Vector space: Let V be a set equipped with two operations and F be a field of scalars.

(i) **Vector addition:** If $u, v \in V \ni u+v \in V$

(ii) **Scalar multiplication:** If $\alpha \in F \ni \alpha v \in V \quad \forall v \in V$.

Then V is called vector space (over F) if the following axioms hold for any vectors $u, v, w \in V$

(i) $(u+v)+w = u+(v+w)$

(ii) $0 \in V$ s.t $u+0 = 0+u = u$ (zero vector)

(iii) $\forall u \in V$, $\exists$ (u^{-1}) or $(-u) \in V$ s.t

$u+(-u) = 0$.

(iv) $v+u=u+v$

(V) $k(u+v)=ku+kv \quad \forall\, k\in F$

(VI) $(a+b)v = av+bv \quad \forall\, a,b\in F$

(VII) $(ab)u = a(bu) \quad \forall\, a,b\in F$

(VIII) $1u = u, \quad 1\in F.$

Ex→ $\mathbb{R}^n(\mathbb{R})$.

Vector addition: $(u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)$

$= (u_1+v_1, u_2+v_2, \dots, u_n+v_n)$

Scalar multiplication: $k(a_1,a_2,\dots,a_n)=(ka_1,ka_2,\dots,ka_n)$

Ex→ Polynomial space $P_n(t)$ Field K

Ex→ Matrix space $\underset{\substack{\downarrow\\ \text{entries from } K}}{M_{m,n}}$ Field K

Ex→ function space $F(x)$

Let X be a non empty set and K be any field. Let $F(X)$ denote the set of all f^n's from X into K. Then $F(X)$ is a v.s over K with respect to the following operations:

(i) Vector addition

$(f+g)(x) = \underset{\substack{\downarrow\\ \in F(X)}}{f(x)+g(x)} \quad \forall\, x\in X$

(ii) Scalar multiplication:

Let $k\in K$ and $f\in F(X)$ then

$(kf)(x) = \underset{\in F(X).}{kf(x)} \quad \forall\, x\in X$

Linear Combinations

Let V be a vector space over field F. A vector $v \in V$ is a linear combination of vectors $u_1, u_2, ---, u_m \in V$ if there exist scalars $c_1, c_2, ---, c_n \in F$ s.t.

$$\boxed{v = c_1 u_1 + c_2 u_2 + --- + c_n u_n}$$

Ex check whether $v = (3,7,-4) \in \mathbb{R}^3$ is a linear combination of $(1,2,3)$, $(2,3,7)$, $(3,5,6)$.

Sol^n:- $v = (3,7,-4)$ is a linear combination of $(1,2,3)$, $(2,3,7)$ and $(3,5,6)$ if we can find c_1, c_2, c_3 in field $\mathbb{R}$ s.t.

$$c_1(1,2,3) + c_2(2,3,7) + c_3(3,5,6) = (3,7,-4)$$

$$\Rightarrow \left.\begin{aligned} c_1 + 2c_2 + 3c_3 &= 3 \\ 2c_1 + 3c_2 + 5c_3 &= 7 \\ 3c_1 + 7c_2 + 6c_3 &= -4 \end{aligned}\right\}$$

If this non-hom. system has a sol^n then v is a linear combination of rest three.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 7 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix}$$

$$\Rightarrow [A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 5 & 7 \\ 3 & 7 & 6 & -4 \end{array}\right] \xrightarrow[\text{Echelon form}]{\text{Convert into}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

$$\Rightarrow \left.\begin{aligned} c_1 + 2c_2 + 3c_3 &= 3 \\ -c_2 - c_3 &= 1 \\ -4c_3 &= -12 \end{aligned}\right\} \Rightarrow \begin{aligned} c_3 &= 4 \\ c_2 &= -4 \\ c_1 &= 2 \end{aligned}$$

$\because$ We have found c_1, c_2, c_3 therform $(3,7,-4)$ is a linear Combination of $(1,2,3)$ $(2,3,7)$ and $(3,5,6)$.

