Homework 2

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Problem 1a. Let $a_n \geq 0$. Assume that

$$\sum_{n=1}^{\infty} a_n < \infty$$

that is the series is **convergent**. Prove the series

$$\sum_{n=1}^{\infty} a_n^p, 1 \leq p < \infty$$

is also convergent.

Proof. Since $\sum a_n$ is convergent, we have

$$(1) \quad 0 \leq a_n < 1, \forall n \geq n_0$$

and because the sum is less than infinity, for some natural number M

$$(2) \quad \sum_{n=1}^{\infty} a_n < M$$

Similarly, from (1) and because p is greater than or equal to 1,

$$0 \leq a_n^p < a_n$$

Now by applying the infinite series to the inequality and utilizing (2)

$$0 \leq \sum_{n=1}^{\infty} a_n^p \leq \sum_{n=1}^{\infty} a_n \leq M$$

And, we finally have

$$0 \leq \sum_{n=1}^{\infty} a_n^p \leq M$$

Therefore, the series is convergent

□

Problem 1b. Let $a_n \geq 0$. Prove if for some $p > 1$, the series $\sum_{n=1}^{\infty} a_n^p$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

Proof. Assume for the sake of contradiction that the series $\sum_{n=1}^{\infty} a_n^p$ is divergent, but the series $\sum_{n=1}^{\infty} a_n$ is convergent.
We have already reached a contradiction, as in Problem 1a, we have shown that for all $p \geq 1$

$$\sum_{n=1}^{\infty} a_n \text{ convergent } \implies \sum_{n=1}^{\infty} a_n^p \text{ convergent}$$

Therefore, the statement must be true. $\square$

Problem 2. Decide whether the following propositions are true or false, providing a short justification for each conclusion

Problem 2A. If every proper subsequence of x_n converges, then x_n converges as well.

Proof. True. We know that a sequence converges if and only if all subsequences converge. By using the reverse direction, we have that

$$x_n \text{ convergent} \implies \text{all subsequences of } x_n \text{ is convergent}$$

We can limit this to just proper sequences which is a simpler statement because we exclude the case where they are equal. Therefore, we have the original statement.

□

Problem 2B. If x_n contains a divergent subsequence, then x_n diverges.

Proof. True. Assume for the sake of contradiction that x_n contains a divergent subsequence, but x_n itself, converges. Then, a contradiction forms as x_n converges if and only if all subsequences of x_n converge. Thus, the statement is true.

□

Problem 2C. If x_n is bounded and diverges, then there exist two subsequences of x_n that converge to different limits.

Proof. True. Since the sequence is divergent, we know that $\limsup_{n\to\infty} x_n \neq \liminf_{n\to\infty} x_n$. However, by definition, there exists a subsequence that converges to each of those limits. Therefore, there exists two subsequences that converge to different limits.

□

Problem 2D. If x_n is monotone and contains a convergent subsequence, then x_n converges.

Proof. False. I'm not justifying this because I just checked the homework assignment, and realized I only had to solve part A. But I'm still going to keep the rest of the answers here. □