Financial Risk Management

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Heavy Tails and High Confidence Level VaR

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THINK IN THE NEXT

Agenda

- Estimating Very High Confidence Level VaRs
- Exponential and Polynomial Tails
- t Distribution
- Power Law

Very High Confidence Level VaRs

- Suppose I have 500 past daily returns to compute historical VaR.
- How do I compute $VaR_{99\%}$?
- What about $VaR_{99.8\%}$? $VaR_{99.9\%}$?
- I have two options for computing VaRs at very high confidence levels:
  - Use a parametric approach to estimate the distribution
  - Use a parametric approach to inflate VaR at a lower confidence level

Results from Historical Simulation

N=500

Scenario Number	Loss ($000s)
494	477.841
339	345.435
349	282.204
329	277.041
487	253.385
227	217.974
131	205.256

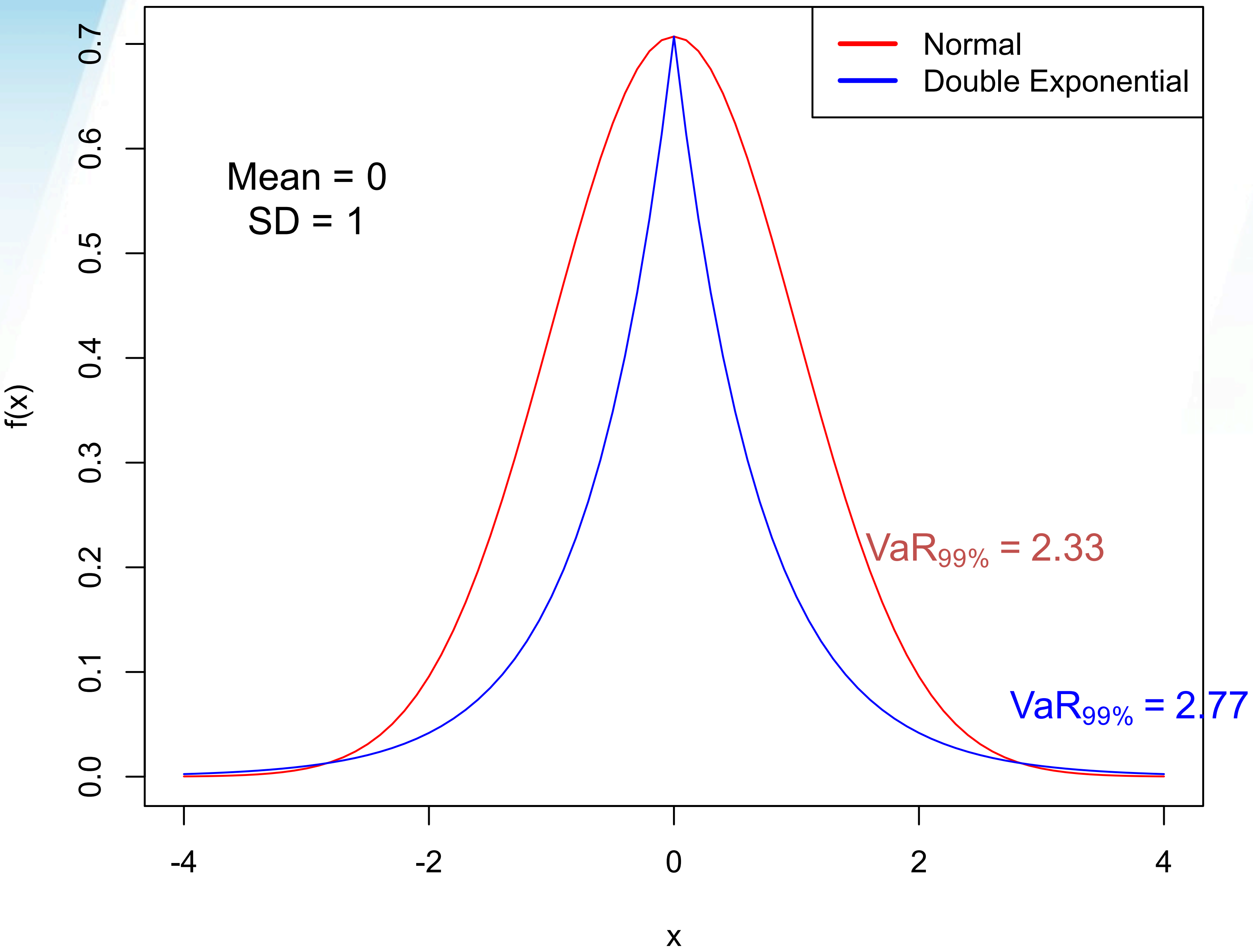
$VaR_{99\%}$? $VaR_{99.8\%}$? $VaR_{99.9\%}$?

Heavy Tails

- Daily exchange rate changes are not normally distributed
  - The distribution has heavier tails than the normal distribution
  - It is more peaked at the center than the normal distribution
- This means that large changes are more likely than the normal distribution would suggest

Exponential Tail

- In order to gauge how heavy is the tail of the distribution, we can look at the density function.
- Consider the density of: $N\left(0,\sigma^2\right)$, $f = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$
- The tail of a distribution with $f \propto e^{-\left|\frac{y}{\theta}\right|}$ will converge slower to 0, and therefore will have a fatter tail.


