# Chapter-04

## Partial Differentiation

If $u = x^3 + y^3 + z^3$, then show that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = 3u$.

Solⁿ: Given,

$$u = x^3 + y^3 + z^3 \quad \text{--- (I)}$$

Diff (I) w.r.t. to $x, y$ and $z$,

$$\frac{\delta u}{\delta x} = 3x^2 + 0 + 0$$

$$\Rightarrow x\frac{\delta u}{\delta x} = 3x^3 \quad \text{--- (II)}$$

Similarly,

$$y\frac{\delta u}{\delta y} = 3y^3 \quad \text{--- (III)}$$

and $z\frac{\delta u}{\delta z} = 3z^3$ --- (IV)

Adding (II), (III) and (IV), we get,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = 3(x^3 + y^3 + z^3)$$

$$\therefore x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = 3u \text{ (showed)}$$

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If $u = \log(x^2+y^2)$, then show that, $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$.

Solⁿ:

Given,

$$u = \log(x^2+y^2) \text{ --- } (1)$$

Diff (1) two times w.r.t. to $x$, we have

$$\frac{\delta u}{\delta x} = \frac{1}{x^2+y^2} \times 2x$$

$$\Rightarrow \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta x}\left(\frac{2x}{x^2+y^2}\right)$$

$$\Rightarrow \frac{\delta^2 u}{\delta x^2} = \frac{(x^2+y^2)\times 2 - 2x \times 2x}{(x^2+y^2)^2} \quad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}\right]$$

$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\delta^2 u}{\delta x^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} \text{ --- } (ii)$$

Again diff. (1) two times w.r.t. to $y$, we have,

$$\frac{\delta u}{\delta y} = \frac{2y}{x^2+y^2}$$

$$\Rightarrow \frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta y}\left(\frac{2y}{x^2+y^2}\right)$$

$$\Rightarrow \frac{\delta^2 u}{\delta y^2} = \frac{(x^2+y^2)\times 2 - 2y \times 2y}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\delta^2 u}{\delta y^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2} \text{ --- } (iii)$$

Adding eq$^n$ (ii) and (iii), we get,

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2+y^2)^2} = 0$$

(showed)

(iii) If $u = a\log(x^2+y^2) + b\tan^{-1}(y/x)$, then show that

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

Sol$^n$: Given,

$$u = a\log(x^2+y^2) + b\tan^{-1}(y/x) \quad \text{--- (1)}$$

Diff (1) two times w.r.t. to $x$, we have,

$$\frac{\delta u}{\delta x} = \frac{a}{x^2+y^2} \times 2x + \frac{b}{1+\frac{y^2}{x^2}} \times y\left(\frac{-1}{x^2}\right)$$

$$= \frac{2ax}{x^2+y^2} - \frac{by}{\frac{x^2+y^2}{x^2} \times x^2}$$

$$= \frac{2ax}{x^2+y^2} - \frac{by}{x^2+y^2}$$

$$\therefore \frac{\delta u}{\delta x} = \frac{2ax - by}{x^2+y^2}$$

$$\Rightarrow \frac{\delta^2 u}{\delta x^2} = \frac{(x^2+y^2)\times 2a - (2ax-by)\times 2x}{(x^2+y^2)^2}$$

$$= \frac{2ax^2 + 2ay^2 - 4ax^2 + 2bxy}{(x^2+y^2)^2}$$

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$$\therefore \frac{\delta^2 u}{\delta x^2} = \frac{2ay^2 - 2ax^2 + 2bxy}{(x^2+y^2)^2} \text{ --- (ii)}$$

Again diff (i) two times w.r.t. to $y$,

$$\frac{\delta u}{\delta y} = \frac{a}{x^2+y^2} \times 2y + \frac{b}{1+\frac{y^2}{x^2}} \times \frac{1}{x}$$

$$= \frac{2ay}{x^2+y^2} + \frac{b}{\frac{x^2+y^2}{x^2} \times x}$$

$$\therefore \frac{\delta u}{\delta y} = \frac{2ay}{x^2+y^2} + \frac{bx}{x^2+y^2}$$

$$= \frac{2ay+bx}{x^2+y^2}$$

$$\Rightarrow \frac{\delta^2 u}{\delta y^2} = \frac{(x^2+y^2) \times 2a - (2ay+bx) \times 2y}{(x^2+y^2)^2}$$

$$= \frac{2ax^2 + 2ay^2 - 4ay^2 - 2bxy}{(x^2+y^2)^2}$$

$$\therefore \frac{\delta^2 u}{\delta y^2} = \frac{2ax^2 - 2ay^2 - 2bxy}{(x^2+y^2)^2} \text{ --- (iii)}$$

Adding eq$^n$ (ii) and (iii), we get,

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{2ay^2 - 2ax^2 + 2bxy + 2ax^2 - 2ay^2 - 2bxy}{(x^2+y^2)^2}$$

$$= 0 \text{ (showed)}$$

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If $v = \frac{1}{\sqrt{x^2+y^2+z^2}}$, then show that $v_{xx} + v_{yy} + v_{zz} = 0$,

or, $\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} + \frac{\delta^2 v}{\delta z^2} = 0$

Sol$^n$: Given,

$$v = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow v = (x^2+y^2+z^2)^{-1/2} \quad \text{--- (i)}$$

Diff (i) two times w.r.t. to $x$,

$$v_x = -\frac{1}{2}(x^2+y^2+z^2)^{-1/2-1} \times 2x$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\therefore v_{xx} = -x(-3/2)(x^2+y^2+z^2)^{-3/2-1} \times 2x - 1 \times (x^2+y^2+z^2)^{-3/2}$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (ii)}$$

Similarly,

$$v_{yy} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \quad \text{--- (iii)}$$

and $v_{zz} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$ --- (iv)

Adding eq$^n$ (ii), (iii) and (iv), we get,

$$v_{xx} + v_{yy} + v_{zz} = (x^2+y^2+z^2)^{-5/2}(3x^2+3y^2+3z^2) - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

(showed)

If $u = \sqrt{x^2+y^2+z^2}$, Prove that, $u_{xx} + u_{yy} + u_{zz} = \frac{2}{u}$.

Sol$^n$: Given,

$$u = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow u = (x^2+y^2+z^2)^{1/2} \quad \text{——— (1)}$$

Diff. u two times w.r.t. to $x$,

$$u_x = \frac{1}{2}(x^2+y^2+z^2)^{-1/2}(2x)$$

$$\Rightarrow u_x = x(x^2+y^2+z^2)^{-1/2}$$

$$\Rightarrow u_{xx} = x(-1/2)(x^2+y^2+z^2)^{-3/2}(2x) + (x^2+y^2+z^2)^{-1/2}(1)$$

$$\therefore u_{xx} = (x^2+y^2+z^2)^{-1/2} - x^2(x^2+y^2+z^2)^{-3/2} \quad \text{——— (2)}$$

Similarly,

$$u_{yy} = (x^2+y^2+z^2)^{-1/2} - y^2(x^2+y^2+z^2)^{-3/2} \quad \text{——— (3)}$$

and, $$u_{zz} = (x^2+y^2+z^2)^{-1/2} - z^2(x^2+y^2+z^2)^{-3/2} \quad \text{——— (4)}$$

Adding eq$^n$ (2), (3) and (4), we get,

$$u_{xx} + u_{yy} + u_{zz} = 3(x^2+y^2+z^2)^{-1/2} - (x^2+y^2+z^2)^{-3/2}(x^2+y^2+z^2)$$

$$= 3(x^2+y^2+z^2)^{-1/2} - (x^2+y^2+z^2)^{-1/2}$$

$$= 2(x^2+y^2+z^2)^{-1/2}$$

$$= \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore u_{xx} + u_{yy} + u_{zz} = \frac{2}{u}$$

(Proved)

If $u = x^2y + y^2z + z^2x$, then show that, $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = (x+y+z)^2$.

Sol$^n$: Given,

$$u = x^2y + y^2z + z^2x \quad \text{--- (1)}$$

Diff. (1) w.r.to $x, y$ and $z$,

$$\frac{\delta u}{\delta x} = 2xy + 0 + z^2 \quad \text{--- (2)}$$

$$\frac{\delta u}{\delta y} = x^2 + 2yz + 0 \quad \text{--- (3)}$$

$$\frac{\delta u}{\delta z} = 0 + y^2 + 2zx \quad \text{--- (4)}$$

Adding eq$^n$ (2), (3) and (4), we get

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$= (x+y+z)^2$$

(showed)

If $u = \ln(x^3 + y^3 + z^3 - 3xyz)$, then show that,

(1) $\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = \frac{3}{x+y+z}$

(2) $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = \frac{-3}{(x+y+z)^2}$

(3) $\left(\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Sol$^n$: Part 1 →

Given,

$$u = \ln(x^3 + y^3 + z^3 - 3xyz) \quad \text{--- (1)}$$

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Diff. (1) w.r.t. to $x, y$ and $z$,

$$\frac{\delta u}{\delta x} = \frac{1}{x^3+y^3+z^3-3xyz} \times (3x^2+0+0-3yz)$$

$$= \frac{3(x^2-yz)}{x^3+y^3+z^3-3xyz} \quad \text{———} \quad (2)$$

Similarly, $$\frac{\delta u}{\delta y} = \frac{3(y^2-zx)}{x^3+y^3+z^3-3xyz} \quad \text{———} \quad (3)$$

and $$\frac{\delta u}{\delta z} = \frac{3(z^2-xy)}{x^3+y^3+z^3-3xyz} \quad \text{———} \quad (4)$$

Adding eq^n (2), (3) and (4), we get,

$$\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$$= \frac{3}{x+y+z}$$ (showed)

Part 2

Diff. (2), (3) and (4) w.r.t. to $x, y$ and $z$ respectively,

$$\frac{\delta^2 u}{\delta x^2} = \frac{(x^3+y^3+z^3-3xyz)\,6x - (3x^2-3yz)(3x^2-3yz)}{(x^3+y^3+z^3-3xyz)^2}$$

$$= \frac{6x^4+6xy^3+6xz^3-18x^2yz - 9x^4 + 18x^2yz - 9y^2z^2}{(x^3+y^3+z^3-3xyz)^2}$$

$$= \frac{-3x^4 + 6xy^3 + 6xz^3 - 9y^2z^2}{(x^3+y^3+z^3-3xyz)^2}$$

$$\therefore \frac{\delta^2 u}{\delta x^2} = \frac{-3(x^4 - 2xy^3 - 2xz^3 + 3y^2z^2)}{(x^3+y^3+z^3-3xyz)^2} \quad \text{--- } (5)$$

Similarly, $\frac{\delta^2 u}{\delta y^2} = \frac{-3(y^4 - 2yz^3 - 2yx^3 + 3z^2x^2)}{(x^3+y^3+z^3-3xyz)^2}$ --- (6)

and $\frac{\delta^2 u}{\delta z^2} = \frac{-3(z^4 - 2zx^3 - 2zy^3 + 3x^2y^2)}{(x^3+y^3+z^3-3xyz)^2}$ --- (7)

Adding eq$^n$ (5), (6) and (7), we get,

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} + \frac{\delta^2 u}{\delta z^2} = \frac{-3(x^2+y^2+z^2-xy-yz-zx)^2}{(x+y+z)^2(x^2+y^2+z^2-xy-yz-zx)^2}$$

$$= \frac{-3}{(x+y+z)^2} \quad \text{(showed)}$$

Part 3,

$$\text{L.H.S.} = \left(\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right)^2 u$$

$$= \left(\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right)\left(\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} + \frac{\delta u}{\delta z}\right)$$

$$= \left(\frac{\delta}{\delta x} + \frac{\delta}{\delta y} + \frac{\delta}{\delta z}\right)\left(\frac{3}{x+y+z}\right) \quad \text{[From Part 1]}$$

$$= \frac{\delta}{\delta x}\left(\frac{3}{x+y+z}\right) + \frac{\delta}{\delta y}\left(\frac{3}{x+y+z}\right) + \frac{\delta}{\delta z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$

(showed)

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Let $T(x,y,z)$ be the temperature at the point $(x,y,z)$ in a metal object. If the temperature doesn't vary with time, then it is known that, $\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} + \frac{\delta^2 T}{\delta z^2} = 0$ —— (1)

Show that the function $T(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ satisfies equation (1).

Solⁿ: Given,

$$T = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow T = (x^2+y^2+z^2)^{-1/2} \text{ —— (i)}$$

Diff (i) w.r.t. to $x$ two times,

$$\frac{\delta T}{\delta x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$$

$$\Rightarrow \frac{\delta T}{\delta x} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\Rightarrow \frac{\delta^2 T}{\delta x^2} = -x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) - (x^2+y^2+z^2)^{-3/2} \text{ —— (ii)}$$

Similarly, $\frac{\delta^2 T}{\delta y^2} = \quad (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$ —— (iii)

and $\frac{\delta^2 T}{\delta z^2} = \quad (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$ —— (iv)

Adding eqⁿ (ii), (iii) and (iv), we get;

$$\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} + \frac{\delta^2 T}{\delta z^2} = (x^2+y^2+z^2)^{-5/2} \times \quad - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-5/2+1} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

Hence, $T(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ satisfies eqⁿ (1)

(Showed)

## Homogeneous Function

A function $f(x_1, x_2, \cdots x_k)$ is a homogeneous function of degree $n$ in $x_1, x_2, \cdots x_k$ if $f(tx_1, tx_2, \cdots tx_k) = t^n f(x_1, x_2, \cdots x_k)$.

