The Kernel and Range of a LT :-

Let $T: V \to W$ be a linear transformation. The kernel of T, denoted $\ker(T)$, is the set of all vectors in V that are mapped by T to 0 in W. That is, $\ker(T) = \{v \in V : T(v) = 0\}$

The range of T, denoted by range(T), is the set of all vectors in W that are images of vectors in V under T. That is

$$\text{range}(T) = \{T(v) : v \in V\}$$
$$= \{w \in W : w = T(v) \text{ for some } v \in V\}$$

Ex. Find the kernel and range of the differential operator $D: P_3 \to P_3$ defined by $D(p(x)) = p'(x)$.

Soln:- Since $D(a+bx+cx^2+dx^3) = b+2cx+3dx^2$

$$\ker(D) = \{a+bx+cx^2+dx^3 : D(a+bx+cx^2+dx^3) = 0\}$$
$$\Rightarrow b+2cx+3dx^2 = 0$$
$$\Rightarrow b=0,\ c=0,\ d=0$$

$\therefore \ker(D) = \{a : a \in \mathbb{R}\}$ i.e. ker of D is the set of constant poly.

Now, $\text{range}(D) = \{D(p(x)) : p(x) \in P_3\}$

$$= \{p'(x) : p(x) \in P_3\}$$
$$= \{b+2cx+3dx^2 : b, c, d \in \mathbb{R}\}$$

or $\text{range}(D) = \{q(x) \in P_3 : q(x) = D(p(x)) \text{ for some } p(x) \in P_3\}$

$$= \{a+bx+cx^2+dx^3 : a+bx+cx^2+dx^3 = D(f+gx+hx^2+ix^3)$$
$$\Rightarrow a+bx+cx^2+dx^3 = g+2hx+3ix^2$$
$$\Rightarrow a=g,\ b=2h,\ c=3i,\ d=0.$$
$$\therefore p(x) = f+ax+\tfrac{b}{2}x^2+\tfrac{c}{3}x^3$$

$$= \{a+bx+cx^2 : a+bx+cx^2 = D(f+ax+\tfrac{b}{2}x^3+\tfrac{c}{3}x^3)$$

Thm:- Let $T: V \to W$ be a linear transformation. Then:

ⓐ The kernel of T is a subspace of V.

ⓑ The range of T is a subspace of W.

Pf:- ⓐ Since $T(0)=0$, the zero vector of V is in $\ker(T)$, so $\ker(T)$ is nonempty. Let u and v be in $\ker(T)$ and let c be a scalar. Then $T(u)=T(v)=0$, so

$$T(u+v) = T(u)+T(v) = 0+0=0$$

$$T(cu) = cT(u) = c \cdot 0 = 0$$

$\therefore$ $u+v$ and cu are in $\ker(T)$, and $\ker(T)$ is a subspace of V.

ⓑ Since $0=T(0)$, the zero vector of W is in range(T), so range(T) is nonempty.

Let $T(u)$ and $T(v)$ be in the range of T and let c be a scalar.

Then, $T(u)+T(v) = T(u+v)$ & $cT(u) = T(cu)$.

$T(u+v)$ is the image of $u+v$. Since u and v are in V, so $u+v$ in V and hence $T(u)+T(v)$ is in range(T).

Similarly, $T(cu)$ is the image of cu. Since u is in V, so cu is in V and hence $cT(u)$ is in range(T).

$\therefore$ $T(u)+T(v)$ and $cT(u)$ are in range(T), and range(T) is a subspace of W.

Def:- Let $T: V \to W$ be a linear transformation. The rank of T is the dimension of the range of T and is denoted by rank(T). The nullity of T is the dimension of the kernel of T and is denoted by nullity(T).