Some special problems

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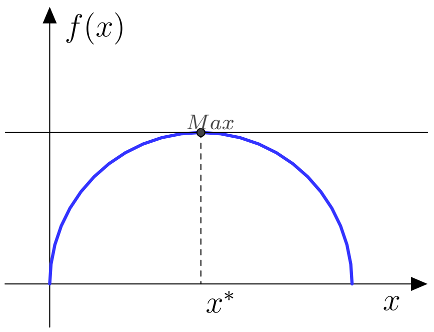
1 Lyrical digression

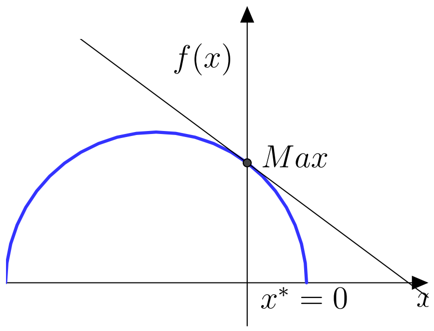
1.1 Kuhn-Tucker conditions: the primer

Consider the following maximization problem:

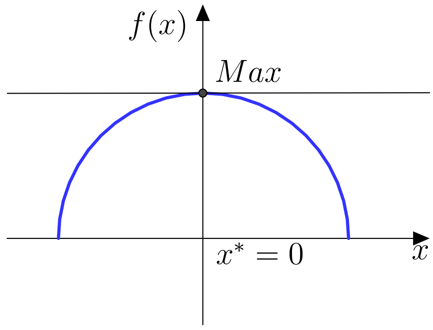
$$f(x) \to \max_{x \geqslant 0} \tag{1}$$



a). Interior solution



b). Corner solution: the first derivative is negative.



c) Corner solution: the first derivative equals to zero.

Figure 1: Optimization problem for nonnegative argument values.

There could be 3 situations:

1. Interior solution $x^* > 0$. This situation is depicted on figure 1.1 a. The condition is $f'_x(x^*) = 0$
2. Corner solution $x^* = 0$. There could be two cases. The first is when the condition $f'_x(x^*) = f'_x(0) = 0$ is holding. The second is when for any value $\forall x \geqslant 0$ $f'_x(x) < 0$, so the maximum value is $x^* = 0$. See figure 1.1 b, c.
3. No solution. There will be no maximum of the function if $\forall x \geqslant 0$ $f'_x(x) > 0$.

All the above-mentioned conditions could be expressed in the following form. If x^* is the solution of the maximization problem $f(x^*) = \max_{x \geqslant 0}(f(x))$ then:

$$\begin{cases} f'_x(x^*) \leqslant 0, \ if \ x^* = 0; \\ f'_x(x^*) = 0, \ if \ x^* > 0. \end{cases} \tag{2}$$

Consider now the following problem of the convex optimization:

$$f(x_1, x_2, \dots, x_n) \to \max_{x_i} \tag{3}$$

$$s.t. \begin{cases} g_s(x_1, x_2, \dots, x_n) \leqslant 0, \ s = 1, \dots, S \\ h_k(x_1, x_2, \dots, x_n) = 0, \ k = 1, \dots, K \\ x_i \geqslant 0 \ i = 1, \dots, n \end{cases} \tag{4}$$

where $f(x_1, x_2, \dots, x_n)$ — convex function, $g_j(x_1, x_2, \dots, x_n)$ and $h_k(x_1, x_2, \dots, x_n)$ — restriction functions. Lagrangian:

$$\mathcal{L} = f(x_1, x_2, \dots, x_n) - \sum_{s=1}^{S} \lambda_s g_s(x_1, x_2, \dots, x_n) - \sum_{k=1}^{K} \mu_k h_k(x_1, x_2, \dots, x_n) \tag{5}$$