For example, $f(x,y) = ax^2 + 2hxy + by^2$ is a homogeneous function of degree two in $x, y$ since

$$f(tx, ty) = a(tx)^2 + 2h(tx)(ty) + b(ty)^2$$
$$= t^2 f(x,y)$$

# Let, $f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + a_3 x^{n-3} y^3 + \cdots$
$$\cdots + a_n x^0 y^n$$

$$= a_0 x^n + a_1 x^n . x^{-1} . y + a_2 . x^n . x^{-2} . y^2 + a_3 x^n . x^{-3} . y^3 + \cdots + a_n y^n$$

$$= x^n \left[ a_0 \left(\frac{y}{x}\right)^0 + a_1 \left(\frac{y}{x}\right)^1 + a_2 \left(\frac{y}{x}\right)^2 + \cdots + a_n \left(\frac{y}{x}\right)^n \right]$$

$$= x^n f\left(\frac{y}{x}\right)$$

## State and prove Euler's Theorem on Homogeneous Function.

* **Statement:** If $u$ be a homogeneous function of degree $n$ in $x$ and $y$, then
$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = nu$$

*Proof: Since $u$ is a homogeneous function of degree $n$ in $x$ and $y$, we can write,

$$u = x^n f(y/x) \text{ ——— } (1)$$

Diff (1) w.r.t. to $x$ and $y$,

$$\frac{\delta u}{\delta x} = x^n f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) + f\left(\frac{y}{x}\right) n x^{n-1}$$

$$= -x^{n-2} y \cdot f'(y/x) + n x^{n-1} f(y/x)$$

$$\Rightarrow x \frac{\delta u}{\delta x} = -x^{n-2+1} \cdot y \cdot f'(y/x) + n x^{n-1+1} f(y/x)$$

$$= -x^{n-1} y f'(y/x) + n x^n f(y/x) \text{ ——— } (2)$$

$$\frac{\delta u}{\delta y} = x^n f'(y/x)(1/x)$$

$$= x^{n-1} f'(y/x)$$

$$\Rightarrow y \frac{\delta u}{\delta y} = x^{n-1} y f'(y/x) \text{ ——— } (3)$$

Adding eq$^n$ (2) and (3), we get,

$$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = n x^n f(y/x)$$

$$= nu$$

(Proved)

☐ If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, then show that, $x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y} = \frac{\tan u}{2}$.

Soln:

Given,

$$u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$\Rightarrow \sin u = \frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y}/\sqrt{x})}$$

$$\Rightarrow \sin u = x^{1/2}\frac{1+y/x}{1+\sqrt{y}/\sqrt{x}}$$

$$\Rightarrow \sin u = x^{1/2} f(y/x)$$

from Euler's theorem,

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y} = nu$$

$$\Rightarrow x\frac{\delta}{\delta x}(\sin u)+y\frac{\delta}{\delta y}(\sin u) = \frac{1}{2}\sin u$$

$$\Rightarrow x\cos u\frac{\delta u}{\delta x}+y\cos u\frac{\delta u}{\delta y} = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y} = \frac{\sin u}{2\times\cos u} = \frac{\tan u}{2}$$ (showed)

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⊞ If $u = \tan^{-1}\frac{x^3+y^3}{x+y}$, show that $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \sin 2u$.

Soln:

Given,

$$u = \tan^{-1}\frac{x^3+y^3}{x+y}$$

$$\Rightarrow \tan u = \frac{x^3+y^3}{x+y}$$

$$\Rightarrow \tan u = \frac{x^3(1+y^3/x^3)}{x(1+y/x)}$$

$$\Rightarrow \tan u = x^2\left\{\frac{1+(y/x)^3}{1+y/x}\right\}$$

$$\Rightarrow \tan u = x^2 f(y/x)$$

From Euler's theorem,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nu$$

$$\Rightarrow x\frac{\delta}{\delta x}(\tan u) + y\frac{\delta}{\delta x}(\tan u) = 2\tan u$$

$$\Rightarrow x.\sec^2 u\frac{\delta u}{\delta x} + y.\sec^2 u.\frac{\delta u}{\delta y} = 2\tan u$$

$$\Rightarrow x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = 2\frac{\tan u}{\sec^2 u} = 2\sin u\cos u$$

$$= \sin 2u$$

(showed)

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⊞ If $\operatorname{cosec}^{-1}\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}} = u$, then prove that

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \frac{-\tan u}{12}.$$

Soln: Given,

$$u = \operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

$$\Rightarrow \operatorname{cosec} u = \left\{\frac{x^{1/2}\left\{1+\left(y/x\right)^{1/2}\right\}}{x^{1/3}\left\{1+\left(y/x\right)^{1/3}\right\}}\right\}^{1/2}$$

$$= \left\{x^{1/2-1/3}\cdot\frac{1+\left(y/x\right)^{1/2}}{1+\left(y/x\right)^{1/3}}\right\}^{1/2}$$

$$= \left(x^{1/6}\right)^{1/2}\cdot\left\{\frac{1+\left(y/x\right)^{1/2}}{1+\left(y/x\right)^{1/3}}\right\}^{1/2}$$

$$\therefore \operatorname{cosec} u = x^{1/12}\cdot f\left(y/x\right)$$

From Euler's theorem,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nu$$

$$\Rightarrow x\frac{\delta}{\delta x}(\operatorname{cosec} u) + y\frac{\delta}{\delta y}(\operatorname{cosec} u) = \frac{\operatorname{cosec} u}{12}$$

$$\Rightarrow -x\cdot\operatorname{cosec} u\cdot\cot u\cdot\frac{\delta u}{\delta x} - y\cdot\operatorname{cosec} u\cdot\cot u\cdot\frac{\delta u}{\delta y} = \frac{\operatorname{cosec} u}{12}$$

$$\Rightarrow x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = \frac{-1}{12 \cot u} = \frac{-\tan u}{12}$$

(proved)

If $u$ is a homogeneous function of degree $n$, then show that $\left(x\frac{\delta}{\delta x} + y\frac{\delta}{\delta u}\right)^2 u = n(n-1)u$ or,

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = n(n-1)u$$

Sol$^n$: If $u$ is a homogeneous f$^n$ of degree $n$, then from Euler's theorem,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nu \quad \text{——} \ (1)$$

Diff. (1) partially w.r. to $x$,

$$x \cdot \frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) + \frac{\delta}{\delta x}(x) \cdot \frac{\delta u}{\delta x} + y\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right) = n\frac{\delta u}{\delta x}$$

$$\Rightarrow x\frac{\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} + y\frac{\delta^2 u}{\delta x \delta y} = n\frac{\delta u}{\delta x}$$

$$\Rightarrow x^2\frac{\delta^2 u}{\delta x^2} + x\frac{\delta u}{\delta x} + xy\frac{\delta^2 u}{\delta x \delta y} = nx\frac{\delta u}{\delta x} \quad \text{——} (2)$$

Diff. (1) partially w.r. to $y$,

$$x\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right) + y\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right) + \frac{\delta}{\delta y}(y) \cdot \frac{\delta u}{\delta y} = n\frac{\delta u}{\delta y}$$

$$\Rightarrow x\frac{\delta^2 u}{\delta x \delta y} + y\frac{\delta^2 u}{\delta y^2} + \frac{\delta u}{\delta y} = n\frac{\delta u}{\delta y}$$

$$\Rightarrow xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + y\frac{\delta u}{\delta y} = ny\frac{\delta u}{\delta y} \quad \text{—— (3)}$$

(2) + (3) $\Rightarrow$

$$x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + \left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right) + y^2\frac{\delta^2 u}{\delta y^2} = n\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y}\right)$$

$$\Rightarrow x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} + nu = n \times nu$$

$$\Rightarrow x^2\frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta x \delta y} + y^2\frac{\delta^2 u}{\delta y^2} = n^2u - nu = nu(n-1)$$

$$\text{or, } \left(x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y}\right)^2 u = n(n-1)u$$ (showed)

田 If $u(x,y) = x^2y + y^2$, then find $\frac{\delta u}{\delta x}$ and $\frac{\delta u}{\delta y}$.

Soln: Given, $u(x,y) = x^2y + y^2$ —— (1)

Diff (1) partially w.r. to $x$,

$$\frac{\delta u}{\delta x} = y\frac{\delta}{\delta x}(x^2) + \frac{\delta}{\delta x}(y^2)$$

$$= 2xy + 0 = 2xy$$ (Ans)

Diff (1) partially w.r. to $y$,

$$\frac{\delta u}{\delta y} = x^2\frac{\delta}{\delta y}(y) + \frac{\delta}{\delta y}(y^2)$$

$$= x^2 + 2y$$ (Ans)

~~Solve~~

# Maximum & Minimum

For what values of $x$, the following expression is maximum and minimum respectively. Also find the maximum and minimum values.

(i) $x^5 - 5x^4 + 5x^3 - 1$ (iii) $5x^6 - 18x^5 + 15x^4 - 10$

(ii) $2x^3 - 21x^2 + 36x - 20$

Solⁿ: (i) Given, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$$\therefore f'(x) = 5x^4 - 20x^3 + 15x^2$$

$$f''(x) = 20x^3 - 60x^2 + 30x$$

$$f'''(x) = 60x^2 - 120x + 30$$

For maxima and minima,

$$f'(x) = 0$$

$$\Rightarrow 5x^4 - 20x^3 + 15x^2 = 0$$

$$\Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$$\cancel{\Rightarrow x^2 - 3x - x + 3 = 0}$$

$$\cancel{\Rightarrow x(x-3) - 1(x-3) = 0}$$

$$\therefore \cancel{x = 1, 3}$$

$$\Rightarrow 5x^2 = 0 \quad \text{or,} \quad x^2 - 4x + 3 = 0$$

$$\therefore x = 0, 1, 3$$

When $x=0$, $f''(0)=0$, $f'''(0)=30>0$.

$\therefore f(x)$ has minimum value at $x=0$.

$\therefore f(0)=-1$

When $x=1$, $f''(1)=20-60+30=-10<0$

$\therefore f(x)$ has maximum value at $x=1$.

$\therefore f(1)=1-5+5-1$

$=0$

When $x=3$, $f''(3)=90>0$.

$\therefore f(x)$ has minimum value at $x=3$.

$\therefore f(3)=(3)^5-5(3)^4+5(3)^3-1$

$=-28$

So, maximum value at $x=1$ and minimum values at $x=0$ and $x=3$. Maximum value $=0$, minimum $=-1,-28$

(Ans)

(ii) Given,

$f(x)=2x^3-21x^2+36x-20$

$\therefore f'(x)=6x^2-42x+36$

$f''(x)=12x-42$

For maxima and minima, $f'(x)=0$

$\therefore 6x^2-42x+36=0$

$\Rightarrow x^2-7x+6=0$

$\Rightarrow x^2-6x-x+6=0$

$\Rightarrow x(x-6)-1(x-6)=0$

$\therefore x = 1, 6$

When $x = 1$, $f''(1) = 12 - 42 = -30 < 0$

and when $x = 6$, $f''(6) = 12(6) - 42 = 30 > 0$

So, $f(x)$ has maximum and minimum values at $x = 1$ and $x = 6$ respectively.

So, maxima $f(1) = 2(1)^3 - 21(1)^2 + 36(1) - 20$

$= -3$

and minima $f(6) = 2(6)^3 - 21(6)^2 + 36(6) - 20$

$= -128$

(Ans)

(iii) Given,

$f(x) = 5x^6 - 18x^5 + 15x^4 - 10$

$\therefore f'(x) = 30x^5 - 90x^4 + 60x^3$

$f''(x) = 150x^4 - 360x^3 + 180x^2$

$f'''(x) = 600x^3 - 1080x^2 + 360x$

$f^{iv}(x) = 1800x^2 - 2160x + 360$

For maxima and minima, $f'(x) = 0$

$\therefore 30x^5 - 90x^4 + 60x^3 = 0$

$\Rightarrow 30x^3(x^2 - 3x + 2) = 0$

Either, $30x^3 = 0$ or, $x^2 - 2x - x + 2 = 0$

$\therefore x = 0$ $\Rightarrow x(x-2) - 1(x-2) = 0$

$\therefore x = 1, 2$

When $x=0$, $f''(0)=0$, $f'''(0)=0$ and $f^{iv}(0)=360>0$

$\therefore f(x)$ has minimum value at $x=0$

~~When $x=$~~ $\therefore f(0)=-10$

When $x=1$, $f''(1) = 150-360+180 = -30<0$

$\therefore f(x)$ has maximum value at $x=1$.

$\therefore f(1) = 5-18+15-10$

$= -8$

When $x=2$, $f''(2) = 150(2)^4 - 360(2)^3 + 180(2)^2$

$= 240 > 0$

$\therefore f(x)$ has minimum value at $x=2$.

$\therefore f(2) = 5(2)^6 - 18(2)^5 + 15(2)^4 - 10$

$= -26$

So, $f(x)$ has maximum value at $x=1$ and minimum at $x=0$ and $x=2$. Maximum value $=-8$ and minimum value $=-10, -26$.

(Ans)

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[4] Show that the maximum value of $x+\frac{1}{x}$ is less than its minimum value.

Solⁿ:

Given, $f(x) = x + \frac{1}{x} = x + x^{-1}$

$\therefore f'(x) = 1 - \frac{1}{x^2}$

$f''(x) = \frac{2}{x^3}$

For maxima and minima, $f'(x) = 0$

$\therefore 1 - \frac{1}{x^2} = 0$

$\Rightarrow \cancel{\frac{x^2-1}{x^2}} = 0$

$\Rightarrow \frac{1}{x^2} = 1$

$\Rightarrow x^2 = 1$

$\therefore x = 1, -1$

When $x = -1$, $f''(-1) = \frac{2}{(-1)^3} = -2 < 0$.

$\therefore f(x)$ has maximum value at $x = -1$

$\therefore f(-1) = -1 + \frac{1}{-1} = -2$

When $x = 1$, $f''(1) = \frac{2}{(1)^3} = 2 > 0$.

$\therefore f(x)$ has minimum value at $x = 1$.

$\therefore f(1) = 1 + \frac{1}{1} = 2$

Here, we can see that the maximum value of $f(x)$ is $-2$ which is less than its minimum value 2.

(Showed)

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田 Show that $x^3-3x^2+6x+5$ has neither maximum nor minimum value.

Sol$^n$:

Given, $f(x) = x^3-3x^2+6x+5$

$\therefore f'(x) = 3x^2-6x+6$

$= 3(x^2-2x+2)$

$= 3\{(x-1)^2+1\}$

For all the real values of $x$, $f'(x) \neq 0$. Since $f'(x)\neq 0$, hence $f(x)$ has neither a maximum nor a minimum value.

(Showed)

田 Prove that, the function $\sqrt{3}\sin x + 3\cos x$ is maximum at $x = \pi/6$ and find the maximum value.

