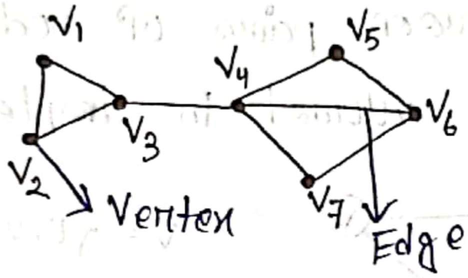
# Graph

⇒ A graph $G = (V, E)$ consists of $V$, a nonempty set of vertices (or nodes), and $E$ a set of edges.

$G = (V, E)$

↓ Vertices ↳ Edges

Example:




$G = (V, E)$ where
$V = \{V_1, V_2, \dots V_7\}$
$E = \{ \{V_1, V_2\}, \{V_1, V_3\}, \{V_2, V_3\}, \dots \{V_6, V_7\} \}$

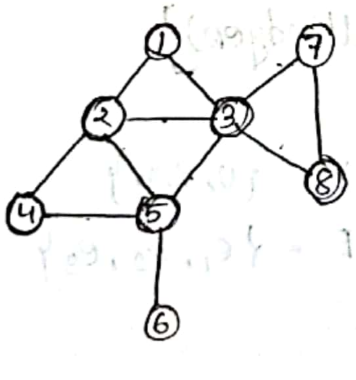
⇒ $V$ = nodes (vertices, points)

⇒ $E$ = edges (links - between pairs of nodes/points)

⇒ Denoted by $G = (V, E)$

⇒ Graph size : $n = |V|$, $m = |E|$

Example:

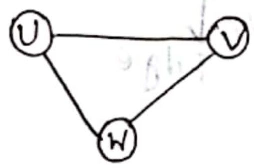



Here, $V = \{1, 2, 3, \dots 7, 8\}$
$E = \{ \{1,2\}, \{1,3\}, \dots \{5, 6\} \}$.
$n = |V| = 8$
$m = |E| = 11$

⊛ Types of Graph: Simple Graphs
Multi Graphs
Pseudographs
Directed graphs

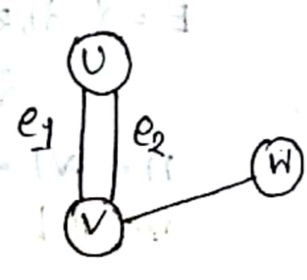
Simple Graphs: A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a simple graph.

⇒ Multiple edges between pairs of nodes and self loops are not allowed in simple graph.



$V = \{U, V, W\}$
$E = \{\{U,V\}, \{V,W\}, \{U,W\}\}$

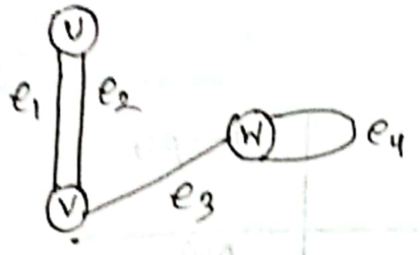
Multigraph: A multigraph is a graph which is permitted to have multiple edges (also called parallel edges), that is edges that have the same end nodes.
[Simple graph + multiple edge (multiedges)]



$V = \{U, V, W\}$
$E = \{e_1, e_2, e_3\}$

Pseudograph: A pseudograph is a non-simple graph in which both graph loops and multiple edges are permitted

⇒ Simple graph + multiedge + loop



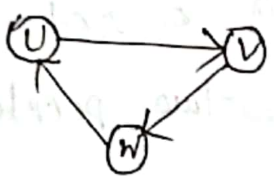

$V = \{U, V, W\}$

$E = \{e_1, e_2, e_3, e_4\}$

**Directed graph:** A set of objects (called vertices or nodes) that are connected together where all the edges are directed from one vertex to another.

→ simple graph with each edge directed

→ Loop are allowed in a directed graph

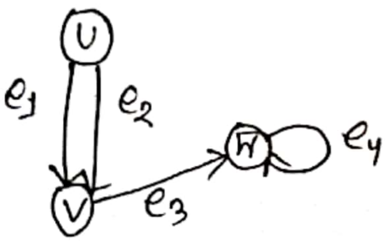




The two edges (U,V), (U,V) are multiedges (direction same)


The two edges (U,V), (V,U) are not multiedges. (direction are not same)

**Directed Multigraphs:** digraph + multiedges.




• The edge $e_1$ and $e_2$ are multiple edge if $f(e_1) = f(e_2)$.

