Proof De-Moivre's theorem, Ex - 1, 2, 5

Set theory: Relation (Reflexive, symmetric, anti-symmetric, transitive), inverse function, Ex - 9, 10, 11, inverse of a function, Cartesian product, proof Demorgan's theorem

Complex Variable: ex - 8, 11, 12, 15, 17, 19, 20, Home work, 22, poles, ex - 2, 3, 4, residue, singular points.

De-Moivre's Theorem

① State and prove De-Moivre's theorem.

<u>Statement</u>: When n is an integer +ve or -ve the value of $(\cos\theta + i\sin\theta)^n$ is ~~$\cos\theta + i\sin\theta$~~ $\cos n\theta + i\sin n\theta$. When n is a fraction +ve or -ve one of the values of $(\cos\theta + i\sin\theta)^n$ is ~~$\cos\theta +$~~ $\cos n\theta + i\sin n\theta$

<u>Proof:</u>

Case-I : When n is a +ve integer, we have,

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= \cos\theta_1 \cdot \cos\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 + i\sin\theta_2 \cdot \cos\theta_1 - \sin\theta_1 \sin\theta_2$ $(i^2 = -1)$

$= \cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cdot \cos\theta_2 + \cos\theta_1 \sin\theta_2)$

$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

Similarly,

$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$= \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}\{\cos\theta_3 + i\sin\theta_3\}$

$= \cos(\theta_1 + \theta_2) \cdot \cos\theta_3 - \sin(\theta_1 + \theta_2)\sin\theta_3 + i\{\sin(\theta_1 + \theta_2)\cos\theta_3 + \cos(\theta_1 + \theta_2)\}\sin\theta_3$

$= \cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)$

Proceeding this way the product of the factor, we have $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + \sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) \ldots (\cos\theta_n + i\sin\theta_n)$

Now, putting $\theta = \theta_1 \cdots = \theta_n = \theta$ we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Case-II when n is a $-$ve integer

Let $n = -m$ where m is a $+$ve integer.

Now, $(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{(\cos m\theta - i\sin m\theta)}{(\cos^2 m\theta - i^2\sin^2 m\theta)} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \quad (i^2 = -1)$$

$$\Rightarrow \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + \sin(-m)\theta$$

$$= \cos n\theta + i\sin n\theta$$

$$\therefore (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Case-III when n is a fraction $+$ve or $-$ve

let, $n = \frac{p}{q}$ where q is a non-zero $+$ve integer and p is any integer $+$ve or $-$ve.

Now, $\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^q = \cos\left(q \cdot \frac{\theta}{q}\right) + i\sin\left(q \cdot \frac{\theta}{q}\right)$

$$= \cos\theta + i\sin\theta$$

Therefore taking the qth root all we have

$$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^{q/q} = (\cos\theta + i\sin\theta)^{1/q}$$

or, $$\left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right) = (\cos\theta + i\sin\theta)^{1/q}$$

Raising both quantities to the power p. we have

$$(\cos\theta + i\sin\theta)^{p/q} = \left(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}\right)^{p}$$
$$= \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)$$

Hence, one of the values of

$$(\cos\theta + i\sin\theta)^{n} = \cos n\theta + i\sin n\theta$$

Thus De-Moivre's theorem is completely establish.

De-Moivre's Theorem:

Ex: 1 If $x_n = \cos\frac{\pi}{3^n} + i\sin\frac{\pi}{3^n}$, Prove that $x_1 x_2 x_3 \ldots \text{inf} = i$.

Solution: Given that, $x_n = \cos\frac{\pi}{3^n} + i\sin\frac{\pi}{3^n}$

Now, putting $n = 1, 2, 3, 4 \ldots \infty$ we have,

$$x_1 = \cos\frac{\pi}{3^1} + i\sin\frac{\pi}{3^1}$$

$$x_2 = \cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}$$

$$x_3 = \cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}$$

$$x_4 = \cos\frac{\pi}{3^4} + i\sin\frac{\pi}{3^4}$$

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$\therefore$ L.H.S $= x_1 x_2 x_3 \ldots\ldots \infty$

$$= \left(\cos\frac{\pi}{3^1} + i\sin\frac{\pi}{3^1}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\left(\cos\frac{\pi}{3^3} + i\sin\frac{\pi}{3^3}\right)\left(\cos\frac{\pi}{3^4} + i\sin\frac{\pi}{3^4}\right)$$

$$= \cos\left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \frac{\pi}{3^4} + \cdots\right) + i\sin\left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \frac{\pi}{3^4} + \cdots\right)$$

$$= \cos\left\{\frac{\pi}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right)\right\} + i\sin\left\{\frac{\pi}{3}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\right)\right\}$$

$$= \cos\left\{\frac{\pi}{3}\left(1 - \tfrac{1}{3}\right)^{-1}\right\} + i\sin\left\{\frac{\pi}{3}\left(1 - \tfrac{1}{3}\right)^{-1}\right\} \quad \left[\because (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots\right]$$

$$= \cos\left\{\frac{\pi}{3}\left(\frac{3-1}{3}\right)^{-1}\right\} + i\sin\left\{\frac{\pi}{3}\left(\frac{3-1}{3}\right)^{-1}\right\}$$

$$= \cos\left\{\frac{\pi}{3}\left(\frac{2}{3}\right)^{-1}\right\} + i\sin\left\{\frac{\pi}{3}\left(\frac{2}{3}\right)^{-1}\right\}$$

$$= \cos\left\{\frac{\pi}{3}\left(\frac{1}{2/3}\right)\right\} + i\sin\left\{\frac{\pi}{3}\left(\frac{1}{2/3}\right)\right\}$$

$$= \cos\left\{\frac{\pi}{3}\left(\frac{3}{2}\right)\right\} + i\sin\left\{\frac{\pi}{3}\left(\frac{3}{2}\right)\right\}$$

$$= \cos\left\{\frac{\pi}{2}\right\} + i\sin\left\{\frac{\pi}{2}\right\}$$

$$= 0 + i.1 = i = \text{R.H.S}$$

[Proved]

2. If $(1+i\frac{x}{a})(1+i\frac{x}{b})(1+i\frac{x}{c})\cdots\cdots \to A+iB$ then prove that,

(i) $(1+\frac{x^2}{a^2})(1+\frac{x^2}{b^2})(1+\frac{x^2}{c^2})\cdots\cdots = A^2+B^2$

(ii) $\tan^{-1}\frac{x}{a} + \tan^{-1}\frac{x}{b} + \tan^{-1}\frac{x}{c} + \cdots\cdots = \tan^{-1}\frac{B}{A}$

Solution: Given that,

(i)
$$(1+i\frac{x}{a})(1+i\frac{x}{b})(1+i\frac{x}{c})\cdots = A+iB \quad \text{——— (1)}$$

Let, $1 = r\cos\alpha$, $\frac{x}{a} = r\sin\alpha$

So, $\frac{\frac{x}{a}}{1} = \frac{r\sin\alpha}{r\cos\alpha}$

$\Rightarrow \frac{x}{a} = \tan\alpha$

$\Rightarrow \tan\alpha = \frac{x}{a}$

$\Rightarrow \alpha = \tan^{-1}\frac{x}{a}$

Again, Let, $1 = r\cos\beta$, $\frac{x}{b} = r\sin\beta$

So, $\frac{\frac{x}{b}}{1} = \frac{r\sin\beta}{r\cos\beta}$

$\Rightarrow \frac{x}{b} = \tan\beta$

$\Rightarrow \tan\beta = \frac{x}{b}$

$\Rightarrow \beta = \tan^{-1}\frac{x}{b}$

Again, Let, $1 = r\cos\gamma$, $\frac{x}{c} = r\sin\gamma$

So $\frac{\frac{x}{c}}{1} = \frac{r\sin\gamma}{r\cos\gamma}$

$\Rightarrow \frac{x}{c} = \tan\gamma$

$\Rightarrow \tan\gamma = \frac{x}{c} \quad \Rightarrow \gamma = \tan^{-1}\frac{x}{c}$

From ① we get,

$$\left(1+i\frac{x}{a}\right)\left(1+i\frac{x}{b}\right)\left(1+i\frac{x}{c}\right)\cdots = A+iB$$

$$\Rightarrow (1+i\tan\alpha)(1+i\tan\beta)(1+i\tan\gamma)\cdots = A+iB$$

$$\Rightarrow \left(1+i\frac{\sin\alpha}{\cos\alpha}\right)\left(1+i\frac{\sin\beta}{\cos\beta}\right)\left(1+i\frac{\sin\gamma}{\cos\gamma}\right)\cdots = A+iB$$

$$\Rightarrow \left(\frac{\cos\alpha+i\sin\alpha}{\cos\alpha}\right)\left(\frac{\cos\beta+i\sin\beta}{\cos\beta}\right)\left(\frac{\cos\gamma+i\sin\gamma}{\cos\gamma}\right)\cdots = A+iB$$

$$\Rightarrow \left(\frac{1}{\cos\alpha}\right)\left(\frac{1}{\cos\beta}\right)\left(\frac{1}{\cos\gamma}\right)\cdots(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)\cdots = A+iB$$

$$\Rightarrow (\sec\alpha)(\sec\beta)(\sec\gamma)\cdots(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)\cdots = A+iB$$

$$\Rightarrow (\sec\alpha)(\sec\beta)(\sec\gamma)\cdots[\cos(\alpha+\beta+\gamma+\cdots)+i\sin(\alpha+\beta+\gamma+\cdots)] = A+iB$$

Equating the real and imaginary part on both sides, we get,

$$(\sec\alpha)(\sec\beta)(\sec\gamma)\cdots[\cos(\alpha+\beta+\gamma+\cdots)] = A \quad \text{②}$$

$$(\sec\alpha)(\sec\beta)(\sec\gamma)\cdots[\sin(\alpha+\beta+\gamma+\cdots)] = B \quad \text{③}$$

Squaring ② and ③ we get,

$$(\sec^2\alpha)(\sec^2\beta)(\sec^2\gamma)\cdots[\cos^2(\alpha+\beta+\gamma+\cdots)] = A^2 \quad \text{④}$$

$$(\sec^2\alpha)(\sec^2\beta)(\sec^2\gamma)\cdots[\sin^2(\alpha+\beta+\gamma+\cdots)] = B^2 \quad \text{⑤}$$

Adding ④ and ⑤

$$(\sec^2\alpha\sec^2\beta\sec^2\gamma\cdots)[\cos^2(\alpha+\beta+\gamma+\cdots)] + (\sec^2\alpha\sec^2\beta\sec^2\gamma+\cdots)\sin^2(\alpha+\beta+\gamma+\cdots) = A^2+B^2$$

$$\Rightarrow (\sec^2\alpha \sec^2\beta \sec^2\gamma \ldots)\{\cos^2(\alpha+\beta+\gamma+\ldots)+\sin^2(\alpha+\beta+\gamma+\ldots)\} = A^2+B^2$$

$$\Rightarrow (\sec^2\alpha \sec^2\beta \sec^2\gamma \ldots)\,1 = A^2+B^2$$

$$\Rightarrow \sec^2\alpha \sec^2\beta \sec^2\gamma = A^2+B^2$$

$$\Rightarrow (1+\tan^2\alpha)(1+\tan^2\beta)(1+\tan^2\gamma) \ldots A^2+B^2$$

$$\therefore \left(1+\frac{x^2}{a^2}\right)\left(1+\frac{x^2}{b^2}\right)\left(1+\frac{x^2}{c^2}\right)\ldots = A^2+B^2$$

[Proved]