Sol$^n$:

Given,

$f(x) = \sqrt{3}\sin x + 3\cos x$

$\therefore f'(x) = \sqrt{3}\cos x - 3\sin x$

$f''(x) = -\sqrt{3}\sin x - 3\cos x$

For maxima and minima, $f'(x)=0$

$\therefore \sqrt{3}\cos x - 3\sin x = 0$

$\Rightarrow \frac{\sin x}{\cos x} = \frac{\sqrt{3}}{3} = 1/\sqrt{3}$

$\Rightarrow \tan x = \tan\frac{\pi}{6}$

$\therefore x = \pi/6$

When $x = \pi/6$, then $f''(\pi/6) = -\sqrt{3}\sin\pi/6 - 3\cos\pi/6$

$$= \frac{-\sqrt{3}}{2} - \frac{3\sqrt{3}}{2}$$

$$= \frac{-\sqrt{3} - 3\sqrt{3}}{2}$$

$$= \frac{-4\sqrt{3}}{2}$$

$$= -2\sqrt{3} < 0$$

So, $f(x)$ has the maximum value at $x = \pi/6$.

$$\therefore f(\pi/6) = \sqrt{3}\sin\pi/6 + 3\cos\pi/6$$

$$= \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}$$

$$= 2\sqrt{3}$$

(Ans)

# Formulae of Integration

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

2) $\int \frac{1}{x} dx = \ln|x| + c$

3) $\int e^x dx = e^x + c$

4) $\int \sin x \, dx = -\cos x + c$

5) $\int \cos x \, dx = \sin x + c$

6) $\int \tan x \, dx = \ln|\sec x| + c$
$= -\ln|\cos x| + c$

7) $\int \cot x \, dx = \ln|\sin x| + c$
$= -\ln|\operatorname{cosec} x| + c$

8) $\int \sec x \, dx = \ln|\sec x + \tan x| + c$
$= \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

9) $\int \operatorname{cosec} x \, dx = \ln|\operatorname{cosec} x - \cot x| + c$
$= \ln\left|\tan\frac{x}{2}\right| + c$

10) $\int uv \, dx = u\int v \, dx - \int\left\{\frac{d}{dx}(u)\int v \, dx\right\} dx$

11) $\int u \, dv = uv - \int v \, du$

12) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

13) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

14) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

15) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

16) $\int \sqrt{x^2+a^2} \, dx = \frac{x\sqrt{x^2+a^2}}{2} \pm \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

17) $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

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# Indefinite Integral

## * Method 1 [ হরে Linear Factor ]

① $\int \frac{dx}{5x+7}$

Let, $u = 5x+7$

$\Rightarrow \frac{du}{dx} = 5$

$\therefore dx = \frac{du}{5}$

$\therefore \int \frac{dx}{5x+7} = \int \frac{\frac{du}{5}}{u}$

$= \frac{1}{5}\int \frac{1}{u}\, du$

$= \frac{1}{5} \ln|u| + c$

$= \frac{1}{5} \ln|5x+7| + c$

(Ans)

② $\int \frac{dx}{(3x+2)^5}$

Let, $u = 3x+2$

$\Rightarrow \frac{du}{dx} = 3$

$\therefore dx = \frac{du}{3}$

$\therefore \int \frac{dx}{(3x+2)^5} = \int \frac{\frac{du}{3}}{u^5}$

$= \frac{1}{3}\int u^{-5}\, du$

$= \frac{1}{3} \frac{u^{-5+1}}{-5+1} + c$

$= \frac{-u^{-4}}{12} + c$

$= \frac{-(3x+2)^{-4}}{12} + c$

$= \frac{-1}{12(3x+2)^4} + c$

(Ans)

* Method 2 [ হরে দ্বিঘাত সমীকরণ ]

$$*\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$*\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

$$*\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$$

(3) $\displaystyle\int \frac{dx}{4x^2+8x+13}$

$$= \int \frac{dx}{4(x^2+2x+13/4)}$$

$$= \frac{1}{4}\int \frac{dx}{x^2+2\cdot x\cdot 1+1+13/4-1}$$

$$= \frac{1}{4}\int \frac{dx}{(x+1)^2+\left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{4}\times\frac{1}{3/2}\tan^{-1}\frac{x+1}{3/2} + c$$

$$= \frac{1}{4}\times\frac{2}{3}\times\tan^{-1}\frac{2(x+1)}{3} + c$$

$$= \frac{1}{6}\tan^{-1}\frac{2(x+1)}{3} + c \quad \text{(Ans)}$$

(4) $\displaystyle\int \frac{dx}{x^2-6x+5}$

$$= \int \frac{dx}{x^2-2\cdot x\cdot 3+3^2-2^2}$$

$$= \int \frac{dx}{(x-3)^2-2^2}$$

$$= \frac{1}{2\times 2}\times\ln\left|\frac{x-3-2}{x-3+2}\right| + c$$

$$= \frac{1}{4}\ln\left|\frac{x-5}{x-1}\right| + c$$

(Ans)

(5) $\int \frac{dx}{16x^2-9}$

$= \int \frac{dx}{(4x)^2 - 3^2}$

$= \cancel{\frac{1}{2\times 3} \ln \left| \frac{4x-3}{4x+3} \right| + c}$

$= \int \frac{dx}{16(x^2 - 9/16)}$

$= \frac{1}{16} \int \frac{dx}{x^2 - (3/4)^2}$

$= \frac{1}{16} \times \frac{1}{2\times 3/4} \times \ln \left| \frac{x - 3/4}{x + 3/4} \right| + c$

$= \frac{1}{16} \times \frac{2}{3} \times \ln \left| \frac{\frac{4x-3}{4}}{\frac{4x+3}{4}} \right| + c$

$= \frac{1}{24} \ln \left| \frac{4x-3}{4x+3} \right| + c$ (Ans)

(6) $\int \frac{dx}{x^2+x-2}$

$= \int \frac{dx}{x^2 + 2\cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - 2 - \frac{1}{4}}$

$= \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$

$= \frac{1}{2\times 3/2} \ln \left| \frac{x + 1/2 - 3/2}{x + 1/2 + 3/2} \right| + c$

$= \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + c$ (Ans)

*Method 3 $\left[ \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)+c| \right]$

(নিজের ফাংশনকে Diff. করে যদি উপরের ফাংশন পাওয়া যায়)

⑦ $\int \frac{2x}{x^2+1}\,dx$

Let, $x^2+1 = z$

$\Rightarrow 2x = \frac{dz}{dx}$

$\therefore 2x\,dx = dz$

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{dz}{z}$$

$$= \ln|z| + c$$

$$= \ln|x^2+1| + c$$

(Ans)

⑧ $\int \frac{\sqrt{1+\ln x}}{x}\,dx$

# Let, $1+\ln x = z$

$\Rightarrow \frac{1}{x} = \frac{dz}{dx}$

$\therefore \frac{dx}{x} = dz$

$$\int \frac{\sqrt{1+\ln x}}{x}\,dx =$$

$$= \int \sqrt{z}\,dz$$

$$= \int z^{1/2}\,dz$$

$$= \frac{z^{1/2+1}}{1/2+1} + c$$

$$= \frac{z^{3/2}}{3/2} + c$$

$$= \frac{2}{3}(1+\ln x)^{3/2} + c$$

(Ans)

(9) $\int \frac{e^x(1+x)}{\cos^2(xe^x)}\,dx$

Let, $xe^x = z$

$\Rightarrow xe^x + e^x = \frac{dz}{dx}$

$\Rightarrow e^x(1+x)\,dx = dz$

$$\int \frac{e^x(1+x)\,dx}{\cos^2(xe^x)} = \int \frac{dz}{\cos^2 z}$$

$$= \int \sec^2 z\,dz$$

$$= \tan z + c \quad \left[\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c\right]$$

$$= \tan(xe^x) + c \text{ (Ans)}$$

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* Method 4 $\left[ \dfrac{\text{একঘাতে } f^n}{\text{দ্বিঘাতে } f^n} \Rightarrow \dfrac{px+q}{ax^2+bx+c} \right]$

(10) $\displaystyle\int \frac{x\,dx}{4x^2+8x+13}$

Let, $4x^2+8x+13 = z$

$\Rightarrow 8x+8 = \dfrac{dz}{dx}$

$\Rightarrow (8x+8)\,dx = dz$

$$= \int \frac{\frac{1}{8}(8x+8) - 1}{4x^2+8x+13}\,dx$$

$$= \frac{1}{8}\int \frac{(8x+8)\,dx}{4x^2+8x+13} - \int \frac{dx}{4x^2+8x+13}$$

$$= \frac{1}{8}\int \frac{dz}{z} - \int \frac{dx}{4(x^2+2x+13/4)}$$

$$= \frac{1}{8}\int \frac{dz}{z} - \frac{1}{4}\int \frac{dx}{x^2+2.x.1+1^2+13/4-1}$$

$$= \frac{1}{8}\int \frac{dz}{z} - \frac{1}{4}\int \frac{dx}{(x+1)^2+\left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{8}\ln|z| - \frac{1}{4}\times\frac{1}{3/2}\tan \frac{x+1}{3/2} + c$$

$$= \frac{1}{8}\ln|4x^2+8x+13| - \frac{1}{6}\tan\frac{2(x+1)}{3} + c$$

(Ans)

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(11) $\int \frac{3x\,dx}{x^2-x-2}$

$= \int \frac{\frac{3}{2}(2x-1)+\frac{3}{2}}{x^2-x-2}\,dx$

$= \frac{3}{2}\int \frac{(2x-1)\,dx}{x^2-x-2} + \frac{3}{2}\int \frac{dx}{x^2-x-2}$

Let, $x^2-x-2 = z$

$\Rightarrow 2x-1 = \frac{dz}{dx}$

$\Rightarrow (2x-1)\,dx = dz$

$= \frac{3}{2}\int \frac{dz}{z} + \frac{3}{2}\int \frac{dx}{x^2-2.x.\frac{1}{2}+(\frac{1}{2})^2-2-\frac{1}{4}}$

$= \frac{3}{2}\int \frac{dz}{z} + \frac{3}{2}\int \frac{dx}{(x-\frac{1}{2})^2-(\frac{3}{2})^2}$

$= \cancel{\frac{3}{2}\int}$ $= \frac{3}{2}\ln|z| + \frac{3}{2} \times \frac{1}{2\times\frac{3}{2}} \times \ln\left|\frac{x-\frac{1}{2}-\frac{3}{2}}{x-\frac{1}{2}+\frac{3}{2}}\right| + c$

$= \frac{3}{2}\ln|x^2-x-2| + \frac{1}{2}\ln\left|\frac{x-2}{x+1}\right| + c$ (Ans)

(12) $\int \frac{2x+3}{3x^2-x+1}\,dx$

$= \int \frac{\frac{1}{3}(6x-1)+3+\frac{1}{3}}{3x^2-x+1}\,dx$

Let, $3x^2-x+1 = z$

$\Rightarrow 6x-1 = \frac{dz}{dx}$

$\therefore (6x-1)\,dx = dz$

$= \frac{1}{3}\int \frac{6x-1}{3x^2-x+1}\,dx + \frac{10}{3}\int \frac{dx}{3x^2-x+1}$

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$$= \frac{1}{3}\int \frac{6x-1}{3x^2-x+1}\,dx + \frac{10}{3}\int \frac{dx}{3\left(x^2-\frac{x}{3}+\frac{1}{3}\right)}$$

$$= \frac{1}{3}\int \frac{dz}{z} + \frac{10}{9}\int \frac{dx}{x^2 - 2\cdot x\cdot\frac{1}{6} + \left(\frac{1}{6}\right)^2 + \frac{1}{3} - \frac{1}{36}}$$

$$= \frac{1}{3}\int \frac{dz}{z} + \frac{10}{9}\int \frac{dx}{\left(x-\frac{1}{6}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2}$$

$$= \frac{1}{3}\ln|z| + \frac{10}{9}\times\frac{1}{\frac{\sqrt{11}}{6}}\times\tan^{-1}\frac{x-\frac{1}{6}}{\frac{\sqrt{11}}{6}} + c$$

$$= \frac{1}{3}\ln|3x^2-x+1| + \frac{20}{3\sqrt{11}}\times\tan^{-1}\frac{6x-1}{\sqrt{11}} + c \quad \text{(Ans)}$$

* **Method 5** [হরে ও লবে দ্বিঘাত $f^n$]

(13) $$\int \frac{x^2-x+1}{x^2+x+1}\,dx$$

$$= \int \frac{(x^2+x+1)-2x}{x^2+x+1}\,dx$$

$$= \int dx - \int \frac{2x\,dx}{x^2+x+1}$$

$$= x - \int \frac{(2x+1)-1}{x^2+x+1}\,dx$$

Let,

$x^2+x+1 = z$

$\Rightarrow 2x+1 = \frac{dz}{dx}$

$\therefore (2x+1)\,dx = dz$

$$= x - \int \frac{2x+1}{x^2+x+1}\,dx + \int \frac{dx}{x^2+x+1}$$

$$= x - \int \frac{1}{z}\,dz + \int \frac{dx}{x^2+2\cdot x\cdot \frac{1}{2}+(\frac{1}{2})^2+1-\frac{1}{4}}$$

$$= x - \ln|z| + \int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= x - \ln|x^2+x+1| + \frac{1}{\sqrt{3}/2}\tan^{-1}\left|\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right| + c$$

$$= x - \ln|x^2+x+1| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + c \quad \text{(Ans)}$$

(14) $\int \frac{x^2+x+2}{x^2-x+1}\,dx$

$$= \int \frac{(x^2-x+1)+2x+1}{x^2-x+1}\,dx$$

$$= \int dx + \int \frac{2x+1}{x^2-x+1}\,dx$$

$$= x + \int \frac{(2x-1)+2}{x^2-x+1}\,dx$$

$$= x + \int \frac{(2x-1)}{x^2-x+1}\,dx + \int \frac{2\,dx}{x^2-x+1}$$

Let, $x^2-x+1 = z$

$\Rightarrow 2x-1 = \frac{dz}{dx}$

$\therefore (2x-1)\,dx = dz$

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$$= x + \int \frac{1}{z}\,dz + 2\int \frac{dx}{x^2 - 2\cdot x\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2 + 1 - \frac{1}{4}}$$

$$= x + \ln|z| + 2\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= x + \ln|x^2 - x + 1| + 2 \times \frac{1}{\sqrt{3}/2} \times \tan^{-1}\frac{x - 1/2}{\frac{\sqrt{3}}{2}} + c$$

$$= x + \ln|x^2 - x + 1| + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}} + c$$ (Ans)

15) $\int \frac{x^2}{x^2-4}\,dx$

$$= \int \frac{(x^2-4)+4}{x^2-4}\,dx$$

$$= \int dx + 4\int \frac{dx}{x^2-4}$$

$$= x + 4\int \frac{dx}{x^2 - 2^2}$$

$$= x + 4 \times \frac{1}{2\times 2} \ln\left|\frac{x-2}{x+2}\right| + c$$

$$= x + \ln\left|\frac{x-2}{x+2}\right| + c$$ (Ans)

