Chapter: 03(04)

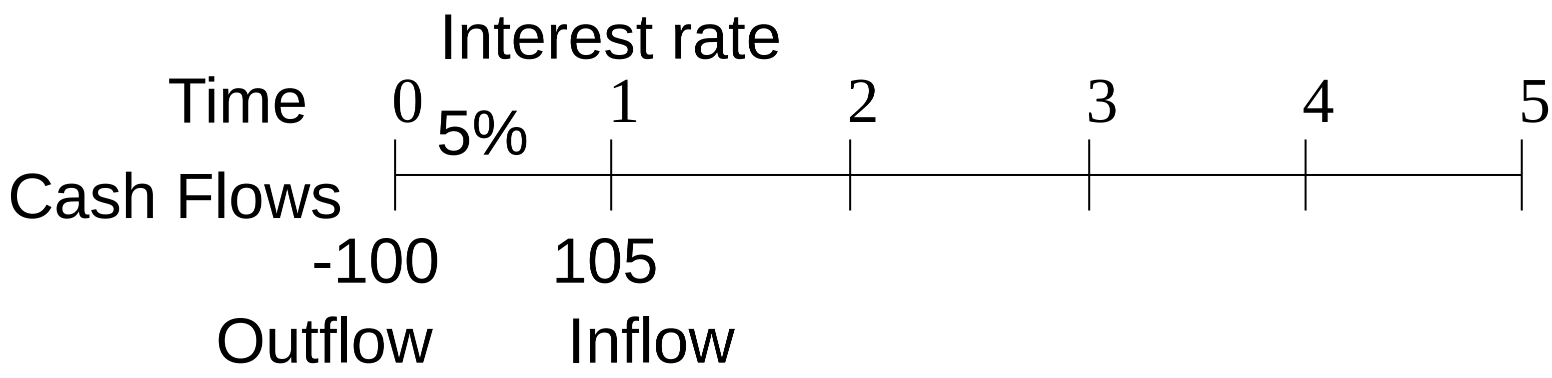
Time Value of Money

What is Time Value of Money?

- Time value of money is based on the belief that a dollar today is worth more than a dollar that will be received at some future date because the money it now can be invested & earn positive return.

Cash Flow Time Lines

- Time 0 is today; Time 1 is one period from today



Future Value

- Compounding
  - The process of determining the value of a cash flow or series of cash flows some time in the future when compound interest is applied.
  - The amount to which a cash flow or series of cash flows will grow over a given period of time when compounded at a given interest rate

Compounded Interest

- Interest earned on interest

$$FV_n = PV(1+i)^n$$

Cash Flow Time Lines

5%

Time	0	1	2	3	4	5
	-100					
Interest		5.00	5.25	5.51	5.79	6.08
Total Value		105.00	110.25	115.76	121.55	127.63

Future Value Interest Factor for i and n ($FVIF_{i,n}$)

- $FV_n = PV(1 + i)^n = PV(FVIF^{i,n})$

Period (n)	4%	5%	6%
1	1.0400	1.0500	1.0600
2	1.0816	1.1025	1.1236
3	1.1249	1.1576	1.1910
4	1.1699	1.2155	1.2625
5	1.2167	1.2763	1.3382
6	1.2653	1.3401	1.4185

For $100 at i = 5% and n = 5 periods

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For \$100 at i = 5% and n = 5 periods

\$100 (1.2763) = \$127.63

Financial Calculator Solution

- Five keys for variable input
  - N = the number of periods
  - I = interest rate per period
    may be I, INT, or I/Y
  - PV = present value
  - PMT = annuity payment
  - FV = future value

Two Solutions

- Find the future value of $100 at 5% interest per year for five years
- 1. Numerical Solution:

Time	0	1	2	3	4	5
	5%					
Cash Flows	-100	5.00	5.25	5.51	5.79	6.08
Total Value		105.00	110.25	115.76	121.55	127.63

