Q. Find the range, rank, ker and nullity of the L.T. $T: \mathbb{R}^4 \to \mathbb{R}^3$ s.t. $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$.

Thm:- The rank Theorem (Rank-Nullity thm)

Let $T: V \to W$ be a linear transformation from a FDVS V into a vector space W. Then

$$\boxed{\text{rank}(T) + \text{nullity}(T) = \dim V}$$

Ex. Find the rank and nullity of the linear transformation $T: P_2 \to P_3$ defined by $T(p(x)) = x\,p(x)$.

Soln:-
$$T(a + bx + cx^2) = x(a + bx + cx^2)$$
$$= ax + bx^2 + cx^3$$

$$\text{ker}(T) = \{p(x) : T(p(x)) = 0\}$$
$$= \{a + bx + cx^2 : T(a + bx + cx^2) = 0\}$$
$$\Rightarrow ax + bx^2 + cx^3 = 0 = 0\cdot x + 0\cdot x^2 + 0\cdot x^3$$
$$\Rightarrow a = 0,\ b = 0,\ c = 0$$

$\therefore \text{ker}(T) = \{0\}$

$\therefore \text{nullity}(T) = 0$

And rank Thm, $\text{rank}(T) = \dim V - \text{nullity}(T)$
$$= 3 - 0 = 3.$$

Ex: Let W be the vector space of all symmetric 2×2 matrices. Define a L.T., $T: W \to P_2$ by

$$T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = (a-b)+(b-c)x+(c-a)x^2$$

Find the rank and nullity of T.

Soln:- The nullity of T is easier to compute directly than the rank, so

$$\ker(T) = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = 0 \right\}$$

$$: (a-b)+(b-c)x+(c-a)x^2 = 0$$

$$: (a-b) = (b-c) = (c-a) = 0$$

$$: a = b = c$$

$$\therefore \ker(T) = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$\therefore \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for the kernel of T, so

$$\text{nullity}(T) = \dim(\ker(T)) = 1.$$

By the rank thm, $\text{rank}(T) = \dim W - \text{nullity}(T) = 3 - 1 = 2.$