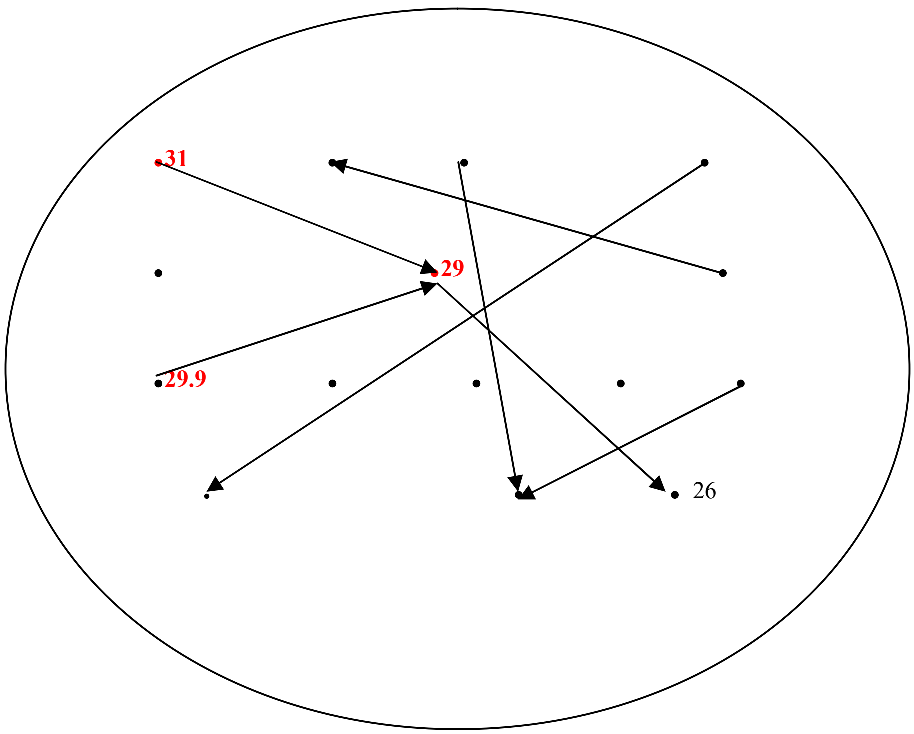
Gradient, Divergence & Curl

Gradient:

- Measures the rate of change in a scalar field; the gradient of a scalar field is a vector field. The derivative/differentiation/rate of change of a scalar field result **in a vector field** called the gradient.

- Computes the gradient of a scalar function. That is, it finds the Gradient, the slope, how fast you change, in any given direction.
- A gradient is applied to a scalar quantity that is a function of a 3D vector field: position. The gradient measures the direction in which the scalar quantity changes the most, as well as the rate of change with respect to position. A common example of this is height as a function of latitude and longitude, often applied to mountain ranges. A measure of the slope, and direction of the slope, is often called the gradient.



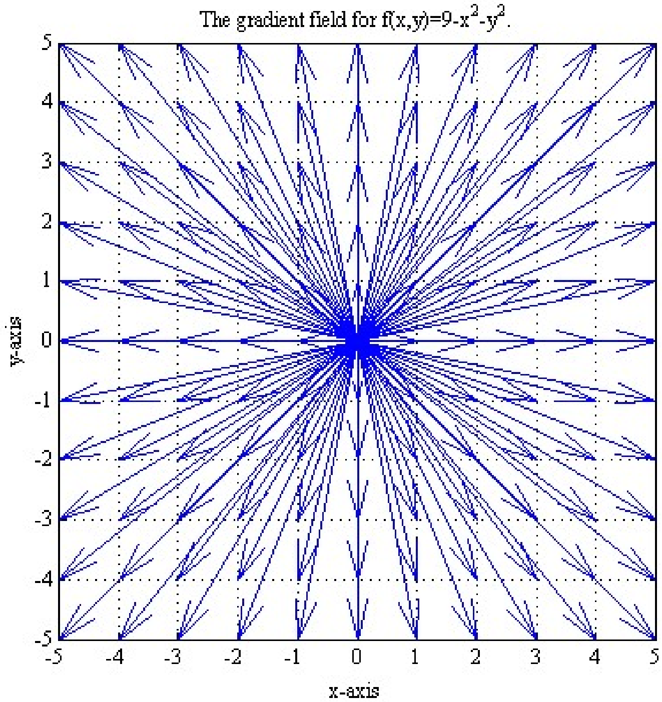


Figure # 58

Divergence:

- Measures a vector field's tendency to originate from or convergent upon a given point.
- Computes the divergence of a vector function. That is, it finds how much "stuff" is leaving a point in space.
- A divergence is applied to a vector as a function of position, yielding a scalar. The divergence actually measures how much the vector function is spreading out. If you are at a location from which the vector field tends to point away in all directions, you will definitely have a positive divergence. If the field points inward toward a point, the divergence in and near that point is negative. If just as much of the vector field points in as out, the divergence will be approximately zero.
- If we again think of $\vec{\mathbf{F}}$ as the velocity field of a flowing fluid then div $\vec{\mathbf{F}}$ represents the net rate of change of the mass of the fluid flowing from the point $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ per unit volume. This can also be thought of as the tendency of a fluid to diverge from a point.

The **divergence** of a vector field is relatively easy to understand intuitively. Imagine that the vector field $\vec{\mathbf{F}}$ below gives the velocity of some fluid flow. It appears that the fluid is exploding outward from the origin

This expansion of fluid flowing with velocity field $\vec{\mathbf{F}}$ is captured by the divergence of $\vec{\mathbf{F}}$, which we denote div $\vec{\mathbf{F}}$. The divergence of the above vector field is positive since the flow is expanding.

In contrast, the below vector field represents fluid flowing so that it compresses as it moves toward the origin. Since this compression of fluid is the opposite of expansion, the divergence of this vector field is negative.

A three-dimensional vector field $\vec{\mathbf{F}}$ showing expansion of fluid flow is shown below. Again, because of the expansion, we can conclude that div $\vec{\mathbf{F}} > \mathbf{0}$.

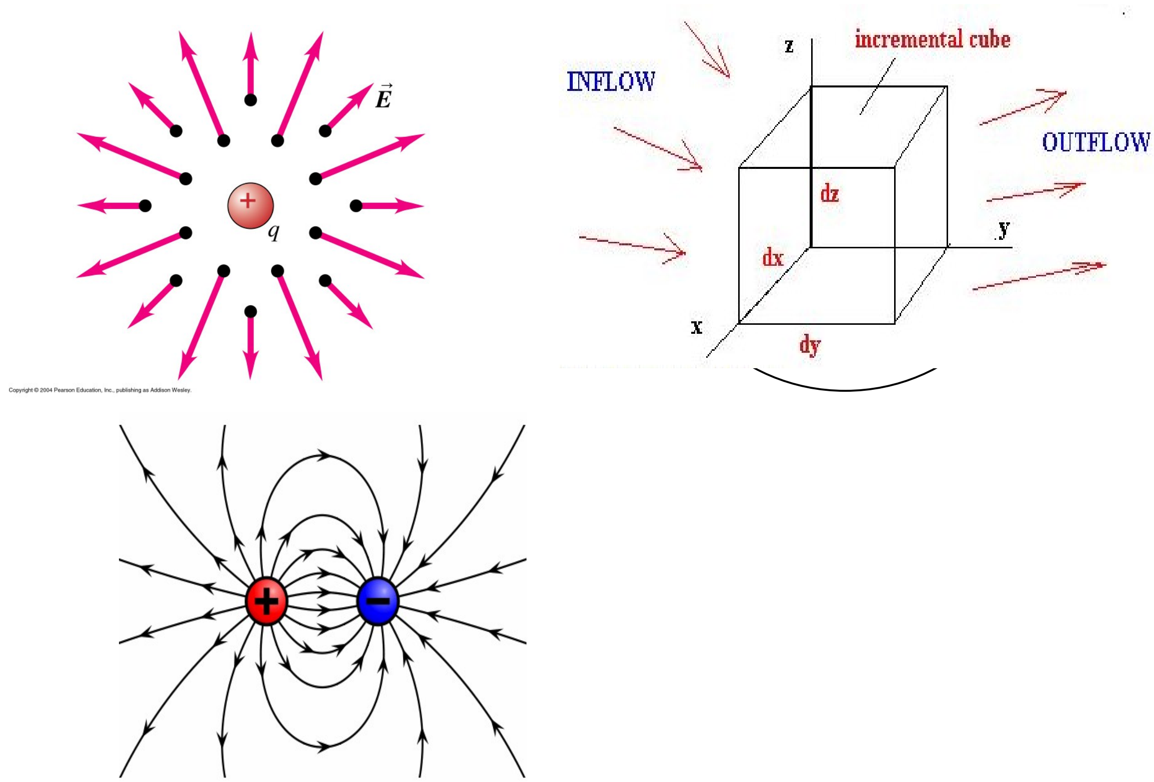


Figure # 59: Divergence of vectors flow field

[কোন একটি point এ চার্জের intensity/effect হচ্ছে divergence যেমন: কোন একটি point এ heat দিলাম। যেমন: কোন একটি point p এ heat দিলে তা চারিদিকে ছড়িয়ে পড়বে। q point এ তার intensity/effect কত? এটাই divergence]

Curl:

- In vector calculus, the **curl** (or **rotor**) is a vector operator that describes the infinitesimal rotation of a 3-dimensional vector field. At every point in the field, the curl is represented by a vector. The attributes of this vector (length and direction) characterize the rotation at that point.
- The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation. If the vector field represents the flow velocity of a moving fluid, then the curl is the **circulation density** of the fluid. A vector field whose curl is zero is called irrotational. The curl is a form of differentiation for vector fields

- measures a vector field's tendency to rotate about a point; the curl of a vector field is another vector field.
- It computes the rotational aspects of a vector function, maybe people thought how vectors "curl" around a center point, like wind curling around a low pressure on a weather map.
- A curl measures just that, the curl of a vector field. Unlike the divergence, a curl yields a vector. A vector field that tends to point around an axis, such as vectors pointing tangential to a circle, will yield a non-zero curl with the axis around which the curl occurs as the direction. Another example is the velocity field of motion spiraling in or out, such as a whirlpool. Point your right-hand thumb along the direction of the curl. Curl your fingers around this axis. They will curl in the same direction as the vector field. I do not know the names of the texts, but I know there are books available with vector fields to illustrate both divergence and curl.

  [একটি field এ কি পরিমান twist/wrapping (প্যাঁচ) আছে তা measurement করাই curling]

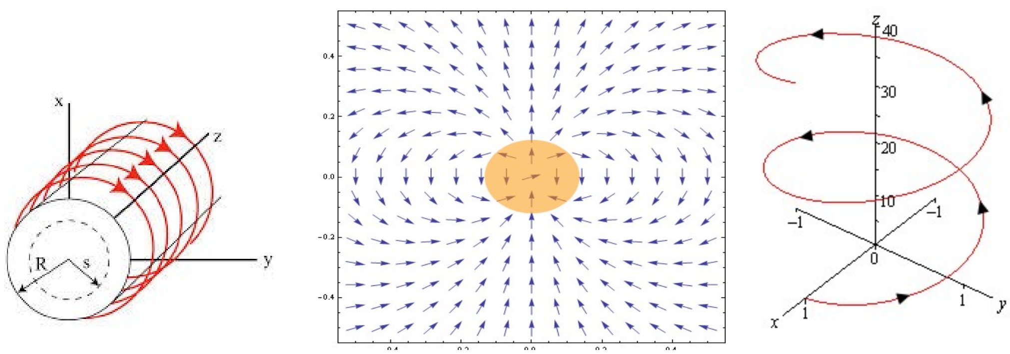


Figure # 60

Mathematical Expression of Gradient, divergence, curl of a Vector Field

Vector differential operator : $\frac{\delta}{\delta x}\hat{i}+\frac{\delta}{\delta y}\hat{j}+\frac{\delta}{\delta z}\hat{k}$ and is denoted by the symbol $\vec{\nabla}$ (pronounced 'del' or sometimes 'Nabla')

That is $\vec{\nabla}=\frac{\delta}{\delta x}\hat{\mathbf{i}}+\frac{\delta}{\delta y}\hat{\mathbf{j}}+\frac{\delta}{\delta z}\hat{\mathbf{k}}$ ----------------------------(i)

Beware! ∇ Cannot exist alone: it is an operator and must operate on a stated scalar function $\phi(x,y,z)$.

If F is a vector, ∇F has no meaning

Grad (*gradient of a scalar function*)

If a scalar function $\phi(x, y, z)$ is continuously differentiable with respect to its variables $x, y, z,$ throughout the region, then the *gradient* of ϕ, Written grad ϕ, is defined as the vector

$$\textbf{grad}\,\phi = \vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi = \frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k} \text{ -------(ii)}$$

where ϕ *is a function of x, y, z*

Note that, while ϕ is a scalar function, grad ϕ is a vector function

Divergence of a vector field: If we form the scalar (dot) product of $\vec{\nabla}$ with a vector function $\vec{A}(x, y, z) = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ we get a scalar result called the divergence of $\vec{\mathbf{A}}$:

$$div\,\vec{A} \equiv \vec{\nabla}.\vec{A} \equiv \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right).\left(A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}\right)$$

$$div\,\vec{A} \equiv \vec{\nabla}.\vec{A} \equiv \frac{\delta A_x}{\delta x} + \frac{\delta A_y}{\delta y} + \frac{\delta A_z}{\delta z} \text{ ------------------------------------(iii)}$$

Curl of a vector field: The curl of a vector field $\vec{\mathbf{A}}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{A_x}\,\hat{\mathbf{i}} + \mathbf{A_y}\,\hat{\mathbf{j}} + \mathbf{A_z}\,\hat{\mathbf{k}}$ is defined by

$$\vec{\nabla} \times \vec{\mathbf{A}} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \times \left(A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_x & A_y & A_z \end{vmatrix} \text{ --------------(iv)}$$

Q# 23: If $\phi(x, y, z) = 3x^2y - y^3z^2$, find $\vec{\nabla}\,\Phi$ (or grad Φ) at the point (1,-2,-1).

