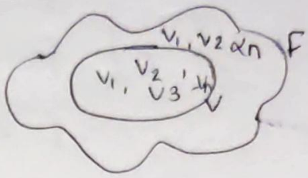
Question Bank - 2

Q1 Define vector space & give eg.

Ans- It's a set or space where all vectors and scalars are in V



Q2 Show that $V = R^2$ is not a vector space over the field R w.r.t the operations of vector addition $(a,b) + (c,d) = (a+c, b+d)$ and scalar multiplication $K(a,b) = K(a,b)$?

Solⁿ $(a,b) + (c,d) = (a+c, b+d)$.

$x = (a,b)$, $y = (c,d)$, $z = (e,f) \in R^2 \mid \alpha, \beta \in R$

$\Rightarrow (\alpha+\beta)\cdot a = \alpha a + \beta a$.

$\Rightarrow ~~\alpha a + \beta a~~$.

Let $\alpha = 1$, $\beta = 2$ and $a = (3,4)$

$L.H.S. = (\alpha+\beta)\cdot a$

$= (1+2)(3,4)$

$= 3(3,4)$

$= (9,4)$

$RHS = \alpha a + \beta a$

$= 1(3,4) + 2(3,4)$

$= (3,4) + (6,4)$

$= (9,8)$

$LHS \neq RHS$.

Q.3 Show that the set of polynomials of degree n with real coff. form a vector space.

Solⁿ A polynomial of degree 'n' is a function in the form of

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

where $a_0 - a_n$ are real no (called coff) and n is a positive integer (called degree of $P(x)$)

$$\cancel{u = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in R^{n+1}} \quad n = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} \in R^{n+1},$$

Q.4 Show that union of two Sub-space of vector space need not to be a Sub-space of the vector Space.

Sol^n. For e.g: Take W_1 to be x-axis and W_2 be the y-axis, both subspaces of R^2. Their union includes both $(3,0)$ and $(0,5)$, whose sum is $(3,5)$, which is not union.
$\therefore$ The union is not a vector space.

Q.5 Discuss whether or not R^2 is a subspace of R^3?

Sol^n R^2 is not a Subspace of R^3 because elements of R^2 have exactly two entries while the elements of R^3 have exactly three entries. That's why R^2 is not a Subspace of R^3.

Q.6 Show that W is a Subspace of $V = R^3$, whose W is the x-y plane which consists of those vectors whose third components is zero i.e.

$$W = \{(a,b,0) : a, b \in R\}.$$

Sol. $x = (a_1, b_1, 0)$, $y_2 = (a_2, b_2, 0) \in V_3(F)$
$a, b \in R$

$$ax + by$$

$$\Rightarrow a(a_1, b_1, 0) + b(a_2, b_2, 0).$$

$$\Rightarrow (a \cdot a_1, a \cdot b_1, 0) + (b \cdot a_2, b \cdot b_2, 0) \in W.$$

$$\subset V_3 F$$

$\therefore$ W is a vector space.

Q.7 ?

Q.8 Is W a Subspace of A? where W is set of all matrices of the form $\begin{bmatrix} a & a+1 \\ 0 & b \end{bmatrix}$ and A is set of matrices of order 2×2:

Sol. Let $x = \begin{bmatrix} a_1 & (a+1)_1 \\ 0 & b_1 \end{bmatrix}$, $y = \begin{bmatrix} a_2 & (a+1)_2 \\ 0 & b_2 \end{bmatrix} \subset M_2(F)$,
$a, b \subset F$.

$ax+by$

$$a\begin{bmatrix} a_1 & (a+1)_1 \\ 0 & b_1 \end{bmatrix} + b\begin{bmatrix} a_2 & (a+1)_2 \\ 0 & b_2 \end{bmatrix}$$

$$\begin{bmatrix} a\cdot a_1 & a\cdot(a+1)_1 \\ 0 & a\cdot b_1 \end{bmatrix} + \begin{bmatrix} b\cdot a_2 & b\cdot(a+1)_2 \\ 0 & b\cdot b_2 \end{bmatrix}$$

$$\begin{bmatrix} a\cdot a_1 + b\cdot a_2 & a\cdot(a+1)_1 + b\cdot(a+1)_2 \\ 0 & (a\cdot b_1) + (b\cdot b_2) \end{bmatrix} \in W$$

$\therefore$ W is a vector space.