Then the First Order Conditions are the following:

$$\forall i = 1, \dots, n \qquad \begin{cases} \mathcal{L}'_{x_i} = 0, \ if \ x_i > 0; \\ \mathcal{L}'_{x_i} \leqslant 0, \ if \ x_i = 0. \end{cases} \tag{6}$$

$$\forall s = 1, \dots, S \qquad \begin{cases} g_s(x_1, x_2, \dots, x_n) = 0, \ if \ \lambda_s > 0; \\ g_s(x_1, x_2, \dots, x_n) \leqslant 0, \ if \ \lambda_s = 0. \end{cases} \tag{7}$$

$$\begin{cases} h_k(x_1, x_2, \dots, x_n) = 0, \ k = 1, \dots, K \\ x_i \geqslant 0 \ i = 1, \dots, n \end{cases} \tag{8}$$

2 The Monopolist

2.1 Two locations with different total costs

A monopolist produces output in two different locations. $TC_1 = c_1 + b_1 q_1 + a_1 q_1^2$ $TC_2 = c_2 + b_2 q_2 + a_2 q_2^2$ The monopolist's problem is the following:

$$TC = TC_1 + TC_2 = c_1 + b_1 q_1 + a_1 q_1^2 + c_2 + b_2 q_2 + a_2 q_2^2 \to \min_{q_1, q_2} \tag{9}$$

$$s.t. \begin{cases} q = q_1 + q_2 \\ q_1 \geqslant 0; \ q_2 \geqslant 0 \end{cases} \tag{10}$$

$$\mathcal{L} = -c_1 - b_1 q_1 - a_1 q_1^2 - c_2 - b_2 q_2 - a_2 q_2^2 + \lambda(q_1 + q_2 - q) \to \max_{q_1, q_2} \tag{11}$$

First Order Conditions:

$$\begin{cases} \mathcal{L}'_{q_1} = -b_1 - 2a_1 q_1 + \lambda = 0, \ if \ q_1 > 0 \\ \mathcal{L}'_{q_1} = -b_1 - 2a_1 q_1 + \lambda \leqslant 0, \ if \ q_1 = 0 \end{cases} \tag{12}$$

$$\begin{cases} \mathcal{L}'_{q_2} = -b_2 - 2a_2 q_2 + \lambda = 0, \ if \ q_2 > 0 \\ \mathcal{L}'_{q_2} = -b_2 - 2a_2 q_2 + \lambda \leqslant 0, \ if \ q_2 = 0 \end{cases} \tag{13}$$

Let's consider 3 cases:

1. $q_1 = 0, \ q_2 > 0$. Then,

$$\begin{cases} -b_2 - 2a_2 q_2 + \lambda = 0 \\ -b_1 - 2a_1 \underbrace{q_1}_{=0} + \lambda \leqslant 0 \end{cases} \Leftrightarrow \begin{cases} b_2 + 2a_2 q_2 = \lambda \\ -b_1 + b_2 + 2a_2 q_2 \leqslant 0 \end{cases} \Leftrightarrow q = q_2 \leqslant \frac{b_1 - b_2}{2a_2} \tag{14}$$

2. $q_1 > 0, \ q_2 = 0$. Then,

$$\begin{cases} -b_1 - 2a_1 q_1 + \lambda = 0 \\ -b_2 - 2a_2 \underbrace{q_2}_{=0} + \lambda \leqslant 0 \end{cases} \Leftrightarrow \begin{cases} b_1 + 2a_1 q_1 = \lambda \\ -b_2 + b_1 + 2a_1 q_1 \leqslant 0 \end{cases} \Leftrightarrow q = q_1 \leqslant \frac{b_2 - b_1}{2a_1} \tag{15}$$

3. $q_1 > 0,\ q_2 > 0$. Then,

$$\begin{cases} -b_1 - 2a_1q_1 + \lambda = 0 \\ -b_2 - 2a_2q_2 + \lambda = 0 \end{cases} \Leftrightarrow \lambda = b_1 + 2a_1q_1 = b_2 + 2a_2q_2 \Leftrightarrow q_1 = \frac{(b_2 - b_1)}{2a_1} + \frac{a_2}{a_1}q_2 \tag{16}$$

