Approximation Algorithms

Analysis of Algorithms

Motivation: Getting around NP-completeness

- Exponential time may be acceptable for small inputs [Brute Force]
- Isolate special cases that can run in polynomial time [Divide-and-Conquer]
- Near-optimal solutions may be acceptable [Approximation Algorithms]

Performance Ratios

- C* is the cost of the optimal solution
- C is the cost of the solution produced by an approximation algorithm
- An algorithm has an approximation ratio of $\rho(n)$ if for an input of size n, C is within a factor of C*

Performance Ratios [cont.]

- Maximization problem
  - 0 < C <= C*
  - C* / C factor by which the cost of the optimal solution is larger than the cost of the approximate solution
- Minimization problem
  - 0 < C* <= C
  - C / C* factor by which the cost of the approximate solution is larger than the cost of the optimal solution

Definitions

- *Approximation Algorithm*
  - produces "near-optimal" solution
- Algorithm has *approximation ratio* $\rho(n)$ if:

$$\boxed{\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \le \rho(n)}$$

C = cost of algorithm's solution

*C** = cost of <u>optimal</u> solution

n = number of inputs = size of instance

- *Approximation Scheme*
  - $(1+\varepsilon)$-approximation algorithm for fixed ε
    - $\varepsilon > 0$ is an input
  - *Polynomial-Time Approximation Scheme*
    - time is polynomial in *n*

How do approximations algorithms work?

- Exploit the nature of the problem
- Use greedy techniques
- Use linear programming
- Use dynamic programming
- Use random assignments

Overview

- VERTEX-COVER
  - Polynomial-time 2-approximation algorithm
- TSP
  - TSP with triangle inequality
    - Polynomial-time 2-approximation algorithm
  - TSP without triangle inequality
    - Negative result on polynomial-time $\rho(n)$-approximation algorithm
- SET-COVER
    - polynomial-time $\rho(n)$-approximation algorithm
      - $\rho(n)$ is a logarithmic function of set size
- SUBSET-SUM
  - Fully polynomial-time approximation scheme
- MAX-3-CNF Satisfiability
  - Randomized $\rho(n)$-approximation algorithm