Note:- If the system is inconsistent then the given vector is not the linear combination of rest vectors.

Q.14:- If possible, write the vector $v = (2,-5,3) \in \mathbb{R}^3$ as linear combination of $v_1 = (1,-3,2)$, $v_2 = (2,-4,-1)$ and $v_3 = (1,-5,7)$.

Sol^n → $v = (2,-5,3)$ is a L.C. of v_1, v_2, v_3 if $\exists$ c_1, c_2, c_3 s.t.

$$c_1(1,-3,2) + c_2(2,-4,-1) + c_3(1,-5,7) = (2,-5,3) \longrightarrow (1)$$

$$\left.\begin{aligned} c_1 + 2c_2 + c_3 &= 2 \\ -3c_1 - 4c_2 - c_3 &= -5 \\ 2c_1 - c_2 + 7c_3 &= 3 \end{aligned}\right\}$$ If this system is consistent then c_1, c_2, c_3 exist and v is a linear comb. of v_1, v_2, v_3.

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ -3 & -4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

Here $\rho[A|B] = 3$ but $\rho[A] = 2$

⇒ System is inconsistent ⇒ No such c_1, c_2, c_3 exists so that

$$c_1(1,-3,2) + c_2(2,-4,-1) + c_3(1,-5,7) = (2,-5,3)$$

⇒ v is not L.C. of v_1, v_2, v_3.

Ex→ Prove that $v = 3t^2 + 5t - 5$ as a L.C. of $P_1 = t^2 + 2t + 1$, $P_2 = 2t^2 + 5t + 4$, $P_3 = t^2 + 3t + 6$.

Sol^n:→ Yes v is a L.C of P_1, P_2, P_3. $\boxed{c_1 = 3, c_2 = 1, c_3 = -2}$.

Spanning set :- Let $V(F)$ be a vector space. Then set of vectors $\{v_1, v_2 --- v_m\}$ will be called spanning set of V if every vector $v \in V$ can be written as linear combination of $v_1, v_2 --- v_m$. i.e if every vector v of V can be written as

$$\boxed{v = c_1 v_1 + c_2 v_2 + --- + c_n v_n}$$

Ex→ Consider the vector space $\mathbb{R}^3(\mathbb{R})$.
Then set of vectors $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a spanning set of V as any vector of $\mathbb{R}^3$ can be written as linear combination of $(1,0,0), (0,1,0), (0,0,1)$.

We can see suppose $(1,2,3) \in \mathbb{R}^3$ then

$$\underbrace{(1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1)}$$

$(1,2,3)$ has been written as L.C of $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Ex→ $\{1, t, t^2\}$ is a spanning set of $P_2(t)$ {set of all polynomials of order less than or equal to 2}.

Linear span :- Let $u_1, u_2, --- u_m$ be vectors of a vector space $V(F)$. Then collection of all linear combinations of $u_1, u_2 --- u_m$ is called linear span of $u_1, u_2 --- u_m$ and is denoted by $span(u_1, u_2 ---, u_m)$.

means generate

Q.18 → Prove that $1, x, x^2$ span $P_2(x)$.

Sol^n → If $1, x, x^2$ spans $P_2(x)$ then any arbitrary element of $P_2(x)$ can be written as L.C of $1, x, x^2$.

Let $ax^2 + bx + c \in P_2(x)$

We can see that

$$ax^2 + bx + c = a(x^2) + b(x) + c(1)$$

$\Rightarrow$ Every element of $P_2(x)$ is a L.C. of $1, x, x^2$.

$\Rightarrow \{1, x, x^2\}$ spans $P_2(x)$ and we write

$$\boxed{\text{span}(1, x, x^2) = P_2(x)}$$

Q.19 → Determine whether $\sin 2x$ is in span$(\sin x, \cos x)$.

