Homework 1

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Problem 1.a. Given $A, B \subseteq \mathbb{R}$, prove that if $\inf(A), \inf(B)$ finite, then $\inf(A+B) = \inf(A) + \inf(B)$.

Solution. Assuming the sets A and B nonempty, we know that $\inf(A)+\inf(B) \leq a+b \ \forall a \in A, b \in B$. Thus, $\inf(A)+\inf(B) \leq \inf(A+B)$.

Now, for all $\epsilon > 0$, there exists an $a \in A$ and $b \in B$ such that

$$a < \inf(A) + \frac{\epsilon}{2}$$

$$b < \inf(B) + \frac{\epsilon}{2}$$

Adding these together yields

$$a + b < \inf(A) + \inf(B) + \epsilon$$

We also know that,

$$\inf(A + B) \leq a + b < \inf(A) + \inf(B) + \epsilon$$

For all $\epsilon > 0$, we have

$$\inf(A + B) - \epsilon \leq \inf(A) + \inf(B)$$

We conclude,

$$\inf(A) + \inf(B) = \inf(A + B)$$

Problem 1.b.1. Prove that (1.a) is true if $\inf(A)$ is infinite and $\inf(B)$ is finite.

Proof. Assume for the sake of contradiction that $\inf(A+B)$ is finite. By definition, we know that $\inf(A+B) \leq a+b$ for all $a \in A$ and $b \in B$. We know that

$$\inf(A+B) \leq a + \inf(B) \quad \forall a \in A$$

$$\inf(A+B) - \inf(B) \leq a \quad \forall a \in A$$

Recall that $\inf(A)$ is negative infinity, so there exists an $a \in A$ such that $a < M$ for any real number, M. Setting $M = \inf(A+B) - \inf(B)$, a contradiction occurs. Thus, $\inf(A+B)$ is infinite. ■

Problem 1.b.2. Prove that (1.a) is true if $\inf(A)$ and $\inf(B)$ is infinite.

Proof. Assume for the sake of contradiction that $\inf(A+B)$ is finite. By definition, we know that $\inf(A+B) \leq a+b$ for all $a \in A$ and $b \in B$. Similarly, we have

$$\inf(A+B) - b \leq a \quad \forall a \in A, b \in B$$

We know that because $\inf(A)$ is negative infinity, there exists an $a \in A$ such that $a < M$ for any real number, M. We can set $M = \inf(A+B) - b$. Now, we have that

$$M \leq a$$

And thus, a contradiction occurs. Therefore, $\inf(A+B)$ is infinite. ■

Problem 2. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a, b]$. Show $\sup(T) = b$.

Proof. If b is rational, then the proof is trivial. If b is irrational, we proceed as follows. We know b is an upper bound for T because

$$\forall x \in \mathbb{Q} \cap [a, b], x \leq b$$

Now, to prove that $b = \sup(T)$, we need only show that

$$\forall \epsilon > 0, \exists r \in T : b - \epsilon < r$$

From the density of the rationals in the reals we know that there exists a rational number, r_0, in the interval $[a, b]$. We can set $a = b - \epsilon$ for all epsilon positive to get

$$b - \epsilon < r_0 < b$$

Thus, we have shown that b is the least upper bound. ■