HONORS REAL ANALYSIS LECTURE 21

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Abstract. This lecture reviews material to prepare for the upcoming second midterm.

Theorem. *Let G be a connected d-regular non-Eulerian graph. If $\bar{G}$ is connected, then $\bar{G}$ is Eulerian.*

Proof. We know that Eulerian is equivalent to being connected and having all vertices of even degree. Since G is connected and d-regular, we only need to show that d must be even. We also know that we cannot have a odd-regular graph on odd vertices, so n must be even. Thus $\bar{G}$ has $n-1-d$ edges, therefore d must be even as the number of vertices cannot be odd. Thus, $\bar{G}$ is Eulerian. □

Theorem. *If $G - |S|$ has more than $|S|$ components, then G is not hamiltonian.*

Proof. If for all non-adjacent vertices u,v in G, $\deg(u) + \deg(v) \geq n$. Then, $\delta(G) \geq \frac{n}{2} \implies G$ is hamiltonian. □

Theorem. *There exists a graph with $\kappa(G) \geq n$ but G is not hamiltonian.*

Proof. Consider $\mathrm{K}_{n,n+1}$, $\kappa(G) = n$ but G is not hamiltonian. □

Theorem. *There exists a graph with $\kappa(G) \geq n$ but G does not contain a hamiltonian path.*

Proof. Consider $\mathrm{K}_{n,n+2}$ where $\kappa(G) = n$ but G does not contain a hamiltonian path. □

Theorem. *Every tree has at most 1 perfect matching.*

Proof. Recall that a perfect matching is a matching that uses every vertex. Suppose for the sake of contradiction that there exists at least 2 perfect matchings. Then, there must exist a vertex that is matched to 2 different vertices in the 2 perfect matchings. □

Theorem. *The number of spanning trees is equal to*

$$\frac{1}{n}\lambda_1\lambda_2\ldots\lambda_{n-1}$$

Example. *Consider the adjacency matrix below:*

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Theorem. *We know that $\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \lfloor \frac{2m}{n} \rfloor$ If G is a tree, then all terms are equal.*

Proof. If G is a tree, then $\kappa(G) = \lambda(G) = \delta(G) = n - 1$. □