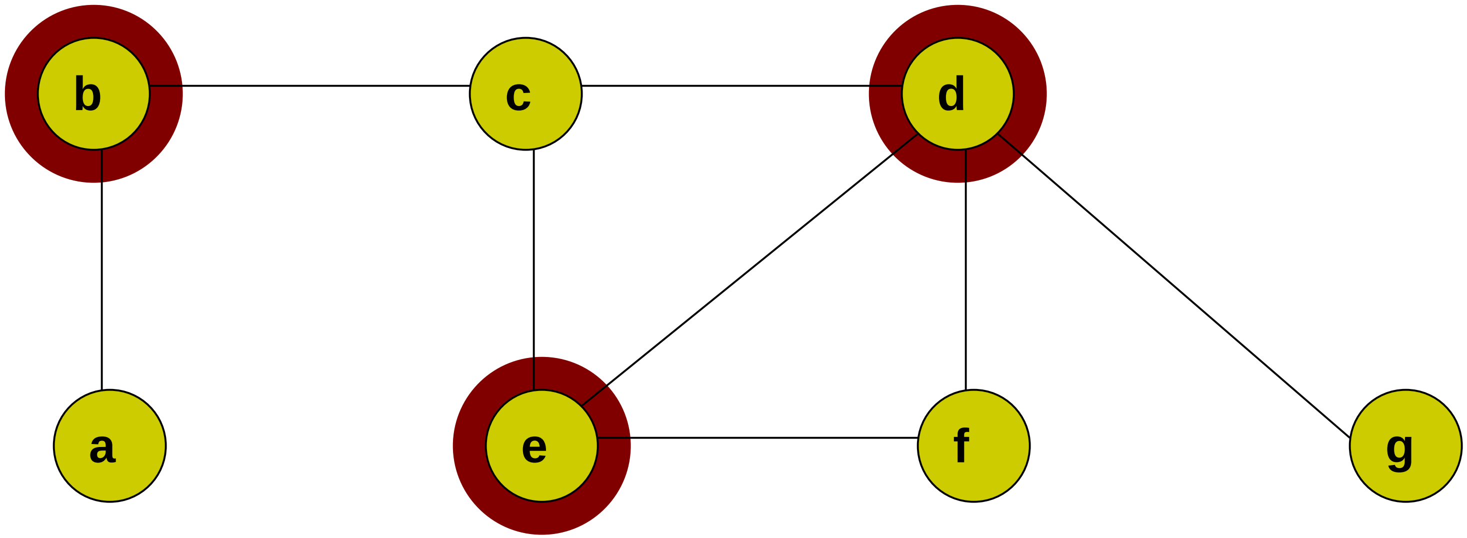
The vertex-cover problem

- Let G=(V,E) be an undirected graph
- A vertex cover
  - subset V' of V
  - such that if (u,v) is an edge of G, then either u belongs in V' or v belongs in V' (or both)
- The size of a vertex cover is the number of vertices in it

The vertex-cover problem [cont.]

- The vertex-cover problem is to find a vertex cover of minimum size
- Such a vertex-cover is called an optimal vertex cover
- **This problem is NP-complete**

Consider the graph:



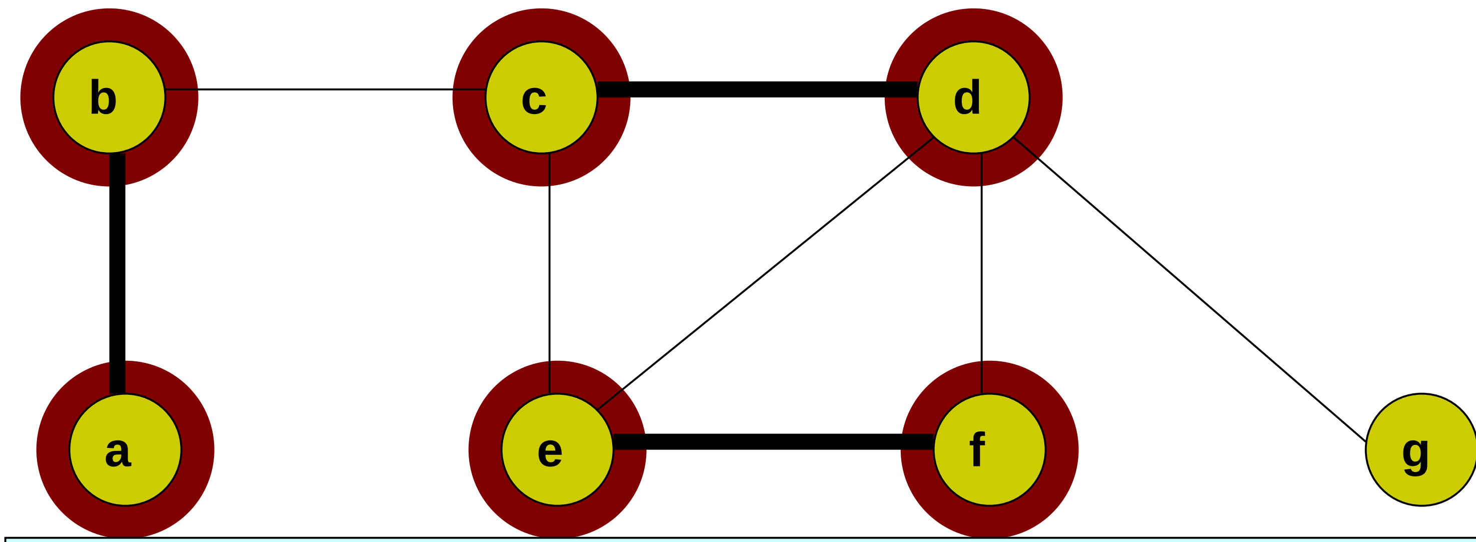
By inspection, optimal vertex cover is {b,d,e}

C* = size of optimal solution = 3

Approx-Vertex-Cover (G)