Answer: $\vec{\nabla}\,\Phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(3x^2y - y^3z^2)$

$$= \hat{i}\frac{\delta}{\delta x}(3x^2y - y^3z^2) + \hat{j}\frac{\delta}{\delta y}(3x^2y - y^3z^2) + \hat{k}\frac{\delta}{\delta z}(3x^2y - y^3z^2)$$

$$= \hat{i}(6xy - 0) + \hat{j}\,(3x^2 - 3y^2z^2) + \hat{k}\,(0 - 2y^3z)$$

$$= 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\,\hat{j} - 2y^3z\,\hat{k}$$

$$= 6(1)(-2)\,\hat{i} + \{3(1)^2 - 3(-2)^2(-1)^2\}\,\hat{j} - 2(-2)^3(-1)\,\hat{k}$$

$$= -12\,\hat{i} - 9\,\hat{j} - 16\,\hat{k} \text{ (Answer).}$$

Q# 24: Find $\vec{\nabla}\,\phi$ if (a) $\phi = \ln|\vec{r}|$ (b) $\phi = \frac{1}{|\vec{r}|}$

(a) Let $\vec{\mathbf{r}} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ Then $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ and $|\vec{r}|^2 = x^2 + y^2 + z^2$

Then $\phi = \ln\left|\vec{r}\right| = \ln\sqrt{x^2+y^2+z^2} = \ln(x^2+y^2+z^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2+z^2)$

$$\therefore \vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i}+\frac{\delta}{\delta y}\hat{j}+\frac{\delta}{\delta z}\hat{k}\right)\phi = \left(\frac{\delta\phi}{\delta x}\hat{i}+\frac{\delta\phi}{\delta y}\hat{j}+\frac{\delta\phi}{\delta z}\hat{k}\right) = \hat{i}\frac{\delta}{\delta x}\phi+\hat{j}\frac{\delta}{\delta y}\phi+\hat{k}\frac{\delta}{\delta z}\phi$$

$$= \frac{1}{2}\hat{i}\frac{\delta}{\delta x}\ln(x^2+y^2+z^2)+\frac{1}{2}\hat{j}\frac{\delta}{\delta y}\ln(x^2+y^2+z^2)+\frac{1}{2}\hat{k}\frac{\delta}{\delta z}\ln(x^2+y^2+z^2)$$

$$= \frac{1}{2}\hat{i}\left(\frac{1}{x^2+y^2+z^2}\right)\left\{\frac{\delta}{\delta x}(x^2+y^2+z^2)\right\}+\frac{1}{2}\hat{j}\left(\frac{1}{x^2+y^2+z^2}\right)\left\{\frac{\delta}{\delta y}(x^2+y^2+z^2)\right\}+\frac{1}{2}$$

$$\hat{k}\left(\frac{1}{x^2+y^2+z^2}\right)\left\{\frac{\delta}{\delta z}(x^2+y^2+z^2)\right\}$$

$$= \frac{1}{2}\hat{i}\left(\frac{1}{x^2+y^2+z^2}\right)(2x+0+0)+\frac{1}{2}\hat{j}\left(\frac{1}{x^2+y^2+z^2}\right)(0+2y+0)+\frac{1}{2}$$

$$\hat{k}\left(\frac{1}{x^2+y^2+z^2}\right)(0+0+2z)$$

$$= \frac{1}{2}\hat{i}\left(\frac{2x}{x^2+y^2+z^2}\right)+\frac{1}{2}\hat{j}\left(\frac{2y}{x^2+y^2+z^2}\right)+\frac{1}{2}\hat{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{1}{2}\times 2\left\{\frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}\right\} = \left\{\frac{x\hat{i}+y\hat{j}+z\hat{k}}{x^2+y^2+z^2}\right\} = \frac{\vec{r}}{\left|\vec{r}\right|^2}$$ Answer

(b) Given, $\phi = \frac{1}{\left|\vec{r}\right|} = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$

$$\vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i}+\frac{\delta}{\delta y}\hat{j}+\frac{\delta}{\delta z}\hat{k}\right)\phi = \left(\frac{\delta\phi}{\delta x}\hat{i}+\frac{\delta\phi}{\delta y}\hat{j}+\frac{\delta\phi}{\delta z}\hat{k}\right) = \hat{i}\frac{\delta}{\delta x}\phi+\hat{j}\frac{\delta}{\delta y}\phi+\hat{k}\frac{\delta}{\delta z}\phi$$

$$= \hat{i}\frac{\delta}{\delta x}(x^2+y^2+z^2)^{-1/2}+\hat{j}\frac{\delta}{\delta y}(x^2+y^2+z^2)^{-1/2}+\hat{k}\frac{\delta}{\delta z}(x^2+y^2+z^2)^{-1/2}$$

$$= \hat{i}\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-1/2-1}\frac{\delta}{\delta x}(x^2+y^2+z^2)\}+\hat{j}\left(-\frac{1}{2}\right)$$

$$(x^2+y^2+z^2)^{-1/2-1}\frac{\delta}{\delta y}(x^2+y^2+z^2)\}+\hat{k}\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-1/2-1}$$

$$\frac{\delta}{\delta z}(x^2+y^2+z^2)\}$$

$$= \hat{i}\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-1/2-1}(2x+0+0)+\hat{j}\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-1/2-1}$$

$$(0+2y+0)+\hat{k}\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-1/2-1}\frac{\delta}{\delta z}(0+0+2z)$$

$$= \hat{i}(-x)\,(x^2+y^2+z^2)^{-3/2} + \hat{j}(-y)\,(x^2+y^2+z^2)^{-3/2} + \hat{k}(-z)\,(x^2+y^2+z^2)^{-3/2}$$

$$= \frac{-x\hat{i}}{(x^2+y^2+z^2)^{3/2}} + \frac{-y\hat{j}}{(x^2+y^2+z^2)^{3/2}} + \frac{-z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$= -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{3/2}} = -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^1.(x^2+y^2+z^2)^{1/2}}$$

$$= -\frac{\vec{r}}{\left|\vec{r}\right|^2.\left|\vec{r}\right|} = -\frac{\vec{r}}{\left|\vec{r}\right|^3} \text{ Answer}$$

Q#25: Find the level curve of $f(x,y) = -x^2 + y^2$ passing through (2, 3). Draw Graph the gradient at the point (2, 3)

Answer: Given, $\mathbf{f(x,y) = -x^2 + y^2}$

$$\mathbf{f(2,3) = -2^2 + 3^2 = -4 + 9 = 5}$$

Hence the level curve is the hyperbola, i.e.

$$\mathbf{f(x,y) = -x^2 + y^2} = 5$$

i.e. $-x^2 + y^2 = 5$

i.e. $x^2 - y^2 = -5$

$\Rightarrow \dfrac{x^2}{-5} - \dfrac{y^2}{-5} = 1$ [This is the equation of a hyperbola, **i. e.** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$] -----------------(i)

From (i),

$\Rightarrow \dfrac{x^2}{-5} - \dfrac{y^2}{-5} = 1 \qquad \Rightarrow x^2 - y^2 = -5 \qquad \Rightarrow y^2 = 5 + x^2 \qquad \Rightarrow y = \pm\sqrt{5+x^2}$

$\Rightarrow y = \pm\sqrt{5+x^2}$ ----------------------------(ii)

x	0	-1	-2	-3	1	2	3	-4	4	
$y = \pm\sqrt{5+x^2}$	$\pm\sqrt{5}$ $= \pm 2.23$	$\pm\sqrt{6}$ ± 2.44	± 3 ± 3	$\pm\sqrt{14}$ ± 3.74	$\pm\sqrt{6}$ ± 2.44	± 3 ± 3	$\pm\sqrt{14}$ ± 3.74	$\pm\sqrt{19}$ ± 4.35	$\pm\sqrt{19}$ ± 4.35	

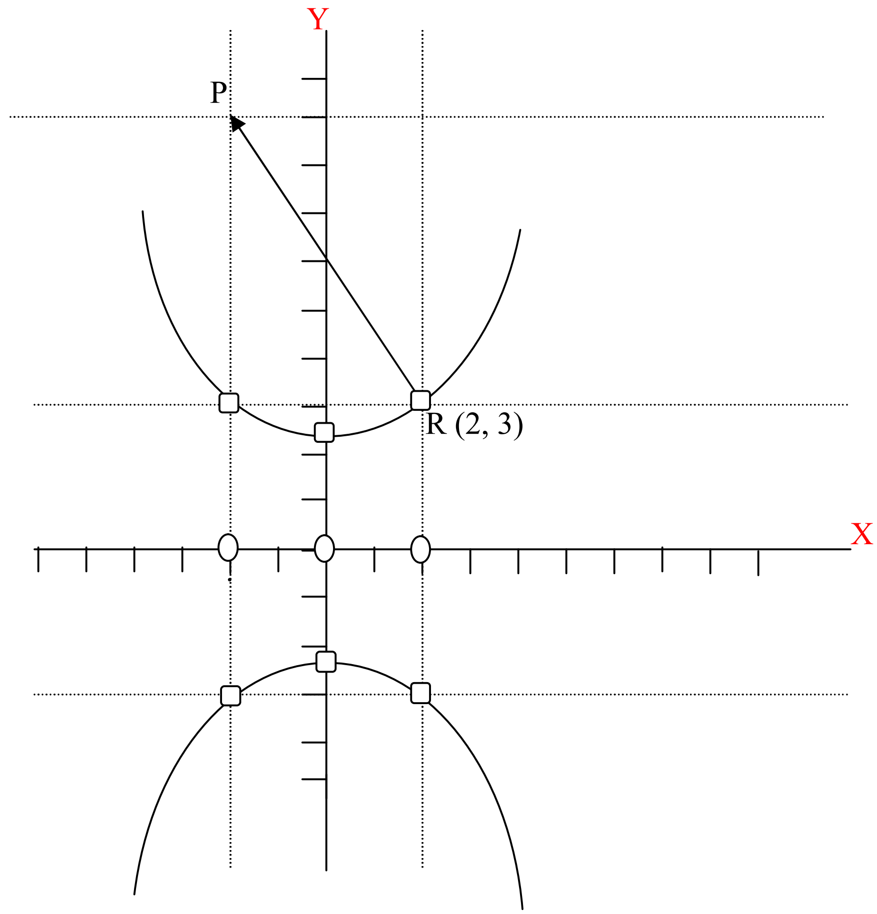


Figure # 61

Given, $f(x,y) = -x^2 + y^2$

Now, Gradient of the function, i.e.

$$\vec{\nabla} f(x,y) = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})(-x^2 + y^2)$$

$$\vec{\nabla} f(x,y) = (\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z})(-x^2 + y^2)$$

$$\vec{\nabla} f(x,y) = \hat{i}\frac{\delta}{\delta x}(-x^2 + y^2) + \hat{j}\frac{\delta}{\delta y}(-x^2 + y^2) + \hat{k}\frac{\delta}{\delta z}(-x^2 + y^2)$$

$$\vec{\nabla} f(x,y) = \hat{i}\frac{\delta}{\delta x}(-x^2) + \hat{i}\frac{\delta}{\delta x}(y^2) + \hat{j}\frac{\delta}{\delta y}(-x^2) + \hat{j}\frac{\delta}{\delta y}(y^2) + \hat{k}\frac{\delta}{\delta z}(-x^2) + \hat{k}\frac{\delta}{\delta z}(y^2)$$

$$\vec{\nabla} f(x,y) = \hat{i}(-2x) + \hat{i}\times 0 + \hat{j}\times 0 + \hat{j}(2y) + \hat{k}\times 0 + \hat{k}\times 0$$

$$\vec{\nabla} f(x,y) = -2x\,\hat{i} + 2y\,\hat{j}$$

$$\vec{\nabla} \mathbf{f}(2,3) = -2\times 2\,\hat{\mathbf{i}} + 2\times 3\,\hat{\mathbf{j}}$$

$$\vec{\nabla} \mathbf{f}(2,3) = -4\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$$

Hence the gradient Vector is $\vec{\mathbf{RP}} = \vec{\nabla}\mathbf{f}(2,3) = -4\,\hat{\mathbf{i}} + 6\,\hat{\mathbf{j}}$ the Answer

Q# 26: Sketch the level curve for the function $\mathbf{f(x,y) = x^2 + y^2}$ through the point (3, 4) and draw the gradient vector at this point.

Answer: Given, the function $\mathbf{f(x,y) = x^2 + y^2}$ through the point (3, 4),

$\mathbf{f(3,4) = 3^2 + 4^2}$

$\mathbf{f(3,4) = 9 + 16 = 25}$

Since $\mathbf{f(3,4) = 25}$, the level curve through the point $\mathbf{(3,4)}$ has the equation $\mathbf{f(x,y) = x^2 + y^2 = 25}$, which is the circle. That is $x^2 + y^2 = 25$ whose centre $\mathbf{(0,0)}$ and radius 5.

Now,

$\mathbf{f(x,y) = x^2 + y^2}$

Now, Gradient of the function,

$$\vec{\nabla}\mathbf{f(x,y)} = \left(\frac{\delta}{\delta x}\hat{\mathbf{i}} + \frac{\delta}{\delta y}\hat{\mathbf{j}} + \frac{\delta}{\delta z}\hat{\mathbf{k}}\right)(\mathbf{x^2 + y^2})$$

$$\vec{\nabla}\mathbf{f(x,y)} = \left(\hat{\mathbf{i}}\frac{\delta}{\delta x} + \hat{\mathbf{j}}\frac{\delta}{\delta y} + \hat{\mathbf{k}}\frac{\delta}{\delta z}\right)(\mathbf{x^2 + y^2})$$

$$\vec{\nabla}\mathbf{f(x,y)} = \hat{\mathbf{i}}\frac{\delta}{\delta x}(\mathbf{x^2 + y^2}) + \hat{\mathbf{j}}\frac{\delta}{\delta y}(\mathbf{x^2 + y^2}) + \hat{\mathbf{k}}\frac{\delta}{\delta z}(\mathbf{x^2 + y^2})$$

$$\vec{\nabla}\mathbf{f(x,y)} = \hat{\mathbf{i}}\frac{\delta}{\delta x}(\mathbf{x^2}) + \hat{\mathbf{i}}\frac{\delta}{\delta x}(\mathbf{y^2}) + \hat{\mathbf{j}}\frac{\delta}{\delta y}(\mathbf{x^2}) + \hat{\mathbf{j}}\frac{\delta}{\delta y}(\mathbf{y^2}) + \hat{\mathbf{k}}\frac{\delta}{\delta z}(\mathbf{x^2}) + \hat{\mathbf{k}}\frac{\delta}{\delta z}(\mathbf{y^2})$$

$$\vec{\nabla}\mathbf{f(x,y)} = \hat{\mathbf{i}}(\mathbf{2x}) + \hat{\mathbf{i}}\times 0 + \hat{j}\times 0 + \hat{j}(2y) + \hat{k}\times 0 + \hat{k}\times 0$$

$$\vec{\nabla}\mathbf{f(x,y)} = \mathbf{2x}\,\hat{\mathbf{i}} + \mathbf{2y}\,\hat{\mathbf{j}} \qquad \text{-------------------------(i)}$$

The gradient vector at $\mathbf{(3,4)}$ is

$$\therefore \vec{\nabla}\mathbf{f(3,4)} = \mathbf{2\times 3\,\hat{i} + 2\times 4\,\hat{j}}$$

$$\vec{\nabla} f(3,4) = 6\,\hat{i} + 8\,\hat{j} \qquad \text{------------------------(ii)}$$

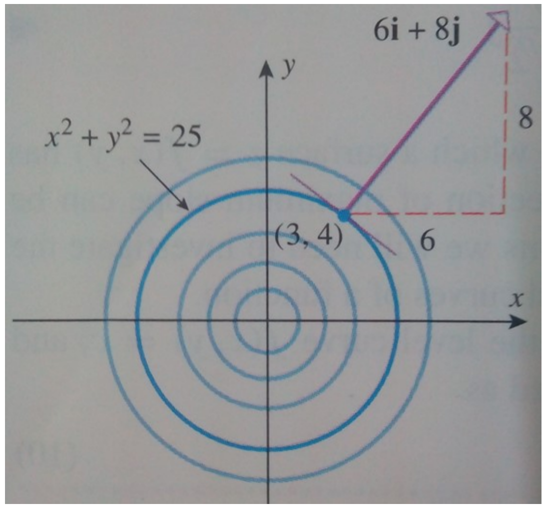


Figure # 62

Hence the gradient vector is perpendicular to the circle at **(3,4)**.

