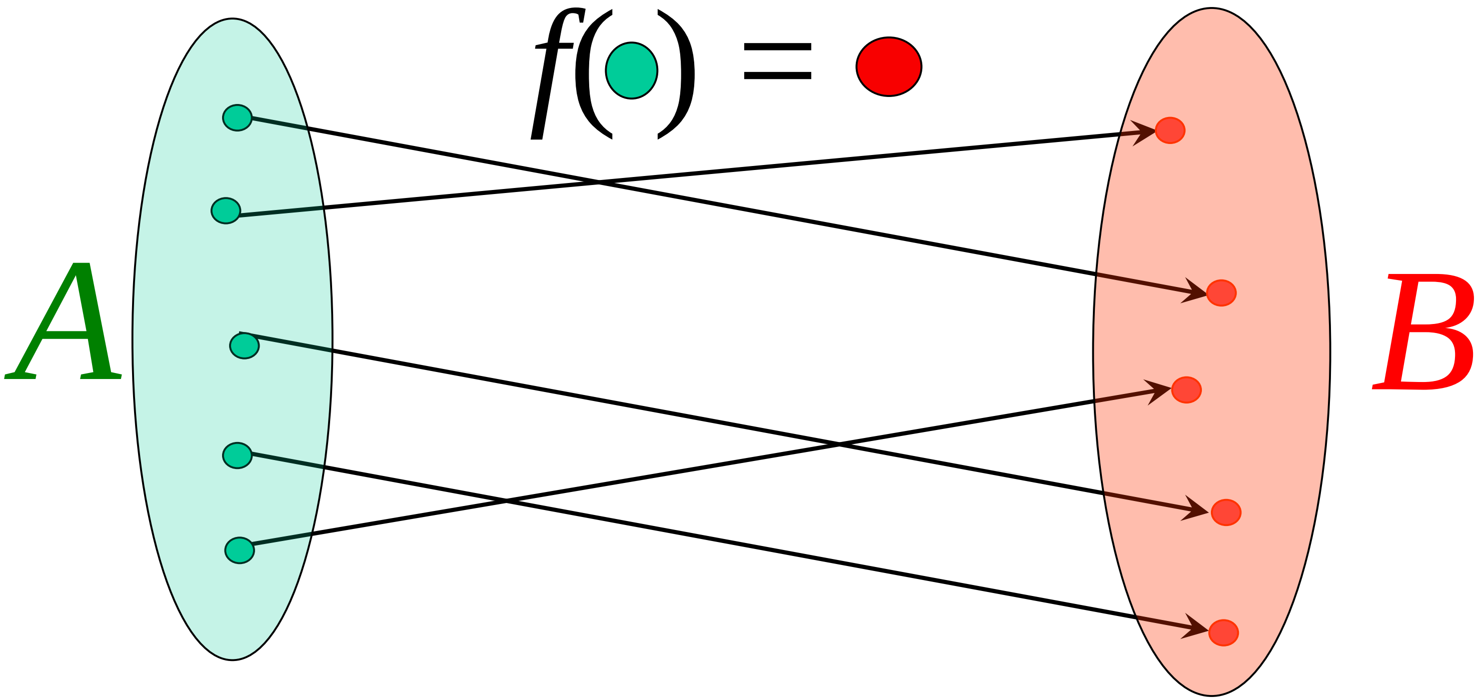
Functions



This Lecture

We will define a function formally, and then
in the next lecture we will use this concept in counting.

We will also study the pigeonhole principle and its applications.

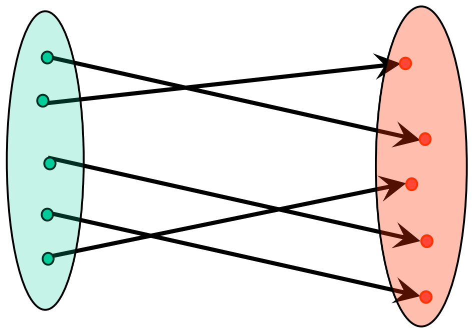
- Examples and definitions (injection, surjection, bijection)
- Pigeonhole principle and applications

Functions

Informally, a function f "maps" the element of an input set *A* to the elements of an output set B.

More formally, we write $f : A \rightarrow B$

to represent that *f* is a function from set *A* to set *B*, which associates an element $f(a) \in B$ with an element $a \in A$.



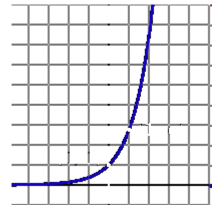
The *domain (input)* of *f* is A.

The *codomain (output)* of *f* is B.

Definition: For every input there is exactly one output.

Functions

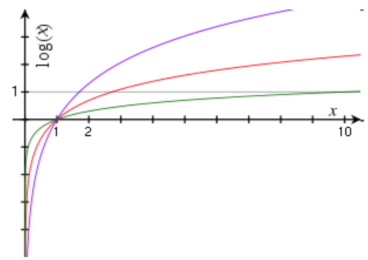
$$f(x) = e^x$$



domain = R

codomain = R^+-{0}

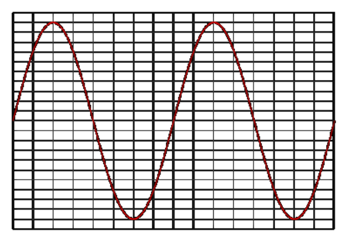
$$f(x) = \log(x)$$



domain = R^+-{0}

codomain = R

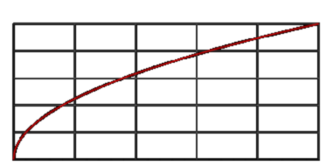
$$f(x) = \sin(x)$$



domain = R

codomain = [0,1]

$$f(x) = \sqrt{x}$$



domain = R^+

codomain = R^+

Functions

$f(S) = |S|$

domain = the set of all sets
codomain = non-negative integers

f(string) = length(string)