Substituting this condition into the restriction one can get:

$$q = q_1 + q_2 = \frac{b_2 - b_1}{2a_1} + \underbrace{\left(\frac{a_2}{a_1} + 1\right)}_{\frac{a_2+a_1}{a_1}} q_2 \tag{17}$$

$$q_2 = \frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q \tag{18}$$

$$q_1 = \frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q \tag{19}$$

To derive the total marginal function one should find the first derivative of the marginal function by total quantity produced q:

$$TC(q) = (c_1 + c_2) + b_2\left(\frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q\right) + b_1\left(\frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q\right) + \\ + a_2\left(\frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q\right)^2 + a_1\left(\frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q\right)^2 \tag{20}$$

$$MC(q) = TC'_q(q) = \frac{b_2a_1}{a_2 + a_1} + \frac{b_1a_2}{a_2 + a_1} + 2a_2\frac{a_1}{a_2 + a_1}\left(\frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q\right) + \\ + 2a_1\frac{a_2}{a_2 + a_1}\left(\frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q\right) = \frac{b_2a_1 + b_1a_2}{a_2 + a_1} + \frac{4a_1a_2}{a_1 + a_2}q \tag{21}$$

Case 1. $b_2 > b_1$.
The monopolist's marginal costs is the following function:

$$MC(q) = \begin{cases} MC \in [0, b_1)\,,\ if\ q = 0; \\ b_1 + 2a_1q,\ if\ 0 < q \leqslant \frac{b_2 - b_1}{2a_2}; \\ \frac{b_2a_1 + b_1a_2}{a_2 + a_1} + \frac{4a_1a_2}{a_1 + a_2}q,\ if\ q > \frac{b_2 - b_1}{2a_2} \end{cases} \tag{22}$$

Example ($b_2 > b_1$). Consider the following total costs functions:

$$\begin{cases} TC_1 = 100 + 5q_1 + 2q_1^2 \\ TC_2 = 200 + 10q_2 + q_2^2 \end{cases} \tag{23}$$

If the monopolist produces only in the first location, the marginal costs will be the following:

$$\begin{cases} MC_1 \in [0, 5),\ if\ q = 0 \\ MC_1 = 5 + 4q,\ if\ q > 0 \end{cases} \tag{24}$$

If the monopolist produces only in the second location, the marginal costs will be the following expression:

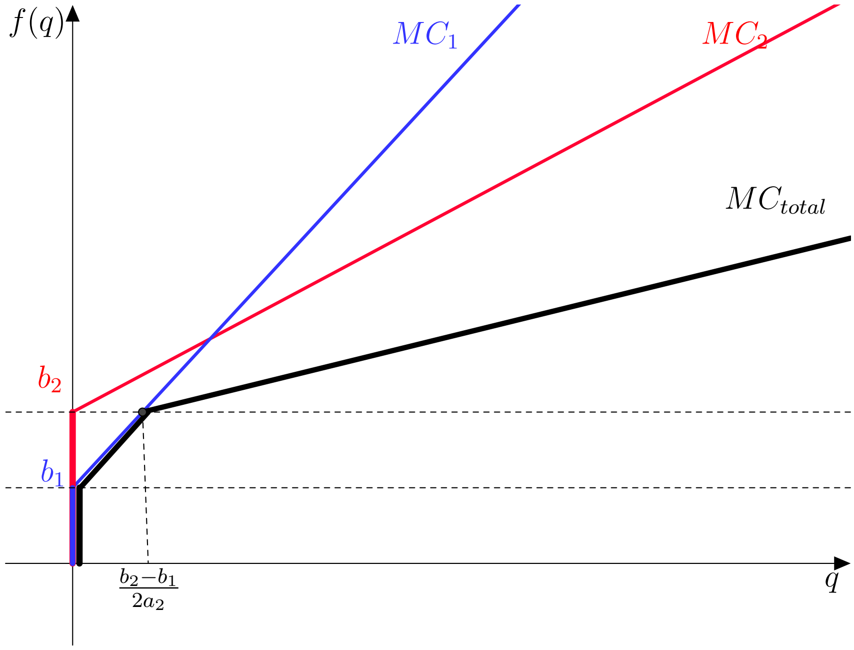


Figure 2: Case 1: $b_2 > b_1$.