## * Method 6

$$\left[\frac{\text{একঘাত } f^n}{\sqrt{\text{দ্বিঘাত } f^n}}\right]$$

(16) $\displaystyle\int \frac{x-2}{\sqrt{2x^2-8x+5}}\,dx$

Let, $2x^2-8x+5 = z$

$\Rightarrow 4x-8 = \dfrac{dz}{dx}$

$\therefore (4x-8)\,dx = dz$

$$= \int \frac{\frac{1}{4}(4x-8)}{\sqrt{2x^2-8x+5}}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{z}}\,dz$$

$$= \frac{1}{4}\int z^{-1/2}\,dz$$

$$= \frac{1}{4}\times\frac{z^{-1/2+1}}{-1/2+1} + c$$

$$= \frac{1}{4}\times\frac{z^{1/2}}{1/2} + c$$

$$= \frac{\sqrt{2x^2-8x+5}}{2} + c \quad \text{(Ans)}$$

## * Method 7 $\left[ \dfrac{dx}{(ax+b)\sqrt{cx+d}} \text{ format} \right]$

(17) $\displaystyle\int \frac{dx}{(2x+1)\sqrt{4x+3}}$

Let, $4x+3 = z^2$

$\Rightarrow 4 = 2z\dfrac{dz}{dx}$

$\Rightarrow dx = \dfrac{2z\,dz}{4} = \dfrac{z\,dz}{2}$

Again, $4x+3 = z^2$

$\Rightarrow x = \dfrac{z^2-3}{4}$

$$= \int \frac{\frac{z\,dz}{2}}{\left(2\frac{z^2-3}{4}+1\right)\sqrt{z^2}}$$

$$= \frac{1}{2}\int \frac{z\,dz}{\left(\frac{z^2-3}{2}+1\right) z}$$

$$= \frac{1}{2}\int \frac{dz}{\frac{z^2-3+2}{2}}$$

$$= \frac{1}{2}\int \frac{2\,dz}{z^2-1}$$

$$= \int \frac{dz}{z^2-1^2}$$

$$= \frac{1}{2\times 1} \times \ln\left|\frac{z-1}{z+1}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}\right| + c$$

(Ans)

(18) $\displaystyle\int \frac{dx}{(3+x)\sqrt{2+x}}$

Let, $2+x = z^2$

$\Rightarrow 1 = 2z\,dz/dx$

$\therefore dx = 2z\,dz$

Again, $2+x = z^2$

$\Rightarrow x = z^2 - 2$

$$= \int \frac{2z\,dz}{(3+z^2-2)\sqrt{z^2}}$$

$$= 2\int \frac{z\,dz}{(z^2+1)\,z}$$

$$= 2\int \frac{dz}{z^2+1^2}$$

$$= 2 \times \frac{1}{1} \times \tan^{-1}\frac{z}{1} + c$$

$$= 2\tan^{-1}(\sqrt{2+x}) + C \quad \text{(Ans)}$$

* Method 8 $\left[ \dfrac{dx}{(px+q)\sqrt{ax^2+bx+c}} \text{ format} \right]$

(19) $\displaystyle\int \frac{dx}{(2x+3)\sqrt{x^2+3x+2}}$

Let, $2x+3 = \frac{1}{z} = z^{-1}$

$\Rightarrow 2 = -\frac{1}{z^2}\cdot\frac{dz}{dx}$

$\Rightarrow dx = \frac{-dz}{2z^2}$

Again, $2x+3 = \frac{1}{z}$

$\Rightarrow x = \left(\frac{1}{z} - 3\right) \times \frac{1}{2}$

$= \frac{1-3z}{2z}$

$$= \int \frac{-dz/2z^2}{\frac{1}{z}\sqrt{\left(\frac{1-3z}{2z}\right)^2 + 3\left(\frac{1-3z}{2z}\right) + 2}}$$

$$= -\frac{1}{2}\int \frac{dz}{2z^2 \times \frac{1}{z}\sqrt{\frac{1-6z+9z^2}{4z^2}+\frac{3-9z}{2z}+2}}$$

$$= -\frac{1}{2}\int \frac{dz}{z\sqrt{\frac{1-6z+9z^2+6z-18z^2+8z^2}{4z^2}}}$$

$$= -\frac{1}{2}\int \frac{dz}{z\sqrt{\frac{1-z^2}{4z^2}}}$$

$$= -\frac{1}{2}\int \frac{dz}{z \times \frac{1}{2z}\sqrt{1-z^2}}$$

$$= -\frac{2}{2}\int \frac{dz}{\sqrt{1-z^2}}$$

$$= -\sin^{-1} z + c \left[ \because \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \right]$$

$$= -\sin^{-1}\left(\frac{1}{2x+3}\right) + c$$ (Ans)

* Method 9 $\left[ \int (px+q)\sqrt{ax^2+bx+c}\, dx \right]$

(20) $\int (4x-3)\sqrt{3x^2-x+1}\, dx$

Here,
$\frac{d}{dx}(3x^2-x+1)$
$= 6x-1$

$(4x-3) = \frac{2}{3}(6x-1) - 3 + \frac{2}{3}$
$= \frac{2}{3}(6x-1) - \frac{7}{3}$

$= \int \left\{ \frac{2}{3}(6x-1) - \frac{7}{3} \right\} \sqrt{3x^2-x+1}\, dx$

$= \frac{2}{3}\int (6x-1)\sqrt{3x^2-x+1}\, dx - \frac{7}{3}\int \sqrt{3x^2-x+1}\, dx$

Now,
$\int (6x-1)\sqrt{3x^2-x+1}\, dx$

Let, $3x^2-x+1 = z$
$\Rightarrow 6x-1 = \frac{dz}{dx}$
$\Rightarrow (6x-1)dx = dz$

$= \int \sqrt{z}\, dz$

$= \int z^{1/2}\, dz$

$= \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= \frac{2}{3} z^{3/2} + c \quad = \frac{2}{3}(3x^2+x-1)^{3/2} + c$

Again, $\int \sqrt{3x^2-x+1}\, dx$

$= \int \sqrt{3\left(x^2 - \frac{x}{3} + \frac{1}{3}\right)}\, dx$

$= \sqrt{3} * \int \sqrt{x^2 - 2 \cdot x \cdot \frac{1}{6} + \left(\frac{1}{6}\right)^2 + \frac{1}{3} - \frac{1}{36}}\, dx$

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$$= \sqrt{3}\int \sqrt{\left\{\left(x-\frac{1}{6}\right)^2+\left(\frac{\sqrt{11}}{6}\right)^2\right\}}\, dx$$

$$= \sqrt{3}\ \frac{\left(x-\frac{1}{6}\right)\sqrt{\left(x-\frac{1}{6}\right)^2+\left(\frac{\sqrt{11}}{6}\right)^2}}{2} \pm \frac{\left(\frac{\sqrt{11}}{6}\right)^2}{2}\sin^{-1}\frac{\left(x-\frac{1}{6}\right)}{\frac{\sqrt{11}}{6}} + c$$

$$= \frac{\sqrt{3}}{2}\left(x-\frac{1}{6}\right)\sqrt{\left(x-\frac{1}{6}\right)^2+\left(\frac{\sqrt{11}}{6}\right)^2} \pm \frac{11}{72}\sin^{-1}\frac{6x-1}{\sqrt{11}} + c$$

$$\therefore \int (4x-3)\sqrt{3x^2-x+1}\, dx$$

$$= \frac{2}{3}\times\frac{2}{3}\left(3x^2-x+1\right)^{\frac{3}{2}} - \frac{7}{3}\left\{\frac{\sqrt{3}}{2}\left(x-\frac{1}{6}\right)\sqrt{\left(x-\frac{1}{6}\right)^2+\left(\frac{\sqrt{11}}{6}\right)^2} \pm \frac{11}{72}\sin^{-1}\frac{6x-1}{\sqrt{11}}\right\}$$

$$+ c \quad \underline{\text{Ans}}.$$

21) $\int (3x-2)\sqrt{x^2-x+1}\, dx$

$$= \int\left\{\frac{3}{2}(2x-1)-\frac{1}{2}\right\}\sqrt{x^2-x+1}\, dx$$

$$= \frac{3}{2}\int (2x-1)\sqrt{x^2-x+1}\,dx - \frac{1}{2}\int\sqrt{x^2-x+1}\, dx$$

Here, $\frac{d}{dx}\left(x^2-x+1\right) = 2x-1$

$$(3x-2) = \frac{3}{2}(2x-1) - 2 + \frac{3}{2}$$

$$= \frac{3}{2}(2x-1) - \frac{1}{2}$$

Here,

$$\int (2x-1)\sqrt{x^2-x+1}\,dx$$

Let, $x^2-x+1 = z$

$\Rightarrow 2x-1 = \frac{dz}{dx}$

$\therefore (2x-1)\,dx = dz$

$$= \int \sqrt{z}\,dz$$

$$= \int z^{\frac{1}{2}}\,dz$$

$$= \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$

$$= \frac{2}{3} z^{3/2} + c \quad = \frac{2}{3}(x^2-x+1)^{3/2} + c$$

Again,

$$\int \sqrt{x^2-x+1}\,dx$$

$$= \int \sqrt{x^2 - 2\cdot x\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2 + 1 - \frac{1}{4}}\;dx$$

$$= \int \sqrt{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\;dx$$

$$= \frac{\left(x-\frac{1}{2}\right)\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}{2} \pm \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\sin^{-1}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{\left(x-\frac{1}{2}\right)\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}{2} \pm \frac{3}{8}\sin^{-1}\frac{2x-1}{\sqrt{3}} + c$$

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$$\therefore \int (3x-2)\sqrt{x^2-x+1}\, dx$$

$$= \frac{3}{2} \times \frac{2}{3} (x^2-x+1)^{3/2} - \frac{1}{2} \left\{ \frac{(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}{2} \pm \frac{3}{8} \sin^{-1}\frac{2x-1}{\sqrt{3}} \right\} + C$$

(Ans)

* Method 11 $\left[ \int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\, dx \text{ format} \right]$

(22) $\int \frac{3\cos x + 4\sin x}{4\cos x + 5\sin x}\, dx$

Let, $3\cos x + 4\sin x = p\left\{\frac{d}{dx}(4\cos x + 5\sin x)\right\} + q(4\cos x + 5\sin x)$

$\Rightarrow 3\cos x + 4\sin x = p(-4\sin x + 5\cos x) + q(4\cos x + 5\sin x)$

$\Rightarrow 3\cos x + 4\sin x = -4p\sin x + 5q\sin x + 5p\cos x + 4q\cos x$

$\Rightarrow 3\cos x + 4\sin x = (5-4p)\sin x + (5p+4q)\cos x$

$\therefore 5q - 4p = 4$

$5p + 4q = 3$

$p = \frac{-1}{41}$

$q = \frac{32}{41}$

$$\therefore\ 3\cos x + 4\sin x = -\frac{1}{41}(-4\sin x + 5\cos x) + \frac{32}{41}(4\cos x + 5\sin x)$$

$$\therefore \int \frac{3\cos x + 4\sin x}{4\cos x + 5\sin x}\,dx = \int \frac{-\frac{1}{41}(-4\sin x + 5\cos x) + \frac{32}{41}(4\cos x + 5\sin x)}{4\cos x + 5\sin x}\,dx$$

$$= -\frac{1}{41}\int \frac{-4\sin x + 5\cos x}{4\cos x + 5\sin x}\,dx + \frac{32}{41}\int \frac{4\cos x + 5\sin x}{4\cos x + 5\sin x}\,dx$$

$$= -\frac{1}{41}\ln|4\cos x + 5\sin x| + \frac{32}{41}x + c$$

(Ans)

## * Method 13

$$* \int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}u \int v\,dx\right\}dx$$

$$* \int u\,dv = uv - \int v\,du$$

(23) $\int xe^x\,dx$

Let, $u = x$

$\Rightarrow \frac{du}{dx} = 1$

$\therefore dx = du$

and, $e^x\,dx = dv$

$\Rightarrow \int e^x\,dx = \int dv$

$\Rightarrow e^x = v$

We know that,

$$\int u\,dv = uv - \int v\,du$$

$$\therefore \int xe^x\,dx = xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c \quad \text{(Ans)}$$

(24) $\int x\ln x\,dx$

Let, $\ln x = u$

$\Rightarrow \frac{1}{x} = \frac{du}{dx}$

$\therefore du = \frac{dx}{x}$

and, $x\,dx = dv$

$\Rightarrow \int x\,dx = \int dv$

$\Rightarrow \frac{x^2}{2} = v$

We know that,

$$\int u\,dv = uv - \int v\,du$$

$$\therefore \int \ln x \; x\,dx = \ln x \times \frac{x^2}{2} - \int \frac{x^2}{2} \times \frac{dx}{x}$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\times\frac{x^2}{2} + c$$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c \quad \text{(Ans)}$$

(25) $\int \tan^{-1}x\,dx$

Let, $\tan^{-1}x = u$

$\Rightarrow \frac{1}{1+x^2} = \frac{du}{dx}$

$\therefore du = \frac{dx}{1+x^2}$

and, $dx = dv$

$\Rightarrow \int dx = \int dv$

$\therefore v = x$

We know that,

$\int u\,dv = uv - \int v\,du$

$\therefore \int \tan^{-1}x\,dx = \tan^{-1}x \times x - \int x \times \frac{dx}{1+x^2}$

$= x\tan^{-1}x - \int \frac{x\,dx}{1+x^2}$

Let, $1+x^2 = z$

$\Rightarrow 2x = \frac{dz}{dx}$

$\therefore 2x\,dx = dz$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x\,dx}{1+x^2}$

$= x\tan^{-1}x - \frac{1}{2}\int \frac{dz}{z}$

$= x\tan^{-1}x - \frac{1}{2}\ln|z| + c$

$= x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + c$ (Ans)

(26) $\int \ln x \, dx$

Let, $\ln x = u$ and, $dx = dv$

$\Rightarrow \frac{1}{x} = \frac{du}{dx}$ $\Rightarrow \int dx = \int dv$

$\Rightarrow du = \frac{dx}{x}$ $\therefore x = v$

We know that,

$$\int u\,dv = uv - \int v\,du$$

$$\therefore \int \ln x \, dx = x\ln x - \int x \frac{dx}{x}$$

$$= x\ln x - x + c \text{ (Ans)}$$

(27) $\int x \sin^{-1}x \, dx$

Let, $\sin^{-1}x = u$ and $x\,dx = dv$

$\Rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{du}{dx}$ $\Rightarrow \int x\,dx = \int dv$

$\Rightarrow \frac{dx}{\sqrt{1-x^2}} = du$ $\Rightarrow \frac{x^2}{2} = v$

We know that,

$$\int u\,dv = uv - \int v\,du$$

$$\therefore \int x \sin^{-1}x \, dx = \frac{x^2}{2}\sin^{-1}x - \int \frac{x^2}{2} \times \frac{dx}{\sqrt{1-x^2}}$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}}$$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

