Logic

Propositional logic

Definition of logic: Logic is the study of the principles and methods that distinguishes between a valid and invalid argument. Logic has numerous applications to computer science.

Definition of proposition: A proposition is a declarative statement that is either true (T) or false (F) but not both.

→ T or 1 if it is true or
F or 0 if it is false

Example:
- 1 + 1 = 2 → Truth value of proposition (T)
- 2+2 = 5 → Truth value of proposition (F)
- Pigs can fly → Truth value of this proposition (F)
- There are 4 fingers in a hand → Truth value of this proposition (F)

Not proposition (not valid logical arguments)

1. What time is it? → Not proposition (Question কে Proposition বলা যাবে না)
2. Read this carefully → Not proposition
3. Do your homework → Not proposition

→ commands/Imperative sentence proposition হতে পারে না।

4. $x+y=2$ → Not proposition (x এর Value define করে দেয়নি, x এর Value কত হবে তা ~~বলা~~ জানা নাই তাই এটা Not proposition)

5. Bangladesh and India → Not proposition (Not statements)

6. Shahin is the best lecturer → Not proposition (because that's opinion)

7. He is a college student → Not proposition (because Person is not defined)

Task-1:

Which of these sentence are proposition?

A. Boston is the capital of Massachusetts. (Proposition)

B. Miami is the capital of Florida. (Proposition)

C. $2+3=5$ (Proposition)

D. $5+7=10$ (Proposition)

E. $x+2=11$ (Not proposition)

F. Answer this question. (Not proposition)

Find the truth value of those that are proposition?

A. True (T)

B. False (F)

C. True (T)

D. False (F)

E. Not proposition

F. Not proposition

Which of these sentence are propositions?

A. Do not pass go. (Not proposition)

B. What time is it? (Not proposition)

C. There are no bad black flies in Maine. (Proposition)

D. $4 + x = 5$ (Not proposition)

E. The moon is made of green cheese. (Proposition)

F. $2^n \geq 100$ (Not proposition)

Propositional logic and variables:

The area of logic that deals with propositions is called the propsitional logic.

It was first developed systematically by the Greek philosopher Aristotle.

Propositional Variables

- Today is Friday – p
- It is raining – q

⇒ The truth value of the propositional variable can be True or false

Compound proposition:

Compound proposition is a proposition formed by combining two or more simple proposition.

The logical operators that are used to form compound proposition is called connectives.

Examples: 1. "3+2 = 5" and "Sylhet is a city in Bangladesh".

2. "The grass is green" or it is hot today".

3. "Discrete Mathmatics is not diffcult to me!"

Logical operator/ connectives : দুই বা ততোধিক Propositions কে একত্রিত করার জন্য logical operator অথবা connectives ব্যবহার করা হয়।

- A operator or connective combines one or more operand expressions into a larger expressions.

Unary operator: It takes only one operand. (-3)

Binary operator: It takes 2 operands. (3×4)

Ternary operator: It takes more than 2 operands. $(a:b)$

Logical connectives: Negation or Not ($\neg a$ / $!a$ / $\bar{a}$)

Conjunction / And ($\wedge$)

Disjunction / or ($\vee$)

Implication ($\Rightarrow$ / $\rightarrow$)

Biconditional ($\Leftrightarrow$ / $\leftrightarrow$)

Symbol	Formal Name	English Name	operator-type
$\neg$	Negation	Not	Unary
$\vee$	Disjunction	OR	Binary
$\wedge$	Conjunction	AND	Binary
$\oplus$	Ex OR	"OR--but no both"	Binary
$\rightarrow$	Implication/ Conditional	If ---then	Binary
$\Leftrightarrow$	Equivalence/ biconditional	"If and only If"	Binary

Negation / Not logical operator:

$P = \neg P$ (called not p)

The truth value of negation of P, $\neg P$ is the opposite of the truth value of P.

- Today is friday

P = Today is friday

$\neg P$ / Negation = Today is not friday /

= It is not the case that today is friday/

= It is not friday today.

- At least 10 inches of rain fell today in Miami

P = At least 10 inches of rain fell today in Miami

$\neg P$ / Negation = It is not the case that at least

10 inches of rain fell today in Miami.

