IIIT-Bangalore,
Mathematics for ML (GEN 512)
Assignment Set 2

(Optimality conditions)

1. Consider the following optimization problem:

$$\begin{aligned} &\textit{minimize: } x_1^2 + (x_2+1)^2 \\ &\textit{subject to: } -1 \le x_1 \le 1, x_2 \ge 0. \end{aligned}$$

   Use the optimality condition to show that the vector $(0,0)$ is a unique optimal solution.

2. Consider the following unconstrained optimization problem:

$$\begin{aligned} &\textit{minimize: } \sum_{j=1}^{m} \|\underline{x} - \underline{v}_j\|^2 \\ &\textit{subject to: } \underline{x} \in \mathbb{R}^n, \end{aligned}$$

   where $\underline{v}_1, \ldots, \underline{v}_n$ are some given vectors in $\mathbb{R}^n$. Use the optimality condition to find an optimal solution to the problem. Is the solution unique? What is the optimal value?

3. Consider the following constrained optimization problem:

$$\begin{aligned} &\textit{minimize: } f(\underline{x}) \\ &\textit{subject to: } \underline{x} \in X \end{aligned}$$

   where f is a convex and continuously differentiable function, and $X \subseteq \mathbb{R}^n$ in a box constraint of the form

$$X = \{\underline{x} \in \mathbb{R}^n \mid a_i \le x_i \le b_i \text{ for all } i, \}$$

   for some scalars a_i and b_i. Using the optimality conditions verify that $\underline{x}^*$ is an optimal solution if and only if $\underline{x}^*$ satisfies the following conditions for all $i = 1, \ldots, n$:

$$\begin{aligned} \frac{\partial f(\underline{x}^*)}{\partial x_i} &\ge 0 \text{ if } x_i^* = a_i, \\ \frac{\partial f(\underline{x}^*)}{\partial x_i} &= 0 \text{ if } a_i < x_i^* < b_i, \\ \frac{\partial f(\underline{x}^*)}{\partial x_i} &\le 0 \text{ if } x_i^* = b_i. \end{aligned}$$

4. Consider the following constrained optimization problem:

$$\begin{aligned} &\textit{minimize: } \|\underline{x}\|^2 \\ &\textit{subject to: } \underline{a}^T \underline{x} = b \end{aligned}$$

   where $\underline{a} \in \mathbb{R}^n$ with $\underline{a} \ne \underline{0}$ and $b \in \mathbb{R}$. Using the optimality conditions, find an optimal solution to the problem. Is the solution unique? What is the optimal value?