(ii) Now, কৰি ③ ÷ ②

$$\frac{(\sec\alpha)(\sec\beta)(\sec\gamma)\ldots[\sin(\alpha+\beta+\gamma+\ldots)]}{(\sec\alpha)(\sec\beta)(\sec\gamma)\ldots[\cos(\alpha+\beta+\gamma+\ldots)]} = \frac{B}{A}$$

$$\Rightarrow \frac{[\sin(\alpha+\beta+\gamma+\ldots)]}{[\cos(\alpha+\beta+\gamma+\ldots)]} = \frac{B}{A}$$

$$\Rightarrow \tan(\alpha+\beta+\gamma+\ldots) = \frac{B}{A}$$

$$\Rightarrow \alpha+\beta+\gamma+\ldots = \tan^{-1}\frac{B}{A}$$

$$\Rightarrow \left(\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}+\ldots\right) = \tan^{-1}\frac{B}{A}$$

[Proved]

3. Using Demoivre's theorem, solve the equation
(CT)

$$x^6+x^5+x^4+x^3+x^2+x^1+1=0$$

Solution: Given that,

$$x^6+x^5+x^4+x^3+x^2+x^1+1=0$$

Multiplying the given equation by $(x-1)$ we get,

$(x-1)(x^6+x^5+x^4+x^3+x^2+x^1+1)=0$

$\Rightarrow x^7-1=0$

$\Rightarrow x^7=1$

$\Rightarrow x=(1)^{\frac{1}{7}}$

$\Rightarrow x=(\cos 0+i\sin 0)^{1/7}$

$\cos(4.\text{Pi}+0)$
$=\cos(8.90+0)$
$=\cos 0$
$=1$

$\Rightarrow x=\{\cos(2n\pi+0)+i\sin(2n\pi+0)\}^{1/7}$

$\Rightarrow x=\cos\frac{2n\pi}{7}+i\sin\frac{2n\pi}{7}$ $\quad [(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta]$

Putting $n=0, 1, 2, 3, 4, 5$ and 6 we get roots of equation as

$\Rightarrow x=\cos 0+i\sin 0$

$\Rightarrow x=\cos\frac{2\pi}{7}+i\sin\frac{2\pi}{7}$

$\Rightarrow x=\cos\frac{4\pi}{7}+i\sin\frac{4\pi}{7}$

$\Rightarrow x=\cos\frac{6\pi}{7}+i\sin\frac{6\pi}{7}$

$\Rightarrow x=\cos\frac{8\pi}{7}+i\sin\frac{8\pi}{7}$

$\Rightarrow x=\cos\frac{10\pi}{7}+i\sin\frac{10\pi}{7}$

$\Rightarrow x=\cos\frac{12\pi}{7}+i\sin\frac{12\pi}{7}$

(Ans)

5 Obtain the quadratic equation having for its roots nth power of the roots of

$x^2 - 2x\cos\theta + 1 = 0$. (HRK - sheet)

Solution: Given that,

$$x^2 - 2x\cos\theta + 1 = 0$$

$$\therefore x = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4.1.1}}{2.1}$$

$$= \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$= \frac{2\cos\theta \pm \sqrt{-4(1-\cos^2\theta)}}{2}$$

$$= \frac{2\cos\theta \pm i2\sin\theta}{2}$$

$$= \frac{2(\cos\theta \pm i\sin\theta)}{2}$$

$$= \cos\theta \pm i\sin\theta$$

Let, α and β be the roots of the given eq^n

$\therefore \alpha = \cos\theta + i\sin\theta$

$\beta = \cos\theta - i\sin\theta$

So, the roots of the required eq^n will be α^n and β^n,

Hence, the required eq^n,

$$x^2 - (\alpha^n + \beta^n)x + \alpha^n\beta^n = 0$$

or, $x^2 - \{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\}x + \{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)\}^n = 0$

or, $x^2 - (\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta)x + (\cos^2\theta + \sin^2\theta)^n = 0$

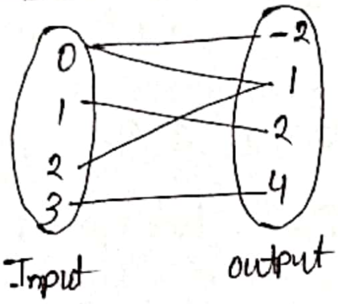
$$x^2 - 2x\cos\theta + 1 = 0$$

(Ans)

Relations and Functions

~~Regular function:~~

Relation: A relation is a set of inputs and outputs, often written as ordered pairs (input, output). We can also represent a relation as a mapping diagram or a graph. For example, the relation can be represented as –



Reflexive Relation: Let $R = (A, A, P(x,y))$ be a relation in a set A. i.e. let R be a subset of $A \times A$. Then R is called a reflexive relation. For every $a \in A$, $(a,a) \in R$. In other words, R is reflexive if every element in A is related to itself.

Symetric Relation: Let R be a subset of $A \times A$. i.e. Let R be a relation in A. Then R is called a symetric relation if $(a,b) \in R$ implies $(b,a) \in R$. That is related to b then b is also related to A.

Anti symetric Relation: A relation R in a set A, i.e. a subset of $A \times A$ is called an anti symetric relation if $(a,b) \in R$ and $(b,a) \in R$ implies $a = b$. In other words if $a \neq b$ then possibly a is related to b or possible b is related to a but never both.

Transitive Relations: A Relation R in a set A is called a transitive relation if $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$. In other words, if a is related to b and b is related to c then a is related to c.

9. Let the function $f: R^{\#} \to R^{\#}$ be defined by $y = f(x) = x^2+1$

Find, ① $f^{-1}(5)$ ② $f^{-1}(0)$ ③ $f^{-1}(10)$ ④ $f^{-1}(-5)$ ⑤ $f^{-1}([10,26])$,

⑥ $f^{-1}([0,5])$, ⑦ $f^{-1}([-5,1])$, ⑧ $f^{-1}([-5,5])$

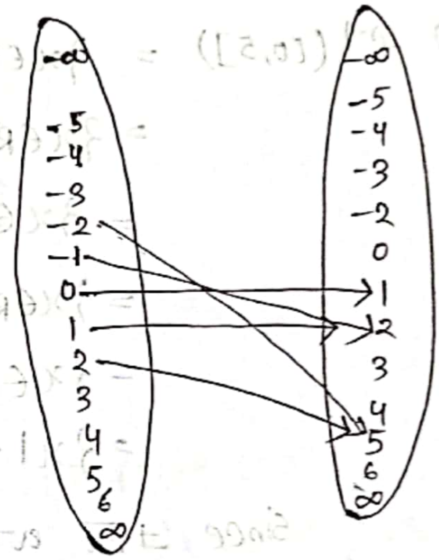
<u>Solution:</u> ① $f^{-1}(5) = \{x \in R^{\#};\ y=5\}$

$= \{x \in R^{\#};\ x^2+1=5\}$ $[f(x) = y = x^2+1]$

$= \{x \in R^{\#};\ x^2=4\}$

$= \{x \in R^{\#};\ x = \pm 2\}$

$= \{-2, 2\}$



② $f^{-1}(10) = \{x \in R^{\#};\ y=0\}$

$= \{x \in R^{\#};\ x^2+1=0\}$

$= \{x \in R^{\#};\ x = \pm\sqrt{-1}\}$

$= \{x \in R^{\#};\ x = \pm\sqrt{i^2}\}$

$= \{x \in R^{\#};\ x = \pm i\}$

$= \phi$ (since $\pm\sqrt{-1}$ are not real numbers)

③ $f^{-1}(10) = \{x \in R^{\#};\ \cancel{x}\ y = 10\}$

$= \{x \in R^{\#};\ x^2+1 = 10\}$

$= \{x \in R^{\#};\ x = \pm 3\}$

$= \{-3, 3\}$

④ $f^{-1}(-5) = \{x \in R^{\#};\ y = -5\}$

$= \{x \in R^{\#};\ x^2+1 = -5\}$

$= \{x \in R^{\#};\ x^2 = -6\}$

$= \{x \in R^{\#};\ x \pm \sqrt{-6}\}$

$= \phi$ (since $\pm\sqrt{-6}$ are not real numbers)

⑤ $f^{-1}([10,26]) = \{x \in R^{\#};\ y \in [10,26]\}$

$= \{x \in R^{\#};\ 10 \le y \le 26\}$

$= \{x \in R^{\#};\ 10 \le x^2+1 \le 26\}$

$= \{x \in R^{\#};\ 9 \le \cancel{x^2+1 \le 25}\ x^2 \le 25\}$

$= \{x \in R^{\#};\ \pm 3 \le x \le \pm 5\}$

$= \{x \mid 3 \le x \le 5,\ -5 \le x \le -3\}$

⑥ $f^{-1}([0,5]) = \{x \in R^{\#};\ f(x) \in [0,5]\}$

$= \{x \in R^{\#};\ 0 \le x^2+1 \le 5\}$

$= \{x \in R^{\#};\ -1 \le x^2 \le 4\}$

$= \{x \in R^{\#};\ \sqrt{-1} \le x \le +2\}$

$= \{x \in R^{\#};\ x \le +2\}$

$= \{x \mid -2 \le x \le 2\}$

Since $\pm\sqrt{-1}$ are not real numbers.

⑦ $f^{-1}([-5,1]) = \{x \in R^{\#};\ f(x) \in [-5,1]\}$

$= \{x \in R^{\#};\ -5 \le x^2+1 \le 1\}$

$= \{x \in R^{\#};\ -6 \le x^2 \le 0\}$

$= \{x \in R^{\#};\ \pm\sqrt{-6} \le x \le \pm 0\}$

$= \{0\}$ [since $\pm\sqrt{-6}$ are not real numbers]

⑧ $f^{-1}([-5,5]) = \{x \in R^{\#};\ f(x) \in [-5,5]\}$

$= \{x \in R^{\#};\ -5 \le x^2+1 \le 5\}$

$= \{x \in R^{\#};\ -6 \le x^2 \le 4\}$

$= \{x \in R^{\#};\ \pm\sqrt{-6} \le x \le \pm 2\}$

$= \{x \in R^{\#};\ x \le \pm 2\}$

$= \{x \mid -2 \le x \le 2\}$

10. Let the function $f: R^{\#} \to R^{\#}$ be defined by $y = f(x) = \sin x$

Find (1) $f^{-1}(0)$, (2) $f^{-1}(1)$, (3) $f^{-1}(2)$, (4) $f^{-1}([-1,1])$

Solution: (1) $f^{-1}(0) = \{x \in R^{\#};\ f(x) = 0\}$

$= \{x \in R^{\#};\ \sin x = 0\}$

$= \{x \in R^{\#};\ \sin x = \sin 0, \sin\pi, \sin 2\pi, -\sin 4\pi, -\sin 2\pi \ldots\}$

$= \{x \in R^{\#};\ x = 0, \pi, 2\pi, -4\pi, -2\pi, \ldots\}$

$= \{-4\pi, -2\pi, -\pi, 0, \pi, 2\pi, 4\pi, \ldots\}$

(2) $f^{-1}(1) = \{x \in R^{\#};\ f(x) = 1\}$

$= \{x \in R^{\#};\ \sin x = 1\}$

$= \{x \in R^{\#};\ \sin x = \sin\frac{\pi}{2}, \sin\frac{5\pi}{2}, \ldots\}$

$= \{x \in R^{\#};\ x = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots\}$

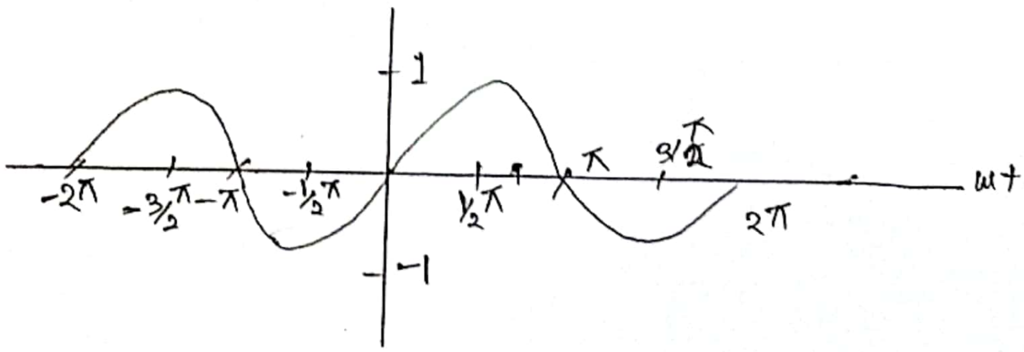
$= \{x \mid x = \frac{\pi}{2} + 2\pi n,$ where, $n \in \mathbb{Z}\}$