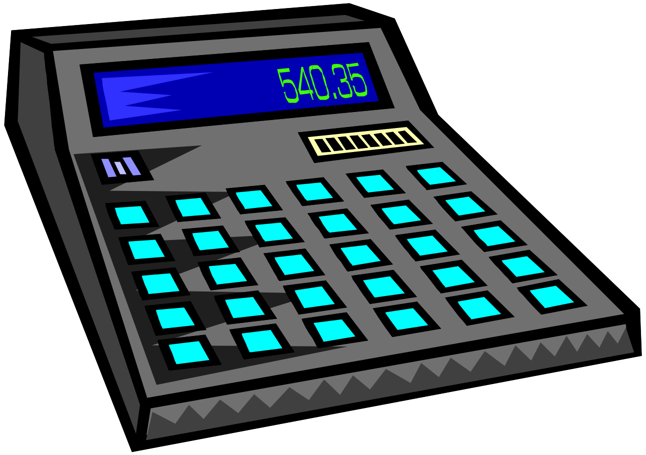
$$FV_5 = \$100(1.05)^5 = \$100(1.2763) = \$127.63$$

Two Solutions

2. Financial Calculator Solution:

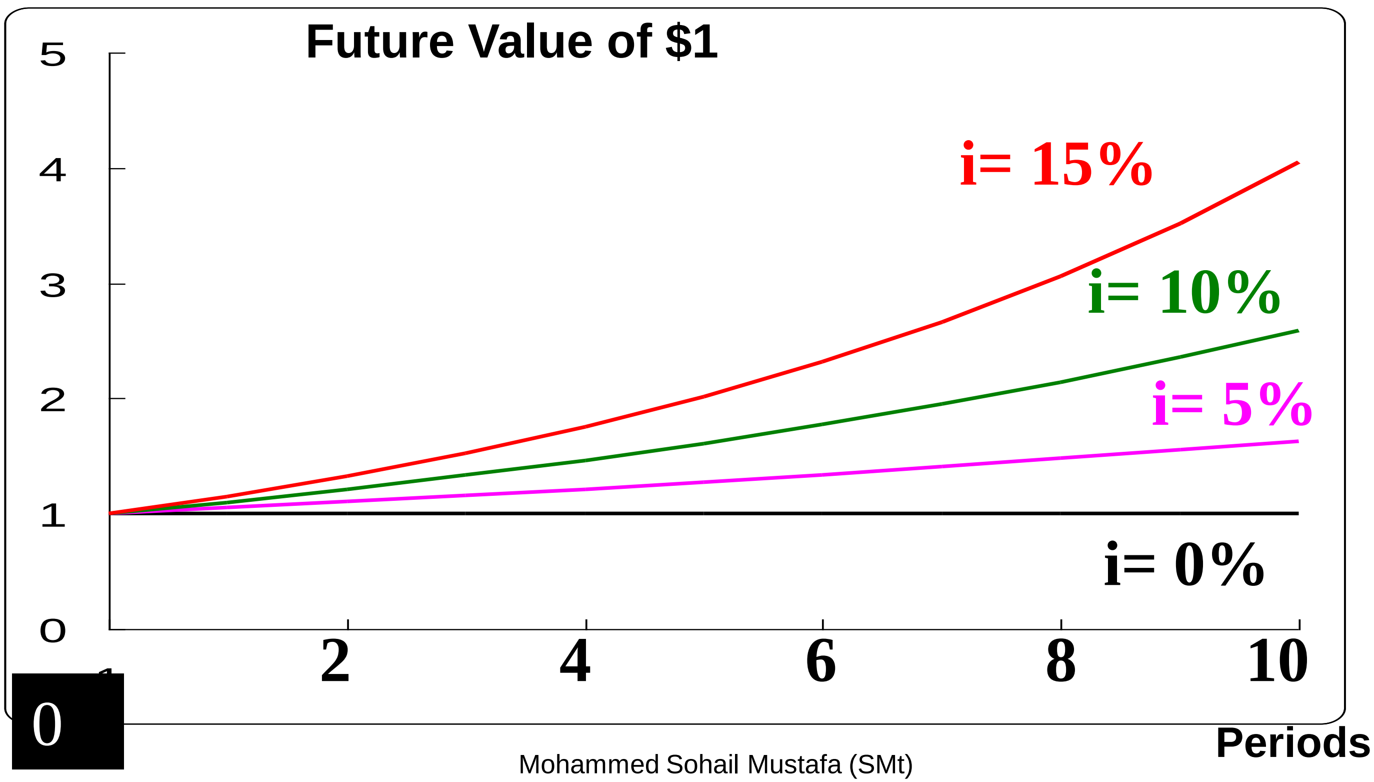
Inputs: N = 5 I = 5 PV = -100 PMT = 0 FV = ?