= Less than 10 inches of rain fell today in Miami.

Truth table :

2^n, where n is number of Variables

Here, $n = 1$, $2^n = 2^1 = 2$

p	$\neg p$
F	T
T	F

1 0

0 1

Find the negation of each of these sentences:

1. 6 sin negative.

   negation: 6 is non-negative

2. $2+1 = 3$; negation : $2+1 \neq 3$

3. There is no pollution in New jersy.

   Negation : There is pollution in New jersy.

4. The summer in marine is hot and sunny.

   Negation: It is not the case that the summer in marine is hot and jersy.

5. Today is Thursday,

   Negation: It is not the case that today is thursday.

Conjunction (And logical operator):

1. Today is friday and it is raining.
   $p \quad \wedge \quad q$

p = Today is friday

q = It is raining

So, $p \wedge q$

$2^n = 2^2 = 4$

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

Note: যখন দুটি বা সব input True (T) হবে তখন output True (T) হবে।

- p = I will have salad for lunch

  q = I will have steak for dinner

  $p \wedge q$ = I will have salad for lunch and I will have steak for dinner.

Disjunction (OR logical operator):

1. Today is friday or It is raining

   p = Today is friday

   q = It is raining

So, $p \vee q$

$2^n = 2^2 = 4$

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Note: যেকোনো একটি input True (T) হলে output True (T) হবে।

- p = My car has a bad engine.

  q = My car has a bad carburetor.

  $p \vee q$ = Either my car has a bad engine or my car has a bad carburetor.

Exclusive OR (Ex OR logical operator):

exclusive OR of p and q denoted by $p \oplus q$.

1. Students who have taken calculus or computer science can take this class.

   (p: Students who have taken calculus; q: computer science)

P = students who have taken calculus

q = computer science

So, $p \vee q$

$2^n = 2^2 = 4$

$\boxed{OR}$

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

$\boxed{\text{Ex-OR}}$

Students who have taken calculus or computer science but not both, can take the class.

p = calculus, q = computer science

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	F

Example: p – Today is friday.

q – It is raining today.

(OR) $p \vee q$ – Today is friday or it is raining today.

(Ex-OR) $p \oplus q$ – Today is friday and it is raining today, but not both.

For each of these sentences, determine whether an exclusive OR or OR is intended:

(a) coffee or tea comes with dinner. (Ex-OR)

p = coffee, q = tea

so, $p \oplus q$

(b) Experience with c++ or Java is required. (OR)

p = c++, q = Java

so, $p \vee q$

(c) Lunch includes $\underset{p}{\underline{\text{soup}}}$ or $\underset{q}{\underline{\text{salad}}}$. (Ex-OR)

so, $p \oplus q$

(d) You can pay $\underset{p}{\underline{\text{using us dollars}}}$ or $\underset{q}{\underline{\text{euros}}}$ (OR/Ex-OR)

Conditional statements:

$p \to q$, is false when p is true and q is false, and true otherwise.

p is called the hypothesis antecedent or premise

q is called the conclusion (or consequence)

$p \to q$: "If (p) ...., then ....(q)"

Example: p : "You study hard".
q : You will get a good grade".

$p \to q$: If you study hard, then you will get a good grade".

- $p \to q$: If you buy your air ticket in advance, it is cheaper.
- $p \to q$: If x is an integer, then $x^2 \geq 0$.
- $p \to q$: If it rains, the grass gets wet.
- $p \to q$: If $2+2 = 5$, then all unicorns are pink.

Truth table: If $\underbrace{\text{today is holiday}}_{p}$ $\underbrace{\text{then}}_{\rightarrow}$ $\underbrace{\text{the store is closed}}_{q}$.

p	q	p→q
F	F	T
F	T	T
T	F	F
T	T	T

Note: True – False = False

Otherwise all true.

Exercises: Determine whether each of the conditional statements is true or false.

(a) If $\underbrace{1+1=2}_{P(T)}$, then $\underbrace{2+2=5}_{q(F)}$ --- False

(b) If $\underbrace{1+1=3}_{P(F)}$, then $\underbrace{2+2=4}_{q(T)}$ – True

(c) If $\underbrace{1+1=3}_{P(F)}$, then $\underbrace{\text{dogs can fly}}_{q(F)}$ – True

(d) If $\underbrace{1+1=2}_{P(T)}$, then $\underbrace{\text{dogs can fly}}_{q(F)}$. – False.