Q.9. Define linear combination in vector space.

Ans: If one vector is equal to the sum of scalar multiples of other vectors.

Q.10 Define linear span of vector space.

Ans - The span is a set of vectors of all linear combinations of the vectors.

Q.11,12

Q.13 State the condition under which set of vectors are
(1) Linearly Independent
(2) Linearly Dependent.

Ans- Linearly Independent:

Let $x_1, x_2 \cdots x_n$ are the set of vectors over the field of scalars $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n$, then vectors are said to be Linearly Independent.

Linearly Dependent:

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots \lambda_n x_n = 0$$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = 0 \cdots \lambda_n = 0.$$

and if atleast one $\lambda_i \neq 0$ then vectors are said to be linearly dependent.

Q.14. If possible, write the vector $v = (2, -5, 3)$ in R^3 as a linear combination of vectors $v_1 = (1, -3, 2)$, $v_2 = (2, -4, -1)$ and $v_3 = (1, -5, 7)$

Sol. Let $\alpha, \beta, \gamma \in R$.

$(1, -3, 2)$, $(2, -4, -1)$ and $(1, -5, 7) \in R^3$.

Linear combination can be written as.

$$\alpha(1, -3, 2) + \beta(2, -4, -1) + \gamma(1, -5, 7).$$

Suppose $(2, -5, 3)$ can be written as.

$$(2, -5, 3) = \alpha(1, -3, 2) + \beta(2, -4, 1) + \gamma(1, -5, 7).$$

$$= \cancel{((\alpha + (2\beta + \gamma) +}$$

$$= (\alpha, -3\alpha, 2\alpha) + (2\beta, -4\beta, \beta) + (\gamma, -5\gamma, 7\gamma)$$

$$(2, -5, 3) = (\alpha + 2\beta + \gamma), (-3\alpha - 4\beta - 5\gamma), (2\alpha + \beta + 7\gamma).$$

$$\alpha_1 + 2\beta_2 + \gamma_3 = 2. \quad \text{—— (1)}$$

$$-3\alpha_1 - 4\beta_2 - 5\gamma_3 = -5 \quad \text{—— (2)}$$

$$2\alpha_1 + \beta_2 + 7\gamma_3 = 3 \quad \text{—— (3)}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & 1 & 7 & 3 \end{array}\right]$$

$\underline{1 \times 3 + 2}$

$$\cancel{3\alpha_1} + 6\beta_2 + 3\gamma_3 = 6$$
$$-\cancel{3\alpha_1} - 4\beta_2 - 5\gamma_3 = -5$$
$$2\beta_2 - 2\gamma_3 = 1$$
$$2(\beta_2 - \gamma_3) = 1$$
$$\beta_2 - \gamma_3 = 1/2 \quad \text{--- } (4)$$

$\underline{2 \times 2 + 3 \times 3}$

$$-\cancel{6\alpha_1} - 8\beta_2 - 10\gamma_3 = -10$$
$$\cancel{6\alpha_1} - 3\beta_2 + 14\gamma_3 = 9$$
$$-11\beta_2 + 11\gamma_3 = -1$$
$$11(-\beta_2 + \gamma_3) = -1$$
$$-\beta_2 + \gamma_3 = -1/11 \quad \text{--- } (5)$$

$\therefore 4 \neq 5$

Eq^n has no sol^n.

$\therefore$ ω can't be expressed in term of given vectors.

Q.15 Determine whether or not the vectors $(1,-2,1)$, $(2,1,-1)$, $(7,-4,1)$ are linearly dependent.

Soln. Let $x_1 = (1,-2,1)$, $x_2 = (2,1,-1)$ and $x_3 = (7,-4,1)$.

