Workshop 2

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Problem 1a. Prove for any $a, b \in \mathbb{R}$

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

Proof. Let $x \in \mathbb{R}$. Trivially, we know $x^2 \geq 0$. Set $x = a - b$ for some real numbers a, b. Thus, we have

$$(a-b)^2 \geq 0$$

Expanding gives us

$$a^2 - 2ab + b^2 \geq 0$$

and equivalently,

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

$\square$

Problem 1b. Prove for any $a_i, b_i \in \mathbb{R}$

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$$

Proof. We begin with our finding from 1a, for any real numbers c,d

$$cd \leq \frac{c^2}{2} + \frac{d^2}{2}$$

Let $c = a_i b_j$, $d = a_j b_i$ where $a_k, b_k \in \mathbb{R}$ for all k.

$$a_i b_j a_j b_i \leq \frac{(a_i b_j)^2}{2} + \frac{(a_j b_i)^2}{2}$$

Now, we can sum equivalently over both the left-hand side and right-hand side. This remains true because there are $\binom{n}{2}$ terms on both sides, all which satisfy 1a's inequality.

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_i b_j a_j b_i \leq \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{(a_i b_j)^2}{2} + \frac{(a_j b_i)^2}{2}$$

We will now multiply by two and add a different term to both sides

$$\left[\sum_{i=1}^{n} (a_i b_i)^2\right] + \left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} 2 a_i b_j a_j b_i\right] \leq \left[\sum_{i=1}^{n} (a_i b_i)^2\right] + \left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} (a_i b_j)^2 + (a_j b_i)^2\right]$$

Notice that, the left-hand side is equivalent to

$$\left[\sum_{i=1}^{n} (a_i b_i)^2\right] + \left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} 2(a_i b_i)(a_j b_j)\right] = \left[\sum_{i=1}^{n} a_i b_i\right]^2$$

Likewise, the right-hand side is equivalent to

$$\left[\sum_{i=1}^{n} (a_i b_i)^2\right] + \left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} (a_i b_j)^2 + (a_j b_i)^2\right] = \left[\sum_{i=1}^{n} \sum_{j=i+1}^{n} (a_i b_j)^2\right] = \left[\sum_{i=1}^{n} a_i^2\right] \left[\sum_{i=1}^{n} b_i^2\right]$$

Combining these, we can conclude that, by definition of absolute value,

$$\left[\sum_{i=1}^{n} a_i b_i\right]^2 \leq \left[\sum_{i=1}^{n} a_i^2\right] \left[\sum_{i=1}^{n} b_i^2\right] \implies \left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$$

$\square$

Problem 1c. Prove for any $a_i \in \mathbb{R}$,

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sqrt{n}\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$$

Proof. We begin with the inequality from 1b,

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$$

Let $\{b_i\}_{i=1}^{n}$ be the constant sequence $\{1, 1, 1, 1, \dots\}$. We can rewrite the inequality as

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sqrt{n}\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$$

$\square$

Lemma (1).

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|}$$

Proof. By the triangle inequality, we know

$$|x+y| \leq |x|+|y|$$

Adding terms to both sides, we have

$$|x+y|+|x||x+y|+|y||x+y| \leq |x|+|y|+|x||x+y|+|y||x+y|$$

Factoring give us

$$|x+y|(1+|x|+|y|) \leq (|x|+|y|)(1+|x+y|)$$

Finally, dividing by $1+|x|+|y|$ and $1+|x+y|$,

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|}$$

□

Problem 2. Prove for any $x, y \in \mathbb{R}$, we have

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

Proof. Take the trivial inequality: $|x|+|y|+|x||y| \leq |x|+|y|+2|x||y|$. We will manipulate both sides to reach our goal. Add one to both sides and factor both sides.

$$1+|x|+|y|+|x||y| \leq 1+|x|+|y|+2|x||y|$$

$$(1+|x|)(1+|y|) \leq 1+|x|(1+|y|)+|y|(1+|x|)$$

We multiply by $|x|+|y|$ on both sides.

$$(|x|+|y|)(1+|x|)(1+|y|) \leq (|x|+|y|)(1+|x|(1+|y|)+|y|(1+|x|))$$

Now, dividing by all the non-$(|x|+|y|)$ terms, we have

$$\frac{|x|}{1+|x|+|y|} + \frac{|y|}{1+|x|+|y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

Our right-hand side matches our goal so we will leave it alone. For the left-hand side, by lemma 1, we know that

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|+|y|}{1+|x|+|y|}$$

Thus, putting it together yields

$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}$$

$\square$