Let,

$$x = \sin\theta$$

$$\Rightarrow 1 = \cos\theta \frac{d\theta}{dx}$$

$$\therefore dx = \cos\theta\, d\theta$$

$$= \int \frac{\sin^2\theta \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \int \frac{\sin^2\theta \cos\theta\, d\theta}{\cos\theta}$$

$$= \int \sin^2\theta\, d\theta$$

$$= \int \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\int (1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)$$

$$= \frac{\theta}{2} - \frac{2\sin\theta\cos\theta}{4}$$

$$= \frac{\sin^{-1}x}{2} - \frac{x\sqrt{1-x^2}}{2}$$

$$\therefore \int x\sin^{-1}x\, dx = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\left(\frac{\sin^{-1}x}{2} - \frac{x\sqrt{1-x^2}}{2}\right) + c$$

$$= \frac{1}{2}\left(x^2\sin^{-1}x - \frac{\sin^{-1}x}{2} + \frac{x\sqrt{1-x^2}}{2}\right) + c$$

Ans

$$\left[\; 1 - 2\sin^2\theta = \cos 2\theta;\quad \Rightarrow 2\sin^2\theta = 1 - \cos 2\theta \quad \therefore \sin^2\theta = \frac{1-\cos 2\theta}{2} \;\right]$$

(28) $\int x \cos^{-1} x \, dx$

Let, $\cos^{-1} x = u$

$\Rightarrow \frac{-1}{\sqrt{1-x^2}} = \frac{du}{dx}$

$\therefore du = \frac{-dx}{\sqrt{1-x^2}}$

and, $x \, dx = dv$

$\Rightarrow \int x \, dx = \int dv$

$\Rightarrow \frac{x^2}{2} = v$

We know that,

$\int u \, dv = uv - \int v \, du$

$= \frac{x^2}{2} \cos^{-1} x - \int \frac{x^2}{2} \times \frac{-dx}{\sqrt{1-x^2}}$

$= \frac{x^2}{2} \cos^{-1} x + \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} dx$

$= \frac{x^2}{2} \cos^{-1} x + \frac{1}{2} \left( \frac{\sin^{-1} x}{2} - \frac{x\sqrt{1-x^2}}{2} \right) + c$

(Ans)

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(29) $\int x \tan^{-1}x \, dx$

Let, $u = \tan^{-1}x$ and $x\,dx = dv$

$\Rightarrow \frac{du}{dx} = \frac{1}{1+x^2}$ $\Rightarrow \int x\,dx = \int dv$

$\therefore du = \frac{dx}{1+x^2}$ $\therefore v = \frac{x^2}{2}$

We know that,

$$\int u\,dv = uv - \int v\,du$$

$$\therefore \int x \tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2} \times \frac{dx}{1+x^2}$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left\{\int 1\,dx - \int \frac{1}{1+x^2}\,dx\right\}$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left\{x - \tan^{-1}x\right\} + c$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{\tan^{-1}x}{2} + C$$

$$= \frac{1}{2}\left\{x^2\tan^{-1}x - x + \tan^{-1}x\right\} + c$$

Ans

* Method 14 $\left[\int e^x\{f(x)+f'(x)\}dx = e^x f(x)+c\right]$

(30) $\int e^x(\sin x+\cos x)\,dx$

Here,
$f(x)=\sin x$
$\therefore f'(x)=\cos x$

We know that,

$$\int e^x\{f(x)+f'(x)\} = e^x f(x)+c$$

$$\therefore \int e^x(\sin x+\cos x) = e^x \sin x + c \quad \text{(Ans)}$$

(31) $\int e^x(\sin x-\cos x)\,dx$

Here,
$f(x)=-\cos x$
$\therefore f'(x)=-(-\sin x)$
$=\sin x$

$$= e^x(-\cos x)+c$$

$$= -e^x\cos x + c \quad \text{(Ans)}$$

(32) $\int \frac{xe^x}{(x+1)^2}\,dx$

Here,
$f(x)=\frac{1}{1+x}=(1+x)^{-1}$
$\therefore f'(x)=-(1+x)^{-2}(0+1)$
$=\frac{-1}{(1+x)^2}$

$$= \int \frac{(x+1)-1}{(x+1)^2}e^x\,dx$$

$$= \int\left\{\frac{x+1}{(x+1)^2}-\frac{1}{(x+1)^2}\right\}e^x\,dx$$

$$= \int\left\{\frac{1}{x+1}-\frac{1}{(x+1)^2}\right\}e^x\,dx$$

$$= e^x \cdot \frac{1}{1+x} + c$$

(Ans)

(33) $\int e^x \frac{1-\sin x}{1-\cos x} dx$

$$= \int e^x \frac{1 - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} dx$$

$$= \int \left\{ \frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}} \right\} e^x dx$$

$$= \int \left\{ \frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \right\} e^x dx$$

$$= \int \left\{ \frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2} \right\} e^x dx$$

$$= \int \left\{ \left(-\cot\frac{x}{2}\right) + \frac{1}{2}\operatorname{cosec}^2\frac{x}{2} \right\} e^x dx$$

$$= e^x\left(-\cot\frac{x}{2}\right) + c$$

$$= -e^x \cot\frac{x}{2} + c$$

(Ans)

Here,

$f(x) = -\cot\frac{x}{2}$

$\therefore f'(x) = -\left(-\operatorname{cosec}^2\frac{x}{2} \times \frac{d}{dx}\left(\frac{x}{2}\right)\right)$

$= \operatorname{cosec}^2\frac{x}{2} \times \frac{1}{2}$

$= \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$

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(34) $\int e^x \left(\frac{1}{1+x^2} + \tan^{-1}x\right) dx$

$= e^x \tan^{-1}x + c$ (Ans)

Here,

$f(x) = \tan^{-1}x$

$\therefore f'(x) = \frac{1}{1+x^2}$

# Definite Integration

* $\int_a^b f(x)\,dx = \int_a^b f(z)\,dz$
* $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \; ; \; a<c<b$
* $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ ; if $f(x)$ is a trigonometric $f^n$
* $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$

① Show that $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx = \frac{\pi}{4}$

Solⁿ:

Let, $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx$ ——— ①

$$= \int_0^{\pi/2} \frac{\sin(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\,dx$$

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\,dx \quad \text{——— (ii)}$$

① + (ii) ⇒ $I + I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$

$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\pi/2} 1\,dx$$

$$= [x]_0^{\pi/2}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4} \text{ (showed)}$$

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② Show that $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$

Solⁿ:

Let,

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx \quad \text{---} \; (1)$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)}+\sqrt{\cos(\pi/2 - x)}}\,dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx \quad \text{---} \; (11)$$

(1) + (11) ⇒

$$I + I = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} dx = [x]_0^{\pi/2}$$

$$\Rightarrow 2I = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4}$$ (showed)

(3) Show that $\int_0^{\pi/2} \frac{dx}{1+\cos^2 x} = \frac{\pi}{2\sqrt{2}}$

Sol$^n$: Let, $I = \int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$

$$= \int_0^{\pi/2} \frac{\frac{1}{\cos^2 x}}{\frac{1+\cos^2 x}{\cos^2 x}} dx$$

$\sec^2 x - \tan^2 x = 1$

$\Rightarrow \sec^2 x = 1 + \tan^2 x$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{1+\sec^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{1+1+\tan^2 x} dx$$

$$\therefore I = \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x} dx$$

Let, $z = \tan x$

$\Rightarrow \frac{dz}{dx} = \sec^2 x$

$\Rightarrow \sec^2 x \, dx = dz$

When,

$x = 0, \; z = \tan 0 = 0$

$x = \pi/2, \; z = \tan\frac{\pi}{2} = \alpha$

$$\therefore I = \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x} dx = \int_0^{\alpha} \frac{1}{2+z^2} dz$$

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$$= \int_0^{\alpha} \frac{dz}{z^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \left[ \tan^{-1} \frac{z}{\sqrt{2}} \right]_0^{\alpha}$$

$$= \frac{1}{\sqrt{2}} \left[ \tan^{-1} \frac{\alpha}{\sqrt{2}} - \tan^{-1} \frac{0}{2} \right]$$

$$= \frac{1}{\sqrt{2}} \left[ \frac{\pi}{2} - 0 \right]$$

$$= \frac{\pi}{2\sqrt{2}}$$

(showed)

~~Sh~~ (4) Show that $\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$

Sol$^n$: Let, $I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx$

$$= \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - \cancel{2}x\right) \log \tan \left(\frac{\pi}{2} - x\right) dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x) \log \tan \left(\frac{\pi}{2} - x\right) dx$$

$$\cancel{= \int_0^{\pi/2} \sin 2x \log \tan \left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \sin 2x \log \cot x \, dx$$

$$= \int_0^{\pi/2} \sin 2x \log \frac{1}{\tan x} \, dx$$

$$= \int_0^{\pi/2} \sin 2x \log (\tan x)^{-1} \, dx$$

$$= -\int_0^{\pi/2} \sin 2x \log \tan x \, dx \quad \left[\because \log x^a = a \log x\right]$$

$$= -I$$

$$\Rightarrow I = -I$$

$$\Rightarrow I + I = 0$$

$$\Rightarrow 2I = 0$$

$$\therefore I = 0 \quad \text{(Showed)}$$

5. Show that $\int_0^{\pi} \cos^{15} x \, dx = 0$

Sol$^n$: Let, $I = \int_0^{\pi} \cos^{15} x \, dx$

$$= \int_0^{\pi} \cos^{15} (\pi - x) \, dx$$

$$= -\int_0^{\pi} \cos^{15} x \, dx$$

$\Rightarrow I = -I$

$\Rightarrow I + I = 0$

$\Rightarrow 2I = 0$

$\therefore I = 0$ (showed)

⑥ Show that $\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}} dx = \frac{\pi}{4}$

Sol$^n$:

Let, $I = \int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}} dx$ ——— (1)

$= \int_0^{\pi/2} \frac{\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}^{3/2}}{\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}^{3/2} + \left\{\cos\left(\frac{\pi}{2} - x\right)\right\}^{3/2}} dx$

$\therefore I = \int_0^{\pi/2} \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} dx$ ——— (2)

(1) + (2) $\Rightarrow$ $I + I = \int_0^{\pi/2} \frac{(\sin x)^{3/2} + (\cos x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}} dx$

$\Rightarrow 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$

$\therefore I = \frac{\pi}{4}$ (showed)

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(17) Show that $\int_0^{\pi} \frac{x}{1+\cos^2 x} dx = \frac{\pi^2}{2\sqrt{2}}$ [Previous]

Sol$^n$:

Let, $I = \int_0^{\pi} \frac{x}{1+\cos^2 x} dx$

$$= \int_0^{\pi} \frac{\pi - x}{1+\{\cos(\pi - x)\}^2} dx$$

$$= \int_0^{\pi} \frac{\pi - x}{1+(-\cos x)^2} dx$$

$$= \int_0^{\pi} \frac{\pi}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x}{1+\cos^2 x} dx$$

$$\Rightarrow I = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x} - I$$

$$\Rightarrow 2I = \pi \int_0^{2\cdot\frac{\pi}{2}} \frac{dx}{1+\cos^2 x}$$

$$\Rightarrow 2I = 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$$

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$$\Rightarrow I = \pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$$

$$= \pi \int_0^{\pi/2} \frac{\frac{1}{\cos^2 x}}{\frac{1+\cos^2 x}{\cos^2 x}} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x}{1+\sec^2 x} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x} dx \qquad [\sec^2 x = 1+\tan^2 x]$$

Let,

$z = \tan x$

$\Rightarrow \frac{dz}{dx} = \sec^2 x$

$\Rightarrow \sec^2 x \, dx = dz$

When $x = 0$,

$z = \tan x = \tan 0 = 0$

$x = \pi/2$,

$z = \tan x = \tan \frac{\pi}{2} = \alpha$

$$\therefore I = \pi \int_0^{\alpha} \frac{dz}{2+z^2}$$

$$= \pi \int_0^{\alpha} \frac{1}{z^2+(\sqrt{2})^2} dz$$

$$= \pi \times \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{z}{\sqrt{2}}\right]_0^{\alpha}$$

$$= \frac{\pi}{\sqrt{2}} \left[\tan^{-1} \frac{\alpha}{\sqrt{2}} - \tan^{-1} \frac{0}{\sqrt{2}}\right]$$

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$$\Rightarrow I = \frac{\pi}{\sqrt{2}}\left[\tan^{-1}\tan\frac{\pi}{2} - \tan^{-1}\tan 0\right]$$

$$\Rightarrow I = \frac{\pi}{\sqrt{2}}\left[\frac{\pi}{2} - 0\right]$$

$$\therefore I = \frac{\pi^2}{2\sqrt{2}} \quad \text{(showed)}$$

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# MULTIPLE INTEGRATION

① Evaluate the triple integral, $\int_0^1 \int_0^{1-x} \int_0^{1-y^2} z \, dz \, dy \, dx$

[Prev]

Sol$^n$:

$$\int_0^1 \int_0^{1-x} \int_0^{1-y^2} z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-y^2} dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{(1-y^2)^2}{2}\right] dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \left[\frac{y^4-2y^2+1}{2}\right] dy \, dx$$

$$= \int_0^1 \frac{1}{2} \int_0^{1-x} (y^4-2y^2+1) \, dy \, dx$$

$$= \int_0^1 \frac{1}{2}\left[\frac{y^5}{5} - 2 \cdot \frac{y^3}{3} + y\right]_0^{1-x} dx$$

$$= \int_0^1 \frac{1}{2}\left\{ \frac{(1-x)^5}{5} - \frac{2}{3}(1-x)^3 + 1 - x \right\} dx$$

$$= \int_0^1 \frac{1}{2}\left\{ \frac{3(1-x)^5 - 10(1-x)^3 + 15(1-x)}{15} \right\} dx$$

$$= \frac{1}{30}\int_0^1 \left\{ 3(1-x)^5 - 10(1-x)^3 + 15(1-x) \right\} dx$$

$$\cancel{= \frac{1}{30}\left[ 3\frac{(1-x)^6}{6x} \right]}$$

$$= \frac{1}{30}\int_0^1 \left\{ 3(1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5) - 10(1 - 3x + 3x^2 - x^3) + (15 - 15x) \right\} dx$$

$$= \frac{1}{30}\int_0^1 (3 - 15x + 30x^2 - 30x^3 + 15x^4 - 3x^5 - 10 + 30x - 30x^2 + 10x^3 + 15 - 15x)\, dx$$

$$= \frac{1}{30}\int_0^1 (8 - 20x^3 + 15x^4 - 3x^5)\, dx$$

$$= \frac{1}{30}\left[8x - 20\frac{x^4}{4} + 15\frac{x^5}{5} - 3\frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{30}\left(8 - 5 + 3 - \frac{1}{2}\right)$$

$$= \frac{1}{30} \times \frac{11}{2}$$

$$= \frac{11}{60} \text{ (Ans)}$$

(2) $\cancel{\int_0^{\pi/2}\int_0^2 (1 - 2r\sin\theta) r \, dr \, d\theta}$

$$= \int_0^{\pi/2}\int_0^2 (r - 2r^2\sin\theta) \, dr \, d\theta$$

$= \int_0^{\pi/2} \left[ \frac{r^2}{2} \right]$

Let, $u = r^2$

$\Rightarrow \frac{du}{dr} = 2r$

$\Rightarrow du = 2r \cdot dr$

and $dv = \sin\theta \, \cancel{d\theta} \, dr$

$\Rightarrow \int dv = \int \sin\theta \, dr$

$\Rightarrow v = \frac{-\cos\theta}{\frac{d\theta}{dr}}$

$= \frac{-\cos\theta \, dr}{d\theta}$

We know,

$\int u \, dv = uv - \int v \, du$

$\therefore r^2 \sin\theta \, dr =$

$r^2 \times \frac{-\cos\theta \, dr}{d\theta} -$

$\int \frac{-\cos\theta \, dr}{d\theta} \times 2r \, dr$

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(2) $\int_0^{\pi/2}\int_0^2 (1-2r\sin\theta)\,r\,dr\,d\theta$