$\lambda_1, \lambda_2, \lambda_3$ are scalars.

$$L.C = x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 = 0$$

$$(1,-2,1)\lambda_1 + (2,1,-1)\lambda_2 + (7,-4,1)\lambda_3 = 0$$

$$\lambda_1 + 2\lambda_2 + 7\lambda_3 = 0.$$
$$-2\lambda_1 + \lambda_2 - 4\lambda_3 = 0.$$
$$\lambda_1 - \lambda_2 + \lambda_3 = 0.$$

$$A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix}$$

$(R_3 \to R_3 - R_1)$ $(R_2 \to R_2 + 2R_1)$.

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & -3 & -6 \end{bmatrix}$$

$(R_3 \to 2R_3 + 3R_1)$

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & 0 & -19 \end{bmatrix}$$

$r(A) = 3 = n$.

Trivial soln.

$\lambda_1 = \lambda_2 = \lambda_3 = 0$

The vectors are linearly Independent.

Q16. Determine whether or not vectors $(2,-5,3)$, $(1,-2,1)$, $(2,1,-1)$, $(7,-4,1)$ are linearly dependent.

Soln: Let $x_1 = (2,-5,3)$, $x_2 = (1,-2,1)$, $x_3 = (2,1,-1)$, $x_4 = (7,-4,1)$

λ_1, λ_2 λ_3 and λ_4 are scalars

$$L.C: x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 + x_4\lambda_4 = 0$$

$$(2,-5,3)\lambda_1 + (1,-2,1)\lambda_2 + (2,1,-1)\lambda_3 + (7,-4,1)\lambda_4 = 0.$$

$$2\lambda_1 + \lambda_2 + 2\lambda_3 + 7\lambda_4 = 0.$$
$$-5\lambda_1 - 2\lambda_2 + \lambda_3 - 4\lambda_4 = 0.$$
$$3\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 = 0.$$

$$A = \begin{bmatrix} 2 & 1 & 2 & 7 \\ -5 & -2 & 1 & -4 \\ 3 & 1 & -1 & 1 \end{bmatrix} \quad 6 \;\; 2 \;\; -2 \;\; 2$$

$(R_3 \to 2R_3 - 3R_1)$ $(R_2 \to 2R_2 + 5R_1)$.

$$\cong \begin{bmatrix} 2 & 1 & 2 & 7 \\ 0 & 1 & 12 & 27 \\ 0 & -1 & -8 & -19 \end{bmatrix}$$

$(R_3 \to R_3 + R_1)$

$$\cong \begin{bmatrix} 2 & 1 & 2 & 7 \\ 0 & 1 & 12 & 27 \\ 0 & 0 & -6 & -12 \end{bmatrix}$$

49
-15
34

$r(A) = 3 = n$.

Trivial solⁿ.

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$

$\therefore$ The vectors are linearly Independent.

Q.17 Determine whether or not the vectors are $(0,0,0)$, $(2,1,-1)$, $(7,-4,1)$ are linearly dependent:

doubt

Solⁿ. Let $x_1 = (0,0,0)$ $x_2 = (2,1,-1)$, $x_3 = (7,-4,1)$.

$\lambda_1, \lambda_2, \lambda_3$ are scalars.

L.C $= x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 = 0$

$(0,0,0)\lambda_1 + (2,1,-1)\lambda_2 + (7,-4,1)\lambda_3 = 0$

$\cancel{0\lambda_1 + 2\lambda_2 + 7\lambda_3 = 0}$
$\cancel{0\lambda_2 + \lambda_2 - 4\lambda_3 = 0}$
$\cancel{0\lambda_1 - \lambda_2 + \lambda_3 = 0}$

$\cancel{0\lambda_1 + 0\lambda_2 + 0\lambda_3 = 0}$
$\cancel{2\lambda_1 + \lambda_2 - \lambda_3 = 0}$
$\cancel{7\lambda_1}$

$0\lambda_1 + 2\lambda_2 + 7\lambda_3 = 0$
$0\lambda_1 + \lambda_2 - 4\lambda_3 = 0$
$0\lambda_1 - \lambda_2 + \lambda_3 = 0$

$\therefore$ det $= 0$

Given vectors are linearly dependent.

