HONORS REAL ANALYSIS LECTURE 15

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Abstract. This lecture covers a review of Chapter 1 and 2 to prepare for the upcoming first midterm.

Recall:

Definition. *A set $K \subseteq \mathbb{R}$ is compact if and only if given any open cover*

$$\{\mathbb{U}_\alpha\}_{\alpha \in I}, \text{ i.e. } K \subseteq \cup_{\alpha\in I}\mathbb{U}_\alpha$$

Where $\mathbb{U}_\alpha$ is open, one can extract a finite sub-cover, i.e.

$$K \subseteq \cup_{i=1}^{n}\mathbb{U}_{\alpha_i}$$

Theorem. ***(Heine-Borel)*** *The following statements are equivalent:*

(1) *K is compact*
(2) *K is closed and bounded*
(3) *(Sequential Compactness) Given any sequence $\{x_n\} \subseteq K$, there exists a convergent subsequence $\{x_{n_k}\}$ that converges to a finite point, x_0, in K.*

Lemma. *Let $F \subseteq K$, F is closed, K is compact. Then F is compact.*

Proof. Let $F \subseteq \cup_{\alpha\in I}\mathbb{U}_\alpha$ be an open cover of F. Now consider $\cup_{\alpha\in I} \cup F^c$, and note that F is closed, so F^c is open. So, $K \subseteq \mathbb{R} \subseteq \cup_{\alpha\in I} \cup F^c$. By compactness of K, we can extract a finite sub-cover, i.e. $K \subseteq \cup_{i=1}^{n}\mathbb{U}_{\alpha_i} \cup F^c$. Because $F \subseteq K$, we have, $F \subseteq \cup_{\alpha\in I}\mathbb{U}_\alpha$. Then F is compact. □

Proof. (2) $\implies$ (1)
We show that if K is closed and bounded, then K is compact. Since K is bounded, we have $K \subseteq [-M, M]$, for some $M \in \mathbb{R}$. We show that $[-M, M]$ is compact. Since K is closed if we show that $[-M, M]$ is closed, the lemma yields that K is compact. Let $[-M, M] \subseteq \cup_{\alpha\in I}\mathbb{U}_\alpha$ Assume that there is no finite sub-cover. Now we subdivide $[-M, M]$ dyadically. At the first stage, we get $[-M, 0]$ and $[0, M]$. By contradiction, we assume that for one of the sub-intervals, it is not covered by a finite sub-cover. Select the sub-interval that is not covered by a finite sub-cover, and subdivide it again. Therefore, we have a sequence of nested, closed intervals, $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$. Where the length of

$I_n = \frac{2M}{2^n}$. By the Nested Interval Property, we have that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. Let $x_0 = \cap_{n=1}^{\infty} I_n$. Furthermore, $x_0 \in I_n \forall n$. Thus for $n \geq n_0$ one has

$$I_n \subseteq \left[x_0 - \frac{2M}{2^n}, x_0 + \frac{2M}{2^n}\right] \subseteq (x_0 - \delta_0, x_0 + \delta_0)$$

Thus for some $n \geq n_0$, $I_n \subseteq \mathbb{U}_{\alpha_0}$, for some $\alpha_0 \in I$. This is a contradiction, as I_n is not covered by a finite sub-cover. □

Proof. (2) $\implies$ (3)
Let $\{x_n\} \subseteq K$. Since K is closed and bounded, by Bolzano-Weierstrass, we have that $\{x_n\}$ has a convergent subsequence, $\{x_{n_k}\}$ that converges to $x_0 \in K$. Since K is closed and bounded, $x_0 \in K$ and x_0 finite. This is (c). □

Proof. (3) $\implies$ (2)
By contradiction, assume that K is not bounded. Then, given any N, one can find x_N such that $x_N > N$ and $x_N \in K$. The sequence $\{x_N\}$ has no convergent subsequence, as it is unbounded. This is a contradiction. □

Definition. *A collection of sets $\{K_\alpha\}_{\alpha \in I}$ are said to have the finite intersection property (FIP) if and only if*

$$\cap_{i=1}^{n} K_{\alpha_i} \neq \emptyset \textit{ for any finite sub-collection, } \{K_{alpha_1}, K_{alpha_2}, \dots\}$$

Lemma. *Let $\{K_\alpha\}_{\alpha \in I}$ be a collection of compact sets having FIP, then*

$$\cap_{\alpha \in I} K_\alpha \neq \emptyset$$

Corollary. *Let*

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$$

be a nested sequence of compact sets. Then

$$\cap_{n=1}^{\infty} K_n \neq \emptyset$$

Theorem. *The arbitrary intersection of compact sets is compact.*

Theorem. *The arbitrary union of compact sets is not compact.*