Echelon form : A matrix is in Echelon form if it satisfies the following properties:

1. Any rows consisting entirely of zeros are at the bottom. leading entry
2. In each nonzero row, the first nonzero entry (called pivot) is in a column to the left of any leading entries (pivots) below it.

e.g. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ $\begin{bmatrix} 0 & 2 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

Elementary Row Operations: There are three elementary row (column) operations —

1. Interchange of two rows (columns) $R_i \leftrightarrow R_j$
2. Multiply a row (columns) by a nonzero constant $R_i \to cR_i$ $(c \neq 0)$
3. Add a multiple of a row (column) to another row (column) $R_i \to R_i + cR_j$

The process of applying elementary row operations to bring a matrix into row echelon form, called row reduction. And the matrix is called echelon form of matrix.

Ex. Reduce the following matrix to echelon form.

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

We begin by introducing zeros into the first column below the leading 1 in the first row:

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

The 1st column is now as we want it,

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$ $R_4 \to R_4 + 3R_2$ $\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$

Column2 is now done.

$R_3 \to \frac{1}{8} R_3$ $\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$ $R_4 \to R_4 - 29R_3$ $\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$

Echelon form

The row echelon form of a matrix is not unique.

Rank of a matrix: The rank of a matrix is the number of nonzero rows in its row echelon form.

Reduced row echelon form: A matrix is in reduced row echelon form if it satisfies the following properties:

1. It is in row echelon form
2. The leading entry in each nonzero row is 1
3. Each column containing a leading 1 has zeros everywhere else.

e.g. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The reduced row echelon form of a matrix is unique.

Inverse of a matrix: If A is an $n \times n$ matrix, an inverse of A is an $n \times n$ matrix B such that $AB = I_n = BA$, where I_n is the $n \times n$ identity matrix.
If such B exists, then A is called invertible.

To check that the matrix is invertible – find determinant
if determinant of A is non-zero then A is invertible.
(Non-Singular)

Gauss-Jordan Method for computing the inverse

Perform row operations on A and I simultaneously by constructing a "super augmented matrix" [A|I].

If A is row equivalent to I then A is invertible and

$$[A|I] \longrightarrow [I|A^{-1}].$$

Example. Find the inverse of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$ if it exists.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{2})R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & +1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow[R_2+3R_3]{R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{bmatrix}$$

QB38: (i) $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ Find A^{-1}.

QB36. (a) $A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} -3 & 4 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} -2 & 1 & 1 \\ 0 & 1 & 1 \\ -3 & 0 & 6 \end{bmatrix}$

find A^{-1}.