Q18 Show that the polynomials 1, x and x^2 span $P_2(x)$?

Sol. By U's definition, a polynomial $p(x) = a + bx + cx^2$ is a linear combination of 1, x and x^2.

$\therefore$ ~~span~~ P_2 = Span $(1, x, x^2)$.

Q19 Determine whether $\sin 2x$ is in Span $(\sin x, \cos x)$

Sol. $c \sin x + d \cos x = \sin 2x$.

Let $x = 0$

$c \sin 0 + d \cos 0 = \sin 2(0)$.

$c(0) + d(1) = 0$.

$\rightarrow$ we see that $d = 0$.

Let $x = \pi/2$

$c \sin(\pi/2) + d \cos(\pi/2) = \sin 2(\pi/2)$.

$c \sin(\pi/2) + d \cos(\pi/2) = \sin(\pi)$.

$c(1) + d(0) = 0$.

$\rightarrow$ we see that $c = 0$.

This implies that

$\sin 2x = 0(\sin x) + 0(\cos x) = 0$ for all x.

$\because$ $\sin 2x$ is not a zero function.

$\therefore$ $\sin 2x$ is not in span $(\sin x, \cos x)$.

Q20 In $P_2(x)$, determine whether $r(x) = 1 - 4x + 6x^2$ is in span$(p(x), q(x))$, where $p(x) = 1 - x + x^2$ and $q(x) = 2 + x - 3x^2$.

Sol: $cp(x) + dq(x) = r(x)$.

$c(1 - x + x^2) + d(2 + x - 3x^2) = 1 - 4x + 6x^2$.

Regrouping according to powers of x.

$(c + 2d) + (-c + d)x + (c - 3d)x^2 = 1 - 4x + 6x^2$.

$c + 2d = 1$

$-c + d = -4$

$c - 3d = 6$.

$\therefore$ $r(x) = 3p(x) - q(x)$, so $r(x)$ is in span $(p(x), q(x))$.

Q.25 In $P_2(x)$ determine whether the set $(1+x, x+x^2 \text{ \& } 1+x^2)$ is linearly independent.

Q.26 Define basis and dimension of vector space.

Sol. Linear span: Let $v_1, v_2 \dots v_K$ be vectors in vector space V.

(a) Span $(v_1, v_2 \dots v_K)$ is a subspace of V.

(b) span $(v_1, v_2 \dots v_K)$ is the smallest subspace of V contain $v_1, v_2 \dots v_K$

Basis: A subset B of vector space V over a field F is said to be a basis of V if:

(i) B is linearly Inde.

(ii) V is generated by B.

Q.27 Show that the set $(1+x, x+x^2), 1+x^2)$ is a basis of $P_2(x)$?

Soln. Let $\alpha, \beta, \gamma \in P$

$(1+x)\ (x+x^2)\ (1+x^2) \in P_2$

L.C. can be written as :

$\alpha(1+x) + \beta(x+x^2) + \gamma(1+x^2) = 0$

$\alpha + \beta x + \gamma x = 0$ $\quad$ $\alpha + x\beta + \gamma = 0$

$\alpha x +$ $\quad$ $x\alpha + x^2\beta + x^2\gamma = 0$

Let $A = \begin{bmatrix} 1 & x & 1 \\ x & x^2 & x^2 \end{bmatrix}$

1

Q 21 In $P_2(x)$ determine whether the set $(1+x, x+x^2, 1+x^2)$ is L.I.

Sol. Let $x_1 = 1+x$, $x_2 = x+x^2$ and $x_3 = 1+x^2$.

λ_1, λ_2 and λ_3 are scalars.

L.C. $= x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3$.

