Workshop 3

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Problem 1. Let γ be a rational number. Compute

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\gamma}$$

Proof. Because the limit variable is only under the exponent, we can rewrite the limit as

$$\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right)^{\gamma}$$

We know that by the archimedean property for all epsilon positive, there exists a natural number N such that

$$\frac{1}{n}<\epsilon, \forall n \geq N$$

Adding absolute value and some terms, we get

$$\left|\left(\frac{1}{n}\right)+1-1\right|<\epsilon, \forall n \geq N$$

$$\left|\left(1+\frac{1}{n}\right)-1\right|<\epsilon, \forall n \geq N$$

And we have reached the limit definition for the original sequence. Therefore, the limit is 1. $\square$

Problem 3. The Fibonacci numbers are given by the natural number sequence $1, 2, 3, 5, 8, \ldots$. The generating recursive relation for the n-th Fibonacci number F_n is given by the formula

$$F_n = F_{n-1} + F_{n-2}, n \geq 3$$

That is each natural number is obtained by summing the previous two natural numbers in the sequence. Form the ratio sequence

$$x_n = \frac{F_n}{F_{n-1}} \geq 1$$

Problem 3a. Prove the recursion relation for the ratio sequence is given by (where $x_1 = 2$)

$$x_n = 1 + \frac{1}{x_{n-1}}, n \geq 2$$

Proof. We begin with the definition of x_n. We have

$$x_n = \frac{F_n}{F_{n-1}}$$

Using the recursive relation for the Fibonacci numbers, we can rewrite this as

$$x_n = \frac{F_{n-1} + F_{n-2}}{F_{n-1}}$$

This simplifies to

$$x_n = 1 + \frac{F_{n-2}}{F_{n-1}}$$

And finally, we can substitute x_{n-1} for F_{n-1}/F_{n-2} to get

$$x_n = 1 + (x_{n-1})^{-1}$$

and we are done. ☐

Problem 3b. Prove we have

$$\frac{6}{5} \leq x_n < 5, n \geq 1$$

Proof. We proceed by induction. We have the base case $x_1 = 2$. We can see that

$$\frac{6}{5} \leq 2 < 5$$

Now, we assume that for some n, we have

$$\frac{6}{5} \leq x_n < 5$$

And use this to prove the following case. We have

$$\frac{6}{5} \leq x_n < 5$$

We know that $\frac{6}{5} \leq x_n < 5$, so we can see

$$\frac{5}{6} \geq \frac{1}{x_n} > \frac{1}{5}$$

Then we can add 1 to each side to get

$$\frac{11}{6} \geq 1 + \frac{1}{x_n} > \frac{6}{5}$$

We know that $x_{n+1} = 1 + \frac{1}{x_n}$, so we can substitute to get

$$\frac{11}{6} \geq x_{n+1} > \frac{6}{5}$$

By extension,

$$5 > x_{n+1} \geq \frac{6}{5}$$

Therefore, by principle of mathematical induction, we have proven the inequality for all $n \geq 1$. ☐

Problem 3c. Prove,

$$|x_n - x_{n-1}| \leq \left(\frac{5}{6}\right)^2 |x_{n-1} - x_{n-2}|, n \geq 4$$

Proof. Using the recursive relation for x_n, we can rewrite the left hand side as

$$\left|1 + \frac{1}{x_{n-1}} - \left(1 + \frac{1}{x_{n-2}}\right)\right|$$

This simplifies to

$$\left|\frac{1}{x_{n-1}} - \frac{1}{x_{n-2}}\right|$$

We can combine the left hand side to get

$$\left|\frac{x_{n-2} - x_{n-1}}{x_{n-1}x_{n-2}}\right|$$

Taking out the denominator, we get

$$\frac{1}{x_{n-1}x_{n-2}} |x_{n-2} - x_{n-1}|$$

From where we started, we know this equals

$$|x_n - x_{n-1}| = \frac{1}{x_{n-1}x_{n-2}} |x_{n-2} - x_{n-1}|$$

Maximizing the fraction using the inequality from 3b, we get this inequality

$$|x_n - x_{n-1}| \leq \frac{1}{\frac{6}{5} \cdot \frac{6}{5}} |x_{n-1} - x_{n-2}|$$

Simplifying, we get

$$|x_n - x_{n-1}| \leq \left(\frac{5}{6}\right)^2 |x_{n-1} - x_{n-2}|$$

And we are done. $\square$

Problem 3d. Prove,

$$|x_n - x_{n-1}| \leq \alpha^{n-5}|x_5 - x_4|, n \geq 7, \alpha = \left(\frac{5}{6}\right)^2$$

Proof. We proceed by induction on n. Let

$$S = \{n \mid |x_n - x_{n-1}| \leq \alpha^{n-5}|x_5 - x_4| \vee n < 7\}$$

Base case

We first need to show that when $n = 7$, the inequality holds. We have

$$|x_7 - x_6| \leq \alpha^{7-5}|x_5 - x_4|$$

We know that $|x_7 - x_6| \leq \alpha^2|x_6 - x_5|$ from 3c. We can substitute to get $|x_7 - x_6| \leq \alpha|x_6 - x_5| \leq \alpha^2|x_5 - x_4|$ And we are done.

Induction hypothesis

Assume that for some n, we have $|x_n - x_{n-1}| \leq \alpha^{n-5}|x_5 - x_4|$

Induction step

We need to show that for $n + 1$, the inequality holds. We have

$$|x_{n+1} - x_n| \leq \alpha^{n-4}|x_5 - x_4|$$

We know that $|x_{n+1} - x_n| \leq \alpha|x_n - x_{n-1}|$ from 3c. We can substitute to get

$$|x_{n+1} - x_n| \leq \alpha|x_n - x_{n-1}| \leq \alpha \cdot \alpha^{n-5}|x_5 - x_4|$$

Simplifying, we get $|x_{n+1} - x_n| \leq \alpha^{n-4}|x_5 - x_4|$ And we are done.

Thus by the principle of mathematical induction, we have proven our theorem. And S is the set of all natural numbers. □

Problem 3e. Prove that the ratio sequence x_n converges and compute the limit, otherwise known as the golden mean.

Proof. We know that the ratio sequence is bounded by 5 and $\frac{6}{5}$ from 3b. We also know that the sequence is monotonically increasing from 3c. Therefore, by the monotone convergence theorem, the sequence converges. We can denote the limit as L. We know that

$$L = 1 + \frac{1}{L}$$

Solving for L, we get

$$L^2 - L - 1 = 0$$

Using the quadratic formula, we get

$$L = \frac{1 \pm \sqrt{5}}{2}$$

We know that $L > 0$, so we can discard the negative root. Therefore, we have

$$L = \frac{1 + \sqrt{5}}{2} = \phi$$

And we are done. $\square$