Problems - Complex Matrices.

7.) Hermitan matrix

$$S^H = S$$

$$(cS)^H = c \cdot S^H = c \cdot S.$$

(c is a real scalar)

$$(iS)^H = -iS^H = -iS$$

$\Rightarrow$ iS is ~~skew symmetric~~ skew Hermitian

8) $$P = \begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} \qquad P^H = \begin{bmatrix} 0 & 0 & -i \\ -i & 0 & 0 \\ 0 & -i & 0 \end{bmatrix}$$

$$= i \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad = -i \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$PP^H = -i^2 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\det P = \pm 1$

P is unitary

$\Rightarrow$ invertible.

$P^2, P^3, P^{100} = ?$ $\qquad\qquad$ $P^H P = I$

$$P = i\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^2 = -1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= -1 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^3 = P^2 P = -i \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= -i \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -iI$$

$$P^{100} = P^{99}P = (P^3)^{33} P = (-iI)^{33} P$$

$$= -1 \times 1 \times i \times I \cdot P$$

$$= \underline{\underline{-iP}}$$

$\det P = \ldots [-1[-1]] = 1$

Trace $P = 0$

$\det|P - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow -\lambda(\lambda^2 - 0) - 1(0 - 1) = 0$$

$-\lambda(\lambda^2) - 1(-1) = 0$

$\lambda^3 = 1$

$\lambda = 1, \omega, \omega^2$

$$\begin{vmatrix} -\lambda & i & 0 \\ 0 & -\lambda & i \\ i & 0 & -\lambda \end{vmatrix} = 0$$

$-\lambda(\lambda^2) - i(-i^2) = 0$

$\lambda^3 = i^3$

$\lambda^3 = -i$

$e^{\left(i\frac{3\pi}{2}\right)\cdot\frac{1}{3}}, e^{i\left(\frac{3\pi}{2}+2\pi\right)\cdot\frac{1}{3}}, e^{i\left(\frac{3\pi}{2}+4\pi\right)\cdot\frac{1}{3}}$

$\frac{3\pi}{2} + 2\pi$ $\frac{7\pi}{2}$ $\frac{2}{3}+\frac{1}{2}$ $\frac{7}{6}$ $\lambda = (-1)^{1/3}$ $(i)^{1/3}$ $\frac{4}{3}+\frac{1}{2}$ $\frac{8+3}{6}$ λ $e^{i2\pi}$

$\frac{19}{6}$ $-\frac{\pi}{2}$ $= i(-1)$ $\frac{11}{2}$ $-1 \times i$ $e^{3\pi} \cdot e^{i\pi/2}$

$\frac{3\pi}{2}$ $= -1$ $\frac{7\pi}{2} \times \frac{1}{3}$ $\frac{3\pi}{2}$

$\left(\frac{3\pi}{2} + 2\pi\right)\frac{1}{3} = e^{-\frac{\pi}{2}+2\pi} \; i \times e^{1/3}$ $\qquad e^{i\left(-\frac{\pi}{2}\right)} = -i$

$i, i\omega, i\omega^2$ are answers

$i, \; i\omega, \; i\omega^2$

a) $Px = \lambda x$

$\lambda = i$

$$\begin{bmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$$

$$\begin{matrix} iy \\ iz \\ ix \end{matrix} = \begin{matrix} ix \\ iy \\ iz \end{matrix}$$

$x = y = z \Rightarrow (1,1,1) \Rightarrow \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\begin{bmatrix} iy \\ iz \\ ix \end{bmatrix} = \begin{bmatrix} i\omega x \\ i\omega y \\ i\omega z \end{bmatrix}$$

$y = \omega x, \quad z = \omega y, \quad x = \omega z$

$\frac{y}{x} = \frac{z}{y} = \frac{x}{z} = \omega \qquad \underline{(1, \omega, \omega^2)}$