Normal
Double Exponential
Mean = 0
SD = 1
VaR99% = 2.33
VaR99% = 2.77
f(x)
x
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
-4
-2
0
2
4

Exponential Tail

- The Normal is part of a family of distributions, with exponential rate of convergence to zero:

$$f(y) \propto e^{-\left|\frac{y}{\theta}\right|^{\alpha}}$$

- α is a shape parameter, θ is a scale parameter.
- In case of the Normal α=2 and θ is σ.
- The lower the α the heavier the tail.
- All absolute moments are finite, i.e. $E\left(|Y|^k\right) < \infty$

Polynomial Tail

- To get heavier tails, we have to consider distributions for which the density has polynomial tails, i.e.

$$f(y) \sim A|y|^{-(a+1)} \quad as \quad |y| \to \infty$$

- a is called the tail index.
- The k^{th} absolute moment, i.e. $E\left[|y|^k\right]$, exists only if the tail index is larger than k.

t - Distribution

- Commonly used way to model polynomial tails.
- The pdf is:

$$f_{t,\nu}(y) = \left[\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{(\pi\nu)^{1/2}\,\Gamma\left(\frac{\nu}{2}\right)} \right] \cdot \left[1 + \left(\frac{y^2}{\nu} \right) \right]^{\frac{-(\nu+1)}{2}}$$

- ν is the degrees of freedom
- It is clear that: $f(y) \propto |y|^{-(\nu+1)} \quad as \quad |y| \to \infty$
- Hence, the tail index is ν. The weight of the tail decreases as ν increases.
- It goes to the Standard Normal as ν goes to ∞.

t – Distribution Moments

- The mean exists and equals 0 only if $\nu>1$.
- The variance exists only if $\nu>2$, and is:

$$\frac{\nu}{\nu-2}$$

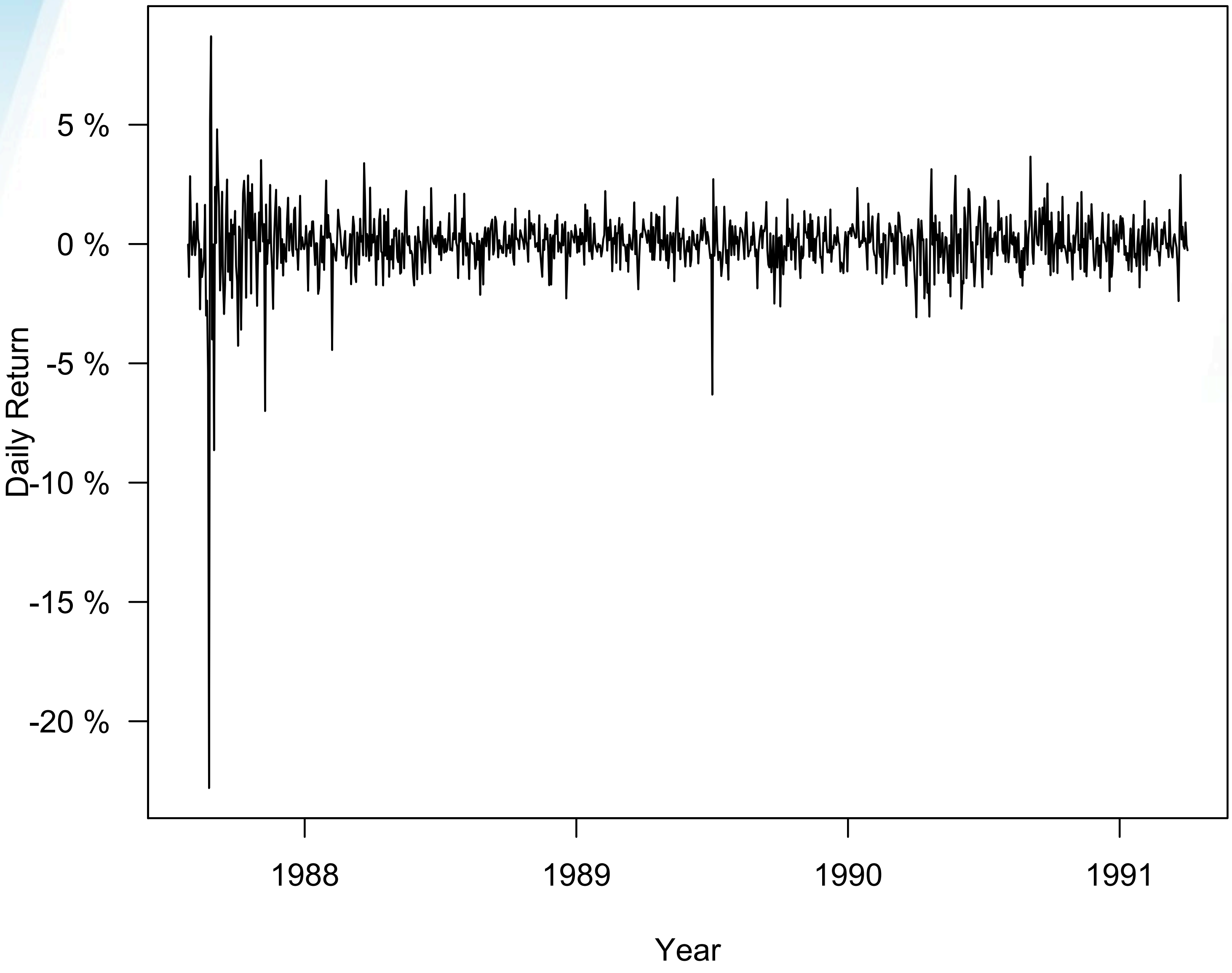
- The distribution is symmetric, and its Skewness is zero.
- The Kurtosis exists for $\nu>4$, and is given by:

