Workshop 1

You are allowed to work in groups and the suggested size is 3 people in a group. Please submit your work as a PDF file which is carefully scanned and legible.

Select **2** out of the **3** problems below and write out a solution for the problems you have selected. A challenge problem is also listed for you to work on but it is **not** for turn in. Each problem is 5 points.

1. Given $a_1, a_2, \cdots a_n > 0$ all positive numbers. Prove

(a) $a_1^2 + a_2^2 \geq 2a_1a_2$.

(b) Prove

$$\frac{a_1}{a_2} + \frac{a_2}{a_1} \geq 2.$$

Prove

$$(a_1 + a_2 + a_3)(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}) \geq 9.$$

(c) Prove the general inequality

$$(\sum_{i=1}^{n} a_i)(\sum_{i=1}^{n} \frac{1}{a_i}) \geq n^2.$$

2. Assume $0 < p < q$. Let $a_i \geq 0$ for all $i = 1, 2, \cdots, n$. Prove

$$(\sum_{i=1}^{n} a_i^q)^{\frac{1}{q}} \leq (\sum_{i=1}^{n} a_i^p)^{\frac{1}{p}}.$$

Hint: First observe that for any b_i such that $0 \leq b_i \leq 1$ and any $\alpha > 0$ one has

$$b_i^\alpha \leq 1. \tag{1}$$

Now consider

$$b_i = \frac{a_i}{(\sum_{i=1}^{n} a_i^p)^{\frac{1}{p}}}.$$

This is similar to making a unit vector out of any vector $\mathbf{v}$ by considering $\frac{\mathbf{v}}{||\mathbf{v}||}$. Now by a careful choice of α which should depend on p, q apply (1) to b_i^α and obtain the desired conclusion.

This is a very important inequality in Analysis. Notice that the inequality does not depend on how many terms n are used.

3. Consider the factorial function $n!$. We wish to show $n!$ grows very fast.

(a) Prove $3^n \leq n!$ for $n \geq N_1$.

(b) Prove $11^n \leq n!$ for $n \geq N_2$.

(c) Prove for **any** $k \in N$, there exists N_3 such that

$$k^n \leq n!, \;\; n \geq N_3.$$

Thus $n!$ grows faster than any polynomial for large enough n.

4. **Challenge Problem:** We proved Euclid's theorem in class that there are infinitely many primes.

Show that the proof actually yields the **quantitative statement**, given the first k primes $p_1, p_2, \cdots p_k$, then the next prime in the list p_{k+1} satisfies

$$p_{k+1} \leq p_1 p_2 \cdots p_k + 1.$$

Take the logarithm of the inequality above to get

$$\ln p_{k+1} \leq \sum_{i=1}^{k} \ln p_i + \ln 2.$$

Iterate this inequality by repeatedly applying it to conclude that

$$\ln p_{k+1} \leq C 2^k,$$

where C is some absolute number maybe 10 say. The number C should not depend on k.

Conclude

$$p_{k+1} \leq \exp(C 2^k).$$

This is a very poor size estimate but illustrates that qualitative statements like done in class can sometimes be converted to a size and growth rate.