Subspace: A subset W of a vector space $V(\mathbb{F})$ is called a subspace of V if W is itself a vector space with same addition, scalar multiplication and scalars as V.

Ex. $W = \{(a,b,0) \mid a,b \in \mathbb{R}\}$ is a vector space and also a subspace of $\mathbb{R}^3(\mathbb{R})$. Geometrically, W is a xy plane and $\mathbb{R}^3$ is 3-D space.

Thm: Let V be a vector space and let W be a non-empty subset of V. Then W is a subspace of V iff the following conditions hold:

(i) If $u, v \in W$ then $u+v \in W$

(ii) If $u \in W$, $c \in \mathbb{F}$ then $cu \in W$.

These 2 conditions can be merged as:- If $u, v \in W$ & $\alpha, \beta \in \mathbb{F}$ then $\alpha u + \beta v \in W$.

Ex. 1 The set of symmetric $n \times n$ matrices is a subspace of $M_{n\times n}$.

2. The subspaces $\{0\}$ and V are called the trivial subspaces of V.

Spanning Sets: If $S = \{v_1, v_2, \dots, v_k\}$ is a set of vectors in a vector space V, then the set of all linear combinations of $v_1, v_2, \dots, v_k$ is called the span of $v_1, v_2, \dots, v_k$ and is denoted by $span(v_1, v_2, \dots, v_k)$ or $span(S)$.
If $V = span(S)$, then S is called a spanning set for V and V is said to be spanned by S.

Ex. 1. The polynomials $1, x$ and x^2 span P_2.

2. The matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ span $M_{2\times 2}$.

Linear combinations of vectors $v_1, v_2, \dots, v_k$ is defined as $c_1v_1 + c_2v_2 + \dots + c_kv_k$ where $c_i \in \mathbb{F}$.

(27) Ex: In P_2, determine wheather $r(x) = 1-4x+6x^2$ is in span$(p(x), q(x))$
where $p(x) = 1-x+x^2$ and $q(x) = 2+x-3x^2$

We are looking for scalars c and d such that $cp(x)+dq(x) = r(x)$.

This means $c(1-x+x^2)+d(2+x-3x^2) = 1-4x+6x^2$

$$\Rightarrow (c+2d)+x(-c+d)+(c-3d)x^2 = 1-4x+6x^2$$

Equating the coefficients of like powers of x gives

$$c+2d = 1$$
$$-c+d = -4$$
$$c-3d = 6$$

which is easily solved to $c = 3$ and $d = -1$.

$\therefore r(x) = 3p(x) - q(x)$.

Hence, $r(x)$ is in span$(p(x), q(x))$.

(28) Ex. Determine whether $\sin 2x$ is in span $(\sin x, \cos x)$

We set $c\sin x + d\cos x = \sin 2x$ and try to determine c and d so that this equation is true. Since these are functions, the equation must be true for all values of x. Setting $x = 0$, we have

$$c\sin 0 + d\cos 0 = \sin 0 \Rightarrow c(0)+d(1) = 0$$

from which we see that $d = 0$.

Setting $x = \pi/2$, we get $c\sin(\pi/2) + d\cos(\pi/2) = \sin 2(\pi/2)$

$$\Rightarrow c(1)+d(0) = 0 \quad \text{giving } c = 0.$$

But this implies that $\sin 2x = 0(\sin x) + 0(\cos x) = 0$ for all x, which is absurd, since $2x$ is not the zero function.

So, $\sin 2x$ is not in span$(\sin x, \cos x)$.

Thm:- Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V.

(a) $\text{span}(v_1, v_2, \ldots, v_k)$ is a subspace of V.

(b) $\text{span}(v_1, v_2, \ldots, v_k)$ is the smallest subspace of V that contains $v_1, v_2, \ldots, v_k$.

Linear Independence:

A set of vectors $\{v_1, v_2, \ldots, v_k\}$ in a vector space V is linearly dependent if there are scalars $c_1, c_2, \ldots, c_k$, at least one of which is not zero, such that $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0$

A set of vectors that is not linearly dependent is said to be linearly independent.

$\{v_1, v_2, \ldots, v_k\}$ is linearly independent in a vector space V iff $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0$ implies $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

Ex. The set $\{1+x+x^2, 1-x+3x^2, 1+3x-x^2\}$ is linearly dependent, since $2(1+x+x^2) - (1-x+3x^2) = 1+3x-x^2$

Ex. In $M_{2\times 2}$, let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Then $A + B - C = 0$, so the set $\{A, B, C\}$ is linearly Dependent.

Ex: In P_2, determine whether the set $\{1+x, x+x^2, 1+x^2\}$ is LI.

Let c_1, c_2, c_3 be scalars s.t.

$$c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) = 0$$

$$\Rightarrow (c_1+c_3) + (c_2+c_1)x + (c_2+c_3)x^2 = 0$$

On comparing coefficients,

$$c_1 + c_3 = 0, \quad c_2 + c_1 = 0, \quad c_2 + c_3 = 0$$

The soln is $c_1 = c_2 = c_3 = 0$.

$\therefore$ The set $\{1+x, x+x^2, 1+x^2\}$ is LI.