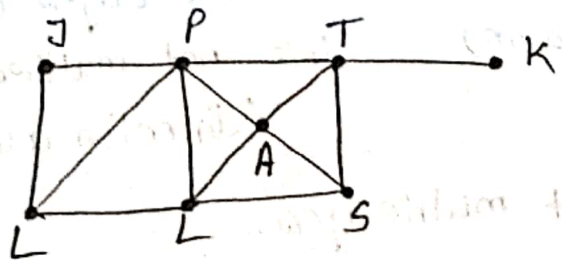
## Graph terminology:

| Type | edges | Multiple edges | loops |
|---|---|---|---|
| ① Simple Graph | undirected | NO | NO |
| ② Multigraph | undirected | Yes | NO |
| ③ Pseudograph | undirected | Yes | Yes |
| ④ Directed graph | directed | NO | Yes |
| ⑤ Directed multigraph | directed | Yes | Yes |

## Graph Model:

**Acquaintanceship graphs:** V (G) is a set of people, and an edge is present if the two people are friends / know each other.

Each person is represented by a vertex. An undirected edge is used to connect two people when they know each other.



## Undirected graph:

**adjacent:** a vertex connected to a vertex.

**incident:** a vertex connected to an edge.


U and v is and edge when they are adjacent.

e is an incident with U and V.

**endpoints:**
U and V are called endpoints of {U, V}

## Degree of graph:

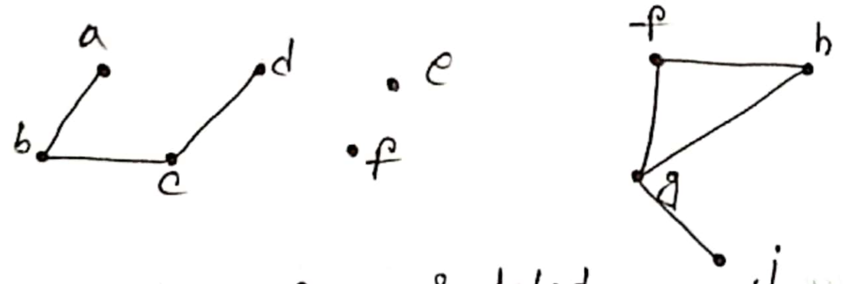
- A vertex of degree 0 is called isolated
- A vertex is pendant if and only if it has degree one.

**isolated:** A vertex with a loop at it has at least degree 2.

• ⟶ isolated

**Pendant:** A vertex of degree 1 is called pendant
It is adjacent to exactly one other vertex.

Examples: which vertices in the following graph are isolated, which are pendant, and what is the maximum degree? What type of graph is it?




Solution: Vertex f is isolated.

and vertices a, d and j are pendant.

The maximum degree(g) = 5)

The graph is pseudograph (undirected, loops).

Degree of Vertex:

The degree of a vertex V, denoted by deg(v) in an undirected graph is the number of edges incident with it.

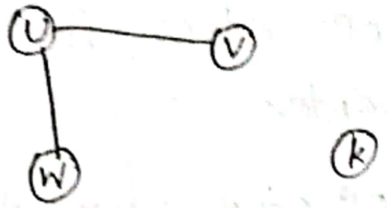
The degree of a node in an undirected graph is the number of edges incident to that node.

A loop adds 2 to the degree.

Pendant vertex: deg (v) = 1

Isolated Vertex: deg (v) = 0

Ex




* A vertex of degree 0 is called isolated, deg(V) = 0
* A vertex of degree 1 is called Pendant. deg(V) = 1
* A loop adds 2 to the degree

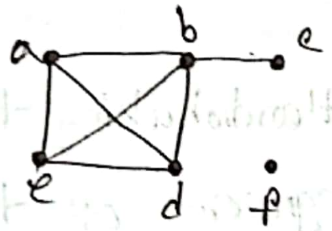
V = {U, V, W},

E = { {U, W}, {U, W}, {U, V} }

deg(U) = 2,
deg(V) = 1
deg(W) = 1
deg(K) = 0

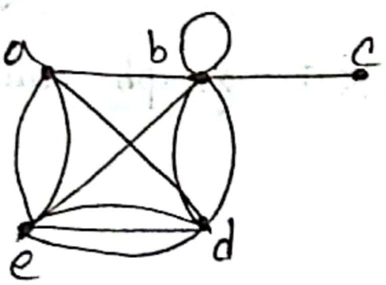
W and V are pendant
K is isolated.