Q# 27: Sketch the gradient field of $\boldsymbol{\phi(x,y) = x + y}$

Answer: Now, the gradient of the function $\boldsymbol{\phi(x,y) = x + y}$

$$\vec{\nabla}\phi(x,y) = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(x+y)$$

$$\vec{\nabla}\phi(x,y) = \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(x+y)$$

$$\vec{\nabla}\phi(x,y) = \hat{i}\frac{\delta}{\delta x}(x+y) + \hat{j}\frac{\delta}{\delta y}(x+y) + \hat{k}\frac{\delta}{\delta z}(x+y)$$

$$\vec{\nabla}\phi(x,y) = \hat{i}(1+0) + \hat{j}(0+1) + \hat{k}(0+0)$$

$$\vec{\nabla}\phi(x,y) = \hat{i} + \hat{j}$$

This is the same at each point. A portion of the vector field is sketched in figure below:

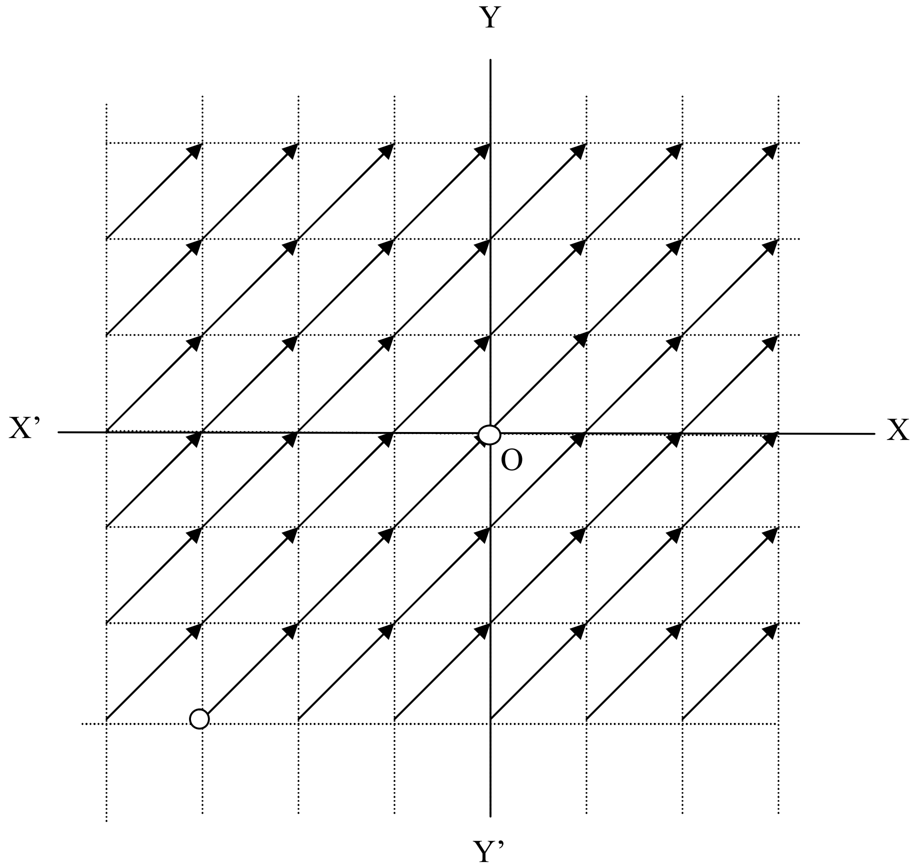


Figure: 63

Q# 28:

Given,

$\varphi(x, y) = x^2 y$

$\dfrac{\delta\phi}{\delta x} = 2xy$ -------------------------(i)

$\dfrac{\delta\phi}{\delta y} = x^2 \times 1$ -------------------------(ii)

We have,

$\varphi(x, y) = x^2 y$

$\therefore \dfrac{d\phi}{dx} = \dfrac{d}{dx}(x^2 y)$

$$= x^2 \frac{d}{dx}(y) + y\frac{d}{dx}(x^2)[\because \frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)]$$

$$= x^2 \frac{dy}{dx} + y \times 2x \times 1$$

$$\therefore \frac{d\phi}{dx} = x^2 \frac{dy}{dx} + 2xy$$

$$\therefore \frac{d\phi}{dx} = 2xy + x^2 \frac{dy}{dx}.$$

Again

$\varphi(\mathrm{x}, \mathrm{y}) = \mathrm{x}^2 y$

$$\therefore \frac{d\phi}{dy} = \frac{d}{dy}(x^2 y)$$

$$= x^2 \frac{d}{dy}(y) + y\frac{d}{dy}(x^2)[\therefore \frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)]$$

$$= x^2 \times 1 + y \times 2x\frac{dx}{dy}$$

$$\therefore \frac{d\phi}{dy} = x^2 + 2xy\frac{dx}{dy}$$

$$\therefore \frac{d\phi}{dy} = 2xy\frac{dx}{dy} + x^2$$

We know,

$$d\phi = \frac{\delta\phi}{\delta x} \times dx + \frac{\delta\phi}{\delta y} \times dy \quad \text{---------------------(iii)}$$

$$\Rightarrow d\phi = 2xy \times dx + x^2 \times dy \quad \text{---------------------(iv)} \quad \text{[From (i) \& (ii)]}$$

$$\Rightarrow \frac{d\phi}{dx} = 2xy \times \frac{dx}{dx} + x^2 \frac{dy}{dx} \quad \text{[Dividing both sides by dx]}$$

$$\therefore \frac{d\phi}{dx} = 2xy + x^2 \frac{dy}{dx}.$$

From equation (iii),

$$d\phi = \frac{\delta\phi}{\delta x} \times dx + \frac{\delta\phi}{\delta y} \times dy$$

$$\Rightarrow d\phi = 2xy \times dx + x^2 \times dy$$

$$\Rightarrow \frac{d\phi}{dy} = 2xy\frac{dx}{dy} + x^2 \frac{dy}{dy} \qquad \text{[Dividing by dy]}$$

$$\therefore \frac{d\phi}{dy} = 2xy\frac{dx}{dy} + x^2$$

Q#29: Show that $\vec{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x,y,z)=c$, where c is a constant.

Answer: Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ **be** the position vector to any point $P(x,y,z)$ **on** the surface.

$\therefore d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ lies in the tangent plane to the surface at P.

Given, $\phi(x,y,z) = c$

$$\Rightarrow d\phi = \frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz = d(c)$$

$$\Rightarrow d\phi = \frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz = 0$$

$$\Rightarrow \frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz = 0 \text{ --(i)}$$

$$\Rightarrow \left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right).(dx\hat{i} + dy\hat{j} + dz\hat{k}) = 0 \text{ -----------------------------(ii)}$$

$$\Rightarrow \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi.(dx\hat{i} + dy\hat{j} + dz\hat{k}) = 0$$

$$\Rightarrow \vec{\nabla}\phi.d\vec{r} = 0 \qquad \left[\because \vec{\nabla} = \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\ ; d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}\right]$$

So that $\vec{\nabla}\phi$ perpendicular to $d\vec{r}$ and therefore to the surface.

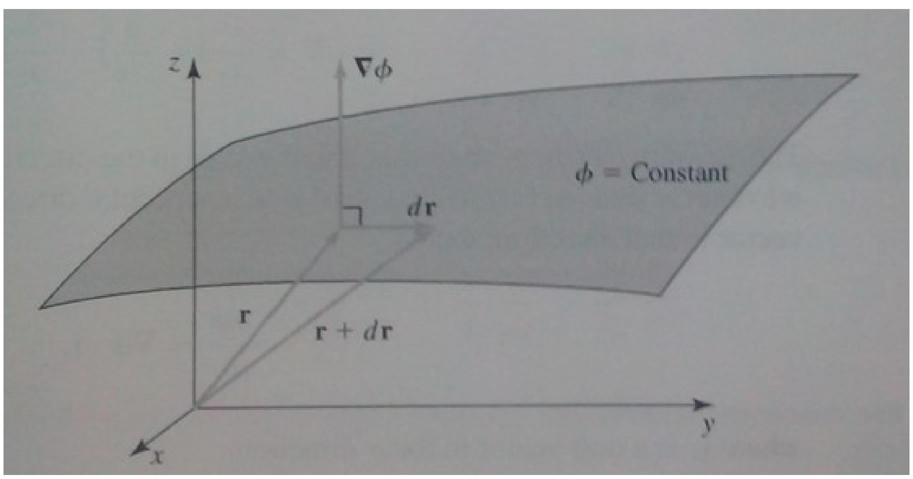


Figure # 64

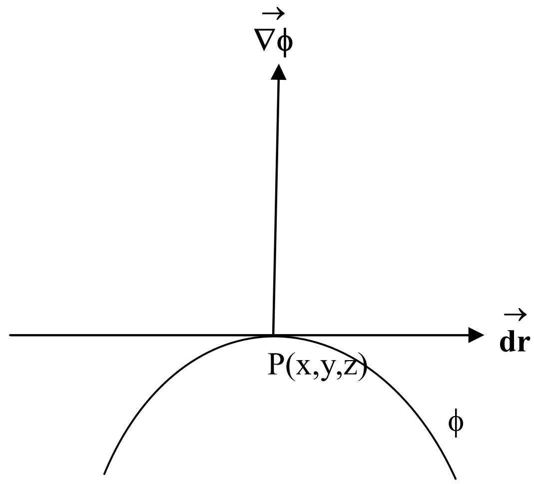


Figure # 65

It is clear that the vector $\vec{\nabla}\phi$ is perpendicular (normal) to the tangent vector $\vec{\mathbf{dr}}$ at a point $\mathbf{P(x,y,z)}$ that is $\vec{\nabla}\phi . d\vec{r} = 0$

So we conclude that $\vec{\nabla}\phi$ is normal (perpendicular) vector to the surface $\phi(x,y,z) = c$ at (x,y,z).

[N.B. We always remember that $\vec{\nabla}\phi$ is perpendicular to the tangent to the surface but not with surface directly and $\vec{\nabla}\phi$ is normal to the surface $\phi(x,y,z) = c$, Where $\vec{\nabla}\phi$ is a vector component that is $\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$]

Q# 30: Find a unit normal to the surface $x^2y + 2xz = 4$ at the point $(2,-2,3)$

Answer: Given, $\phi(x,y,z) = x^2y + 2xz = 4$

$$\therefore \vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi = \left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right) = \hat{i}\frac{\delta}{\delta x}\phi + \hat{j}\frac{\delta}{\delta y}\phi + \hat{k}\frac{\delta}{\delta z}\phi$$

$$= \hat{i}\frac{\delta}{\delta x}(x^2y + 2xz) + \hat{j}\frac{\delta}{\delta y}(x^2y + 2xz) + \hat{k}\frac{\delta}{\delta z}(x^2y + 2xz)$$

$$= (2xy + 2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

$$= (2\times 2\times(-2) + 2\times 3)\hat{i} + 2^2\hat{j} + 2\times 2\hat{k} \quad \text{at the point } (2,-2,3)$$

$$= -2\hat{i} + 4\hat{j} + 4\hat{k}$$

Then a unit normal to the surface $= \dfrac{-2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{(-2)^2 + (4)^2 + (4)^2}} = -\dfrac{1}{3}\hat{i} + \dfrac{2}{3}\hat{j} + \dfrac{2}{3}\hat{k}$ Answer

Q# 31:

Find the level surface of $F(x,y,z) = x^2 + y^2 + z^2$ passing through $(1,1,1)$. Graph the gradient at the point.

Answer: Given, $F(x,y,z) = x^2 + y^2 + z^2$

$$\therefore F(1,1,1) = 1^2 + 1^2 + 1^2 = 3$$

Hence $F(x,y,z) = x^2 + y^2 + z^2 = 3$

Because $F(1,1,1) = 3$, **the level surface passing through (1,1,1) is the sphere** $x^2 + y^2 + z^2 = 3$.

The gradient of the function is

$F(x,y,z) = x^2 + y^2 + z^2$

$$F(x,y,z) = x^2 + y^2 + z^2$$

$$\therefore \nabla F(x,y,z) = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)F$$

$$\therefore \nabla F(x,y,z) = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(x^2 + y^2 + z^2)$$

$$\therefore \nabla F(x,y,z) =$$

$$\hat{i}\frac{\delta}{\delta x}(x^2 + y^2 + z^2) + \hat{j}\frac{\delta}{\delta y}(x^2 + y^2 + z^2) + \hat{k}\frac{\delta}{\delta z}(x^2 + y^2 + z^2)$$

$$\therefore \nabla F(x,y,z) = \hat{i}\frac{\delta}{\delta x}(2x + 0 + 0) + \hat{j}\frac{\delta}{\delta y}(0 + 2y + 0) + \hat{k}\frac{\delta}{\delta z}(0 + 0 + 2z)$$

$$\therefore \nabla F(x,y,z) = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

$$\therefore \nabla F(x,y,z) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

And so, at the given point

$$\therefore \nabla F(1,1,1) = 2.1\hat{i} + 2.1\hat{j} + 2.1\hat{k}$$

$$\therefore \nabla F(1,1,1) = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

The level surface and $\nabla F(1,1,1)$ are illustrated in figure no 63

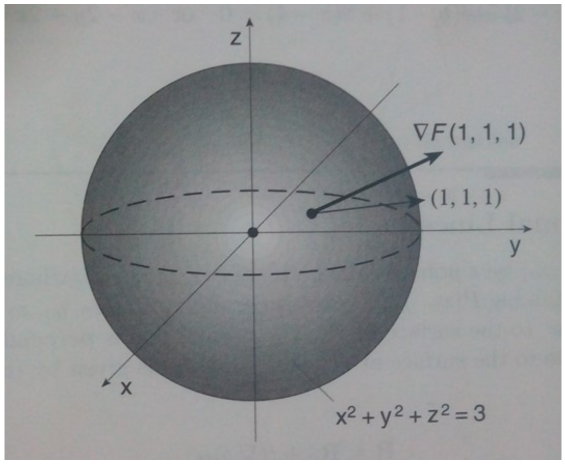


Figure # 66

Q#32: Prove that the angle between the surfaces at the point is equal to the angle between **the normals** to the surfaces at the point.

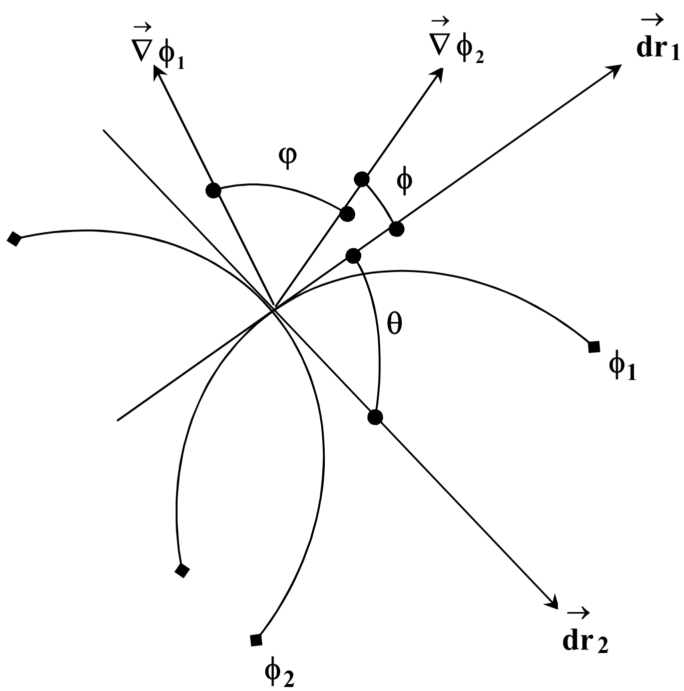


Figure # 67

Here, $\hat{\eta}_1$ is the unit vector of $\vec{\nabla}\phi_1$ and $\hat{\eta}_2$ is the unit vector of $\vec{\nabla}\phi_2$

We can write,

$$\hat{\eta}_1 = \frac{\vec{\nabla}\phi_1}{\left|\vec{\nabla}\phi_1\right|} \text{ and } \hat{\eta}_2 = \frac{\vec{\nabla}\phi_2}{\left|\vec{\nabla}\phi_2\right|}$$

We have,

$$\vec{\mathbf{a}}.\vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\cos\theta$$

$$\therefore \hat{\eta}_1 . \hat{\eta}_2 = \left|\hat{\eta}_1\right|\left|\hat{\eta}_2\right|\cos\varphi \qquad [\because \varphi \text{ be the angle between the normals to the surfaces } \phi_1 \text{ and } \phi_2]$$

$$\therefore \hat{\eta}_1 . \hat{\eta}_2 = \mathbf{1.1\cos\varphi} \qquad [\because \text{The length or magnitude of unit vector is 1}]$$

$$\Rightarrow \frac{\vec{\nabla}\phi_1}{\left|\vec{\nabla}\phi_1\right|} . \frac{\vec{\nabla}\phi_2}{\left|\vec{\nabla}\phi_2\right|} = \mathbf{1.1\cos\varphi}$$

$$\Rightarrow \vec{\nabla}\phi_1 . \vec{\nabla}\phi_2 = \left|\vec{\nabla}\phi_1\right|\left|\vec{\nabla}\phi_2\right|\cos\varphi \text{-------------------------(i)}$$