1. C ← NULL //C contains results of vertex cover
2. E’ ← E[G] //Initially all edges are stored here
3. while E’ != NULL //all edges that are not considered yet
4. do let (u,v) be an edge of E’ //select an edge from E’
5. C ← C U {u,v} //add vertices to C of the selected edge
6. remove from E’ all edges incident on u or v
   //delete all edges related to the vertices “u” and “v”
7. return C

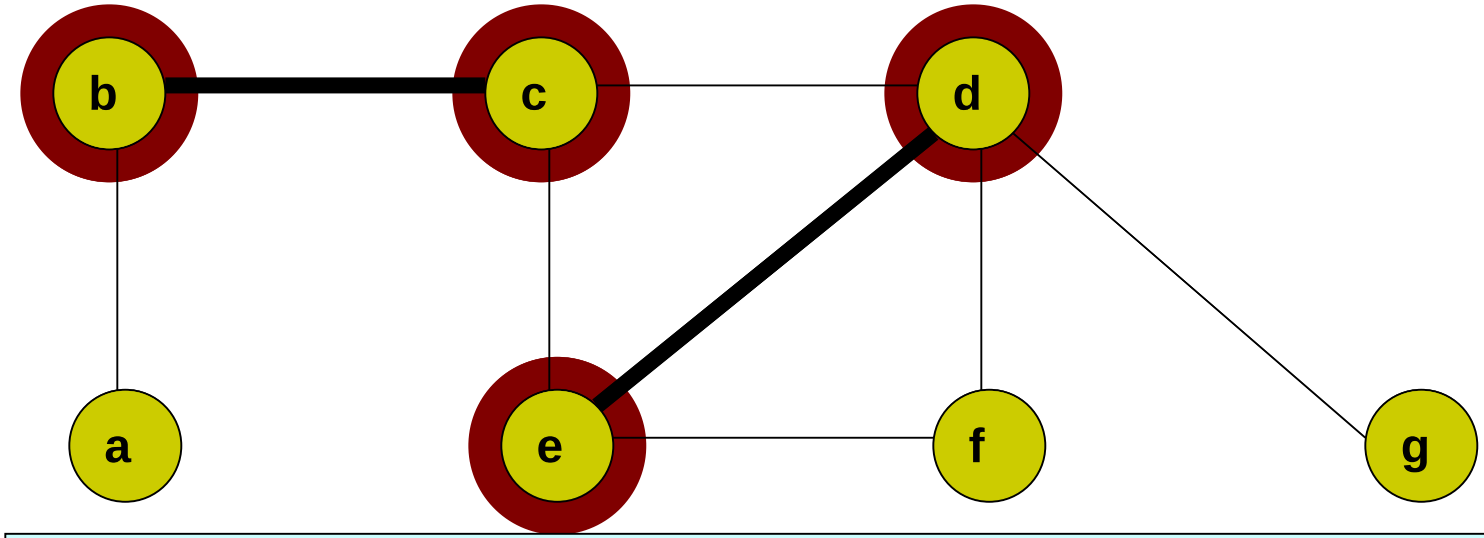
Back to our graph:



C : a b c d e f Approximate vertex cover. C=6

E' : ~~(a,b)~~ ~~(b,c)~~ ~~(c,d)~~ ~~(c,e)~~ ~~(d,e)~~ ~~(d,f)~~ ~~(d,g)~~ ~~(e,f)~~

