Final (Group-B)

Relativity

Outline:

1. Definition of inertial frame and non inertial frame
2. Postulates of special Relativity
3. Lorentz transformation

$\}$ Defn

4. Variation of mass with velocity
5. Time dilation
6. Length contraction
7. $E = mc^2$

$\}$ Exp

Relativity: In classical physics it was supposed that space, time and mass are constant But according to Einstein space, time and mass are not constant but they are relative. This is called einstein's theory of relativity.

Relativity is divided into two parts. There are

(a) Special theory of relativity

(b) General theory of relativity

(Q) **Inertial reference frame:** Any system moving at constant speed with respect to each other and in which Newton's laws can be achieved are called inertial frame of reference. In this frame of reference a free object exhibits no acceleration.

(Q) **Non inertial reference frame:** The reference frames which are not moving with constant velocity with each other that is the reference frames which have acceleration are called non inertial frame.

Postulates of special theory of relativity:

(P) *** The special theory of relativity is based on the two Postulates.

1st postulate: The fundamentals laws of physics are the same in all inertial frames of reference.

2nd postulate: The velocity of light in free space has the same value for all inertial frames of reference.

(P) Lorentz transformation: Einstein deduced some transformation equations which satisfies the conditions of theory of relativity. These equations came to existence through the electromagnetic theory of A H Lorentz in 1930. That is why these equations are called Lorentz transformation.

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z$$

and $$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

The above equation are known as Lorentz transformation equation.

Inverse lorentz equation are,

$$x = \frac{x' + vt}{\sqrt{1 - v^2/c^2}}$$

$$y = y'$$

$$z = z'$$

$$t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1 - v^2/c^2}}.$$

(e) **Time dilation:** | Show that "A moving clock always appears to go slow".
***

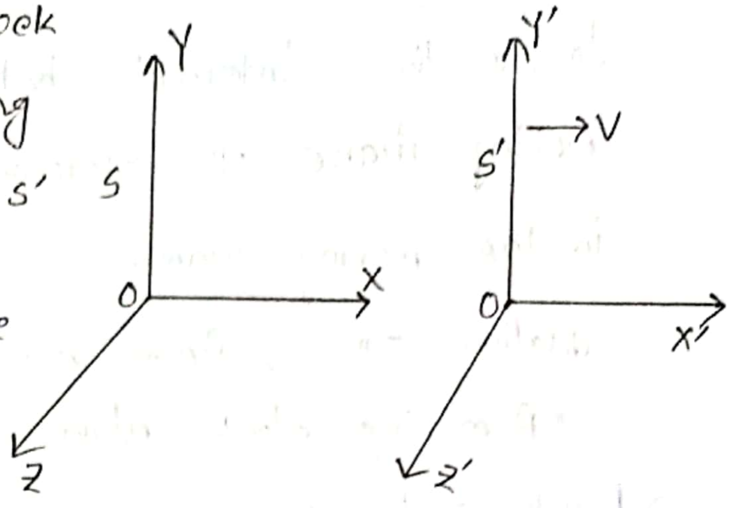
An interval of time observed in a moving frame of reference will be less than the same interval of time observed in a stationary frame of reference. This effect is called the time dilation.

Equation of time dilation is, $t = \frac{t_0}{\sqrt{1 - v^2/c^2}}$

where, t = the interval of time measured by an observer in stationary frame.

t_0 = the interval of time measured by an observer in moving frame.

