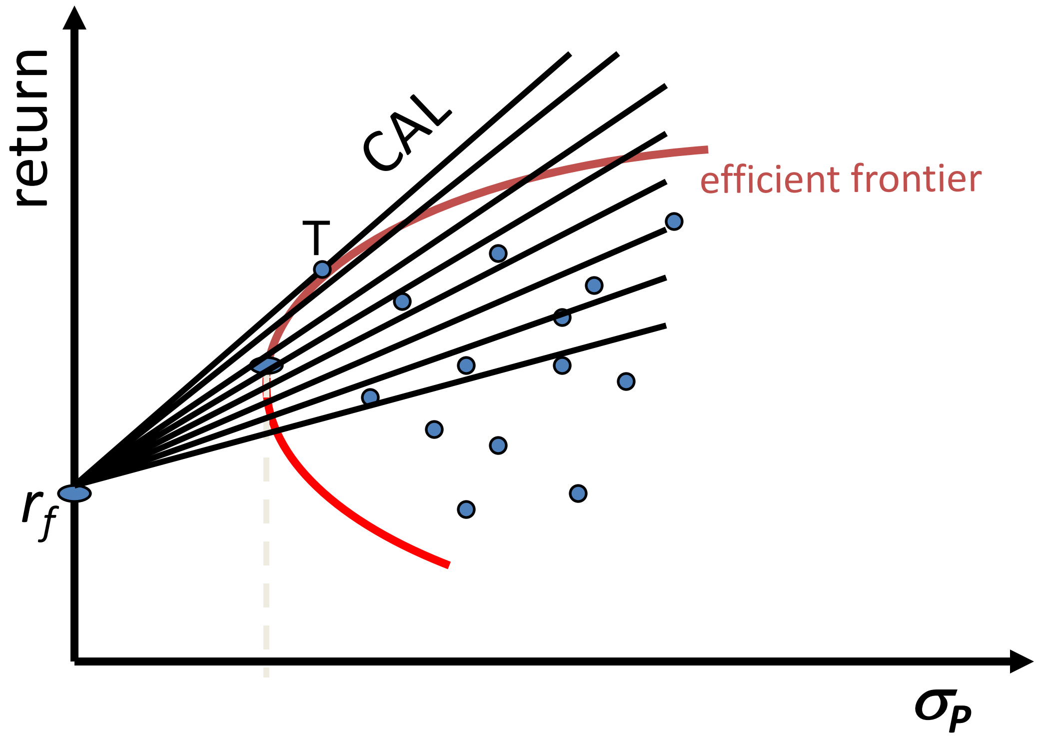
$$Cov(r_1, w_1 r_1 + w_2 r_2) = w_1 \sigma_1^2 + w_2 \sigma_{12}$$

$In\ general:$

$$Cov(r_k, r_P) = \sigma_{kP} = Cov(r_k, \sum_{i=1}^{N} w_i r_i) = \sum_{i=1}^{N} w_i \sigma_{ki}$$

The cov of a stock with a portfolio is equal to the weighted average of the cov between the stock and each stock in the portfolio

Determining Tangency Portfolio



With a risk-free asset available and the efficient frontier identified, we choose the capital allocation line with the steepest slope because that line always has a portfolio with a higher return for the same risk. We want to find the feasible w with the highest possible Sharp Ratio.

Determining Tangency Portfolio

- Tangency portfolio will be the one with the highest Sharp Ratio
  - Slope of the investment opportunity set

$$\sum_i w_i r_f = r_f$$

$$\max_{\{w_k\}} SR_P = \max_{\{w_k\}} \frac{\mu_p - r_\mathrm{f}}{\sigma_\mathrm{p}} = \max_{\{w_k\}} \frac{\sum_i w_i\left(\mu_i - r_\mathrm{f}\right)}{\left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{1/2}}$$

Tangency portfolio

- First-order conditions for any asset k

$$\frac{\partial SR}{\partial w_k} = 0 \quad k = 1, \cdots N \quad , \quad \sum_{k=1}^{N} w_k = 1$$

- First-order conditions imply that for asset k

$$\frac{\mu_k - r_{\mathrm{f}}}{\sum_j w_j^* \sigma_{kj}} = \frac{\sum_i w_i^* \left(\mu_i - r_{\mathrm{f}} \right)}{\sum_i \sum_j w_i^* w_j^* \sigma_{ij}}$$

$$\Leftrightarrow \frac{\mu_k - r_{\mathrm{f}}}{\sigma_{kT}} = \frac{\mu_T - r_{\mathrm{f}}}{\sigma_T^2} \quad , \quad k = 1 \cdots N$$

