Workshop 1

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Problem 1.a. Given $a_1, a_2, \ldots, a_n > 0$ all positive numbers.
Prove that $a_1^2 + a_2^2 \geq 2a_1a_2$

Proof. We begin by subtracting $2a_1a_2$ from both sides of the inequality to get $a_1^2 + a_2^2 - 2a_1a_2 \geq 0$. We can factor this to get $(a_1 - a_2)^2 \geq 0$. Since a_1 and a_2 are both positive, we know that $(a_1 - a_2)^2$ is also positive. Thus, we have proven that $a_1^2 + a_2^2 \geq 2a_1a_2$. ∎

Problem 1.b.1. Prove that $\frac{a_1}{a_2} + \frac{a_2}{a_1} \geq 2$

Proof. We begin by multiplying both sides of the inequality by a_1a_2 to get $a_1^2 + a_2^2 \geq 2a_1a_2$. This is identical to the inequality we proved in part (a), so we know that it is true. ∎

Problem 1.b.2. Prove that $(a_1 + a_2 + a_3)(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}) \geq 9$

Proof. We begin by expanding the left hand side.

$$3 + \frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_2}{a_1} + \frac{a_2}{a_3} + \frac{a_3}{a_1} + \frac{a_3}{a_2} \geq 9$$

We can simplify and group terms to get

$$(\frac{a_1}{a_2} + \frac{a_2}{a_1}) + (\frac{a_1}{a_3} + \frac{a_3}{a_1}) + (\frac{a_2}{a_3} + \frac{a_3}{a_2}) \geq 6$$

We can use the inequality we proved in part (1.b.1) to show that each of the terms on the left hand side is greater than or equal to 2. Thus, each of the terms on the left hand side is greater than or equal to 2, and the sum of the terms is greater than or equal to 6. Therefore, we have proven that $(a_1 + a_2 + a_3)(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}) \geq 9$. ∎

Problem 1.c. Prove that $(\Sigma_{i=1}^{n} a_i)(\Sigma_{j=1}^{n} \frac{1}{a_j}) \geq n^2$

Proof. We can rewrite the two sums as $\Sigma_{i=1}^{n}\Sigma_{j=1}^{n}\frac{a_i}{a_j}$. We can group two terms in the sum by rewriting it as follows:

$$n + \Sigma_{i=1}^{n}\Sigma_{j=i+1}^{n}\left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right)$$

The term of n appears for the excluded terms of the sum where $i = j$. We can use the inequality we proved in part (1.b.1) to show that each of the terms in the sum is greater than or equal to 2. Therefore,

$$n + \Sigma_{i=1}^{n}\Sigma_{j=i+1}^{n}\left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right) \geq n + \Sigma_{i=1}^{n}\left(\Sigma_{j=i+1}^{n} 2\right)$$

We can count the number of terms in the sum on the right hand side by counting the number of pairs of i and j such that $i < j$. This is, in fact, $\binom{n}{2}$, because, we need to choose two distinct a_k's. Because each term is 2, we multiply this by 2 to get $n(n-1)$. Therefore,

$$n + \Sigma_{i=1}^{n}\Sigma_{j=i+1}^{n}\left(\frac{a_i}{a_j} + \frac{a_j}{a_i}\right) \geq n + n(n-1) = n^2$$

And we are done. ■

Problem 3.c. Prove for any $k \in \mathbb{N}$, there exists N_3 such that $k^n \geq n!$.

Proof. Let $N_3 = 3k$. We proceed by induction on n. Let

$$S = \{n \mid k^n \leq n! \vee n < N_3\}$$

Our goal is to prove $S = \mathbb{N}$.

Base case

We first need to show $N_3 \in S$. We can rewrite the inequality as $k^{3k} \leq (3k)!$. On the LHS, we have 3k factors of k. On the RHS, we have $3k$ factors, $2k + 1$ of which is greater than or equal to k. Because there are double as many terms greater than k than less than k, we can reasonably assume that this statement is true.

Induction hypothesis

Now, we assume that $n \in S : n \geq N_3$ and will show that $n + 1 \in S$.

Induction step

From our assumption, we know

$$k^n \leq n!$$

Multiplying both sides by k gives us

$$k^{n+1} \leq k(n!)$$

We also know that, because $n > 3k$, then

$$k(n!) \leq (n+1)!$$

Thus, we have

$$k^{n+1} \leq (n+1)!$$

Therefore, $n + 1 \in S$.

Thus by the principle of mathematical induction, we have proven our theorem. ■

Note: Because we have proven this statement for all k, Problem 3.a and 3.b are also proven.