Homogeneous Linear System :

A homogeneous Linear system

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0$$

$$\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = 0$$

always has the trivial solution $x_1 = 0, \dots, x_n = 0$.

If the rank of coefficient matrix A is equal to no. of variables then system has only trivial soln.

If the rank of coefficient matrix A less than no of variables then system has trivial and non-trivial solns both.

Solutions of Homo. Linear System Gauss-Elimination Method :

Find the rank of the coefficient matrix by reducing into Echelon form.

For trivial soln, nothing to do.

For non-trivial soln, back substitution applied with some parameters.

Ex. Solve $x+2y+3z=0$; $3x+4y+4z=0$; $7x+10y+12z=0$

The coefficient matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Now, reduce to echelon form,

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$ $R_3 \to R_3 + 2R_2$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & -19 \end{bmatrix}$

The reduced matrix is in echelon form and it has 3 non-zero rows. The rank of A is 3.

But the rank $(A) = 3 =$ no. of variables $= 3$.

$\therefore$ There exist only trivial soln $x=0, y=0, z=0$.

Ex: Solve $2x+y+2z=0$; $x+y+3z=0$; $4x+3y+8z=0$

The coefficient matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix}$

By row operations, reduce A into echelon form;

$R_2 \leftrightarrow R_1 \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 8 \end{bmatrix}$ $\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix}$ $\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix}$

$R_3 \to R_3 - R_2 \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

The reduced matrix is in Echelon form and it has 2 non-zero rows.

$\therefore$ rank$A = 2$. This is less than no. of variables 3.

Thus, this system has non-trivial soln too.

The system in reduced form;

$$x + y + 3z = 0$$

$$-y - 4z = 0$$

Let $z = t$ then $y = -4z = -4t$

and $x = -y - 3z = 4t - 3t = t$.

$\therefore$ The non-trivial sol is $\begin{bmatrix} t \\ -4t \\ t \end{bmatrix}$.

[For getting trivial soln, Put $t = 0$]