Technical Note

$$\max_{\{w_k\}} SR_P = \max_{\{w_k\}} \frac{\mu_p - r_\mathrm{f}}{\sigma_\mathrm{p}} = \max_{\{w_k\}} \frac{\sum_i w_i (\mu_i - r_\mathrm{f})}{\left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{1/2}}$$

$$SR = \sum_i w_i (\mu_i - r_\mathrm{f}) \times \left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-1/2}$$

$$\frac{\partial SR}{\partial w_k} = (\mu_k - r_\mathrm{f}) \times \left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-1/2} + \sum_i w_i (\mu_i - r_\mathrm{f}) \times (-\frac{1}{2}) \times$$

$$\left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-3/2} \times 2 \times \sum_j w_j \sigma_{kj} = 0$$

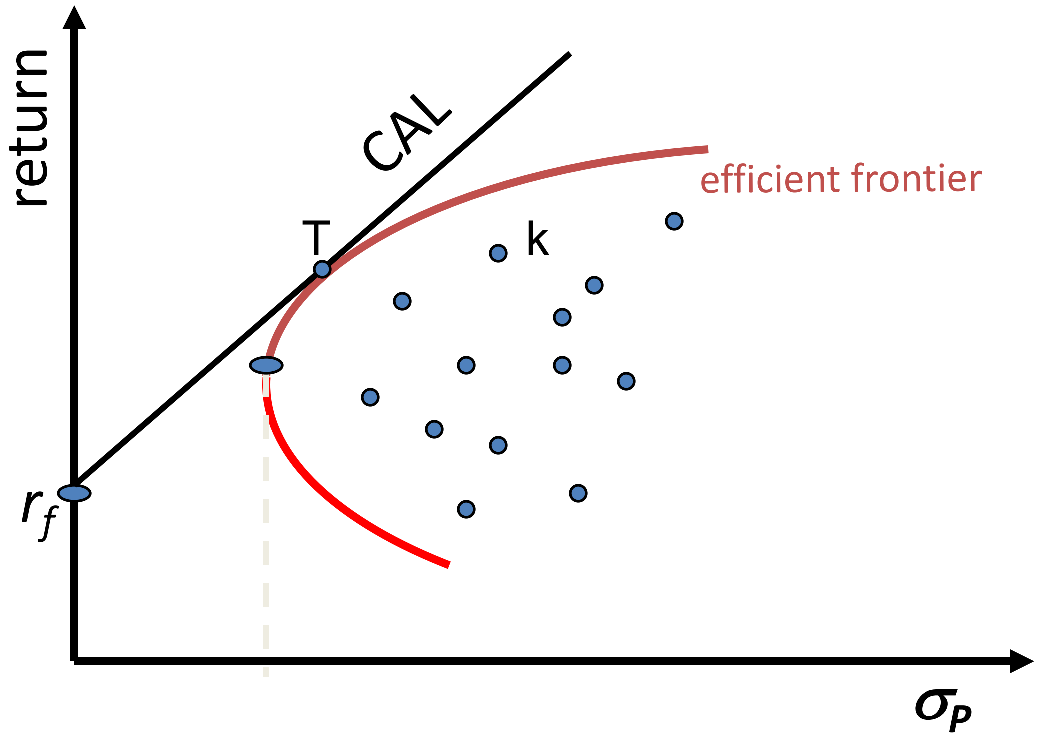
$$(\mu_k - r_\mathrm{f}) - \sum_i w_i (\mu_i - r_\mathrm{f}) \times \left(\sum_i \sum_j w_i w_j \sigma_{ij}\right)^{-1} \times \sum_j w_j \sigma_{kj} = 0$$

First order condition

$$\frac{\mu_k - r_\mathrm{f}}{\sum_j w_j^* \sigma_{kj}} = \frac{\sum_i w_i^* \left(\mu_i - r_\mathrm{f}\right)}{\sum_i \sum_j w_i^* w_j^* \sigma_{ij}} = \frac{\mu_T - r_\mathrm{f}}{\sigma_T^2} \quad , \quad k = 1 \cdots N$$

$$\Leftrightarrow \frac{\mu_k - r_\mathrm{f}}{\sigma_{kT}} = \frac{\mu_i - r_\mathrm{f}}{\sigma_{iT}} = \frac{\mu_j - r_\mathrm{f}}{\sigma_{jT}} = \frac{\mu_T - r_\mathrm{f}}{\sigma_T^2} = \frac{1}{K}$$

Property of the Tangency Portfolio



The excess return on any stock (k) divided by its covariance with the tangency portfolio (T) is constant for all stocks and equal to the excess return on T divided by its variance!