$= \int_0^{\pi/2}\int_0^2 (r-2r^2\sin\theta)\,dr\,d\theta$

$= \int_0^{\pi/2}\left[\frac{r^2}{2}-2\sin\theta\frac{r^3}{3}\right]_0^2 d\theta$

$= \int_0^{\pi/2}\left(\frac{2^2}{2}-2\sin\theta\frac{2^3}{3}\right)d\theta$

$= \int_0^{\pi/2}\left(2-\frac{16}{3}\sin\theta\right)d\theta$

$= \left[2\theta-\frac{16}{3}(-\cos\theta)\right]_0^{\pi/2}$

$= 2\left(\frac{\pi}{2}-0\right)+\frac{16}{3}\left(\cos\frac{\pi}{2}-\cos 0^\circ\right)$

$= \pi+\frac{16}{3}(0-1)$

$= \pi-\frac{16}{3}$

$= \frac{3\pi-16}{3}$ (Ans)

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③ $\int_1^2\int_0^1\int_{-1}^1 (x^2+y^2+z^2)\,dx\,dy\,dz$

Sol$^n$: $\int_1^2\int_0^1 \left[\frac{x^3}{3}+xy^2+xz^2\right]_{-1}^{1} dy\,dz$

$$= \int_1^2\int_0^1 \left[\frac{1}{3}\{1^3-(-1)^3\}+y^2(1+1)+z^2(1+1)\right] dy\,dz$$

$$= \int_1^2\int_0^1 \left(\frac{2}{3}+2y^2+2z^2\right) dy\,dz$$

$$= \int_1^2 \left[\frac{2}{3}y+2\frac{y^3}{3}+2z^2y\right]_0^1 dz$$

$$= \int_1^2 \left(\frac{2}{3}+\frac{2}{3}+2z^2\right) dz$$

$$= \left[\frac{4}{3}z+2\frac{z^3}{3}\right]_1^2$$

$$= \frac{4}{3}(2-1)+\frac{2}{3}(2^3-1^3)$$

$$= \frac{4}{3}+\frac{2}{3}(8-1)$$

$$= \frac{4}{3}+\frac{14}{3}$$

$$= \frac{18}{3}$$

$$= 6 \text{ (Ans)}$$

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④ $$\int_1^2\int_3^4 \frac{dx\,dy}{(x+y)^2}$$

$$= \int_1^2\int_3^4 (x+y)^{-2}\,dx\,dy$$

$$= \int_1^2 \left[\frac{(x+y)^{-2+1}}{(-2+1)\times\frac{d}{dx}(x+y)}\right]_3^4 dy$$

$$= \int_1^2 \left[\frac{(x+y)^{-1}}{-1\times(1+0)}\right]_3^4 dy$$

$$= \int_1^2 \left[\frac{1}{-(x+y)}\right]_3^4 dy$$

$$\cancel{= \int_1^2 \left(\frac{1}{-(4-3+y)}\right) dy}$$

$$\cancel{= \int_1^2 \frac{1}{-1-y}\,dy}$$

$$= \int_1^2 \left(-\frac{1}{4+y} - \left(-\frac{1}{3+y}\right)\right) dy$$

$$= \int_1^2 \left(\frac{1}{3+y} - \frac{1}{4+y}\right) dy$$

$$= \int_1^2 \left(\frac{4+y-3-y}{y^2+7y+12}\right) dy$$

$$= \int_1^2 \frac{1}{y^2+7y+12}\, dy$$

$$= \int_1^2 \frac{1}{y^2 + 2 \cdot y \cdot \frac{7}{2} + \left(\frac{7}{2}\right)^2 + 12 - \frac{49}{4}}\, dy$$

$$= \int_1^2 \frac{1}{\left(y+\frac{7}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\, dy$$

$$= \frac{1}{2 \times \frac{1}{2}} \left[ \ln \left| \frac{y + \frac{7}{2} - \frac{1}{2}}{y + \frac{7}{2} + \frac{1}{2}} \right| \right]_1^2$$

$$\left[ \because \int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| \right]$$

$$= \left[ \ln \left| \frac{y+3}{y+4} \right| \right]_1^2$$

$$\boxed{\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right|}$$

$$= \ln \left| \frac{2+3}{2+4} \right| - \ln \left| \frac{1+3}{1+4} \right|$$

$$= \ln \left| \frac{5}{6} \right| - \ln \left| \frac{4}{5} \right|$$

$$= \ln \left| \frac{5}{6} \times \frac{5}{4} \right| = \ln \left| \frac{25}{24} \right| \text{ (Ans)}$$

# Geometrical Interpretation

(I) Evaluate $\int_0^1 x\,dx$ geometrically.

Soln:

Here, $f(x) = x$ and $a = 0,\ b = 1$

$\therefore f(a+rh) = a+rh$

$\Rightarrow f(0+rh) = 0+rh \quad [\because a = 0]$

$\therefore f(rh) = rh$

We have, $\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h f(a+rh)$

$$\therefore \int_0^1 x\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h f(0+rh)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h \times f(rh)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h \times rh$$

$$= \lim_{h\to 0} \sum_{r=1}^{n} h \times rh$$

$$= \lim_{h \to 0} h(h + 2h + 3h + \cdots + nh)$$

$$= \lim_{h \to 0} h^2 \times \frac{n(n+1)}{2} \quad \left[\because 1+2+3+\cdots+n = \frac{n(n+1)}{2}\right]$$

$$\cancel{= \lim_{h \to 0} \frac{h(nh+h)}{2}}$$

$$= \lim_{h \to 0} \frac{n^2h^2 + nh^2}{2}$$

$$= \lim_{h \to 0} \frac{nh(nh+h)}{2}$$

$$= \lim_{h \to 0} \frac{1(1+h)}{2} \quad [\because nh = b - a = 1 - 0 = 1]$$

$$= \lim_{h \to 0} \frac{1+h}{2}$$

$$= \frac{1+0}{2}$$

$$\therefore \int_0^1 x\,dx = \frac{1}{2}$$

(Ans)

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(2) Evaluate $\int_0^1 x^3\,dx$

Here, $a=0$, $b=1$ and

$f(x) = x^3$

$\therefore f(a+rh) = (a+rh)^3$

$= (0+rh)^3$

$= r^3h^3$

We have, $\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h f(a+rh)$

$\therefore \int_0^1 x^3\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h \times r^3h^3$

$= \lim_{h\to 0} h^4 \sum_{r=1}^{n} r^3$

$= \lim_{h\to 0} h^4 (1^3+2^3+3^3+\cdots+n^3)$

$= \lim_{h\to 0} h^4 \cdot \frac{n^2(n+1)^2}{4}$

$= \lim_{h\to 0} \frac{nh^2(n+1) \times nh^2(n+1)}{4}$

$= \lim_{h\to 0} \frac{nh(nh+h) \times nh(nh+h)}{4}$

$$= \frac{1(1+0)\times 1(1+0)}{4} \quad [\because nh = b-a = 1-0 = 1]$$

$$= \frac{1}{4} \text{ (Ans)}$$

③ Evaluate $\int_2^3 x^3\,dx$ geometrically. [Prov]

Here, $a = 2$, $b = 3$

$\therefore nh = b - a$
$= 3 - 2$
$= 1$

$f(x) = x^3$

$\therefore f(a+rh) = (a+rh)^3$

$\therefore f(2+rh) = (2+rh)^3$

$= 2^3 + 3\times 2^2 \times rh + 3\times 2\times r^2h^2 + r^3h^3$

$= 8 + 12rh + 6r^2h^2 + r^3h^3$

We have, $\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\, f(a+rh)$

$\therefore \int_2^3 x^3\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h(8 + 12rh + 6r^2h^2 + r^3h^3)$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} (8h + 12rh^2 + 6r^2h^3 + r^3h^4)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} 8h + \lim_{h\to 0} \sum_{r=0}^{n-1} 12rh^2 + \lim_{h\to 0} \sum_{r=0}^{n-1} 6r^2h^3 + \lim_{h\to 0} \sum_{r=0}^{n-1} r^3h^4$$

$$= \lim_{h\to 0} \sum_{r=1}^{n} 8h + \lim_{h\to 0} 12h^2 \sum_{r=1}^{n} r + \lim_{h\to 0} 6h^3 \sum_{r=1}^{n} r^2 + \lim_{h\to 0} h^4 \sum_{r=1}^{n} r^3$$

$$= \lim_{h\to 0} 8h(1+1+1+1\cdots+1) + \lim_{h\to 0} 12h^2(1+2+3+\cdots+n) + \lim_{h\to 0} 6h^3(1^2+2^2+3^2+\cdots+n^2) + \lim_{h\to 0} h^4(1^3+2^3+3^3+\cdots+n^3)$$

$$= \lim_{h\to 0} 8h(n\times 1) + \lim_{h\to 0} 12h^2 \frac{n(n+1)}{2} + \lim_{h\to 0} 6h^3 \frac{n(n+1)(2n+1)}{6} + \lim_{h\to 0} h^4 \left\{\frac{n(n+1)}{2}\right\}^2$$

$$= \lim_{h\to 0} 8nh + \lim_{h\to 0} \frac{12nh(nh+h)}{2} + \lim_{h\to 0} \frac{6nh(nh+h)(2nh+h)}{6}$$

$$+ \lim_{h\to 0} \left\{ \frac{h^2 n(n+1)}{2} \right\}^2$$

$$= \lim_{h\to 0} (8\times 1) + \lim_{h\to 0} \frac{12\times 1\times(1+h)}{2} + \lim_{h\to 0} 1\times(1+h)(2\times 1+h)$$

$$+ \lim_{h\to 0} \left\{ \frac{nh(nh+h)}{2} \right\}^2$$

$$= 8 + \lim_{h\to 0} 6(1+h) + \lim_{h\to 0} (1+h)(2+h) +$$

$$\lim_{h\to 0} \frac{1^2(1+h)^2}{4}$$

$$= 8 + 6(1+0) + (1+0)(2+0) + \frac{1}{4}$$

$$= 8+6+2+\frac{1}{4}$$

$$= \frac{65}{4}$$

(Ans)

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(4) Evaluate $\int_0^{\pi/2} \sin x \, dx$ geometrically

Here, $a = 0$, $b = \pi/2$ ; $\therefore nh = b - a = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$f(x) = \sin x$

$\therefore f(a + rh) = \sin(a + rh)$

$\therefore f(0 + rh) = \sin(0 + rh)$

$= \sin rh$

We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} \sum_{r=0}^{n-1} h f(a + rh)$$

$$= \lim_{h \to 0} \sum_{r=1}^{n} h \sin rh$$

$$= \lim_{h \to 0} h(\sin h + \sin 2h + \sin 3h + \dots + \sin nh) \quad \text{---(1)}$$

Let, $S = \sin h + \sin 2h + \sin 3h + \dots + \sin nh$

$\Rightarrow S \times 2\sin\frac{h}{2} = 2\sin h \sin\frac{h}{2} + 2\sin 2h \sin\frac{h}{2} + 2\sin 3h \sin\frac{h}{2} + \dots + 2\sin nh \sin\frac{h}{2}$

[Multiplying both sides by $2\sin\frac{h}{2}$]

$$= \left\{\cos\left(h-\frac{h}{2}\right) - \cos\left(h+\frac{h}{2}\right)\right\} + \left\{\cos\left(2h-\frac{h}{2}\right) - \cos\left(2h+\frac{h}{2}\right)\right\} + \left\{\cos\left(3h-\frac{h}{2}\right) - \cos\left(3h+\frac{h}{2}\right)\right\} + \cdots + \left\{\cos\left(nh-\frac{h}{2}\right) - \cos\left(nh+\frac{h}{2}\right)\right\}$$

$$\left[\because \cos(A-B) - \cos(A+B) = 2\sin A \sin B\right]$$

$$= \cos\frac{h}{2} - \cos\frac{3h}{2} + \cos\frac{3h}{2} - \cos\frac{5h}{2} + \cos\frac{5h}{2} - \cos\frac{7h}{2} + \cdots + \cos\left(nh-\frac{h}{2}\right) - \cos\left(nh+\frac{h}{2}\right)$$

$$= \cos\frac{h}{2} - \cos\left(nh+\frac{h}{2}\right)$$

$$\therefore S = \frac{\cos\frac{h}{2} - \cos\left(nh+\frac{h}{2}\right)}{2\sin\frac{h}{2}}$$

From (1),

$$\int_0^{\pi/2} \sin x \, dx = \lim_{h\to 0} h \, \frac{\cos\frac{h}{2} - \cos\left(nh+\frac{h}{2}\right)}{2\sin\frac{h}{2}}$$

$$= \lim_{h\to 0} \frac{h/2}{\sin h/2} \times \lim_{h\to 0} \left(\cos h/2 - \cos\left(nh + h/2\right)\right)$$

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$$= 1 \times \cos\left(\frac{0}{2}\right) - \cos\left(\frac{\pi}{2} + \frac{0}{2}\right)$$

$$\left[\lim_{x \to 0} \frac{x}{\sin x} = 1 \; ; \; nh = \frac{\pi}{2}\right]$$

$$= \cos 0^\circ - \cos \frac{\pi}{2}$$

$$= 1 - 0$$

$$\therefore \int_0^{\pi/2} \sin x \, dx = 1 \quad \text{(Ans)}$$

# GAMMA($\Gamma$) & BETA($\beta$) FUNCTION

## Define Gamma & Beta function

→ Gamma $f^n$: The integral $\int_0^\alpha x^{n-1} e^{-x} dx$ is known as 2nd Eulerian integral which is called Gamma function.