Output: = 127.63



Graphic View of the Compounding Process: Growth

- Relationship among Future Value, Growth or Interest Rates, and Time



Present Value

- The present value is the value today of a future cash flow or series of cash flows
- The process of finding the present value is discounting, and is the reverse of compounding
- Opportunity cost becomes a factor in discounting

Present Value

- Start with future value:
- $FV_n = PV(1 + i)^n$

$$\mathbf{PV} = \frac{\mathbf{FV_n}}{(\mathbf{1+i})^{\mathbf{n}}} = \mathbf{FV_n}\left[\frac{\mathbf{1}}{(\mathbf{1+i})^{\mathbf{n}}}\right]$$

Two Solutions

- Find the present value of $127.63 in five years when the opportunity cost rate is 5%
- 1. Numerical Solution:

0	1	2	3	4	5
5%					
PV = ?	÷ 1.05	÷ 1.05	÷ 1.05	÷ 1.05	127.63
-100.00	105.00	110.25	115.76	121.55	

$$\mathrm{PV} = \frac{\$127.63}{(1.05)^5} = \frac{\$127.63}{1.2763} = \$127.63(0.7835) = \$100$$

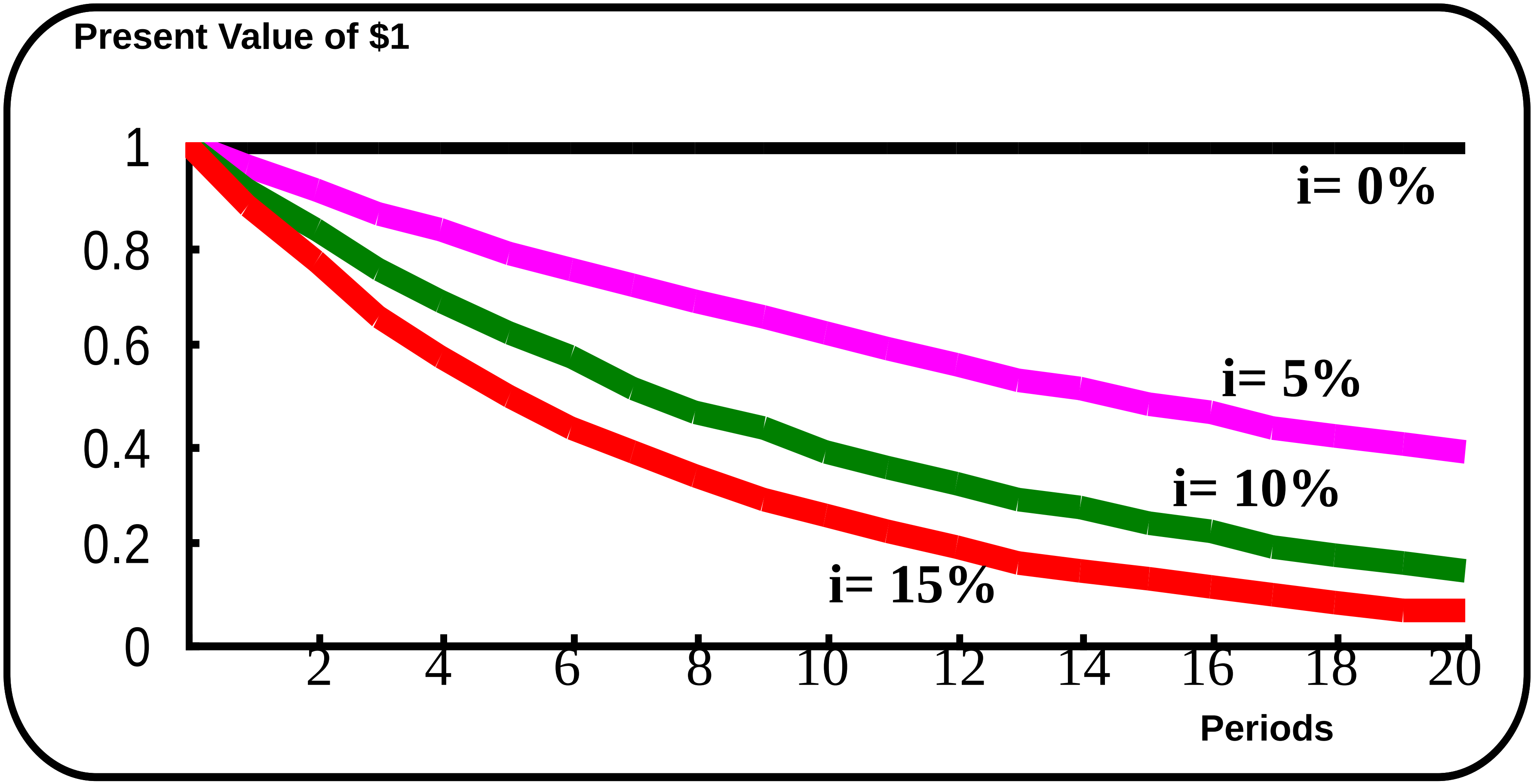
Two Solutions

- Find the present value of $127.63 in five years when the opportunity cost rate is 5%
- 2. Financial Calculator Solution:

Inputs: N = 5 I = 5 PMT = 0 FV = 127.63 PV = ?

Output: = -100

Graphic View of the Discounting Process



Annuity

- An annuity is a series of payments of an equal amount at fixed intervals for a specified number of periods
- Ordinary (deferred) annuity has payments at the end of each period
- Annuity due has payments at the beginning of each period
- FVA_n is the future value of an annuity over n periods

Future Value of an Annuity

- The future value of an annuity is the amount received over time plus the interest earned on the payments from the time received until the future date being valued
- The future value of each payment can be calculated separately and then the total summed

Future Value of an Annuity

- If you deposit $100 at the end of each year for three years in a savings account that pays 5% interest per year, how much will you have at the end of three years?

0	5%	1	2	3	
		100	100	100.00	$= 100\,(1.05)^0$
				105.00	$= 100\,(1.05)^1$
				110.25	$= 100\,(1.05)^2$
				315.25	

Future Value of an Annuity

$$\mathbf{FVA_n = PMT(1+i)^0 + PMT(1+i)^1 + \cdots + PMT(1+i)^{n\text{-}1} = PMT \sum_{t=0}^{n\text{-}1} (1+i)^t}$$

$$\mathbf{= PMT\left[\sum_{t=1}^{n} (1+i)^{n-t}\right] = PMT\left[\frac{(1+i)^n - 1}{i}\right]}$$

