Some special problems

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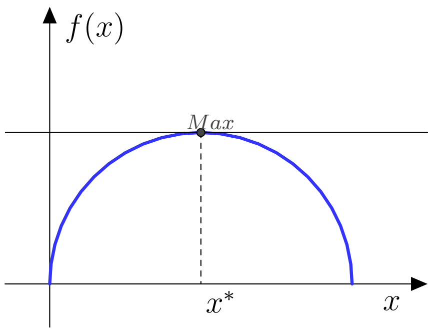
1 Lyrical digression

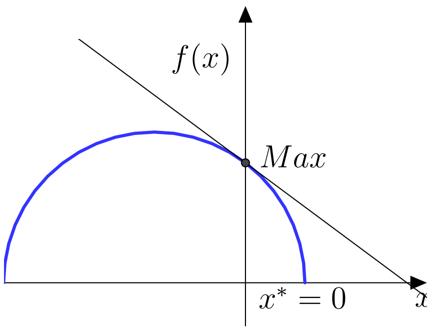
1.1 Kuhn-Tucker conditions: the primer

Consider the following maximization problem:

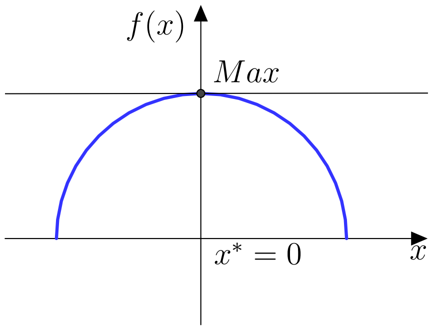
$$f(x) \to \max_{x \geqslant 0} \tag{1}$$



a). Interior solution



b). Corner solution: the first derivative is negative.



c) Corner solution: the first derivative equals to zero.

Figure 1: Optimization problem for nonnegative argument values.

There could be 3 situations:

1. Interior solution $x^* > 0$. This situation is depicted on figure 1.1 a. The condition is $f'_x(x^*) = 0$
2. Corner solution $x^* = 0$. There could be two cases. The first is when the condition $f'_x(x^*) = f'_x(0) = 0$ is holding. The second is when for any value $\forall x \geqslant 0$ $f'_x(x) < 0$, so the maximum value is $x^* = 0$. See figure 1.1 b, c.
3. No solution. There will be no maximum of the function if $\forall x \geqslant 0$ $f'_x(x) > 0$.

All the above-mentioned conditions could be expressed in the following form. If x^* is the solution of the maximization problem $f(x^*) = \max_{x \geqslant 0}(f(x))$ then:

$$\begin{cases} f'_x(x^*) \leqslant 0, \ if \ x^* = 0; \\ f'_x(x^*) = 0, \ if \ x^* > 0. \end{cases} \tag{2}$$

Consider now the following problem of the convex optimization:

$$f(x_1, x_2, \dots, x_n) \to \max_{x_i} \tag{3}$$

$$s.t. \begin{cases} g_s(x_1, x_2, \dots, x_n) \leqslant 0, \ s = 1, \dots, S \\ h_k(x_1, x_2, \dots, x_n) = 0, \ k = 1, \dots, K \\ x_i \geqslant 0 \ i = 1, \dots, n \end{cases} \tag{4}$$

where $f(x_1, x_2, \dots, x_n)$ — convex function, $g_j(x_1, x_2, \dots, x_n)$ and $h_k(x_1, x_2, \dots, x_n)$ — restriction functions. Lagrangian:

$$\mathcal{L} = f(x_1, x_2, \dots, x_n) - \sum_{s=1}^{S} \lambda_s g_s(x_1, x_2, \dots, x_n) - \sum_{k=1}^{K} \mu_k h_k(x_1, x_2, \dots, x_n) \tag{5}$$