Ex What are the degrees of the vertices in the graph H.




H

deg(a) = 4
deg(b) = 6
deg(c) = 1
deg(d) = 5
deg(e) = 6
deg(f) = 0




## Terminology - Directed graph

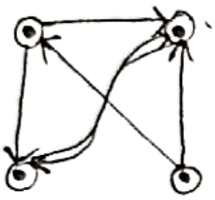
For the edges (U, V), u is adjacent to V or V is adjacent from U

→ U is initial vertex
→ V is terminal Vertex

**In degree (deg(u)):** number of edges for which u is terminal vertex.

**out degree (deg(u)):** number of edges for which u is initial vertex.

**Example:** What are the in-degrees and out degrees of the vertices a, b, c, d in this graph



deg(a) = 1
deg (a) = 2

deg(b) = 4
deg (b) = 2

deg(d) = 2
deg(d) = 1

deg(c) = 0
deg (c) = 2

**The handshaking theorem:** ~~Le~~ Handshaking theorem states that the sum of degrees of the vertices of a graph is twice the number of edges.

Let G = (V, E) be an undirected graph with e edges. Then,

$$2e = \Sigma \deg(V)$$

**Example:** How many edges are there in a graph with 10 vertices, each of degree 6?

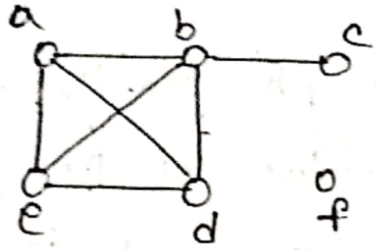
$$2e = \Sigma \deg(V)$$

$$\Rightarrow 2e = 6 \times 10$$

$$\Rightarrow 2e = 60$$

$$\therefore e = 30 \text{ edges.}$$

Example:




The graph has 11 edges.

$\therefore 2e = \Sigma \deg(v)$

$\Rightarrow 2 \cdot 11 = \Sigma \deg(v)$

$\therefore \Sigma \deg(v) = 22.$

Example: How many edges are there in a graph with 10 vertices each of degree 6.

$2e = \Sigma \deg(v)$

$\Rightarrow 2e = 6 \times 10$

$\Rightarrow e = 30$

Theorem: 1

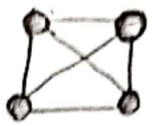
An undirected graph has an even number of vertices of odd degree.

Theorem: 2

Let $G = (V, E)$ be a graph with directed edge;

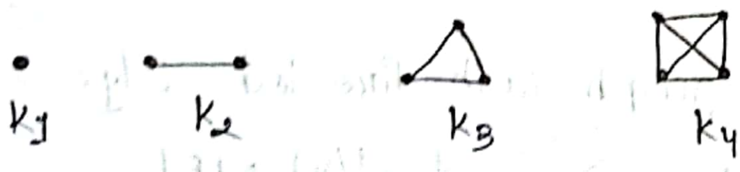
Then: $\Sigma_{v \in V} \deg^-(v) = \Sigma_{v \in V} \deg^+(v) = |E|$

**Regular graph:** A simple graph $G = (V, E)$ is called regular graph if every vertex of the graph has the same degree. A regular graph is called n-regular if $\deg(v) = n$



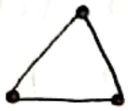
is 3 regular

**Complete graph:** The complete graph on $n$ vertices, denoted by $K_n$ is the simple graph that contains exactly one edge between each pair of distinct vertices.

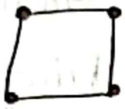


**Cycle:** The cycle consists of n vertices $v_1, v_2$ and $v_3$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, ---- $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$

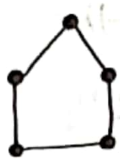
Cycle → $C_n$ → $C_3$ থেকে start হয়।



$C_3$



$C_4$

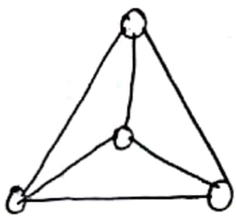


$C_5$

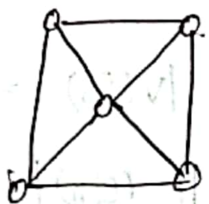
$C_n$ is 2-regular $\quad |V(C_n)| = n$

$|E(C_n)| = n$

**wheel:** when we add an additional vertex to the cycle $C_n$ and connect this new vertex to each of the n vertices in C by new edges.



