Workshop 2

Problem 1 is mandatory for all. Select one of the problems from 2 and 3 and write complete solutions for that problem. You are not supposed to work by yourself, but in groups of suggested size three people. Write ups of the problem should be an independent effort.

1. (5) (a) Prove for any $a, b \in \mathbf{R}$

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}.$$

(b) Prove for any $a_i,\ b_i \in \mathbf{R}$

$$\Big|\sum_{i=1}^{n} a_i b_i\Big| \leq \Big(\sum_{i=1}^{n} a_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{n} b_i^2\Big)^{\frac{1}{2}}.$$

Your strategy should be similar to Problem 2, workshop 1 where the problem was converted to the study of unit vectors. That is consider

$$c_i = \frac{a_i}{\big(\sum_{i=1}^{n} a_i^2\big)^{\frac{1}{2}}},\quad d_i = \frac{b_i}{\big(\sum_{i=1}^{n} b_i^2\big)^{\frac{1}{2}}}.$$

Apply part (a) to some form of $c_i d_i$ (you may have to consider $|c_i d_i|$) and conclude the inequality.

(c) Prove for any $a_i \in \mathbf{R}$,

$$\Big|\sum_{i=1}^{n} a_i\Big| \leq \sqrt{n}\Big(\sum_{i=1}^{n} a_i^2\Big)^{\frac{1}{2}}.$$

Notice that the inequality in (c) goes opposite to that obtained in Problem 2 in workshop 1. However now we pay a price as the inequality has a constant $\sqrt{n}$ and so depends on the number of terms in the sum. In workshop 1 the inequality had no such constant and the constant in the inequality did not depend on how many terms were present in the sum.

2.(5) Prove for any $x, y \in \mathbf{R}$ we have

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$

3.(5) Given the function

$$f(x) = \frac{x^3 + 3x - 9}{x^4 + 5x^2 + 2}.$$

(a) Establish rigorously that for **any** $x \in [0,1]$ one can find $0 < u \leq v$ such that

$$u \leq |f(x)| \leq v.$$

What is the largest value of u and smallest value of v that your proof allows? Your proof should not be based on any numerical calculator driven scheme and should not use any concept like derivatives or maxima, minima since we do not know these concepts yet. Your proof should solely use properties of absolute values and the triangle inequality.

(b) Find $0 < u \leq v$ for $x \in [10, 11]$. That is repeat part (a) with $x \in [10, 11]$.