① "if p, then q"

"If I am elected, then I will lower Taxes."

$\underbrace{\text{If I am elected}}_{p}$, then $\underbrace{\text{I will lower Taxes}}_{q}$

$p \to q$

② "If p, q"

"If it is below freezing, It is also snowing.

$p \to q$

③ "p is sufficient for q"

"Driving over 65 miles per hour is sufficient for getting a speeding ticket.

$\underbrace{\text{Driving over 65 miles per hour}}_{p}$ is sufficient for $\underbrace{\text{getting a speeding ticket}}_{q}$

$p \to q$

④ "q if p"

"Maria will get a good job if she learns Discrete mathematics.

$\underbrace{\text{Maria will get a good job}}_{p}$ if $\underbrace{\text{she learns Discrete mathematics}}_{q}$

$p \to q$

⑤ "q whenever p"

⑥ "q is necessary for p".

⑦ "q follows from p"

⑧ "a sufficient condition for q is p."

⑨ "q when p"

} $p \to q$

Maria will find a good job when she learns

$p \to q$

⑩ "q unless $\neg p$"

"Maria will find a good job unless she does not learn discrete mathmatics" $\rightarrow p \rightarrow q$

p $\neg q$

⑪ "p only if" $\rightarrow$ $p \rightarrow q$

Converse, inverse, contrapositive

Conditional statement $\longrightarrow$ new conditional statement

Converse   Inverse   Contrapositive

$p \rightarrow q$ (conditional statement)

$q \rightarrow p$ (converse)

$\neg p \rightarrow \neg q$ (Inverse)

$\neg q \rightarrow \neg p$ (contrapositive)

$\downarrow$

converse + inverse

Truth table:

p	¬p	q	¬q	Converse	Inverse	Contra Positive	Conditional
F	T	F	T	T	T	T	T
F	T	T	F	F	F	T	T
T	F	F	T	T	T	F	F
T	F	T	F	T	T	T	T

Biconditional statement

① $\underbrace{\text{You can take the flight}}_{p}$ $\underbrace{\text{If and only if}}_{\Leftrightarrow}$ $\underbrace{\text{you buy a ticket}}_{q}$.

$p \Leftrightarrow$

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Biconditional truth table এ উভয় condition ই True হলে output True আবার ২টা input false হলে output True. অন্যথায় output false.

- Same value output true.

② $\underbrace{\text{That it is below freezing}}_{p}$ is $\underbrace{\text{necessary and sufficient}}_{\Leftrightarrow}$ for $\underbrace{\text{it to be snowing}}_{q}$.

p = That it is below freezing

q = for it to be snowing

$p \Leftrightarrow q$

p	q	$p \Leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Determine whether these bioconditionals are true ~~or~~ or false

① $\underbrace{2+2=4}_{T}$ $\underbrace{\text{if and only if}}_{\Leftrightarrow}$ $\underbrace{1+1=2}_{T}$ (T) $T \Leftrightarrow T = T$

② $\underbrace{1+1=2}_{T}$ $\underbrace{\text{if and only}}_{\Leftrightarrow}$ if $\underbrace{2+3=4}_{F}$ (F) $T \Leftrightarrow F = F$

③ $\underbrace{1+1=3}_{F}$ $\underbrace{\text{if and only}}_{\Leftrightarrow}$ if $\underbrace{\text{monkeys can fly}}_{F}$ (T)

$F \Leftrightarrow F = T$

④ $\underbrace{0>1}_{F}$ $\underbrace{\text{if and only}}_{\Leftrightarrow}$ if $\underbrace{2>1}_{T}$ $F \Leftrightarrow T = F$

Truth table of compound propositions

Precendence of logical operator: $\neg, \wedge, \vee, \rightarrow, \Leftrightarrow$

① $(p \vee \neg q) \rightarrow (p \wedge q)$ $2^n = 2^2 = 4$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
F	F	T	T	F	F
F	T	F	F	F	T
T	F	T	T	F	F
T	T	F	T	T	T