$$Kurt = 3 + \frac{6}{\nu-4}$$

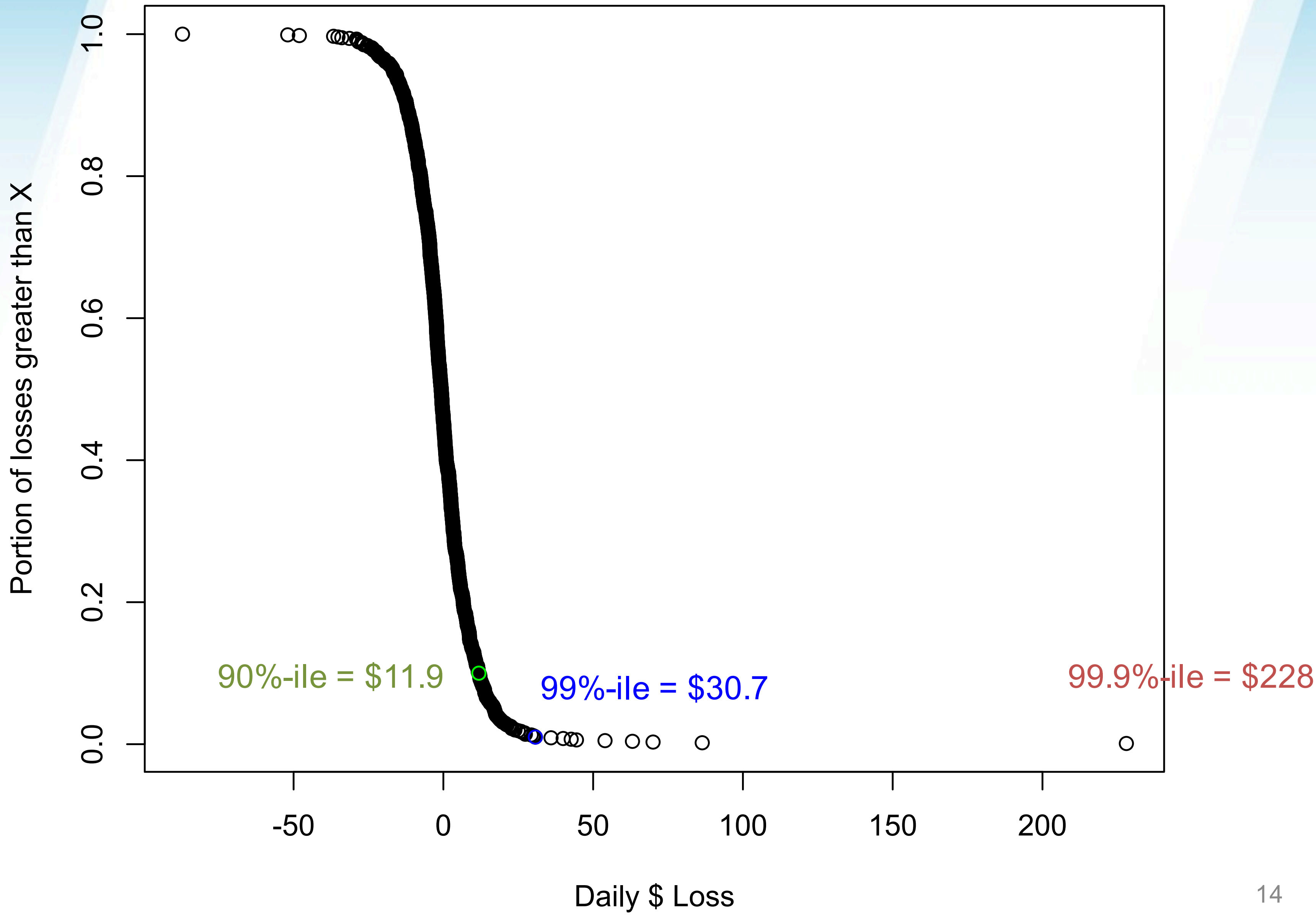
t-Distribution

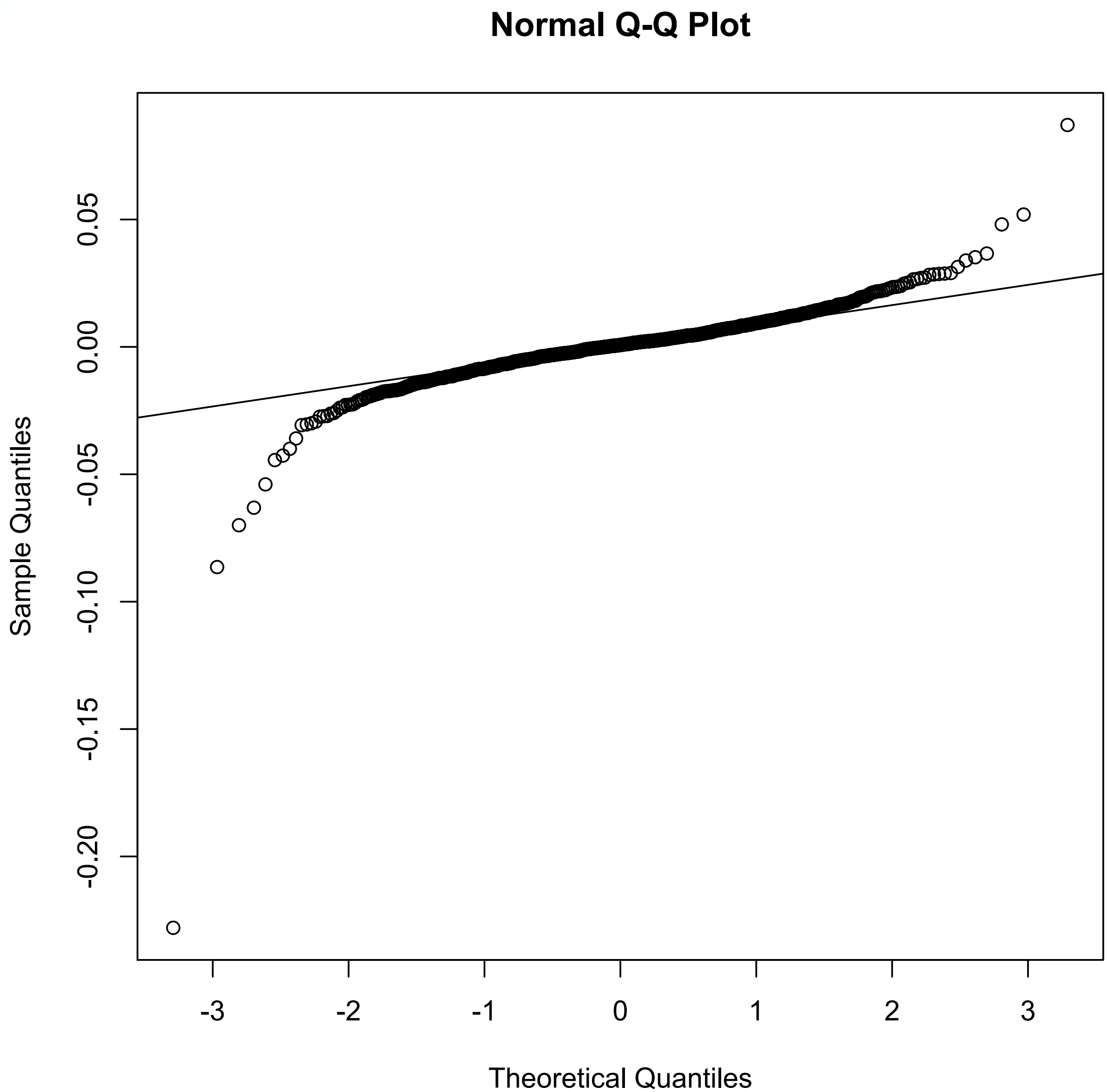
- The classic t-distribution has mean zero, and variance defined by ν
- We can shift and scale it.
- If Y has classic t-distribution with ν degrees of freedom then: $\mu + \lambda Y \sim t_\nu\left(\mu, \lambda^2\right)$