Proof: Let us consider a clock at the point ~~o~~ x' in the moving frame S', when an observer in S' finds that the line is t_1'. an observer in S will find it to be t_1,



Now according to inverse lorentz transformation,

$$t_1 = \frac{t_1' + \frac{Vx'}{c^2}}{\sqrt{1-V^2/c^2}} \quad \text{—— ①}$$

After a time interval ~~to~~ t_0, the observer in the moving system finds that the time is now t_2' according to his clock. That is, $t_0 = t_2' - t_1'$ —— ②

$$t_2 = \frac{t_2' + \frac{Vx'}{c^2}}{\sqrt{1-V^2/c^2}} \quad \text{—— ③}$$

So, the duration of the interval t is

$$t = t_2 - t_1 = \Delta t$$

$$\Delta t = \frac{t_2' - t_1'}{\sqrt{1-V^2/c^2}} \Rightarrow \Delta t = \frac{t_2' - t_1'}{\sqrt{1-V^2/c^2}} \Rightarrow \frac{\Delta t'}{\sqrt{1-V^2/c^2}}$$

$$\therefore t = \frac{t_0}{\sqrt{1-\frac{V^2}{c^2}}} \qquad \text{i.e. } \Delta t > \Delta t'$$

$$\therefore t_0 = t\sqrt{1-V^2/c^2} \quad \text{—— ④}$$

From eq ④ it is proved that $t > t_0$. i.e, time is longer in a moving frame A stationary clock measures a longer time interval between events occuring in a moving frame of reference ~~frame~~ than does a clock in the moving frame. This effect is called time dilation. Thus, from above reasoning we may say "A moving clock always appears to go slow."

(8) **Length contraction:** The length of a stationary object with respect to an observer in motion is shorter than the length measured by the observer a rest. This effect is called length contraction.

The equation of length contraction is;

$$L = L_0 \sqrt{1 - v^2/c^2}$$

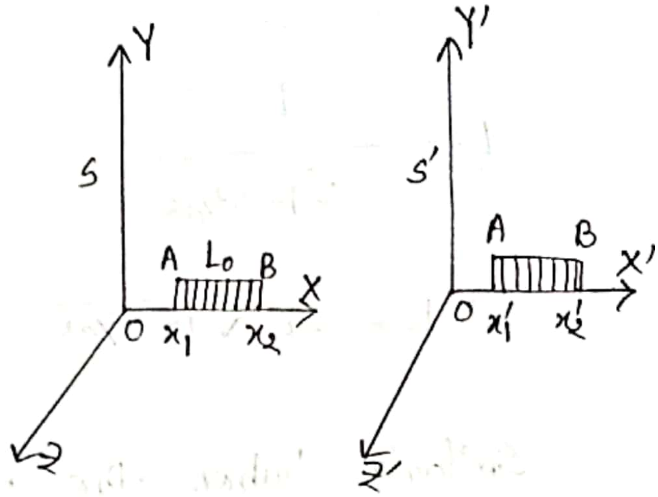
where, L = length observed by moving observer

L_0 = length observed by stationary observer

v = relative velocity between the two frames.

c = speed of light.

Proof: Let us consider a rod AB of length L_0 parallel to x-axis and having the coordinates x_1 and x_2 in the reference frame S. An observer in the frame S measures the length of the rod as,



$$L_0 = x_2 - x_1 \quad \text{——} \ (1)$$

Again, consider a second reference frame S' moving with velocity v along x-axis with respect to the frame S. An observer in the frame S' measure the end coordinates of the rod as x_1' and x_2'. The observed length by the observer in S' is,

$$L = x_2' - x_1' \quad \text{——} \ (2)$$

Now, according to the inverse lorentz transformation,

$$x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} \quad \text{and}$$

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$$

Then we get, $L_0 = x_2 - x_1$

$$= \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} - \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$= \frac{x_2' - x_1'}{\sqrt{1 - v^2/c^2}}$$

$$L_0 = \frac{L}{\sqrt{1-v^2/c^2}}$$

$$\therefore L = L_0 \sqrt{1-v^2/c^2}$$

Similarly, when the object is in motion with respect to a stationary observer. again the object is shortened ways i.e. whether object is in motion or observer is in motion. This is called length contraction.

Einstein's mass energy relation: ($E=mc^2$)

Let us suppose that force F is applied to an object to bring it to ~~d~~ its state of motion and the object covers a distance ds. the kinetic energy is Fds.