Then the First Order Conditions are the following:

$$\forall i = 1, \dots, n \qquad \begin{cases} \mathcal{L}'_{x_i} = 0, \ if \ x_i > 0; \\ \mathcal{L}'_{x_i} \leqslant 0, \ if \ x_i = 0. \end{cases} \tag{6}$$

$$\forall s = 1, \dots, S \qquad \begin{cases} g_s(x_1, x_2, \dots, x_n) = 0, \ if \ \lambda_s > 0; \\ g_s(x_1, x_2, \dots, x_n) \leqslant 0, \ if \ \lambda_s = 0. \end{cases} \tag{7}$$

$$\begin{cases} h_k(x_1, x_2, \dots, x_n) = 0, \ k = 1, \dots, K \\ x_i \geqslant 0 \ i = 1, \dots, n \end{cases} \tag{8}$$

2 The Monopolist

2.1 Two locations with different total costs

A monopolist produces output in two different locations. $TC_1 = c_1 + b_1 q_1 + a_1 q_1^2$ $TC_2 = c_2 + b_2 q_2 + a_2 q_2^2$ The monopolist's problem is the following:

$$TC = TC_1 + TC_2 = c_1 + b_1 q_1 + a_1 q_1^2 + c_2 + b_2 q_2 + a_2 q_2^2 \to \min_{q_1, q_2} \tag{9}$$

$$s.t. \begin{cases} q = q_1 + q_2 \\ q_1 \geqslant 0; \ q_2 \geqslant 0 \end{cases} \tag{10}$$

$$\mathcal{L} = -c_1 - b_1 q_1 - a_1 q_1^2 - c_2 - b_2 q_2 - a_2 q_2^2 + \lambda(q_1 + q_2 - q) \to \max_{q_1, q_2} \tag{11}$$

First Order Conditions:

$$\begin{cases} \mathcal{L}'_{q_1} = -b_1 - 2a_1 q_1 + \lambda = 0, \ if \ q_1 > 0 \\ \mathcal{L}'_{q_1} = -b_1 - 2a_1 q_1 + \lambda \leqslant 0, \ if \ q_1 = 0 \end{cases} \tag{12}$$

$$\begin{cases} \mathcal{L}'_{q_2} = -b_2 - 2a_2 q_2 + \lambda = 0, \ if \ q_2 > 0 \\ \mathcal{L}'_{q_2} = -b_2 - 2a_2 q_2 + \lambda \leqslant 0, \ if \ q_2 = 0 \end{cases} \tag{13}$$

Let's consider 3 cases:

1. $q_1 = 0, \ q_2 > 0$. Then,

$$\begin{cases} -b_2 - 2a_2 q_2 + \lambda = 0 \\ -b_1 - 2a_1 \underbrace{q_1}_{=0} + \lambda \leqslant 0 \end{cases} \Leftrightarrow \begin{cases} b_2 + 2a_2 q_2 = \lambda \\ -b_1 + b_2 + 2a_2 q_2 \leqslant 0 \end{cases} \Leftrightarrow q = q_2 \leqslant \frac{b_1 - b_2}{2a_2} \tag{14}$$

2. $q_1 > 0, \ q_2 = 0$. Then,

$$\begin{cases} -b_1 - 2a_1 q_1 + \lambda = 0 \\ -b_2 - 2a_2 \underbrace{q_2}_{=0} + \lambda \leqslant 0 \end{cases} \Leftrightarrow \begin{cases} b_1 + 2a_1 q_1 = \lambda \\ -b_2 + b_1 + 2a_1 q_1 \leqslant 0 \end{cases} \Leftrightarrow q = q_1 \leqslant \frac{b_2 - b_1}{2a_1} \tag{15}$$

3. $q_1 > 0,\ q_2 > 0$. Then,

$$\begin{cases} -b_1 - 2a_1q_1 + \lambda = 0 \\ -b_2 - 2a_2q_2 + \lambda = 0 \end{cases} \Leftrightarrow \ \lambda = b_1 + 2a_1q_1 = b_2 + 2a_2q_2 \ \Leftrightarrow \ q_1 = \frac{(b_2 - b_1)}{2a_1} + \frac{a_2}{a_1}q_2 \tag{16}$$