- μ is the mean, λ is the scale, the variance is equal to: $\lambda^2 \frac{\nu}{\nu - 2}$

Daily Returns on S&P 500 (1987-1991)

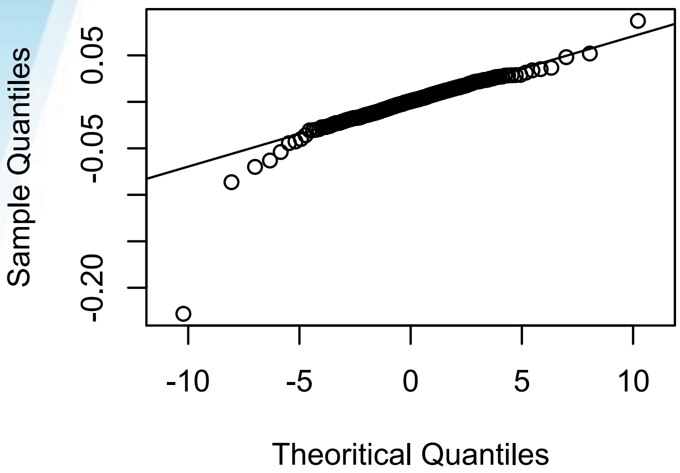


Daily losses on $1000 invested in S&P 500 (1987-1991)

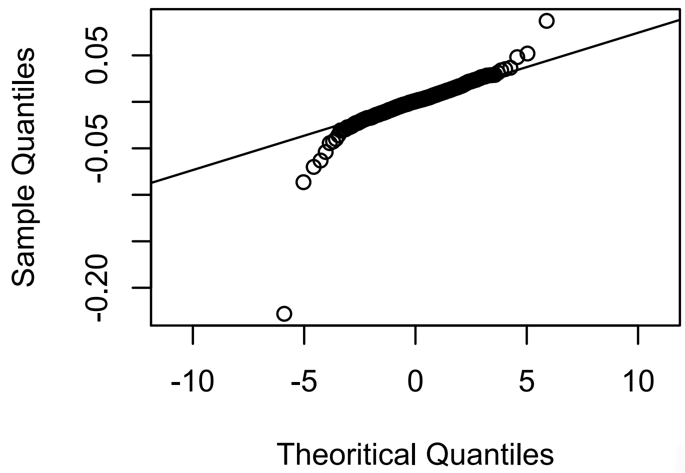



Normal Q-Q Plot
Sample Quantiles
Theoretical Quantiles
0.05
0.00
-0.05
-0.10
-0.15
-0.20
-3
-2
-1
0
1
2
3