If the object covers the total distance s then the total kinetic energy.

$$T = \int_0^s F\,ds \quad \text{——} \; (1)$$

we know that, $F = \frac{d}{dt}(mv)$

and, $\frac{ds}{dt} = v$

$\therefore ds = v\,dt$

Now, if we put the value of F and ds we get,

$$T = \int_0^{mv} \frac{d}{dt}(mv)\ v\,dt$$

$$= \int_0^{mv} v\,d(mv)$$

$$= \int_0^{mv} v\,(v\,dm + m\,dv)$$

$$= \int_0^{mv} (v^2 dm + mv\,dv) \quad \text{— (2)}$$

From the relativity of mass we know that,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

By squaring this equation we get,

$$m^2 = \frac{m_0^2}{1 - \frac{v^2}{c^2}}$$

$$\Rightarrow \left(1 - v^2/c^2\right) m^2 = m_0^2$$

$$\Rightarrow \left(\frac{c^2 - v^2}{c^2}\right) m^2 = m_0^2$$

$$\Rightarrow (c^2 - v^2) m^2 = m_0^2 c^2$$

$$\Rightarrow m^2 c^2 - m^2 v^2 = m_0^2 c^2$$

By differentiating this equation we get,

$$2mc^2 dm - (2m\,v^2 dm + 2vm^2 dm) = 0$$

Dividing the above equation by 2m we get

$$c^2 dm - v^2 dm = mv\,dv$$

$\Rightarrow c^2 dm = v^2 dm + mv\,dv$

Putting the value of $mv.dv$ in equation ② we get,

$$T = \int_{m_0}^{m} c^2 dm$$

Here, m_0 is the rest mass.

$$T = c^2 \int_{m_0}^{m} dm$$

$$= c^2 [m]_{m_0}^{m}$$

$$= mc^2 - m_0 c^2$$

$$\therefore T = mc^2 - m_0 c^2 \quad \text{———} ③$$

$\therefore$ But total energy = kinetic energy + potential energy

$$\therefore E = T + m_0 c^2 \quad [\because m_0 c^2 = \text{rest mass energy}]$$

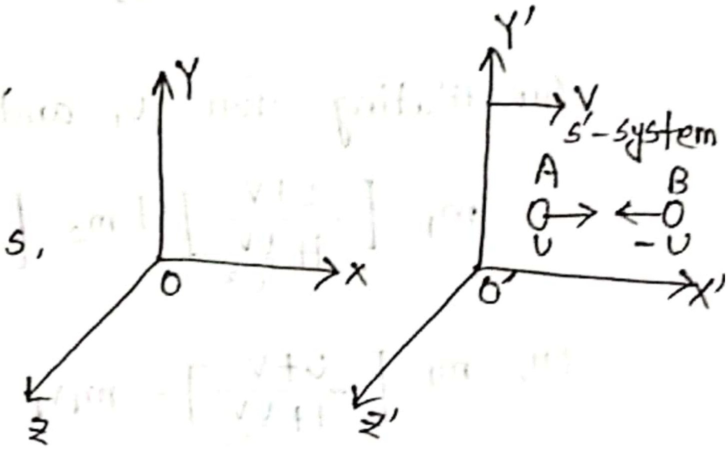
$$= mc^2 - m_0 c^2 + m_0 c^2$$

$$= mc^2$$

$$\therefore E = mc^2$$

$\therefore$ which is known as Einstein's Mass-energy relation.

Variation of mass with velocity:



Consider two system S and S', S' is moving with a constant velocity V relative to the system S, in the positive X-direction.

Let the mass of each ball be m in S', two exactly similar elastic balls A and B approach each other at equal speeds.

Momentum of ball A + momentum of ball B = momentum of coalesced mass

or $mu + (-mu) = $ momentum of coalesced mass $= 0$

Let u_1, and u_2 be the velocities of the balls relative to S. Then,

$$U_1 = \frac{U+V}{1+\frac{UV}{c^2}} \quad \text{—①}$$

$$U_2 = \frac{-U+V}{1-\frac{UV}{c^2}} \quad \text{—②}$$

After collision, velocity of the coalesced mass is V relative to the system S.