(3) $f^{-1}(2) = \{x \in R^{\#};\ f(x) = 2\}$

$= \{x \in R^{\#};\ \sin x = 2\}$

$= \phi$

(4) $f^{-1}([-1,1])$ = The set of all real numbers



11. Let the function $f: R^{\#} \to R^{\#}$ be defined by $f(x) = x^2 + x - 2$

Find: (1) $f^{-1}(10)$

(2) $f^{-1}(4)$

(3) $f^{-1}(-5)$

Solution: (1) $f^{-1}(10) = \{x \in R^{\#};\ x^2 + x - 2 = 10\}$

$= \{x \in R^{\#};\ x^2 + x - 12 = 0\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm \sqrt{1+48}}{2}\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm 7}{2}\}$

$= \{x \in R^{\#};\ x = -4, 3\}$

$= \{-4, 3\}$

(2) $f^{-1}(4) = \{x \in R^{\#};\ x^2 + x - 2 = 4\}$

$= \{x \in R^{\#};\ x^2 + x - 6 = 0\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm \sqrt{1+24}}{2}\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm \sqrt{25}}{2}\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm 5}{2}\}$

$= \{x \in R^{\#};\ x = -3, 2\}$

$= \{-3, 2\}$

(3) $f^{-1}\cancel{(5)}(-5)$

$\therefore f^{-1}(-5) = \{x \in R^{\#};\ x^2 + x - 2 = -5\}$

$= \{x \in R^{\#};\ x^2 + x + 3 = 0\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm \sqrt{1-12}}{2}\}$

$= \phi$

[since $\frac{-1 \pm \sqrt{-11}}{2}$ are not real numbers].

(Ans)

Inverse of a function: Let f be a function of A into B and let $b \in B$. Then the inverse of b denoted by $f^{-1}(b)$ consists of those elements in A which are mapped onto b that is those elements in A which have b as their image. $f: A \to B$ then $f^{-1}(b)$

$= \{x \mid x \in A, f(x) = b\}$

Cartesian product: Two set A and B denoted by A×B is the set of all ordered pairs when a is in A is in B.

Ex: $A = \{1, 2\}$, $B = \{1, 3\}$

$A \times B = \{(1,1), (1,3), (1,2), (2,3)\}$

Question-7 Prove De-Morgan's theorem -

1. $(A \cup B)' = A' \cap B'$

Let, $x \in (A \cup B)'$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in (A' \cap B')$

By definition of subset we get $(A \cup B)' \subseteq A' \cap B'$ — ①

Again,

Let, $y \in A' \cap B'$

$\Rightarrow y \overset{\in}{\cancel{\notin}} A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

By definition of subset we get $A' \cap B' \subseteq (A \cup B)'$ — ②

From ① and ② we get,

$(A \cup B)' = A' \cap B'$

[Proved]

2. $(A \cap B)' = A' \cup B'$

Let, $x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in (A' \cup B')$

By defn of subset we get $(A \cap B)' \subseteq A' \cup B'$ — ①

Again, Let,

$$y \in A' \cup B'$$

Then, $y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

By definition of subset we get $A' \cup B' \subseteq (A \cap B)'$ —— (2)

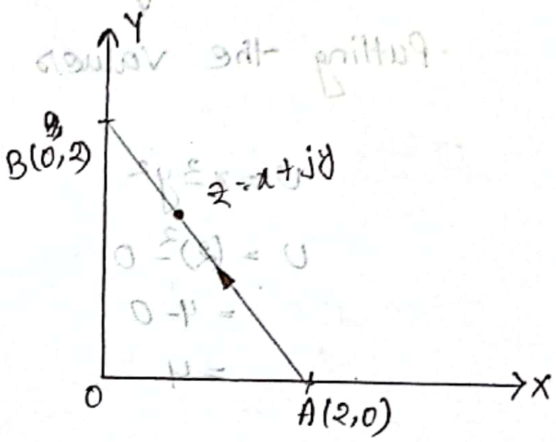
From (1) and (2) we get,

$$(A \cap B)' = A' \cup B'$$

[Proved]

Example-8

If $w = z^2$, find the path traced out by w as z moves along the straight line joining $A(2+0.j)$ and $B(0+2j)$.



Solution:

Given that, $w = f(z) = z^2$

we know that, $w = u + jv$

$z = x + jy$

we have, $w = f(z) = z^2$

or, $u + jv = f(z) = (x+jy)^2$

or, $u + jv = x^2 + 2xjy + j^2y^2$

or, $u + jv = x^2 + j2xy - y^2$ $[\because j^2 = -1]$

or, $u + jv = x^2 - y^2 + j2xy$ ——(i)

Equating the coefficient of real and imaginary part,

$u = x^2 - y^2$ ——(ii)

$v = 2xy$ ——(iii)

Given the point A $(2+0.j)$

(That is, A $(2,0)$ —— (IV)

Here, $x=2,\ y=0$

Putting the values of x and y in (II) and (III)

$$U = x^2 - y^2 \qquad V = 2xy$$
$$U = (2)^2 - 0 \qquad = 2.2.0$$
$$= 4 - 0 \qquad = 4$$
$$= 4$$

$\therefore\ W = U + jV = 4 + j.0$

The image of A is $A'(w = 4 + j.0)$

That is $A'(4,0)$ —— (V)

Again, $B(z = 0 + j2)$

That is $B(0,2)$ —— (VI)

Putting the values of x and y in (II) and (III)

$$U = x^2 - y^2 \qquad V = 2xy$$
$$= 0^2 - 2^2 \qquad = 2.0.2$$
$$= 0 - 4 \qquad = 0$$
$$= -4 \quad \text{(I)}$$

$\therefore\ U = -4$

$\therefore\ W = U + jV = -4 + j.0$

The image of B is $B'(w + jv = -4 + j.0)$

That is $B'(-4,0)$ —— (VII)

We have, $A(2,0)$ and $B(0,2)$

$A: x_1 = 2, y_1 = 0$

$B: x_2 = 0, y_2 = 2$

The equation of the line AB is

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

or, $\frac{y-0}{0-2} = \frac{x-2}{2-0}$

or, $\frac{y}{-2} = \frac{x-2}{2}$

or, $2y = -2(x-2)$

or, $2y = \frac{-2(x-2)}{2}$

or, $y = -(x-2)$

or, $y = -x+2$

$\therefore$ $y = 2-x$ —— (VII)

Putting the value of y in (II)

$\therefore$ $U = x^2 - y^2$

or, $U = x^2 - (2-x)^2$

or, $U = x^2 - (4-4x+x^2)$

or, $U = x^2 - 4 + 4x - x^2$

or, $U = 4x - 4$

or, $U + 4 = 4x$

or, $x = \frac{U+4}{4}$ —— (IX)

Putting the value of y in (iii)

$V = 2xy$

$V = 2x(2-x)$

$V = 4x - 2x^2$ —— (x)

Now, putting the value of x in eqn (x)

$V = 4x - 2x^2$

$\therefore V = 4\left(\frac{U+4}{4}\right) - 2\left(\frac{U+4}{4}\right)^2$

or, $V = U + 4 - 2\left(\frac{U^2+8U+16}{16}\right)$

or, $V = U + 4 - \frac{1}{8}(U^2+8U+16)$

or, $V = U + 4 - \frac{1}{8}U^2 - U - 2$

or, $V = 2 - \frac{1}{8}U^2$

or, $V = \cancel{-\frac{1}{8}(U^2} \; \frac{16-U^2}{8}$

or, $V = -\frac{1}{8}(U^2-16)$

or, $8V = -(U^2-16)$

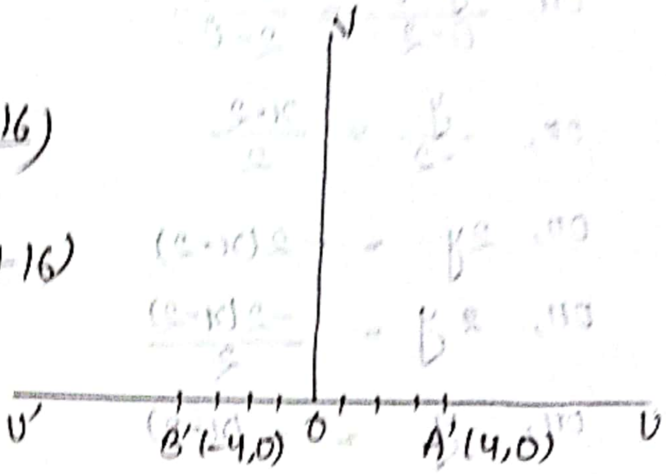
or, $8V = -U^2 + 16$

~~or, $8V = -U^2$~~

or, $-U^2 = 8V - 16$

or, $U^2 = -8(V-2)$

or, $U^2 = -4.2(V-2)$ —— (xi)



The equation (xi) represents an equation of a parabola.

Let, $U = u$ and $V = v - 2$ —— (xii)

From (xii),

when,

$U = 0$ then $u = 0$

and $V = 0$ then

$\Rightarrow 0 = v - 2$

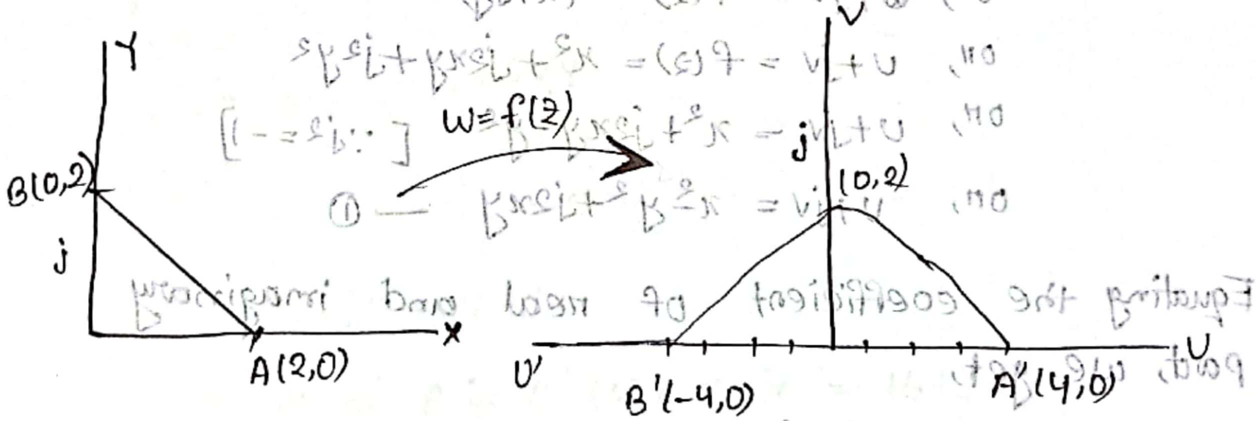
$\therefore v = 2$

$\therefore$ Vertex $= (U, V) = (0, 2)$

And latus rectum $A'B' = 4a$

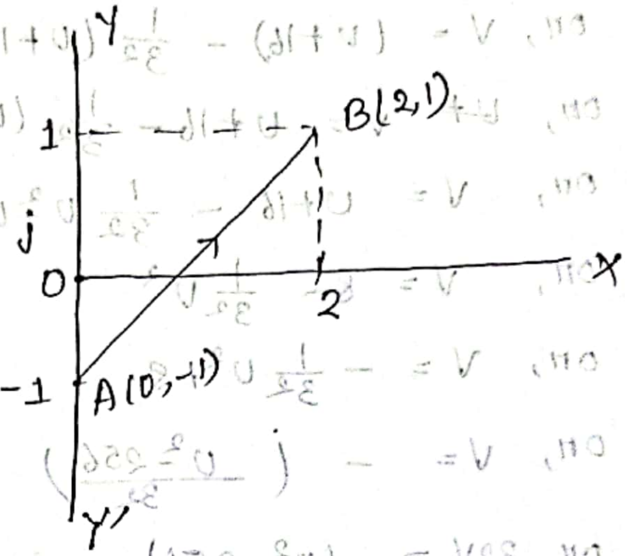
$= 4 \cdot 2$

$= 8$



Example-11 A straight line joining $A(-j)$ and $B(2+j)$ in the z-plane is mapped onto the w-plane by the transformation equation $w = \frac{1}{z}$.