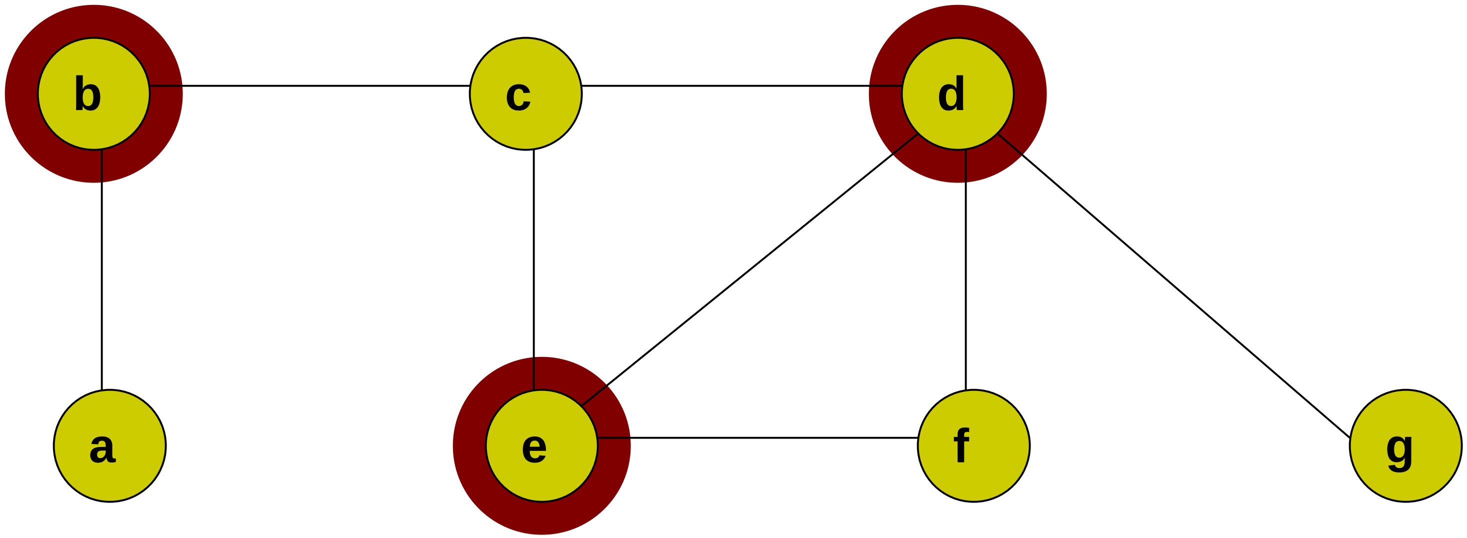
Improved version:



C :   d e b c   Approximate vertex cover. C=4

E' : (d,e) (c,d) (c,e) (d,f) (d,g) (e,f) (b,c) (a,b)

Is it possible to approximate the optimal solution?



In this case, and with our algorithm, NO!

Analysis of Algorithm

- Running time:
  - O(V+E): if we use adjacency list.
- 2-Approximation Algorithm
  - Minimization problem $C^* = 3$; $C = 6$
  - Factor = $C/C^* = 2$

Theorem: APPROX-VERTEX-COVER is a polynomial-time 2-approximation algorithm

- Proof:
  - Polynomial time algorithm because : Complexity = O(V+E)
  - Let, A = set of edges picked by approximation algorithm
  - No 2 edges in A share an endpoint.
  - Thus no 2 edges in A are covered by the same vertex in C*
  - **So, Lower bound: |C*| >= |A|**
  - We pick an edge for which neither of its endpoints are already in C
  - **So, Upper bound: |C| = 2|A|**
  - **Therefore: |C| = 2|A| <= 2|C*|**

Traveling Salesman

<u>TSP with triangle inequality</u>

Polynomial-time 2-approximation algorithm

<u>TSP without triangle inequality</u>

Negative result on polynomial-time $\rho(n)$-approximation algorithm

The traveling-salesperson problem.

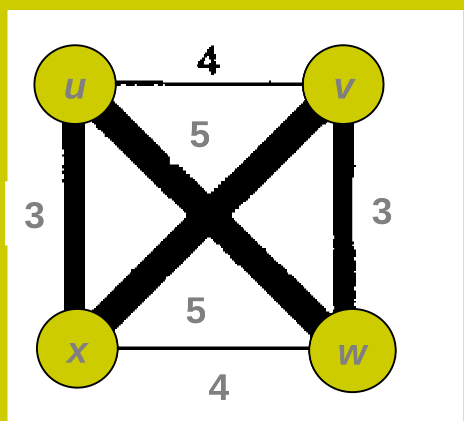
- Given a complete undirected graph G=(V,E)
- Each edge has a nonnegative integer cost c(u,v)
- Find a Hamiltonian cycle of G with minimum cost.

