IIIT-Bangalore,
Mathematics for ML (GEN 512)
Assignment Set 1

(Convex sets)

1. Show that:

   (a) Given a family $\{C_a \mid a \in \mathcal{A}\}$ of convex sets, their intersection $\cap_{a\in\mathcal{A}} C_a$ is convex.

   (b) Given a finite collection $C_1, \dots, C_m$ of convex sets, their sum $C_1 + \dots + C_m$ is convex.

2. Let A be an $m \times n$ matrix.

   (a) Let $X \subseteq \mathbb{R}^n$ be a convex set. Show that the forward image AX is convex, where

   $$AX = \{\mathbf{y} \in \mathbb{R}^m \mid \underline{y} = A\underline{x} \text{ for some } \mathbf{x} \in X\}.$$

   (b) Let $Y \subseteq \mathbb{R}^m$ be a convex set. Show that the inverse image $A^{-1}Y$ is convex, where

   $$A^{-1}Y = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{y} \in Y\}.$$

3. Let $P = \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ be a set of m points in $\mathbb{R}^n$. Show that:

   (a) linear span $LS(P)$ is convex.

   (b) affine hull $\mathrm{aff}(P)$ is convex.

   (c) convex hull $\mathrm{conv}(P)$ is convex.

   (d) conic hull $\mathrm{conic}(P)$ is convex.

4. Show that (i) a hyperplane $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = b\} \quad (\mathbf{a} \neq \mathbf{0})$ is convex, (ii) a half-space $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} \leq b\} \quad (\mathbf{a} \neq \mathbf{0})$ is convex.

5. Consider the function

   $$f(\mathbf{x}) = -\sum_{i=1}^{n} \ln(1 + x_i)$$

   where $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$. What is the domain of the function? Is the domain convex?

6. Show that the hyperbolic set $\{\mathbf{x} \in \mathbb{R}^2_+ \mid x_1x_2 \geq 1\}$ is convex. As a generalization, show that $\{\mathbf{x} \in \mathbb{R}^n_+ \mid \Pi_{i=1}^n x_i \geq 1\}$ is convex.

7. (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $\lambda > 0$ be scalar. Show that function λf is convex.

   (b) Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be convex functions. Show that their sum $f_1 + f_2$ is a convex function.

   (c) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and A be an $m \times n$ matrix. Show that their composition $f \circ A$ is a convex function.

   (d) Let $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ be convex functions. Show that the pointwise-max function

   $$F(\mathbf{x}) = \max\{f_1(\mathbf{x}), \dots, f_m(\mathbf{x})\}$$

   is convex.

8. Consider the function

$$f(\mathbf{x}) = -\sum_{i=1}^{n} \ln(1+x_i) \text{ where } \mathbf{x} = (x_1, \dots, x_n).$$

What is the domain of the function? Show that the function is convex.

9. Check whether the following quadratic functions are convex or not:
(i) $p(\mathbf{x}) = 4x_1^2 + 2x_2^2 + 3x_1x_2 + 4x_1 + 5x_2 + 10^5$
(ii) $q(\mathbf{x}) = 4x_1^2 - 2x_2^2 + 3x_1x_2 + 4x_1 + 5x_2 + 10^5$.

10. Show that $f(X) = -\log\det X$ is convex over the domain of symmetric, positive definite matrices, $\operatorname{dom} f = \mathbf{S}_{++}^n$.