→ Beta $f^n$: The integral $\int_0^1 x^{m-1} (1-x)^{n-1} dx$ is known as 1st Eulerian integral which is called Beta function.

i.e. $$\Gamma n = \int_0^\alpha x^{n-1} e^{-x} dx$$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

## Formulae

① $\Gamma n = (n-1)!$

$\Rightarrow \Gamma 6 = (6-1)! = 5! = 120$

② $\lfloor n = n(n-1)(n-2)(n-3) \ldots 4.3.2.1$

③ $\Gamma n+1 = n \Gamma n$

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④ $\Gamma\left(\frac{n}{2}\right) = \frac{n-2}{2} \cdot \frac{n-4}{2} \cdot \frac{n-6}{2} \cdots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right),$

When $n$ is odd.

⑤ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

⑥ $\beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$

⑦ $\beta(m,n) = \dfrac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

⑧ $\Gamma 1 = 1$

⑨ $\beta(m,n) = \displaystyle\int_0^{\pi/2} x^{m-1}(1-x)^{n-1}\,dx = \dfrac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

⑩ $\beta(p,q) = \displaystyle\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

1. Prove that $\Gamma 1 = 1$.

We know that the gamma $f^n$,

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} \, dx \quad \text{--- (1)}$$

Putting $n = 1$ in $eq^n$ (1),

$$\Gamma 1 = \int_0^{\infty} x^{1-1} e^{-x} \, dx$$

$$= \int_0^{\infty} x^{0} e^{-x} \, dx$$

$$= \int_0^{\infty} e^{-x} \, dx$$

$$= -\left[ e^{-x} \right]_0^{\infty}$$

$$= -\left( e^{-\infty} - e^{-0} \right)$$

$$= -\left( \frac{1}{e^{\infty}} - \frac{1}{e^{0}} \right)$$

$$= -\left( \frac{1}{\infty} - \frac{1}{1} \right) \quad [\because e^{\infty} = \infty, \; e^{0} = 1]$$

$$= -(0 - 1)$$

$\therefore \Gamma 1 = 1$ (proved)

② Prove that $\Gamma(n+1) = n\Gamma(n)$

We know that gamma $f^n$

$$\Gamma n = \int_0^\alpha x^{n-1} e^{-x} dx \quad \text{—— (1)}$$

Putting $n = n+1$ in $eq^n$ (1),

$$\Gamma(n+1) = \int_0^\alpha x^{n+1-1} e^{-x} dx$$

$$\Rightarrow \Gamma(n+1) = \int_0^\alpha x^n e^{-x} dx \quad \text{—— (2)}$$

Now we'll find $\int x^n e^{-x} dx$

Let, $u = x^n$

$$\therefore \frac{du}{dx} = nx^{n-1}$$

$$\Rightarrow du = nx^{n-1} dx$$

and $dv = e^{-x} dx$

$$\Rightarrow \int dv = \int e^{-x} dx$$

$$\Rightarrow v = -e^{-x}$$

We know that,

$$\int u dv = uv - \int v du$$

$$\therefore \int x^n e^{-x} dx = -x^n e^{-x} - \int -e^{-x} \cdot nx^{n-1} dx$$

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$$= -x^n e^{-x} + n\int x^{n-1} e^{-x}\,dx$$

From eq$^n$ (2),

$$\Gamma(n+1) = \int_0^{\alpha} x^n e^{-x}\,dx$$

$$= \left[-x^n e^{-x}\right]_0^{\alpha} + n\int_0^{\alpha} x^{n-1} e^{-x}\,dx$$

$$= \left[-\alpha^n e^{-\alpha} + 0^n e^{-0}\right] + n\Gamma n \quad \left[\because \Gamma n = \int_0^{\alpha} x^{n-1} e^{-x}\,dx\right]$$

$$= \left(-\alpha^n \frac{1}{e^{\alpha}} + 0^n \frac{1}{e^0}\right) + n\Gamma n$$

$$= \left(-\alpha^n \frac{1}{\alpha} + 0^n \frac{1}{1}\right) + n\Gamma n \quad \left[\because e^{\alpha} = \alpha,\ e^0 = 1\right]$$

$$= (0 + 0) + n\Gamma n$$

$$= n\Gamma n$$

$$\therefore \Gamma(n+1) = n\Gamma n$$ (Proved)

③ Prove that $\Gamma n = (n-1)!$

We know that,

$$\Gamma(n+1) = n\,\Gamma n \quad \text{——— (1)}$$

Putting $n = n-1$ in $eq^n$ (1),

$$\Gamma(n-1+1) = (n-1)\,\Gamma(n-1)$$

$$\Rightarrow \Gamma n = (n-1)\,\Gamma(n-1) \quad \text{——— (2)}$$

Putting $n = n-2$ in $eq^n$ (1),

$$\Gamma(n-2+1) = (n-2)\,\Gamma(n-2)$$

$$\Rightarrow \Gamma(n-1) = (n-2)\,\Gamma(n-2) \quad \text{——— (3)}$$

Putting $n = n-3$ in $eq^n$ (1),

$$\Gamma(n-3+1) = (n-3)\,\Gamma(n-3)$$

$$\Rightarrow \Gamma(n-2) = (n-3)\,\Gamma(n-3) \quad \text{——— (4)}$$

Putting $n = 1$ in $eq^n$ (1),

$$\Gamma(1+1) = 1\,\Gamma 1$$

$$\Rightarrow \Gamma 2 = 1\,\Gamma 1 \quad \text{——— (5)}$$

From $eq^n$ (1),

$$\overline{|n+1} = n\overline{|n}$$

$$= n(n-1)\overline{|n-1} \quad [\text{from } eq^n (2)]$$

$$= n(n-1)(n-2)\overline{|n-2} \quad [\text{from } eq^n (3)]$$

$$= n(n-1)(n-2)(n-3)\overline{|n-3} \quad [\text{from } eq^n (4)]$$

$$= n(n-1)(n-2)(n-3) \ldots\ldots 2\overline{|2}$$

$$= n(n-1)(n-2)(n-3) \ldots\ldots 2 \cdot 1\overline{|1} \quad [\text{from } eq^n (5)]$$

$$= n(n-1)(n-2)(n-3) \ldots\ldots 2 \cdot 1 \cdot 1 \quad [\overline{|1} = 1]$$

$$= n(n-1)(n-2)(n-3) \ldots 2 \cdot 1$$

$$\therefore \overline{|n+1} = n! \text{ ——— } (6)$$

Putting $n = n-1$ in $eq^n$ (6),

$$\overline{|n-1+1} = (n-1)!$$

$$\therefore \overline{|n} = (n-1)!$$

(proved)

④ Evaluate $\int_0^\alpha x^7 e^{-x} dx$

We can write the function

$$\int_0^\alpha x^7 e^{-x} dx = \int_0^\alpha x^{8-1} e^{-x} dx$$

We know that,

$$\Gamma n = \int_0^\alpha x^{n-1} e^{-x} dx$$

$$\therefore \int_0^\alpha x^{8-1} e^{-x} dx = \Gamma 8 = (8-1)! = 7! = 5040$$

Ans

⑤ Evaluate $\int_0^\alpha x^3 e^{-4x} dx$

Let, $y = 4x$

$$\Rightarrow \frac{dy}{dx} = 4$$

$$\therefore dx = \frac{dy}{4} \quad \text{and} \quad x = \frac{y}{4}$$

When, $x = 0$, $y = 4 \times 0 = 0$

$x = \alpha$, $y = 4 \times \alpha = \alpha$

$$\therefore \int_0^\alpha x^3 e^{-4x} dx = \int_0^\alpha \left(\frac{y}{4}\right)^3 e^{-y} \frac{dy}{4} = \int_0^\alpha \frac{y^3}{64} \cdot e^{-y} \cdot \frac{dy}{4}$$

$$= \frac{1}{256} \int_0^{\alpha} y^{4-1} e^{-y} \, dy$$

$$= \frac{1}{256} \times \sqrt{4}$$

$$= \frac{1}{256} \times 3!$$

$$= \frac{6}{256}$$

$$= \frac{3}{128} \quad \text{(Ans)}$$

(6) Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

We know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \quad \text{---} \quad (1)$$

Let, $x = \sin^2\theta$

$$\Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\therefore dx = 2\sin\theta\cos\theta \, d\theta$$

$\therefore \sin\theta = \sqrt{x}$

$\Rightarrow \theta = \sin^{-1}\sqrt{x}$

When $x = 0$,

$\theta = \sin^{-1}\sqrt{0} = 0$

When $x = 1$,

$\theta = \sin^{-1}\sqrt{1} = \frac{\pi}{2}$

From eqⁿ (1),

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \cdot \sin\theta\,(\cos^2\theta)^{n-1} \cdot \cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

(proved)

⑦ Evaluate using Beta $f^n$ : $\int_0^{\pi/2} \sin^5\theta\cos^4\theta\, d\theta$

Let, $I = \int_0^{\pi/2} \sin^5\theta\cos^4\theta\, d\theta$

$$= \beta(5,4)$$

$$= \frac{\Gamma\frac{5+1}{2}\,\Gamma\frac{4+1}{2}}{2\,\Gamma\frac{5+4+2}{2}}$$

$$= \frac{\Gamma 3\,\Gamma\frac{5}{2}}{2\,\Gamma\frac{11}{2}}$$

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$$= \frac{2! \times \left(\frac{5-2}{2} \cdot \frac{5-4}{2} \cdot \Gamma\frac{1}{2}\right)}{2 \times \left(\frac{11-2}{2} \cdot \frac{11-4}{2} \cdot \frac{11-6}{2} \cdot \frac{11-8}{2} \cdot \frac{11-10}{2} \cdot \Gamma\frac{1}{2}\right)}$$

$$= \frac{\frac{3\sqrt{\pi}}{4}}{\frac{945\sqrt{\pi}}{32}}$$

$$= \frac{3\sqrt{\pi}}{4} \times \frac{32}{945\sqrt{\pi}}$$

$$= \frac{8}{315}$$ (Ans)

(8) Evaluate $\int_0^{\pi/2} \cos^8 x \sin^6 x \, dx$ using Beta fn.

Let, $I = \int_0^{\pi/2} \cos^8 x \sin^6 x \, dx$

$= \beta(8,6)$

$$= \frac{\Gamma\frac{8+1}{2} \, \Gamma\frac{6+1}{2}}{2\Gamma\frac{8+6+2}{2}}$$

$$= \frac{\Gamma\frac{9}{2} \, \Gamma\frac{7}{2}}{2\Gamma 8}$$

$$= \frac{\left(\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\right)\left(\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}\right)}{2\times 7!}$$

$$= \frac{\frac{105\sqrt{\pi}}{16}\times\frac{15\sqrt{\pi}}{8}}{10080} \qquad \left[\because \Gamma\frac{1}{2} = \sqrt{\pi}\right]$$

$$= \frac{1575\pi}{128}\times\frac{1}{10080}$$

$$= \frac{5\pi}{4096}$$

(Ans)

9) Show that $\int_0^1 \frac{x^5}{\sqrt{1-x^2}}\,dx = \frac{8}{15}$

Let, $x = \sin\theta$

$\Rightarrow \frac{dx}{d\theta} = \cos\theta$

$\therefore dx = \cos\theta\, d\theta$

~~Whe~~ $\therefore \theta = \sin^{-1}x$

When $x = 0$, $\theta = \sin^{-1}0 = 0$

$x = 1$, $\theta = \sin^{-1}1 = \frac{\pi}{2}$

Let, $I = \int_0^1 \frac{x^5}{\sqrt{1-x^2}}\,dx$

$$= \int_0^{\pi/2} \frac{(\sin\theta)^5}{\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta$$

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$$= \int_0^{\pi/2} \frac{\sin^5\theta}{\sqrt{\cos^2\theta}} \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^5\theta}{\cos\theta} \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^5\theta \, d\theta$$

$$= \int_0^{\pi/2} \cos^0\theta \, \sin^5\theta \, d\theta$$

$$= \beta(0,5)$$

$$= \frac{\Gamma\frac{0+1}{2} \, \Gamma\frac{5+1}{2}}{2\,\Gamma\frac{0+5+2}{2}}$$

$$= \frac{\Gamma\frac{1}{2} \, \Gamma 3}{2\,\Gamma\frac{7}{2}}$$

$$= \frac{\sqrt{\pi} \times \cancel{2}!}{\cancel{2} \times \left(\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{\frac{15\sqrt{\pi}}{8}} = \frac{8}{15} \quad \text{(Showed)}$$

(10) Evaluate $\int_0^1 x^4 \sqrt{1-x^2}\, dx$

Let, $x = \sin\theta$ $\therefore \theta = \sin^{-1} x$

$\Rightarrow \frac{dx}{d\theta} = \cos\theta$

When $x = 0$, $\theta = \sin^{-1} 0 = 0$

$\therefore dx = \cos\theta \, d\theta$

$x = 1$, $\theta = \sin^{-1} 1 = \frac{\pi}{2}$

Let, $I = \int_0^1 x^4 \sqrt{1-x^2}\, dx$

$= \int_0^{\pi/2} \sin^4\theta \sqrt{1-\sin^2\theta}\, \cos\theta\, d\theta$

$= \int_0^{\pi/2} \sin^4\theta \sqrt{\cos^2\theta}\, \cos\theta\, d\theta$

$= \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta$

$= \beta(4,2)$

$= \dfrac{\Gamma\frac{4+1}{2}\,\Gamma\frac{2+1}{2}}{2\,\Gamma\frac{4+2+2}{2}}$

$= \dfrac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma 4}$

$$= \frac{\left(\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\right)\left(\frac{1}{2}\Gamma\frac{1}{2}\right)}{2\times 3!}$$

$$= \frac{\frac{3\sqrt{\pi}}{4}\times\frac{\sqrt{\pi}}{2}}{12}$$

$$= \frac{3\pi}{8}\times\frac{1}{12}$$

$$= \frac{\pi}{32} \text{ (Ans)}$$

(11) Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$.