domain = the set of all strings
codomain = non-negative integers

f(student-name) = student-ID

not a function,
since one input could have
more than one output

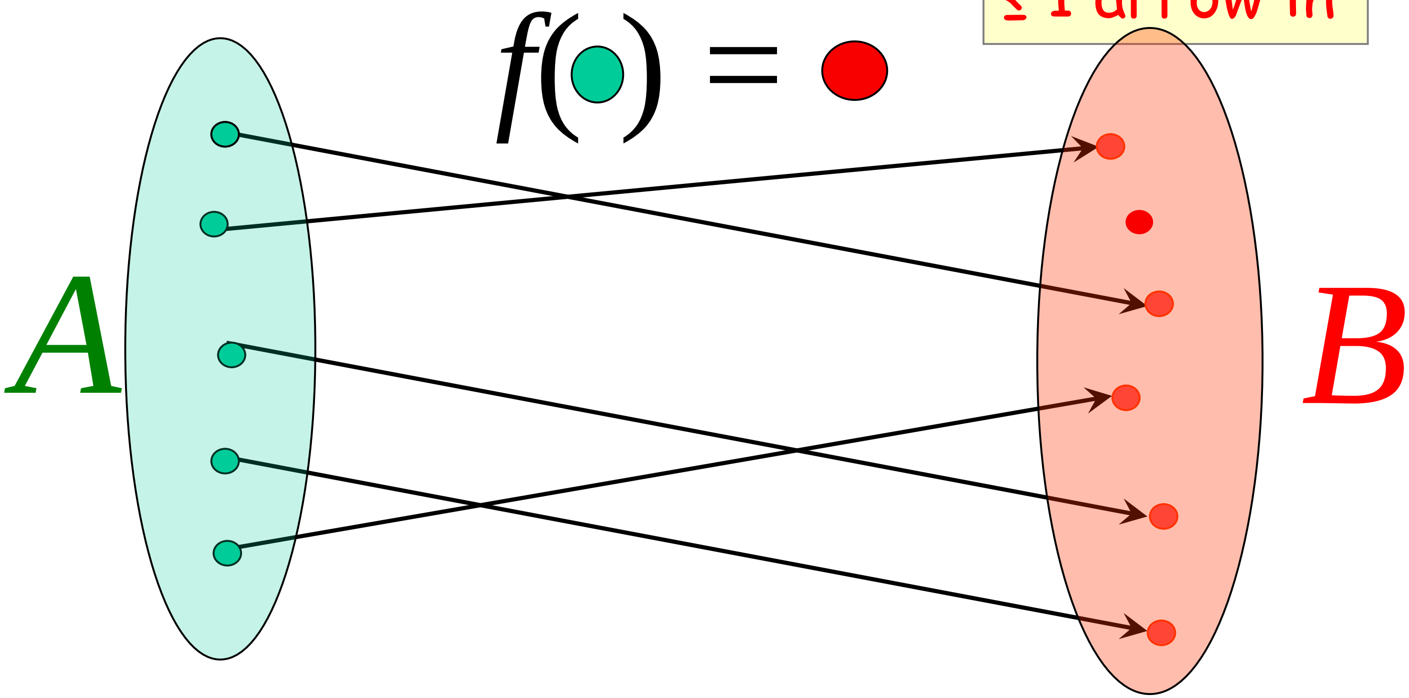
f(x) = is-prime(x)

domain = positive integers
codomain = {T,F}

Injections (One-to-One)

$f : A \to B$ is an *injection* iff no two inputs have the same output.

≤ 1 arrow in



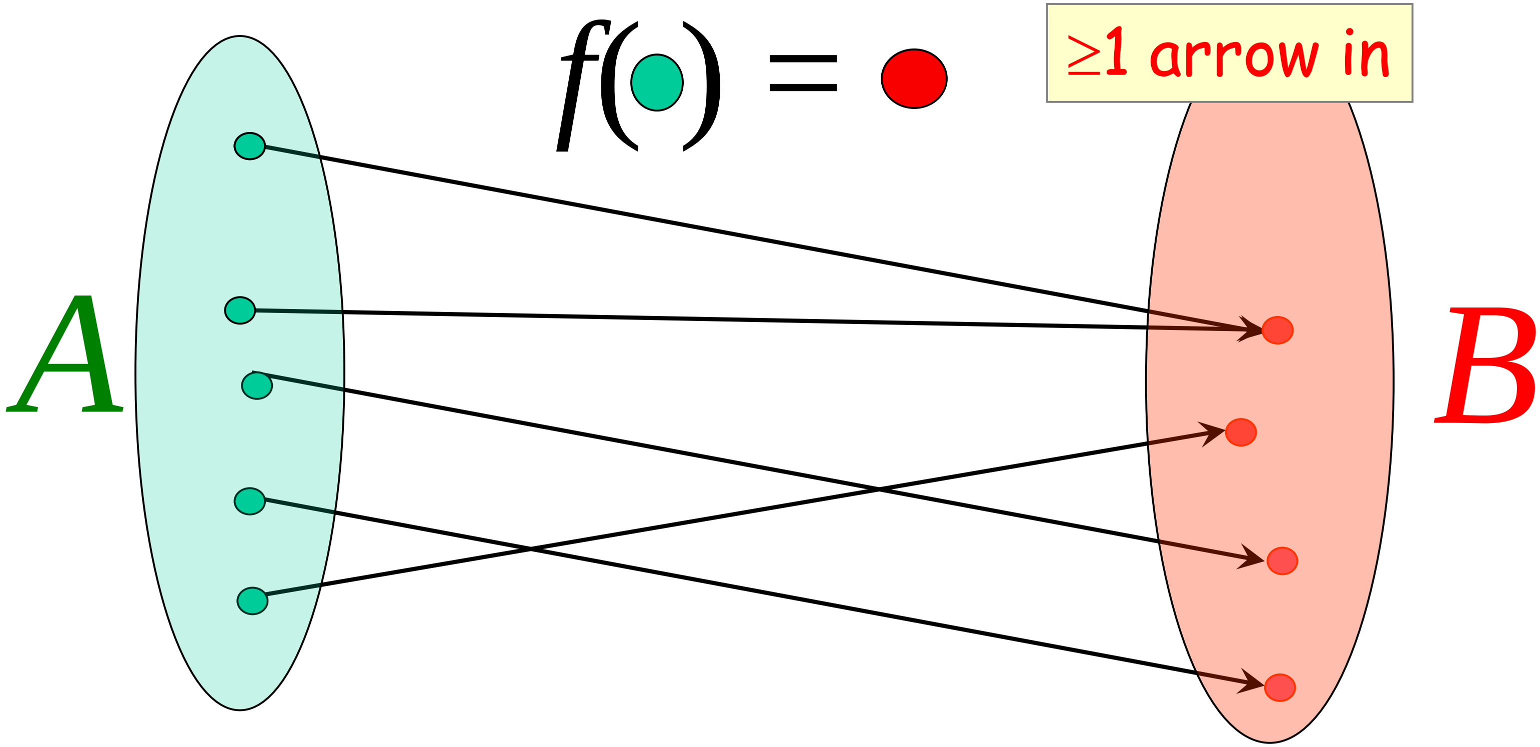
$\forall a, a' \in A.$

$$(f(a) = f(a')) \to (a = a')$$

$$|A| \leq |B|$$

Surjections (Onto)

$f : A \to B$ is a *surjection* iff every output is possible.



