Q.2:- $V = \mathbb{R}^2$ is not a vector space over the field $\mathbb{R}$ w.r. to the operations of vector addition

$(a,b)+(c,d) = (a+c, b+d)$

and scalar multiplication

$k(a,b) = (ka, b)$.

Solⁿ→ If $\mathbb{R}^2$ over $\mathbb{R}$ is a vector space w.r. to the operations

$(a,b)+(c,d) = (a+c, b+d)$ and

$k(a,b) = (ka, b)$

then It must satisfy all the 9 properties of vector space.

If any one property does not hold it can't be vector space.

Now, see $(\alpha+\beta)u = \alpha u + \beta u \quad \forall \alpha, \beta \in F$ and $u \in V$

~~$\alpha(u+v) = \alpha u + \alpha v \quad \forall u, v \in V$ and $\forall \alpha \in F$~~

This is one of the property of vector space.

Therefore, we choose two elements from $\mathbb{R}^2$ and one element from $\mathbb{R}$

Let $\alpha = 2$, $\beta = 3$ and $u = (1,2)$

$(\alpha+\beta)u = (2+3)(1,2) = 5(1,2) = (5,2)$

$\alpha u + \beta u = 2(1,2) + 3(1,2) = (2,2)+(3,2) = (5,4)$.

L.HS ≠ RHS. Property does not hold.

Hence $\mathbb{R}^2(\mathbb{R})$ is not a vector space w.r to the operations

Q5.- $\mathbb{R}^2$ is not a ~~vector~~ sub space of $\mathbb{R}^3$ as $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$.

Q.31:- If $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ x & y & z \end{bmatrix}$

$V = \{(x,y,z) \in \mathbb{R}^3 ;\ \det(A) = 0\}$

Find the $\dim(V)$.

$Sol^n \rightarrow$ $|A| = 1(2z-3y) - 1(2z-3x) + 1(2y-2x)$

Now,

$|A| = 0$ (Given)

$\Rightarrow \cancel{2z} - 3y - \cancel{2z} + 3x + 2y - 2x = 0$

$\Rightarrow x - y = 0 \Rightarrow x = y$

$\Rightarrow$ V contains elements of the type (x, x, z)

$\Rightarrow$ dimension of $V = 2$.

Q.30:- Find the dimension of the subspace

$W = \{(x_1, x_2, x_3, x_4, x_5) : 3x_1 - x_2 + x_3 = 0\}$ of $\mathbb{R}^5$.

$Sol^n \rightarrow$ W contains the elements of the type $\{(x_1, 3x_1 + x_3, x_3, x_4, x_5)\}$

$$(x_1, 3x_1 + x_3, x_3, x_4, x_5) = x_1(1,3,0,0,0) + x_3(0,1,1,0,0) + x_4(0,0,0,1,0) + x_5(0,0,0,0,1)$$

$\Rightarrow$ $(1,3,0,0,0)$, $(0,1,1,0,0)$, $(0,0,0,1,0)$ and $(0,0,0,0,1)$ are spanning W. Now, If they are LI then they will form basis of W.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 4 =$ no. of vectors

$\Rightarrow$ They all are LI and will form basis of W.

$\Rightarrow$ dim $W = 4$.

(32) If $V = \{(x, y, z, w) \in \mathbb{R}^4 : x + y - z = 0,\ y + z + w = 0,\ 2x + y - 3z - w = 0\}$

find basis of V.

Sol^n: First we shall find relation in x, y, z

$$\begin{bmatrix} 1 & \cancel{0}\ 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $x + y - z = 0 \Rightarrow x = w + 2z$

$y + z + w = 0 \Rightarrow y = -w - z$

$V = \{(w + 2z, -w - z, z, w)\}$

$(w + 2z, -w, -z, z, w) = w(1, -1, 0, 0, 1)$
$+ z(2, 0, -1, 1, 0)$

Means $(1, -1, 0, 0, 1)$ and $(2, 0, -1, 1, 0)$ will from basis of V.

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & -1 \\ 0 & 1 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 2$ = no. of vectors

Hence $\{(1, -1, 0, 0, 1), (2, 0, -1, 1, 0)\}$ is a basis of V.