$W_3$



$W_4$

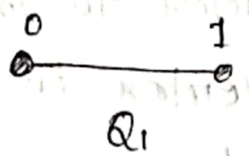
☐ $|V(W_n)| = n+1$, $|E(W_n)| = 2n$.

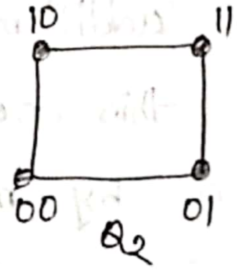
$W_n$ is not regular graph if $n \neq 3$

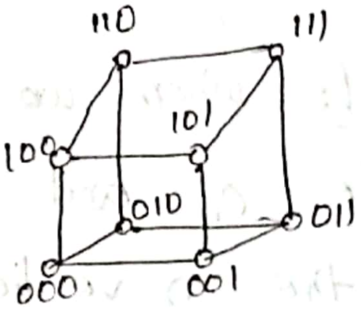
$|V(W_n)| = n+1$

$|E(W_n)| = 2n$

**N-cube:** The n-dimensional hypercube, or n-cube, denoted by $Q_n$ is the graph that has vertices representing the $2^n$ bit strings of length n. Two vertices are adjacent if and only if the bit strings ~~of length n~~ that they represent differ ~~Two vertices are adjacent~~ in exactly one bit position.



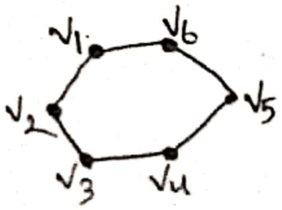



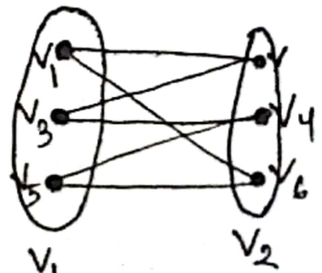

$Q_n$ is regular, $|V(Q_n)| = 2^n$

$|E(Q_n)| = (2^n n)/2$

$= 2^{n-1} n$

**Bipartite Graphs:** A simple graph $G = (V, E)$ is called bipartite if V can be partitioned into $V_1$ and $V_2$, $V_1 \cap V_2 = \emptyset$,

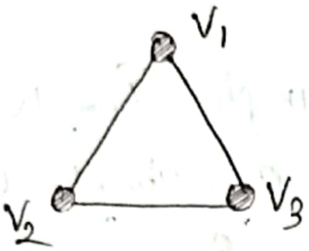


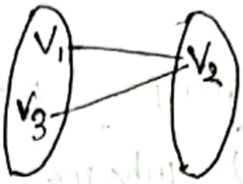




$C_6$ is bipartite

## Is $C_3$ bipartite?

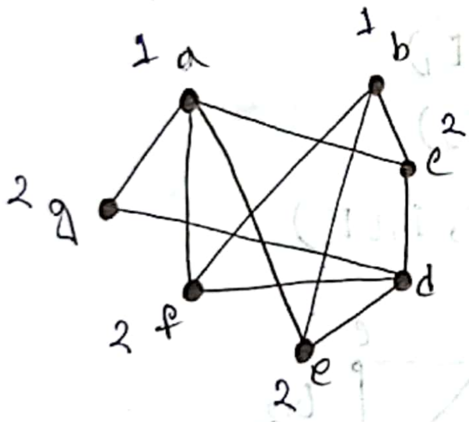
→ No, because there is no way to partition the vertices into two sets so that there are no edges with both endpoints in the same set.



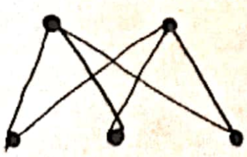




not bipartite

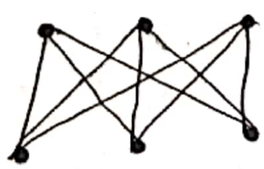
**Bipartite theorem:** A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph




**Complete bipartite graphs:** $K_{m,n}$ is the graph that has its vertex set portioned into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertices is in the first subset and the other vertex is in the second vertex.