$(1+x)\lambda_1 + (x+x^2)\lambda_2 + (1+x^2)\lambda_3 = 0$

$(\lambda_1+\lambda_3)1 + (\lambda_1+\lambda_2)x + (\lambda_2+\lambda_3)x^2 = 0.$

On comparing coeffecients.

$\lambda_1+\lambda_3 = 0.$

$\lambda_1+\lambda_2 = 0$

$\lambda_2+\lambda_3 = 0.$

The soln of $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

$\therefore$ The set is L.I.

Q.22 Determine whether the set $\{\sin x, \cos x\}$ is linearly Independent.

Soln. Let $x_1 = \sin x$ and $x_2 = \cos x$.

λ_1, λ_2 are scalars.

L.C. $x_1\lambda_1 + x_2\lambda_2 = 0$.

$\sin x\,\lambda_1 + \cos x\,\alpha_2 = 0$.

For the value of x, putting $x=0$, we get

$0 + \alpha_2 = 0$.

$\alpha_2 = 0$ — ①.

For the value of x, putting $x = \pi/2$.

$\alpha_1 + 0 = 0$.

$\alpha_1 = 0$ — ②.

From eqn ① & ②.

Given set is L.I.

Q 24. Determine whether the set $\{\cos^2 x, \sin^2 x, \cos 2x\}$ is L.D.

Solⁿ. Let $x_1 = \cos^2 x$, $x_2 = \sin^2 x$, $x_3 = \cos 2x$.

λ_1, λ_2 and λ_3 are Scalars.

$$x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 = 0$$

$$\cos^2 x\,\lambda_1 + \sin^2 x\,\lambda_2 + \cos 2x\,\lambda_3 = 0.$$

For the value of x, putting $x = 0$.

$$\lambda_1 + 0 + \lambda_3 = 0.$$

$$\lambda_1 + \lambda_3 = 0 \quad \text{—(1)}.$$

For the value of x, putting $x = \pi/2$.

$$0 + \lambda_2 + 0 = 0$$

$$\lambda_2 = 0 \quad \text{—(2)}$$

For the value of x, putting $x = \pi$.

$$\lambda_1 + 0 + \lambda_3 = 0$$

$$\lambda_1 + \lambda_3 = 0 \quad \text{—(3)}$$

From eqⁿ (1) & (3).

∴ Given set is L.D.

Q.25 Show that the set $\{1, x, x^2, x^3 \ldots\}$ L.I.

Sol. If coefficient $a_0 \ldots a_n$ exist such that

$$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots + a_n x^n = 0,$$

$$\forall x \in [a,b].$$

Then since a polynomial is continuous function. which means that

$$p(x) = 0 \; ; \; \forall x \in R.$$

∴ Given set is L.I.

Q.27 Show that the set $(1+x, x+x^2, 1+x^2)$ is a basis of $P_2(x)$?

Sol. Let $x_1 = 1+x$, $x_2 = x+x^2$, $x_3 = 1+x^2$

λ_1, λ_2 and λ_3 are Scalars.

$x_1\lambda_1 + x_2\lambda_2 + x_3\lambda_3 = 0$

$(1+x)\lambda_1 + (x+x^2)\lambda_2 + (1+x^2)\lambda_3 = 0.$

$(\lambda_1+\lambda_3)1 + (\lambda_1+\lambda_2)x + (\lambda_2+\lambda_3)x^3 = 0$

$\lambda_1 + \lambda_3 = 0.$

$\lambda_1 + \lambda_2 = 0.$

$\lambda_2 + \lambda_3 = 0$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$|A| = 1(1-0) - 0(1-0) + 1(1-0).$

$= 1 - 0 + 1$

$= 2$

$|A| \neq 0.$

$\therefore$ gi's L.I.

$\therefore$ vector is a basis of $P_2(x)$.

Q.28. Find the coordinate vector $[p(x)]$ of $p(x) = 1-4x+6x^2$ w.r.t the basis $\{1+x, x+x^2, 1+x^2\}$.