Tangency portfolio weights

- We can rewrite the first-order conditions as

$$\sigma_{kT} = (\mu_k - r_\mathrm{f})\frac{\sigma_T^2}{\mu_T - r_\mathrm{f}}$$

$$\Leftrightarrow \sum_j w_j \sigma_{kj} = (\mu_k - r_\mathrm{f})\frac{\sigma_T^2}{\mu_T - r_\mathrm{f}}$$

$$\Leftrightarrow w_1\sigma_{k1} + w_2\sigma_{k2} + \ldots + w_N\sigma_{kN} = (\mu_k - r_\mathrm{f}) \times \mathrm{K}$$

- Where the constant K is the same for all assets and ensures that weights add up to one

Tangency portfolio weights

- The first order conditions define a system of equations that can be solved for the portfolio weights of the tangency portfolio (set K=1)

$$
\begin{aligned}
& w_1\sigma_{11} + w_2\sigma_{12} + \ldots + w_N\sigma_{1N} = (\mu_1 - r_\mathrm{f}) \\
& w_1\sigma_{21} + w_2\sigma_{22} + \ldots + w_N\sigma_{2N} = (\mu_2 - r_\mathrm{f}) \\
& \ldots \\
& w_1\sigma_{N1} + w_2\sigma_{N2} + \ldots + w_N\sigma_{NN} = (\mu_N - r_\mathrm{f})
\end{aligned}
$$

- Need to rescale the solution so the weights add up to one

In Matrix Notation

$$\vec{\mu} - \vec{r}_f : \quad vector\, of\ excess\ returns$$

$$\Omega \vec{w}_T \propto (\vec{\mu} - \vec{r}_f)$$

$$\vec{w}_T \propto \Omega^{-1}(\vec{\mu} - \vec{r}_f)$$

$$weights\ need\ to\ add\ to\ 1$$

$$\vec{w}_T = \frac{\Omega^{-1}(\vec{\mu} - \vec{r}_f)}{\vec{1}^{tr}\Omega^{-1}(\vec{\mu} - \vec{r}_f)}$$

Determining Tangency Portfolio

- Suppose expected returns of Stocks A, B and C are 14%, 8% and 20%, and the associated standard deviations are respectively 6%, 3% and 15%. Let the correlation coefficients be A-B 0.5, A-C 0.2, and B-C 0.4. The Variance-Covariance Matrix is:

36	9	18
9	9	18
18	18	225

Determining Tangency Portfolio

- We need to solve the system of 3 equations with three unknowns

$$w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13} = (\mu_1 - r_{\mathrm{f}})$$
$$w_1\sigma_{21} + w_2\sigma_{22} + w_3\sigma_{23} = (\mu_2 - r_{\mathrm{f}})$$
$$w_1\sigma_{31} + w_2\sigma_{32} + w_3\sigma_{33} = (\mu_3 - r_{\mathrm{f}})$$
$$\Omega\vec{w}_T = (\vec{\mu} - \vec{r}_f) \quad \textit{in vector / matrix notation}$$
$$\vec{w}_T = \Omega^{-1}(\vec{\mu} - \vec{r}_f)$$

- Then we need to rescale the solution so the weights add up to one

Determining Tangency Portfolio

- Suppose that the risk free rate is 5%

$$36w_1 + 9w_2 + 18w_3 = 14 - 5 = 9$$
$$9w_1 + 9w_2 + 18w_3 = 8 - 5 = 3$$
$$18w_1 + 18w_2 + 225w_3 = 20 - 5 = 15$$
$$w_1 = \frac{14}{63} \quad , \quad w_2 = \frac{1}{63} \quad , \quad w_3 = \frac{3}{63}$$
$$\textit{finally we normalize the weights to add to } 1$$
$$w_1 = \frac{14}{18} \quad , \quad w_2 = \frac{1}{18} \quad , \quad w_3 = \frac{3}{18}$$