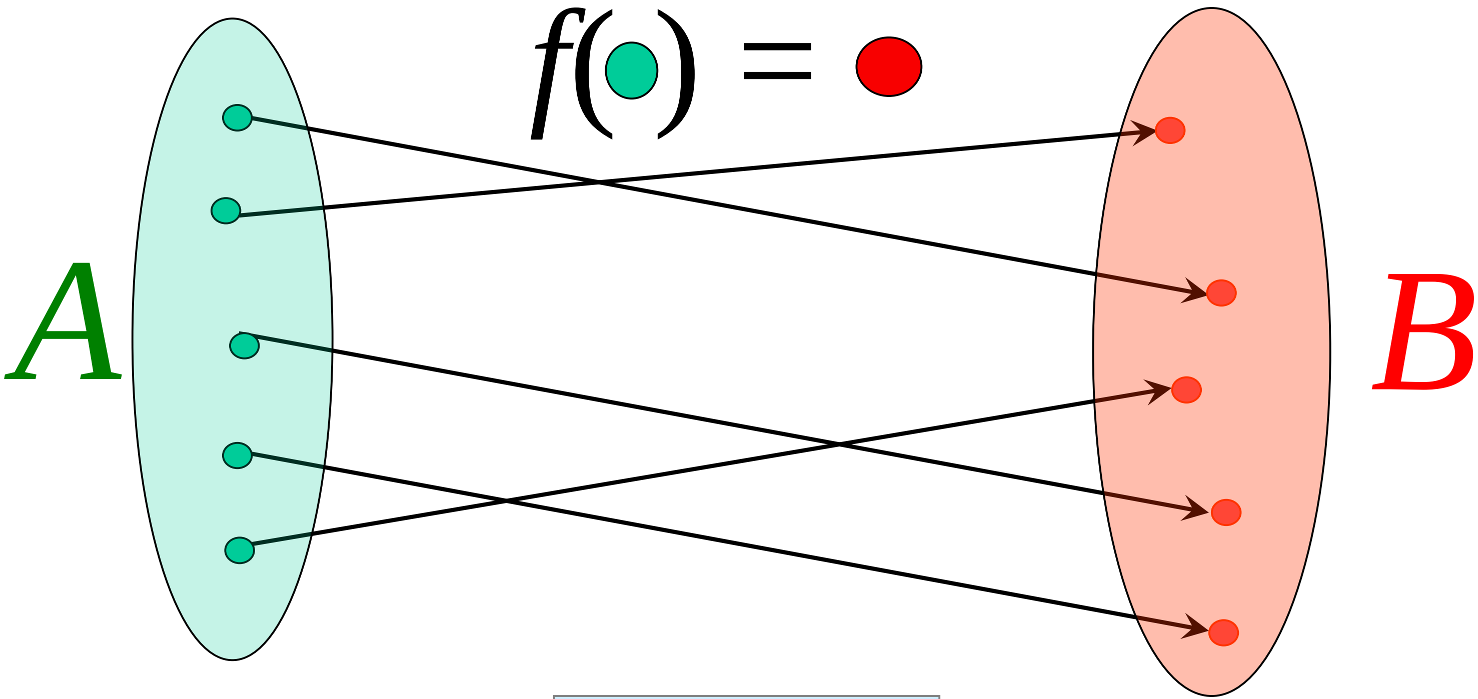
$\forall b \in B \, \exists a \in A. \, f(a) = b$

$\boxed{|A| \geq |B|}$

Bijections

$f : A \rightarrow B$ is a *bijection* iff it is surjection and injection.

