Change of basis

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

it is possible to change the basis from $|0\rangle, |1\rangle$ to some other set

for ex, new computational basis can be

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$\downarrow$

denoted by $|+\rangle$, $|-\rangle$

$$|\psi\rangle = \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$$

$|\psi\rangle$ can be result $|+\rangle$ with a probability $\frac{|\alpha+\beta|^2}{2}$, can be $|-\rangle$ with a probability $\frac{|\alpha-\beta|^2}{2}$.

if new basis are orthonormal, it is possible to measure the probabilities.

like

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1/\sqrt{2} \end{pmatrix}$$

New basis has to be orthonormal it seems! (yet to be proved).

Proof. $\underbrace{\alpha|0\rangle + \beta|1\rangle}_{1\times 1}$ can be written as $\alpha \cdot \begin{bmatrix}1\\0\end{bmatrix}^T \begin{bmatrix}|0\rangle\\|1\rangle\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}^T\begin{bmatrix}|0\rangle\\|1\rangle\end{bmatrix}$

$$= \underbrace{\left[\alpha\begin{bmatrix}1\\0\end{bmatrix}^T + \beta\begin{bmatrix}0\\1\end{bmatrix}^T\right]}_{1\times 2} \cdot \underbrace{\begin{bmatrix}|0\rangle\\|1\rangle\end{bmatrix}}_{2\times 1}$$

Now the idea of change of basis is $\begin{bmatrix}1\\0\end{bmatrix}^T$ & $\begin{bmatrix}0\\1\end{bmatrix}^T$ can be replaced by other vector Transposes, say A^T, B^T. We need to prove $A^TB = 0$ (orthonormal)

Generalised rep. can be written as

$$\underbrace{\left[\alpha A^T + \beta B^T\right]}_{\text{this has to be of unit length (normalisation cond}^n\text{)}} \begin{bmatrix}|0\rangle\\|1\rangle\end{bmatrix}$$

$$\underbrace{\left[\alpha A^T + \beta B^T\right]}_{1\times 2} \cdot \underbrace{\left[\alpha A^T + \beta B^T\right]^T}_{2\times 1} = 1$$

$\Rightarrow (\alpha A^T + \beta B^T)(\alpha A + \beta B) = 1 \Rightarrow$ $\alpha^2 A^TA + \beta^2 B^TB + \alpha\beta(A^TB + B^TA) = 1$

$\alpha^2 + \beta^2 + \alpha\beta(A^TB + B^TA) = 1$

$(A^TA = 1,\ B^TB = 1)$ Cont..

CONT of orthonormal proof

$$\alpha^2 A^T A + \beta^2 B^T B + \alpha\beta[A^T B + B^T A] = 1.$$

We choose A, B s.t they are of unit length

(Why?, look at ~~Hence~~ $|0\rangle \to [1 \;\; 0]^T$, $|+\rangle \to [1/\sqrt{2} \;\; 1/\sqrt{2}]^T$)

$$\alpha^2 + \beta^2 + \alpha\beta[A^T B + B^T A] = 1.$$

when we choose Basis such as $[1 \;\; 0]^T \begin{bmatrix} |0\rangle \\ |1\rangle \end{bmatrix}$, $[0 \;\; 1]^T \begin{bmatrix} |0\rangle \\ |1\rangle \end{bmatrix}$

$[1/\sqrt{2} \;\; 1/\sqrt{2}] \begin{bmatrix} |0\rangle \\ |1\rangle \end{bmatrix}$,

Say B_1, B_2

we write $\alpha B_1 + \beta B_2$

which means getting B_1 with prob $|\alpha|^2$

getting B_2 with prob $|\beta|^2$

$$\alpha^2 + \beta^2 = 1$$

$$\Rightarrow \quad A^T B + B^T A = 0$$

$A^T B = 0 = B^T A \Rightarrow A, B$ are ortho normal