Quantum gates

intuitively

$|0\rangle \xrightarrow{NOT} |1\rangle$

$|1\rangle \xrightarrow{NOT} |0\rangle$

but this doesn't tell anything abt individual superposition

rather

$\alpha|0\rangle + \beta|1\rangle \xrightarrow{NOT} \alpha|1\rangle + \beta|0\rangle$

exchanging the amplitudes

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

NOT transformation matrix

interesting pt

this linear behaviour is general property of quantum mech

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}$$

Transformation matrix

New states

We shld not forget the new states shld also obey normalisation condition

$$|\alpha'|^2 + |\beta'|^2 = 1$$

[Vector wise $x^H x = 1$]

So there must be some limit on what kind of transformation matrix to use.

that is A must be unitary, let's prove ($A^H A = I$)

(2)

$Ax = Y$

↓ Transpose in complex world (H)

$x^H A^H = Y^H$ (Multiply X)

$x x^H A^H = x Y^H$

$$\boxed{A^H = x Y^H}$$

$Ax = Y$

↓ Multiply A^{-1}

$A^{-1} A x = A^{-1} Y$

$x = A^{-1} Y$

$x Y^H = A^{-1} Y Y^H$

$$\boxed{A^{-1} = x Y^H}$$

$$A^H = A^{-1} \Rightarrow \underline{A^H A = I}$$

(1) Things assumed,

$$\underset{\substack{\text{gate} \\ A}}{\begin{bmatrix} & \end{bmatrix}} \underset{x}{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}} = \underset{Y}{\begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}} \qquad Ax = Y$$

x, Y shld be normalised, that means $x^H x = Y^H Y = 1$

$x x^H = Y Y^H = I$

other popular gates

Z-gate $\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{Z-gate}} \alpha|0\rangle - \beta|1\rangle$

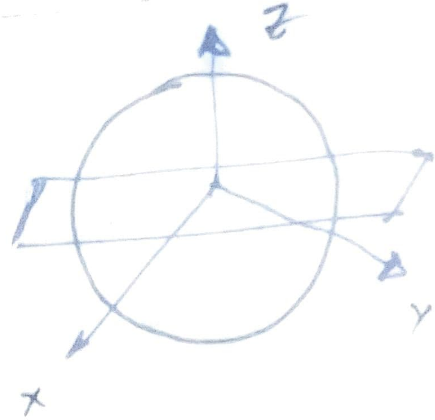
$$Z \text{ gate} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hadamard gate

$$\alpha|0\rangle + \beta|1\rangle \xrightarrow[\text{gate}]{\text{Hadamard}} \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}|1\rangle$$

$$H \text{ gate} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Visualising gates on Bloch sphere:



1. X - Gate.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \cancel{\beta|1\rangle + \alpha|0\rangle}.\ \beta|0\rangle + \alpha|1\rangle$$

Reflection abt x-y plane

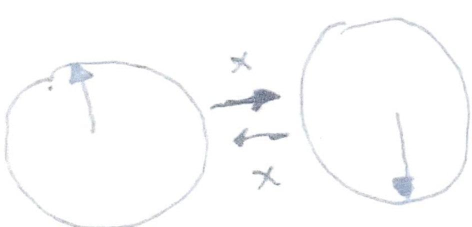
$|0\rangle \longrightarrow |1\rangle$

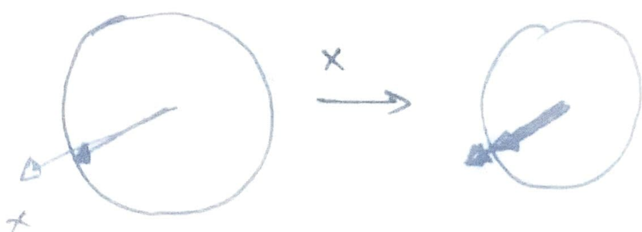
$|1\rangle \longrightarrow |0\rangle$.

$|0\rangle \to |1\rangle$

~~introducing~~

$|+\rangle$.