$$\begin{cases} MC_2 \in [0, 10), \; if \;\; q = 0 \\ MC_2 = 10 + 2q, \; if \;\; q > 0 \end{cases} \tag{25}$$

If the monopolist produces in two locations simultaneously, the relations between q_i and $q = q_1 + q_2$ are the following (see equations (19) and (18)):

$$q_2 = \frac{b_1 - b_2}{2(a_2 + a_1)} + \frac{a_1}{a_2 + a_1}q = -\frac{5}{6} + \frac{2}{3}q \tag{26}$$

$$q_1 = \frac{b_2 - b_1}{2(a_2 + a_1)} + \frac{a_2}{a_2 + a_1}q = \frac{5}{6} + \frac{1}{3}q \tag{27}$$

From the equation (21) we can get:

$$MC = TC'_q(q) = \frac{b_2 a_1 + b_1 a_2}{a_2 + a_1} + \frac{4a_1 a_2}{a_1 + a_2}q = 5 + \frac{8}{3}q \tag{28}$$

This is the case only if $q > \frac{b_2 - b_1}{2a_2} = 2.5$. From the equation (29), one can get (see figure 2.1):

$$MC(q) = \begin{cases} MC \in [0, 5)\,, \; if \;\; q = 0; \\ 5 + 4q, \; if \;\; 0 < q \leqslant 2.5; \\ 5 + (8/3)q, \; if \;\; q > 2.5 \end{cases} \tag{29}$$

Case 2. $b_1 > b_2$. The same logic holds for this case. See figure 2.1.

$$MC(q) = \begin{cases} MC \in [0, b_2), \; if \;\; q = 0; \\ b_2 + 2a_2 q, \; if \;\; q \leqslant \frac{b_1 - b_2}{2a_1} \\ \frac{b_2 a_1 + b_1 a_2}{a_2 + a_1} + \frac{4a_1 a_2}{a_1 + a_2}q, \; if \;\; q > \frac{b_1 - b_2}{2a_1} \end{cases} \tag{30}$$

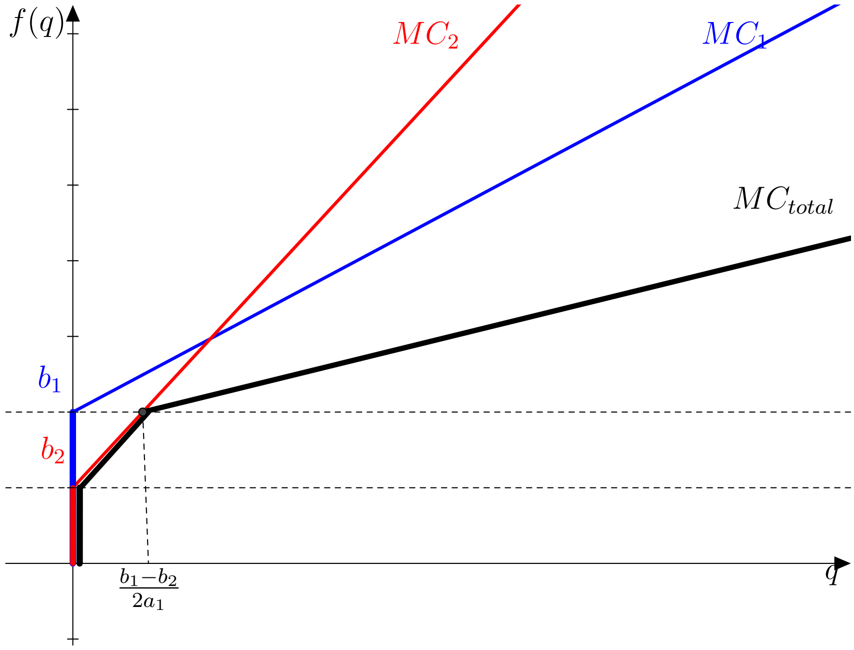


Figure 3: Case 2: $b_2 > b_1$.

