Problem 2: Probability Distributions

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Question 1

Solution:

The four given probability density functions are:

1. Pdf 1:

$$\begin{aligned} f_1(x) &= 1 \qquad ; -\frac{1}{2} \le x \le \frac{1}{2} \\ &= 0 \qquad ; otherwise \end{aligned}$$

2. Pdf 2:

$$\begin{aligned} f_2(x) &= e^{-x} \qquad ; x \ge 0 \\ &= 0 \qquad ; otherwise \end{aligned}$$

3. Pdf 3:

$$f_3 = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

4. If we replace x by $x/\sqrt{2}$ this will give standard normal density,

$$\begin{aligned} f_3 &= \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}} \; |J| \\ &= \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}} \; \frac{1}{\sqrt{2}} \end{aligned}$$

5. Therefore,

$$f_3 = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad ; -\infty \le x \le \infty$$

6. Pdf 4:

$$\begin{aligned} f_4 &= x^{-2} \qquad ; x \ge 1 \\ &= 0 \qquad ; otherwise \end{aligned}$$

The cumulative distribution function of the above density functions are:

1. CDF 1:

$$\begin{aligned} F_1(x) &= \int_{\frac{1}{2}}^{x} 1 \quad dx \\ &= x \mid_{\frac{1}{2}}^{x} \\ &= x + \frac{1}{2} \end{aligned}$$

2. CDF 2:

$$\begin{aligned} F_2(x) &= \int_{0}^{x} e^{-x} \quad dx \\ &= -e^{-x} \mid_{0}^{x} \\ &= 1 - e^{-x} \end{aligned}$$

3. CDF 3:

$$F_3(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad dx$$

4. Note that the above integral does not exist in a simple closed formula. It can be computed numerically.

5. CDF 4:

$$\begin{aligned} F_4(x) &= \int_{1}^{x} x^{-2} \quad dx \\ &= -\frac{1}{x} \mid_{1}^{x} \\ &= 1 - \frac{1}{x} \end{aligned}$$

Question 2

Solution:

To make line plots of all eight functions (four PDFs and four CDFs), we created objects for observation and plotted the resulting four PDFs and four CDFs in the same plotting space.

```
# Observation for PDF:1
x1<-seq(-0.5,0.5,by=0.01)
y1<-rep(1,length(x1))
# Observation for PDF:2
x2<-seq(0,4,0.01)
y2<-exp(-x2)
# Observation for PDF:3
x3<-seq(-4,4,0.01)
y3<-(1/sqrt(pi))*exp(-x3^2)
# Observation for PDF:4
x4<-seq(1,4,0.01)
y4<-x4^-2
```

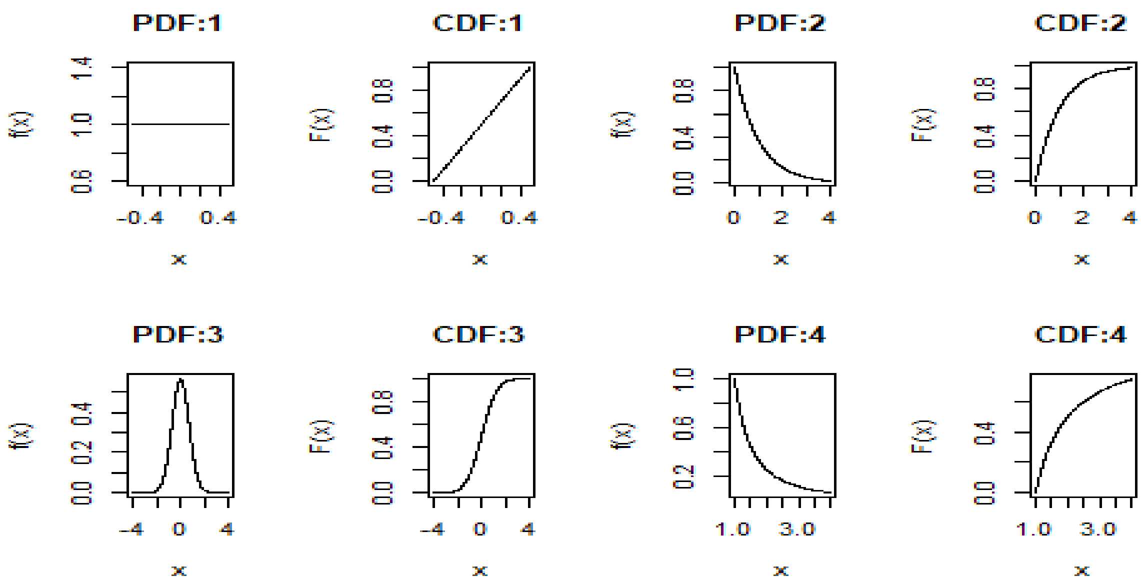
```r
# Combined plot for all PDF and CDF
par(mfrow=c(2,4))

plot(x1,y1,type="l",xlab="x",ylab="f(x)",main="PDF:1")
plot(x1,x1+0.5,type="l",xlab="x",ylab="F(x)",main="CDF:1")

plot(x2,y2,type="l",xlab="x",ylab="f(x)",main="PDF:2")
plot(x2,1-y2,type="l",xlab="x",ylab="F(x)",main="CDF:2")

plot(x3,y3,type="l",xlab="x",ylab="f(x)",main="PDF:3")
plot(x3,pnorm(x3),type="l",xlab="x",ylab="F(x)",main="CDF:3")

plot(x4,y4,type="l",xlab="x",ylab="f(x)",main="PDF:4")
plot(x4,1-1/x4,type="l",xlab="x",ylab="F(x)",main="CDF:4")
```