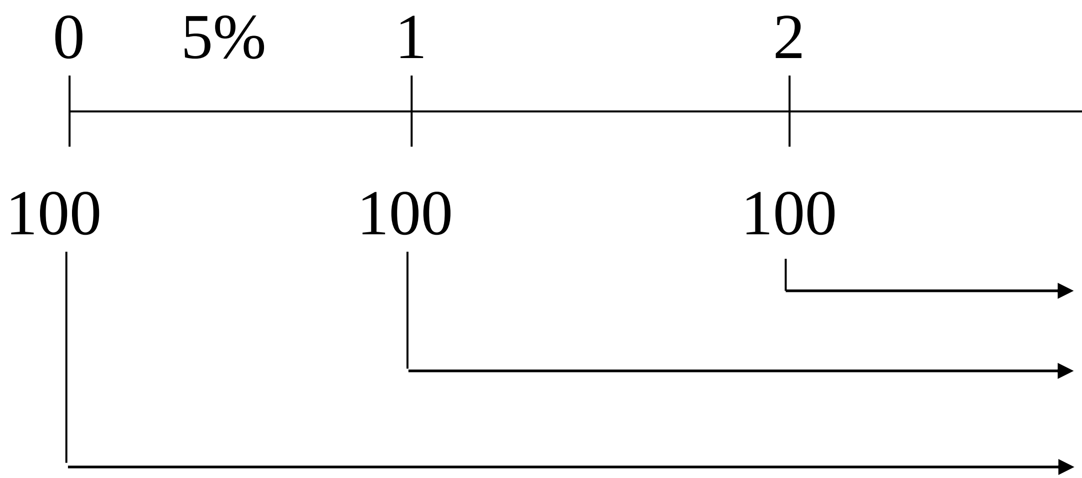
$$\mathbf{FVA_3 = \$100\left[\frac{(1.05)^3 - 1}{0.05}\right] = \$100(3.1525) = \$315.25}$$

Annuities Due

- If the three $100 payments had been made at the beginning of each year, the annuity would have been an annuity due.
- Each payment would shift to the left one year and each payment would earn interest for an additional year (period).

Future Value of an Annuity Due

- $100 at the start of each year



$$105.00 = 100\,(1.05)^1$$

$$110.25 = 100\,(1.05)^2$$

$$\underline{115.7625} = 100\,(1.05)^3$$

$$\underline{\underline{\mathbf{331.0125}}}$$

Future Value of an Annuity Due

- Numerical solution:

$$\mathrm{FVA(DUE)}_n = \mathrm{PMT}\left[\sum_{t=1}^{n}(1+i)^t\right]$$

$$= \mathrm{PMT}\left[\left\{\sum_{t=1}^{n}(1+i)^{n-t}\right\}\times(1+i)\right]$$

$$= \mathrm{PMT}\left[\left\{\frac{(1+i)^n - 1}{i}\right\}\times(1+i)\right]$$

Future Value of an Annuity Due

- Numerical solution:

$$\mathbf{FVA(DUE)_n} = \$100\left[\left\{\frac{(1.05)^3 - 1}{0.05}\right\} \times (1.05)\right]$$

$$= \$100[(3.1525) \times 1.05]$$

$$= \$331.0125$$

Present Value of an Annuity

- If you were offered a three-year annuity with payments of $100 at the end of each year
- Or a lump sum payment today that you could put in a savings account paying 5% interest per year
- How large must the lump sum payment be to make it equivalent to the annuity?

Present Value of an Annuity

$$\frac{100}{(1.05)^1} = 95.238$$

$$\frac{100}{(1.05)^2} = 90.703$$

$$\frac{100}{(1.05)^3} = 86.384$$

272.325

Present Value of an Annuity

- Numerical solution:

$$PVA_n = PMT\left[\frac{1}{(1+i)^1}\right] + PMT\left[\frac{1}{(1+i)^2}\right] + \cdots + PMT\left[\frac{1}{(1+i)^n}\right]$$

$$= PMT\left[\sum_{t=1}^{n} \frac{1}{(1+i)^t}\right]$$

Present Value of an Annuity

$$PVA_n = PMT\left[\frac{1}{(1+i)^1}\right] + PMT\left[\frac{1}{(1+i)^2}\right] + \cdots + PMT\left[\frac{1}{(1+i)^n}\right]$$