$y = \omega^2 x, \quad z = \omega^2 y, \quad x = \omega^2 z$

$(1, \omega^2, \omega^4)$

q. cont. Eigen vecs of ~~any~~ unitary matrix are orthogonal

$$Ax = \lambda_1 x$$

$$\cancel{y^H A x = \lambda_1 y^H x}$$

eigen values have absolute value = 1 for a unitary matrix

$$\lambda \bar{\lambda} = 1$$

$$\bar{\lambda} = \frac{1}{\lambda}$$

$$Ay = \lambda_2 y$$

H

$$y^H A^H = \bar{\lambda}_2 y^H$$

$$y^H A^H x = \bar{\lambda}_2 y^H x$$

$$y^H A^{-1} x = \frac{1}{\lambda_2} y^H x$$

$$\frac{y^H x}{\lambda_1} = \frac{y^H x}{\lambda_2}$$

$$\because \lambda_1 \neq \lambda_2 \Rightarrow y^H x = 0$$

Diagnolising a matrix

13) $Z^H A^H A Z$

$(AZ)^H AZ = \|AZ\|^2 \longrightarrow$ Hence +ve

14) $S = \begin{bmatrix} 0 & 1-i \\ i+1 & 1 \end{bmatrix}$

eigen values

$(-\lambda)(1-\lambda) - (2) = 0$

$\lambda^2 - \lambda - 2 = 0$

$\lambda = \dfrac{1 \pm \sqrt{1 - 4(1)(-2)}}{2}$

$= 2, -1$

$$\begin{pmatrix} 0 & 1-i \\ i+1 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\begin{pmatrix} y(1-i) \\ i(x) + x+y \end{pmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$y(1-i) = 2x,$ $\quad i(x) + x = y.$

$y(1-i) = 2x.$ $\quad \cancel{x(x} \; x(i+1) = y.$

$$\frac{x}{1} = \frac{y}{i+1}$$

$$\begin{bmatrix} 1 \\ i+1 \end{bmatrix}$$

$$\text{length}^2 = [1 \;\; 1-i] \begin{pmatrix} 1 \\ i+1 \end{pmatrix}$$

$$= 1 + 2$$

$$\text{len} = \sqrt{3}$$

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

$$\begin{pmatrix} 0 & 1-i \\ i+1 & 1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

$$y(1-i) = -x$$

$$\frac{x}{1-i} = \frac{y}{-1}$$

$$\begin{pmatrix} 1-i \\ -1 \end{pmatrix}$$

$$len^2 = \begin{bmatrix} 1+i & -1 \end{bmatrix} \begin{pmatrix} 1-i \\ -1 \end{pmatrix} = 3$$

$len = \sqrt{3}$

$$\underline{\frac{1}{\sqrt{3}} \begin{pmatrix} 1-i \\ -1 \end{pmatrix}}.$$

Columns are orthonormal to eachother then we get a unitary matrix
$A^H A = I$

$$\underset{(J)}{E_{Basis}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$\underset{(J)}{E_{Basis}^{-1}} = E_{Bais}^{H} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}.$$

($\because$ they are orthonormal $\rightarrow A^H A = I \Rightarrow A^H = A^{-1}$.
(eigenvectors of Hermition matrix are orthogonal)

$J^{-1} S J$ = diagonal matrix.

$$\frac{1}{3} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} 0 & 1-i \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$\frac{1}{3} \begin{pmatrix} \cancel{2} & 2-2i \\ -1-i & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-i \\ 1+i & -1 \end{pmatrix}$$

$$\frac{1}{3} \begin{pmatrix} 6 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$2(1)$
$+(2-2i)(1+i)$

$2+2+2$
$\cancel{2i} - \cancel{2i}$

$-(2) \quad -1$
-3

$$J^{-1} S J = \text{diagonalmatrix}$$

$$S = J \, \text{dia} \, J^{-1}$$

$$= J \text{ diagonalmatrix } J^{-1}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