② $(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

p	q	$\neg q$	$p \leftrightarrow q$	$p \leftrightarrow \neg q$	$(p \to q) \oplus (p \leftrightarrow \neg q)$
F	F	T	T	F	T
F	T	F	F	T	T
T	F	T	F	T	T
T	T	F	T	F	T

③ $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$ $\quad 2^n = 2^3 = 8$

p (4)	q (2)	r (1)	$\neg q$	$\neg p$	$\neg p \leftrightarrow \neg q$	$q \leftrightarrow r$	$(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	F	F
F	T	F	F	T	F	F	T
F	T	T	F	T	F	T	F
T	F	F	T	F	F	T	F
T	F	T	T	F	F	F	T
T	T	F	F	F	T	F	F
T	T	T	F	F	T	T	T

(4) $\neg p \wedge (q \vee \neg r)$

p	q	r	$\cancel{p\vee}\ \neg r$	$p \vee \neg r$	$\neg p$	$\neg p \wedge (p \vee \neg r)$
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	F	T	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	T	T	T

In case of ExOR, if odd no of input is true then TRUE

Translating English sentence into proposition

Let p and q be the propositions.

p : It is below freezing

q : It is snowing

Write these propositions using p and q and logical connectives.

ⓐ It is below freezing and snowing.

p $\wedge$ q

$p \wedge q$

ⓑ It is below freezing but not snowing.

p $\wedge$ $\neg$ q

$p \wedge (\neg q)$

ⓒ It is either snowing or freezing but not both.

p q $\oplus$

$p \oplus q$

ⓓ If it is below freezing, it is also snowing.

p q

$p \rightarrow q$

ⓔ It is either below freezing or it is snowing, but it is not snowing if it is below freezing.

p $\vee$ q $\wedge$ p $\neg$ $\neg q$

$[(p \vee q) \wedge (p \rightarrow \neg q)]$

ⓕ That is below $\underset{p}{\underline{\text{freezing}}}$ is $\underline{\text{necessary and}}$ $\underline{\text{sufficient}}$ for it to be $\underset{q}{\underline{\text{snowing}}}$.

$\leftrightarrow$

$p \leftrightarrow q$

ⓖ $\underset{p}{\underline{\text{You can access the internet from campus}}}$ $\underline{\text{only if}}$ $\underset{q}{\underline{\text{you are a computer science major}}}$ $\underline{\text{or}}$ $\underset{r}{\underline{\text{you are}}}$ $\boxed{\text{not}}$ $\underline{\text{a freshman}}$.

$p \rightarrow (q \lor \neg r)$

Translating proposition into english sentence

p: "Swimming at the New jersy is allowed".

q = "Sharks have been spotted near the shore."

a. $\neg p$

⇒ Swimming at the new jersy is not allowed.

b. $p \lor q$

⇒ Swimming at the new jersy is allowed or sharks have been spotted near the shore.

c. $p \rightarrow q$

⇒ If swimming at the new jersy is allowed then sharks have been spotted near the shore.

d. $p \leftrightarrow q$

⇒ Swimming at the new jersy is allowed if and only if sharks have been spotted near the shore.

e. $\neg p \rightarrow \neg q$

⇒ swimming at the new jersy is not allowed if and only if sharks have not been spotted near the shore.

f. $\neg p \vee (p \wedge q)$

⇒ Swimming at the new jersy is not allowed or Swimming at the new jersy is allowed and sharks have been spotted near the shore.

Propositional equivalence: It is used extansively in the construction of mathmatical arguments. By using these equivalences, we can substitute. Propositions with other propositions with the same truth value. This proves to be very useful in different types of situations.

Symbol "$\equiv$"

Compound proposition:
① Tautology
② Contradiction
③ Contingency

Tauotology: A compound proposition that is always true, no matter what the truth values of the propositions that occur in it is called a Tautology.

Contradiction: A compound proposition that is always ~~fastor~~ false, no matter what the truth values of the propositions that occur in it, is called a contradiction.

Contingency: A compound proposition that is neither tautology nor contradiction is called contingency.

There are two ways of proving logical equivalance.