exactly one arrow in



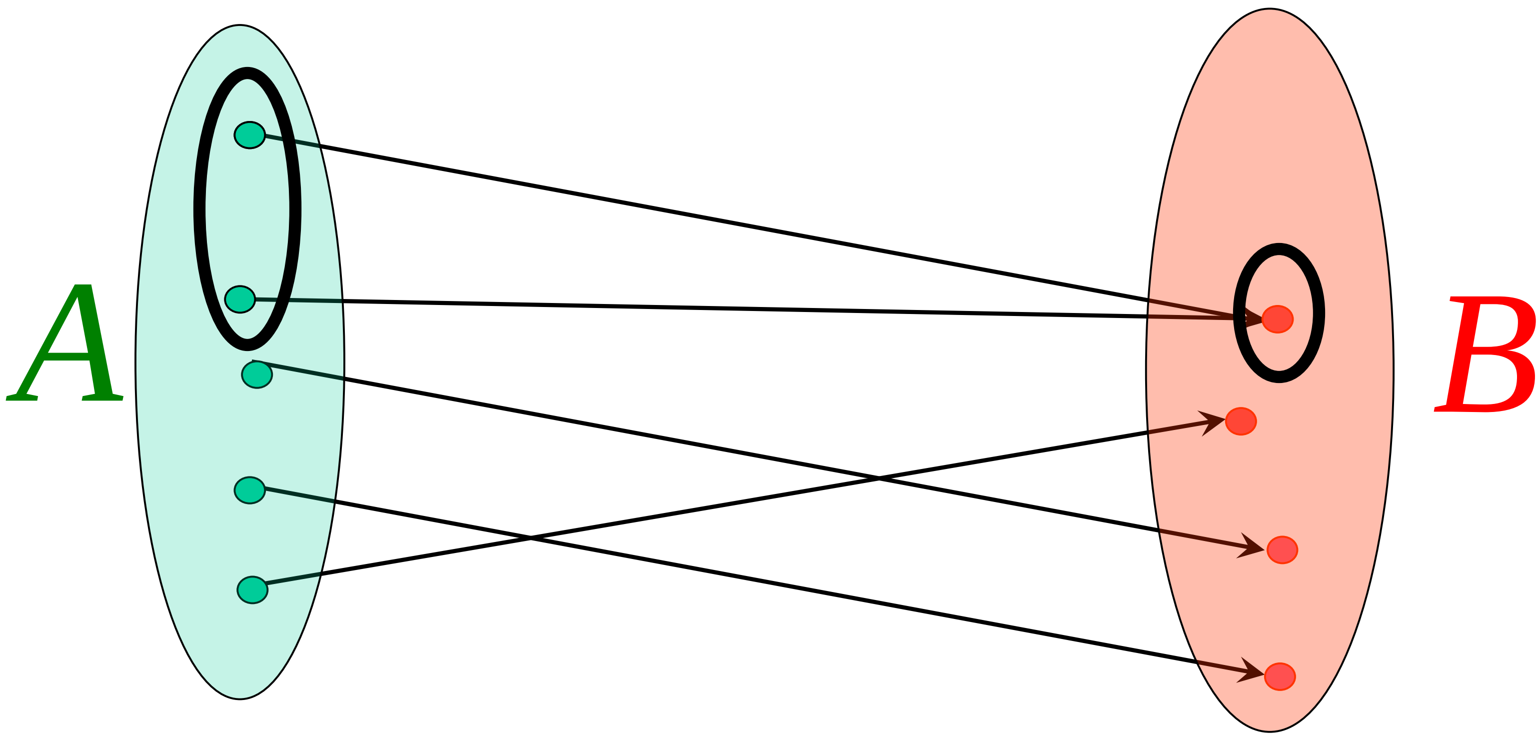
$$|A| = |B|$$

In-Class Exercises

Function	Domain	Codomain	Injective?	Surjective?	Bijective?
$f(x)=\sin(x)$	Real	Real	No	No	No
$f(x)=2^x$	Real	Positive real	Yes	Yes	Yes
$f(x)=x^2$	Real	Non-negative real	No	Yes	No
Reverse string	Bit strings of length n	Bit strings of length n	Yes	Yes	Yes

Whether a function is injective, surjective, bijective depends on its domain and the codomain.

Inverse Sets



Given an element y in B, the inverse set of y := $f^{-1}(y)$ = {x in A | f(x) = y}.

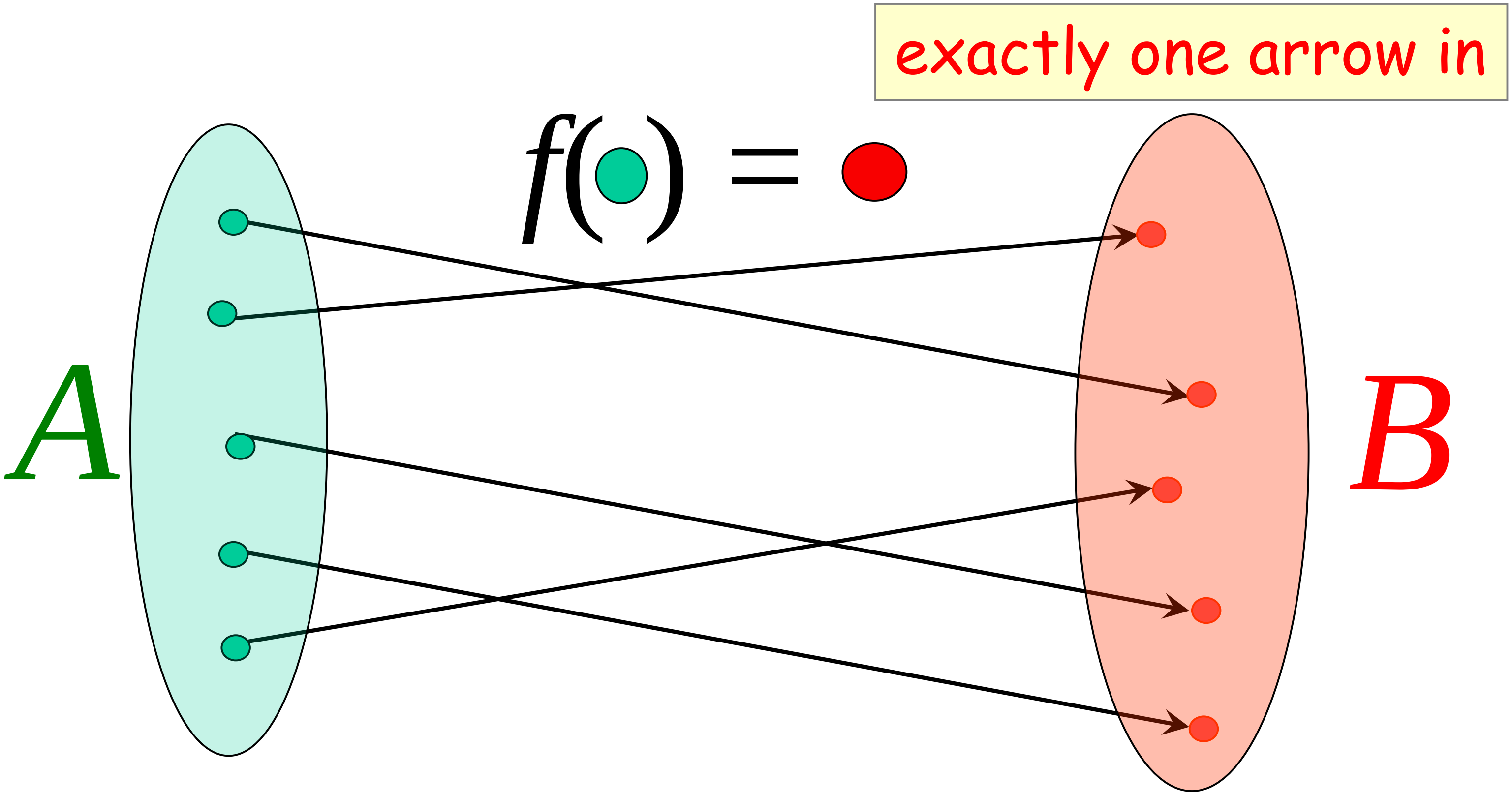
In words, this is the set of inputs that are mapped to y.

More generally, for a subset Y of B,

the inverse set of Y := $f^{-1}(Y)$ = {x in A | f(x) in Y}.

Inverse Function

Informally, an inverse function f^{-1} is to "undo" the operation of function f.