Sol^n → $\sin 2x$ will be in span$(\sin x, \cos x)$ iff $\sin 2x$ can be written as L.C of $\sin x$ and $\cos x$.

i.e.

$$\sin 2x = c_1 \cos x + c_2 \sin x$$

If we can find c_1 and c_2 in field $\mathbb{R}$ then L.C. otherwise not.

$\cancel{\text{For } x = \frac{\pi}{4} \Rightarrow 1 = \frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}}}$

$\cancel{\text{For } x = \frac{\pi}{3} \Rightarrow \frac{\sqrt{3}}{2} = \frac{c_1}{2} + c_2\frac{\sqrt{3}}{2}}$

We know that $\sin 2x = 2 \sin x \cos x$

$\Rightarrow$ No c_1 and c_2 will exist so that

$$c_1 \cos x + c_2 \sin x = \sin 2x$$

$\Rightarrow \sin 2x$ is not in span$(\sin x, \cos x)$.

Note

↳ Q21 and 22 are same as above

Linear dependence and Independence

Vectors in $\mathbb{R}^n(\mathbb{R})$ are L.D. if rank of the matrix formed by these vectors is less than no. of given vectors and if rank of the matrix is less than no. of given vectors then vectors are L.I.

Q.15:- Determine whether or not the vectors $(1,-2,1),(2,1,-7),(7,-4,1)$ are L.I.

Sol^n→

$$A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -7 & 1 \end{bmatrix}$$

We find the rank of A.

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & -9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & 10 \\ 0 & 0 & 12 \end{bmatrix}$$

$\Rightarrow \rho(A) = 3 =$ no. of given vectors

$\Rightarrow$ Given vectors are L.I.

Q16, 17 are same as above.

Q 23:- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ is LD or not.

Sol^n→ $c_1\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_2\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} + c_3\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 + 2c_3 &= 0 \\ c_1 - c_2 + 0c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_1 + c_3 &= 0 \end{aligned}\right\}$$

After solving this system if $c_1, c_2, c_3 = 0 \Rightarrow$ A, B, C are L.I.

if atleast one c_i is non zero

$\Rightarrow$ A, B, C are LD

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 2 < 3$ (no. of unknowns)

$\Rightarrow$ System has non zero sol^n (∞ sol^n) or non trivial sol^n.

$\Rightarrow$ atleast one c_i will be non-zero

$\Rightarrow$ A, B, C are L.D.

Linear dependance and independence of vectors:

Vectors ~~$a_1, a_2 ---- a$~~ $v_1, v_2, ---, v_n$ in V are said to be L.D. if $\exists$ scalars $c_1, c_2, ---, c_n$ (not all zero) in F s.t

$$c_1 v_1 + c_2 v_2 + --- + c_n v_n = 0.$$

If all c_i's $= 0$ Then $v_1, v_2, ---, v_n$ are called L.I.

Remark 1: Suppose 0 is one of the vectors $v_1, v_2, ---, v_n$, say $v_k = 0$.

Then vectors are L.D.

$\because$ $1.v_1 + 0.v_2 + ---- + 0 v_n = 0$.

Rem. 2 $\rightarrow$ Suppose v is a non-zero vector. Then v, by itself is L.I.

Rem 3 $\rightarrow$ Two vectors v_1 and v_2 can be L.D iff one is scalar multiple of other.

Ex $\rightarrow$ Let $u = (1,1,0)$, $v = (1,3,2)$, $w = (4,9,5)$. Then u, v, w are LD.

Sol^n $\rightarrow$ $c_1(1,1,0) + c_2(1,3,2) + c_3(4,9,5) = (0,0,0)$

$$\Rightarrow \left.\begin{array}{l} c_1 + c_2 + 4c_3 = 0 \\ c_1 + 3c_2 + 9c_3 = 0 \\ 0c_1 + 2c_2 + 5c_3 = 0 \end{array}\right\} \Rightarrow c_1 = 3, \ c_2 = 5, \ c_3 = -2.$$

Ex $\rightarrow$ show that $(1,2,3)$ $(2,5,7)$ $(1,3,5)$ are LI.