1. Truth table
2. logic law

1. Show that $p \to q$ and $(\neg p \vee q)$ are logically equivalent.

p	q	$\neg p$		
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

$\therefore (p \to q) \equiv (\neg p \vee q)$

Logic law: Logic laws are such formula which are used for compound and complex propositions to simply them.

① Identity laws : $p \wedge T \equiv p$
$p \vee F \equiv p$

② Domination laws: $p \vee T \equiv T$
$p \wedge F \equiv F$

③ Idempotent laws: $p \vee p \equiv p$
$p \wedge p \equiv p$

④ Double negation laws : $\neg(\neg p) \equiv p$

⑤ Commutative laws : $p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$

⑥ Associative laws: $(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

⑦ Distributive laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

⑧ De Morgan's law : $\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

⑨ Absorption laws: $p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

⑩ Negation laws: $p \vee \neg p \equiv T$

$p \wedge \neg p \equiv F$

Logical equivalance Involving conditional statements

① $p \rightarrow q \equiv \neg p \vee q$

② $p \rightarrow q \equiv \neg q \rightarrow \neg p$

③ $p \vee q \equiv \neg p \rightarrow q$

④ $p \wedge q \equiv \neg (p \rightarrow \neg q)$

⑤ $\neg (p \rightarrow q) \equiv p \wedge \neg q$

⑥ $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

⑦ $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

⑧ $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

⑨ $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical equivalance Involving Biconditional statements.

① $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

② $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

③ $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

④ $\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

⊞ DE Morgan's laws

① $\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
F	F	T	T	F	T	T
F	T	T	F	F	T	T
T	F	F	T	F	T	T
T	T	F	F	T	F	F

② $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
F	F	T	T	F	T	T
F	T	T	F	T	F	F
T	F	F	T	T	F	F
T	T	F	F	T	F	F

Proving tautology using logic law:

① $(p \wedge q) \rightarrow p$

$(p \wedge q) \rightarrow p$

$\equiv \neg (p \wedge q) \vee p$ [conditional law]

$\equiv \neg p \vee \neg q \vee p$ [De Morgan's law]

$\equiv (\neg p \vee p) \vee \neg q$ [Associative law]

$\equiv T \vee \neg q$ [Negation law]

$\equiv T$ [Domination law]

[Proved]

② $(p \wedge q) \rightarrow (p \vee q)$

$(p \wedge q) \rightarrow (p \vee q)$

$\equiv \neg (p \wedge q) \vee (p \vee q)$ [conditional law]

$\equiv \neg p \vee \neg q \vee (p \vee q)$ [De morgan's law]

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ [Associative law]

$\equiv T \vee T$ [Negation law]

$\equiv T$

[Proved]

Proving logical equivalalance using logic law:

① $\neg(p \to q) \equiv p \wedge q$

L.H.S $= \neg(p \to q)$

$\equiv \neg(\neg p \vee q)$ [conditional law]

$\equiv \neg(\neg p) \wedge \neg q$ [De Morgan's law]

$\equiv p \wedge \neg q$ [Double Negation law]

$=$ R.H.S

② $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

LHS $= \neg(p \vee (\neg p \wedge q))$ [De morgan's law]

$\equiv \neg p \wedge \neg(\neg p \wedge q)$ [De morgan's law]

$\equiv \neg p \wedge \neg(\neg p) \vee \neg q)$ [Double Negation law]

$\equiv \neg p \wedge (p \vee \neg q)$ [distributive law]

$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ [negation law]

$\equiv F \vee (\neg p \wedge \neg q)$

$\equiv \neg p \wedge \neg q$

$=$ R.H.S.

③ $\neg(R\wedge S\wedge T)\wedge\neg(R\vee S\vee T)$

<u>Solution:</u>

$\neg(R\wedge S\wedge T)\wedge\neg(R\vee S\vee T)$

$\equiv (\neg R\vee\neg S\vee\neg T)\wedge\neg(R\vee S\vee T)$ [DE Morgan's Law]

$\equiv (\bar{R}+\bar{S}+\bar{T}).(\bar{R}.\bar{S}.\bar{T})$ [changing into algebraic form)

$\equiv \bar{R}.\bar{R}..\bar{S}.\bar{T}+\bar{R}.\bar{S}.\bar{T}.\bar{S}+\bar{R}.\bar{S}.\bar{T}.\bar{T}$

$\equiv \bar{R}.\bar{S}.\bar{T}+\bar{R}.\bar{S}.\bar{T}+\bar{R}.\bar{S}.\bar{T}$ [Idempotent law]

$\equiv \bar{R}.\bar{S}.\bar{T}$

$\equiv \neg R\wedge\neg S\wedge\neg T$

Use De Morgan's law to find the negation of each of the following statements:

<u>(a)</u> John is rich and happy.