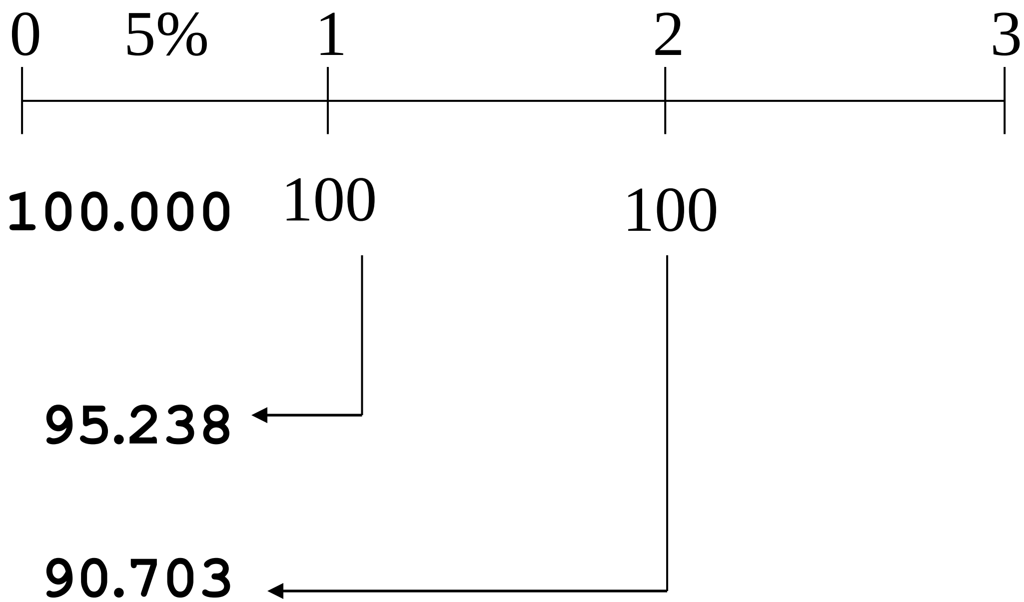
$$= PMT\left[\sum_{t=1}^{n}\frac{1}{(1+i)^t}\right] = PMT\left[\frac{1-\frac{1}{(1+i)^n}}{i}\right]$$

$$= \$100\left[\frac{1-\frac{1}{(1.05)^3}}{0.05}\right] = \$100(2.7232) = \$272.32$$

Present Value of an Annuity Due

- Payments at the beginning of each year
- Payments all come one year sooner
- Each payment would be discounted for one less year
- Present value of annuity due will exceed the value of the ordinary annuity by one year’s interest on the present value of the ordinary annuity

Present Value of an Annuity Due



$$\frac{100}{(1.05)^1} \times (1.05) = \frac{100}{(1.05)^0} = 100.000$$

$$\frac{100}{(1.05)^1} \times (1.05) = \frac{100}{(1.05)^1} = \quad 95.238$$

$$\frac{100}{(1.05)^2} \times (1.05) = \frac{100}{(1.05)^2} = \quad 90.703$$

285.941

Present Value of an Annuity Due

- Numerical solution:

$$\mathbf{PVA(DUE)_n = PMT}\left[\sum_{t=0}^{n-1}\frac{1}{(1+i)^1}\right] = \mathbf{PMT}\left[\left\{\sum_{t=1}^{n}\frac{1}{(1+i)^t}\right\}\times(1+i)\right]$$

$$= \mathbf{PMT}\left[\left\{\frac{1-\dfrac{1}{(1+i)^n}}{i}\right\}\times(1+i)\right]$$