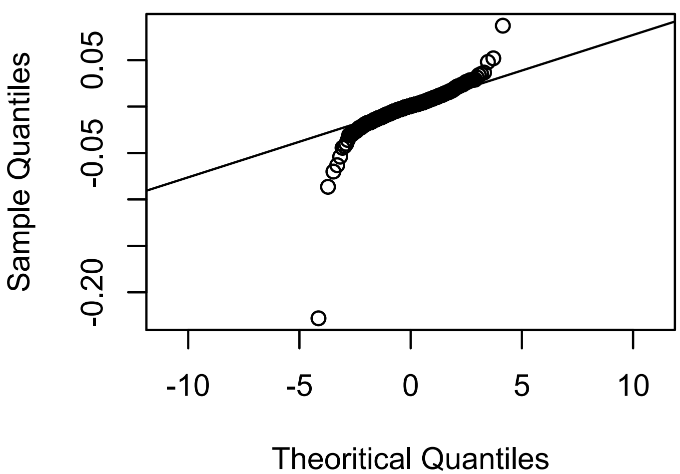
t-distribution, nu= 3



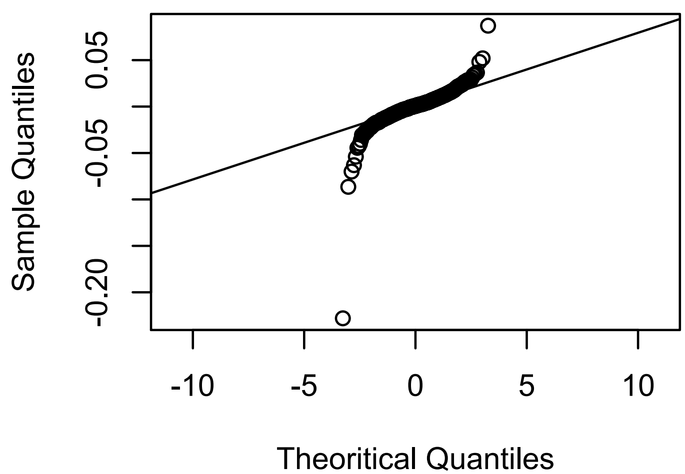
t-distribution, nu= 5



t-distribution, nu= 10



t-distribution, nu= 50



Fitting t-distribution

- To fit a t-distribution, we can use MLE.

$$f_{t,\nu}(X) = \left[\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{(\pi\nu)^{1/2}\,\Gamma\left(\frac{\nu}{2}\right)}\right]\prod_{i=1}^{n}\left[1+\left(\frac{\left(\frac{x_i-\mu}{\lambda}\right)^2}{\nu}\right)\right]^{\frac{-(\nu+1)}{2}}$$

- Use R:

```
fitt = fitdistr(X,"t")
param = as.numeric(fitt$estimate)
mean = param[1]        lambda = param[2]
nu= param[3]
sd = lambda*sqrt((nu)/(nu-2))
```

Using MLE: **v = 2.98, λ = 7.16**

Estimating VaR 99.9%

- Is \$228 a good estimator for $\text{VaR}_{99.9\%}$?
- The average daily loss/gain is roughly 0.
- If we use the Normal distribution, with the sample standard deviation, σ=13.54:

$$VaR_{99.9\%} = \Phi^{-1}\left(0.999\right) \cdot \hat{\sigma} = \$42$$

- If we use the MLE t-distribution estimates:

$$VaR_{99.9\%} = t_{\nu}^{-1}\left(0.999\right) \cdot \hat{\lambda} = \$74$$

GARCH with t-errors

- We can also fit a GARCH with t-distribution for the errors: $U_t = \sigma_t \varepsilon_t$

$$\sigma_t^2 = \omega + \alpha \cdot U_{t-1}^2 + \beta \cdot \sigma_{t-1}^2$$

- Where: $\varepsilon_t \sim f_{t,\nu}$

The Power Law

- For many variables in practice, it is approximately true that, when *X* is large enough: $\Pr\left(X > x\right) = Kx^{-\alpha}$
- *K* and α are parameters to be estimated.
- To find $\text{VaR}_{1\text{-}p}$: $p = \Pr\left(Loss > VaR_{1-p}\right) = K\left[VaR_{1-p}\right]^{-\alpha}$

$$\Rightarrow \left[VaR_{1-p}\right]^{\alpha} = \frac{K}{p} \Rightarrow VaR_{1-p} = \left(\frac{K}{p}\right)^{1/\alpha}$$

- For example: $VaR_{99\%} = \left(\frac{K}{0.01}\right)^{1/\alpha}$

Power Law – Example

- Q: Suppose we know that $\text{VaR}_{95\%}$ is \$10M, and $\alpha = 3$, what is the probability of the loss being greater than \$20M?
- A:

$$VaR_{95\%} = \left(\frac{K}{0.05}\right)^{1/\alpha}$$

$$0.05 = K \cdot 10^{-3}$$

$$K = 50$$

$$p = 50 \cdot 20^{-3} = 0.00625$$

Using the Power Law to Estimate Higher Confidence Level VaR

- Since, $VaR_{1\text{-}p} = (K/p)^{1/\alpha}$
- If we feel more confident estimating a lower percentile, we can derive higher percentiles if we know alpha:

$$\frac{VaR_{1-p_1}}{VaR_{1-p_0}} = \left(\frac{p_0}{p_1}\right)^{1/\alpha}$$

- Q: Suppose we know that $VaR_{95\%}$=\$10M, and α = 3, what is $VaR_{99\%}$?
- A: $$\frac{VaR_{99\%}}{VaR_{95\%}} = \left(\frac{0.05}{0.01}\right)^{1/3} \Rightarrow VaR_{99\%} = 10 \cdot 5^{1/3} = \$17.10M$$

Compare to Normal: $$\frac{VaR_{99\%}}{VaR_{95\%}} = \frac{N^{-1}(0.99)}{N^{-1}(0.95)} \Rightarrow VaR_{99\%} = 10 \cdot 1.414 = \$14.14M$$

Expected Shortfall for Power Law

Power Law: $\mathrm{P}\left[X > x\right] = Kx^{-\alpha}$

CDF of X: $F\left(x\right) = P\left[X \le x\right] = 1 - Kx^{-\alpha}$

PDF of X: $f\left(x\right) = \alpha K x^{-(\alpha+1)}$

Let d=$\mathrm{VaR}_{1\text{-}p}$
Conditional PDF: $f\left(x \mid x \ge d\right) = \dfrac{\alpha K x^{-(\alpha+1)}}{Kd^{-\alpha}} = \alpha d^{\alpha} x^{-(\alpha+1)}$