$K_{2,3}$



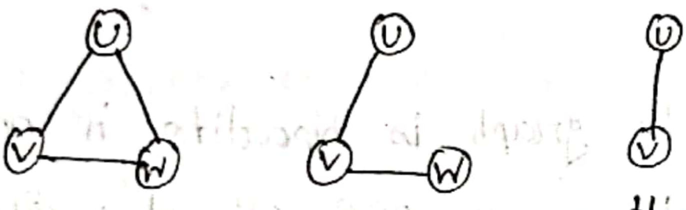
$K_{3,3}$

* $|V(K_{m,n})| = m+n$

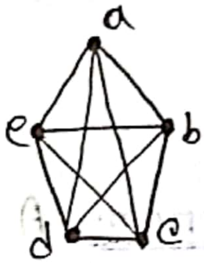
$|E(K_{m,n})| = mn$,

$K_{mn}$ is regular if and only if $m=n$

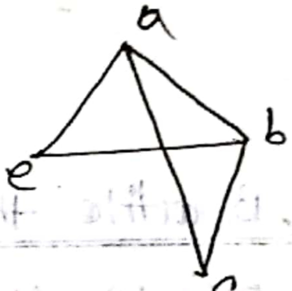
**Subgraph:** A subgraph of a Graph $G = (V,E)$ is a graph $H = (V', E')$ where $V'$ is a subset of $V$ and $E'$ is a subset of $E$




$G$ $H_1$ $H_2'$




$K_5$

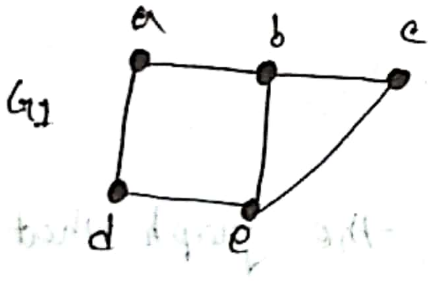


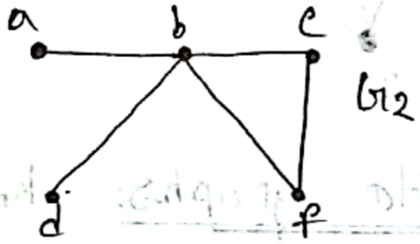

Subgraph of $K_5$

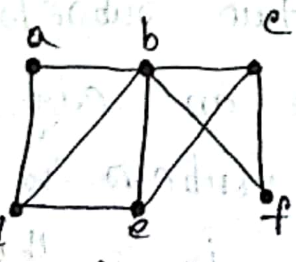
**Union of two graph:** $G_1 = (V_1, E_1)$

$G_2 = (V_2, E_2)$

$G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$



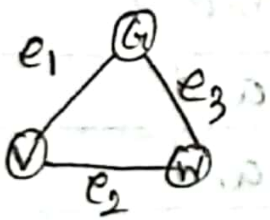

$G_1 \cup G_2$

**Incidence Matrix:** If an undirected graph G consists of n vertices and m edges, then the incidence matrix is an $n \times m$ matrix $C = [c_{ij}]$ and defined by

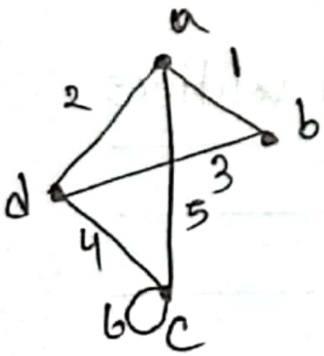
$$c_{ij} = \begin{cases} 1, & \text{if the matrix } v_i \text{ incident by edge } e_j \\ 0, & \text{otherwise} \end{cases}$$

- There is an row for vertex
  column for edge

**Example:**




| | $e_1$ | $e_2$ | $e_3$ |
|---|---|---|---|
| V | 1 | 1 | 0 |
| U | 1 | 0 | 1 |
| W | 0 | 1 | 1 |



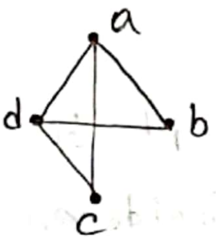

$$\longrightarrow \quad M = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

**Adjacency Matrix Representation:** If an undirected graph G consists of n vertices, then the adjacency matrix of a graph is an $n \times n$ matrix $A = [a_{ij}]$ and defined by

$$a_{ij} = \left[ \begin{array}{l} 1, \text{ if } \{v_i, v_j\} \text{ is an edge i.e, } v_i \text{ is adjacent to} \\ 0, \text{ if there is no edge between } v_i \text{ and } v_j \end{array} \right]$$

⊞ Adjacency matrix of undirected graphs are always symmetric.