For the Tangency Portfolio

- Recall from the first order conditions, that for every asset k:

$$\frac{\mu_k - r_\mathrm{f}}{\sigma_{kT}} = \frac{\mu_T - r_\mathrm{f}}{\sigma_T^2} \quad , \quad k = 1 \cdots N$$

$$\mu_k - r_\mathrm{f} = \frac{\sigma_{kT}}{\sigma_T^2}\left(\mu_T - r_\mathrm{f}\right)$$

$$\mu_k = r_\mathrm{f} + \frac{\sigma_{kT}}{\sigma_T^2}\left(\mu_T - r_\mathrm{f}\right) \quad , \quad k = 1 \cdots N$$

For the Tangency Portfolio

$$\mu_k = r_{\mathrm{f}} + \frac{\sigma_{kT}}{\sigma_T^2}\left(\mu_T - r_{\mathrm{f}}\right) \quad , \quad k = 1 \cdots N$$

$$let \quad \beta_{kT} = \frac{\sigma_{kT}}{\sigma_T^2}$$

$$\mu_k = r_{\mathrm{f}} + \beta_{kT}\left(\mu_T - r_{\mathrm{f}}\right) \quad , \quad k = 1 \cdots N$$

Risk premium

Relevant measure for risk is the covariance with the tangency portfolio.
Beta is a standardized measure of covariance.
This is a mathematical relation.

For the Tangency Portfolio

- This relation also applies to portfolios

$$\mu_k = r_{\mathrm{f}} + \beta_{kT}\left(\mu_T - r_{\mathrm{f}}\right)$$

$$\sum_k w_k \mu_k = \sum_k w_k r_{\mathrm{f}} + \sum_k w_k \beta_{kT}\left(\mu_T - r_{\mathrm{f}}\right) = r_{\mathrm{f}} \sum_k w_k + \left(\mu_T - r_{\mathrm{f}}\right) \sum_k w_k \beta_{kT}$$

$$\mu_P = r_{\mathrm{f}} + \beta_{PT}\left(\mu_T - r_{\mathrm{f}}\right)$$

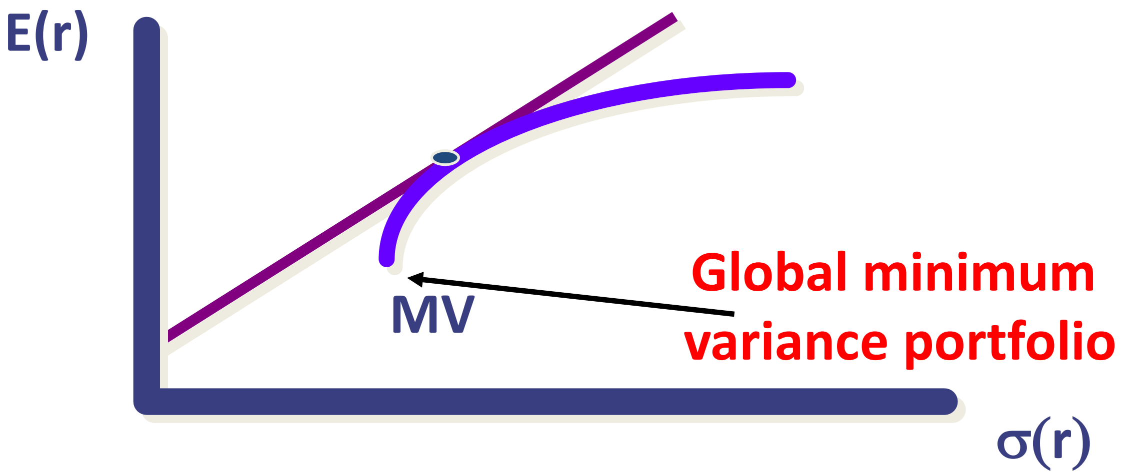
- The beta of a portfolio is a weighted average of the betas of the individual assets.

$$\beta_{PT} = \sum_k w_k \beta_{kT}$$

- What is the beta of the tangency portfolio?