Ex $\rightarrow$ Prove that $f(t) = \sin t$, $g(t) = e^t$, $h(t) = t^2$ are LI.

Sol^n $\rightarrow$ $c_1 \sin t + c_2 e^t + c_3 t^2 = 0 \quad \forall t$

$\Rightarrow$ $t = 0$ gives $\quad c_2 = 0$

$\Rightarrow$ $t = \pi$ gives $\quad c_2 e^\pi + c_3 \pi^2 = 0 \quad \Rightarrow c_3 = 0$

$\Rightarrow$ $t = \pi/2$ gives $\quad c_1 + c_2 e^{\pi/2} + c_3 \pi^2/4 = 0 \Rightarrow c_1 = 0.$

Non-zero Rows of a matrix (in echelon form) are L.I.

Basis and dimension:

Def → A set $S = \{u_1, u_2 - - -, u_n\}$ of vectors is a basis of V if it has the following two properties:

(i) S is L.I.

(ii) S spans V.

Def → A set $S = \{u_1, u_2 - - - u_n\}$ of vectors is a basis of V if every $v \in V$ is a unique linear combination of basis vectors.

Let V be a vector space s.t one basis has m elements and another basis has n. Then $m = n$.

Dimension: A vector space V is called n-dimensional if the basis of V has n elements and we write $\dim V = n$.

Ex of Bases →

(i) $\mathbb{R}^n(\mathbb{R})$: $\overset{e_1}{(1, 0 - - - 0)}, \overset{e_2}{(0, 1, - - -, 0)} - - - \overset{e_n}{(0, 0, - - - 1)}$

$\Rightarrow \dim \mathbb{R}^n = n$

(ii) Vector space of all ~~mxn~~ 2×3 matrices:

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\dim V = 6$

(iii) $P_n(t)$ [all polynomials of degree $\leq n$]

$\{1, t, t^2, - - -, t^n\}$ $\Rightarrow \dim(P_n(t)) = n+1$

Name : Deptt. :

Q.27:- Show that the set $\{1+x, x+x^2, 1+x^2\}$ is a basis for $P_2(x)$?

Sol^n: The set $\{1+x, x+x^2, 1+x^2\}$ will be basis for $P_2(x)$ if all the elements of $P_2(x)$ can be generated by this set and this set should be L.I.

Let Ax^2+Bx+C be any arbitrary element of $P_2(x)$.

Now

$$Ax^2+Bx+C = c_1(1+x)+c_2(x+x^2)+c_3(1+x^2)$$
$$= (c_2+c_3)x^2+(c_1+c_2)x+c_1+c_3$$

$$\Rightarrow \left.\begin{aligned} A &= c_2+c_3 \\ B &= c_1+c_2 \\ C &= c_1+c_3 \end{aligned}\right\}$$ If the value of c_1, c_2, c_3 will exist then $P_2(x)$ can be generated by $\{1+x, x+x^2, 1+x^2\}$.

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & A \\ 1 & 1 & 0 & B \\ 1 & 0 & 1 & C \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & B \\ 0 & 1 & 1 & A \\ 1 & 0 & 1 & C \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & B \\ 0 & 1 & 1 & A \\ 0 & -1 & 1 & C-B \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & B \\ 0 & 1 & 1 & A \\ 0 & 0 & 2 & C-B+A \end{array}\right]$$

Here $\rho(A|B) = 3 = \rho(A) =$ no. of unknowns

$\Rightarrow$ any element of $P_2(x)$ can be written as L.C of $1, x, x^2$

$\Rightarrow$ $\text{span}\{1, x, x^2\} = P_2(x)$

Now $\{1, x, x^2\}$ is L.I also as no scalar can convert x into x^2 or 1 into x^2.

$\Rightarrow$ This set $\{1, x, x^2\}$ is a basis of $P_2(x)$.