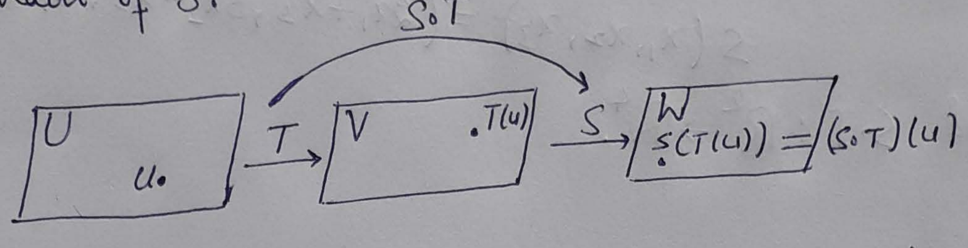
Composition of Linear maps :

Def:- If $T: U \to V$ and $S: V \to W$ are linear transformations, then the composition of S with T is the mapping $S \circ T$, defined by $(S \circ T)(u) = S(T(u))$ where $u \in U$.

Observe that $S \circ T$ is a mapping from $U \to W$ and the definition makes sense when the range of T must be contained in the domain of S.



Ex. Let $T: \mathbb{R}^2 \to P_1$ and $S: P_1 \to P_2$ be the L.T. defined by

$T(a, b) = a + (a+b)x$ and $S(p(x)) = x\,p(x)$.

Find $(S \circ T)(3, -2)$ and $(S \circ T)(a, b)$.

Soln:- $(S \circ T)(3, -2) = S(T(3, -2)) = S(3 + (3-2)x) = S(3+x)$

$= x(3+x) = 3x + x^2$.

$(S \circ T)(a, b) = S(T(a, b)) = S(a + (a+b)x) = x(a + (a+b)x)$

$= ax + (a+b)x^2$.

Thm: If $T: U \to V$ and $S: V \to W$ are linear transformations, then $S \circ T: U \to W$ is a linear transformation.

Pf:- Let u and v be in U and let c be a scalar. Then

$(S \circ T)(u+v) = S(T(u+v)) = S(T(u) + T(v))$ [$\because$ T is L.T.]

$= S(T(u)) + S(T(v))$ [$\because$ S is L.T.]

$= (S \circ T)(u) + (S \circ T)(v)$

And
$$(S \circ T)(cu) = S(T(cu))$$
$$= S(cT(u)) \quad [\because T \text{ is L.T.}]$$
$$= cS(T(u)) \quad [\because S \text{ is L.T.}]$$
$$= c(S \circ T)(u)$$

$\therefore$ $(S \circ T)$ is a L.T.

Ex. Consider the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$, $S : \mathbb{R}^3 \to \mathbb{R}^4$, defined by $T(x_1, x_2) = (x_1, 2x_1 - x_2, 3x_1 + 4x_2)$

$$S(x_1, x_2, x_3) = (2x_1 + x_3, 3x_2 - x_3, x_1 - x_2, x_1 + x_2 + x_3).$$

Find $S \circ T : \mathbb{R}^2 \to \mathbb{R}^4$.