Minimum variance portfolio (MV)

- Has the lowest possible variance
- Weights also solve a system of equations



Technical Note

$$\sigma_P^2 = \vec{w}^{Tr}\Omega\vec{w}$$

$$\vec{w}^{Tr}\mathbf{1} = 1$$

$$\min_w L = \vec{w}^{Tr}\Omega\vec{w} - 2\lambda(\vec{w}^{Tr}\mathbf{1} - 1)$$

$$\frac{\partial L}{\partial \vec{w}} = 2\Omega\vec{w} - 2\lambda\mathbf{1} = 0$$

$$\vec{w} = \lambda\Omega^{-1}\mathbf{1}$$

$$\vec{w}_{MVP} \propto \Omega^{-1}\mathbf{1}$$

MVP and Tangency – matrix solution

- Systems of equations for T and MVP better solved with matrix algebra

$$\vec{w}_{MVP} \propto \Omega^{-1}\mathbf{1}$$

$$\vec{w}_{T} \propto \Omega^{-1}\left(\vec{\mu} - \vec{r}_f\right)$$

Ω : Covariance matrix

μ-r_f : Vector of mean excess returns

1 : Vector of ones

We still need to normalize weights so they add up to one

- Once we have two efficient portfolios, we are back to the two asset case

Limitations of Markowitz Approach

- Large number of inputs are needed.
  - Suppose we have N stocks. We need to estimate
    - N expected returns (one for each asset) 100
    - N variances (one for each asset) 100
    - N(N-1)/2 covariances (for each pair of assets)

      = 9,900/2 = 4950

- Estimates of inputs are difficult

Inputs

- Expected returns
  - Past may not be a good indicator of future performance
  - Need to look at today's circumstances (dividend yields, earning-price ratios)
  - Shrinkage
  - Use asset pricing models such as the CAPM!
- Standard deviations and covariances
  - Somewhat more stable over time
  - Estimating from past data is OK: ARCH/GARCH models
  - Shrinkage
  - Use factor models

Issues

- Sometimes recommendations of MV analysis seem unreasonable (with many assets)
  – Optimization is very sensitive to inputs
  – Large short (or long) positions
- Portfolio constraints
  – Use numerical methods like Solver in Excel
- What if investor cares about higher order moments?

Efficient frontier with constraints

- Parametric quadratic programming problem

$$\min_w \sigma_P^2 - \lambda \mu_P$$

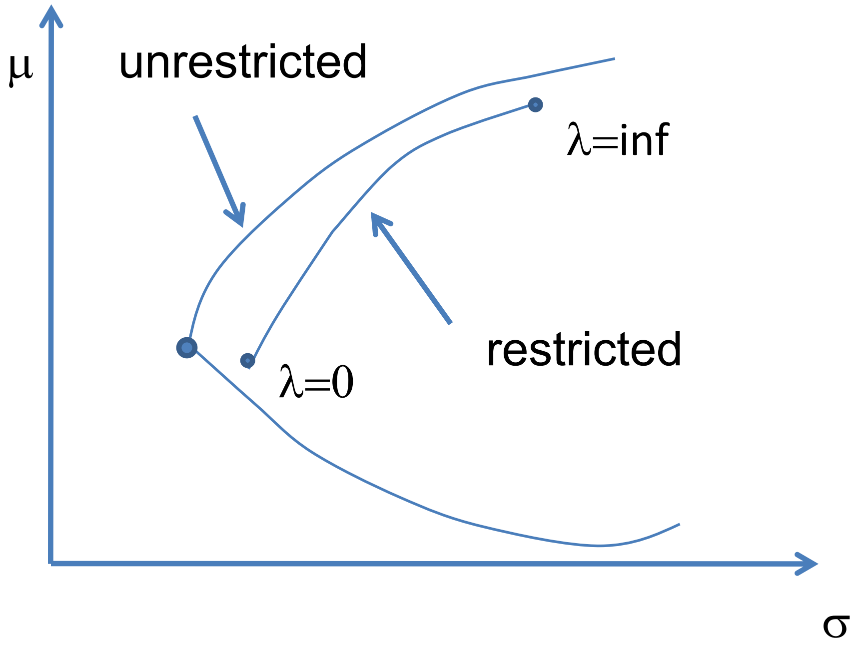
$$\mu_P = \sum_{i=1}^{N} w_i \mu_i \qquad \sigma_P^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

$$\sum_{i=1}^{N} w_i = 1$$

- Subject to linear constraints

$$w_i \geq 0 \quad , \quad w_i \leq M$$

Efficient frontier with constraints



$$\min_w \sigma_P^2 - \lambda \mu_P$$

$$\lambda \min_w \frac{\sigma_P^2}{\lambda} - \mu_P$$

$$\max_w \mu_P - \frac{\sigma_P^2}{\lambda}$$

$$\lambda = 0 \quad \min \sigma_P^2$$

$$\lambda = \infty \quad \max \mu_P$$