(CT)



Solution: Given that, $w = \left(\frac{1}{z}\right)$

we know that, $z = x + jy$

$w = u + jv$

we have, $w = \frac{1}{z}$

or, $w = \frac{1}{x+jy}$

or, $w = \frac{x-jy}{(x+jy)(x-jy)}$ [Multiplying $x-jy$]

or, $w = \frac{x-jy}{x^2 - jxy + jxy - j^2y^2}$

or, $w = \frac{x-jy}{x^2+y^2}$ [$j^2 = -1$]

or, $u + jv = \frac{x-jy}{x^2+y^2}$

or, $u + jv = \frac{x}{x^2+y^2} - j\frac{y}{x^2+y^2}$ —— (1)

Equating the coefficient of real and imaginary part, we get,

$$U = \frac{x}{x^2+y^2} \quad \text{—— (II)}$$

$$V = \frac{-y}{x^2+y^2} \quad \text{—— (III)}$$

Given, $A(0, -j.1)$

That is $A(0, -1)$ —— (IV)

Here, $x = 0$, $y = -1$

Putting the value of x and y in (II) and (III)

$$U = \frac{x}{x^2+y^2} \qquad V = \frac{-y}{x^2+y^2}$$

$$= \frac{0}{(0)^2+(-1)^2} \qquad = \frac{-(-1)}{(0)^2+(-1)^2}$$

$$= \frac{0}{1} \qquad = \frac{1}{1}$$

$$= 0 \qquad = 1$$

$W = U + jV = 0 + j.1$

The image of A is $A'(W = 0+j1)$ (IV) ——

That is $A'(0,1)$ —— (V)

Again, $B(z = 2+j.1)$

That is $B(2,1)$ —— (VI)

Here, $x = 2$, $y = 1$

Putting the value of x and y in (II) and (III)

$$U = \frac{x}{x^2+y^2} \qquad V = \frac{-y}{x^2+y^2}$$

$$= \frac{2}{(2)^2+(1)^2} \qquad = \frac{-1}{(2)^2+(1)^2}$$

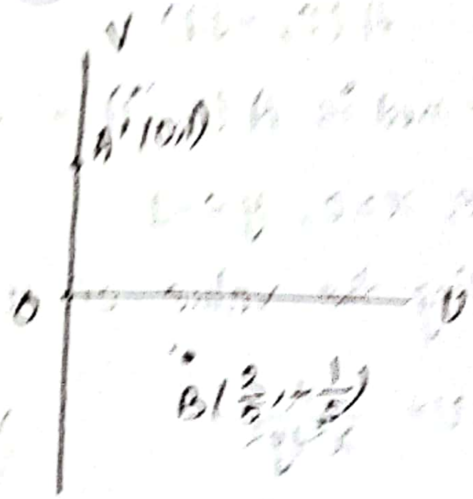
$$= \frac{2}{5} \qquad = -\frac{1}{5}$$

$\therefore w = u + jv$

$= \frac{2}{5} - j\frac{1}{5}$

The image of B is $B'\left(w = \frac{2}{5} - j\frac{1}{5}\right)$

That is $B'\left(\frac{2}{5}, -\frac{1}{5}\right)$ —— (iii)



Given, A(0,−1) and B (2,1)

The equation of the line AB is,

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

or, $\frac{y-(-1)}{-1-1} = \frac{x-0}{0-2}$

or, $\frac{y+1}{-2} = \frac{x-\cancel{0}}{-2}$

or, $y+1 = x$

$\therefore y = x-1$ —— (vii)

Again, Given, $w = \frac{1}{z}$

$\therefore z = \frac{1}{w}$

or, $z = \frac{1}{u+jv}$

or, $z = \frac{u-jv}{(u+jv)(u-jv)}$

or, $z = \frac{u-jv}{u^2-(jv)^2}$

or, $z = \frac{u-jv}{u^2+v^2}$ $(\because j^2 = -1)$

or, $x+jy = \frac{u-jv}{u^2+v^2}$ $[z=x+jy]$

or, $x+jy = \frac{u}{u^2+v^2} - j\frac{v}{u^2+v^2}$ —— (ix)

Equating the coefficient of real and imaginary part, we get

$$x = \frac{u}{u^2+v^2} \,;\quad y = \frac{-v}{u^2+v^2} \text{ —— (x)}$$

Putting the value of x and y in (viii)

$$y = x-1$$

or, $\frac{-v}{u^2+v^2} = \frac{u}{u^2+v^2} - 1$

or, $\frac{-v}{u^2+v^2} = \frac{u-u^2-v^2}{u^2+v^2}$

or, $-v = u-u^2-v^2$

or, $u-u^2-v^2+v=0$

or, $-u+u^2+v^2-v=0$

or, $u^2-u+v^2-v=0$

or, $(u^2-u)+(v^2-v)=0$

or, $u^2-2\cdot u\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+v^2-2\cdot v\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2=0$

or, $u^2-2\cdot u\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2+v^2-2\cdot v\cdot\frac{1}{2}+\left(\frac{1}{2}\right)-\frac{1}{4}-\frac{1}{4}=0$

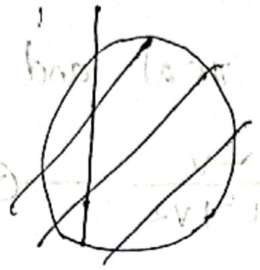
or, $\left(u-\frac{1}{2}\right)^2+\left(v-\frac{1}{2}\right)^2-\frac{2}{4}=0$

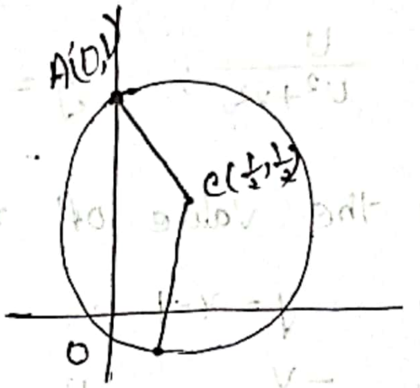
or, $\left(u-\frac{1}{2}\right)^2+\left(v-\frac{1}{2}\right)^2-\frac{1}{2}=0$

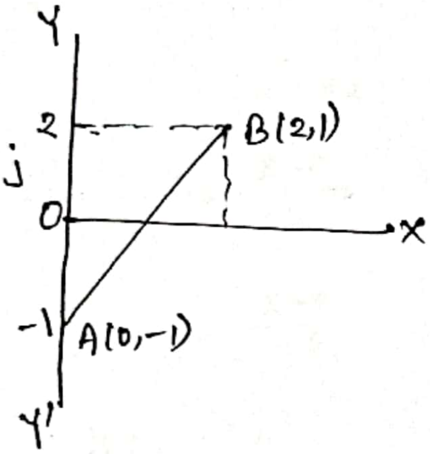
or, $\left(u-\frac{1}{2}\right)^2+\left(v-\frac{1}{2}\right)^2=\frac{1}{2}$

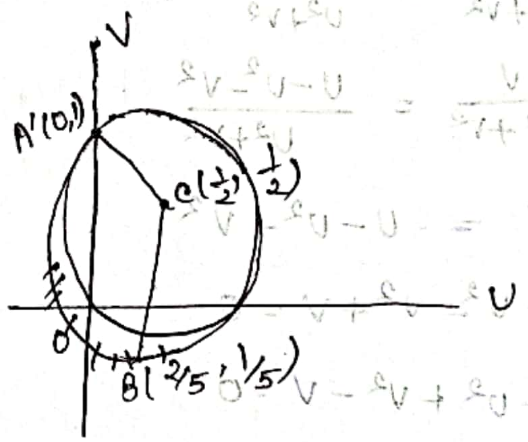
or, $\left(u-\frac{1}{2}\right)^2+\left(v-\frac{1}{2}\right)^2=\left(\frac{1}{\sqrt{2}}\right)^2$ —— (xi)

The equation (xi) represents an equation of a circle whose centre $c\left(\frac{1}{2},\frac{1}{2}\right)$ and radius $=\frac{1}{\sqrt{2}}$









Justification of radius of the circle:

we have $A'(w = 0 + j.1)$ that is the coordinate of $A'(0,1)$ and the center $c\left(\frac{1}{2},\frac{1}{2}\right)$

$$\therefore A'c = \sqrt{(U_1-U_2)^2+(V_1-V_2)^2}$$

$$A'c = \sqrt{(0-\tfrac{1}{2})^2+(1-\tfrac{1}{2})^2}$$

$$A'c = \sqrt{(\tfrac{1}{2})^2+(\tfrac{1}{2})^2}$$

$$A'C = \sqrt{\frac{1}{4}+\frac{1}{4}}$$

$$= \sqrt{2/4}$$

$$= \sqrt{1/2}$$

$$\therefore A'C = \frac{1}{\sqrt{2}}$$

[Proved]

$$\therefore \text{Radius} = \frac{1}{\sqrt{2}}$$

we have $B'\left(\frac{3}{5}, -\frac{1}{5}\right)$ and $C\left(\frac{1}{2}, \frac{1}{2}\right)$

$$\therefore B'C = \sqrt{(u_1-u_2)^2+(v_1-v_2)^2}$$

$$= \sqrt{\left(\frac{3}{5}-\frac{1}{2}\right)^2+\left(-\frac{1}{5}-\frac{1}{2}\right)^2}$$

$$= \sqrt{\left(\frac{4-5}{10}\right)^2+\left(\frac{-2-5}{10}\right)^2}$$

$$= \sqrt{\left(\frac{-1}{10}\right)^2+\left(\frac{-7}{10}\right)^2}$$

$$= \sqrt{\frac{1}{100}+\frac{49}{100}}$$

$$= \sqrt{\frac{50}{100}}$$

$$= \sqrt{\frac{1}{2}}$$

$$\therefore B'C = \frac{1}{\sqrt{2}}$$

[Proved]

$$\therefore \text{Radius} = \frac{1}{\sqrt{2}}.$$

12. A circle in the z-plane has its centre at $z=3$ and a radius of 2 units. Determine its image in the w-plane when transformation by $w=\frac{1}{z}$.