Again, from figure # 64

$$\varphi + \phi = \mathbf{90}$$
$$\underline{\pm\theta \pm \phi = \pm\mathbf{90}}$$
$$\varphi - \theta = \mathbf{0}$$
$$\therefore \varphi = \theta \text{---(ii)} \textbf{\textit{(Proved)}}$$

Q# 33: Find the angle between the surfaces $\mathbf{x^2 + y^2 + z^2 = 9}$ and $\mathbf{z = x^2 + y^2 - 3}$ at the point (2,-1, 2)

Answer:

Given, $\mathbf{z = x^2 + y^2 - 3}$

$$\Rightarrow \mathbf{x^2 + y^2 - z = 3}$$

Let, $\phi_1(\mathbf{x,y,z}) = \mathbf{x^2 + y^2 + z^2 = 9}$ **and** $\phi_2(\mathbf{x,y,z}) = \mathbf{x^2 + y^2 - z = 3}$

$$\therefore \nabla\phi_1 = \left(\frac{\delta}{\delta x}\hat{\mathbf{i}} + \frac{\delta}{\delta y}\hat{\mathbf{j}} + \frac{\delta}{\delta z}\hat{\mathbf{k}}\right)\phi_1 = \left(\frac{\delta\phi_1}{\delta x}\hat{\mathbf{i}} + \frac{\delta\phi_1}{\delta y}\hat{\mathbf{j}} + \frac{\delta\phi_1}{\delta z}\hat{\mathbf{k}}\right) = \hat{\mathbf{i}}\frac{\delta}{\delta x}\phi_1 + \hat{\mathbf{j}}\frac{\delta}{\delta y}\phi_1 + \hat{\mathbf{k}}\frac{\delta}{\delta z}\phi_1$$

$$= \hat{i}\frac{\delta}{\delta x}(x^2+y^2+z^2)+\hat{j}\frac{\delta}{\delta y}(x^2+y^2+z^2)+\hat{k}\frac{\delta}{\delta z}(x^2+y^2+z^2)$$

$$= 2x\hat{i}+2y\hat{j}+2z\hat{k}$$

$$= 4\hat{i}-2\hat{j}+4\hat{k} \text{ at the point } (2,-1,2)$$

A normal to $x^2+y^2+z^2=9$ at $(2,-1,2)$ is $\nabla\phi_1 = 4\hat{i}-2\hat{j}+4\hat{k}$

Again,

$\phi_2(x,y,z) = x^2+y^2-z=3$

$$\therefore \nabla\phi_2 = (\frac{\delta}{\delta x}\hat{i}+\frac{\delta}{\delta y}\hat{j}+\frac{\delta}{\delta z}\hat{k})\phi_2 = (\frac{\delta\phi_2}{\delta x}\hat{i}+\frac{\delta\phi_2}{\delta y}\hat{j}+\frac{\delta\phi_2}{\delta z}\hat{k}) = \hat{i}\frac{\delta}{\delta x}\phi_2+\hat{j}\frac{\delta}{\delta y}\phi_2+\hat{k}\frac{\delta}{\delta z}\phi_2$$

$$= \hat{i}\frac{\delta}{\delta x}(x^2+y^2-z)+\hat{j}\frac{\delta}{\delta y}(x^2+y^2-z)+\hat{k}\frac{\delta}{\delta z}(x^2+y^2-z)$$

$$= 2x\hat{i}+2y\hat{j}-\hat{k}$$

$$= 4\hat{i}-2\hat{j}-\hat{k} \text{ at the point } (2,-1,2)$$

A normal to $x^2+y^2-z=3$ at $(2,-1,2)$ is $\nabla\phi_2 = 4\hat{i}-2\hat{j}-\hat{k}$

Let φ be the angle between the normals to the surfaces at the point $(2,-1,2)$

Then we have,

$\nabla\phi_1 . \nabla\phi_2 = |\nabla\phi_1|\,|\nabla\phi_2|\cos\varphi$, **[from equation no (i), page no 48]**

$$\Rightarrow (4\hat{i}-2\hat{j}+4\hat{k}).(4\hat{i}-2\hat{j}-\hat{k}) = |4\hat{i}-2\hat{j}+4\hat{k}|\,|4\hat{i}-2\hat{j}-\hat{k}|\cos\varphi$$

$$\Rightarrow 16+4-4 = \sqrt{(4)^2+(-2)^2+(4)^2}\sqrt{(4)^2+(-2)^2+(-1)^2}\cos\varphi$$

$$\Rightarrow 16 = 6\sqrt{21}\cos\varphi$$

$$\Rightarrow \cos\varphi = \frac{16}{6\sqrt{21}}$$

$$\therefore \varphi = \cos^{-1}(\frac{16}{6\sqrt{21}}) \text{ Answer } [\because \varphi=\theta]$$

Q# 34: Prove that $\underset{-}{\nabla}.[\frac{f(r)}{r}\vec{r}] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$

Here $|\vec{r}| = r$

L.H.S.

$$\underset{-}{\nabla}.[\frac{f(r)}{r}\vec{r}]$$

$$= (\hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}+\hat{k}\frac{\delta}{\delta z}).[\frac{f(r)}{r}(x\hat{i}+y\hat{j}+z\hat{k})] \quad [\because \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}]$$

$$=(\hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}+\hat{k}\frac{\delta}{\delta z}).(\hat{i}\frac{f(r)}{r}x+\hat{j}\frac{f(r)}{r}y+\hat{k}\frac{f(r)}{r}z)$$

$$=\frac{\delta}{\delta x}\frac{f(r)}{r}x+\frac{\delta}{\delta y}\frac{f(r)}{r}y+\frac{\delta}{\delta z}\frac{f(r)}{r}z \text{-----------------------------------}(i)$$

$$[\because \hat{i}.\hat{i}=1;\hat{j}.\hat{j}=1;\hat{k}.\hat{k}=1]$$

Now, $\frac{\delta}{\delta x}\frac{f(r)}{r}x$

$$=\frac{\delta}{\delta x}\{\frac{f(r)}{r}x\}$$

$$=\frac{f(r)}{r}\frac{\delta}{\delta x}(x)+x\frac{\delta}{\delta x}\frac{f(r)}{r}$$

$$=\frac{f(r)}{r}.1+x\frac{\delta}{\delta x}\frac{f(r)}{r}$$

$$=\frac{f(r)}{r}+x\frac{\delta}{\delta x}\frac{f(r)}{r}$$

$$=\frac{f(r)}{r}+x\frac{\delta}{\delta x}\{f(r)r^{-1}\}$$

$$=\frac{f(r)}{r}+x[f(r)\frac{\delta}{\delta x}(r^{-1})+r^{-1}\frac{\delta}{\delta x}\{f(r)\}]$$

$$=\frac{f(r)}{r}+x[f(r)(-1)(r^{-2})\frac{\delta r}{\delta x}+r^{-1}\frac{\delta}{\delta x}\{f(r)\}]$$

$$=\frac{f(r)}{r}+x[f(r)(-1)(r^{-2})\frac{\delta r}{\delta x}+r^{-1}\{f^{'}(r)\}\frac{\delta r}{\delta x}]$$

$$=\frac{f(r)}{r}+x[-f(r)(r^{-2})\frac{\delta r}{\delta x}+r^{-1}\{f^{'}(r)\}\frac{\delta r}{\delta x}]$$

$$=\frac{f(r)}{r}+x[r^{-1}\{f^{'}(r)\}\frac{\delta r}{\delta x}-f(r)(r^{-2})\frac{\delta r}{\delta x}]$$

$$=\frac{f(r)}{r}+x[\frac{1}{r}\{f^{'}(r)\}\frac{\delta r}{\delta x}-f(r)\frac{1}{r^2}\frac{\delta r}{\delta x}]$$

$$=\frac{f(r)}{r}+x[\frac{f^{'}(r)}{r}-\frac{f(r)}{r^2}]\frac{\delta r}{\delta x}$$

$$=\frac{f(r)}{r}+x[\frac{f^{'}(r)}{r}-\frac{f(r)}{r^2}]\frac{\delta}{\delta x}(x^2+y^2+z^2)^{1/2}$$

$$[\because \left|\vec{r}\right|=r=\sqrt{x^2+y^2+z^2}=(x^2+y^2+z^2)^{1/2}]$$

$$=\frac{f(r)}{r}+x[\frac{f^{'}(r)}{r}-\frac{f(r)}{r^2}]\frac{1}{2}(x^2+y^2+z^2)^{-1/2}.\frac{\delta}{\delta x}(x^2+y^2+z^2)$$

$$= \frac{f(r)}{r} + x[\frac{f'(r)}{r} - \frac{f(r)}{r^2}]\frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}.(2x)$$

$$= \frac{f(r)}{r} + x[\frac{f'(r)}{r} - \frac{f(r)}{r^2}](x^2 + y^2 + z^2)^{-1/2}.(x)$$

$$= \frac{f(r)}{r} + x[\frac{f'(r)}{r} - \frac{f(r)}{r^2}]\frac{x}{(x^2 + y^2 + z^2)^{1/2}}$$

$$= \frac{f(r)}{r} + x[\frac{f'(r)}{r} - \frac{f(r)}{r^2}]\frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{f(r)}{r} + x[\frac{f'(r)}{r} - \frac{f(r)}{r^2}]\frac{x}{r}$$

$$= \frac{f(r)}{r} + \frac{x^2 f'(r)}{r^2} - \frac{x^2 f(r)}{r^3} \text{------------------------------------(ii)}$$

Similarly,

$$\frac{\delta}{\delta y}\frac{f(r)}{r}y = \frac{f(r)}{r} + \frac{y^2 f'(r)}{r^2} - \frac{y^2 f(r)}{r^3} \text{-----------------------(iii)}$$

and

$$\frac{\delta}{\delta z}\frac{f(r)}{r}z = \frac{f(r)}{r} + \frac{z^2 f'(r)}{r^2} - \frac{z^2 f(r)}{r^3} \text{-----------------------(iv)}$$

Putting the value of (ii), (iii) and (iv) in (i)

$$\underset{-}{\nabla}.[\frac{f(r)}{r}\vec{r}] = \frac{\delta}{\delta x}\frac{f(r)}{r}x + \frac{\delta}{\delta y}\frac{f(r)}{r}y + \frac{\delta}{\delta z}\frac{f(r)}{r}z$$

$$= \frac{f(r)}{r} + \frac{x^2 f'(r)}{r^2} - \frac{x^2 f(r)}{r^3} + \frac{f(r)}{r} + \frac{y^2 f'(r)}{r^2} - \frac{y^2 f(r)}{r^3} + \frac{f(r)}{r} + \frac{z^2 f'(r)}{r^2} - \frac{z^2 f(r)}{r^3}$$

$$= 3\frac{f(r)}{r} + \frac{x^2 f'(r)}{r^2} - \frac{x^2 f(r)}{r^3} + \frac{y^2 f'(r)}{r^2} - \frac{y^2 f(r)}{r^3} + \frac{z^2 f'(r)}{r^2} - \frac{z^2 f(r)}{r^3}$$

$$= 3\frac{f(r)}{r} + \frac{f'(r)}{r^2}(x^2 + y^2 + z^2) - \frac{f(r)}{r^3}(x^2 + y^2 + z^2)$$

$$= 3\frac{f(r)}{r} + \frac{f'(r)}{r^2}r^2 - \frac{f(r)}{r^3}r^2$$

$$= 3\frac{f(r)}{r} + f'(r) - \frac{f(r)}{r}$$

$$= 2\frac{f(r)}{r} + f'(r)$$

$$= \frac{1}{r^2}.r^2[2\frac{f(r)}{r} + f'(r)]$$

$$= \frac{1}{r^2}[2rf(r) + r^2 f'(r)]$$

$$= \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)] \; [\because \frac{d}{dr}[r^2 f(r)] = 2rf(r) + r^2 f'(r)]$$

(Proved)

Directional derivatives

I. **MMMMMMMMMM**

II.

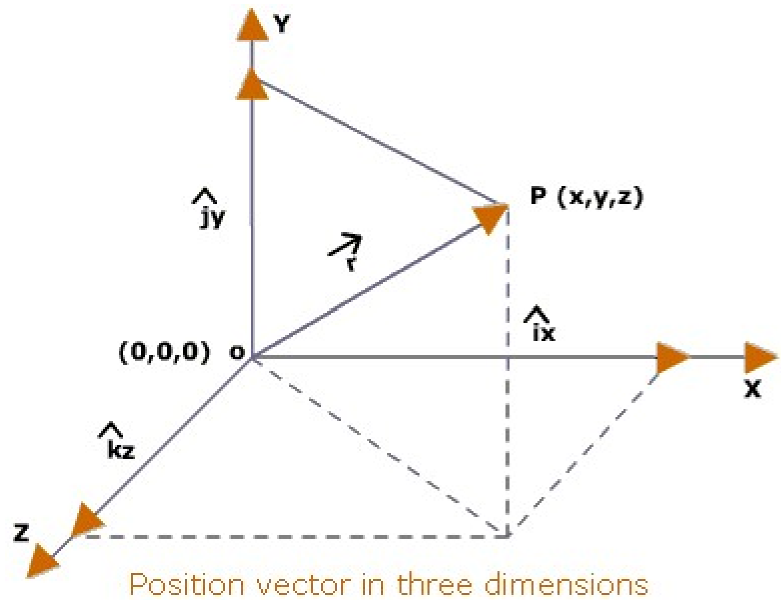


Figure # 68

Let $\overrightarrow{OP}$ is a position vector $\vec{r}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $d\vec{r}$ is a small displacement corresponding to changes dx, dy, dz in x, y, z respectively, then

$\overrightarrow{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ **…………………….....……..(i)**

If $\phi(x, y, z)$ is a scalar function at P, then the *gradient* of ϕ

$grad\ \phi = \vec{\nabla}\phi = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})\phi$ **…………………**(ii)

Then $\text{grad}\,\phi . d\vec{r} = \vec{\nabla}\phi . d\vec{r} = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})\phi . (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$$= (\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}) . (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\delta\phi}{\delta x}dx + \frac{\delta\phi}{\delta y}dy + \frac{\delta\phi}{\delta z}dz$$

$$= d\phi$$

= **The total differential $d\phi$ of ϕ**

$$\text{grad}\,\phi . d\vec{r} = d\phi$$

$$d\phi = \text{grad}\,\phi . d\vec{r}$$

$$d\phi = \vec{\nabla}\phi . d\vec{r} \text{ ----------------------------------(iii)}$$

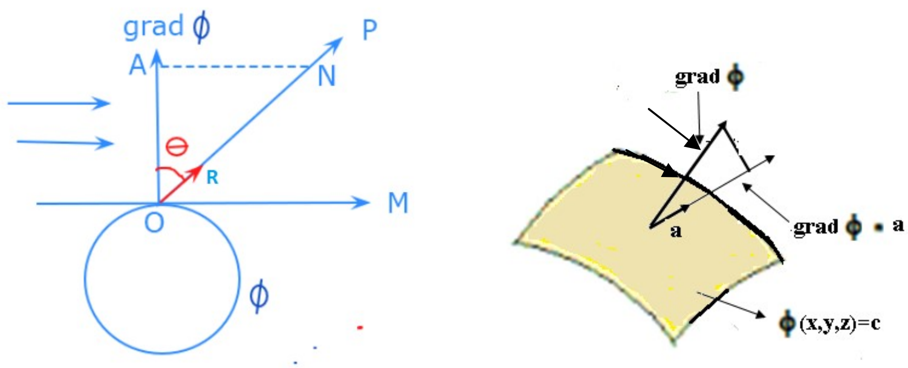


Figure # 69

We have just established that

$\boxed{\text{A}}$

$$d\phi = dr \,.\, \text{grad}\,\phi$$

If ds is the small element of arc between P(r) and Q(r+dr) then ds= $\left|d\vec{r}\right|$

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{\left|d\vec{r}\right|}$$

and $\frac{d\vec{r}}{ds}$ is thus a unit vector in the direction of $d\vec{r}$.

$$\therefore \frac{d\phi}{ds} = \frac{d\vec{r}}{ds} \,.\, \text{Grad}\ \phi$$

If we denote this unit vector by $\hat{a}$, i.e. $\frac{dr}{ds} = \hat{a}$, the result becomes

$$\frac{d\phi}{ds} = \hat{a} \,.\, \text{Grad}\ \phi$$

$\frac{d\phi}{ds}$ is the projection of grad ϕ on the unit vector $\hat{a}$ is called the directional derivative of ϕ in the direction of $\hat{a}$ .It gives the rate of change of ϕ with distance measured in the direction of $\hat{a}$ and $\frac{d\phi}{ds} = \hat{a}$. Grad ϕ will be a maximum when $\hat{a}$ and grad ϕ have the same direction, since then, $\hat{a}\,.\,\boldsymbol{grad}\,\phi = \left|\hat{a}\right|\left|\boldsymbol{grad}\,\phi\right|\cos\theta$ and θ will be zero

Thus direction of grad ϕ gives the direction in which the maximum rate of change of ϕ occurs.

