$\therefore$ V is spanned by u, v where $u=(1,0,1,-1)$ & $v=(0,1,1,-2)$

And u & v are L.I. as rank of the matrix $\begin{bmatrix}1&0\\0&1\\1&1\\-1&-2\end{bmatrix}$ is 2.

$\therefore$ Basis of V is $\{u, v\}$.

Coordinates:

Let V be a vector space and let B be a basis for V. For every vector v in V, there is exactly one way to write v as a linear combination of the basis vectors in B. [Converse of this statement is also true]

Since representation of a vector w.r.t. a basis is unique, the next definition make sense,

Let $B=\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V. Let v be a vector in V and write $v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$.

Then $c_1, c_2, \ldots, c_n$ are called the coordinates of v w.r.t. B, and the colum vector $[v]_B = \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$ is called the coordinate vector of v w.r.t. B.

Ex: Find the coordinate vector $[p(x)]_B$ of $p(x)=2-3x+5x^2$ w.r.t. the standard basis $B=\{1, x, x^2\}$ of P_2.

Soln:- The polynomial $p(x) = (2)1 + (-3)x + (5)x^2$ is already a linear combination of $1, x, x^2$ so the coordinate vector

$$[p(x)]_B = \begin{bmatrix}2\\-3\\5\end{bmatrix}.$$

[Note: The order in which the basis vectors appear in B affects the order of the entries in a coordinate vector.

QB(29) Find the coordinate vector $[p(x)]$ of $p(x) = 1-4x+6x^2$ w.r.t. the basis $\{1+x, x+x^2, 1+x^2\}$.

Soln:- We need to find c_1, c_2 and c_3 s.t.

$$c_1(1+x)+c_2(x+x^2)+c_3(1+x^2) = 1-4x+6x^2$$

$$\Rightarrow (c_1+c_3)+(c_1+c_2)x+(c_2+c_3)x^2 = 1-4x+6x^2$$

$$\Rightarrow c_1+c_3 = 1;\ c_1+c_2=-4,\ c_2+c_3=6$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & -4 \\ 0 & 1 & 1 & 6 \end{array}\right] \xrightarrow{R_2 \to R_1} \begin{bmatrix} 1 & 1 & 0 & -4 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & -1 & 1 & +5 \\ 0 & 1 & 1 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3+R_2} \begin{bmatrix} 1 & 1 & 0 & -4 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 2 & 11 \end{bmatrix}$$

$\Rightarrow$ unique soln, $c_3 = \frac{11}{2}$, $c_2 = -5+\frac{11}{2} = \frac{1}{2}$

$c_1 = -4-\frac{1}{2} = -\frac{9}{2}$.

$$\therefore\ [p(x)] = \begin{bmatrix} -9/2 \\ 1/2 \\ 11/2 \end{bmatrix}.$$

Thm: Let $B = \{v_1, v_2, \dots, v_n\}$ be a basis for a vector space V.

(a) Any set of more than n vectors in V must be linearly dependent.

(b) Any set of fewer than n vectors in V cannot span V.

Thm: If a vector space V has a basis with n vectors, then every basis for V has exactly n vectors.

Def:- The dimension of vector space V, denoted by $\dim V$, is the number of vectors in a basis for V.

A vector space V is called finite-dimensional (FDVS) if it has a basis consisting of finitely many vectors.

The dimension of the zero vector space $\{0\}$ is defined to be zero.