Example 2 ($b_1 > b_2$). Show, please, that for the following total costs functions:

$$\begin{cases} TC_1 = 1000 + 20q_1 + 3q_1^2 \\ TC_2 = 2100 + 10q_2 + q_2^2 \end{cases} \tag{31}$$

the marginal costs function looks like this:

$$MC(q) = \begin{cases} MC \in [0, 10), \; if \;\; q = 0; \\ 10 + 2q, \; if \;\; q \leqslant 5/3 \\ 12.5 + 3q, \; if \;\; q > 5/3 \end{cases} \tag{32}$$

Case 3. $b_1 = b_2$.

$$MC(q) = \begin{cases} MC \in [0, b1), \; if \;\; q = 0; \\ \frac{b_2a_1+b_1a_2}{a_2+a_1} + \frac{4a_1a_2}{a_1+a_2}q = b_1 + \frac{4a_1a_2}{a_1+a_2}q, \;\; \forall q > 0 \end{cases} \tag{33}$$

Example 3 ($b_1 = b_2$). Show, please, that for the following total costs functions:

$$\begin{cases} TC_1 = 1000 + 10q_1 + 3q_1^2 \\ TC_2 = 2100 + 10q_2 + q_2^2 \end{cases} \tag{34}$$

the marginal costs function looks like this:

$$MC(q) = \begin{cases} MC \in [0, 10), \; if \;\; q = 0; \\ 10 + 3q, \; if \;\; q > 0 \end{cases} \tag{35}$$

2.2 Third Degree Price Discrimination with Non-constant Marginal Costs

A seller faces the following demand curves by two identifiable groups of consumers: $q_H = H - hp_H$ and $q_L = L - lp_L$, where $H > L$. The monopoly produces units in a single plant and the total cost is $TC(q) = aq^3 + bq$.

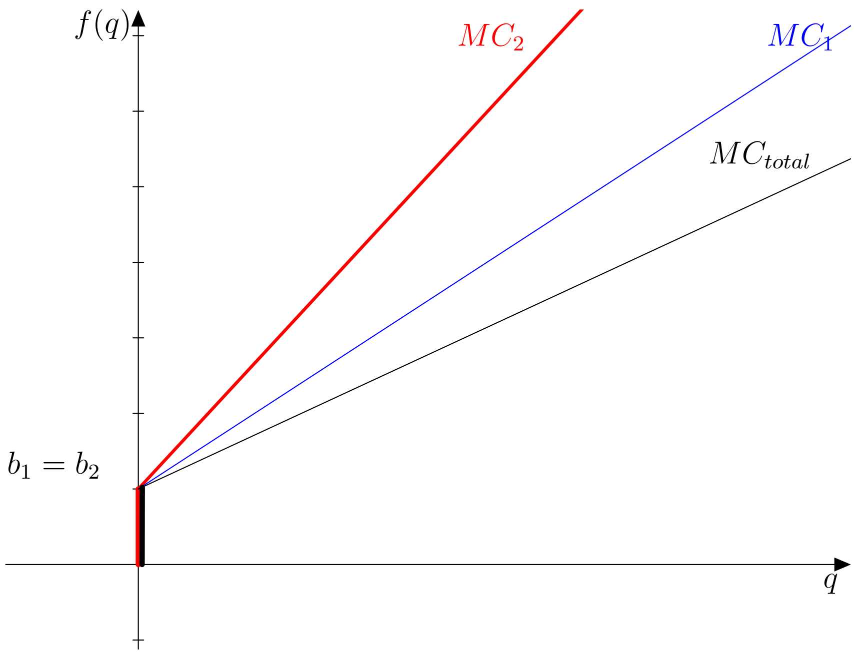


Figure 4: Case 3: $b_1 = b_2$.