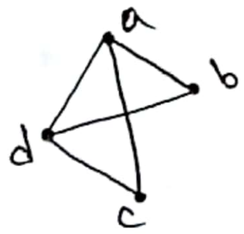
Ex: 1



$\rightarrow A_G = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

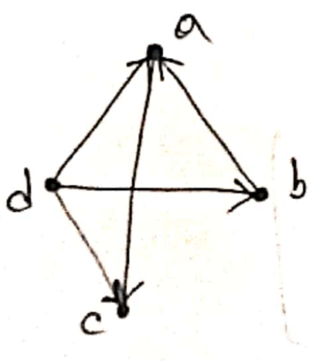
**Adjacency list:** Each ~~mt~~ node (vertex) has a list of which nodes (vertex) it is adjacent.

Ex:

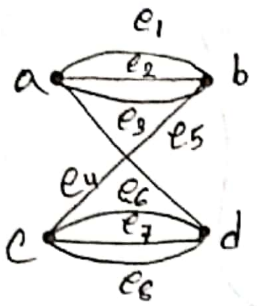


$\rightarrow$

| vertex | Adjacent vertices |
|---|---|
| a | b, c, d |
| b | a, d |
| c | a, d |
| d | a, b, c |



| Initial vertex | Terminal vertices |
|---|---|
| a | c |
| b | a |
| c | |
| d | a, b, c |




Adj'acency:

$$A = \begin{bmatrix} 0 & 3 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \end{bmatrix}$$

Incidence:

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

## Isomorphism of Graph:

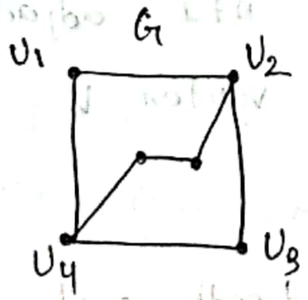
$G1 = (V1, E1)$ and $G2 = (V2, E2)$ are isomorphic

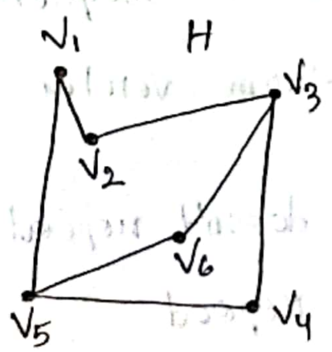
if : There is a one to one and onto function f from V1 and V2 with property that

- a and b are adjacent in G1 if and only if f(a) and f(b) are adjacent in G2, for all a and b in V1

function f is called isomorphism.

Ex:




Solution: $f(U_1) = V_6$, $f(U_2) = V_3$, $f(U_3) = V_4$, $f(U_4) = V_5$, $f(U_5) = V_1$, $f(U_6) = V_2$

→ Yes.

To check Isomorphism: No of Vertex
No of edges
Degree sequence
Mapping

**Connectivity:** In a graph reachability among vertices by traversing the edges.

**Path:** A path is a sequence of edges that begins at a vertex of a graph and travels along edge of the graph, always connecting pairs of adjacent vertices.

[path : vertex can repeat
edges cannot repeat]

* In an undirected graph, a path of length ~~n~~ n from u to v is a sequence of n+1 adjacent vertices going from vertex u to vertex v.

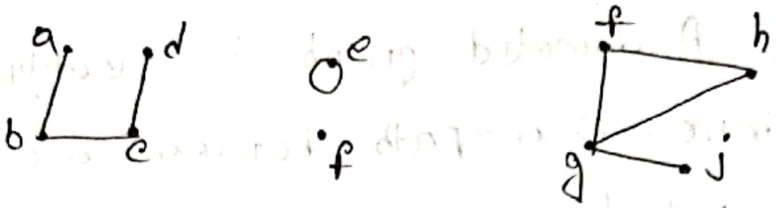
**Path:** Vertex → doesn't repeat
edges → repeat
} closed path = cycle

**Trail:** Vertex → repeat
edges → doesn't repeat
} closed trail - Circuit

Def An undirected graph is connected if there exists in a simple path between every pair of vertices.