- This is a NP complete problem

Triangle Inequality

- The cost function satisfies the triangle inequality for all vertices u, v, w in V
  - $c(u,w) <= c(u,v) + c(v,w)$

TSP with Triangle Inequality



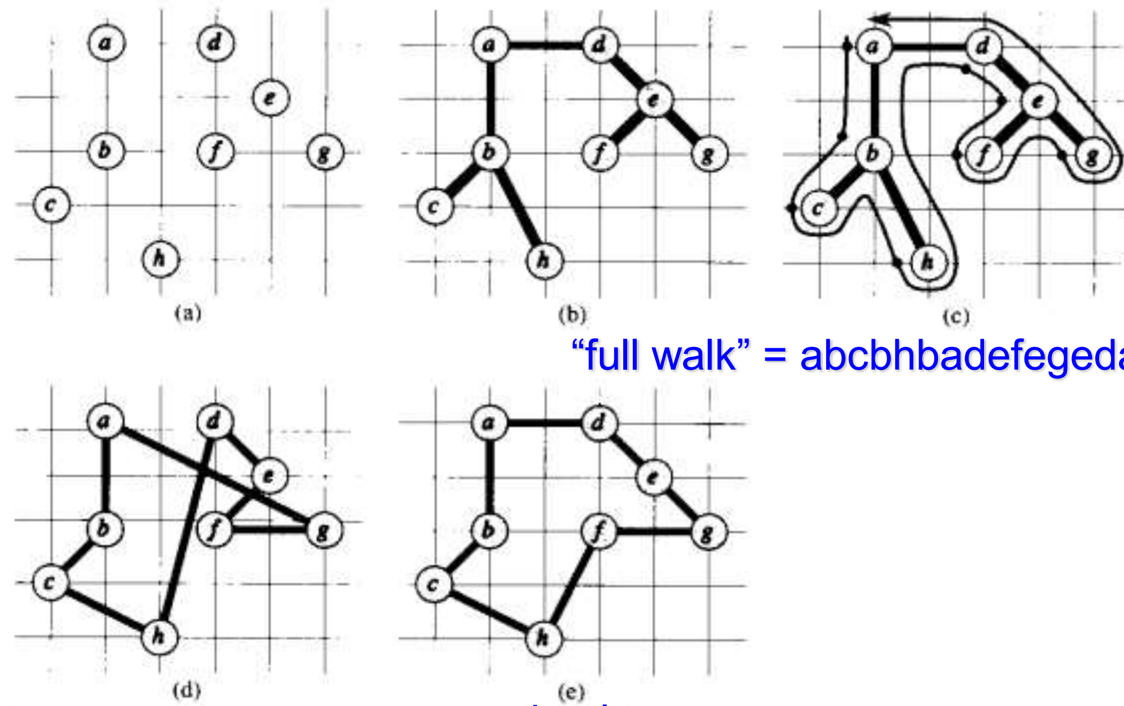
Cost Function Satisfies

Triangle Inequality

$$\forall u, v, w \in V$$

$$c(u,w) \le c(u,v) + c(v,w)$$

```
APPROX-TSP-TOUR(G, c)
select a vertex r ∈ V[G] to be a "root" vertex
compute a minimum spanning tree T for G from root r
        using MST-PRIM(G, c, r)
let L be the list of vertices visited in a preorder tree walk of T
return the hamiltonian cycle H that visits the vertices in the order L
```



"full walk" = abcbhbadefegeda

final approximate tour

optimal tour

source: 91.503 textbook Cormen et al.

TSP with Triangle Inequality

- Time Complexity:
  - MST-Prim()'s running time: $O(V^2)$
  - So, APPROX-TSP-TOUR's running time: $O(V^2)$

TSP with Triangle Inequality

<u>**Theorem**</u>**:** ***APPROX-TSP-TOUR*** *is a* ***polynomial-time 2-approximation algorithm for TSP with triangle inequality****.*

<u>***Proof***</u>***:***

Algorithm runs in time polynomial in *n*.

