Inverse of Linear Transformations :

A linear transformation $T: V \to W$ is invertible if there is a linear transformation $T': W \to V$ such that

$$T' \circ T = I_V \text{ and } T \circ T' = I_W$$

In this case, T' and T are inverse of each other.

Thm:- If T is an invertible L.T., then its inverse is unique.

Since every T has a matrix A, so inverse of A, if exists, is unique.

Ex: Verify that the mappings $T: \mathbb{R}^2 \to \mathbb{R}^2$ and $T': \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(a,b) = (a, a+b)$ and $T'(c,d) = (c, d-c)$ are inverses.

$$(T' \circ T)(a,b) = T'(T(a,b)) = T'(a, a+b) = (a, a+b-a) = (a,b)$$

$$(T \circ T')(c,d) = T(T'(c,d)) = T(c, d-c) = (c, c+d-c) = (c,d)$$

$\therefore (T' \circ T)(a,b) = (a,b) = I(a,b)$ [$\because I$ is identity L.T.]

& $(T \circ T')(c,d) = (c,d) = I(c,d)$

$\Rightarrow (T' \circ T) = I$ & $(T \circ T') = I$. Verified.

Thm: A L.T. $T: V \to W$ is called non-singular (invertible) if $\mathrm{Ker}(T) = \{0\}$ i.e. T is one-one.

Ex: Let T be a linear trans. from $\mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

Find T^{-1}!

Soln:- Let, $T(x_1, x_2, x_3) = (z_1, z_2, z_3)$. To find T^{-1} s.t.

$$T^{-1}(z_1, z_2, z_3) = (x_1, x_2, x_3).$$

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$$T(x_1, x_2, x_3) = (z_1, z_2, z_3)$$

$$\Rightarrow (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3) = (z_1, z_2, z_3)$$

$$\Rightarrow \quad 3x_1 = z_1 \qquad \Rightarrow x_1 = \frac{z_1}{3}$$

$$x_1 - x_2 = z_2 \qquad \Rightarrow x_2 = x_1 - z_2 = \frac{z_1}{3} - z_2 =$$

$$2x_1 + x_2 + x_3 = z_3 \Rightarrow x_3 = (z_3 - 2x_1 - x_2) = \left(z_3 - \frac{2z_1}{3} - \frac{z_1}{3} + z_2\right)$$

$$= (z_3 - z_1 + z_2)$$

$$\therefore T^{-1}(z_1, z_2, z_3) = \left(\frac{z_1}{3}, \frac{z_1}{3} - z_2, -z_1 + z_2 + z_3\right).$$

\# $T : V \to W$ is onto then $\text{rank}(T) = \dim(W)$ $[= \dim V$, if $V = W]$

" is 1-1 then $\text{nullity}(T) = 0$

\# $S \circ T : V \to W$, $S \circ T$ is one-one then T is one-one

$S \circ T$ is onto then S is onto

\# $\text{rank}(\underset{n \times m}{A}\,\underset{m \times p}{B}) = \text{rank}(A) + \text{rank}(B) - m$

\# Composition of L.T. $[S \circ T]_{B,C} = AB$ (product of matrices if $[S] = A$, $[T] = B$).