Abstract Vector Space

("Abstraction is price of generality")

Linear Transformations are not just operations on arrows in space, The idea can be abstracted and ~~other popular~~ it turns out the some of the regular operations (like differation) we have been doing are Linear transformation.

ex: Consider a vector space of all 3-degree polynomials.

Basis vectors would be $1, x, x^2, x^3$

All vectors in the vector space, are formed by Linear Combination of the above basis vector.

$p(x) = c_1(1) + c_2(x) + c_3x^2 + c_4x^3$

We can think of $\frac{d}{dx}$ → differentiation operation as Linear transformation, bcoz it obeys the property

$$L(a\vec{c} + b\vec{d}) = aL(\vec{c}) + bL(\vec{d})$$

a, b → scalars

$\vec{c}, \vec{d}$ → vectors

What would be the matrix representing this (differentiation) transformation

- This matrix can be generated by stacking columns side by side where each column is differentiation of a basis vector

$$\frac{d}{dx}(1) = 0 = 0(1) + 0(x) + 0(x^2) + 0(x^3) \rightarrow \text{Col 1}$$

$$\frac{d}{dx}(x) = 1 = 1(1) + 0(x) + 0(x^2) + 0(x^3) \rightarrow \text{col 2}$$

$$\frac{d}{dx}(x^2) = 2x = 0(1) + 2(x) + 0(x^2) + 0(x^3) \rightarrow \text{col 3}$$

$$\frac{d}{dx}(x^3) = 3x^2 = 0(1) + 0(x) + 3(x^2) + 0(x^3) \rightarrow \text{col 4}$$

$$\frac{d}{dx} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{d}{dx}(5x^3 + 3x^2 + 1) = 15x^2 + 6x + 0.$$

$$\Big\Vert \text{ equivalent to}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 15 \\ 0 \end{bmatrix}$$

$$0(1) + 6(x) + 15(x^2) + 0(x^3)$$

$$= 6x + 15x^2$$

→ Hence vector space idea is generalised as nice properties like above can be used

→ To whether or not decide - something is vector space or not, there are axioms but ultimately they boil down to "linear combination of any 2 vecs in vec space is also in vec space" provided basic props like Commutative hold (which are true)