Let mass of the ball A travelling with velocity U_1 be m_1 and that of B with velocity U_2 be m_2 in the system S. Total momentum of the ball is conserved.

$$m_1 u_1 + m_2 u_2 = (m_1+m_2) V$$

Substituting for u_1 and u_2 from ① and ② we have,

$$m_1\left[\frac{u+V}{1+\frac{uV}{c^2}}\right] + m_2\left[\frac{-u+V}{1-\frac{uV}{c^2}}\right] = (m_1+m_2)V$$

or, $$m_1\left[\frac{u+V}{1+\frac{uV}{c^2}}\right] - m_1 V = m_2 V - m_2\left[\frac{-u+V}{1-\frac{uV}{c^2}}\right]$$

or, $$m_1\left[\frac{u+V}{1+\frac{uV}{c^2}} - V\right] = m_2\left[V - \frac{-u+V}{1-\frac{uV}{c^2}}\right]$$

or, $$m_1\left[\frac{u+V-V-\frac{uV^2}{c^2}}{1+\frac{uV}{c^2}}\right] = m_2\left[\frac{V-\frac{uV^2}{c^2}+u-V}{1-\frac{uV}{c^2}}\right]$$

or, $$m_1\left[\frac{u\left(1-\frac{V^2}{c^2}\right)}{1+\frac{uV}{c^2}}\right] = m_2\left[\frac{u\left(1-\frac{V^2}{c^2}\right)}{1-\frac{uV}{c^2}}\right]$$

or, $$\frac{m_1}{m_2} = \frac{1+\frac{uV}{c^2}}{1-\frac{uV}{c^2}} \quad \text{——— ④}$$

Also, $$1-\frac{u^2}{c^2} = 1 - \frac{\left\{\frac{u+V}{c}\right\}^2}{\left\{1+\frac{uV}{c}\right\}^2}$$

$$= \frac{1+\frac{u^2V^2}{c^2}+\frac{2uV}{c^2}-\frac{u^2}{c^2}-\frac{V^2}{c^2}-\frac{2uV}{c^2}}{\left(1+\frac{uV}{c^2}\right)^2}$$

$$= \frac{\left(1+\frac{u^2}{c^2}\right)-\frac{V^2}{c^2}-\left(1-\frac{u^2}{c^2}\right)}{\left(1+\frac{uV}{c^2}\right)^2}$$

$$1-\frac{u_1^2}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{V^2}{c^2}\right)}{\left(1+\frac{uV}{c^2}\right)^2} \quad \text{——— ⑤}$$

Similarly, $1-\frac{u_2^2}{c^2} = \frac{(1-\frac{u^2}{c^2})(1-\frac{v^2}{c^2})}{(1+\frac{uv}{c^2})^2}$ —— (6)

Dividing equation (6) by equation (5)

$$\frac{1-u_2^2/c^2}{1-\frac{u_1}{c^2}} = \frac{(1+\frac{uv^2}{c^2})^2}{(1-\frac{uv^2}{c^2})^2}$$

or, $\frac{\sqrt{1-u_2^2/c^2}}{\sqrt{1-\frac{u_1^2}{c^2}}} = \frac{1+\frac{uv^2}{c^2}}{1-\frac{uv^2}{c^2}}$ —— (7)

From equations (7) and (4)

$$\frac{m_1}{m_2} = \frac{\sqrt{1-\frac{u_2^2}{c^2}}}{\sqrt{1-\frac{u_1^2}{c^2}}}$$

or, $m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}}$ —— (8)

$$m_1\sqrt{1-\frac{u_1^2}{c^2}} = m_2\sqrt{1-\frac{u_2^2}{c^2}} = m_0$$

The constant denoted by m_0 is called the rest mass of the body and corresponds to zero velocity.