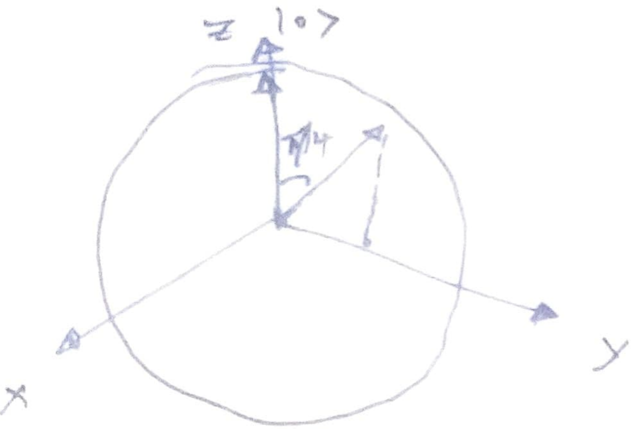




introducing coordinates

introducing complex states

$\theta = \frac{\pi}{4}, \quad \varphi = \frac{\pi}{2}$



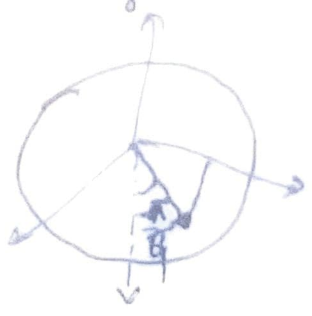
$|0\rangle \left(\cos\frac{\pi}{8}\right) + \left(i\sin\frac{\pi}{8}\right)|1\rangle \xrightarrow{x\text{-gate}} \left(i\sin\frac{\pi}{8}\right)|0\rangle + \left(\cos\frac{\pi}{8}\right)|1\rangle$

Multiply by i
(as multiplying by a num of norm = 1 doesn't change)

$-\sin\frac{\pi}{8}|0\rangle + i\cos\frac{\pi}{8}|1\rangle$

$\sin\left(-\frac{\pi}{8}\right)|0\rangle + i\cos\left(-\frac{\pi}{8}\right)|1\rangle$

