Counting

Basic counting principle - (The sum rule)

If a task can be done with either in one of n_1 ways or in one of n_2, where none of the set of n_1 ways is the same as any of the n_2 ways, then there are n_1+n_2 ways to do the task.

→ Same time then sum rule

Generalized form of the sum rule:

If the tasks $T_1, T_2 \cdots T_m$ can be done in $n_1, n_2 \cdots, n_m$ ways respectively and no two of those tasks can be done at the same time. Then the number of ways to do one of these tasks is $n_1+n_2+\cdots+n_m$

Exp: A student can choose a project from one of three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from ⇒ ?

⇒ $23+15+19 = 57$ projects to choose form.

Sum rule in terms of sets:

$|A \cup B| = |A| + |B|$ | A, B are disjoint set

→ $|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|$

when $A_i \cap A_j = \Phi$ for all i, j

The product rule: A procedure can be broken down into a sequence of two tasks. There are n_1 ways to do the 1st task and n_2 ways to do the 2nd task. Then there are $n_1 . n_2$ ways to do the Procedure.

→ at different time

Exp: How many bit strings of length seven are there?

→ Since each of the seven bits is either a 0 or a 1, ∴ $2^7 = 128$ $(2\times2\times2\times2\times2\times2\times2)$

Generalized from of the product rule:

Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$. If task T_i can be done in n_i ways after tasks $T_1, T_2 \ldots T_{i-1}$ have been done, then there are $n_1 . n_2 . n_3 \ldots n_m$ ways to carry out the procedure.

Exp: Suppose a license plate contains two letters followed by three digits with the first digit not zero. How many different license plates can be printed?

→ Each letter can be printed in 26 different ways, the first digit in 9 ways and each of the other two digit in two ways.

∴ $26.26.9.10.10 = 608400$ different plates can be

Printed.

In term of sets:

$$|A_1 \times A_2 \times A_3 \cdots \times A_m| = |A_1| \cdot |A_2| \cdots |A_m|$$

Combining the sum and product rule:

Many counting problem cannot be solved using sum rule or just product rule. Complicated counting problem can be solved using both of those rules.

Exp: Suppose statements labels in a programming language can be either a single letter or a letter followed by a digit. Find the number of possible labels.

(Solution: use product rule,

$$26 + 26 \cdot 10 = 286$$

Counting password:

Exp: Each user on a computer system has a password which is six to eight characters long, where each characte-rs is an uppercase letter or a digit. Each password must containt at list one digit. How many possible ~~character~~ passwords are there?

Solution: Let p be the number of passwords and P_6, P_7 and P_8 be the passwords of length 6, 7, 8.

By sum rule, $P = P_6 + P_7 + P_8$

$\left.\begin{array}{l}\text{26 letter}\\ \text{10 digit}\end{array}\right\}36$

$P_6 = 36^6 - 26^6$

$= 2,176,782,336 - 308,915,776$

$= 1,867,866,560$

$P_7 = 36^7 - 26^7 = 78,364,164,~~20~~096 - 8,031,810,176$

$= 70,332,253,920$

$P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576$

$= 2,612,282,842,880$

$P = P_6 + P_7 + P_8$

$= 2,684,483,063,360$

Subtraction rule / Inclusion - Exclusion principle:

If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

$|A \cup B| = |A| + |B| - |A \cap B|$

Generalized principle of Inclusion - Exclusion:

The principle of inclusion - exclusion can be generalized to find the number of ways to do one of n different tasks or equivalently to find the number of elements in the union of sets.

Let $A_1, A_2, \ldots, A_n$ be finite sets. Then,

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \Sigma |A_i| - \Sigma |A_i \cap A_j| + \Sigma |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Exp: Give a formula for the number of elements in the union of four sets.

⇒ The inclusion - exclusion principle shows that

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| - \cancel{A_1 A_2} |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

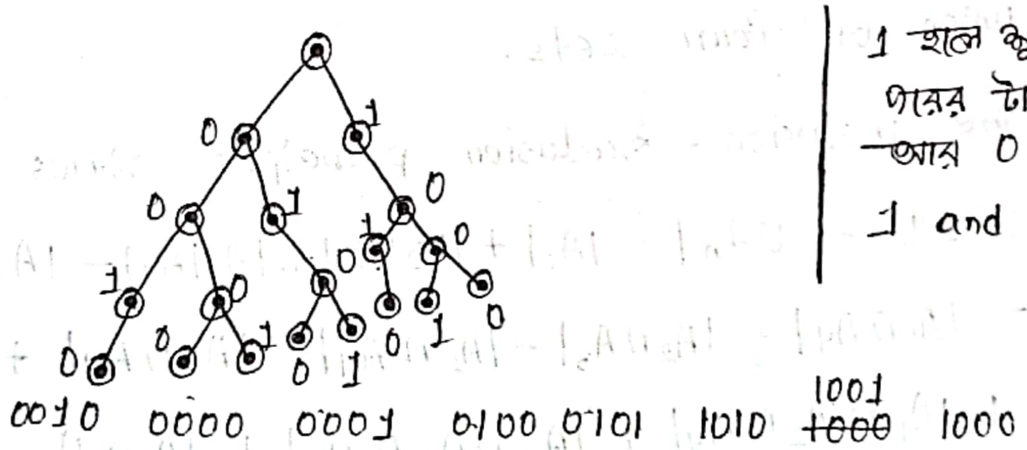
∴ This formula contains 15 different items

∴ i.e, $2^4 - 1$

Tree diagrams: Many counting problems can solve through the use of tree diagrams, where a branch represents a possible choice and the leaves represent possible outcomes.

