Basis & Dimension :

A subset B of a vector space V is a basis for V if

1. B spans V and
2. B is linearly independent.

Ex: 1. $B = \{(1,0),(0,1)\}$ is standard basis of $\mathbb{R}^2(\mathbb{R})$.

2. $B = \{1\}$ is standard basis of $\mathbb{R}(\mathbb{R})$

3. $B = \{(1,0,0),(0,1,0),(0,0,1)\}$ is standard basis of $\mathbb{R}^3(\mathbb{R})$.

4. $B = \{(1,0,\dots,0),(0,1,\dots,0),\dots,(0,0,\dots,1)\}$ is standard basis of $\mathbb{R}^n(\mathbb{R})$.

5. $B = \{1, x, x^2, \dots, x^n\}$ is a standard basis of P_n.

Ex. Show that $B = \{1+x, x+x^2, 1+x^2\}$ is a basis of P_2.

We have show that B is LI. To show that B spans P_2, let $a+bx+cx^2$ be an arbitrary polynomial in P_2. We must show that there are scalars c_1, c_2 and c_3 such that

$$c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = a+bx+cx^2$$

$$\Rightarrow (c_1+c_3) + (c_2+c_1)x + (c_2+c_3)x^2 = a+bx+cx^2$$

$$\Rightarrow \begin{aligned} c_1 + c_3 &= a \\ c_2 + c_1 &= b \\ c_2 + c_3 &= c \end{aligned} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

The rank of coefficient matrix is 3, so has unique soln

$\therefore c_1, c_2, c_3$ has non-zero values. So, B spans P_2

$\therefore B$ is a basis of P_2.

Ex. Find bases for the three vector spaces:

ⓐ $W_1 = \left\{ \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \right\}$ ⓑ $W_2 = \{a+bx-bx^2+ax^3\}$ ⓒ $W_3 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right\}$

ⓐ Since $\begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow W_1 = \text{span}(u,v)$ where $u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

u & v are LI as rank of matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$ is 2.

$\therefore \{u,v\}$ is a basis of W_1.

ⓑ Since $a+bx-bx^2+ax^3 = a(1+x^3)+b(x-x^2) \Rightarrow W_2 = \text{span}(u(x), v(x))$

where $u(x) = 1+x^2$, $v(x) = x-x^2$.

This $u(x)$ and $v(x)$ are LI as $a(1+x^3)+b(x-x^2)=0 \Rightarrow a=b=0$.

$\therefore \{u(x), v(x)\}$ is a basis of W_2.

ⓒ Since $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow W_3 = \text{span}(A, B)$ where

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Here A and B are L.I. as

$\alpha A + \beta B = 0 \Rightarrow \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \alpha = 0 = \beta$. $\therefore \{A, B\}$ is a basis of W_3.

Q(33). If $V = \{(x,y,z,w) \in \mathbb{R}^4 ; x+y-z=0, y+z+w=0, 2x+y-3z-w=0\}$ is a subspace of $\mathbb{R}^4$ then find basis of V?

$x+y-z=0 \Rightarrow z = x+y$

$y+z+w=0 \Rightarrow w = -y-z = -y-(x+y) = -x-2y$

Put z & w in (iii) eqn

$2x+y-3(x+y)-(-x-2y)=0 \Rightarrow 2x+y-3x-3y+x+2y=0 \Rightarrow 0=0$.

$\therefore$ The values of z & w satisfy (iii) eqn, so

$V = \{(x, y, x+y, -x-2y) = x(1,0,1,-1)+y(0,1,1,-2)\}$