18)

$$U = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

K

$\rightarrow$ Columns are orthogonal (orthonormal)

we get a unitary Matrix. $U^H U = I$

$K \times n$ matrix $\qquad Z = K \times 1$ vector

$v_i = K \times 1$ vector $(i : 1 \rightarrow n)$

$$Z = \underset{n\times k}{\underline{U^H}} \; \underset{k\times n}{\underline{U}} \; \underset{k\times 1}{\underline{z}} \quad \times$$

$$= \underline{n \times k \quad k \times n} \quad k \times 1$$

$$= \underset{k\times n}{U} \; \underset{n\times k}{U^H} \; \underset{k\times 1}{z} \rightarrow k \times 1 \quad \checkmark$$

$$Z = U U^H Z$$

$$= \begin{bmatrix} | & | & | & & | \\ v_1 & v_2 & v_3 & \cdots & v_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} — & v_1^H & — \\ — & v_2^H & — \\ & \vdots & \\ — & v_n^H & — \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}$$

$K \times n$ $\quad$ $n \times k$ $\quad$ $k \times 1$

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} v_1^H \cdot z \\ v_2^H \cdot z \\ v_n^H \cdot z \end{bmatrix}$$

$K \times n$ $\quad$ $n \times 1$

observation that $\begin{bmatrix} x \\ y \end{bmatrix}$ vector transformation is linear combination of new Basis, which are columns of transformation matrix.

$$(v_1^H z) v_1 + (v_2^H z) v_2 \ldots (v_n^H z) v_n$$

$$(v_1^H z) v_1 + (v_2^H z) v_2 + \cdots (v_n^H z) v_n$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} x + 3y \\ 2x + 4y \end{bmatrix}$$

open up things & see How you can multiply

18)

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

Let v_i is $k\times 1$ dimension vector

A dim $k \times n$

$\because$ ~~dimensions of~~ columns of A are orthonormal

$\Rightarrow A$ is unitary matrix

$$A^H A = I.$$

Let 'z' be a $k\times 1$ vector

$$\underset{n\times k}{\;} Z = \underset{n\times k}{A^H}\,\underset{k\times n}{A}\,\underset{k\times 1}{Z} \quad \times$$

$$= \underbrace{\underset{k\times n}{A}\,\underset{n\times k}{A^H}}_{k\times k}\,\underset{k\times 1}{Z} = k\times 1$$

$$\underset{k\times n}{\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}} \underset{n\times k}{\begin{bmatrix} - & v_1^H & - \\ - & v_2^H & - \\ & \vdots & \\ - & v_n^H & - \end{bmatrix}} \underset{k\times 1}{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix}}$$

$v_1^H \to [1\times k]$, $v_2^H \to [1\times k]$, $v_n^H \to [1\times k]$

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \underset{n\times 1}{\begin{bmatrix} v_1^H z \\ v_2^H z \\ v_n^H z \end{bmatrix}}$$

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} v_1^H z \\ v_2^H z \\ \vdots \\ v_n^H z \end{bmatrix}$$

$k \times n \qquad n \times 1$

The above reminds of Applying a transformation A on vector x

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} =$$ Linear combination of individual columns being scaled

$$= x \begin{bmatrix} a \\ b \end{bmatrix} + y \begin{bmatrix} c \\ d \end{bmatrix}$$

$$= \begin{bmatrix} xa + cy \\ xb + yd \end{bmatrix}$$

Similarly in our case,

$$z = (v_1^H z) v_1 + (v_2^H z) v_2 \cdots (v_n^H z) v_n$$

25) $A + iB$ is unitary, then

$$Q = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \text{ is orthogonal}$$

$(A+iB)$ is unitary $\Rightarrow (A+iB)^H (A+iB) = I$

if Q is orthogonal

$$Q^H Q = I$$

$$Q^H = \begin{bmatrix} A^H & B^H \\ -B^H & A^H \end{bmatrix}$$

$$Q^H Q = \begin{bmatrix} A^H & B^H \\ -B^H & A^H \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

$$= \begin{bmatrix} A^H A + B^H B & -A^H B + B^H A \\ -B^H A + A^H B & -B^H B + A^H A \end{bmatrix}$$

$$(A+iB)^H (A+iB) = I$$

$$(A^H + -iB^H)(A+iB) = I$$

$$A^H A + B^H B + i(A^H B - B^H A) = I$$

$$A^H A + B^H B = I$$

$$B^H A - A^H B = 0$$

$$Q^H Q = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

26) $A + iB \to$ Hermitian, $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ symmetric.