Soln. Let $x_1 = 1+x$, $x_2 = x+x^2$, $x_3 = 1+x^2$.

λ_1, λ_2 and λ_3 are scalars.

$(1+x)\lambda_1 + (x+x^2)\lambda_2 + (1+x^2)\lambda_3 = 0.$

$(\lambda_1+\lambda_3)1 + (\lambda_1+\lambda_2)x + (\lambda_2+\lambda_3) = 0$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

$|A| = 1[1] - 0 + 1[1].$

$= 1 + 1$

$= 2$

$|A| \neq 0.$

$\therefore$ gi's L.I

$1 - 4x + 6x^2 = \cancel{(\alpha_1+\alpha_3)}\ (\lambda_1+\lambda_3) + (\lambda_1+\lambda_2)x + (\lambda_2+\lambda_3)x^2$

On equating we get.

$\lambda_1 + \lambda_3 = 1$ — (1)

$\lambda_1 + \lambda_2 = -4$ — (2)

$\lambda_2 + \lambda_3 = 6$ — (3)

1 - 3.

$\lambda_1 + \lambda_3 = 1$

$\lambda_2 + \lambda_3 = 6$

$-\quad -\quad -$

$\lambda_1 - \lambda_2 = -5$ — (4)

2 + 4.

$\lambda_1 + \lambda_2 = -4$

$\lambda_1 - \lambda_2 = -5$

$2\lambda_1 = -9$.

$\boxed{\lambda_1 = -9/2}$

Solving eq^n (1).

$-9/2 + \lambda_3 = 1$

$\lambda_3 = 1 + 9/2$

$\lambda_3 = 11/2$.

Solving eq^n — (2).

$-9/2 + \lambda_2 = -4$.

$\lambda_2 = -4 + 9/2$

$\lambda_2 = \frac{-8+9}{2}$

$\boxed{\lambda_2 = 1/2}$

Q29 Find the coordinate vector $[p(x)]$ of $p(x) = 2-3x+5x^2$ with respect to the basis $\{1, x, x^2\}$.

Ans. we know that $1, x, x^2$ is linear independent.

let λ_1, λ_2 and λ_3 are scalars

$$2-3x-5x^2 = \lambda_1 + \lambda_2 x + \lambda_3 x^2$$

$$\boxed{\lambda_1 = 2}$$

$$\lambda_2 \cancel{x} = -3\cancel{x}$$

$$\boxed{\lambda_2 = -3}$$

$$\lambda_3 \cancel{x^2} = -5\cancel{x^2}$$

$$\boxed{\lambda_3 = -5}$$

$\lambda_1 = 2$, $\lambda_2 = -3$ and $\lambda_3 = -5$ are the coordinates.

Q30 Find the dimensions of Subspace $\{(x_1, x_2, x_3, x_4, x_5) : 3x_1 - x_2 + x_3 = 0\}$ of R^5

Ans dimension $(5-2) = 5-1$

$= (3) \Rightarrow 4$

(31)

Q32 If $V = \{(x, y, z, \omega) \in R^4 ; x+y-z=0, y+z+\omega=0; 2x+y-3z-\omega=0\}$ then find basis of V?

Soln: $x+y-z=0$

$$\boxed{z = x+y}$$

$$y+z+\omega=0$$

$$\omega = -y-z$$

$$\omega = -y-(x+y)$$

$$\boxed{\omega = -x-2y}$$

Put value of z and ω in $2x+y-3z-\omega=0$

$$2x+y-3(x+y)-(-x-2y)=0$$

$$2x+y-3x-3y+x+2y=0$$

$$\cancel{3x}+\cancel{3y}-\cancel{3x}-\cancel{3y}=0$$

$$0=0$$

$\therefore$ The value of z and ω satisfies eq^n $2x+y-3x-\omega=0$

So;

$V - \{(x, y, x+y, -x-2y) = x(1,0,1,-1) + y(0,1,1,-2)$

$\therefore$ It's ~~Ba~~ Linearly Independent

$\Rightarrow$ It forms basis.