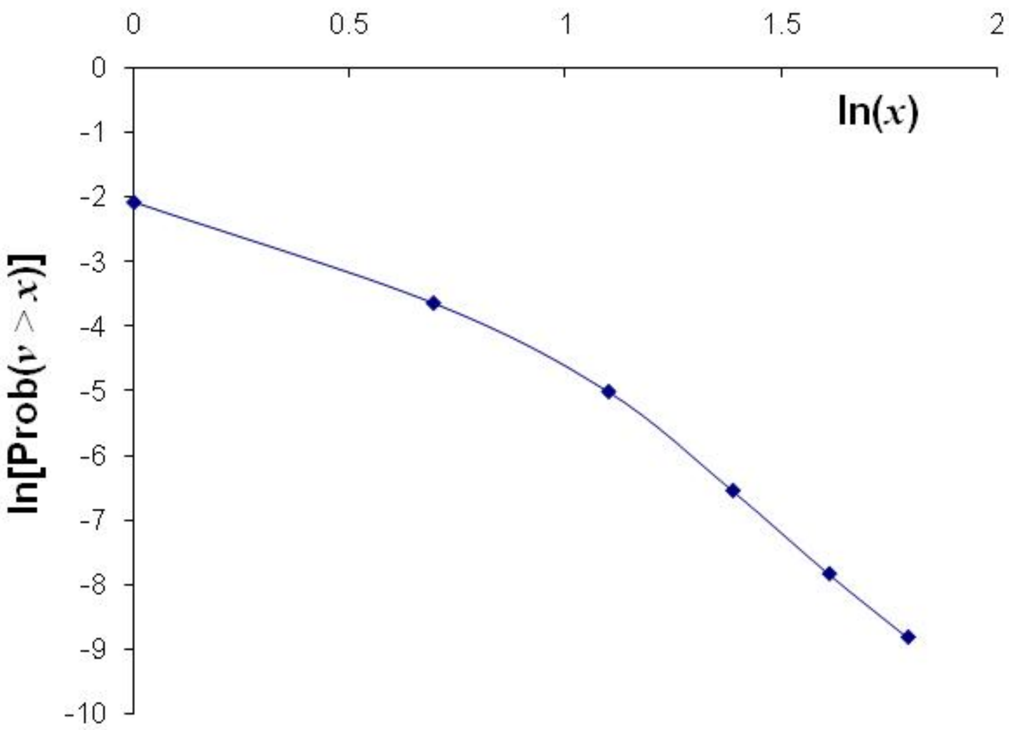
$$E\left[x \mid x \ge d\right] = \int_{d}^{\infty} \alpha d^{\alpha} x^{-(\alpha+1)} \cdot x\,dx = \alpha d^{\alpha} \int_{d}^{\infty} x^{-\alpha} dx$$

$$E\left[x \mid x \ge d\right] = \frac{\alpha}{\alpha - 1} d$$

$$ES_{1-p} = \frac{\alpha}{\alpha - 1} VaR_{1-p}$$

The lower the alpha, the higher the ratio ES/VaR.

Log-Log Plot for Estimating Power Law



Far enough in the tail, we can run a regression to find K and α:

$$\Pr(v > x) = Kx^{-\alpha} \Rightarrow \text{Ln[Prob}(v > x)] = \ln K - \alpha \ln x$$

Estimating Power Law

Suppose daily returns for S&P are given in *SPreturn* array, with length *n*:

x=sort(SPreturn) #Sort the returns

m=100 #Consider the first 100, i.e. greatest losses

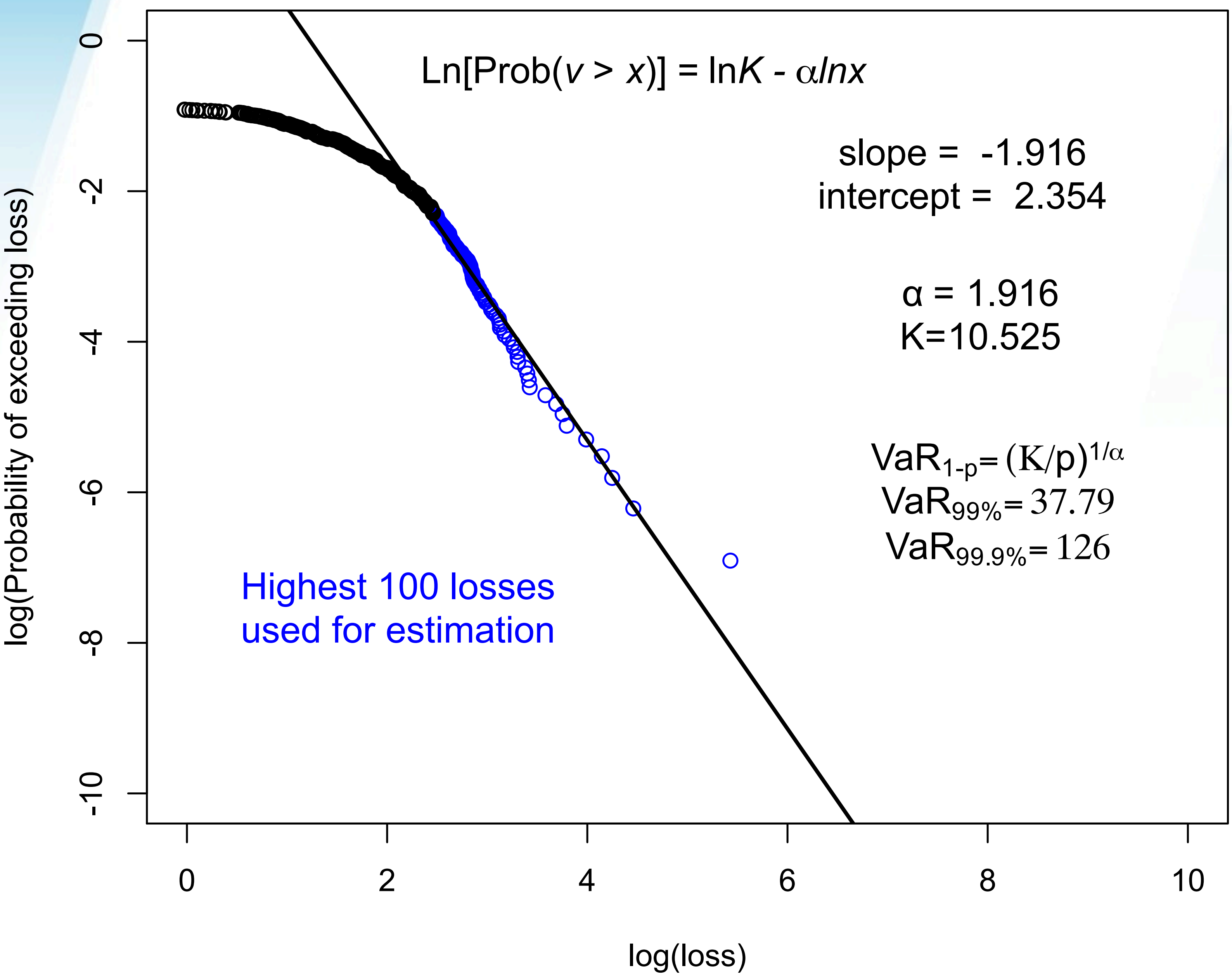
xx=log(-x[1:m]) #Take log of absolute value of losses

yy=log((1:m)/n) #yy[i] is the log of the portion of losses greater than xx[i]

fit =lm(yy~xx) #Estimate the regression

paste("slope = ", fit$coef[2])

Daily returns on $1000 invested in S&P 500 (1987-1991)



Hill's Alpha

- An estimator for alpha.

Conditional distribution: $$f\left(x \mid x \geq d\right) = \alpha d^{\alpha} x^{-(\alpha+1)}$$

$$E\left[\ln(x) \mid x \geq d\right] = \int_{d}^{\infty} \alpha d^{\alpha} x^{-(\alpha+1)} \cdot \ln(x)dx$$

Integration by parts: $$E\left[\ln(x) \mid x \geq d\right] = \ln(d) + \frac{1}{\alpha}$$

$$\alpha = \left(E\left[\ln(x) - \ln(d) \mid x \geq d\right]\right)^{-1}$$

- We can use a conditional sample average as an estimator of the conditional expectation.

Hill's Alpha (cont.)

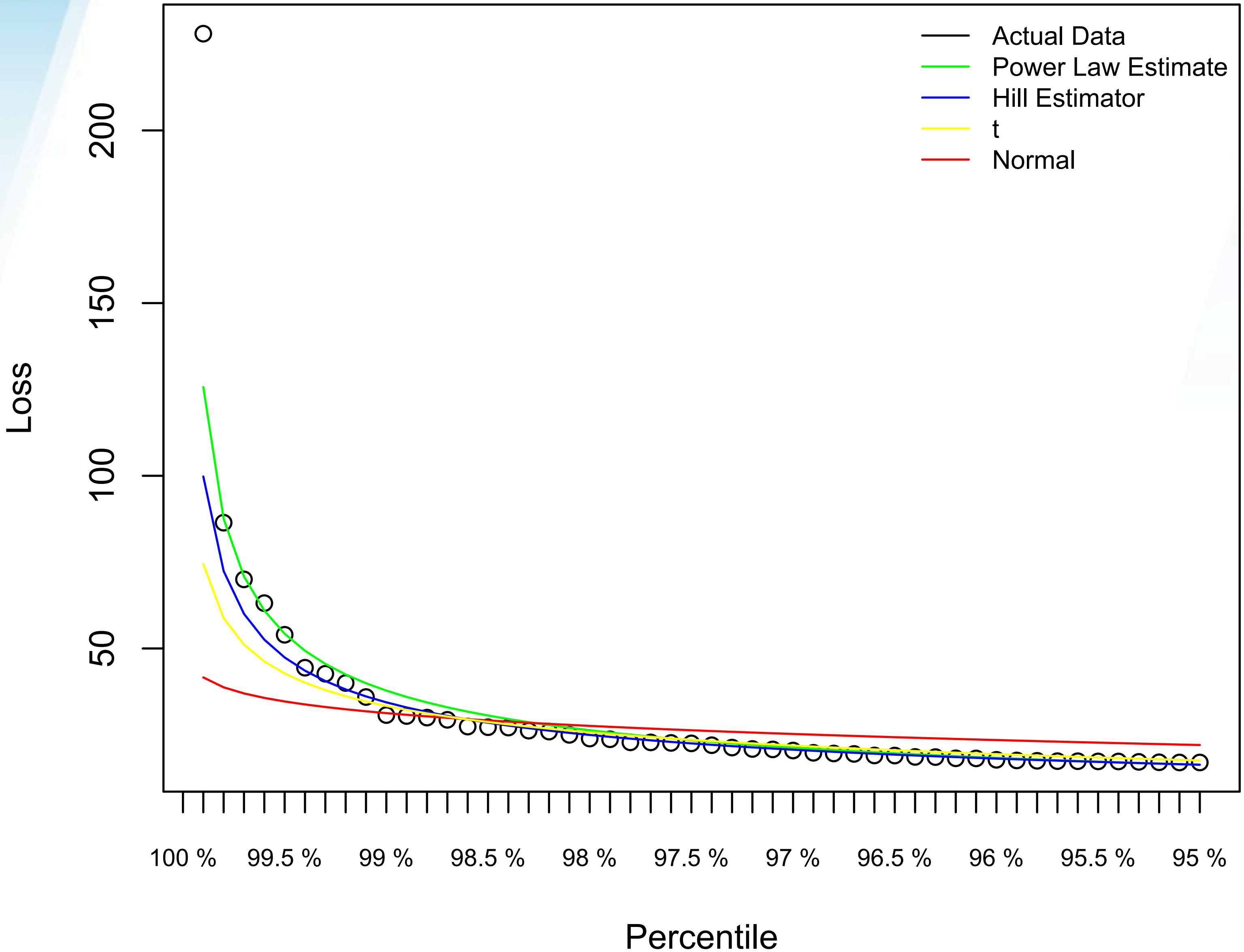
- Select a high loss level, *d*. Consider all the losses that are greater than *d.* Suppose there are *n(d)* such losses, call them $x_{(i)}$ then:

