Vector space

Let V and $\mathbb{F}$ be non-empty sets. If the following axioms hold for all u, v and w in V and for all c and d in $\mathbb{F}$, then V is called a vector space over $\mathbb{F}$:-

(i) $u+v \in V$ ⟶ Closure under addition

(ii) $u+v = v+u$ ⟶ Commutativity

(iii) $(u+v)+w = u+(v+w)$ ⟶ Associativity

(iv) There exist zero vector (0) in V, such that $u+0 = u$

(v) For each u in V, there is an element $-u$ in V such that $u+(-u) = 0$

(vi) $cu \in V$ ⟶ Closure under scalar multiplication

(vii) $c(u+v) = cu+cv$

(viii) $(c+d)u = cu+du$

(ix) $c(du) = (cd)u$

(x) $1u = u$ $(1 \in F)$.

The elements of V are called vectors and elements of $\mathbb{F}$ are called scalars. When scalars are real number, i.e., $\mathbb{F} = \mathbb{R}$ then vector space over real numbers also called real vector space and denoted by $V(\mathbb{R})$. The complex vector space has scalars as complex numbers, $V(\mathbb{C})$.

Ex.1. The real numbers is a vector space over real numbers $\mathbb{R}(\mathbb{R})$.

2. The $\mathbb{R}^2$ (plane) is also a vector space over real numbers $\mathbb{R}^2(\mathbb{R})$.

3. The $\mathbb{R}^n$ (Euclidean space) is also a vector space over real numbers $\mathbb{R}^n(\mathbb{R})$.

4. The set of real numbers is not a vector space over complex numbers.

5. The set of real numbers is a vector space over rational numbers. $\mathbb{R}(\mathbb{Q})$.

6. The set of all $m \times n$ matrices (having entries real no. or complex no.) forms a vector space with the usual operations of matrix addition and matrix scalar multiplication over real numbers or complex numbers resp. $(M_{m \times n})$

7. Let P_2 denote the set of all polynomials of degree 2 or less with real coefficients. Define addition and scalar multiplication as follows: If $p(x) = a_0 + a_1 x + a_2 x^2$ & $q(x) = b_0 + b_1 x + b_2 x^2$ are in P_2 then

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$ has degree at most 2 and is in P_2. If c is a scalar, then

$$c\,p(x) = ca_0 + ca_1 x + ca_2 x^2$$ is also in P_2.

The set P_2 is also a vector space over real numbers.

In general, for any fixed $n \geq 0$, the set P_n of all polynomials of degree less than or equal to n is a vector space.

8. The set $\mathbb{Z}$ of integers with the usual operations is not a vector space over $\mathbb{R}$ [as $\frac{1}{3}(2) = \frac{2}{3} \notin \mathbb{Z}$].

9. The set of complex numbers with their usual operations forms a vector space over real numbers and over complex numbers both. $\underline{\mathbb{C}(\mathbb{R})}$, $\underline{\mathbb{C}(\mathbb{C})}$

10. If p is prime, the set $\mathbb{Z}_p^n$ over $\mathbb{Z}_p$ is a vector space for all $n \geq 1$. $\mathbb{Z}_p = \{0, 1, 2, \dots, p-1\}$

addition means $a \oplus_p b = a + b$, if $a + b < p$.
$= p - (a+b)n$, if $a + b \geq p$

$ca = ca$, if $ca < p$; $c \in \mathbb{Z}_p$
$= p - nca$, if $ca \geq p$