Connected compound: maximal connected subgraph.

Ex:

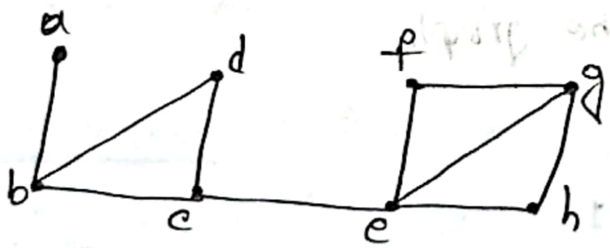



The connected components are the graph with vertices $\{a, b, c, d\}$, $\{e\}$, $\{f\}$, $\{f, g, h, j\}$

Cut vertex: A cut vertex separates one connected component into several components if it is removed.

cut edges: A cut edges separets are connected component into two components if it is removed.

Ex:




Cut vertices : b, c, e

cut edges: $\{a, b\}$, $\{c, e\}$

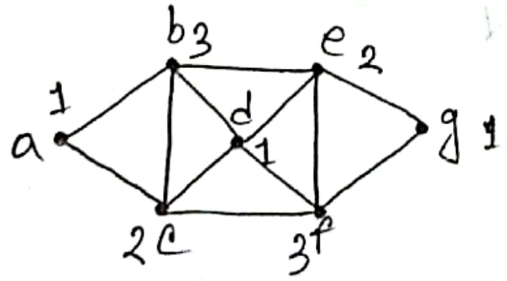
**Strongly connected:** A directed graph is strongly connected if there is a path from a to b for any two vertices a, b.

**Weakly connected:** A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

**Graph colouring:** A colouring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

**Chromatic number:** The chromatic number of a graph is the least number of colors needed for a coloring of the graph.

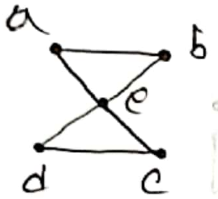
Ex:



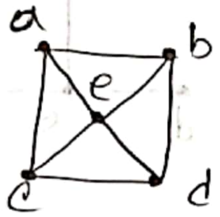
**Euler path:** An eulerian path in a graph is a path that uses each edges precisely once. If such path exist, the graph is called the graph is called traversable.

**Eulerian cycle circuit:** An eulerian circuit in a graph is a circuit that uses each edges precisely one. If such circuit exist, the graph is called Eulerian.

Ex:

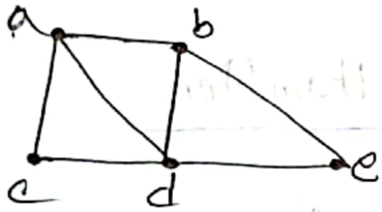



abedcea




Nothing

Euler circuit




acdebdab
path

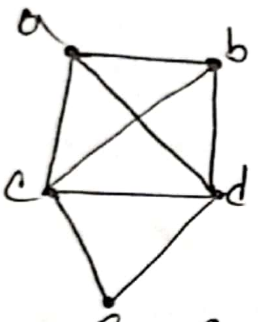
**Euler's theorem:** ① A connected graph G is Eulerian if and only if G is connected and has no vertices of odd degree.

② A connected graph G is has an Euler trail from node 'a' to some other node b if and only if G is connected and $a \neq b$ are the only two nodes of odd degree.

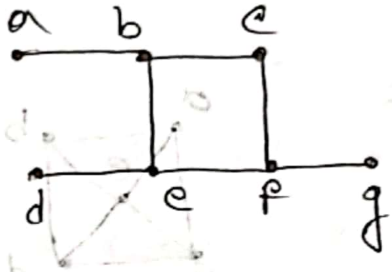
**Hamilton path:** A hamilton path is a path that traverses each vertex in a graph G exactly once.

**Hamilton circuit:** A hamilton circuit is a circuit that traverses each vertex in graph G exactly once.

Ex:




circuit




path

| Property | Euler | Hamilton |
|---|---|---|
| Repeated visits to a given node allowed? | Yes | No |
| Repeated traversals of a given edge allowed? | No | No |
| Omitted nodes allowed? | No | No |
| Omitted edges allowed? | No | Yes |