Four PDFs and their corresponding CDFs

Question 3

Solution:

Suppose r be the q^{th} quantiles of the PDFs, therefore, we can write $F(x) = r$. Now, we can find median, first, and third quartiles in terms of r as $r = 1/2, 1/4$, and $3/4$, respectively. We can write quantile equations as follows:

1. PDF 1:

$$
\begin{aligned}
F_1(x) &= r \\
x + \tfrac{1}{2} &= r \\
x &= r - \tfrac{1}{2}
\end{aligned}
$$

2. PDF 2:

$$
\begin{aligned}
F_2(x) &= r \\
1 - e^{-x} &= r \\
x &= -\log(1 - r)
\end{aligned}
$$

3. PDF 3:

$$
\begin{aligned}
F_3(x) &= r \\
\phi(x) &= r \\
x &= \phi^{-1}(r)
\end{aligned}
$$

4. PDF 4:

$$
\begin{aligned}
F_4(x) &= r \\
1 - \frac{1}{x} &= r \\
x &= \frac{1}{1-r}
\end{aligned}
$$

```
med_pdf1 <-1 /2 - 1/2
Q1_pdf1 <- 1/4 - 1/2
Q2_pdf1 <- 3/4 - 1/2
IGR_pdf1 <- Q2_pdf1-Q1_pdf1
summary_pdf1<-c(med_pdf1,Q2_pdf1,IGR_pdf1)

med_pdf2<--log(1-(1/2))
Q1_pdf2<--log(1-(1/4))
Q2_pdf2<--log(1-(3/4))
IGR_pdf2<-Q2_pdf2-Q1_pdf2
summary_pdf2<-c(med_pdf2,Q1_pdf2,Q2_pdf2,IGR_pdf2)

med_pdf3<-qnorm(1/2)
Q1_pdf3<-qnorm(1/4)
Q2_pdf3<-qnorm(3/4)
IGR_pdf3<-Q2_pdf3-Q1_pdf3
summary_pdf3<-c(med_pdf3,Q1_pdf3,Q2_pdf3,IGR_pdf3)

med_pdf4<-1/(1-1/2)
Q1_pdf4<- 1/(1-1/4)
Q2_pdf4<- 1/(1-3/4)
IGR_pdf4<-Q2_pdf4-Q1_pdf4
summary_pdf4<-c(med_pdf4,Q1_pdf4,Q2_pdf4,IGR_pdf4)

sum_pdf<-rbind(summary_pdf1,summary_pdf2,summary_pdf3,summary_pdf4)

## Warning in rbind(summary_pdf1, summary_pdf2, summary_pdf3, summary_pdf4):
number
## of columns of result is not a multiple of vector length (arg 1)

rownames(sum_pdf)<-paste("PDF",1:4,sep="_")
colnames(sum_pdf)<-c( "Median", "Q1", "Q3", "IQR")
sum_pdf

##          Median         Q1        Q3      IQR
## PDF_1 0.0000000  0.2500000 0.5000000 0.000000
## PDF_2 0.6931472  0.2876821 1.3862944 1.098612
## PDF_3 0.0000000 -0.6744898 0.6744898 1.348980
## PDF_4 2.0000000  1.3333333 4.0000000 2.666667
```