$$\hat{\alpha}^{Hill}(d) = \frac{n(d)}{\sum_{i=1}^{n(d)} \ln\left(\frac{x_{(i)}}{d}\right)}$$

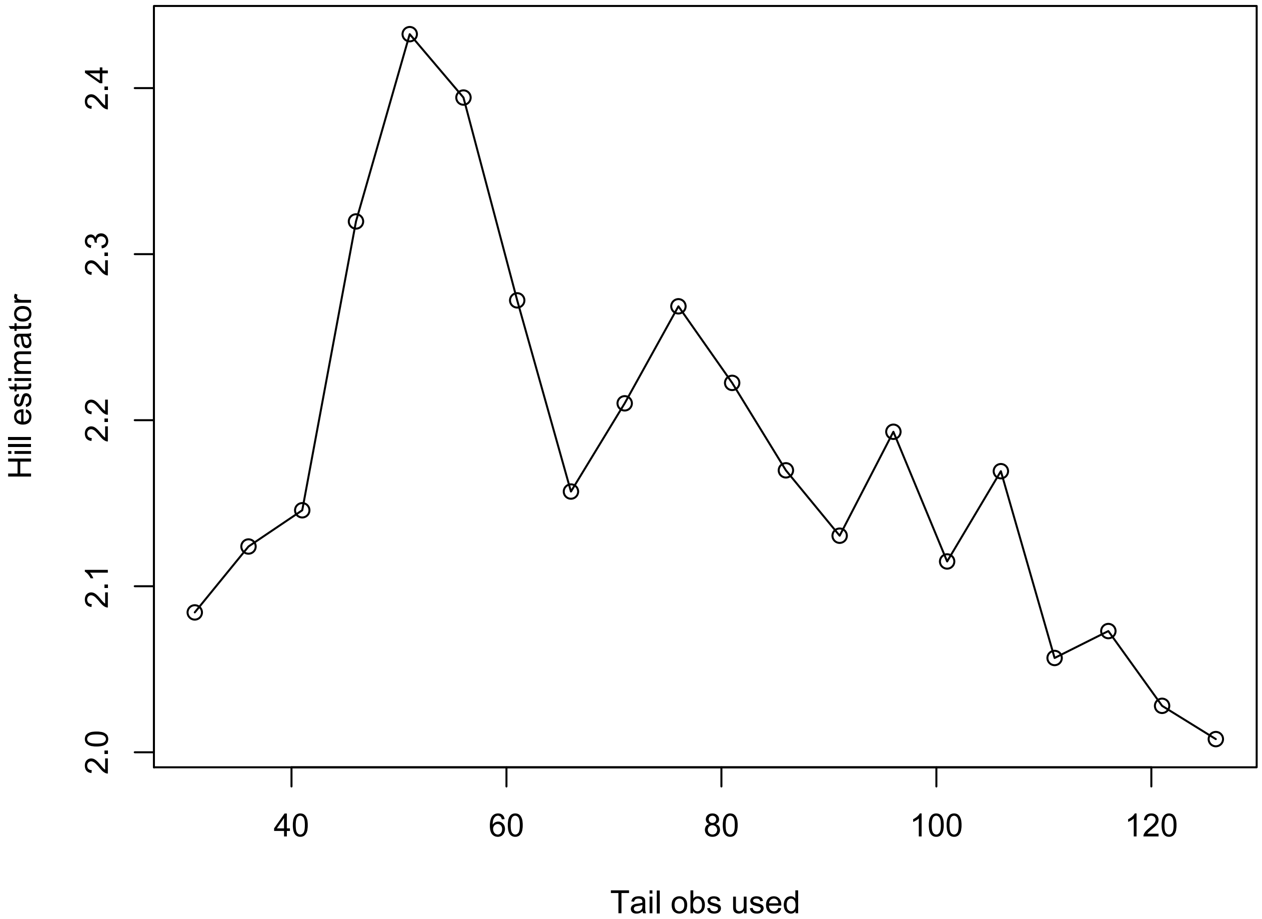
- Extend code to estimate $\hat{\alpha}^{Hill}$ using largest 100 losses: recall: x=sort(SPreturn) and m=100

  hill.alpha = m/sum(log(x[1:m]/x[m]))

Top percentiles of daily losses on $1000 S&P position



Hill Estimates based on different cutoffs (d) for 1000 S&P Daily Returns



Thanks