HONORS REAL ANALYSIS LECTURE 1

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Abstract. This lecture covers a review of Chapter 1 and 2 to prepare for the upcoming first midterm.

Proposition. *Except for finitely many,* $a_n \leq a + \epsilon, \forall n \geq N$.

Proof. Assume there are infinitely many a_n, such that $a_n > a + \epsilon$. So we have $a_{n_1}, a_{n_2}, \dots, a_{n_k}, \dots$ such that (1): $a_{n_k} > a + \epsilon$ We proceed by cases: **Case 1**: $a_{n_k} \leq M$
$a_{n_k} > a + \epsilon$ is given By Bolzano-Weierstrass, there exists

$$\lim_{j \to \infty} a_{n_k} = a_0$$

Using (1) and order theorem,

$$\lim_{j \to \infty} a_{n_k} = a_0 \geq a + \epsilon$$

Now $a_0 \in S$

$$a = \sup S \geq a_0 \geq a + \epsilon$$

Thus, a contradiction forms as epsilon must be positive. $\square$

Proposition.

$$\limsup_{n \to \infty}(a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$$

This is an equality if both sequences converge. For example:

$$a_n = (-1)^n, b_n = (-1)^{n+1}$$

$$a_n + b_n = 0 + 0 + \dots$$

$$\limsup_{n \to \infty}(a_n + b_n) = 0$$

$$0 \leq 1 + 1 = 2$$

Proof. We proceed by cases.
Case 1: Assume $a = \limsup_{n \to \infty} a_n, b = \limsup_{n \to \infty} b_n$ and a, b both finite.
We recall, for every $\epsilon > 0$

$$a_n \leq a + \epsilon, \forall n \geq N_1$$

$$b_n \leq b + \epsilon, \forall n \geq N_2$$

So for $n \geq \max N_1, N_2 = N$

$$a_n + b_n \leq a + b + 2\epsilon, \forall n \geq N$$

So for any subsequence

$$a_{n_k} + b_{n_k} \leq a + b + 2\epsilon, \forall k \geq k_0$$

Thus,

$$\lim_{k\to\infty} (a_{n_k} + b_{n_k} \leq a + b + 2\epsilon)$$

via order theorem, we have

$$\sup S \leq a + b + 2\epsilon$$

$$\lim_{n\to\infty} (a_n + b_n) \leq a + b + 2\epsilon, \forall \epsilon > 0$$

Case 2: Given any $M > 0$,

$$a_{n_k} > M, \forall k \geq k_0 \iff \limsup_{n\to\infty} a_n = +\infty$$

Assume $b_n \leq M_1$. Then, note that

$$a_{n_k} + b_{n_k} \geq M - M_1 \geq \frac{M}{2}$$

If M is very large, $M \geq 2M_1$. Thus,

$$a_{n_k} + b_{n_k} > \frac{M}{2}, \forall k \geq k_0$$

And thus,

$$\limsup(a_n + b_n) = \infty$$

Case 3: Given any $M > 0$,

$$a_{n_k}, b_{n_k} > M, \forall k \geq k_0 \iff \limsup_{n\to\infty} a_n = +\infty, \limsup_{n\to\infty} b_n = +\infty$$

The proof is left as an exercise to the reader.

□

Theorem. *Let $S \subseteq \mathbb{N}$. Then S is countable*

Proof. Use the well-ordering theorem. There exists $x_{i_1} \in S$, a least element. Note S has no repetition. Now consider $S \setminus x_{i_1}$. Then, there exists a least element, $x_{i_2} \in S \setminus x_{i_1}$. And continue to form $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}, \ldots\}$. Construct $f : \mathbb{N} \to S$. f is injective as we threw repetitions out, therefore each input has a unique output. f is surjective as we can achieve every natural number by looking at the index of i. Likewise, we can get every index of i with every natural number. Thus, we have a bijective function that maps the naturals to S. Therefore, S is countable.

□

Proposition. *This series diverges:*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{6n^2+10n}+\sqrt{n^2+3}}$$

Proof.

$$\frac{1}{\sqrt{6n^2+10n}+\sqrt{n^2+3}} \geq \frac{1}{\sqrt{6n^2+10n^2}+\sqrt{n^2+3n^2}}$$

□

Tests we can use to determine convergence and divergence for series:

(1) Comparison Test
(2) Cauchy Condensation Theorem ($\sum r^k a_{r^k}$)
(3) Ratio Test (Utilize comparison test with a geometric series)
(4) Root Test (Utilize comparison test with a geometric series)
(5) Partial Summation $\rightarrow$ Dirichlet's Test $\rightarrow$ Alternating Series Test.

How to solve for $\sum \sin(kx)$:

$$\sum_{k=0}^{n} e^{ikx} = \left[\sum \cos(kx)\right] + \left[i \sum \sin(kx)\right]$$

$$\frac{1-e^{i(n+1)x}}{1-e^{ix}}$$