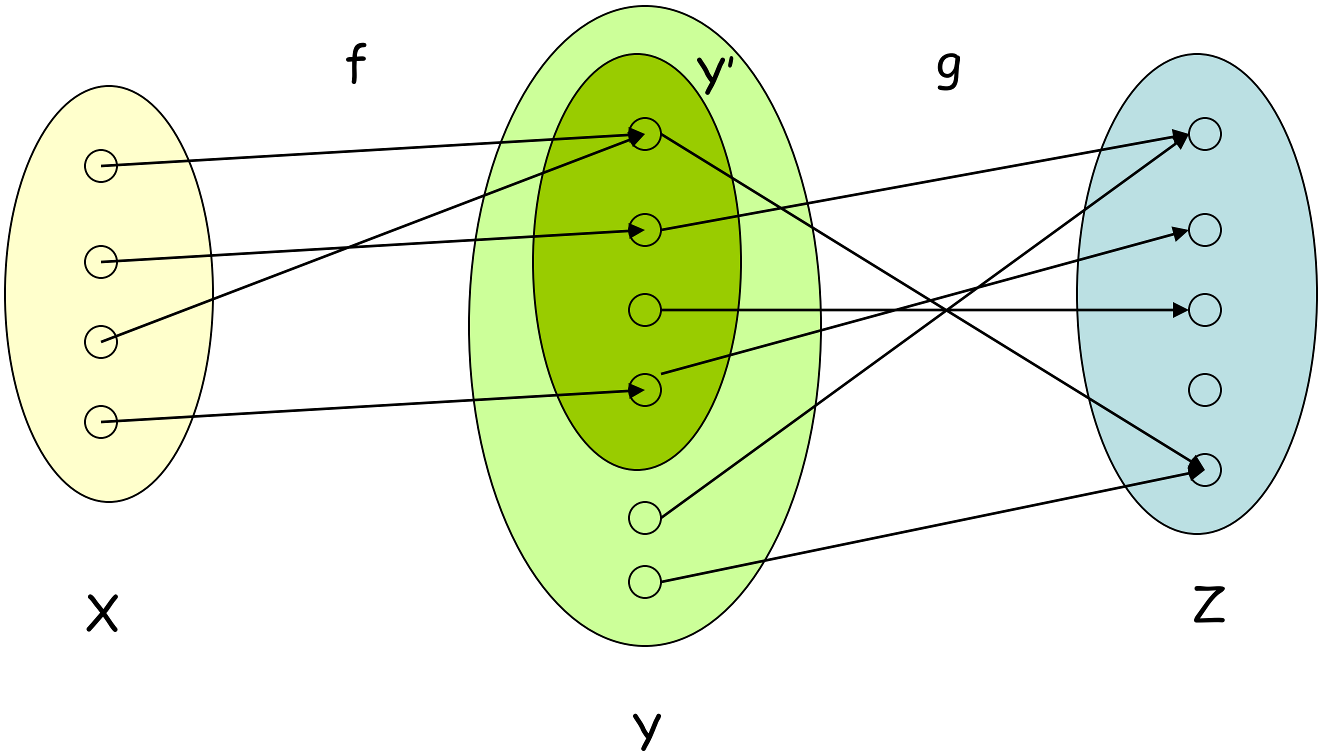


There is an inverse function f^{-1} for f if and only if f is a bijection.

Composition of Functions

Two functions $f:X \to Y'$, $g:Y \to Z$ so that Y' is a subset of Y, then the composition of f and g is the function $g \circ f: X \to Z$, where

$$g \circ f(x) = g(f(x)).$$



In-Class Exercises

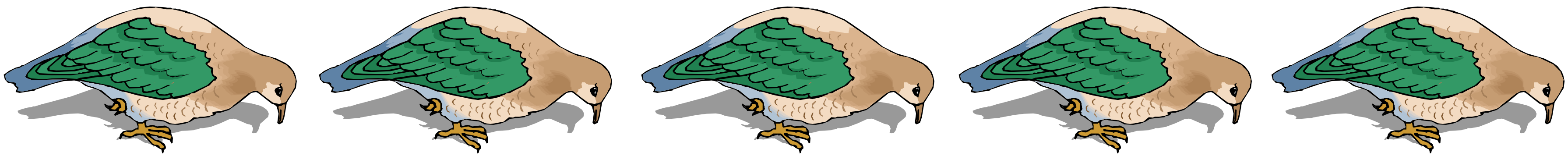
Function f	Function g	$g \circ f$ injective?	$g \circ f$ surjective?	$g \circ f$ bijective?
f:X->Y f surjective	g:Y->Z g injective	No	No	No
f:X->Y f surjective	g:Y->Z g surjective	No	Yes	No
f:X->Y f injective	g:Y->Z g surjective	No	No	No
f:X->Y f bijective	g:Y->Z g bijective	Yes	Yes	Yes
f:X->Y	f^{-1}:Y->X	Yes	Yes	Yes

This Lecture

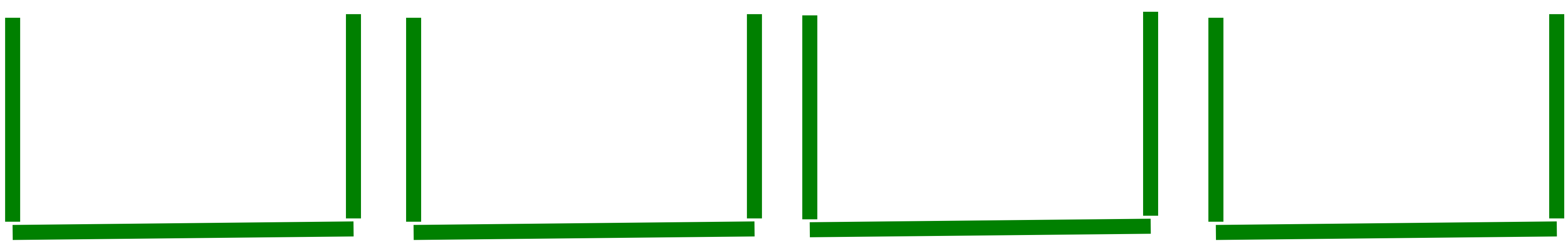
- Examples and definitions (injection, surjection, bijection)
- Pigeonhole principle and applications