Imp Q.34 Let Mapping $T: R^2 \to R^3$ be defined by $T(x,y) = (x, 2x-y, 3x+4y)$

Is T a linear Transformation?

Sol^n $(x_1, y_1)\ (x_2, y_2) \in R^2$, $\alpha, \beta \to R$

$$LHS: \{\alpha x + \beta y\} = T\{\alpha(x_1, y_1) + \beta(x_2, y_2)\}$$

$$= T\{(\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)\}$$

$$= T\{(\alpha x_1 + \beta x_2); (\alpha y_1 + \beta y_2)\}$$

$$\underline{LHS} = \{(\alpha x_1 + \beta x_2), 2(\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2), 3(\alpha x_1 + \beta x_2) + 4(\alpha y_1 + \beta y_2)\}$$

[From $\to x, 2x-y, 3x+4y$].

RHS: $(x, 2x-y, 3x+4y)$.

$$= [\alpha x_1 + \beta x_2,\ \alpha(2x_1 - y_1) + \beta(2x_2 - y_2),\ \alpha(3x_1 + 4y_1) + \beta(3x_2 + 4y_2)]$$

$$= [\alpha\{x_1, (2x_1 - y_1), (3x_1 + 4y_1)\} + \beta\{x_2, (2x_2 - y_2), 3x_2 + 4y_2\}]$$

$$= [\alpha T(x_1, y_1) + \beta T(x_2, y_2)]$$

RHS.

$\therefore$ Thus T is linear Transformation.

Q 35 Let mapping $T: R^2 \to R^3$ be defined by $T(x,y) = (x, 2x-y, 3x+4y)$. Then Find range space, rank, kernel and nullity of F

Sol^n $\text{kernel}(T) = \{(x,y) : T(x,y) = 0\}$

$= \{(x,y) : (x, 2x-y, 3x+4y) = (0,0,0)\}$

$\therefore x = 0$

$2x - y = 0 \Rightarrow 2\times 0 - y = 0 \; ; \; y = 0$

$3x + 4y = 0 \Rightarrow 3\times 0 + 4\times 0 \Rightarrow 0 = 0$

$\therefore \text{Ker}(T) = \{0, 0\}$

$\text{Nullity}(T) = 0$

$T(x,y) = (x, 2x-y, 3x+4y)$

$$\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, \; R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 4 \end{bmatrix}$$

$R_3 \to R_3 + 4R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$$

$\boxed{\therefore \text{Rank} = 2}$

$\text{Range}(T) = \{x, 2x-y, 3x+4y\}, \; x, y \in R$

$= \{x(1,2,3) + y(0,-1,4)\}$

Rank + Nullity = dim V

$2 + 0 = 2$ $\quad \therefore \dim V = 2$

Q 36. Let the linear transformation $H: R^3 \to R^3$ be defined by
$H(x,y,z) = (2x+z), (3x-z), (x+y+z)$ Then find nullspace of P

Soln $Ker(T) = \{(x,y,z) : (x,y,z) = 0\}$.
$= \{(x,y,z) : (2x+z, 3x-z, x+y+z) = (0,0,0)\}$

$2x+z=0$
$2x=-z$
x

$2x = 0$
$\boxed{x = 0}$.

$3x - z = 0$.
$3(0) - z = 0$
$\boxed{z = 0}$.

$x + y + z = 0$.
$0 + y + 0 = 0$.
$\boxed{y = 0}$.

$Ker(T) = (0,0,0)$

Nullspace = ~~2~~ 1

Q 37 Let the linear Transformation $T: R^3 \to R^3$ be defined by
$T(x,y,z) = \{x + 3y + 2z, 3x + 4y + z, 2x + y - z\}$
Then find the dimensions of the range space and null space of T, T^2 and T^3.

Soln Dimension of Range Space = Rank.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & -\cancel{2}1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$; $R_2 \to R_2 - 3R_1$.