$\Rightarrow \neg(P\wedge q)=\neg P\vee\neg q$

John is not rich or John is not happy.

<u>(b)</u> Charlos will ~~bye~~ bicycle or run tomorrow.

$\Rightarrow \neg(P\vee q)=\neg P\wedge\neg q$

Carlos will not bicycle and not run tomorrow.

Predicates and Quantifiers :

Predicate logic:
1. Unary (1 variable)
2. Binary (2 Variable)
3. Ternary (3 Variable)

Examples: x is greater than 3 $\Rightarrow P(x)$ [unary]
x is greater than y $\Rightarrow Q(x,y)$ [Binary]
x is greater than y and z $\Rightarrow R(x,y,z)$ [Ternary]
x can speak in English $\Rightarrow E(x)$ [unary]

① Let $P(x)$ denote "x is greater than 3", What are the truth values of $P(4)$ and $P(2)$?

Sol: If $x=2$; $P(2) \Rightarrow$ 2 is greater than 3 (False Value)
If $x=4$; $P(4) \Rightarrow$ 4 is greater than 3 (True Value)

② Let $Q(x,y)$ denote "$x=y+3$". What are the truth values of $Q(1,2)$ and $Q(3,0)$?

Sol: If $x=1$ and $y=2$, $Q(1,2) \Rightarrow 1 = 2+3$ [False Value]
If $x=3$ and $y=0$, $Q(3,0) \Rightarrow 3 = 0+3$ [True Value]

③ Let $R(x,y,z)$ denote "$x+y=z$". What are the truth Value of $R(1,2,3)$ and $R(0,0,1)$?

Sol: If $x=1$ and $y=2$ and $z=3$; $R(1,2,3) \Rightarrow 1+2=3$ (True)
If $x=0$ and $y=0$ and $z=1$; $R(0,0,1) \Rightarrow 0+0=1$ (False)

- Propositional logic cannot adequately express the meaning of all statements in mathematics and in Natural language processing. So a logic is used to express the meaning of a wide range of statements in mathmatics. This types of logic is called predicate logic.

Propositional function: Once a Value has been assigned to x, the statement of $p(x)$ becomes a proposition and has a truth Value p is called proposition function.

Quantifiers: When the variables in a propositional function are ~~jus~~ assigned values, the resulting statement becomes a proposition with a certain Value.

" Quantification expresses the extent to which a Predicate is true over a range of elements.

In english the words all, some, many, none, few etc are used in quantification.

Example: $P(x)$ = x can speak English

[Possible values for x ⇒ some, a student, All people, ...]

Predicate calculus: The area of logic that deals with Predicates and quantifiers is called predicate calculus.

Types of quantifiers: ① Universal quantifier

② Existential quantifier

Universal: which tells us that a predicate is true for every element under consideration.

Existential: which tells us that a predicate is true for one or more element under consideration.

Denoted by "$\forall$"

Denoted by "$\exists$"

Universal quantification: The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain"

"$\forall x\, P(x)$ read as "for all x $P(x)$" / "for every x $P(x)$"

Exp: $P(x) = x$ can speak in English.

$\Rightarrow$ $\underbrace{\text{Everyone}}_{\forall x}$ $\underbrace{\text{can speak in English.}}_{P(x)}$

Existential quantifiers: The existential quantification of $P(x)$ is the statement " there exists an element x in the domain such that $P(x)$".

" $\exists x P(x)$" read as

" There is an x such that $P(x)$" | " There is at least one x such that $P(x)$" / " for some x $P(x)$'

Exp: $P(x) = x$ can speak in English.