Q# 35: Find the directional derivative of the function $\phi = x^2 z + 2xy^2 + yz^2$ at the point of (1, 2,-1) in the direction of the vector $\vec{A} = 2\hat{i} + 3\hat{j} + \hat{k}$.

We start off with $\phi = x^2 z + 2xy^2 + yz^2$

$$\therefore \nabla\phi = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})\phi$$

$$\therefore \nabla\phi = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})(x^2 z + 2xy^2 + yz^2)$$

$$\therefore \nabla\phi =$$

$$\hat{i}\frac{\delta}{\delta x}(x^2 z + 2xy^2 + yz^2) + \hat{j}\frac{\delta}{\delta y}(x^2 z + 2xy^2 + yz^2) + \hat{k}\frac{\delta}{\delta z}(x^2 z + 2xy^2 + yz^2)$$

$$\therefore \nabla\phi = \hat{i}(2xz + 2.1.y^2 + 0) + \hat{j}(0 + 2x.2y + 1.z^2) + \hat{k}(x^2.1 + 0 + y.2z)$$

$$\therefore \nabla\phi = \hat{i}(2xz + 2y^2) + \hat{j}(4xy + z^2) + \hat{k}(x^2 + 2yz)$$

At the point $(1,2,-1)$

$$\nabla\phi = \hat{i}(2xz + 2y^2) + \hat{j}(4xy + z^2) + \hat{k}(x^2 + 2yz)$$

$$\therefore \nabla\phi = \hat{i}[2.1.(-1) + 2(2^2)] + \hat{j}[4.1.2 + (-1)^2] + \hat{k}[(1^2 + 2.2.(-1)]$$

$$\therefore \nabla\phi = \hat{i}[-2 + 8] + \hat{j}[8 + 1)] + \hat{k}[(1 - 4]$$

$$\therefore \nabla\phi = \hat{i}[6] + \hat{j}[9)] + \hat{k}(-3)$$

$$\therefore \nabla\phi = 6\hat{i} + 9\hat{j} - 3\hat{k}$$

Next we have to find out the unit vector $\hat{a}$ where $\vec{A} = 2\hat{i} + 3\hat{j} - 4\hat{k}$

$$\therefore \hat{a} = \frac{\vec{A}}{|A|}$$

$$\therefore \hat{a} = \frac{2\hat{i} + 3\hat{j} - 4\hat{k}}{\left|2^2 + 3^2 + (-4)^2\right|}$$

$$\therefore \hat{a} = \frac{2\hat{i} + 3\hat{j} - 4\hat{k}}{|4+9+16|}$$

$$\therefore \hat{a} = \frac{2\hat{i} + 3\hat{j} - 4\hat{k}}{|29|}$$

Hence $\dfrac{d\phi}{ds} = \hat{a} . \vec{\nabla}\phi$

$$\frac{d\phi}{ds} = \frac{2\hat{i} + 3\hat{j} - 4\hat{k}}{|29|}.(6\hat{i} + 9\hat{j} - 3\hat{k})$$

$$\frac{d\phi}{ds} = \frac{1}{|29|}(2\hat{i} + 3\hat{j} - 4\hat{k}).(6\hat{i} + 9\hat{j} - 3\hat{k})$$

$$\frac{d\phi}{ds} = \frac{1}{|29|}(12 + 27 + 12)$$

$$\frac{d\phi}{ds} = \frac{51}{|29|} \textit{ Answer}$$

Q#36: Find the directional derivative of the function $\phi = x^2 y + y^2 z + z^2 x$ at the point of (1, -1, 2) in the direction of the vector $\vec{A} = 4\hat{i} + 2\hat{j} - 5\hat{k}$.

We have $\phi = x^2 y + y^2 z + z^2 x$

$$\therefore \nabla\phi = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})\phi$$

$$\therefore \nabla\phi = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})(x^2 y + y^2 z + z^2 x)$$

$$\therefore \nabla\phi = \hat{i}\frac{\delta}{\delta x}(x^2 y + y^2 z + z^2 x) + \hat{j}\frac{\delta}{\delta y}(x^2 y + y^2 z + z^2 x) + \hat{k}\frac{\delta}{\delta z}(x^2 y + y^2 z + z^2 x)$$

$$\therefore \nabla\phi = \hat{i}(2xy + 0 + z^2.1) + \hat{j}(x^2.1 + 2yz + 0) + \hat{k}(0 + y^2.1 + 2zx)$$

$$\therefore \nabla\phi = \hat{i}(2xy + z^2) + \hat{j}(x^2 + 2yz) + \hat{k}(y^2 + 2zx)$$

At the point $(1,-1,2)$

$$\therefore \nabla\phi = \hat{i}(2xy + z^2) + \hat{j}(x^2 + 2yz) + \hat{k}(y^2 + 2zx)$$

$$\therefore \nabla\phi = \hat{i}[2.1.(-1) + 2^2)] + \hat{j}[1^2 + 2(-1).2] + \hat{k}[(-1)^2 + 2.2.1]$$

$$\therefore \nabla\phi = \hat{i}[-2 + 4] + \hat{j}[1 - 4] + \hat{k}[1 + 4]$$

$$\therefore \nabla\phi = 2\hat{i} - 3\hat{j} + 5\hat{k}$$

Next we have to find out the unit vector $\hat{a}$ where $\vec{A} = 4\hat{i} + 2\hat{j} - 5\hat{k}$

$$\therefore \hat{\boldsymbol{a}} = \frac{\vec{\boldsymbol{A}}}{|\boldsymbol{A}|}$$

$$\therefore \hat{a} = \frac{4\hat{i}+2\hat{j}-5\hat{k}}{\sqrt{4^2+2^2+(-5)^2}}$$

$$\therefore \hat{a} = \frac{4\hat{i}+2\hat{j}-5\hat{k}}{\sqrt{45}}$$

Hence $\frac{\boldsymbol{d}\phi}{\boldsymbol{ds}} = \hat{\boldsymbol{a}}.\vec{\nabla}\phi$

$$\frac{d\phi}{ds} = \frac{(4\hat{i}+2\hat{j}-5\hat{k})}{\sqrt{45}}.(2\hat{i}-3\hat{j}+5\hat{k})$$

$$\frac{d\phi}{ds} = \frac{1}{\sqrt{45}}(4\hat{i}+2\hat{j}-5\hat{k}).(2\hat{i}-3\hat{j}+5\hat{k})$$

$$\frac{d\phi}{ds} = \frac{1}{\sqrt{45}}(8-6-25)$$

$$\frac{d\phi}{ds} = \frac{1}{\sqrt{45}}(-23) \textit{ Answer}$$

Q#37: Find the directional derivative of the function $\phi = (x,y,z) = x^2 - y^2 + 2z^2$ at the point of (1,2,3) in the direction of the vector $\vec{A} = 4\hat{i} - 2\hat{j} + \hat{k}$.

Answer: Let, $\phi(x,y,z) = x^2 - y^2 + 2z^2$

$$\frac{\delta\phi}{\delta x} = \frac{\delta}{\delta x}(x^2 - y^2 + 2z^2)$$

$$\frac{\delta\phi}{\delta x} = (2x - 0 + 0)$$

$$\frac{\delta\phi}{\delta x} = 2x$$

$$\phi = (x,y,z) = x^2 - y^2 + 2z^2$$

$$\frac{\delta\phi}{\delta y} = \frac{\delta}{\delta y}(x^2 - y^2 + 2z^2)$$

$$\frac{\delta\phi}{\delta y} = (0 - 2y + 0)$$

$$\frac{\delta\phi}{\delta y} = -2y$$

$$\phi(x,y,z) = x^2 - y^2 + 2z^2$$

$$\frac{\delta\phi}{\delta z}=\frac{\delta}{\delta z}(x^2-y^2+2z^2)$$
$$\frac{\delta\phi}{\delta z}=(0-0+4z)$$
$$\frac{\delta\phi}{\delta z}=4z$$

$$\phi(x,y,z)=x^2-y^2+2z^2$$
$$\text{grad } \phi=\vec{\nabla}\phi=\left(\frac{\delta}{\delta \boldsymbol{x}}\hat{\boldsymbol{i}}+\frac{\delta}{\delta \boldsymbol{y}}\hat{\boldsymbol{j}}+\frac{\delta}{\delta \boldsymbol{z}}\hat{\boldsymbol{k}}\right)\phi$$
$$\text{grad } \phi=\vec{\nabla}\phi=\left(\frac{\delta}{\delta x}\hat{i}+\frac{\delta}{\delta y}\hat{j}+\frac{\delta}{\delta z}\hat{k}\right)(x^2-y^2+2z^2)$$
$$\text{grad } \phi=\vec{\nabla}\phi=\left(\hat{i}\frac{\delta}{\delta x}+\hat{j}\frac{\delta}{\delta y}+\hat{k}\frac{\delta}{\delta z}\right)(x^2-y^2+2z^2)$$
$$\text{grad } \phi=\vec{\nabla}\phi=\hat{i}\frac{\delta}{\delta x}(x^2-y^2+2z^2)+\hat{j}\frac{\delta}{\delta y}(x^2-y^2+2z^2)+\hat{k}\frac{\delta}{\delta z}(x^2-y^2+2z^2)$$
$$\text{grad } \phi=\vec{\nabla}\phi=\hat{i}\,2x+\hat{j}(-2y)+\hat{k}\,4z$$
$$\text{grad } \phi=\vec{\nabla}\phi(1,2,3)=\hat{i}(2\times1)+\hat{j}(-2\times2)+\hat{k}(4\times3)$$
$$\text{grad } \phi=\vec{\nabla}\phi(1,2,3)=2\hat{i}-4\hat{j}+12\hat{k}$$

Given ,

$$\vec{A}=4\hat{i}-2\hat{j}+\hat{k}$$
$$\hat{A}=\left|\frac{\vec{A}}{|\vec{A}|}\right|=\frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{(4)^2+(-2)^2+(1)^2}}=\frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{16+4+1}}=\frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{21}}$$

$\therefore$ The directional derivative of the function $\phi=(x,y,z)=x^2-y^2+2z^2$ at the point of (1,2,3) in the direction of the vector $\vec{A}=\vec{\nabla}\phi.\,\hat{A}=(2\hat{i}-4\hat{j}+12\hat{k}).\,\frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{21}}$

$$=\frac{8+8+12}{\sqrt{21}}=\frac{28}{\sqrt{21}}$$

Q# 38 Suppose that over a certain region of space the electrical potential V is given by $\phi(x,y,z)=5x^2-3xy+xyz$

(i) Find the rate of change (derivative) of the potential at P(3,4,3) in the direction of the vector $\vec{v}=\hat{i}+\hat{j}-\hat{k}$

(ii) In which direction does ϕ changes most rapidly at P?

(iii) What is the maximum rate of change at P?

Answer: grad $\phi = \vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi$

$$= \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)(5x^2 - 3xy + xyz)$$

$$= \left(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}\right)(5x^2 - 3xy + xyz)$$

$$= \hat{i}\frac{\delta}{\delta x}(5x^2 - 3xy + xyz) + \hat{j}\frac{\delta}{\delta y}(5x^2 - 3xy + xyz) + \hat{k}\frac{\delta}{\delta z}(5x^2 - 3xy + xyz)$$

$$= \hat{i}(10x - 3y + yz) + \hat{j}(-3x + xz) + \hat{k}(xy)$$

At P (3, 4, 3),

$$\vec{\nabla}\phi = \hat{i}(10x - 3y + yz) + \hat{j}(-3x + xz) + \hat{k}(xy)$$

$$\vec{\nabla}\phi = \hat{i}(10\times 3 - 3\times 4 + 4\times 3) + \hat{j}(-3\times 3 + 3\times 3) + \hat{k}(3\times 4)$$

$$\vec{\nabla}\phi = \hat{i}(30 - 12 + 12) + \hat{j}(-9 + 9) + \hat{k}(12)$$

$$\vec{\nabla}\phi = \hat{i}(30) + 12\hat{k}$$

Given, $\vec{v} = \hat{i} + \hat{j} - \hat{k}$; the unit vector of $\vec{v} = \hat{a} = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{i} + \hat{j} - \hat{k}}{\left|\sqrt{1^2 + 1^2 + (-1)^2}\right|} = \frac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}$

i)

$$\vec{\nabla}\phi \,.\, \hat{a} = [\hat{i}(30) + 12\hat{k}].\left[\frac{\hat{i} + \hat{j} - \hat{k}}{\sqrt{3}}\right]$$

$$\vec{\nabla}\phi \,.\, \hat{a} = [\hat{i}(30) + 12\hat{k}].\left[\frac{\hat{i}}{\sqrt{3}} + \frac{\hat{j}}{\sqrt{3}} - \frac{\hat{k}}{\sqrt{3}}\right]$$

$$\vec{\nabla}\phi \,.\, \hat{a} = \left[\frac{30}{\sqrt{3}} - \frac{12}{\sqrt{3}}\right]$$

$$\vec{\nabla}\phi \,.\, \hat{a} = \frac{30 - 12}{\sqrt{3}} = \frac{18}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$$

ii)

$$\vec{\nabla}\phi = \hat{i}(30) + 12\hat{k}$$

iii)

$$\left|\vec{\nabla}\phi\right| = \sqrt{(30)^2 + (12)^2} = \sqrt{900 + 144} = \sqrt{1044}$$

Q# 39: If $\vec{\mathbf{V}}(\mathbf{x},\mathbf{y},\mathbf{z}) = \mathbf{xz}\,\hat{\mathbf{i}} + \mathbf{xyz}\,\hat{\mathbf{j}} - \mathbf{y}^2\,\hat{\mathbf{k}}$ Find divergence of $\vec{\mathbf{V}}$ that is $\mathbf{div}\,\vec{\mathbf{V}}$

Answer: $\mathbf{div\,\vec{V} = \nabla.\vec{V} = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).\,(xz\hat{i} + xyz\hat{j} - y^2\hat{k})}$

$$= \frac{\delta}{\delta x}(xz) + \frac{\delta}{\delta y}(xyz) - \frac{\delta}{\delta z}(y^2)\ [\therefore \hat{i}.\hat{i} = 1,\ \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{k}.\hat{i} = 0\ \text{etc.}]$$

$= \mathbf{z + xz}$ Answer

Q# 40: Let $\vec{V}$ be a constant vector field. Show that $div\,\vec{V} = 0$

Answer: Let, $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$, where a, b, c are constants, Then

$$div\,\vec{V} = \nabla.\vec{V} = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).\,(a\hat{i} + b\hat{j} + c\hat{k})$$

$$= \frac{\delta}{\delta x}(a) + \frac{\delta}{\delta y}(b) + \frac{\delta}{\delta z}(c) \qquad [\therefore \hat{i}.\hat{i} = 1,\ \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{k}.\hat{i} = 0\ \text{etc.}]$$

$= 0$

Q# 41: what is solenoidal?

