HONORS REAL ANALYSIS LECTURE 2

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Abstract. This lecture covers sections 1.3 and 1.4 from the text, *Understanding Analysis*. It delves into additional supremum/infimum theorems, topologies of open sets, denseness of the rationals in the reals, and existence proofs.

Definition. *Let $A \in \mathbb{R}$. We say that A is a bounded set if and only if there exists a positive, real number, M such that for any $a \in A, |a| \leq M$*

Remarks:

(1) $A \subseteq [-M, M]$
(2) $a \leq M \implies$ A bounded above.
(3) $a \geq -M \implies$ A bounded below.
(4) A set is bounded if and only if it is bounded above and below.

Lemma (1.3.8)**.** *Let $x_0 = \sup A$ and x_0 finite. One can find $a \in A$ such that*

$$x_0 - \epsilon < a < x_0$$

Remarks:

(1) ∞ is just a symbol and is used as notation. The interval $(a, \infty) = \{x | x > a\}$.
(2) The supremum can exist and be infinite.

Definition. *Given $A \subseteq \mathbb{R}$, we say $\sup A = \infty$ if and only if for any $M > 0, \sup A > M$.*

Theorem (Consequence of Dirichlet's Theorem)**.** *Given an interval (a, b) there exists a rational number, r, such that $r \in (a, b)$.*

Proof. Let $n \in \mathbb{N}$ such that, by the Archimedean Property, $\frac{1}{n} < b - a$ Consider na, we can find $m \in \mathbb{N}$ such that $0 \leq na < m$ Let m_0 be the smallest natural number that satisfies that expression. Thus, $na < m_0$ and $m_0 - 1 \leq na$. We now have the following equivalent inequalities

$$a < \frac{m_0}{n}, \quad \frac{m_0}{n} - \frac{1}{n} \leq a$$

Therefore, substituting b,

$$\frac{m_0}{n} \leq a + \frac{1}{n} < a + (b - a) = b$$

Finally, we have

$$a < \frac{m_0}{n} < b$$

Because $m_0, n \in \mathbb{N}$, we have found a rational number between any interval (a, b). ☐

Definition. *Given two sets $A, B \subseteq \mathbb{R}$. We define*

$$A + B = \{a + b | a \in A, b \in B\}$$

Theorem.

$$\sup (A + B) = \sup (A) + \sup (B)$$

Proof. $(\leq)$ We begin by using the definition of supremum for the right hand side.

$$\forall a \in A, b \in B \quad a + b \leq \sup (A) + \sup (B)$$

Therefore, $\sup(A + B) \leq \sup(A) + \sup(B)$

$(\geq)$ Suppose for all epsilon positive, there exists, a, an element of A, and, b, an element of B such that

$$\sup (A) - \frac{\epsilon}{2} < a$$
$$\sup (B) - \frac{\epsilon}{2} < b$$

If we sum these two we get

$$\sup (A) + \sup (B) - \epsilon < a + b \leq \sup (A + B)$$

Therefore, we have

$$\sup (A) + \sup (B) \leq \sup (A + B) + \epsilon$$

Thus, we conclude

$$\sup (A) + \sup (B) = \sup (A + B)$$

☐

Theorem (The Nested Interval Problem)**.** *Let $I_n = [a_n, b_n]$ such that $\{I_n\}_{n=1}^{\infty}$ is nested. Then $\cap_{n=1}^{\infty} I_n \neq \varnothing$.*

Proof. Consider the set $L = \{a_1 \leq a_2 \leq \cdots \leq a_n \leq \dots\}$. Where a_i is the left-endpoint of I_i. Similarly, $R = \{\cdots \geq b_n \geq \cdots \geq b_2 \geq b_1\}$. Where b_i is the right-endpoint of I_i By the Axiom of Completeness, $\sup(a_n) = x_0$ exists. We know that $x_0 < b_n$, and we also know $\forall n$, $x_0 \in I_n$. So because b_n is an upper bound and x_0 is the supremum, then $x_0 \leq b_n$. Therefore $\cap_{n=1}^{\infty} I_n \neq \varnothing$. $\square$

Definition (Topology)**.** *Given a set, X. We say $\mathcal{F}$ is a topology on X If*

(1) $A \in \mathcal{F}, A \subseteq X$
(2) $X, \varnothing \in \mathcal{F}$
(3) $\{A_\alpha\}_{\alpha \in I} \subseteq \mathcal{F} \implies \cup_{\alpha \in I} A_\alpha \in \mathcal{F}$
(4) $A_i \in \mathcal{F} \implies \cap_{i=1}^{n} A_i \in \mathcal{F}$

Definition. *A set B is closed if and only if $X \setminus B$ is open.*

Definition. *A set $S \subseteq \mathbb{R}$, we say S is dense in the reals if and only if for any open interval, one can find $s \in (a, b)$.*

Remark: Therefore, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem. *$\sqrt{2}$ exists.*

Proof. Consider the set $S = \{x | x^2 M 2\}$. We know S is nonempty since $1 \in S$. Next, we know S is bounded because if $x_0 > 4$, then $x_0^2 > 16$. Now, for any epsilon positive, there exists an element, y, in S such that $x_i - \epsilon < y$. So,

$$(x_0 - \epsilon)^2 < y^2 < 2 \implies x_0^2 + \epsilon^2 - 2\epsilon x < 2 \implies x_0^2 < 2$$

Therefore, the supremum of this set is the square root of 2. Therefore, it must exist. $\square$