we know that,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{———} (1)$$

Putting $m = n = \frac{1}{2}$ in $eq^n$ (1),

$$\beta\left(\tfrac{1}{2}\cdot\tfrac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$= \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx$$

$$\therefore \beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}} \quad \text{———} (2)$$

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Let, $x = \sin^2\theta$

$\Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta$

$\therefore dx = 2\sin\theta\cos\theta\, d\theta$

$\therefore \sin\theta = \sqrt{x}$

$\Rightarrow \theta = \sin^{-1}\sqrt{x}$

When $x = 0$,

$\theta = \sin^{-1}\sqrt{0} = 0$

When $x = 1$,

$\theta = \sin^{-1}\sqrt{1} = \pi/2$

From eq$^n$ (2),

$$\beta\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}}$$

$$\Rightarrow \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\, d\theta}{(\sin^2\theta)^{1/2}(1-\sin^2\theta)^{1/2}}$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{\Gamma 1} = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\, d\theta}{\sin\theta \times (\cos^2\theta)^{1/2}}$$

$$\Rightarrow \frac{\left(\Gamma\frac{1}{2}\right)^2}{1} = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta\, d\theta}{\sin\theta\cos\theta} = \int_0^{\pi/2} 2\, d\theta$$

$$\Rightarrow \left(\Gamma\tfrac{1}{2}\right)^2 = 2\left[\theta\right]_0^{\pi/2} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

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$\therefore \Gamma\frac{1}{2} = \sqrt{\pi}$ (proved)

(12) Prove that $\int_0^{2\pi} \cos^4 x \, dx = \frac{3\pi}{4}$

Let, $I = \int_0^{2\pi} \cos^4 x \, dx$

$= \int_0^{4\times\frac{\pi}{2}} \cos^4 x \, dx$

$= 4\int_0^{\pi/2} \cos^4 x \, dx$

$= 4 \times \frac{1}{2} \times \sqrt{\pi} \times \frac{\Gamma\frac{4+1}{2}}{\Gamma\frac{4+2}{2}}$

$= 2\sqrt{\pi} \, \frac{\Gamma\frac{5}{2}}{\Gamma 3}$

$= 2\sqrt{\pi} \, \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}}{2!}$

$= \sqrt{\pi} \times \frac{3\sqrt{\pi}}{4}$

$= \frac{3\pi}{4}$ (proved)

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(13) Show that $\int_0^{\pi} x\cos^4 x\,dx = \frac{3\pi^2}{16}$

Let, $I = \int_0^{\pi} x\cos^4 x\,dx$

$= \int_0^{\pi} (\pi - x)\cos^4(\pi - x)\,dx$

$= \int_0^{\pi} (\pi - x)\{-\cos x\}^4\,dx$

$= \int_0^{\pi} (\pi - x)\cos^4 x\,dx$

$= \int_0^{\pi} \pi\cos^4 x\,dx - \int_0^{\pi} x\cos^4 x\,dx$

$\Rightarrow I = \pi\int_0^{2\frac{\pi}{2}} \cos^4 x\,dx - I$

$\Rightarrow I + I = 2\times\pi\int_0^{\pi/2} \cos^4 x\,dx$

$\Rightarrow 2I = 2\pi\int_0^{\pi/2} \cos^4 x\,dx$

$\Rightarrow I = \pi\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma\frac{4+1}{2}}{\Gamma\frac{4+2}{2}}$

$= \frac{\pi\sqrt{\pi}}{2}\times\frac{\Gamma\frac{5}{2}}{\Gamma 3}$

$$= \frac{\pi\sqrt{\pi}}{2} \times \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\frac{1}{2}}}{2!}$$

$$= \frac{\pi\sqrt{\pi} \times \frac{3\sqrt{\pi}}{4}}{4}$$

$$= \frac{3\pi^2}{16}$$ (showed)

# REDUCTION

(1) Prove that $I_n = \frac{n-1}{n} I_{n-2}$ where $I_n = \int_0^{\pi/2} \sin^n x \, dx$

Sol$^n$:

First, we'll find $\int \sin^n x \, dx$

Let, $I = \int \sin^n x \, dx$

$$= \int \sin^{n-1+1} x \, dx$$

$$\Rightarrow I = \int \sin^{n-1} x \cdot \sin x \, dx$$

Let, $u = \sin^{n-1} x$ and $dv = \sin x \, dx$

$$\Rightarrow \frac{du}{dx} = (n-1)\sin^{n-2} x \cdot \cos x \qquad \Rightarrow \int dv = \int \sin x \, dx$$

$$\Rightarrow v = -\cos x$$

$$\therefore du = (n-1)\sin^{n-2} x \cdot \cos x \cdot dx$$

We know that,

$$\int u \, dv = uv - \int v \, du$$

$$\therefore \int \sin^{n-1} x \cdot \sin x \, dx = -\sin^{n-1} x \cdot \cos x - \int (-\cos x)\{(n-1) \cdot \sin^{n-2} x \cdot \cos x \, dx\}$$

$$\Rightarrow I = -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot \cos^2 x \, dx$$

$$= -\sin^{n-1} x \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot (1 - \sin^2 x) \, dx$$

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$$\Rightarrow -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^{n-2+2}x\,dx$$

$$= -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^{n}x\,dx$$

$$\Rightarrow I = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx - (n-1)I$$

$$\left[\because I = \int \sin^{n}x\,dx\right]$$

$$\Rightarrow I + (n-1)I = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$\Rightarrow (1+n-1)I = -\sin^{n-1}x\cos x + (n-1)\int \sin^{n-2}x\,dx$$

$$\therefore I = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx + C \quad \text{---- (1)}$$

Given that,

$$I_n = \int_0^{\pi/2} \sin^{n}x\,dx$$

$$= \left[-\frac{1}{n}\sin^{n-1}x\cos x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

$$[\text{from } (1)]$$

$$= -\frac{1}{n}\sin^{n-1}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + \frac{1}{n}\sin^{n-1}(0)\cos(0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

$$= \frac{-1}{n}(1)^{n-1}\times 0 + \frac{1}{n}\times(0)^{n-1}\times 1 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

$$\Rightarrow I_n = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x \, dx$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \quad \text{(proved)}$$

* Special Case

$$I_n = \frac{n-1}{n} I_{n-2} \text{ ——— (1)}$$

Putting $n = n-2$ in eq$^n$ (1),

$$I_{n-2} = \frac{n-2-1}{n-2} I_{n-2-2}$$

$$= \frac{n-3}{n-2} I_{n-4} \text{ ——— (2)}$$

Putting $n = n-4$ in eq$^n$ (1),

$$I_{n-4} = \frac{n-4-1}{n-4} I_{n-4-2}$$

$$= \frac{n-5}{n-4} I_{n-6} \text{ ——— (3)}$$

Putting $n = n-6$ in eq$^n$ (1),

$$I_{n-6} = \frac{n-6-1}{n-6} I_{n-6-2}$$

$$= \frac{n-7}{n-6} I_{n-8} \text{ ——— (4)}$$

. . . . . . . .

. . . . .

From eq$^n$ (1)

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot I_{n-4} \quad \text{[from eq}^n\text{ (2)]}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot I_{n-6} \quad \text{[from eq}^n\text{ (3)]}$$

$$\Rightarrow I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdot I_{n-8} \quad \text{[from eq}^n\text{ (4)]}$$

When $n$ is odd, say $n = 9$,

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \frac{n-7}{n-6} \cdot \cdots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

(showed)

② Prove that $I_n = \frac{n-1}{n} I_{n-2}$ where $I_n = \int_0^{\pi/2} \cos^n x \, dx$

Let, $I = \int \cos^n x \, dx$ ——— (1)

$= \int \cos^{n-1+1} x \, dx$

$\Rightarrow I = \int \cos^{n-1} x \cdot \cos x \, dx$

Let,

$u = \cos^{n-1} x$

$\Rightarrow \frac{du}{dx} = (n-1)\cos^{n-2} x (-\sin x)$

$\therefore du = -(n-1)\cos^{n-2} x \sin x \, dx$

and, $dv = \cos x \, dx$

$\Rightarrow \int dv = \int \cos x \, dx$

$\Rightarrow v = \sin x$

We know that,

$\int u \, dv = uv - \int v \, du$

$\therefore I = \int \cos^{n-1} x \cdot \cos x \, dx = \cos^{n-1} x \cdot \sin x - \int \sin x \cdot -(n-1) \cdot \cos^{n-2} x \sin x \, dx$

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \cdot \sin^2 x \, dx$

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x (1 - \cos^2 x) \, dx$

$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^{n-2+2} x \, dx$

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$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1)\int \cos^n x\,dx$$

$$\Rightarrow I = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx - (n-1) I$$

[from eq$^n$ (1)]

$$\Rightarrow I + (n-1) I = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$\Rightarrow (1+n-1) I = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$\therefore I = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx + C \quad \text{--- (2)}$$

Given,

$$I_n = \int_0^{\pi/2} \cos^n x\,dx$$

$$= \left[\frac{1}{n}\cos^{n-1}x \sin x\right]_0^{\pi/2} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

[from (2)]

$$= \frac{1}{n}\cos^{n-1}\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - \frac{1}{n}\cos^{n-1}(0)\sin(0) + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

$$\Rightarrow I_n = 0 - 0 + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

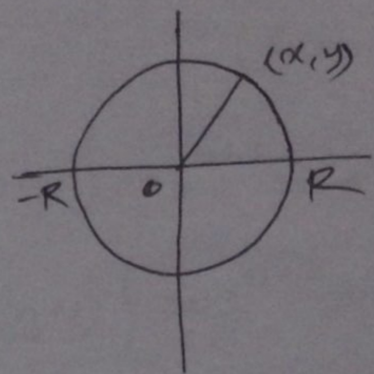
$$\therefore I_n = \frac{n-1}{n} I_{n-2}$$

(proved)

# ARC LENGTH OF CURVE

① Find the circumference of a circle $x^2+y^2=R^2$ of radius $R$.

Sol$^n$: Given,




$$x^2+y^2=R^2$$

$$\Rightarrow y^2 = R^2-x^2$$

$$\therefore y = \pm\sqrt{R^2-x^2}$$

i.e. as a graph, the upper semi circle is $y=+\sqrt{R^2-x^2}$ and lower semi circle $y=-\sqrt{R^2-x^2}$

we will find the length along the upper semi circle of the circle, which is half the circumference of the circle.

We have,

$$s=\int ds = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{—— (1)}$$

Given, $y=\sqrt{R^2-x^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(R^2-x^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}(R^2-x^2)^{\frac{1}{2}-1}\frac{d}{dx}(R^2-x^2)$$

$$= \frac{1}{2}(R^2-x^2)^{-1/2}(-2x)$$

$$= \frac{1}{2(R^2-x^2)^{1/2}} \times (-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{(R^2-x^2)^{1/2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \left\{\frac{-x}{(R^2-x^2)^{1/2}}\right\}^2$$

$$= \frac{x^2}{R^2-x^2}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{R^2-x^2} = \frac{R^2-x^2+x^2}{R^2-x^2}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = \frac{R^2}{R^2-x^2} \quad \text{——— (2)}$$

From (1),

$$S = \int ds = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$\Rightarrow S = \int_{-R}^{R} \sqrt{\frac{R^2}{R^2-x^2}}\, dx \quad [\text{from } (2)]$$

$$= \int_{-R}^{R} \frac{R}{\sqrt{R^2-x^2}}\, dx$$

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$$= R\int_{-R}^{R} \frac{1}{\sqrt{R^2-x^2}}\,dx$$

$$= R \times \sin^{-1}\left[\frac{x}{R}\right]_{-R}^{R} \qquad \left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}\right]$$

$$= R\left\{\sin^{-1}\left[\frac{R}{R}\right] - \sin^{-1}\left[\frac{-R}{R}\right]\right\}$$

$$= R\left\{\sin^{-1}1 - \sin^{-1}(-1)\right\}$$

$$= R\left\{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right\}$$

$$\Rightarrow s = \pi R$$

Hence, the length of half circle is $\pi R$. So the length of the full circle is $2\pi R$. i.e, the circumference of the circle is $2\pi R$ (Ans)

(2) Find the area of revolution obtained by revolving the graph of $y = f(x) = 2x$ from $x = -3$ to $x = -1$ about the $x$-axis.

Soln:

Let S be the desired area.

We have,

$$S = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{--- (1)}$$

Given, the function $y = f(x) = 2x$

$$\therefore \frac{dy}{dx} = f'(x) = 2$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \{f'(x)\}^2 = 4$$

From (1),

$$S = \int_{-3}^{-1} 2\pi . 2x \times \sqrt{1+4}\, dx$$

$$= 4\sqrt{5}\pi \int_{-3}^{-1} x\, dx$$

$$= \frac{4\sqrt{5}\pi}{2} \left[x^2\right]_{-3}^{-1}$$

$$= 2\sqrt{5}\pi \{(-1)^2 - (-3)^2\}$$

$$= 2\sqrt{5}\pi (1 - 9)$$

$$= -16\sqrt{5}\pi \text{ unit}^2$$

(Ans)

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(3) Find the area of the surface generated by revolving the arc of $y = x^3$ from $x = 0$ to $x = 1$ about the x-axis.

Soln: Let S be the desired area.

We have,

$$S = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{——— (1)}$$

Given, the function $y = x^3$

$$\Rightarrow \frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 9x^4$$

$$\Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + 9x^4 \quad \text{——— (2)}$$

From (1),

$$S = \int_0^1 2\pi \times x^3 \sqrt{1 + 9x^4}\, dx \quad \text{——— (3)}$$

Let, $1 + 9x^4 = z$

$$\Rightarrow \frac{dz}{dx} = 36x^3$$

$$\Rightarrow x^3 dx = \frac{dz}{36}$$

When $x = 0$,

$$z = 1 + 9(0)^4 = 1$$

When $x = 1$,

$$z = 1 + 9(1)^4 = 10$$

From (3)

$$s = \int_0^1 2\pi x^3 \sqrt{1+9x^4}\, dx$$

$$= \int_1^{10} 2\pi \sqrt{z}\, \frac{dz}{36}$$

$$= \frac{\pi}{18} \int_1^{10} z^{1/2}\, dz$$

$$= \frac{\pi}{18} \left[ \frac{z^{1/2+1}}{1/2+1} \right]_1^{10}$$

$$= \frac{\pi}{18} \left[ \frac{z^{3/2}}{3/2} \right]_1^{10}$$

$$= \frac{\pi}{18} \times \frac{2}{3} \left( 10^{3/2} - 1^{3/2} \right)$$

$$= \frac{\pi}{27} \left( 10^{3/2} - 1 \right) \text{ unit}^2$$

(Ans)