Pigeonhole Principle

If more pigeons

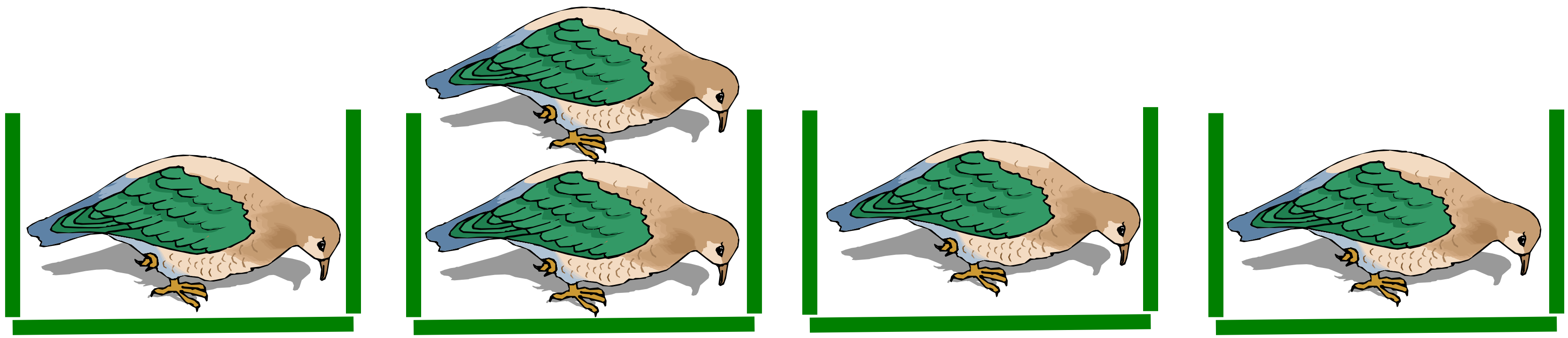


than pigeonholes,



Pigeonhole Principle

then some hole must have at least two pigeons!



Pigeonhole principle

A function from a larger set to a smaller set cannot be injective.

(There must be at least two elements in the domain that have the same image in the codomain.)

Example 1

Question: Let $A = \{1,2,3,4,5,6,7,8\}$

If five integers are selected from A,
must a pair of integers have a sum of 9?

Consider the pairs $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$.
The sum of each pair is equal to 9.
If we choose 5 numbers from this set,
then by the pigeonhole principle,
both elements of some pair will be chosen,
and their sum is equal to 9.

Example 2

Question: In a party of n people, is it always true that there are two people shaking hands with the same number of people?

Everyone can shake hand with 0 to n-1 people, and there are n people, and so it does not seem that it must be the case, but think about it carefully:

Case 1: if there is a person who does not shake hand with others, then any person can shake hands with at most n-2 people, and so everyone shakes hand with 0 to n-2 people, and so the answer is "yes" by the pigeonhole principle.

Case 2: if everyone shakes hand with at least one person, then any person shakes hand with 1 to n-1 people, and so the answer is "yes" by the pigeonhole principle.

Birthday Paradox

In a group of 366 people, there must be two people having the same birthday.

Suppose n <= 365, what is the probability that in a random set of n people, some pair of them will have the same birthday?

We can think of it as picking n random numbers from 1 to 365 without repetition.

There are 365^n ways of picking n numbers from 1 to 365.

There are $365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365-n+1)$ ways of picking n numbers from 1 to 365 without repetition.

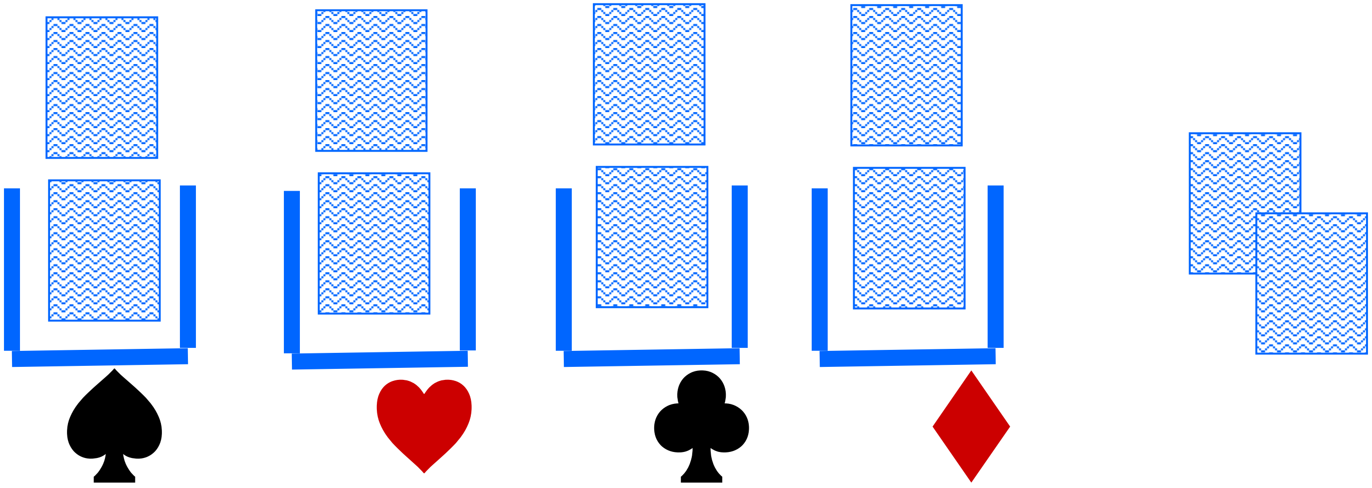
So the probability that no pairs have the same birthday is equal to $365 \cdot 364 \cdot 363 \cdot \ldots \cdot (365-n+1) / 365^n$

This is smaller than 50% for 23 people, smaller than 1% for 57 people.

Generalized Pigeonhole Principle

Generalized Pigeonhole Principle

If n pigeons and h holes,
then some hole has at least $\left\lceil \frac{n}{h} \right\rceil$ pigeons.



Cannot have < 3 cards in every hole.