Solution: Here, $z=3$

radius $=2$

where c is the circle, $|z-3|=2$

we have,

$$z=x+jy$$

or, $z-3=x+jy-3$

or, $z-3=\sqrt{(x-3)^2+y^2}$

Given, $|z-3|=2$

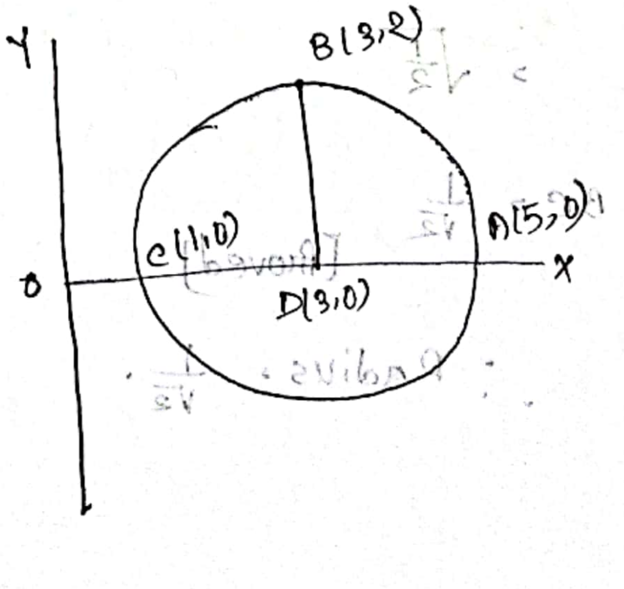
or, $|z-3|=\sqrt{(x-3)^2+y^2}=2$

$\therefore \sqrt{(x-3)^2+y^2}=2$

$\therefore (x-3)^2+y^2=2^2$

$\therefore (x-3)^2+(y-0)^2=2^2$

This is the equation of the circle whose centre (3,0) and radius 2.



Given that,

$z=3$

$x+jy=3$

$\Rightarrow x+jy=3+0.j$ —— ①

Equating the coefficient of real and imaginary part, we get,

$x=3$

$y=0$

$\therefore (x,y)=(3,0)$

and, given, radius $=2$

So, Equation of the circle, —— ②

$(x-3)^2+(y-0)^2=2^2$ $[\because (x-a)^2+(y-b)^2=r^2]$

That is centre of the circle is (3,0) and radius 2

$(x-3)^2+y^2=4$

or, $x^2-6x+9+y^2=4$

or, $x^2+y^2-6x+5=0$ —— ③

Again, Given,

$w=\frac{1}{z}$

or, $w=\frac{1}{x+jy}$ $[z=x+jy]$

or, $w=\frac{x-jy}{(x+jy)(x-jy)}$ [Multiplying by $x-jy$]

or, $w=\frac{x-jy}{x^2-jxy+jxy-j^2y^2}$

or, $w=\frac{x-jy}{x^2-j^2y^2}$

or, $w=\frac{x-jy}{x^2+y^2}$ $[j^2=-1]$

or, $u+jv = \dfrac{x-jy}{x^2+y^2}$ $[w=u+jv]$

or, $u+jv = \dfrac{x}{x^2+y^2} - j\dfrac{y}{x^2+y^2}$ —(4)

Equating the coefficient of real and imaginary part, we get,

$$u = \frac{x}{x^2+y^2} \quad \text{—(5)}$$

$$v = \frac{-y}{x^2+y^2} \quad \text{—(6)}$$

Again, Given

$$w = \frac{1}{z}$$

or, $z = \dfrac{1}{w}$

or, $z = \dfrac{1}{u+jv}$ $[w=u+jv]$

$$= \frac{u-jv}{(u+jv)(u-jv)}$$

$$= \frac{u-jv}{u^2-(jv)^2}$$

$$= \frac{u-jv}{u^2+v^2} \quad [j^2=-1]$$

or, $x+jy = \dfrac{u-jv}{u^2+v^2}$

or, $x+jy = \dfrac{u}{u^2+v^2} - j\dfrac{v}{u^2+v^2}$ —(7)

Equating the coefficient of real and imaginary part we get,

$$x = \frac{u}{u^2+v^2}\ ;\ y = \frac{-v}{u^2+v^2} \quad \text{—(8)}$$

Substituting the values of x and y in ③

$$x^2+y^2-6x+5=0$$

or, $\left(\frac{U}{U^2+V^2}\right)+\left(\frac{-V}{U^2+V^2}\right)^2-6\left(\frac{U}{U^2+V^2}\right)+5=0$

or, $\frac{U^2}{(U^2+V^2)^2}+\frac{V^2}{(U^2+V^2)^2}-\frac{6U}{U^2+V^2}+5=0$

or, $\frac{U^2+V^2}{(U^2+V^2)^2}-\frac{6U}{U^2+V^2}+5=0$

or, $\frac{1}{U^2+V^2}-\frac{6U}{U^2+V^2}+5=0$

or, $\frac{1-6U+5(U^2+V^2)}{U^2+V^2}=0$

or, $1-6U+5(U^2+V^2)=0$

or, $5(U^2+V^2)-6U+1=0$

or, $U^2+V^2-\frac{6}{5}U+\frac{1}{5}=0$ [Dividing by 5]

or, $U^2+V^2-2\cdot\frac{3}{5}U+2\cdot0\cdot V+\frac{1}{5}=0$

or, $U^2+V^2+2\left(-\frac{3}{5}\right)U+2\cdot0\cdot V+\frac{1}{5}=0$ ——— ⑨

We know the general equation of the circle is

$$x^2+y^2+2gx+2fy+c=0$$

whose centre is $(-g,-f)$ and radius is $\sqrt{g^2+f^2-c}$

Hence from ⑨

Here, $g=-\frac{3}{5}$, $f=0$ and $c=\frac{1}{5}$

The centre of the new circle of ⑨ is $(-g,-f)$..

$$\therefore(-g,-f)=\left(-\left(-\frac{3}{5}\right),-0\right)=\left(\frac{3}{5},0\right)$$

That is, centre of new circle in the w plane is $D\left(\frac{3}{5}, 0\right)$.

Radius is $\sqrt{g^2+f^2-c}$

$$= \sqrt{\left(-\frac{3}{5}\right)^2+0^2-\frac{1}{5}}$$

$$= \sqrt{\frac{9}{25}+0-\frac{1}{5}}$$

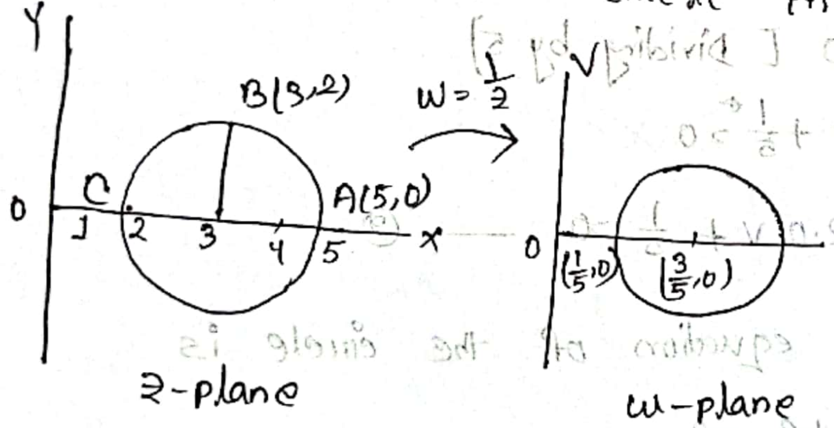
$$= \sqrt{\frac{9}{25}-\frac{1}{5}}$$

$$= \sqrt{\frac{9-5}{25}}$$

$$= \sqrt{\frac{4}{25}}$$

$$= \frac{2}{5}$$

That is, radius of new circle in the w-plane is $\frac{2}{5}$.



Taking three sample points A, B, C as shown, that is:

A(5, 0), B(3, 2), C(1, 0)

Putting the values of A(5, 0), B(3, 2), C(1, 0) in ~~(4)~~ ~~&~~ (5) and (6)

$U = \frac{x}{x^2+y^2}$, $V = \frac{-y}{x^2+y^2}$

For, $A(5,0)$; $U = \frac{x}{x^2+y^2} = \frac{5}{5^2+0^2} = \frac{5}{25+0} = \frac{1}{5}$

For, $A(5,0)$; $V = \frac{-y}{x^2+y^2} = \frac{-0}{5^2+0^2} = \frac{0}{25+0} = 0$

$\therefore$ For $A(5,0)$; $W = U+jV = \frac{1}{5}+j\cdot 0$

The image of A is $A'(W=U+jV = \frac{1}{5}+j\cdot 0) = \frac{1}{5}+j\cdot 0$

That is $A'(\frac{1}{5}, 0)$ —— (10)

For $B(3,2)$; $U = \frac{x}{x^2+y^2} = \frac{3}{3^2+2^2} = \frac{3}{9+4} = \frac{3}{13}$

For $B(3,2)$; $V = \frac{-y}{x^2+y^2} = \frac{-2}{3^2+2^2} = \frac{-2}{9+4} = \frac{-2}{13}$

$\therefore$ For $B(3,2)$; $W = U+jV = \frac{3}{13}+j\left(\frac{-2}{13}\right)$

The image of B is $B'(W = U+jV = \frac{3}{13}+j\left(-\frac{2}{13}\right)) = \frac{3}{13} - j\cdot\frac{2}{13}$

That is $B'(\frac{3}{13}, -\frac{2}{13})$ —— (11)

For $C(1,0)$; $U = \frac{x}{x^2+y^2} = \frac{1}{1^2+0^2} = \frac{1}{1} = 1$

For $C(1,0)$; $V = \frac{-y}{x^2+y^2} = \frac{-0}{1^2+0^2} = \frac{-0}{1} = 0$

For $C(1,0)$; $W = U+jV$

$= 1+j(0)$

The image of C is $C'(W = U+jV = 1+j\cdot 0 = 1+j\cdot 0)$

That is $C'(1,0)$ —— (12)

Cauchy–Riemann Test

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \cancel{\frac{\delta v}{\delta x} = \frac{\delta u}{\delta y}} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

Example-15

Determine the function $w = f(z) = z^2 - 4$ is analytic or not.

Solution: Given that,

$$w = f(z) = z^2 - 4$$

or, $u + jv = (x+jy)^2 - 4 \quad [w = u+jv\ ;\ z = x+jy]$

or, $u + jv = x^2 + 2\cdot x\cdot iy + (jy)^2 - 4$

or, $u + jv = x^2 + 2xjy - y^2 - 4$

or, $u + jv = x^2 - y^2 - 4 + j2xy$ — ①

Now, equating real and imaginary parts from both sides we get,

$u = x^2 - y^2 - 4$ — ②

and $v = 2xy$ — ③

From, ②

$$u = x^2 - y^2 - 4$$

$$\therefore \frac{\delta u}{\delta x} = \frac{\delta}{\delta x}(x^2 - y^2 - 4)$$

$$\therefore \frac{\delta u}{\delta x} = 2x$$

Again, $\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}(x^2 - y^2 - 4)$

$$= -2y$$

From ③,

$$v = 2xy$$

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}(2xy)$$

$$\therefore \frac{\delta v}{\delta y} = 2x$$

$$\therefore \frac{\delta v}{\delta x} = \frac{\delta}{\delta x}(2xy)$$

$$= 2y$$

we have, cauchy-Riemann equation are

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \cancel{\frac{\delta y}{\delta x} \quad \frac{\delta u}{} \quad \delta y} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

$$\text{or, } 2x = 2x \quad \text{and} \quad \text{or } -2y = -2y$$

$\therefore$ L.H.S = R.H.S

As the given function satisfies the cauchy-Riemann

So, the function is analytic.

