Ex. Find rank and nullity of the L.T. $D: P_3 \to P_3$ defined by $D(p(x)) = p'(x)$.

Soln:- In previous example (page 13), we have computed

$\ker(D) = \{a : a \in \mathbb{R}\} = \{a \cdot 1 : a \in \mathbb{R}\} = \text{span}(1)$.

Hence, $\{1\}$ is a basis of $\ker(D)$. So nullity$(D) = 1$.

Also, $\text{range}(D) = \{a + bx + cx^2\} = \text{span}(\{1, x, x^2\})$

And $\{1, x, x^2\}$ is L.I.

$\therefore$ $\{1, x, x^2\}$ is a basis of range$(D)$. So, $\dim(\text{range}(D)) = 3$

$\therefore$ rank$(D) = 3$.

Q36. Let the L.T. $F: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $F(x,y) = (x, 2x-y, 3x+4y)$

Find rank$(T)$ and nullity$(T)$?

Soln:- $\text{Range}(T) = \{(x, 2x-y, 3x+4y) : x, y \in \mathbb{R}\}$

$= \{x(1,2,3) + y(0,-1,4) : x, y \in \mathbb{R}\}$

$= \text{span}(\{(1,2,3), (0,-1,4)\})$

And $(1,2,3), (0,-1,4)$ are L.I. as $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}$ has rank 2

because $\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \neq 0$.

$\therefore$ $\{(1,2,3), (0,-1,4)\}$ is a basis of range$(T)$, and therefore

rank$(T) = 2$.

$\text{Kernel}(T) = \{(x,y) : T(x,y) = 0\} = \{(x,y) : (x, 2x-y, 3x+4y) = 0\}$

$\Rightarrow x = 0,\ 2x - y = 0$
$3x + 4y = 0$

$\Rightarrow x = 0$ & $y = 0$

$\therefore$ $\ker(T) = \{(0,0)\}$

$\therefore$ nullity$(T) = 0$.

Ex: Find the range, Rank, ker and nullity of the L.T. $T: \mathbb{R}^3 \to \mathbb{R}^3$ s.t.

$$T(x,y,z) = (x+z,\ x+y+2z,\ 2x+y+3z)$$

Soln:- $\ker(T) = \{(x,y,z) : T(x,y,z) = (0,0,0)\}$

$\Rightarrow (x+z,\ x+y+2z,\ 2x+y+3z) = (0,0,0)$

$\Rightarrow x+z=0,\ x+y+2z=0,\ 2x+y+3z=0$

$\Rightarrow x=-z,\ -z+y+2z=0,\ 2(-z)+(-z)+3z=0$

$\Rightarrow y=-z \qquad \Rightarrow -3z+3z=0$

$\Rightarrow 0=0$

$\therefore x=-z,\ y=-z.$

$\therefore \ker(T) = \{(-z,-z,z) : z \in \mathbb{R}\} = \text{span}(\{-1,-1,1\}).$

[Note, $(-1,-1,1) \in \ker(T)$]

And one vector is always L.I.

$\therefore$ nullity$(T) = 1$

Range$(T) = \{T(x,y,z)\} = \{(x+z,\ x+y+2z,\ 2x+y+3z)\}$

$= \{x(1,1,2) + y(0,1,1) + z(1,2,3)\}$

$= \text{span}(\{(1,1,2),(0,1,1),(1,2,3)\}).$

Now, check L.I, $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Range$(T) = (\{(1,1,2),(0,1,1)\})$ as both vectors are L.I.

$\therefore$ rank$(T) = 2.$