Answer: If $\vec{A}$ is solenoidal then $\vec{\nabla}.\vec{A} = 0$

Q# 42:Show that the vector field $\vec{v} = \frac{-x\hat{i} - y\hat{j}}{\sqrt{x^2 + y^2}}$ is a "sink field". Plot and give a physical interpretation. [Here V is a unit vector; Since it is divided by its length]

Answer: given,

$$\vec{v} = \frac{-x\hat{i} - y\hat{j}}{\sqrt{x^2 + y^2}}$$

$$\vec{v} = \frac{-x\hat{i} - y\hat{j}}{\sqrt{(-x)^2 + (-y)^2}}$$

$$\vec{v} = \frac{-x\hat{i} - y\hat{j}}{\sqrt{x^2 + y^2}}$$

$$\vec{v} = \frac{-x}{\sqrt{x^2 + y^2}}\hat{i} + \frac{-y}{\sqrt{x^2 + y^2}}\hat{j}$$

$$\therefore div\,\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).\,(\frac{-x}{\sqrt{x^2 + y^2}}\hat{i} + \frac{-y}{\sqrt{x^2 + y^2}}\hat{j} + 0.\hat{k})$$

$$div\,\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{\delta}{\delta x}(\frac{-x}{\sqrt{x^2 + y^2}}) + \frac{\delta}{\delta y}(\frac{-y}{\sqrt{x^2 + y^2}})$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{\delta}{\delta x}\left(\frac{-x}{\sqrt{x^2+y^2}}\right) + \frac{\delta}{\delta y}\left(\frac{-y}{\sqrt{x^2+y^2}}\right)$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{(\sqrt{x^2+y^2})\frac{\delta}{\delta x}(-x) - (-x)\frac{\delta}{\delta x}(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^2} + \frac{(\sqrt{x^2+y^2})\frac{\delta}{\delta y}(-y) - (-y)\frac{\delta}{\delta y}(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{(\sqrt{x^2+y^2})(-1) + x\frac{\delta}{\delta x}(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^2} + \frac{(\sqrt{x^2+y^2})(-1) + y\frac{\delta}{\delta y}(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-(\sqrt{x^2+y^2}) + x\frac{\delta}{\delta x}(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^2} + \frac{-(\sqrt{x^2+y^2}) + y\frac{\delta}{\delta y}(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-(\sqrt{x^2+y^2}) + x\frac{\delta}{\delta x}(x^2+y^2)^{1/2}}{(\sqrt{x^2+y^2})^2} + \frac{-(\sqrt{x^2+y^2}) + y\frac{\delta}{\delta y}(x^2+y^2)^{1/2}}{(\sqrt{x^2+y^2})^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-(\sqrt{x^2+y^2}) + x\frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1}.\frac{\delta}{\delta x}(x^2+y^2)}{(\sqrt{x^2+y^2})^2} + \frac{-(\sqrt{x^2+y^2}) + y\frac{1}{2}(x^2+y^2)^{\frac{1}{2}-1}.\frac{\delta}{\delta y}(x^2+y^2)}{(\sqrt{x^2+y^2})^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-(\sqrt{x^2+y^2}) + x\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}.(2x)}{(\sqrt{x^2+y^2})^2} + \frac{-(\sqrt{x^2+y^2}) + y\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}.(2y)}{(\sqrt{x^2+y^2})^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-(\sqrt{x^2+y^2}) + x^2(x^2+y^2)^{-\frac{1}{2}}}{(\sqrt{x^2+y^2})^2} + \frac{-(\sqrt{x^2+y^2}) + y^2(x^2+y^2)^{-\frac{1}{2}}}{(\sqrt{x^2+y^2})^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-(\sqrt{x^2+y^2}) + \frac{x^2}{(x^2+y^2)^{\frac{1}{2}}}}{x^2+y^2} + \frac{-(\sqrt{x^2+y^2}) + \frac{y^2}{(x^2+y^2)^{\frac{1}{2}}}}{x^2+y^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-(\sqrt{x^2+y^2}) + \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{-(\sqrt{x^2+y^2}) + \frac{y^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{\frac{-x^2-y^2+x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{\frac{-x^2-y^2+y^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{\frac{-y^2}{\sqrt{x^2+y^2}}}{x^2+y^2} + \frac{\frac{-x^2}{\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-y^2}{(x^2+y^2)(\sqrt{x^2+y^2})} + \frac{-x^2}{(x^2+y^2)(\sqrt{x^2+y^2})}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-y^2-x^2}{(x^2+y^2)(\sqrt{x^2+y^2})}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-(x^2+y^2)}{(x^2+y^2)(\sqrt{x^2+y^2})}$$

$$\operatorname{div}\vec{v} \equiv \vec{\nabla}.\vec{v} \equiv \frac{-1}{\sqrt{x^2+y^2}} < 0$$

So, $\vec{v}$ a “sink field”

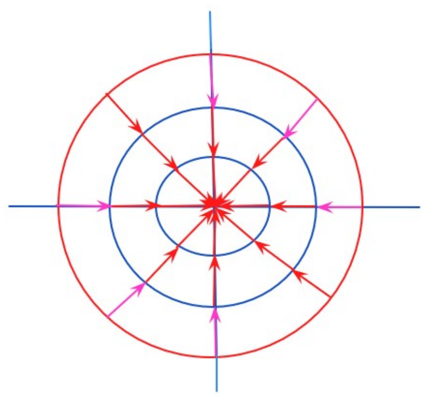


Figure # 70

Q# 43: Let $\vec{V}$ be a constant vector field. Show that $\mathbf{curl}\,\vec{V} = 0$

Answer Let, $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$**, where a, b, c are constants, Then**

$$\mathbf{curl}\,\vec{V} = \nabla \times \vec{V} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \times (a\hat{i} + b\hat{j} + c\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ a & b & c \end{vmatrix}$$

$$= \hat{i}\left(\frac{\delta c}{\delta y} - \frac{\delta b}{\delta z}\right) - \hat{j}\left(\frac{\delta c}{\delta x} - \frac{\delta a}{\delta z}\right) + \hat{k}\left(\frac{\delta b}{\delta x} - \frac{\delta a}{\delta y}\right)$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(0-0)$$
$$= 0$$

Q# 44: If $\vec{v}(x,y,z) = xz\,\hat{i} + xyz\,\hat{j} - y^2\,\hat{k}$ **Find curl** $\vec{V}$

Answer: The curl of a vector field $\vec{v}(x,y,z) = xz\,\hat{i} + xyz\,\hat{j} - y^2\,\hat{k}$ is defined by

$$\nabla \times \vec{v} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \times (xz\,\hat{i} + xyz\,\hat{j} - y^2\,\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xz & xyz & -y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}(-y^2) - \frac{\delta}{\delta z}(xyz)\right] - \hat{j}\left[\frac{\delta}{\delta x}(-y^2) - \frac{\delta}{\delta z}(xz)\right] + \hat{k}\left[\frac{\delta}{\delta x}(xyz) - \frac{\delta}{\delta y}(xz)\right]$$

$$= \hat{i}[-2y - xy] - \hat{j}[0 - x] + \hat{k}[yz - 0]$$

$$= -[2y + xy]\hat{i} + \hat{j}x + \hat{k}yz$$

$$= -[2y + xy]\hat{i} + x\hat{j} + yz\hat{k}$$

Q# 45: What is i**rrotational Field or conservative vector field?**

A vector field $\vec{V}$ for which the curl vanishes, that is: $\vec{V} \times \vec{V} = 0$

Q# 46: Determine $\vec{F}$ is a conservative vector field or not where

$\vec{F} = x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}$

Answer:

So all that we need to do is compute the curl and see if we get the zero vector or not.

The curl of a vector field $\vec{F} = x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}$ is defined by

$$\nabla \times \vec{F} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \times (x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2y & xyz & -x^2y^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}(-x^2y^2) - \frac{\delta}{\delta z}(xyz)\right] - \hat{j}\left[\frac{\delta}{\delta x}(-x^2y^2) - \frac{\delta}{\delta z}(x^2y)\right]$$
$$+ \hat{k}\left[\frac{\delta}{\delta x}(xyz) - \frac{\delta}{\delta y}(x^2y)\right]$$

$$= \hat{i}[-2x^2y - xy] - \hat{j}[-2xy^2 - 0] + \hat{k}[yz - x^2]$$

$\neq \mathbf{0}$

So, the curl isn't the zero vectors and so this vector field is not conservative.

Q# 47: If $\vec{\mathbf{A}} = (\mathbf{xz^3}\,\hat{\mathbf{i}} - \mathbf{2x^2yz}\,\hat{\mathbf{j}} + \mathbf{2yz^4}\,\hat{\mathbf{k}})$, find $\nabla \times \vec{\mathbf{A}}$ (or curl A) at the point (1,-1, 1).

Answer:

$$\nabla \times \vec{\mathbf{A}} = (\frac{\delta}{\delta x}\hat{\mathbf{i}} + \frac{\delta}{\delta y}\hat{\mathbf{j}} + \frac{\delta}{\delta z}\hat{\mathbf{k}}) \times (\mathbf{xz^3}\,\hat{\mathbf{i}} - \mathbf{2x^2yz}\,\hat{\mathbf{j}} + \mathbf{2yz^4}\,\hat{\mathbf{k}})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{\mathbf{i}}[\frac{\delta}{\delta y}(2yz^4) - \frac{\delta}{\delta z}(-2x^2yz)] - \hat{\mathbf{j}}[\frac{\delta}{\delta x}(2yz^4) - \frac{\delta}{\delta z}(xz^3)] + \hat{\mathbf{k}}[\frac{\delta}{\delta x}(-2x^2yz) - \frac{\delta}{\delta y}(xz^3)]$$

$$= \hat{\mathbf{i}}[2z^4 + 2x^2y] - \hat{\mathbf{j}}[0 - 3xz^2] + \hat{\mathbf{k}}[-4xyz - 0]$$

$$= \hat{\mathbf{i}}[2.1^4 + 2.1^2.(-1)] - \hat{\mathbf{j}}[0 - 3.1.1^2] + \hat{\mathbf{k}}[-4.1.(-1).1 - 0]$$

$$= \hat{\mathbf{i}}[2 - 2] - \hat{\mathbf{j}}[0 - 3] + \hat{\mathbf{k}}[4]$$

$$= 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

Q# 48: Prove that; $\mathbf{curl}(\phi\,\vec{\mathbf{F}}) = \mathbf{grad}\phi \times \vec{\mathbf{F}}$; if $\vec{\mathbf{F}}$ is irrotational and $\phi(x, y, z)$ is a Scalar function.

Answer: Let, $\vec{\mathbf{F}} = \mathbf{F_1}\,\hat{\mathbf{i}} + \mathbf{F_2}\,\hat{\mathbf{j}} + \mathbf{F_3}\,\hat{\mathbf{k}}$

$$\therefore \mathbf{curl}(\phi\,\vec{\mathbf{F}}) = \underline{\nabla} \times (\phi\,\vec{\mathbf{F}})$$

$$= (\hat{\mathbf{i}}\frac{\delta}{\delta x} + \hat{\mathbf{j}}\frac{\delta}{\delta y} + \hat{\mathbf{k}}\frac{\delta}{\delta z}) \times [\phi\,(F_1\,\hat{\mathbf{i}} + F_2\,\hat{\mathbf{j}} + F_3\,\hat{\mathbf{k}})]$$

$$= (\hat{\mathbf{i}}\frac{\delta}{\delta x} + \hat{\mathbf{j}}\frac{\delta}{\delta y} + \hat{\mathbf{k}}\frac{\delta}{\delta z}) \times (\phi F_1\,\hat{\mathbf{i}} + \phi F_2\,\hat{\mathbf{j}} + \phi F_3\,\hat{\mathbf{k}})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \phi F_1 & \phi F_2 & \phi F_3 \end{vmatrix}$$

$$= \hat{i}[\frac{\delta}{\delta y}(\phi F_3) - \frac{\delta}{\delta z}(\phi F_2)] - \hat{j}[\frac{\delta}{\delta x}(\phi F_3) - \frac{\delta}{\delta z}(\phi F_1)] + \hat{k}[\frac{\delta}{\delta x}(\phi F_2) - \frac{\delta}{\delta y}(\phi F_1)]$$

$$= \hat{i}[\phi\frac{\delta}{\delta y}(F_3) + F_3\frac{\delta\phi}{\delta y} - \phi\frac{\delta}{\delta z}(F_2) - F_2\frac{\delta\phi}{\delta z}] - \hat{j}[\phi\frac{\delta}{\delta x}(F_3) + F_3\frac{\delta\phi}{\delta x} - \phi\frac{\delta}{\delta z}(F_1) - F_1\frac{\delta\phi}{\delta z}]$$

$$+ \hat{k}[\phi\frac{\delta}{\delta x}(F_2) + F_2\frac{\delta\phi}{\delta x} - \phi\frac{\delta}{\delta y}(F_1) - F_1\frac{\delta\phi}{\delta y}]$$

$$= \phi[\hat{i}\{\frac{\delta}{\delta y}(F_3) - \frac{\delta}{\delta z}(F_2)\} - \hat{j}\{\frac{\delta}{\delta x}(F_3) - \frac{\delta}{\delta z}(F_1)\} + \hat{k}\{\frac{\delta}{\delta x}(F_2) - \frac{\delta}{\delta y}(F_1)\}]$$

$$+ \hat{i}[F_3\frac{\delta\phi}{\delta y} - F_2\frac{\delta\phi}{\delta z}] - \hat{j}[F_3\frac{\delta\phi}{\delta x} - F_1\frac{\delta\phi}{\delta z}] + \hat{k}[F_2\frac{\delta\phi}{\delta x} - F_1\frac{\delta\phi}{\delta y}]$$

$$= \phi\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta\phi}{\delta x} & \frac{\delta\phi}{\delta y} & \frac{\delta\phi}{\delta z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \phi(\underline{\nabla} \times \underline{F}) + \underline{\nabla}\,\phi \times \underline{F}$$

$$= \phi.0 + \underline{\nabla}\,\phi \times \underline{F}\ [\because (\underline{\nabla} \times \underline{F}) = 0 \text{ for irrotational}]$$

$$= \underline{\nabla}\,\phi \times \underline{F}$$

$$= \mathbf{grad}\phi \times \underline{F} \text{ (Proved)}$$

Q# 49: Prove that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} . \vec{c})\vec{b} - (\vec{a} . \vec{b})\vec{c}$

Answer: Let $\vec{a}, \vec{b}, \vec{c}$ are three vectors

Then $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$(\vec{b} \times \vec{c}) = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \times (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= 0 + b_1c_2\hat{k} + b_1c_3(-\hat{j}) + b_2c_1(-\hat{k}) + 0 + b_2c_3\hat{i} + b_3c_1\hat{j} + b_3c_2(-\hat{i}) + 0$$

$$= b_1c_2\hat{k} - b_1c_3\hat{j} - b_2c_1\hat{k} + b_2c_3\hat{i} + b_3c_1\hat{j} - b_3c_2\hat{i}$$

$$= \hat{i}(b_2c_3 - b_3c_2) + \hat{j}(b_3c_1 - b_1c_3) + \hat{k}(b_1c_2 - b_2c_1)$$