(Ans)

Example-17 Determine the function $w = f(z) = e^z = e^x(\cos y + i\sin y)$ is analytic or not. Also find its derivative that is $f'(z) = ?$

Solution: We have,

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Put $x = ix$,

$$e^{ix} = 1 + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots$$

$[i^2 = -1;\ i^3 = i^2 i = -i;\ i^4 = i^2 \cdot i^2 = (-1)(-1) = +1;\ i^5 = i^4 \cdot i = i]$

$$= 1 + \frac{ix}{1!} + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \left(\frac{ix^1}{1!} - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \cdots\right)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\left[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right.$$
$$\left. \sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$\therefore e^{ix} = \cos x + i\sin x$

$e^{iy} = \cos y + i\sin y$ —— ①

we have,

$w = f(z) = e^z = e^{x+iy}$ $[z = x+iy]$

$\Rightarrow w = f(z) = e^z = e^x \cdot e^{iy}$

$\Rightarrow w = f(z) = e^z = e^x \cdot (\cos y + i\sin y)$ [From ① $e^{iy} = \cos y + i\sin y$]

$\Rightarrow U+iV = e^x(\cos y + i\sin y)$ [$w = U+iV$]

$\Rightarrow U+iV = e^x\cos y + i\,e^x\sin y$ —— (ii)

Equating real and imaginary part we get,

$U = e^x\cos y$ —— (iii)

$V = e^x\sin y$ —— (iv)

From (iii) $U = e^x\cos y$

Differentiating (iii) partially with respect to x

$$\frac{\delta U}{\delta x} = \frac{\delta}{\delta x}(e^x\cos y)$$

$$\Rightarrow \frac{\delta U}{\delta x} = e^x\frac{\delta}{\delta x}\cos y + \cos y\frac{\delta}{\delta x}e^x$$

$$\Rightarrow \frac{\delta U}{\delta x} = e^x.0 + \cos y\, e^x$$

$$\Rightarrow \frac{\delta U}{\delta x} = \cos y\, e^x.$$

$$\therefore \frac{\delta U}{\delta x} = e^x\cos y \quad \text{—— (v)}$$

Again, differentiating (iii) partially with respect to y

$$\frac{\delta U}{\delta y} = \frac{\delta}{\delta y}(e^x\cos y)$$

$$\Rightarrow \frac{\delta U}{\delta y} = e^x\frac{\delta}{\delta y}\cos y + \cos y\frac{\delta}{\delta y}e^x$$

$$\Rightarrow \frac{\delta U}{\delta y} = e^x(-\sin y) + 0$$

$$\therefore \frac{\delta U}{\delta y} = -e^x\sin y \quad \text{—— (vi)}$$

From (iv), $V = e^{x} \sin y$

Differentiating (iv) partially with respect to x

$$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}(e^{x} \sin y)$$

$$\Rightarrow \frac{\delta v}{\delta x} = e^{x} \frac{\delta}{\delta x} \sin y + \sin y \frac{\delta}{\delta x} e^{x}$$

$$\Rightarrow \frac{\delta v}{\delta x} = 0 + e^{x} \sin y$$

$$\therefore \frac{\delta v}{\delta x} = e^{x} \sin y \quad \text{— (vii)}$$

Again, Differentiating (iv) partially with respect to y

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}(e^{x} \sin y)$$

$$\Rightarrow \frac{\delta v}{\delta y} = e^{x} \frac{\delta}{\delta y} \sin y + \sin y \frac{\delta}{\delta y} e^{x}$$

$$\Rightarrow \frac{\delta v}{\delta y} = e^{x} \cos y + 0$$

$$\therefore \frac{\delta v}{\delta y} = e^{x} \cos y \quad \text{— (viii)}$$

We have cauchy - Riemann equation,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \quad \text{— (ix)}$$

Putting the values of $\frac{\delta u}{\delta x}$, $\frac{\delta v}{\delta y}$, $\frac{\delta u}{\delta y}$ and $\frac{\delta v}{\delta x}$ in eqn (ix)

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

$$\Rightarrow e^{x} \cos y = e^{x} \cos y \qquad \Rightarrow -e^{x} \sin y = -e^{x} \sin y$$

$\therefore$ L.H.S = R.H.S $\qquad$ L.H.S = R.H.S

Since the Cauchy-Riemann equations are satisfied by the function $w = f(z) = e^x(\cos y + i\sin y)$. Hence the function $w = f(z) = e^x(\cos y + i\sin y)$ is analytic.

2nd part.

We have from (ii).

$$w = f(z) = u + iv = e^x\cos y + i e^x \sin y$$

$$w = f(z) = u + iv \quad \text{—— (x)}$$

Differentiating (x) with respect to x

$$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$$

$$\Rightarrow f'(z) = e^x\cos y + i e^x \sin y$$

$$\Rightarrow f'(z) = e^x(\cos y + i\sin y)$$

$$\Rightarrow f'(z) = e^x e^{iy} \quad [\text{From (i)}]$$

$$\Rightarrow f'(z) = e^{x+iy}$$

$$\Rightarrow f'(z) = e^z$$

(Ans)

Example-19 Derive Laplace's equation from cauchy-Riemann equations.

Solution:

We have cauchy Riemann equations are,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{and} \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

That is,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad \text{--- (i)}$$

$$\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \quad \text{--- (ii)}$$

Differenting (i) with respect to x we get,

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta x}\left(\frac{\delta v}{\delta y}\right)$$

$$\Rightarrow \frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 v}{\delta x \delta y} \quad \text{--- (iii)}$$

Differentiating (i) with respect to y we get,

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta y}\left(\frac{\delta v}{\delta y}\right)$$

$$\Rightarrow \frac{\delta^2 u}{\delta y \delta x} = \frac{\delta^2 v}{\delta y^2}$$

$$\Rightarrow \frac{\delta^2 v}{\delta y^2} = \frac{\delta^2 u}{\delta y \delta x} \quad \text{--- (iv)}$$

Again, differentiating (ii) with respect to x we get

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta x}\left(-\frac{\delta v}{\delta x}\right)$$

$$\Rightarrow \frac{\delta^2 u}{\delta x \delta y} = -\frac{\delta^2 v}{\delta x^2} \quad \text{--- } \cancel{\text{(v)}}$$

$$\Rightarrow \frac{\delta^2 v}{\delta x^2} = -\frac{\delta^2 u}{\delta x \delta y} \quad \text{--- (v)}$$

Again, differentiating (II) with respect to y we get

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta y}\left(-\frac{\delta v}{\delta x}\right)$$

$$\Rightarrow \frac{\delta^2 u}{\delta y^2} = -\frac{\delta^2 v}{\delta y \delta x} \quad \text{—— (VI)}$$

Adding (III) and $\cancel{\text{(IV)}}$ (VI)

$$\cancel{\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} =}$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{\delta^2 v}{\delta x \delta y} - \frac{\delta^2 v}{\delta x \delta y}$$

$$\therefore \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \quad \text{—— (VII)}$$

Adding (IV) and (V)

$$\frac{\delta^2 v}{\delta y^2} + \frac{\delta^2 v}{\delta x^2} = \frac{\delta^2 u}{\delta y \delta x} - \frac{\delta^2 u}{\delta x \delta y}$$

$$\therefore \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0 \quad \text{—— (VIII)}$$

The equation no (VII) and (VIII) are called Laplace's equation.

Harmonic function: Any function which Satisfies the Laplace's equation is known as a harmonic function.

If $f(z) = u+jv$ is any analytic function, then u and v are both harmonic function.

A function $f(x,y,z)$ is called a harmonic function if its second-order partial derivatives exist and if it Satisfies LapLace's equation $\frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} + \frac{\delta^2 f}{\delta z^2} = 0$

Example-20 Prove that $U = x^2-y^2$ and $V = \frac{y}{x^2+y^2}$ are harmonic functions of (x,y).

Solution: Given that,

$$U = x^2-y^2 \text{ --- (i)}$$

Differentiating (i) with respect to x

$$\frac{\delta U}{\delta x} = \frac{\delta}{\delta x}(x^2-y^2)$$

$$\Rightarrow \frac{\delta U}{\delta x} = 2x \text{ --- (ii)}$$

Again differentiating (ii) with respect to x

$$\frac{\delta}{\delta x}\left(\frac{\delta U}{\delta x}\right) = \frac{\delta}{\delta x}(2x)$$

$$\text{or, } \frac{\delta^2 U}{\delta x^2} = 2 \text{ --- (iii)}$$

Now, differentiating (i) with respect to y

$$\frac{\delta U}{\delta y} = \frac{\delta}{\delta y}(x^2-y^2)$$

$$\Rightarrow \frac{\delta v}{\delta y} = -2y \quad \text{— (iv)}$$

Again, differentiating (iv) with respect to y

$$\frac{\delta}{\delta y}\left(\frac{\delta v}{\delta y}\right) = \frac{\delta}{\delta y}(-2y)$$

$$\Rightarrow \frac{\delta^2 v}{\delta y^2} = -2 \quad \text{— (v)}$$

Adding (iii) and (v)

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 2 - 2 = 0$$

Since, $u(x,y) = x^2 - y^2$ satisfies Laplace's equation. Here,
$u(x,y) = x^2 - y^2$ is a harmonic function.

Now, given,

$$V = \frac{y}{x^2+y^2} \quad \text{— (vi)}$$

Differentiating (vi) with respect to x

$$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{d}{dx}(y) - y\frac{d}{dx}(x^2+y^2)}{(x^2+y^2)^2} \quad \left[\frac{d}{dx}\left(\frac{U}{V}\right) = \frac{V\frac{d}{dx}U - U\frac{d}{dx}V}{V^2}\right]$$

$$= \frac{(x^2+y^2).0 - y(2x+0)}{(x^2+y^2)^2}$$

$$= \frac{0-2xy}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2} \quad \text{— (vii)}$$

Differentiating (VII) with respect to x

$$\frac{\delta}{\delta x}\left(\frac{\delta v}{\delta x}\right) = \frac{\delta}{\delta x}\left\{\frac{-2xy}{(x^2+y^2)^2}\right\}$$

$$\Rightarrow \frac{\delta^2 v}{\delta x^2} = -\left[\frac{(x^2+y^2)^2\frac{\delta}{\delta x}(2xy) - (2xy)\frac{\delta}{\delta x}(x^2+y^2)^2}{\{(x^2+y^2)^2\}^2}\right]$$

$$= -\left[\frac{(x^2+y^2)^2 \cdot 2y - (2xy)\times 2(x^2+y^2)^{2-1}\frac{\delta}{\delta x}(x^2+y^2)}{(x^2+y^2)^4}\right]$$

$$= \frac{-(x^2+y^2)^2 \cdot 2y + 4xy(x^2+y^2)(2x)}{(x^2+y^2)^4}$$

$$= \frac{-(x^2+y^2)^2 \cdot 2y + 8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\left[-(x^2+y^2)2y + 8x^2y\right]}{(x^2+y^2)^4}$$

$$= \frac{-2x^2y - 2y^3 + 8x^2y}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3} \quad \text{—— (VIII)}$$

Again,

Differentiating (VI) with respect to y

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}\left(\frac{y}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\delta}{\delta y}(y) - y\frac{\delta}{\delta y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{——— (ix)}$$

Again, differentiating (ix) with respect to y

$$\frac{\delta}{\delta y}\left(\frac{\delta v}{\delta y}\right) = \frac{\delta}{\delta y}\left[\frac{x^2-y^2}{\{(x^2+y^2)^2\}^2}\right]$$

$$\Rightarrow \frac{\delta^2 v}{\delta y^2} = \frac{(x^2+y^2)^2\frac{\delta}{\delta y}(x^2-y^2) - (x^2-y^2)\frac{\delta}{\delta y}(x^2+y^2)^2}{\{(x^2+y^2)^2\}^2}$$

$$= \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)\frac{\delta}{\delta} \times 2(x^2+y^2)^{2-1}\frac{\delta}{\delta y}(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)\cdot 2(x^2+y^2)\cdot 2y)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y) - 4y(x^2-y^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)(-2y) - 4y(x^2-y^2)}{(x^2+y^2)^3}$$

$$= \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2+y^2)^3}$$

$$= \frac{-6x^2y + 2y^3}{(x^2+y^2)^3} \quad \text{——— (x)}$$

Adding (viii) and (x)

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3} + \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3 - 6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$\therefore \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0$$

Since, $v(x,y) = \frac{y}{x^2+y^2}$ satisfies the Laplace's equation

Hence, $v(x,y) = \frac{y}{x^2+y^2}$ is a harmonic function.

