Cramer's rule

for

$$AX = B,$$

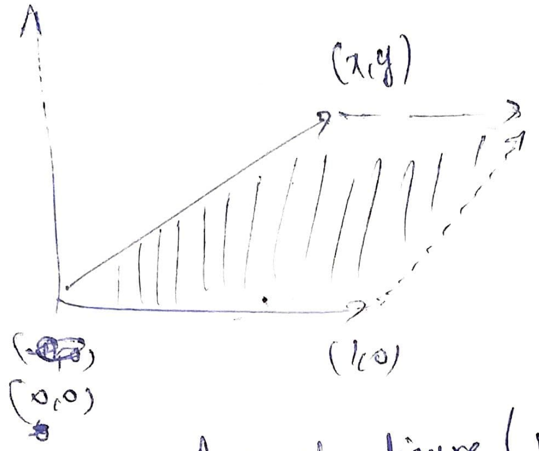
$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$$x = \begin{vmatrix} x' & c \\ y' & d \end{vmatrix} \Big/ \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$y = \begin{vmatrix} a & x' \\ b & y' \end{vmatrix} \Big/ \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

Cramer's rule - geometric view

1). Seeing co-ordinates as area



Area of figure (parallelogram) formed by a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with first basis vector is

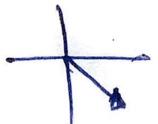
$= 1 \times y$

$= y$ co-ordinate.

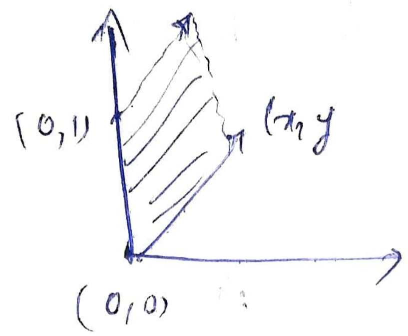
And this area, can be cleverly calculated using determinant

$$\begin{vmatrix} 1 & x \\ 0 & y \end{vmatrix} = Y\text{-co-ordinate}$$

As determinant can be negative also we can say that we can get the "signed" area. That means, we can get -ve 'y' co-ordinate too if the vector is something like this



Similarly,



Area of ||gm formed by $\begin{bmatrix} x \\ y \end{bmatrix}$ vector with second basis vector

$= 1 \times x$

$= \underline{\underline{x}}$

Similarly that can be calculated using

$$\begin{vmatrix} x & 0 \\ y & 1 \end{vmatrix} = x \text{ coordinate}$$

Now when we apply the transformation 'A', we get a different parallelogram, and the area of this parallelogram is scaled (or squished) by det(A). [That is what determinant means by definition]

The bigger picture is

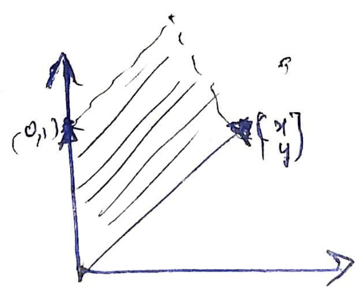
for $AX = B$.

A is the transformation

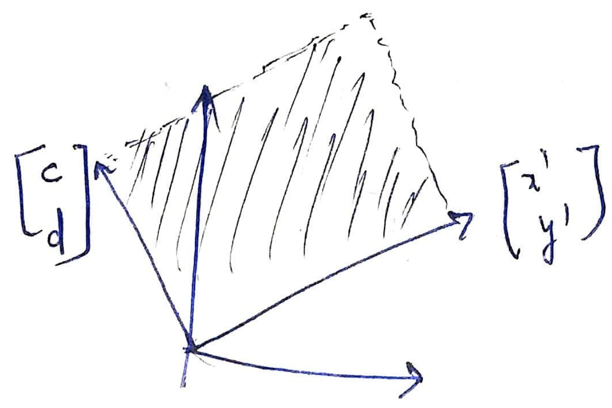
$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

on applying 'A' transformation $\begin{bmatrix} x \\ y \end{bmatrix}$ falls on $\begin{bmatrix} x' \\ y' \end{bmatrix}$

Coming back to the parallelogram formed by 2nd basis vector and $\begin{bmatrix} x \\ y \end{bmatrix}$



A transformation applied



Area of this new ||gm

$$\begin{vmatrix} x' & c \\ y' & d \end{vmatrix}$$

We know, the area is scaled (or squished) by det(A) times, that is

Area of New parallelogram = Area of old parallelogram X det(A)