Now,

$$\vec{a} \times (\vec{b} \times \vec{c})$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \{\hat{i}(b_2c_3 - b_3c_2) + \hat{j}(b_3c_1 - b_1c_3) + \hat{k}(b_1c_2 - b_2c_1)\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ (b_2c_3 - b_3c_2) & (b_3c_1 - b_1c_3) & (b_1c_2 - b_2c_1 \end{vmatrix}$$

$$= \hat{i}\{a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3)\} - \hat{j}\{a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - b_3c_2)\}$$

$$+ \hat{k}\{a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2)\}$$

$$= \hat{i}(a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3) + \hat{j}(-a_1b_1c_2 + a_1b_2c_1 + a_3b_2c_3 - a_3b_3c_2)$$

$$+ \hat{k}(a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2)$$

$$= \hat{i}(a_1b_1c_1 - a_1b_1c_1 + a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3) +$$

$$\hat{j}(a_2b_2c_2 - a_2b_2c_2 - a_1b_1c_2 + a_1b_2c_1 + a_3b_2c_3 - a_3b_3c_2)$$

$$+ \hat{k}(a_3b_3c_3 - a_3b_3c_3 + a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2)$$

$$= \hat{i}\{(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1\} + \hat{j}\{(a_2c_2 + a_1c_1 + a_3c_3)b_2$$

$$- (a_2b_2 + a_1b_1 + a_3b_3)c_2\} + \hat{k}\{(a_3c_3 + a_1c_1 + a_2c_2)b_3 - (a_3b_3 + a_1b_1 + a_2b_2)c_3\}$$

$$= \hat{i}\{(a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1\} + \hat{j}\{(a_1c_1 + a_2c_2 + a_3c_3)b_2$$

$$- (a_1b_1 + a_2b_2 + a_3b_3)c_2\} + \hat{k}\{(a_1c_1 + a_2c_2 + a_3c_3)b_3 - (a_1b_1 + a_2b_2 + a_3b_3)c_3\}$$

$$= \hat{i}(a_1c_1 + a_2c_2 + a_3c_3)b_1 + \hat{j}(a_1c_1 + a_2c_2 + a_3c_3)b_2 + \hat{k}(a_1c_1 + a_2c_2 + a_3c_3)b_3$$

$$- \hat{i}(a_1b_1 + a_2b_2 + a_3b_3)c_1 - \hat{j}(a_1b_1 + a_2b_2 + a_3b_3)c_2 - \hat{k}(a_1b_1 + a_2b_2 + a_3b_3)c_3$$

$$= \hat{i}(a_1c_1 + a_2c_2 + a_3c_3)b_1 + \hat{j}(a_1c_1 + a_2c_2 + a_3c_3)b_2 + \hat{k}(a_1c_1 + a_2c_2 + a_3c_3)b_3$$

$$- \{\hat{i}(a_1b_1 + a_2b_2 + a_3b_3)c_1 + \hat{j}(a_1b_1 + a_2b_2 + a_3b_3)c_2 + \hat{k}(a_1b_1 + a_2b_2 + a_3b_3)c_3\}$$

$$= (a_1c_1 + a_2c_2 + a_3c_3)(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - (a_1b_1 + a_2b_2 + a_3b_3)(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= \{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) - \{(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).$$

$$(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\}(c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} \text{ Proved}$$

Q# 50:

a) Prove that $\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = -\nabla^2 \vec{A} + \underline{\nabla}(\underline{\nabla}.\vec{A})$

b) Prove that $\vec{\nabla}\times(\phi\vec{A}) = (\vec{\nabla}\phi)\times\vec{A} + \phi(\vec{\nabla}\times\vec{A})$

Answer a)

Let, $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$$\therefore \nabla\times\vec{A} = (\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z})\times(A_1\hat{i} + A_2\hat{j} + A_3\hat{k})]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{i}[\frac{\delta}{\delta y}(A_3) - \frac{\delta}{\delta z}(A_2)] - \hat{j}[\frac{\delta}{\delta x}(A_3) - \frac{\delta}{\delta z}(A_1)] + \hat{k}[\frac{\delta}{\delta x}(A_2) - \frac{\delta}{\delta y}(A_1)]$$

$$= \hat{i}[\frac{\delta}{\delta y}(A_3) - \frac{\delta}{\delta z}(A_2)] + \hat{j}[\frac{\delta}{\delta z}(A_1) - \frac{\delta}{\delta x}(A_3)] + \hat{k}[\frac{\delta}{\delta x}(A_2) - \frac{\delta}{\delta y}(A_1)]$$

$$\therefore \underline{\nabla}\times(\underline{\nabla}\times\underline{A}) = (\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z})\times\{\hat{i}[\frac{\delta}{\delta y}(A_3) - \frac{\delta}{\delta z}(A_2)] + \hat{j}[\frac{\delta}{\delta z}(A_1) - \frac{\delta}{\delta x}(A_3)]$$
$$+ \hat{k}[\frac{\delta}{\delta x}(A_2) - \frac{\delta}{\delta y}(A_1)]\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z} & \frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x} & \frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y} \end{vmatrix}$$

$$= \hat{i}[\frac{\delta}{\delta y}(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}) - \frac{\delta}{\delta z}(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x})] - \hat{j}[\frac{\delta}{\delta x}(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y})$$
$$- \frac{\delta}{\delta z}(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z})] + \hat{k}[\frac{\delta}{\delta x}(\frac{\delta A_1}{\delta z} - \frac{\delta A_3}{\delta x}) - \frac{\delta}{\delta y}(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z})]$$

$$= \hat{i}[\frac{\delta^2 A_2}{\delta y\delta x} - \frac{\delta^2 A_1}{\delta y^2}) - (\frac{\delta^2 A_1}{\delta z^2} - \frac{\delta^2 A_3}{\delta z\delta x})] - \hat{j}[(\frac{\delta^2 A_2}{\delta x^2} - \frac{\delta^2 A_1}{\delta x\delta y}) - (\frac{\delta^2 A_3}{\delta z\delta y} - \frac{\delta^2 A_2}{\delta z^2})]$$
$$+ \hat{k}[(\frac{\delta^2 A_1}{\delta x\delta z} - \frac{\delta^2 A_3}{\delta x^2}) - (\frac{\delta^2 A_3}{\delta y^2} - \frac{\delta^2 A_2}{\delta y\delta z})]$$

$$= \hat{i}[-\frac{\delta^2 A_1}{\delta y^2} - \frac{\delta^2 A_1}{\delta z^2}] + \hat{j}[-\frac{\delta^2 A_2}{\delta x^2} - \frac{\delta^2 A_2}{\delta z^2}] + \hat{k}[-\frac{\delta^2 A_3}{\delta x^2} - \frac{\delta^2 A_3}{\delta y^2}] + \hat{i}[\frac{\delta^2 A_2}{\delta y\delta x}$$
$$+ \frac{\delta^2 A_3}{\delta z\delta x}] + \hat{j}[\frac{\delta^2 A_1}{\delta x\delta y} + \frac{\delta^2 A_3}{\delta z\delta y}] + \hat{k}[\frac{\delta^2 A_1}{\delta x\delta z} + \frac{\delta^2 A_2}{\delta y\delta z}]$$

$$= \hat{i}[-\frac{\delta^2 A_1}{\delta x^2} - \frac{\delta^2 A_1}{\delta y^2} - \frac{\delta^2 A_1}{\delta z^2}] + \hat{j}[-\frac{\delta^2 A_2}{\delta y^2} - \frac{\delta^2 A_2}{\delta x^2} - \frac{\delta^2 A_2}{\delta z^2}]$$

$$+ \hat{k}[-\frac{\delta^2 A_3}{\delta z^2} - \frac{\delta^2 A_3}{\delta x^2} - \frac{\delta^2 A_3}{\delta y^2}] + \hat{i}[\frac{\delta^2 A_1}{\delta x^2} + \frac{\delta^2 A_2}{\delta y \delta x} + \frac{\delta^2 A_3}{\delta z \delta x}] + \hat{j}[\frac{\delta^2 A_1}{\delta x \delta y}$$

$$+ \frac{\delta^2 A_2}{\delta y^2} + \frac{\delta^2 A_3}{\delta z \delta y}] + \hat{k}[\frac{\delta^2 A_3}{\delta z^2} + \frac{\delta^2 A_1}{\delta x \delta z} + \frac{\delta^2 A_2}{\delta y \delta z}]$$

$$= -(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})A_1 \hat{i} - (\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})A_2 \hat{j} - (\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}$$

$$+ \frac{\delta^2}{\delta z^2})A_3 \hat{k} + \hat{i}\frac{\delta}{\delta x}[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}] + \hat{j}\frac{\delta}{\delta y}[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}]$$

$$+ \hat{k}\frac{\delta}{\delta z}[\frac{\delta A_3}{\delta z} + \frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y}]$$

$$= -(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})A_1 \hat{i} - (\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})A_2 \hat{j} - (\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}$$

$$+ \frac{\delta^2}{\delta z^2})A_3 \hat{k} + \hat{i}\frac{\delta}{\delta x}[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}] + \hat{j}\frac{\delta}{\delta y}[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}]$$

$$+ \hat{k}\frac{\delta}{\delta z}[\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z}]$$

$$= -(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) + (\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z})(\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y}$$

$$+ \frac{\delta A_3}{\delta z})$$

$$= -\nabla^2 (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) + \underline{\nabla}(\frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z})$$

$$= -\nabla^2 \vec{A} + \underline{\nabla}\, [(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}).\, (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})]$$

$$= -\nabla^2 \vec{A} + \underline{\nabla}(\underline{\nabla} . \vec{A}) \quad \textit{Proved}$$

Answer b)

Let, $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$

$\phi \vec{A} = \phi(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})$

$\phi \vec{A} = \phi A_1 \hat{i} + \phi A_2 \hat{j} + \phi A_3 \hat{k}$

L.H.S. $\vec{\nabla} \times (\phi \vec{A}) = (\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}) \times (\phi A_1 \hat{i} + \phi A_2 \hat{j} + \phi A_3 \hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \phi A_1 & \phi A_2 & \phi A_3 \end{vmatrix}$$

$$= \hat{i}[\frac{\delta}{\delta y}(\phi A_3) - \frac{\delta}{\delta z}(\phi A_2)] - \hat{j}[\frac{\delta}{\delta x}(\phi A_3) - \frac{\delta}{\delta z}(\phi A_1)] + \hat{k}[\frac{\delta}{\delta x}(\phi A_2) - \frac{\delta}{\delta y}(\phi A_1)]$$

$$= \hat{i}[\phi\frac{\delta}{\delta y}(A_3) + A_3\frac{\delta\phi}{\delta y} - \phi\frac{\delta}{\delta z}(A_2) - A_2\frac{\delta\phi}{\delta z}] - \hat{j}[\phi\frac{\delta}{\delta x}(A_3) + A_3\frac{\delta\phi}{\delta x} - \phi\frac{\delta}{\delta z}(A_1) - A_1\frac{\delta\phi}{\delta z}]$$

$$+ \hat{k}[\phi\frac{\delta}{\delta x}(A_2) + A_2\frac{\delta\phi}{\delta x} - \phi\frac{\delta}{\delta y}(A_1) - A_1\frac{\delta\phi}{\delta y}]$$

$$= \phi[\hat{i}\{\frac{\delta}{\delta y}(A_3) - \frac{\delta}{\delta z}(A_2)\} - \hat{j}\{\frac{\delta}{\delta x}(A_3) - \frac{\delta}{\delta z}(A_1)\} + \hat{k}\{\frac{\delta}{\delta x}(A_2) - \frac{\delta}{\delta y}(A_1)\}]$$

$$+ \hat{i}[A_3\frac{\delta\phi}{\delta y} - A_2\frac{\delta\phi}{\delta z}] - \hat{j}[A_3\frac{\delta\phi}{\delta x} - A_1\frac{\delta\phi}{\delta z}] + \hat{k}[A_2\frac{\delta\phi}{\delta x} - A_1\frac{\delta\phi}{\delta y}]$$

$$= \phi\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta\phi}{\delta x} & \frac{\delta\phi}{\delta y} & \frac{\delta\phi}{\delta z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \phi(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}) \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + (\hat{i}\frac{\delta}{\delta x}\phi + \hat{j}\frac{\delta}{\delta y}\phi + \hat{k}\frac{\delta}{\delta z}\phi) \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$= \phi(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z}) \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) + (\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z})\phi \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k})$$

$$= \phi(\vec{\nabla} \times \vec{A}) + (\vec{\nabla}\phi) \times \vec{A} \text{ (Proved)}$$

Q# 51: If $r^2 = x^2 + y^2 + z^2$ then find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}$

We have, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\therefore |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2}$$

$$\therefore |\vec{r}|^2 = r^2 = \left\{(x^2 + y^2 + z^2)^{1/2}\right\}^2 = x^2 + y^2 + z^2$$

$r^2 = x^2 + y^2 + z^2$ -------------------------(i)

Differentiating (i) with respect to x partially,

$$\therefore 2r\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2)$$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x + 0 + 0$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r},$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

Q# 52: Show that $\vec{\nabla} \cdot \vec{r} = 3$

Also, $\vec{\nabla} \cdot \vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$[\because \hat{i}.\hat{i} = 1,\ \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0\ ]$

Q# 53: Show that $\vec{r}.\vec{r} = r^2$

Also, $\vec{r}.\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k}).(x\hat{i} + y\hat{j} + z\hat{k})$

$$\vec{r}.\vec{r} = x^2 + y^2 + z^2\ [\because \hat{i}.\hat{i} = 1; \hat{j}.\hat{j} = 1; \hat{k}.\hat{k} = 1]$$

$$= r^2\ [\because r^2 = x^2 + y^2 + z^2]$$

Similarly,

$\vec{A}.\vec{A}. = A^2$

$\vec{\nabla}.\vec{\nabla} = \nabla^2$

Q# 54: Show that, $\vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}$

Proof: **L.H.S** $= \vec{\nabla}$

$$= \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}\frac{\partial r}{\partial r} + \hat{j}\frac{\partial}{\partial y}\frac{\partial r}{\partial r} + \hat{k}\frac{\partial}{\partial z}\frac{\partial r}{\partial r}$$

$$= \hat{i}\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \hat{j}\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \hat{k}\frac{\partial r}{\partial z}\frac{\partial}{\partial r}$$

$$= \left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)\frac{\partial}{\partial r}$$

$$= \left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right)\frac{\partial}{\partial r}\ [\because \frac{\partial r}{\partial x} = \frac{x}{r}; \frac{\partial r}{\partial y} = \frac{y}{r}; \frac{\partial r}{\partial z} = \frac{z}{r}]$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} \frac{\partial}{\partial r}$$

$$= \frac{\vec{r}}{r} \frac{\partial}{\partial r}$$

Q# 55: Show that $\vec{\nabla}.(\phi \vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}$

Solution

Let $\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$ is a vector and ϕ is a function of a variable or variables