Home Task - I, II, III, IV

① Prove that $f(z)=|z|^2$ is not harmonic function but $f(z)=\ln(|z|)^2$ is harmonic.

Solution:

$$z=x+iy$$

$$|z|=|x+iy|=\sqrt{x^2+y^2}$$

$$|z|^2=|(x+iy)|^2=(\sqrt{x^2+y^2})^2=x^2+y^2$$

Let, $u=x^2+y^2$ —— ①

Now differentiating eqn ① with respect to x.

we get, $u=x^2+y^2$

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2)$$

$$\Rightarrow \frac{\partial u}{\partial x}=2x \text{ —— (II)}$$

Again Differentiating eqn (II) with respect to x we get,

$$\frac{\partial u}{\partial x}=2x$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}(2x)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}=2 \text{ —— (III)}$$

Now differentiating eqn ① with respect to y. we get.

$$u=x^2+y^2$$

$$\Rightarrow \frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2)$$

$$\Rightarrow \frac{\partial u}{\partial y}=2y \text{ —— (IV)}$$

Again Differentiating eqn (iv) with respect to y. we get,

$$\frac{\partial u}{\partial y} = 2y$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(2y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = 2 \qquad \text{(v)}$$

Now adding eqn (iii) and (v), we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2+2=4 \neq 0$$

Since the function $u(x,y) = x^2+y^2$ doesn't satisfied the Laplace's equation. Since this function is not harmonic.

Again, Given that,

$$f(z) = \ln(|z|)^2$$

$\Rightarrow \;\; = \ln(|x+iy|)^2$

$= \ln(x^2+y^2)$

let $v = \ln(x^2+y^2)$ ———— (VI)

Now differentiating eq^n (VI) with respect to x.

we get,

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\ln(x^2+y^2)$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} \text{ ———— (VII)}$$

Again differentiating eq^n (VII) with respect to x.

we get,

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2}$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right)$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{(x^2+y^2)\frac{\partial}{\partial x}(2x) - 2x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{2(x^2+y^2) - 2x \cdot 2x}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} = \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} \quad \text{——— (viii)}$$

Now differentiating eq^n (vi) with respect to y.

we get,

$$v = \ln(x^2+y^2)$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial}{\partial y} \ln(x^2+y^2)$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} \quad \text{——— (ix)}$$

Again Differentiating eq^n (ix) with respect to y.

we get,

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2}$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2} \text{ ——— (x)}$$

Now adding eqn (viii) and (x) we get,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2(x^2+y^2)-4x^2}{(x^2+y^2)^2} + \frac{2(x^2+y^2)-4y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2x^2+2y^2-4x^2+2x^2+2y^2-4y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Since the function $f(z) = \ln(|z|)^2$ satisfied the Laplace's equation. Hence the function $f(z) = \ln(|z|)^2$ is a harmonic function.

[Proved]

(11) If $f(x,y,z) = x^2+y^2-2z^2$ harmonic? what about $f(x,y,z) = x^2-y^2+z^2$?

Given that,

$$f(x,y,z) = x^2+y^2-2z^2 \text{ ——— (1)}$$

$$\text{let, } u = x^2+y^2-2z^2 \text{ ——— (1)}$$

Now differentiating eqn (1) with respect to ~~x we get~~, x, we get,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2-2z^2)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x \text{ ——— (11)}$$

Again Differentiating eqn (11) with respect to x. we get,

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(2x)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 2 \text{ ——— (111)}$$

Now Differentiating eqn (1) with respect to y. we get.

$$\frac{\partial u}{\partial y} = 2y \text{ ——— (IV)}$$

Again Differentiating eqn (IV) with respect to y.

we get,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(2y)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = 2 \text{ ——— (V)}$$

Now differentiating eqn (I) with respect to z.

we get.

$$\frac{\partial u}{\partial z} = -4z \text{ ——— (VI)}$$

Again differentiating eqn (VI) with respect to z

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}(-4z)$$

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} = 4 \text{ ——— (VII)}$$

Now adding eqn (III), (V) and (VII) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 2+2-4$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

So this function $f(x,y,z)=x^2+y^2-2z^2$ statisfied the Laplace's equation. Hence this function $f(x,y,z)=x^2+y^2-2z^2$ is harmonic.

Again, Given that,

$$f(x,y,z)=x^2-y^2+z^2$$

Let, $v=x^2-y^2+z^2$ ——— (VIII)

Now Differentiating eqn (VIII) with respect to x. we get,

$$\frac{\partial v}{\partial x}=2x \text{ ——— (IX)}$$

Again Differentiating eqn (IX) with respect to x,

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right)=\frac{\partial}{\partial x}(2x)$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2}=2 \text{ ——— (X)}$$

Now differentiating eqn (IX) with respect to y. we get,

$$\frac{\partial v}{\partial y}=-2y \text{ ——— (XI)}$$

Again differentiating eqn (xi) with respect to y we get,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}(-2y)$$

$$\Rightarrow \frac{\partial^2 v}{\partial y^2} = -2 \text{ ——— (xii)}$$

Now Differentiating eqn (viii) with respect to z

$$\frac{\partial v}{\partial z} = 2z \text{ ——— (xiii)}$$

Again Differentiating eqn (xiii) with respect to z. we get,

$$\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right) = \frac{\partial}{\partial z}(2z)$$

$$\Rightarrow \frac{\partial^2 v}{\partial z^2} = 2 \text{ ——— (xiv)}$$

Now adding eqn (x), (xii) and (xiv)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 2-2+2$$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 2 \neq 0$$

(III) Show that the function $u(x,y) = 3x^3 - 9xy^2$ is harmonic.

Solution:

Given that,

$$u(x,y) = 3x^3 - 9xy^2 \qquad \text{(I)}$$

Differentiating eqⁿ (I) with respect to x, we get,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(3x^3 - 9xy^2)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 9x^2 - 9y^2 \qquad \text{(II)}$$

Again Differentiating eqⁿ (II) with respect to x. we get,

$$\frac{\partial u}{\partial x} = 9x^2 - 9y^2$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(9x^2 - 9y^2)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 18x \quad \text{(III)}$$

Differentiating eqn (I) with respect to y. we get,

$$u = 3x^3 - 9xy^2$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(3x^3 - 9xy^2)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -18xy \quad \text{(IV)}$$

Again differentiating eqn (IV) with respect to y.

we get,

$$\frac{\partial u}{\partial y} = -18xy$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}(18xy)$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -18x \quad \text{(V)}$$

Now Adding eqn (III) and (V). we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 18x - 18x$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since $u(x,y) = 3x^3 - 9xy^2$ satisfies Laplace's equation

Hence $u(x,y) = 3x^3 - 9xy^2$ is a harmonic function.

[Proved]

(iv) verify $u(x,y) = x^3 - 3xy^2 - 5y$ is harmonic?

Solution:

Given that,

$$u(x,y) = x^3 - 3xy^2 - 5y \quad \text{--- (i)}$$

~~Differen~~ Differentiating eqn (i) with respect to x

we get,

$$u = x^3 - 3xy^2 - 5y$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^3 - 3xy^2 - 5y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \text{--- (ii)}$$

Again Differentiating eqn (ii) with respect to x.

we get,

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 - 3y^2)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x, \quad \text{--- (iii)}$$

Now Differentiating eqn (i) with respect to y

we get,

$$u = x^3 - 3xy^2 - 5y$$

$$\Rightarrow \frac{\partial u}{\partial y} = -6xy - 5 \quad \text{--- (iv)}$$

Again ⇒ differentiating eqⁿ (iv) with respect to y. we get,

$$\frac{\partial u}{\partial y} = -6xy - 5$$

$$\Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(-6xy-5)$$

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$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x \quad \text{——— (v)}$$

Now adding eqⁿ (iii) and (v) we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x$$

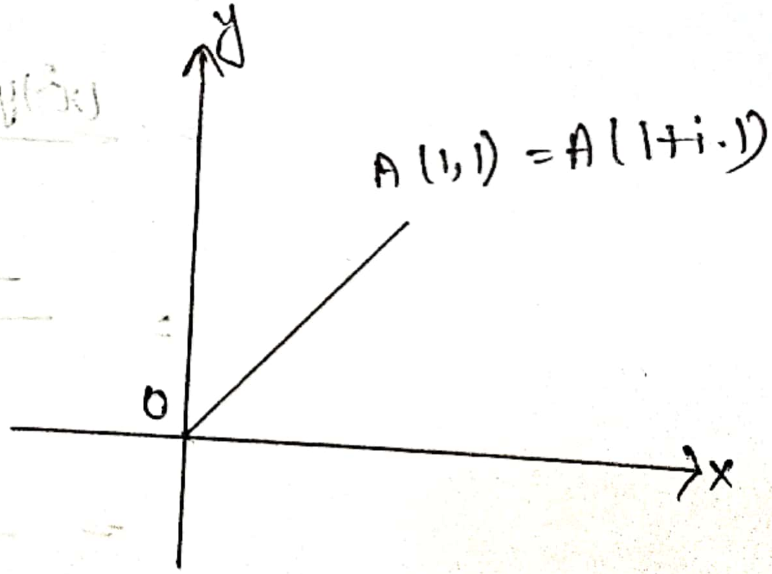
$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since $u(x,y) = x^3 - 3xy^2 - 5y$ satisfies Laplace's equation. Hence $u(x,y) = x^3 - 3xy^2 - 5y$ is a harmonic function.

(22) Find the value of integral $\int_0^{1+i} (x - y + ix^2)\, dz$

(a) Along straight line form $z = 0$ to $z = 1+i$

(b) Along real Axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1+i$



(a) Along OA line:

Given that,

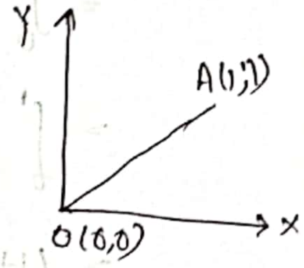
$$z = 0$$
$$x + iy = 0 \qquad [z = x+iy]$$
$$x + iy = 0 + i.0$$

Equating real and imaginary part, $x = 0$, $y = 0$

That is coordinate of $O(0,0)$

Again, Given that

$$z = 1+i$$
$$x+iy = 1+i1 \qquad [z = x+iy]$$



Equating real and imaginary part, $x = 1$ and $y = 1$

That is coordinate of $A(1,1)$

The equation of straight line OA passing through $O(0,0)$ and $A(1,1)$ is,

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

$$\Rightarrow \frac{y-0}{0-1} = \frac{x-0}{0-1}$$

$$\Rightarrow \frac{y}{-1} = \frac{x}{-1}$$

$$\Rightarrow y = x \qquad \text{——— (1)}$$

we have,

$$z = x+iy$$
$$z = x+ix \qquad [y = x] \text{ from (1)}$$

$\Rightarrow \frac{dz}{dx} = \frac{d}{dx}(x+ix)$

$\Rightarrow \frac{dz}{dx} = 1 + i.1$

$\Rightarrow dz = (1+i)\,dx$ —— (ii)

Now, $\int_{OA} (x - y + ix^2)\, dz$

$= \int_0^1 (x - x + ix^2)(1+1)\,dx$ [From (i) and (ii) ; $y = x$, $dz = (1+i)dx$]

$= \int_0^1 (ix^2)(1+i)\,dx$

$= (1+i)\int_0^1 ix^2\,dx$

$= (1+i)i\int_0^1 x^2\,dx$

$= (1+i)\, i\, \left[\frac{x^3}{3}\right]_0^1$

$= i(1+i)\left[\frac{1}{3} - \frac{0}{3}\right]$

$= i(1+i)\left[\frac{1}{3}\right]$

$= \frac{1}{3}(i-1)$ $[i^2 = -1]$

(b) Along OB and then along BA; Along OB from $z=0$ to $z=1$ and then along BA from $z=1$ to $z=i+1$