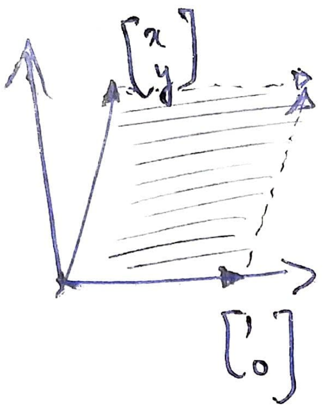
$$\begin{vmatrix} x' & c \\ y' & d \end{vmatrix} = \begin{vmatrix} x & 0 \\ y & 1 \end{vmatrix} \times \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$\downarrow$$

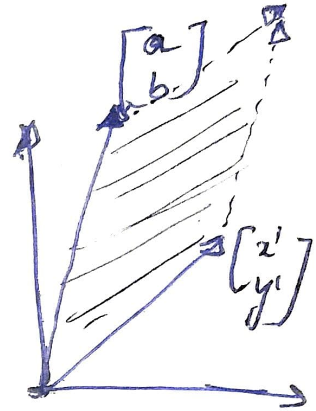
$$\begin{vmatrix} x' & c \\ y' & d \end{vmatrix} = X\text{-coordinate} \times \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$X\text{-co-ordinate} = \frac{\begin{vmatrix} x' & c \\ y' & d \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$$

Repeating this exercise for Y-co-ordinate



$\xrightarrow[\text{transformation}]{A}$



Area of this parallelogram gives Y-coordinate

$$\begin{vmatrix} 1 & x \\ 0 & y \end{vmatrix} = \text{Y-coordinate}$$

$$= y$$

Area of this parallelogram

$$\begin{vmatrix} a & x' \\ b & y' \end{vmatrix}$$

But Area of New ||gm = old parallelogram Area × det(A)

$$\begin{vmatrix} a & x' \\ b & y' \end{vmatrix} = \text{Y-coordinate} \times \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$\text{Y-coordinate} = \frac{\begin{vmatrix} a & x' \\ b & y' \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$$

beautiful parts of this view

1) Seeing Co-ordinates as area

2) Seeing that the lines forming the area are transformed to new - coordinates

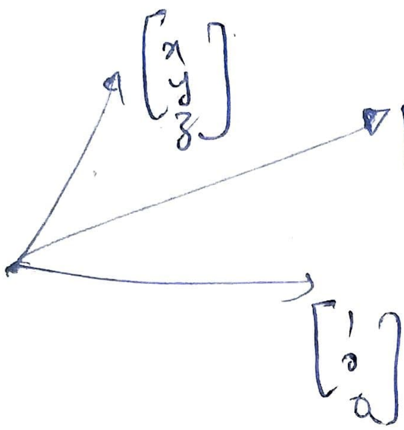
3.) Seeing that new area is the old area scaled (or squished) by determinant.

Q. Why take determinant?
Why Not Cross product's magnitude?

A. Bcoz determinant gives you signed area, but cross product's magnitude will always be +ve. Hence you may get the sign wrong (if after transformation orientation changes)

~~[illegible]~~

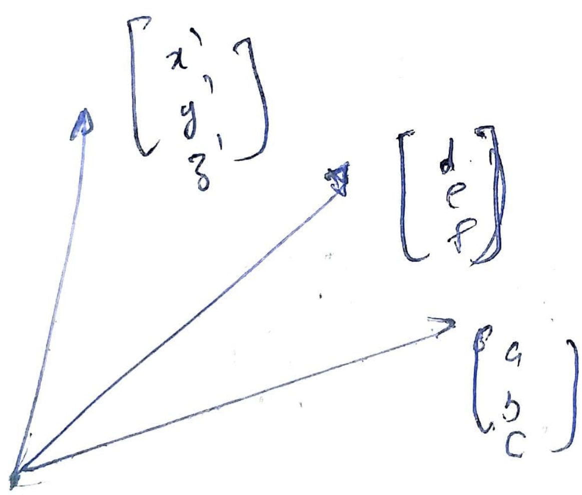
if u Consider 3D then its the volume of parallelopiped.



volume of this parallelopiped

$$= Z\text{-coordinate} \quad \begin{vmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{vmatrix}$$

A transformation $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$



volume of this parallelopiped is det(A) times old parallelopiped.

To Calculate volume of this parallelopiped

$$\begin{vmatrix} x' & d & g \\ y' & e & h \\ z' & f & i \end{vmatrix}$$

$$\begin{vmatrix} x' & d & g \\ y' & e & h \\ z' & f & i \end{vmatrix} = \begin{array}{c}\text{old} \\ \text{Area} \\ \text{or} \\ (Z\text{-coordinate})\end{array} \times \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$$