Substituting this condition into the restriction one can get:

$$q = q_1 + q_2 = \frac{b_2 - b_1}{2a_1} + \underbrace{\left(\frac{a_2}{a_1} + 1\right)}_{\frac{a_2+a_1}{a_1}} q_2 \tag{17}$$

$$q_2 = \frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q \tag{18}$$

$$q_1 = \frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q \tag{19}$$

To derive the total marginal function one should find the first derivative of the marginal function by total quantity produced q:

$$TC(q) = (c_1 + c_2) + b_2\left(\frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q\right) + b_1\left(\frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q\right) + \\ + a_2\left(\frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q\right)^2 + a_1\left(\frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q\right)^2 \tag{20}$$

$$MC(q) = TC'_q(q) = \frac{b_2a_1}{a_2 + a_1} + \frac{b_1a_2}{a_2 + a_1} + 2a_2\frac{a_1}{a_2 + a_1}\left(\frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q\right) + \\ + 2a_1\frac{a_2}{a_2 + a_1}\left(\frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q\right) = \frac{b_2a_1 + b_1a_2}{a_2 + a_1} + \frac{4a_1a_2}{a_1 + a_2}q \tag{21}$$

Case 1. $b_2 > b_1$.
The monopolist's marginal costs is the following function:

$$MC(q) = \begin{cases} MC \in [0, b_1)\,,\ if\ q = 0; \\ b_1 + 2a_1q,\ if\ 0 < q \leqslant \frac{b_2 - b_1}{2a_2}; \\ \frac{b_2a_1 + b_1a_2}{a_2 + a_1} + \frac{4a_1a_2}{a_1 + a_2}q,\ if\ q > \frac{b_2 - b_1}{2a_2} \end{cases} \tag{22}$$

Example ($b_2 > b_1$). Consider the following total costs functions:

$$\begin{cases} TC_1 = 100 + 5q_1 + 2q_1^2 \\ TC_2 = 200 + 10q_2 + q_2^2 \end{cases} \tag{23}$$

If the monopolist produces only in the first location, the marginal costs will be the following:

$$\begin{cases} MC_1 \in [0, 5),\ if\ q = 0 \\ MC_1 = 5 + 4q,\ if\ q > 0 \end{cases} \tag{24}$$

If the monopolist produces only in the second location, the marginal costs will be the following expression:

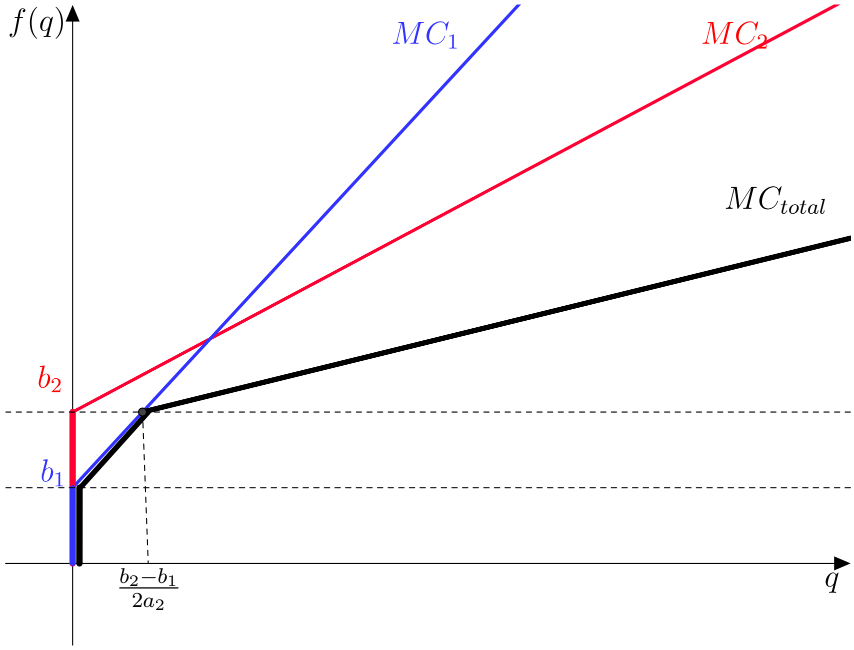


Figure 2: Case 1: $b_2 > b_1$.

$$\begin{cases} MC_2 \in [0, 10), \ if \ q = 0 \\ MC_2 = 10 + 2q, \ if \ q > 0 \end{cases} \tag{25}$$

If the monopolist produces in two locations simultaneously, the relations between q_i and $q = q_1 + q_2$ are the following (see equations (19) and (18)):

$$q_2 = \frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1} q = -\frac{5}{6} + \frac{2}{3} q \tag{26}$$

$$q_1 = \frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1} q = \frac{5}{6} + \frac{1}{3} q \tag{27}$$

From the equation (21) we can get:

$$MC = TC'_q(q) = \frac{b_2 a_1 + b_1 a_2}{a_2 + a_1} + \frac{4 a_1 a_2}{a_1 + a_2} q = 5 + \frac{8}{3} q \tag{28}$$

This is the case only if $q > \frac{b_2 - b_1}{2a_2} = 2.5$. From the equation (29), one can get (see figure 2.1):

$$MC(q) = \begin{cases} MC \in [0, 5), \ if \ q = 0; \\ 5 + 4q, \ if \ 0 < q \leqslant 2.5; \\ 5 + (8/3) q, \ if \ q > 2.5 \end{cases} \tag{29}$$

Case 2. $b_1 > b_2$. The same logic holds for this case. See figure 2.1.

$$MC(q) = \begin{cases} MC \in [0, b_2), \ if \ q = 0; \\ b_2 + 2a_2 q, \ if \ q \leqslant \frac{b_1 - b_2}{2a_1} \\ \frac{b_2 a_1 + b_1 a_2}{a_2 + a_1} + \frac{4 a_1 a_2}{a_1 + a_2} q, \ if \ q > \frac{b_1 - b_2}{2a_1} \end{cases} \tag{30}$$

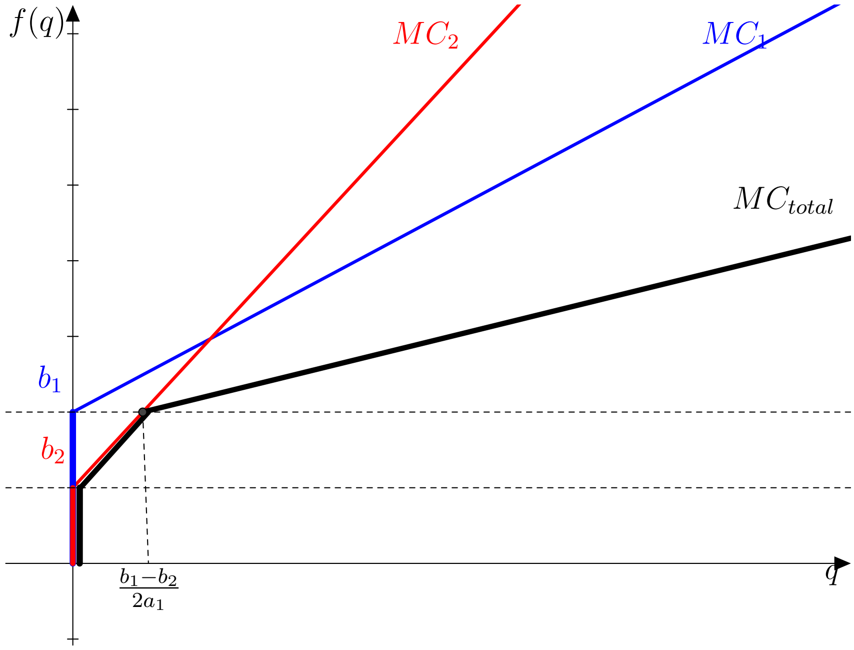


Figure 3: Case 2: $b_2 > b_1$.