Solution: Along line OB

Given that,

$z = 0$

$x + iy = 0$ $[z = x+iy]$

$x+iy = 0+i.0$

Equating real and imaginary part,

$x=0$ and $y=0$

That is coordinate of $O(0,0)$

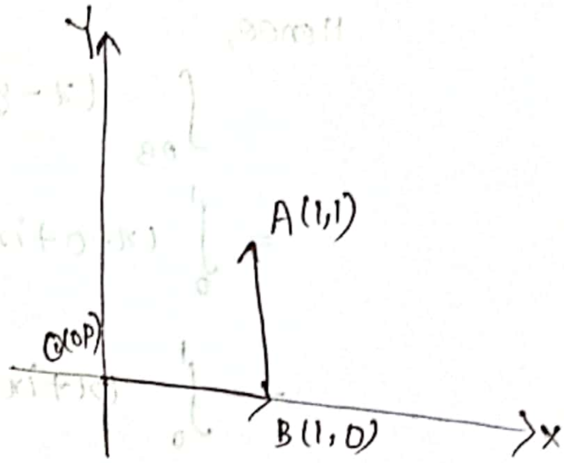
Again,

Given $z=1$

$x+iy = 1$

$x+iy = 1+0$

$x+iy = 1+i.0$



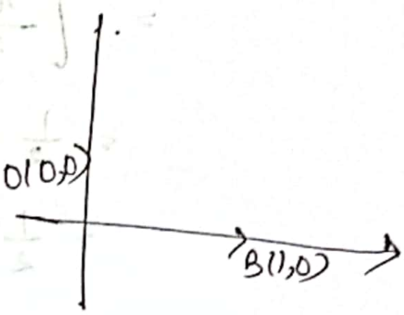
Equating real and imaginary part,

$x=1$ and $y=0$

That is coordinate of $B(1,0)$

Required integral,

$$\int_{OB}(x-y+ix^2)dz+\int_{BA}(x-y+ix^2)dz \quad \text{--- (iii)}$$



The equation of OB is,

$$\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$$

$$\Rightarrow \frac{y-0}{0-0} = \frac{x-0}{0-1}$$

$$\Rightarrow -y = 0$$

$$\therefore y = 0 \quad \text{--- (iv)}$$

we have,

$z = x+iy$

$z = x+i.0$ [$y=0$]

$\therefore z = x$

$$\Rightarrow \frac{dz}{dx} = \frac{d}{dx}(x)$$

$$\Rightarrow \frac{dz}{dx} = 1$$

$$\Rightarrow dz = dx \quad \text{——— (v)}$$

Now 1st part of (iii)

Hence,

$$\int_{OB} (x - y + ix^2)\, dz$$

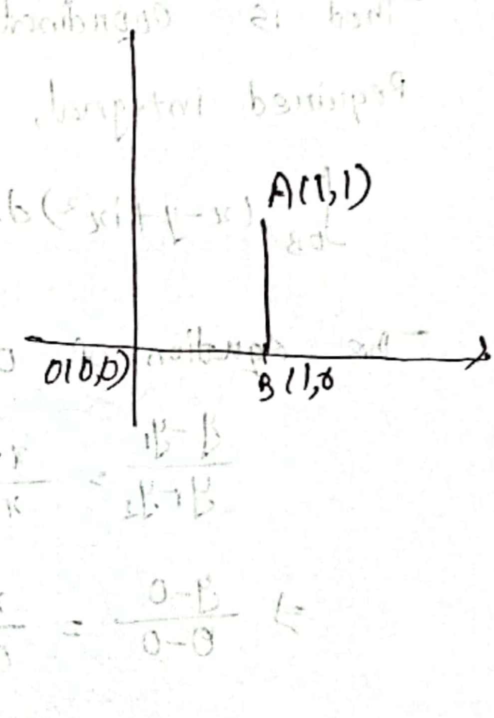
$$= \int_0^1 (x - 0 + ix^2)\, dx \qquad \text{[From (iv) and (v) } y = 0,\ dz = dx]$$

$$= \int_0^1 (x + ix^2)\, dx$$

$$= \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} + i\left(\frac{1}{3}\right) - \frac{0}{2} - i\frac{0}{3}$$

$$= \frac{1}{2} + i\frac{1}{3}$$



Along Line BA:

The equation of BA is

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\Rightarrow \frac{y - 0}{0 - 1} = \frac{x - 1}{1 - 1}$$

$$\Rightarrow \frac{y}{-1} = \frac{x - 1}{0}$$

$$\Rightarrow y \cdot 0 = (x - 1)(-1)$$

$$\Rightarrow 0 = -x + 1$$

$-x = -1$

$\therefore x = 1$ —— (vi)

$\therefore dx = 0$ —— (vii)

we have,

$$z = x + iy$$

$$z = 1 + iy \quad [\text{From (vi)}]$$

$$\Rightarrow \frac{dz}{dy} = \frac{d}{dy}(1+iy)$$

$$\Rightarrow \frac{dz}{dy} = 0 + i\frac{dy}{dy}$$

$$\Rightarrow \frac{dz}{dy} = 0 + i.1$$

$$\Rightarrow dz = i\,dy \quad \text{—— (viii)}$$

Second part of (iii)

Now,

$$\int_{BA} (x - y + ix^2)\,dz$$

$$= \int_0^1 (1 - y + i.1^2)\,i\,dy \quad [\text{From vi, viii, } x = 1,\ dz = i\,dy]$$

$$= \int_0^1 (1 - y + i)\,i\,dy$$

$$= i\int_0^1 (1 - y + i)\,dy$$

$$= i\left[y - \frac{y^2}{2} + iy\right]_0^1$$

$$= \cancel{i\left[y - \frac{y^2}{2} + iy\right]_0^1}$$

$$= i\left[1 - \frac{1}{2} + i - (0 - \frac{0}{2} + 0)\right]$$

$= i\left[-\frac{1}{2}+i\right]$

$= \frac{i}{2} - 1$ ——— (ix)

Putting result in (ii)

$$\int_{OB} (x-y+ix^2)\,dz + \int_{BA} (x-y+ix^2)\,dz$$

$$= \left(\frac{1}{2}+i\frac{1}{3}\right) \left(\frac{i}{2}-1\right)$$

$$= \frac{1}{2} + \frac{i}{3} + \frac{i}{2} - 1$$

$$= -\frac{1}{2} + \frac{5}{6}i$$

(Ans)

Pole: Pole is a certain type of singularity of a function can be found by substituting the denominator of the function equal to zero.

Poles represent the points where a complex function cease to be analytic.

② Evaluate $\int_c \frac{z}{z^2-3z+2}\, dz$ where c is the circle $|z-2| = 1/2$

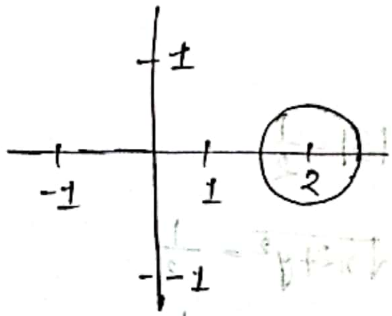
Solution: we have, $z = x+jy$

$$\Rightarrow z-2 = x+jy-2$$
$$\Rightarrow z-2 = x-2+jy$$
$$\therefore |z-2| = \sqrt{(x-2)^2+y^2}$$

Given that, $|z-2| = \frac{1}{2}$

$$\Rightarrow \sqrt{(x-2)^2+y^2} = \frac{1}{2}$$
$$\Rightarrow (x-2)^2+y^2 = \frac{1}{4}$$
$$\Rightarrow (x-2)^2+(y-0)^2 = \left(\frac{1}{2}\right)^2$$

which is the equation of circle whose center $(2,0)$ Radius $1/2$.



Poles: $z^2-3z+2 = 0$

$$\Rightarrow z^2-2z-z+2=0$$
$$\Rightarrow z(z-2)-1(z-2)=0$$
$$\Rightarrow (z-2)(z-1)=0$$

That is, $z = 1, 2$

There is only one pole at $z=2$ inside the given circle.

3. Evaluate $\int_c \frac{2z+1}{z^2+z}\,dz$ where c is the circle $|z| = 1/2$

<u>Solution:</u> we have,

$$z = x + iy$$

$$\therefore |z| = \sqrt{x^2+y^2}$$

$$\therefore |z| = 1/2$$

$$\therefore \sqrt{x^2+y^2} = 1/2$$

$$\Rightarrow x^2+y^2 = 1/4$$

$$\Rightarrow (x-0)^2 + (y-0)^2 = \left(\tfrac{1}{2}\right)^2 \quad \text{--- (1)}$$

[we have $(x-a)^2 + (y-b)^2 = r^2$]

which is the equation of a circle whose center $(0,0)$ Radius $1/2$.

poles: $z^2+z = 0$

$$\Rightarrow z(z+1) = 0$$

$$\therefore z = 0, -1$$

There is only one pole at $z = 0$ inside the given circle.

$$\int_c \frac{2z+1}{z^2+z}\,dz$$

$$\Rightarrow \int_c \frac{2z+1}{z(z+1)}\,dz$$

$$= \int_c \frac{\frac{2z+1}{z+1}}{z}\,dz \quad \text{x ?}$$

Here, $f(z) = \dfrac{2z+1}{z+1}$

$\therefore f(0) = \dfrac{1}{1} = 1$

From cauchy's integral Formula

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i \times f(a)$$

$[a=0]$

$$\Rightarrow \int_c \frac{f(z)}{z-0}\,dz = 2\pi i \times f(0)$$

$$\Rightarrow \int_c \frac{f(z)}{z}\,dz = 2\pi i \times f(0)$$

$$\Rightarrow \int_c \frac{\frac{2z+1}{z+1}}{z}\,dz = 2\pi i \times f(0)$$

$$\Rightarrow \int_c \frac{\frac{2z+1}{z+1}}{z}\,dz = 2\pi i \times 1$$

$$\Rightarrow \int_c \frac{\frac{2z+1}{z+1}}{z}\,dz = 2\pi i$$

(Ans)

(4) Find the residue at pole of $\dfrac{1-2z}{z(z-1)(z-2)}$

Solution: Let, $f(z) = \dfrac{1-2z}{z(z-1)(z-2)}$

Poles, $z(z-1)(z-2) = 0$

$z=0$; $(z-1)=0$; $(z-2)=0$

$z=0$; $z=1$; $z=2$

Residue of $f(z)$ at $(z=0)$ $= \lim_{z\to 0} (z-0) f(z)$

$$= \lim_{z\to 0} z \frac{1-2z}{z(z-1)(z-2)}$$

$$= \frac{1}{(-1)(-2)}$$

$$= 1/2$$

Residue of $f(z)$ at $(z=1)$ $= \lim_{z=1} (z-1) f(z)$

$$= \lim_{z=1} (z-1) \frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z=1} \frac{1-2z}{z(z-2)}$$

$$= \frac{1-2}{1(1-2)}$$

$$= 1$$

Residue of $f(z)$ at $(z=2)$ $= \lim_{z \to 2} (z-2) f(z)$

$$= \lim_{z \to 2} (z-2) \frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z \to 2} \frac{1-2z}{z(z-1)}$$

$$= \frac{1-2\times 2}{2\times(2-1)}$$

$$= -3/2$$

(Ans)

Singular point: A point at which a function $f(z)$ is not analytic is known as a singular point or singularity of the function.

Example: The function $\frac{1}{z-2}$ has a singular point at $z=2$

$$\because \left[\frac{1}{z-2} = \frac{1}{2-2} = \frac{1}{0} = \infty\right]$$

points in a region where $f(z)$ ceases to be regular are called singular points or singularities.

The point at which the function is not differentiable is called a singular point of the function.