Subset Sum

2048013538550296444803 8	3171004832173501394113017	5763257331083479647409398	8247331000042995311646021
4894459918669156762409 92	3208234421597368647019265	5800949123548989122628663	8496243997123475922766310
1082662032430379651370981	3437254656355157864869113	6042900801199280218026001	8518399140676002660747477
1178480894769706178994993	3574883393058653923711365	6116171789137737896701405	8543691283470191452333763
1253127351683239693851327	3644909946040480189969149	6144868973001582369723512	8675309258374137092461352
1301505129234077811069011	3790044132737084094417246	6247314593851169234746152	8694321112363996867296665
1311567111143866433882194	3870332127437971355322815	6814428944266874963488274	8772321203608477245851154
1470029452721203587686214	4080505804577801451363100	6870852945543886849147881	8791422161722582546341091
1578271047286257499433886	4167283461025702348124920	6914955508120950093732397	9062628024592126283973285
1638243921852176243192354	4235996831123777788211249	6949632451365987152423541	9137845566925526349897794
1763580219131985963102365	4670939445749439042111220	7128211143613619828415650	9153762966803189291934419
1826227795601842231029694	4815379351865384279613427	7173920083651862307925394	9270880194077636406984249
1843971862675102037201420	4837052948212922604442190	7215654874211755676220587	9324301480722103490379204
2396951193722134526177237	5106389423855018550671530	7256932847164391040233050	9436090832146695147140581
2781394568268599801096354	5142368192004769218069910	7332822657075235431620317	9475308159734538249013238
2796605196713610405408019	5181234096130144084041856	7426441829541573444964139	9492376623917486974923202
2931016394761975263190347	5198267398125617994391348	7632198126531809327186321	9511972558779880288252979
2933458058294405155197296	5317592940316231219758372	7712154432211912882310511	9602413424619187112552264
3075514410490975920315348	5384358126771794128356947	7858918664240262356610010	9631217114906129219461111
3111474985252793452860017	5439211712248901995423441	7898156786763212963178679	9908189853102753335981319
3145621587936120118438701	5610379826092838192760458	8147591017037573337848616	9913237476341764299813987
3148901255628881103198549	5632317555465228677676044	8149436716871371161932035	
3157693105325111284321993	5692168374637019617423712	8176063831682536571306791	

Two different subsets of the 90 25-digit numbers shown above have the same sum.

Subset Sum

Let A be the set of the 90 numbers, each with at most 25 digits.

So the total sum of the 90 numbers is at most 90×10^{25}.

Let 2^A be the set of all subsets of the 90 numbers. **(pigeons)**

Let B be the set of integers from 0 to 90×10^{25}. **(pigeonholes)**

Let $f: 2^A \to B$ be a function mapping each subset of A into its sum.

If we could show that $|2^A| > |B|$, then by the pigeonhole principle,

the function f must map two elements in 2^A into the same element in B.

This means that there are two subsets with the same sum.

Subset Sum

90 numbers, each with at most 25 digits.

So the total sum of the 90 numbers is at most 90×10^{25}

Let 2^A be the set of all subsets of the 90 numbers. **(pigeons)**

Let B be the set of integers from 0 to 90×10^{25}. **(pigeonholes)**

$$|2^A| = 2^{90} \geq 1.237 \times 10^{27}$$

$$|B| = 90 \times 10^{25} + 1 \leq 0.901 \times 10^{27}$$

So, $|2^A| > |B|$.

By the pigeonhole principle, there are two different subsets with the same sum.

Club vs Strangers

Let's agree that given any two people, either they have met or not.

If every people in a group has met, then we'll call the group a club.

If every people in a group has not met, then we'll call a group of strangers.

Theorem: Every collection of 6 people includes a club of 3 people, or a group of 3 strangers.

Let x be one of the six people.

By the (generalized) pigeonhole principle, we have the following claim.

Claim: Among the remaining 5 people, either 3 of them have met x, or 3 of them have not met x.

Club vs Strangers

Theorem: Every collection of 6 people includes a club of 3 people, or a group of 3 strangers.

Claim: Among the remaining 5 people, either 3 of them have met x, or 3 of them have not met x.

Case 1: "3 people have met x"

Case 1.1: No pair among those people met each other.
Then there is a group of 3 strangers. **OK!**

Case 1.2: Some pair among those people have met each other.
Then that pair, together with x, form a club of 3 people. **OK!**

Club vs Strangers

Theorem: Every collection of 6 people includes a club of 3 people,
or a group of 3 strangers.

Claim: Among the remaining 5 people, either 3 of them have met x,
or 3 of them have not met x.

Case 2: “3 people have not met x”

Case 2.1: Every pair among those people met each other.
Then there is a club of 3 people. **OK!**

Case 2.2: Some pair among those people have not met each other. **OK!**
Then that pair, together with x, form a group of 3 strangers.

Club vs Strangers

Theorem: Every collection of 6 people includes a club of 3 people, or a group of 3 strangers.

Theorem: For every k, if there are enough people, then either there exists a club of k people, or a group of k strangers.

A large enough structure cannot be totally disorder.

More applications 1

For every integer n there is a multiple of n that has only 0s and 1s in its decimal representation.

Consider n different numbers $1, 11, 111, \ldots$ Note that the last number has n 1s in its decimal representation. These are the pigeons.

If either of these numbers is a multiple of n we are done. So assume otherwise

Consider the remainder obtained when these numbers are divided by n. This can take values from 1 to n-1. These are the holes.

Two of the n numbers will have the same remainder when divided by n. Hence the difference of these numbers (which contains only 0s and 1s) is a multiple of n

More applications 2

Team DD plays 45 games in April and at least one game each day. Show that there must be a period of consecutive days during which DD plays exactly 14 games.

a_j is the number of games played till day j. Note $1 \leq a_1 < a_2 < \cdots < a_{30} = 45$

Let $b_j = a_j + 14$. Then $15 \leq b_1 < b_2 < \cdots < b_{30} = 59$

Consider the 60 numbers a_j and b_j as pigeons. Since these numbers take values between 1 and 59, two of them have the same value.

a_is and b_is are distinct. So it can only be that for some $i, j, a_i = b_j = a_j + 14$.

Hence from day j+1 to day i the team played 14 games.

More applications 3

Among n+1 positive integers not exceeding 2n there must be an integer that divides one of the other integers.

Let $1 \leq a_1 < a_2 < \cdots < a_{n+1} \leq 2n$ be the $n+1$ integers. No loss of generality to assume they are distinct as else the stmt is trivially true.

Let $b_i = a_i/2^{k_i}$ be odd. Then $1 \leq b_1, b_2, \ldots, b_{n+1} \leq 2n-1$ are $n+1$ odd integers.

Since there are only n odd integers between 1 and $2n-1$, by PHP for some $i, j, b_i = b_j$.