Example 2 ($b_1 > b_2$). Show, please, that for the following total costs functions:

$$\begin{cases} TC_1 = 1000 + 20q_1 + 3q_1^2 \\ TC_2 = 2100 + 10q_2 + q_2^2 \end{cases} \tag{31}$$

the marginal costs function looks like this:

$$MC(q) = \begin{cases} MC \in [0, 10), \ if \ q = 0; \\ 10 + 2q, \ if \ q \leqslant 5/3 \\ 12.5 + 3q, \ if \ q > 5/3 \end{cases} \tag{32}$$

Case 3. $b_1 = b_2$.

$$MC(q) = \begin{cases} MC \in [0, b1), \ if \ q = 0; \\ \frac{b_2a_1+b_1a_2}{a_2+a_1} + \frac{4a_1a_2}{a_1+a_2}q = b_1 + \frac{4a_1a_2}{a_1+a_2}q, \ \forall q > 0 \end{cases} \tag{33}$$

Example 3 ($b_1 = b_2$). Show, please, that for the following total costs functions:

$$\begin{cases} TC_1 = 1000 + 10q_1 + 3q_1^2 \\ TC_2 = 2100 + 10q_2 + q_2^2 \end{cases} \tag{34}$$

the marginal costs function looks like this:

$$MC(q) = \begin{cases} MC \in [0, 10), \ if \ q = 0; \\ 10 + 3q, \ if \ q > 0 \end{cases} \tag{35}$$

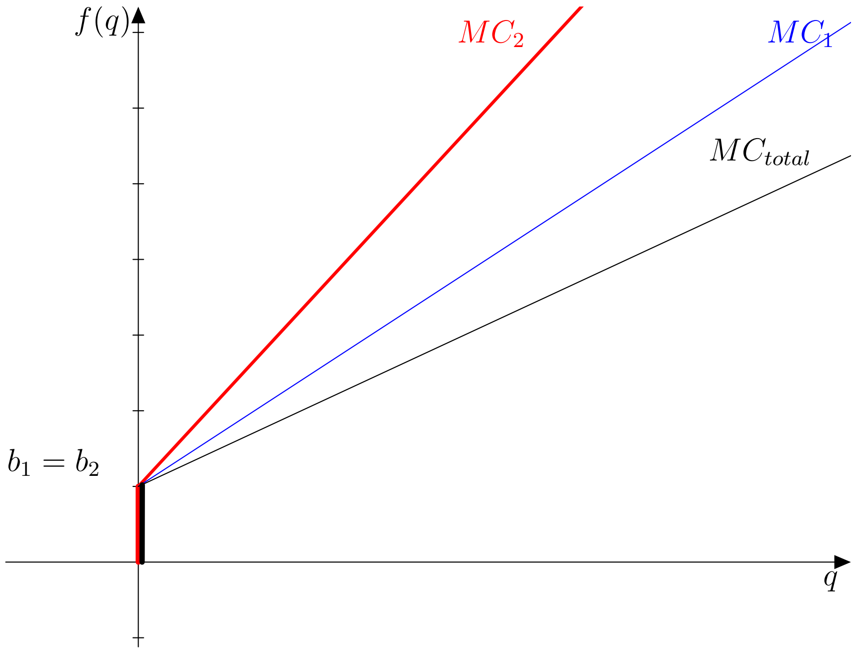


Figure 4: Case 3: $b_1 = b_2$.