(1). Assume that the seller can charge a different two part tariff in each group of the form $P_i = p_i q_i + A_i$, where $i \in \{L, H\}$ indexes the group. Find P_H and P_L that maximize each group's profit.

The maximum profit in the case of two-part tariff is achieved when a price equals to marginal costs, and entry fee equals to each group consumer surplus.

It is obvious that if either the monopolist serves both groups or not, the marginal costs are equal to $MC(q) = 3aq^2 + b$. It means that the marginal costs have the same value for both groups. Hence, the price for both groups should also be the same: $p_H = p_L = p$. Let's find it.

First of all it is necessary to find a market demand. To derive it one should sum quantities demanded by both groups subject to market price. Hence, we get:

$$Q_m = \begin{cases} 0, \ if \ p \in [H, \infty); \\ H - hp, \ if \ p \in [L, H); \\ (L + H) - (h + l)p \ if \ p \in [0, L). \end{cases} \tag{36}$$

Figures 2.2 and 2.2 depict the following three cases:

(a). If $b > H$ there is no way a monopolist could serve both groups. So the equilibrium is empty set.

(b). If $b \in (L - 3a(H - hL)^2, H]$[1] the only "High" group should be served due to high marginal costs. So the price will be the following:

$$p = \frac{H}{h} - \frac{q}{h} = 3aq^2 + b$$

q Hence, to derive quantity produced, one should solve the following equation:

$$q^2 + \frac{q}{3ah} + \frac{3ab - H}{3ah} = 0$$

It is obvious that for $q > 0$ the solution is the following:

$$q^* = q^*_H = \frac{-(3ah)^{-1} + \sqrt{D}}{2},$$

[1]To derive the lower bound the quantity $= H - hL$ should be substituted in the "High" demand function.

where $D = (3ah)^{-2} - 4(3ab - H)/(3ah)$.

Optimal price should be determined as:

$$p^* = \frac{H}{h} - \frac{q^*}{h},$$

since only "High" group is served.

To derive an entry fee one should simply derive $A_H = CS_H = \frac{1}{2}(H - p^*)q^*$, since in this case only "High" group is served.

Overall,

$$A_H + p_H q_H = \frac{1}{2}(H - p^*)q^* + p^* q^* = \frac{1}{2}(H + p^*)q^* \qquad A_L + p_L q_L = \emptyset \tag{37}$$

(c). If $b \leqslant L - 3a(H - hL)^2$. The optimal price is the following:

$$p = \frac{H + L}{h + l} - \frac{q}{h + l} = 3aq^2 + b \tag{38}$$

The quantity produced is the solution of the following equation:

$$q^2 + \frac{q}{3a(h + l)} + \frac{b(h + l) - (H + L)}{3a(h + l)} = 0$$

Hence,

$$q^* = q_H^* + q_L^* = \frac{-3a(h + l) + \sqrt{D}}{2},$$

where $D = (3a(h + l))^2 - 4(b(h + l) - (H + L))/(3a(h + l))$.

Optimal price is the following:

$$p^* = \frac{L + H}{h + l} - \frac{q^*}{h + l}$$

To derive the quantity demanded by the "High" group, one should substitute the optimal price found above into the "High" group demand function:

$$q_H^* = H - hp^* \tag{39}$$

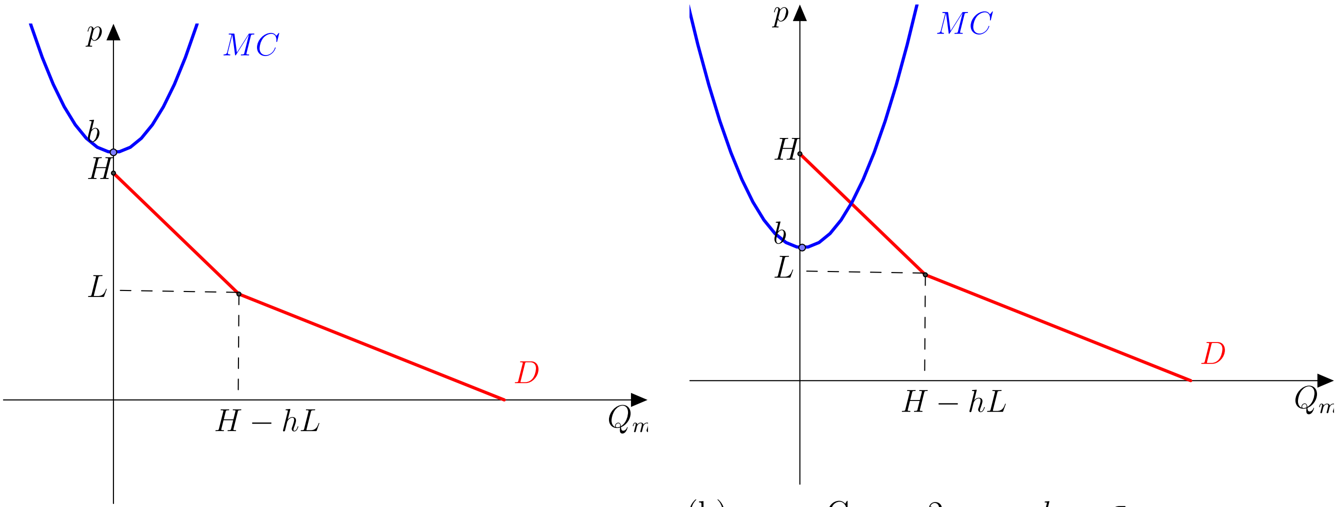
The same is true for the "Low" group:

$$q_L^* = L - lp^* \tag{40}$$

The entry fees for two consumers could be derived as $A_H = CS_H = 0.5(H - p^*)q_H^*$[2] and $A_L = CS_L = 0.5(L - p^*)q_L^*$. Hence, we get:

$$A_H + p^* q_H^* = 0.5(H - p^*)q_H^* + p^* q_H^* = 0.5(H + p^*)q_H^* \tag{41}$$
$$A_L + p^* q_L^* = 0.5(L - p^*)q_L^* + p^* q_L^* = 0.5(L + p^*)q_L^* \tag{42}$$

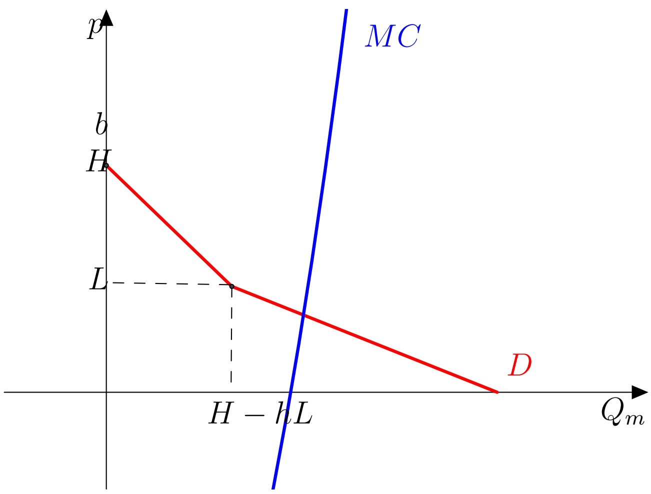
[2]Note that the optimal quantity demanded for only "High" group q_H^* is used.



(a). Case 1: $b > H$.

(b). Case 2. $b \in (L - 3a(H - hL)^2, H]$

Figure 5: Price Determination.



(c). Case 3. $b \leqslant L - 3a(H - hL)^2$.

Figure 6: Price Determination (continued).

(2). If now the seller must charge a single tariff for both groups, $A + pq$, how should she set A and p?

The price would be the same as found in the previous task. The only difference is that in the case of both groups served the entry fee should be set at the value of the "Low" group's consumer surplus: $A = CS_L = 0.5(L - p^*)q_L$, but the firm's total revenue is equal to $2A + p^*(q_H^* + q_L^*)$.