HONORS REAL ANALYSIS LECTURE 1

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Abstract. This lecture covers sections 1.2 from the text, *Understanding Analysis*. We begin with a proof of the extended triangle inequality by induction. Then, move on to supremums and infimums. We also cover the Axiom of Completeness and the Archimedean Property. Lastly, we introduce the denseness of the rational numbers.

Proposition (Extended Triangle Inequality)**.**

$$|\sum_{k=1}^{n} \pm a_k| \leq \sum_{k=1}^{n} |a_k|$$

Proof. We proceed by induction. Our base step is n = 2, which is true from the triangle inequality.

$$|a+b| \leq |a| + |b|$$

Let $P(n)$ be the statement, $|\sum_{k=1}^{n} \pm a_k| \leq \sum_{k=1}^{n} |a_k|$. We will prove that for any natural number, n, $P(n) \implies P(n+1)$. Let us assume $P(n)$ is true and show $P(n+1)$ follows.
Letting $x = \pm a_{n+1}$ and $y = \sum_{k=1}^{n+1} \pm a_k$, we have

$$|\sum_{k=1}^{n+1} \pm a_k| = |x+y| \leq |x| + |y| = | \pm a_{n+1}| + |\sum_{k=1}^{n} \pm a_k|$$

By our induction hypothesis, we know

$$| \pm a_{n+1}| + |\sum_{k=1}^{n} \pm a_k| = |a_{n+1}| + \sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n+1} |a_k|$$

Thus, we have $P(n+1)$

$$|\sum_{k=1}^{n+1} \pm a_k| \leq \sum_{k=1}^{n+1} |a_k|$$

∎

Proposition.

$$||a| - |b|| \leq |a + b|$$

Proof. By definition,

$$||a| - |b|| = \begin{cases} |a| - |b|, |a| \geq |b| \\ |b| - |a|, |b| \geq |a| \end{cases}$$

We need only check that $|a| - |b| \leq |a + b|$ and $|b| - |a| \leq |a + b|$. Start with

$$|a| = |a + b - b|$$

$$|a + b - b| \leq |a + b| + |-b| = |a + b| + |b|$$

Thus, we have

$$|a| \leq |a + b| + |b| \implies |a| - |b| \leq |a + b|$$

Likewise,

$$|b| = |b + a - a| \leq |b + a| + |a| \implies |b| - |a| \leq |a + b|$$

Because we have covered all cases, it follows that

$$||a| - |b|| \leq |a + b|$$

□

Corollary.

$$||a| - |b|| \leq |a - b|$$

Proof. From the previous proposition, we have

$$||a| - |b|| \leq |a + b|$$

Let $b = -b$

$$||a| - |-b|| \leq |a + (-b)|$$

Therefore,

$$||a| - |b|| \leq |a - b|$$

□

Definition. *Let A be a non-empty subset of the real numbers. The supremum of A is the least upper bound of the set A.*

Remark: $\sup A/\inf A$ need not be an element of A. Take, for example, $A = (0, 1)$: $\sup A = 1, \sup A \notin A$.

Proposition. *Let $A \subseteq \mathbb{R}$. Let x_0 be an upper bound for A. Then $x_0 = \sup A \iff$ for any $\epsilon > 0$, there exists an element $a \in A$ such that for $x_0 - \epsilon \leq a$.*

Proof. ($\implies$) Let $x_0 = \sup A$. For the sake of contradiction, assume that there is an ϵ_0 such that for all $a \in A$, $x_0 - \epsilon \leq a$. Therefore, by definition, $x_0 - \epsilon_0$ is an upper bound. However, x_0 is strictly less than $x_0 - \epsilon$. Thus, a contradiction is formed because we assumed x_0 was the least upper bound.

($\impliedby$) We know for any $a \in A, a \leq x_0$ since x_0 is an upper bound. Next, let us pick a number smaller than x_0, say $x_0 - \epsilon$. Our hypothesis says, there exists $a \in A$ such that $x_0 - \epsilon < a$. Therefore, it cannot be an upper bound. Because x_0 is an upper bound and any number smaller than x_0 is not an upper bound, x_0 is $\sup A$. □

Proposition.

$$\alpha = \sup(-A) = -\inf(A) = \beta$$

Proof. Our strategy is to first show (1) $\alpha \leq \beta$, then show that (2) $\alpha \geq \beta$. Therefore, $\alpha = \beta$.

(1): For any $\epsilon > 0$, one finds $x_0 \in A$ such that

$$\alpha - \epsilon < -x_0$$

$$\epsilon - \alpha > x_0$$

But $\inf A \leq x_0 \leq \epsilon - \alpha$. Therefore,

$$-\alpha + \epsilon > \inf(A)$$

$$\epsilon - \alpha < -\inf(A) = \beta$$

So, $\alpha - \beta < \epsilon$.

(2): By definition,

$$\forall x \in A, -x \leq \alpha \implies x \geq \alpha$$

So, $-\alpha$ is a lower bound of A. Therefore, $\inf(A) \geq -\alpha$ Equivalently, $\alpha \geq \beta$.

Therefore, by (1) and (2), $\alpha = \beta$. □

Axiom (Axiom of Completeness). *Given a non-empty subset of* $\mathbb{R}$*. If that set is bounded above, then its supremum exists.*

Theorem (Archimedean Property).
(1) Let $x \in \mathbb{R}$*, then there exists a natural number,* n*, such that* $x < n$
(2) Let $x \in \mathbb{R}$*, then there exists an integer such that* $n < x < n + 1$
(3) Let $x > 0$*, then there exists a natural number,* n*, such that* $\frac{1}{n} < x$

Proof.
(1): Assume, for the sake of contradiction, that for any natural number, n, $n \leq x$. Define the set S as follows,

$$S = \{n | n \leq x\}$$

S is bounded above, so by completeness, the supremum exists. Let $x_0 = \sup S$ So, there exists an $n_0 \in \mathbb{N}$, such that

$$x_0 - 1 < n_0$$

This also shows that

$$x_0 < n_0 + 1 \in \mathbb{N}$$

Thus a contradiction forms as there exists a natural number greater than x_0.

(2): Given any x and, without loss of generality, let $x > 0$.
From (1), we know there exists a natural number, n, such that $x < n$. Let $S = \{n | n \leq x\}$. By the well-ordering principle, we know that n_0 is the smallest element of S. So,

$$n_0 - 1 < x < n_0$$

(3): Consider $\frac{1}{x} = z \in \mathbb{R}$ From (1) we have $n \in \mathbb{N}$ so that

$$\frac{1}{x} = z < n \implies \frac{1}{n} < x$$

□