Matrix associated with linear map :

Let $U(\mathbb{F})$, $V(\mathbb{F})$ be vector spaces of dimension n and m respectively. Let $B = \{u_1, u_2, \ldots, u_n\}$ and $C = \{v_1, v_2, \ldots, v_m\}$ be their ordered basis respectively.

Suppose $T : U \to V$ is a linear transformation.

Since $T(u_1), T(u_2), \ldots, T(u_n) \in V$ and $\{v_1, v_2, \ldots, v_m\}$ spans V, each $T(u_i)$ is a linear combination of vectors $v_1, \ldots, v_m$.

Let
$$T(u_1) = \alpha_{11} v_1 + \alpha_{21} v_2 + \cdots + \alpha_{m1} v_m$$
$$T(u_2) = \alpha_{12} v_1 + \alpha_{22} v_2 + \cdots + \alpha_{m2} v_m$$
$$\cdots$$
$$T(u_n) = \alpha_{1n} v_1 + \alpha_{2n} v_2 + \cdots + \alpha_{mn} v_m$$

where each $\alpha_{ij} \in \mathbb{F}$. Then the $m \times n$ matrix

$$[T]_{B,C} = A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \\ \alpha_{m1} & \alpha_{m2} & & \alpha_{mn} \end{bmatrix}$$

is called matrix of T w.r.t. ordered bases B and C.

The word ordered basis is very significant, for as the order of basis is changed, the entries α_{ij} will change their positions and so the corresponding matrix will be different.

Ex: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the L.T. defined by
$$T(x, y, z) = (x - 2y, x + y - 3z)$$
and let $B = \{e_1, e_2, e_3\}$ and $C = \{e_1, e_2\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively. Find the matrix of T w.r.t. B and C.

Soln:- $B = \{\underbrace{(1,0,0)}_{e}, \underbrace{(0,1,0)}_{e_2}, \underbrace{(0,0,1)}_{e_3}\}$ and $C = \{\underbrace{(1,0)}_{e_1}, \underbrace{(0,1)}_{e}\}$

$T(e_1) = (1, 1)$, $T(e_3) = (0, -3)$.

$T(e_2) = (-2, 1)$

Now, write images as linear combination of basis elements of codomain –

$$(1,1) = 1(1,0) + 1(0,1)$$

$$(-2,1) = -2(1,0) + 1(0,1)$$

$$(0,-3) = 0(1,0) + (-3)(0,1)$$

$\therefore$ The matrix A of T w.r.t. B and C is

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -3 \end{bmatrix}$$

Ex: Let $D : P_3 \to P_2$ be the Diff. operator $D(p(x)) = \frac{d}{dx}(p(x))$. Let $B = \{1, x, x^2, x^3\}$ and $C = \{1, x, x^2\}$ be bases for P_3 and P_2 respectively.

(a) Find the matrix A of D w.r.t. B and C.

(b) Find the matrix A′ of D w.r.t. $B' = \{x^3, x^2, x, 1\}$ and C.