Homework 3

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Problem 1a.

Solution. Here's an adjacency matrix, A, of G:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Problem 1b.

Solution. Let the first row and column of the matrix A signify the vertex a. Likewise, let the third row and column signify the vertex b(such that the distance between a and b is 2). To find the number of walks of length 30, we can just find the $(1, 3)$ or $(3, 1)$ element of the matrix A^{30}. Using a computer, the matrix A^{30} is

$$\begin{bmatrix} 215492564 & 214146295 & 214978335 & 214978335 & 214146295 \\ 214146295 & 215492564 & 214146295 & 214978335 & 214978335 \\ 214978335 & 214146295 & 215492564 & 214146295 & 214978335 \\ 214978335 & 214978335 & 214146295 & 215492564 & 214146295 \\ 214146295 & 214978335 & 214978335 & 214146295 & 215492564 \end{bmatrix}$$

Thus, the number of walks of length 30 from a to b is $214\,978\,335$.

Problem 2.

Proof. Let T be the transition probability matrix of G. To prove that $\frac{\deg(v)}{2m}$ is a steady state, we need to show that

$$Tx = x, \quad x = \begin{bmatrix} \frac{\deg(v_1)}{2m} \\ \frac{\deg(v_2)}{2m} \\ \vdots \\ \frac{\deg(v_n)}{2m} \end{bmatrix}$$

Now, we know that

$$Tx = \begin{bmatrix} \sum_{k=1}^{n} T_{1,i} \cdot \frac{\deg(v_1)}{2m} \\ \sum_{k=1}^{n} T_{2,i} \cdot \frac{\deg(v_2)}{2m} \\ \vdots \\ \sum_{k=1}^{n} T_{n,i} \cdot \frac{\deg(v_n)}{2m} \end{bmatrix}$$

By construction, we also know that the (i, j)-element of T is equal to $\frac{1}{\deg(v_i)}$ if $i \neq j$ and v_i is connected to v_j. Let $f(u, v)$ signify that the vertices u and v are connected. Therefore, we can simplify Tx to

$$\begin{bmatrix} \sum_{k:f(v_1,v_k)} \frac{1}{\deg(v_1)} \cdot \frac{\deg(v_1)}{2m} \\ \sum_{k:f(v_2,v_k)} \frac{1}{\deg(v_2)} \cdot \frac{\deg(v_2)}{2m} \\ \vdots \\ \sum_{k:f(v_n,v_k)} \frac{1}{\deg(v_n)} \cdot \frac{\deg(v_n)}{2m} \end{bmatrix}$$

This can be written as

$$\begin{bmatrix} \sum_{k:f(v_1,v_k)} \frac{1}{2m} \\ \sum_{k:f(v_2,v_k)} \frac{1}{2m} \\ \vdots \\ \sum_{k:f(v_n,v_k)} \frac{1}{2m} \end{bmatrix}$$

Because there are $\deg(v_i)$ terms of the i^{th} sum, this equals

$$\begin{bmatrix} \frac{\deg(v_1)}{2m} \\ \frac{\deg(v_2)}{2m} \\ \vdots \\ \frac{\deg(v_n)}{2m} \end{bmatrix}$$

Thus, is equivalent to x and we are done.

□

Problem 3a.

Proof. We know that because G has 6 vertices and non-bipartite, it must contain an odd cycle. Notably, a 3-cycle (triangle) or a 5-cycle because G has only 6 vertices. Assume for the sake of contradiction that G doesn't contain a triangle. We can form G by starting with C_5 as it must contain the 5-cycle. However, to make G 3-regular, we must connect more edges. But there is no possible edge you can add between vertices that will not create a 3-cycle. Thus, a contradiction forms. Therefore G must contain a triangle. ☐

Problem 3b.

Proof. 3 edges because U needs at least 3 edges to make the triangle. U also has at most 3 edges connected to each of its vertices as a triangle is also the complete graph on 3 vertices. Therefore, there is exactly 3 edges.

Because each vertex has degree 3, and the vertices in U connect to two other vertices in U, each of those vertices must connect to exactly 1 other vertex in W. Therefore there is 3 edges that connect a vertex in U and in W, 1 for each vertex in U.

We have sorted that all the vertices in U are connected to two vertices in U and 1 in W. Now, let's say all of the vertices in U connected to the same vertex in W. Then it would be impossible for the other vertices in W to be degree 3, as they can only connect to each other and are max degree 2. Now, suppose that 2 vertices in U connected to the same vertex in W. Then, the vertex in W with no connection to U could only connect to the other two vertices in W, which would be max degree 2. Therefore, we have shown that each vertex in U connects to a unique vertex in W. Because of this, it follows that each vertex in W must connect to the other two vertices in W to be degree 3.

We know W forms a triangle because each of the three vertices in W is connected to two other vertices in W. This creates a 3-cycle/triangle. ☐

Problem 3c.

Proof. Let the smaller triangle in the drawing be U. Let the bigger triangle be W. We can see that the drawing follows all the same rules as G does. This includes, 3-regular, non-bipartite, and all the rules we showed in 3b. We can see the 3 edges that connect vertices in U which form a triangle. We can see the 3 edges that connect a vertex in U with a unique vertex in W (and vice-versa). We can see the 3 edges that connect vertices in W which form a triangle. $\square$