Exp: How many bit strings of length four do not have two consecutive 1's.

⇒ The following tree diagram displays all bit strings if length four without two consecutive.



1 হলে শুধু 0 এর পরের টা,
আর 0 হলে পরেরটা
1 and 0

Pigeonhole principle: If k is a positive integer and $k+1$ objects are placed into k boxes, then at least one box contains two or more objects.

~~Statements: The pigeonhole principle states that if we must put $N+1$ or more~~

⇒ If there are more pigeons than pigeonholes then there must be at least one pigeonhole with at least two pigeons in it.

Corollary 1: A function 'f' from a set with $k+1$ elements to a set with 'k' elements is not one to one.

Proof: → Create a box for each element y in the codomain of f

⇒ $f(x) = y$

⇒ there are $k+1$ elements and only k boxes, at least one box has two or more elements.

∴ f can't be one to one.

- Pigeonhole principle → Another name is Dirichlet drawer Principle

The Generalized pigeonhole principle:

If N objects are placed into 'k' boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

⇒ If none of boxes contains more than $\lceil N/k \rceil - 1$ objects then the total number of object is at most,

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

Permutations: A permutation is a set of distinct object is an ordered arrangement of those objects, An ordered arrangement of r elements of a set is called an r-permutations.

Formula:

Theorem 1 – If n is a positive integer and r is an integer $1 \le r \le n$ then there are

$$P(n,r) = n(n-1)(n-2) \cdots (n-r+1)$$

Proof: Use the product rule, The first element can be choosen in 'n' ways. The 2nd in $n-1$ ways. and so on untill there are $(n-(r-1))$

ways to ~~d~~ choose the last element.

If n and r are integers with $1 \leq r \leq n$ then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Permutation with Repetition:

Theorem : 2

The number of r- permutations of a set of n objects with repetation allowed is n^r.

With indistinguishable objects:

Theorem : 03

The number of diff. permutations of n objects where there are n_1 indistiguishable objects of type 1, n_2 . indistinguishable objects of type 2 ...... and n_k. indistinguishable objects of type k is:

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Combination: An r- combinations of element of a set is an unordered selection of r elements from the set. Thus an r combinations is simply a subset of the set with 'r' elements.

→ r- combination of a set with n- distinct element is denoted by $C(n, r)$ / $C\binom{n}{r}$.

Theorem: 1

The number of r- combinations of a set with n elements where $n > r \geq 0$ equals,

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Proof: By the product rule,

$$P(n,r) = C(n,r) \cdot P(r,r)$$

$$C(n,r) = \frac{P(n,r)}{P(r,r)}$$

$$= \frac{n!/(n-r)!}{r!/(r-r)!}$$

$$= \frac{n!}{(n-r)!\,r!}$$

Corollary 2:

Let n and r be the nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n-r)$

Proof: From theorem, $C(n,r) = \frac{n!}{(n-r)!\,r!}$

and, $C(n, n-r) = \frac{n!}{(n-r)!\,[n-(n-r)]!}$

$= \frac{n!}{(n-r)!\,r!}$

Hence, $C(n,r) = C(n, n-r)$

Combination with repetation:

Theorem: There are $C(n+r-1, r)$ r-combinations from a set with n elements when repetation of elements is allowed.

Formula:

Type	Repetition	Formula
r-permutations	NO	$\frac{n!}{(n-r)!}$
r-~~permutations~~ combinations	NO	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Binomial Expression: A binomial expression is the sum of two terms, such as $(x+y)$ [more generally, these terms can be products of constants and variables]

Pascal's Identity:

If 'n' and 'k' are integers with $n \geq k \geq 0$.

then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

The n^{th} row in the triangle consists of the binomial coefficients $\binom{n}{k}$; $k = 0, 1, \ldots, n$

Theorem: Let n be a positive integer, Then

$$\Sigma\, C(n,k) = 2^n$$

Theorem: Let m, n and r be non negative integers with r not exceeding either 'm' or 'n'. Then $C(m+n, r) = \Sigma\, (m, r-k)\, C(n,k)$

Binomial theorem: Let 'x' and 'y' be variables and 'n' a nonnegative integer.

Then,

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots \binom{n}{n-1}xy^{n-1}$$
$$+ \binom{n}{n}y^n$$

Exp: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

$\Rightarrow$ $C(25,13)\ 2^{12}(-3)^{13} = -(25!\,/\,13!\,12!)\,2^{12}\cdot 3^{13}$

Theorem: Let n be a positive integer,

Then $\sum (-1)^k\, C(n,k) = 0$