Let $a_i < a_j$. Then $k_i < k_j$ and so $a_i = b_i 2^{k_i}$ divides $a_j = b_j 2^{k_j}$

More applications 4

Every sequence of $n^2 + 1$ distinct numbers contains a subsequence of length n+1 that is strictly increasing or strictly decreasing

e.g. 8, 11, 9, 1, 4, 6, 12, 10, 5, 7 contains the increasing subsequence 1,4,6,10 of length 4.

Let $a_1, a_2, a_3, \dots, a_{n^2+1}$ be the sequence.
Let $I(k)$ be the length of the longest increasing subsequence starting from a_k.
Similarly, $D(k)$ is the length of the longest decreasing subsequence starting from a_k.

In the above example: $I(3) = 2, D(3) = 3, I(5) = 3, D(5) = 1$

More applications 4 (contd.)

$a_1, a_2, a_3, \dots, a_{n^2+1}$ is the sequence.
$I(k)$: length of the longest increasing subsequence starting from a_k.
$D(k)$: length of the longest decreasing subsequence starting from a_k.

If for some k, $I(k)$ or $D(k)$ exceeds n then we are done (there is an inc./dec. subsequence of length $n+1$).

For contradiction, assume $\forall k, 1 \leq I(k), D(k) \leq n$.

Consider the n^2+1 tuples $\big(I(1), D(1)\big), \big(I(2), D(2)\big), \dots, \big(I(n^2+1), D(n^2+1)\big)$.
Since these takes n^2 different values, by PHP $\exists r, s$: $\big(I(r), D(r)\big) = \big(I(s), D(s)\big)$.

No loss of generality to assume that $r < s$.

More applications 4 (contd.)

$a_1, a_2, \dots, a_r, \dots, a_s, \dots, a_{n^2+1}$ is the sequence.
Longest increasing sequence starting from a_r = Longest increasing sequence starting from $a_s = x$.
Longest decreasing sequence starting from a_r = Longest decreasing sequence starting from $a_s = y$.

Two possibilities:

1. $a_r < a_s$: Then we have an increasing subsequence of length $x + 1$ starting from a_r – a contradiction.
2. $a_r > a_s$: Then we have a decreasing subsequence of length $y + 1$ starting from a_r – a contradiction.

This implies tat our assumption that "there is no increasing or decreasing subsequence of length more than n" is incorrect.

Cardinality

Functions are useful to compare the sizes of two different sets.

Question: Do all infinite sets have the same cardinality?

Two sets A and B have the same cardinality if and only if there is a bijection between A and B.

A set, S, is countable if there exists an injective mapping from S to the set of positive integers.

Integers vs Positive Integers

Is the set of integers countable?

Define an injection from the set of all integers to the positive integers.

1	2	3	4	5	6	7	8	. . .
0	1	–1	2	–2	3	–3	4	. . .

$$f(n) = \begin{cases} n/2, \text{ if } n \text{ is even;} \\ -(n-1)/2, \text{ if } n \text{ is odd.} \end{cases}$$

So, the set of integers is countable.

Rational Numbers vs Positive Integers

Question: Is the set of **rational numbers** countable?

We want to show that the set rational numbers is countable, by defining an **injective mapping** to the set of positive integers.

The mapping is defined by visiting the rational numbers in a specific order such that all numbers appear.

Rational Numbers vs Positive Integers

(0, 0),	(0, 1),	(0,-1),	(0, 2),	(0,-2),	(0, 3),	(0,-3),	. . .
(1, 0),	(1, 1),	(1,-1),	(1, 2),	(1,-2),	(1, 3),	(1,-3),	. . .
(-1, 0),	(-1, 1),	(-1,-1),	(-1, 2),	(-1,-2),	(-1, 3),	(-1,-3),	. . .
(2, 0),	(2, 1),	(2,-1),	(2, 2),	(2,-2),	(2, 3),	(2,-3),	. . .
(-2, 0),	(-2, 1),	(-2,-1),	(-2, 2),	(-2,-2),	(-2, 3),	(-2,-3),	. . .

If we first visit all numbers in first row/column then we will never reach the second row/column.

The trick is to visit the rational numbers diagonal by diagonal.

Each diagonal is finite, so eventually every pair will be visited.

Therefore, there is an injective mapping from the rationals to the set of positive integers,and so the set of rational numbers is countable.

Real Numbers vs Positive Integers

Question: Is the set of **real numbers** countable?

Theorem: **No injective** mapping from real numbers to positive integers.

The string map to the first natural number

The string map to the fifth natural number

$$
\begin{array}{lcccccccccccccc}
E_0 = & m & m & m & m & m & m & m & m & m & m & m & m & \cdots \\
E_1 = & w & w & w & w & w & w & w & w & w & w & w & w & \cdots \\
E_2 = & m & w & m & w & m & w & m & w & m & w & m & w & \cdots \\
E_3 = & w & m & w & m & w & m & w & m & w & m & m & w & \cdots \\
E_4 = & w & m & m & w & w & m & m & w & m & w & m & w & \cdots \\
E_5 = & m & w & m & w & w & m & w & m & w & m & w & m & \cdots \\
E_6 = & m & w & m & w & w & m & w & w & m & w & m & w & \cdots \\
E_7 = & w & m & m & w & m & w & m & w & m & w & m & w & \cdots \\
E_8 = & m & m & w & m & w & m & w & m & w & m & w & m & \cdots \\
E_9 = & w & m & w & m & m & w & w & m & w & w & m & w & \cdots \\
E_{10} = & w & w & m & w & m & w & m & w & m & m & w & m & \cdots \\
E_{11} = & m & w & m & w & w & m & w & m & m & w & m & m & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
E_u \neq & w & m & w & w & m & w & m & m & m & m & m & w & \cdots
\end{array}
$$

It can not be in any row i because its i-th bit is different, and so this string is not mapped!

The opposite of the diagonal

Diagonalization Argument

Similarly, power sets can be shown to be **uncountable**.

This argument is called Cantor's diagonal argument.

http://en.wikipedia.org/wiki/Cantor's_diagonal_argument

This has been used in many places; for example the Russell's paradox.

Cardinality and Computability

The set of all computer programs in a given computer language is countable.

The set of all functions is uncountable.

There must exist a non-computable function!

Quick Summary

Make sure you understand basic definitions of functions.

These will be used in the next lecture for counting.

The pigeonhole principle is very simple,

but there are many clever uses of it to prove non-trivial results.