Thus, $m_1 = \frac{m_0}{\sqrt{1-\frac{u_1^2}{c^2}}}$

If m denotes the mass of a body when it is moving with a velocity v,

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

→ This is the relativistic formula for the variation of mass with velocity.

① The length of a spaceship is measured to be exactly half its actual length. calculate (i) the speed of the spaceship and (ii) the time dilation corresponding to one second on the spaceship.

Solution: (i) $L = L_0\sqrt{1-v^2/c^2}$

$\Rightarrow \frac{L}{L_0} = \sqrt{1-v^2/c^2}$

$\Rightarrow 0.5 = \sqrt{1-v^2/c^2}$ $\quad \left[\frac{L}{L_0} = 0.5\right]$

$\Rightarrow (0.5)^2 = 1 - v^2/c^2$

$\Rightarrow v^2/c^2 = 1-(0.5)^2$

$\Rightarrow \frac{v^2}{c^2} = 0.75$

$\Rightarrow v^2 = 0.75\times c^2$

$\Rightarrow v = \sqrt{0.75}\times c$

$\Rightarrow v = 0.866c$

$\Rightarrow v = 0.866\times(3\times10^8)$

$\Rightarrow v = 2.598\times10^8$ m/sec (Ans)

(ii) The time t as observed from the stationary frame corresponding to the time $t_0 = 1$ sec on the spaceship is given by,

$t_0 = t\sqrt{1-v^2/c^2}$

or, $t = \frac{t_0}{\sqrt{1-v^2/c^2}}$

Here,
$t_0 = 1$ second
$v^2/c^2 = 0.75$

$= \frac{1}{\sqrt{1-0.75}}$

$\therefore t = 2$ sec

(Ans)

② A particle is moving with a speed of 0.5c. Calculate the ratio of its rest mass and the mass while in motion.

Solution: $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

or, $\frac{m_0}{m} = \sqrt{1 - v^2/c^2}$

or, $\frac{m_0}{m} = \sqrt{1 - \frac{(0.5c)^2}{c^2}}$ | $v = 0.5c$

or, $\frac{m_0}{m} = \sqrt{1-(0.5)^2}$

or, $\frac{m_0}{m} = 0.866$

(Ans)

③ Calculate the velocity that one atomic mass unit will have if it has a kinetic energy equal to twice the rest mass energy.

Solution: We have, $E = mc^2 = m_0c^2 + T$

Here, $T = 2m_0c^2$

$\therefore mc^2 = m_0c^2 + 2m_0c^2$

or, $3m_0c^2 = mc^2$

or, $m = 3m_0$

But, $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

or, $3m_0 = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

or, $\sqrt{1-v^2/c^2} = \frac{m_0}{3m_0}$

or, $1-\frac{v^2}{c^2} = \frac{1}{9}$

or, $\frac{v^2}{c^2} = 1-\frac{1}{9}$

or, $\frac{v^2}{c^2} = \frac{8}{9}$

or, $v^2 = (\frac{8}{9}) \times c^2$

or, $v = \sqrt{8/9} \times c$

or, $v = 0.943\, c$

(Ans)

(4) The total energy of a particel is exactly twice its rest energy. Calculate the speed.

Solution: We have, $E = T + m_0c^2$

or, $2m_0c^2 = T + m_0c^2$ [As $E = 2m_0c^2$]

or, $2m_0c^2 = mc^2 - m_0c^2 + m_0c^2$ [Here, $T = mc^2 - m_0c^2$]

or, $2m_0c^2 = mc^2$

or, $\frac{m}{m_0} = 2$

But,

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

or, $\frac{m}{m_0} = \frac{1}{\sqrt{1-v^2/c^2}}$

or, $\sqrt{1-v^2/c^2} = \frac{1}{2}$

or, $1-\frac{v^2}{c^2} = \frac{1}{4}$

or, $\frac{v^2}{c^2} = 1 - \frac{1}{4}$

or, $v^2 = 0.75 \times c^2$

or, $v = \sqrt{0.75} \times c$

$\therefore$ $v = 0.866c$ (Ans)