$(A+iB)^H = A + iB$

$A^H - iB^H = A + iB.$

$B^H = -B$

B is real

$\Rightarrow B = 0.$

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}^T = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

$$= \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

$$= A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$= AI$

I is symmetric.

27) $S^H = S$

$I = (S^H)^{-1} S$

$S^{-1} = (S^H)^{-1}.$

Proof

- Can $(A^T)^{-1} = (A^{-1})^T$ be true

$(A^T)^{-1} A^T = (A^{-1})^T A^T$

$I = (A A^{-1})^T$

$I = I \rightarrow$ true.

---

proof for $(A^T)^{-1} = (A^{-1})^T$

$(A^{-1})^T = (A^{-1})^T I$

$= (A^{-1})^T A^T (A^T)^{-1}$

$= (A A^{-1})^T (A^T)^{-1}$

$= I^T (A^T)^{-1}$

$= I (A^T)^{-1}$

$= (A^T)^{-1}$

So, $S^{-1} = (S^H)^{-1}$ can be written as

$$S^{-1} = (S^{-1})^H.$$

S^{-1} is symmetric, infact hermitian

proof 2 → different/easy way to prove.

$$SS^{-1} = I$$

(one liner)

$$(S^{-1})^H S^H = I$$

$$(S^{-1})^H = (S^H)^{-1}$$

$$(S^{-1})^H = (S)^{-1} \Rightarrow \underline{(S^{-1})^H = S^{-1}}$$

28)

$$N = Q \Lambda Q^{-1}$$

Q has orthonormal vectors (bcoz N is Hermitian or unitary)

$\Rightarrow Q^H Q = I$

$N N^H =$

$$Q^{-1} = Q^H$$

$$N = Q \Lambda Q^H$$

$N N^H$

$= (Q \Lambda Q^H) \cdot (Q \Lambda Q^H)^H$

$= (Q \Lambda Q^H) \cdot (Q^H{}^H \Lambda^H Q^H)$

$Q^H Q = I$

$= Q \Lambda Q^H Q \Lambda^H Q^H$

$= Q \Lambda \Lambda^H Q^H$

$N^H \cdot N$

$=$ ~~$Q \Lambda (Q \Lambda)^H$~~

$=$ ~~$Q^H Q \Lambda (Q \Lambda)^H$~~

$N^H = (Q \Lambda Q^H)^H$

$= Q \Lambda^H Q^H$

$=$ ~~$Q \Lambda Q^H$~~

$=$

$= Q \Lambda^H \Lambda Q^H$

$= Q \Lambda^H I \Lambda Q^H$

$= Q \Lambda^H Q^H Q \Lambda Q^H$

$= (Q \Lambda Q^H)^H Q \Lambda Q^H$

$= N^H \cdot N$

realisation that $\Lambda^H \Lambda = \Lambda \Lambda^H$

Λ is a diagonal matrix

and hence multiplication would be commutative as diagonal elements remain in place and hence $x \bar{x}$ is ~~the~~ same as $\bar{x} x$.

Review of key ideas

Hermition matrix

→ $A^H = A$.

→ eigen values are real

→ eigen ~~vals~~ vectors are orthogonal

Stack orthonormal vectors (orthogonal unit vectors) Column wise and you get a matrix 'A' such that $A^H A = I$ or $A^H = A^{-1}$, it is called unitary matrix

Unitary Matrix

→ $A^H A = I$ or $A^H = A^{-1}$

→ eigen values have abs. value = 1

→ eigen vectors are othogonal.