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -5 \\ 0 & -5 & +3 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -5 \\ 0 & 0 & 8 \end{bmatrix}$$

$-5-(-5)$
$-5+5$

$\therefore$ Rank $= 3$.

$\therefore$ dimension of range space $= 3$

$\cancel{Ker(T) = \{(x,y,z) : (x,y,z) = 0\}}$

$\cancel{= \{(x,y,z) : (x+3y+2z, 3x+4y+z, 2x+y-z) = (0,0,0)\}}$

$\cancel{x+3y+2z = 0}$.

$x + 3y + 2z = 0$ — (1)

$-5y - 5z = 0$ — (2)

$8z = 0$ — (3)

Solving - 3.

$\boxed{z = 0}$

Putting value of z in 2.

$-5y - 5(0) = 0$

$-5y = 0$

$\boxed{y = 0}$

Putting value of y & z in (1)

$x + 3(0) + 2(0) = 0$

$\boxed{x = 0}$

$\therefore$ Null space is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Null space $= 1$

Q.38 If the Nullity of matrix $A = \begin{bmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{bmatrix}$ is 1, then find the value of k.

Sol: nullity of matrix = Total no. of elements − Rank of matrix

$1 = 3 - r$

$\boxed{r = 2}$

If rank = 2, then $|A| = 0$

$$\begin{vmatrix} k & 1 & 2 \\ 1 & -1 & -2 \\ 1 & 1 & 4 \end{vmatrix} = 0$$

$k(-4+2) - 1(4+2) + 2(1+1) = 0$

$k(-2) - 1(6) + 2(2) = 0$

$-2k - 6 + 4 = 0$

$-2k - 2 = 0$

~~$-2(k+1) = 0$~~

$-2k = 2$

$k = \frac{-\cancel{2}}{\cancel{2}}$

$\boxed{k = -1}$

Q.39 Let the linear Transformation $T: R^2 \to R^3$ be defined by $T(x,y) = (x, x+y, y)$, then find nullity of T ?

Soln. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$x = 0$ — (1)

$y = 0$ — (2)

$N(T) = 0$.

$\text{Ker}(T) = \{(x,y) : (x,y) = 0\}$.

$\{(x,y) : (x, x+y, y) = \cancel{0} (0,0,0)\}$.

$x = 0$

$x + y = 0 \Rightarrow 0 + y = 0$

$y = 0$.

$y = 0$

$\therefore x = 0, y = 0$.

$\text{Ker}(T) = \{0, 0\}$

Nullity $= 0$.

Q.40 Let the linear Transformation $T : R^4 \rightarrow R^4$ be defined by $T(x,y,z,\omega) = (x + z, 2x + y + 3z, 2y + 2z, \omega)$. Then find the dimensions of the range space and null space of T.

Sol^n

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \cancel{3}1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rank $= 3$

$\therefore$ dimension of range space = 3

$Ker(T) = \{(x, y, z, w) : (x, y, z, w) = (0, 0, 0, 0)\}$

$= \{(x, y, z, w) : (x + z, 2x + y + 3z, 2y + 2z, w) = (0, 0, 0, 0)\}$

$$x + z = 0 \quad — (1)$$
$$2x + y + 3z = 0 \quad — (2)$$
$$2y + 2z = 0 \quad — (3)$$
$$w = 0 \quad — (4)$$

Solving (1).

$\boxed{x = -z}$.

~~Putting value of x in 3.~~

~~2y + 2~~ Solving - 3.

$$2y + 2z = 0$$
$$2y = -2z$$
$$y = \frac{-\cancel{2}z}{\cancel{2}}$$
$$\boxed{y = -z}$$

Solving - (2).

$$2x + y + 3z = 0$$

~~2(-z) + y + 3~~ $2(-z) - z + 3z = 0$

$$-2z - z + 3z = 0$$
$$-\cancel{3z} + \cancel{3z} = 0$$
$$0 = 0$$

$\therefore x = -z,\ y = -z,\ z = z$

Null space $\begin{bmatrix} -z \\ -z \\ z \end{bmatrix}$.