Present Value of an Annuity Due

$$PV(DUE)_3 = \$100\left[\left\{\frac{1-\frac{1}{(1.05)^3}}{0.05}\right\}\times(1.05)\right]$$

$$= \$100\,[(2.72325)(1.05)]$$

$$= \$100\,(2.85941)$$

$$= \$285.941$$

Perpetuities

- Perpetuity - a stream of equal payments expected to continue forever
- Consol - a perpetual bond issued by the British government to consolidate past debts; in general, and perpetual bond

$$\text{PVP} = \frac{\text{Payment}}{\text{Interest Rate}} = \frac{\text{PMT}}{\text{i}}$$

Uneven Cash Flow Streams

- Uneven cash flow stream is a series of cash flows in which the amount varies from one period to the next
- Payment (PMT) designates constant cash flows
- Cash Flow (CF) designates cash flows in general, including uneven cash flows

Present Value of Uneven Cash Flow Streams

- PV of uneven cash flow stream is the sum of the PVs of the individual cash flows of the stream

$$\mathbf{PV} = \mathbf{CF_1}\left[\frac{1}{(1+\mathbf{i})^1}\right] + \mathbf{CF_2}\left[\frac{1}{(1+\mathbf{i})^2}\right] + \cdots + \mathbf{CF_n}\left[\frac{1}{(1+\mathbf{i})^\mathbf{n}}\right]$$

$$= \sum_{\mathbf{t}=1}^{\mathbf{n}} \mathbf{CF_t}\left[\frac{1}{(1+\mathbf{i})^\mathbf{t}}\right]$$

Future Value of Uneven Cash Flow Streams

- Terminal value is the future value of an uneven cash flow stream

$$
\begin{aligned}
FV_n &= CF_1(1+i)^{n-1} + CF_2(1+i)^{n-2} + \cdots + CF_n(1+i)^0 \\
&= \sum_{t=1}^{n} CF_t(1+i)^{n-t}
\end{aligned}
$$

Amortized Loans

- Loans that are repaid in equal payments over its life
- Borrow $15,000 to repay in three equal payments at the end of the next three years, with 8% interest due on the outstanding loan balance at the beginning of each year

Amortized Loans

0   8%   1   2   3

15,000   PMT   PMT   PMT

$$PVA_3 = \frac{PMT}{(1+i)^1} + \frac{PMT}{(1+i)^2} + \frac{PMT}{(1+i)^3}$$

$$= \sum_{t=1}^{3} \frac{PMT}{(1+i)^t}$$

$$\$15{,}000 = \sum_{t=1}^{3} \frac{PMT}{(1.08)^t}$$

Amortized Loans

- Numerical Solution:

$$\$15{,}000 = \sum_{t=1}^{3} \frac{\text{PMT}}{(1.08)^t} = \text{PMT}\left[\sum_{t=1}^{3} \frac{1}{(1.08)^t}\right] = \text{PMT}\left[\frac{1 - \dfrac{1}{(1.08)^3}}{0.08}\right]$$

$$\$15{,}000 = \text{PMT}(2.5771)$$

$$\text{PMT} = \frac{\$15{,}000}{2.5771} = \$5{,}820.50$$

Amortized Loans

Year	Beginning Amount (1)	Payment (2)	Interest[a] (3)	Repayment of Principal[b] (2)-(3)=(4)	Remaining Balance (1)-(4)=(5)
1	$ 15,000.00	$ 5,820.50	$ 1,200.00	$ 4,620.50	$ 10,379.50
2	10,379.50	5,820.50	830.36	4,990.14	5,389.36
3	5,389.36	5,820.50	431.15	5,389.35	0.01 [c]

[a]Interest is calculated by multiplying the loan balance at the beginning of the year by the interest rate. Therefore, interest in Year 1 is $15,000(0.08) = $1,200; in Year 2, it is $10,379.50(0.08)=$830.36; and in Year 3, it is $5,389.36(0.08) = $431.15 (rounded).

[b]Repayment of principal is equal to the payment of $5,820.50 minus the interest charge for each year.

[c]The $0.01 remaining balance at the end of Year 3 results from rounding differences.