$\theta = -\frac{\pi}{4}, \quad \varphi = \frac{\pi}{2}$

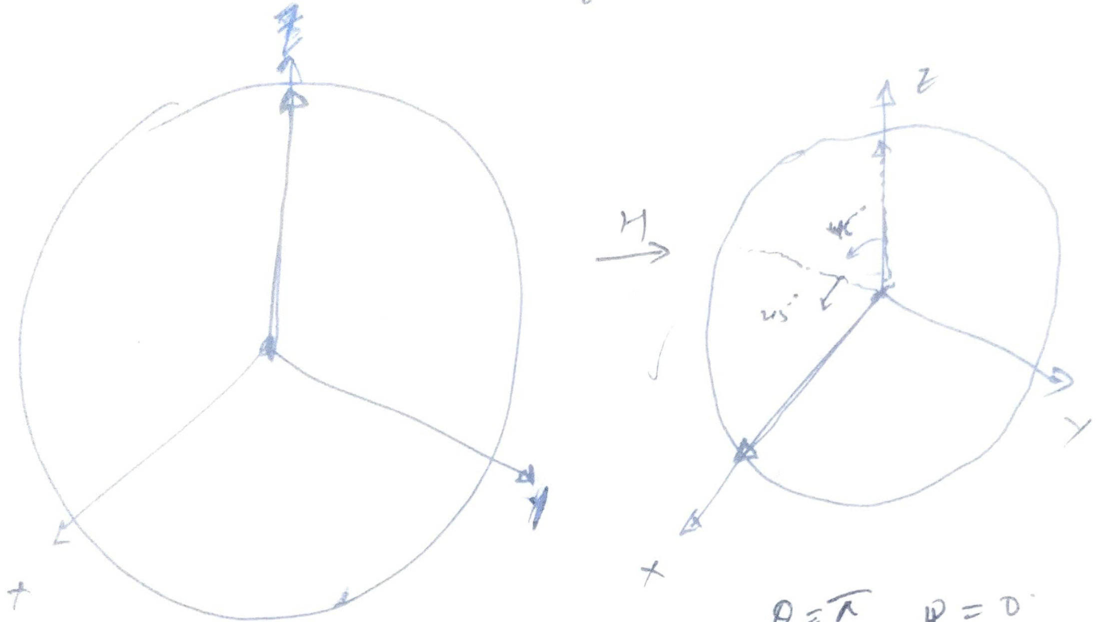


2. Hardamard gate

~~reflection~~ rotation ~~rotation~~ of 90° abt x-z diagonal plane

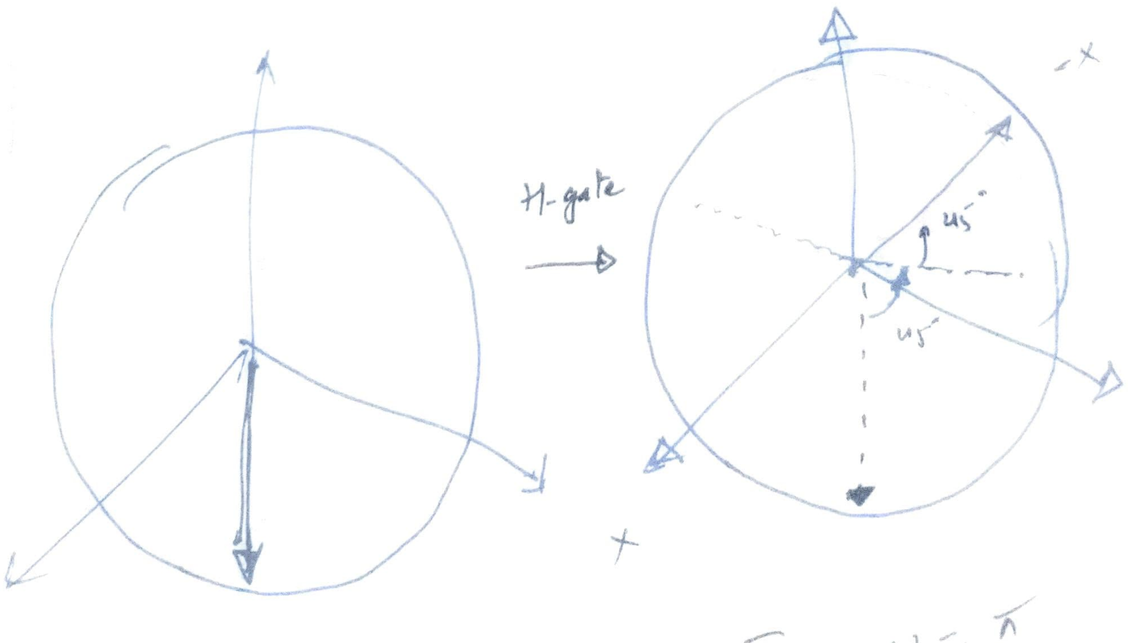
1 -

$|0\rangle \rightarrow \frac{|0\rangle + |1\rangle}{\sqrt{2}}$



$\theta = \frac{\pi}{2} \quad \varphi = 0$

$|1\rangle \longrightarrow \frac{|0\rangle - |1\rangle}{\sqrt{2}}$



$\theta = \frac{\pi}{2}, \quad \varphi = \pi$

Multiple Qubit gates

CNOT

$$|A, B\rangle \xrightarrow{CNOT} |A, A \text{ XOR } B\rangle$$

$$|00\rangle \rightarrow |0, 0 \text{ XOR } 0\rangle = |00\rangle$$

$$|01\rangle \rightarrow |0, 0 \text{ XOR } 1\rangle = |01\rangle$$

$$|10\rangle \rightarrow |1, 0 \text{ XOR } 1\rangle = |11\rangle$$

$$|11\rangle \rightarrow |1, 1 \text{ XOR } 1\rangle = |10\rangle$$

$$U_{CN} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$|00\rangle \quad |01\rangle \quad |10\rangle \quad |11\rangle$$

CNOT gate operation can be thought this way

if 1st bit is '0' keep the whole thing as it is

$$|00\rangle \rightarrow |00\rangle$$

$$|01\rangle \rightarrow |01\rangle$$

if the 1st bit is 1, then flip the 2nd bit (keeping 1st bit as it is)

$$|10\rangle \rightarrow |11\rangle$$

$$|11\rangle \rightarrow |10\rangle$$

Hence the 1st bit is given name - Control bit

2nd bit is given name - target bit.

Coming to vector operations:

2-qubit system amplitudes can be thought of as a vector say $\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$

where $|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$

A gate would mean applying a matrix transformation to transform the vector.

$$\underbrace{\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}}_{4\times4.} \underbrace{\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}}_{4\times1} = \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{4\times1}$$

Let's construct the CNOT gate ~~ma~~ matrix.

$$\begin{array}{c|cccc|c} & |00\rangle & |01\rangle & |10\rangle & |11\rangle & \\ \hline & 1 & 0 & 0 & 0 & |00\rangle \\ & 0 & 1 & 0 & 0 & \\ & 0 & 0 & 0 & 1 & \\ & 0 & 0 & 1 & 0 & \end{array}$$

remains as it is

flipped the second bit

$|10\rangle \rightarrow |11\rangle$

$|11\rangle \rightarrow |10\rangle.$

Why there is no equialent of AND or OR gate in quantum?

→ By applying such gates you loose info you can't back to the original states.

→ But here, you can inverse the Gate-Matrix & multiply with it.

Loss of information doesn't happen in Quantum computing? (atleast here!)