$\Rightarrow$ $\underbrace{\text{There is a student}}_{\exists x}$ $\underbrace{\text{can speak in English.}}_{P(x)}$

Example-1: Let $Q(x)$ be the statement " $x<2$". What is/are the truth values of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: Here is the quantification is $\forall x Q(x)$

Domain = [real numbers] = $\{ ----, -1, 0, 1, 2 \}$

if $x=-1$; $x<2 \Rightarrow -1<2$; true

if $x=0$; $x<2 \Rightarrow 0<2$; true

if $x=1$; $x<2 \Rightarrow 1<2$; true

if $x=2$; $x<2 \Rightarrow 2<2$; false

Therefore as in case of all values the $Q(x)$ is not true so the truth value of the quantification is False.

Example: 2: What is the truth Value of the quantification $\exists x(P(x))$. where Let $P(x)$ be the statement "$x^2 > 10$" and the domain consists of the positive integers not exceeding 4?

Solution: Given that,

$P(x)$ be the statement "$x^2 > 10$".

Here the quantification is: $\exists x P(x)$

Domain, $x = \{1, 2, 3, 4\}$

If $x = 0$; $x^2 < 10 \Rightarrow 0^2 > 10$; False

If $x = 1$; $x^2 < 10 \Rightarrow 1^2 > 10$; False

If $x = 2$; $x^2 < 10 \Rightarrow 2^2 > 10$; False

If $x = 3$; $x^2 < 10 \Rightarrow 3^2 > 10$; False

If $x = 4$; $x^2 < 10 \Rightarrow 4^2 > 10$; True

Therefore as in case of all Values the $P(x)$ is not false but for some Values it is true so the truth Value of quantification is True.

Translate the following sentences from English into logical connectives using predicate and quantifiers:

(a) Every student in the class has studied calculus.

Here, Domain, x = student

$\underbrace{\text{Every}}_{\forall}$ $\underbrace{\text{student}}_{x}$ $\underbrace{\text{in the class has studied calculus.}}_{c(x)}$

Ans: $\forall x \; c(x)$

(b) For every person x, if person x is a student in the class then x has studied calculus.

Domain, x = person,

$\underbrace{\text{For every}}_{\forall}$ $\underbrace{\text{person } x}_{x}$, if person $\underbrace{x \text{ is a student in the class}}_{S(x)}$ $\underbrace{\text{then}}_{\rightarrow}$ $\underbrace{x \text{ has studied calculus.}}_{c(x)}$

Ans: $\forall x \, (S(x) \rightarrow c(x))$

(c) Some student in this class has visited Mexico

Domain, x = Student

$\underbrace{\text{Some}}_{\exists}$ $\underbrace{\text{student}}_{x}$ $\underbrace{\text{in this class has visited Mexico.}}_{M(x)}$

Ans: $\exists x\, M(x)$

(d) There is a person x, having the properties that x is a student in the class and x has visited Mexico.

Domain, x = person

$\underbrace{\text{There is a}}_{\exists}$ $\underbrace{\text{person } x}_{(x)}$, having the properties that $\underbrace{x \text{ is a student in the class}}_{S(x)}$ $\underbrace{\text{and}}_{\wedge}$ $\underbrace{x}_{}$ $\underbrace{\text{has visited Mexico.}}_{M(x)}$

Ans: $\exists x\, (S(x) \wedge M(x))$

Translating from logical expression into English using Predicates and quantifiers :

$C(x)$ be the statement " x is a comedian" and

$F(x)$ be the statement " x is funny"

Here domain consists of " all people".

(a) $\forall x(C(x) \to F(x))$

$$\forall x\,(C(x) \to F(x))$$

For all x (all people) $\boxed{\text{if}}$ x is a comedian $\boxed{\text{then}}$ x is funny

Ans: Every comedian is funny.

De morgan's law for negation of quantifiers:

original statement	$\forall\_x P(x)$	$\exists\_x P(x)$
Negation	$(\neg\forall)\_x P(x)$	$(\neg\exists)\_x P(x)$
After negation Statement	$\exists\_x \neg P(x)$	$\forall\_x \neg P(x)$

Exercises: (a) Every student in your class has taken a course in calculus.

Sol: Logical expression : $\forall x\, C(x)$

Negation : $\neg(\forall x\, C(x))$

After negation : $\exists x \neg C(x)$

Ans: There is a student in your class who has not taken a course in calculus.