Let H^* be an optimal tour and T be MST

$c(T) \le c(H^*)$ **(since deleting 1 edge from a tour creates a spanning tree)**

Let W be a full walk of T **(lists vertices when they are first visited and when returned to after visiting subtree)**

$c(W) = 2c(T)$ **(since full walk traverses each edge of T twice)**

$c(W) \le 2c(H^*)$

Now make W into a tour H using triangle inequality.

$c(H) \le c(W)$ **(New inequality holds since H is formed by deleting duplicate vertices from W.)**

$$c(H) \le 2c(H^*)$$

TSP without Triangle Inequality

- If P $\neq$ NP, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general traveling-salesman problem
- Prove: try yourself

MAX-3-CNF Satisfiability

3-CNF Satisfiability Background

Randomized Algorithms

Randomized MAX-3-CNF SAT Approximation Algorithm

MAX-3-CNF Satisfiability

- Background on Boolean Formula Satisfiability
  - Boolean Formula Satisfiability: Instance of language **SAT** is a boolean formula ϕ consisting of:
    - n boolean variables: $x_1, x_2, \ldots, x_n$
    - m boolean connectives: boolean function with 1 or 2 inputs and 1 output
      - e.g. AND, OR, NOT, implication, iff

        $\vee \quad \wedge \quad \neg \quad \rightarrow \quad \leftrightarrow$
    - parentheses
    - truth, satisfying assignments notions apply

$$SAT = \{<\phi>: \phi \text{ is a satisfiable boolean formula}\}$$

source: 91.503 textbook Cormen et al.

MAX-3-CNF Satisfiability

(continued)

- Background on 3-CNF-Satisfiability
  - Instance of language **SAT** is a boolean formula ϕ consisting of:
    - *literal:* variable or its negation
    - CNF = conjunctive normal form
      - *conjunction*: AND of clauses
      - *clause*: OR of literal(s)
    - 3-CNF: each clause has **exactly 3** distinct literals

MAX-3-CNF Satisfiability: optimization version of 3-CNF-SAT

- Maximization: satisfy as many clauses as possible

- Input Restrictions:

  - exactly 3 literals/clause

  - no clause contains both variable and its negation

source: 91.503 textbook Cormen et al.

Randomized approximation algorithm for MAX-3-CNF-SAT

- Each clause has 3 distinct literals. Also assume that no clause contains a variable and its negation.
- **Problem**: Return an assignment of variables which maximizes the number of clauses which evaluate to true.

The random approximation algorithm

- Randomly assign a 0 or a 1 value to all variables.

Analysis

- We have set each variable to 1 with a ½ probability.
- We have set each variable to 0 with a ½ probability.
- We define an indicator random variable
  - Y_i = I {clause i is satisfied}
- Y_i = 1 as long as at least one of the literals in the i^{th} clause has been set to 1.
- A clause is not satisfied iff all it's literals are 0.
  - Pr {clause i is not satisfied} = $(½)^3$ = 1/8

Analysis [cont.]

- So Pr {clause i is satisfied} = 1 – 1/8 = 7/8
- Lemma 5.1 (CLRS)
  - Given a sample space S and an event A in the sample space, let $X_A = I\{A\}$. Then $E[X_A] = Pr\{A\}$
- $E[Y_i] = 7/8$

Analysis [cont.]

- Let Y = number of satisfied clauses.
  - $Y = Y_1 + Y_2 + \ldots\ldots + \ldots\ldots\ Y_m$
- Then we have:

$$
\begin{aligned}
E[Y] &= E\left[\sum_{i=1}^{m} Y_i\right] \\
&= \sum_{i=1}^{m} E[Y_i] \\
&= \sum_{i=1}^{m} 7/8 \\
&= 7m/8
\end{aligned}
$$

- Since m is the upper bound on the number of satisfied clauses, the approximation ratio is m/(7m/8) = 8/7
- Therefore this is a 8/7 approximation algorithm