L.H.S $\vec{\nabla}.(\phi \vec{A}) = (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).(\phi \vec{A})$

$$= (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).\phi(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})$$

$$= (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).(\phi A_1 \hat{i} + \phi A_2 \hat{j} + \phi A_3 \hat{k})$$

$$= \frac{\delta}{\delta x}(\phi A_1) + \frac{\delta}{\delta y}(\phi A_2) + \frac{\delta}{\delta z}(\phi A_3) \qquad [\because \hat{i}.\hat{i} = 1; \hat{j}.\hat{j} = 1; \hat{k}.\hat{k} = 1]$$

$$= \phi\frac{\delta}{\delta x}(A_1) + A_1\frac{\delta}{\delta x}(\phi) + \phi\frac{\delta}{\delta y}(A_2) + A_2\frac{\delta}{\delta y}(\phi) + \phi\frac{\delta}{\delta z}(A_3) + A_3\frac{\delta}{\delta z}(\phi)$$

$$[\because \frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u]$$

$$= A_1\frac{\delta}{\delta x}(\phi) + A_2\frac{\delta}{\delta y}(\phi) + A_3\frac{\delta}{\delta z}(\phi) + \phi\frac{\delta}{\delta x}(A_1) + \phi\frac{\delta}{\delta y}(A_2) + \phi\frac{\delta}{\delta z}(A_3)$$

$$= A_1\frac{\delta}{\delta x}(\phi) + A_2\frac{\delta}{\delta y}(\phi) + A_3\frac{\delta}{\delta z}(\phi) + \phi\left\{\frac{\delta}{\delta x}(A_1) + \frac{\delta}{\delta y}(A_2) + \frac{\delta}{\delta z}(A_3)\right\}$$

$$= A_1\frac{\delta\phi}{\delta x} + A_2\frac{\delta\phi}{\delta y} + A_3\frac{\delta\phi}{\delta z} + \phi\left\{\frac{\delta}{\delta x}(A_1) + \frac{\delta}{\delta y}(A_2) + \frac{\delta}{\delta z}(A_3)\right\}$$

$$= \frac{\delta\phi}{\delta x}A_1 + \frac{\delta\phi}{\delta y}A_2 + \frac{\delta\phi}{\delta z}A_3 + \phi\left\{\frac{\delta}{\delta x}(A_1) + \frac{\delta}{\delta y}(A_2) + \frac{\delta}{\delta z}(A_3)\right\}$$

$$= \phi\left\{\frac{\delta}{\delta x}(A_1) + \frac{\delta}{\delta y}(A_2) + \frac{\delta}{\delta z}(A_3)\right\} + \frac{\delta\phi}{\delta x}A_1 + \frac{\delta\phi}{\delta y}A_2 + \frac{\delta\phi}{\delta z}A_3$$

$$= \phi(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) + (\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}).(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})$$

$$[\because \hat{i}.\hat{i} = 1; \hat{j}.\hat{j} = 1; \hat{k}.\hat{k} = 1]$$

$$= \phi(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) + (\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})\phi.(A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k})$$

$$= \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A} \qquad [\because \vec{\nabla} = \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}]$$

$$\therefore \vec{\nabla}.(\phi\vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}$$

Q# 56: Show that $\nabla^2(\ln r) = \frac{1}{r^2}$

$$\text{L.H.S} = \nabla^2(\ln r)$$

$$= \vec{\nabla}.\vec{\nabla}(\ln r) \qquad [\because \vec{\nabla}.\vec{\nabla} = \nabla^2]$$

$$= \vec{\nabla}.\left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}\ln r\right] \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \vec{\nabla}.\left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}(\ln r)\right]$$

$$= \vec{\nabla}.\left[\frac{\vec{r}}{r}\frac{1}{r}\right] \qquad [\because \frac{\partial}{\partial r}(\ln r) = \frac{1}{r}]$$

$$= \vec{\nabla}.\left[\frac{\vec{r}}{r^2}\right]$$

$$= \vec{\nabla}.\left[\frac{1}{r^2}\vec{r}\right]$$

$$= \frac{1}{r^2}\left[\vec{\nabla}.\vec{r}\right] + \left[\vec{\nabla}\left(\frac{1}{r^2}\right)\right].\vec{r} \qquad [\because \vec{\nabla}.(\phi\vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}]$$

$$= \frac{3}{r^2} + \left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right].\vec{r} \qquad [\because \vec{\nabla}.\vec{r} = 3 \ \& \because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \frac{3}{r^2} + \left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(r^{-2}\right)\right].\vec{r}$$

$$= \frac{3}{r^2} + \frac{\vec{r}}{r}\left(-2r^{-2-1}\right).\vec{r}$$

$$= \frac{3}{r^2} + \frac{\vec{r}}{r}\left(-\frac{2}{r^3}\right).\vec{r}$$

$$= \frac{3}{r^2} - \frac{2}{r^4}\left(\vec{r}.\vec{r}\right)$$

$$= \frac{3}{r^2} - \frac{2}{r^4} \times r^2 \qquad [\because \vec{r}.\vec{r}. = r^2]$$

$$= \frac{3}{r^2} - \frac{2}{r^2}$$

$$= \frac{\mathbf{1}}{\mathbf{r^2}}.$$

Q# 57: Find the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$

Answer: Let $\phi = \frac{1}{r}$

Therefore, the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$ $= \vec{\nabla}\phi.\hat{r}$

We can write, $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$

Here, $\phi = \frac{1}{r}$

$$\Rightarrow \vec{\nabla}\phi = \vec{\nabla}\frac{1}{r}$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}\frac{\partial}{\partial r}\frac{1}{r} \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}\frac{\partial}{\partial r}(r^{-1})$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}(-1)(r^{-1-1})$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}(-1)(r^{-2})$$

$$\Rightarrow \vec{\nabla}\phi = -\frac{\vec{r}}{r}\frac{1}{r^2}$$

$$\Rightarrow \vec{\nabla}\phi = -\frac{\vec{r}}{r^3}$$

Therefore, the directional derivative of $\frac{1}{r}$ in the direction of $\vec{r}$

$$= \vec{\nabla}\phi.\hat{r}$$

$$= -\frac{\vec{r}}{r^3}.\hat{r} \qquad [\vec{\nabla}\phi = -\frac{\vec{r}}{r^3}]$$

$$= -\frac{\vec{r}}{r^3}.\frac{\vec{r}}{r} \qquad [\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}]$$

$$= -\frac{1}{r^4}(\vec{r}.\vec{r})$$

$$= -\frac{1}{r^4}r^2 \qquad [\because \vec{\mathbf{r}}.\vec{\mathbf{r}}. = \mathbf{r^2}]$$

$$= -\frac{1}{r^2}$$

Q# 58: Show that $\nabla^2 \mathbf{r}^n = \mathbf{n}(\mathbf{n}+1)\mathbf{r}^{n-2}$

$$\mathbf{L.H.S} = \nabla^2 \mathbf{r}^n = \vec{\nabla}.\vec{\nabla}(\mathbf{r}^n) \qquad [\because \vec{\nabla}.\vec{\nabla} = \nabla^2]$$

$$= \vec{\nabla}.\left[\frac{\vec{\mathbf{r}}}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}\mathbf{r}^n\right] \qquad [\because \vec{\nabla} = \frac{\vec{\mathbf{r}}}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}]$$

$$= \vec{\nabla}.\left[\frac{\vec{\mathbf{r}}}{\mathbf{r}}\mathbf{n}\mathbf{r}^{n-1}\right]$$

$$= \vec{\nabla}.\left[\mathbf{n}\vec{\mathbf{r}}\,\mathbf{r}^{n-1}\,\mathbf{r}^{-1}\right]$$

$$= \vec{\nabla}.\left[\mathbf{n}\vec{\mathbf{r}}\,\mathbf{r}^{n-1-1}\right]$$

$$= \vec{\nabla}.\left[\mathbf{n}\vec{\mathbf{r}}\,\mathbf{r}^{n-2}\right]$$

$$= \mathbf{n}\left[\vec{\nabla}.(\vec{\mathbf{r}}\,\mathbf{r}^{n-2})\right]$$

$$= \mathbf{n}\left[\vec{\nabla}.(\mathbf{r}^{n-2}\vec{\mathbf{r}})\right]$$

$$= \mathbf{n}\left[\mathbf{r}^{n-2}(\vec{\nabla}.\vec{\mathbf{r}}) + \overline{\nabla}(\mathbf{r}^{n-2}).\vec{\mathbf{r}}\right] \qquad [\because \vec{\nabla}.(\phi\vec{\mathbf{A}}) = \phi(\vec{\nabla}.\vec{\mathbf{A}}) + (\vec{\nabla}\phi).\vec{\mathbf{A}}]$$

$$= \mathbf{n}\left[3\mathbf{r}^{n-2} + \frac{\vec{\mathbf{r}}}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}(\mathbf{r}^{n-2}).\vec{\mathbf{r}}\right] \qquad [\because \vec{\nabla} = \frac{\vec{\mathbf{r}}}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}]\ \&\ [\vec{\nabla}\cdot\vec{\mathbf{r}} = 3]$$

$$= \mathbf{n}\left[3\mathbf{r}^{n-2} + \frac{1}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}(\mathbf{r}^{n-2})(\vec{\mathbf{r}}.\vec{\mathbf{r}})\right]$$

$$= \mathbf{n}\left[3\mathbf{r}^{n-2} + \frac{1}{\mathbf{r}}(\mathbf{n}-2)(\mathbf{r}^{n-2-1})(\vec{\mathbf{r}}.\vec{\mathbf{r}})\right]$$

$$= \mathbf{n}\left[3\mathbf{r}^{n-2} + \frac{1}{\mathbf{r}}(\mathbf{n}-2)\mathbf{r}^{n-3}\,\mathbf{r}^2\right] \qquad [\because \vec{\mathbf{r}}.\vec{\mathbf{r}} = \mathbf{r}^2]$$

$$= \mathbf{n}\left[3\mathbf{r}^{n-2} + (\mathbf{n}-2)\mathbf{r}^{n-3}\mathbf{r}\right]$$

$$= \mathbf{n}\left[3\mathbf{r}^{n-2} + (\mathbf{n}-2)\mathbf{r}^{n-3+1}\right]$$

$$= \mathbf{n}\left[3\mathbf{r}^{n-2} + (\mathbf{n}-2)\mathbf{r}^{n-2}\right]$$

$$= \mathbf{n}[3+\mathbf{n}-2]\mathbf{r}^{n-2}$$

$$= \mathbf{n}(\mathbf{n}+1)\mathbf{r}^{n-2}$$

Q# 59: Show that $\vec{\nabla}.(\mathbf{r}^3\vec{\mathbf{r}}) = 6\mathbf{r}^3$

$$\mathbf{L.H.S} = \vec{\nabla}.(\mathbf{r}^3\vec{\mathbf{r}})$$

$$= \mathbf{r}^3(\vec{\nabla}.\vec{\mathbf{r}}) + \vec{\nabla}(\mathbf{r}^3).\vec{\mathbf{r}} \qquad [\because \vec{\nabla}.(\phi\vec{\mathbf{A}}) = \phi(\vec{\nabla}.\vec{\mathbf{A}}) + (\vec{\nabla}\phi).\vec{\mathbf{A}}]$$

$$= 3\mathbf{r}^3 + \left[\frac{\vec{\mathbf{r}}}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}\mathbf{r}^3\right]\cdot\vec{\mathbf{r}} \qquad [\because \vec{\nabla}\cdot\vec{\mathbf{r}} = 3\ \&\ \vec{\nabla} = \frac{\vec{\mathbf{r}}}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}]$$

$$= 3r^3 + \left[\frac{\vec{r}}{r}3r^{3-1}\right].\vec{r}$$

$$= 3r^3 + \frac{1}{r}3r^2(\vec{r}\cdot\vec{r})$$

$$= 3r^3 + 3r(r^2) \qquad [\because \vec{r}.\vec{r} = r^2]$$

$$= 3r^3 + 3r^3$$

$$= 6r^3$$

Q# 60: Show that $\vec{\nabla}.\left[r\vec{\nabla}\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$.

$$\text{L.H.S} = \vec{\nabla}.\left[r\vec{\nabla}\left(\frac{1}{r^3}\right)\right]$$

$$= \vec{\nabla}.\left[r\left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\right\}\right] \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \vec{\nabla}.\left[r\left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}r^{-3}\right\}\right]$$

$$= \vec{\nabla}.\left[\vec{r}\frac{\partial}{\partial r}r^{-3}\right]$$

$$= \vec{\nabla}.\left[\vec{r}(-3)r^{-3-1}\right]$$

$$= \vec{\nabla}.\left[\vec{r}(-3)r^{-4}\right]$$

$$= \vec{\nabla}.\left[-3r^{-4}\vec{r}\right]$$

$$= -3r^{-4}(\vec{\nabla}\cdot\vec{r}) + \left\{\vec{\nabla}(-3r^{-4})\right\}\cdot\vec{r} \qquad [\because \vec{\nabla}.(\phi\vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}]$$

$$= -3r^{-4}(\vec{\nabla}\cdot\vec{r}) - 3\left\{\vec{\nabla}(r^{-4})\right\}\cdot\vec{r}$$

$$= -9r^{-4} - 3\left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}r^{-4}\right\}\cdot\vec{r} \qquad [\because \vec{\nabla}\cdot\vec{r} = 3] \;\&\; [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= -9r^{-4} - 3\left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}r^{-4}\right\}\cdot\vec{r}$$

$$= -9r^{-4} - 3\left\{\frac{\vec{r}}{r}(-4)r^{-4-1}\right\}\cdot\vec{r}$$

$$= -9r^{-4} - 3\left\{(-4)r^{-4-1}\times\frac{1}{r}\right\}(\vec{r}\cdot\vec{r})$$

$$= -\frac{9}{r^4} - 3(-4)r^{-5}r^{-1}(\vec{r}\cdot\vec{r})$$

$$= -\frac{9}{r^4} - 3(-4)r^{-6}(\vec{r}\cdot\vec{r})$$

$$= -\frac{9}{r^4} + \frac{12}{r^6} \cdot r^2 \qquad [\because \vec{r}.\vec{r} = r^2]$$

$$= -\frac{9}{r^4} + \frac{12}{r^4}$$

$$= \frac{3}{r^4}.$$

Q# 61: Show that $\nabla^2\left[\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^2}\right)\right] = \frac{2}{r^4}$.

$$\text{L.H.S} = \nabla^2\left[\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^2}\right)\right]$$

$$= \nabla^2\left[\vec{\nabla} \cdot \left(\frac{1}{r^2}\vec{r}\right)\right]$$

$$= \nabla^2\left[\frac{1}{r^2}\left(\vec{\nabla} \cdot \vec{r}\right) + \left\{\vec{\nabla}\left(\frac{1}{r^2}\right)\right\} \cdot \vec{r}\right] \qquad [\because \vec{\nabla}.(\phi \vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}]$$

$$= \nabla^2\left[\frac{3}{r^2} + \left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right\} \cdot \vec{r}\right] \qquad [\because \vec{\nabla} \cdot \vec{r} = 3]\ \&\ [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \nabla^2\left[\frac{3}{r^2} + \left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}(r^{-2})\right\} \cdot \vec{r}\right]$$

$$= \nabla^2\left[\frac{3}{r^2} + \left\{\frac{\vec{r}}{r}(-2)(r^{-2-1})\right\} \cdot \vec{r}\right]$$

$$= \nabla^2\left[\frac{3}{r^2} + \left\{\frac{\vec{r}}{r}(-2)(r^{-3})\right\} \cdot \vec{r}\right]$$

$$= \nabla^2\left[\frac{3}{r^2} + \left\{\vec{r}.(-2)(r^{-3}.r^{-1})\right\} \cdot \vec{r}\right]$$

$$= \nabla^2\left[\frac{3}{r^2} + \left\{\vec{r}.(-2)(r^{-4})\right\} \cdot \vec{r}\right]$$

$$= \nabla^2\left[\frac{3}{r^2} + \frac{-2}{r^4}\left(\vec{r} \cdot \vec{r}\right)\right]$$

$$= \nabla^2\left[\frac{3}{r^2} + \frac{-2}{r^4}r^2\right] \qquad [\because \vec{r}.\vec{r} = r^2]$$

$$= \nabla^2\left[\frac{3}{r^2} - \frac{2}{r^2}\right]$$

$$= \nabla^2\left(\frac{1}{r^2}\right)$$

$$= \vec{\nabla} \cdot \vec{\nabla}\left(\frac{1}{r^2}\right) \qquad [\because \vec{\nabla}.\vec{\nabla} = \nabla^2]$$

$$= \vec{\nabla} \cdot \left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\right] \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \vec{\nabla} \cdot \left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}(r^{-2})\right]$$

$$= \vec{\nabla} \cdot \left[\frac{\vec{r}}{r}(-2)(r^{-2-1})\right]$$

$$= \vec{\nabla} \cdot \left[\frac{\vec{r}}{r}(-2)(r^{-3})\right]$$

$$= \vec{\nabla} \cdot \left[\frac{\vec{r}}{r}\left(-\frac{2}{r^3}\right)\right]$$

$$= \vec{\nabla} \cdot \left[\vec{r}\left(-\frac{2}{r^4}\right)\right]$$

$$= \vec{\nabla} \cdot \left[\left(-\frac{2}{r^4}\right)\vec{r}\right]$$

$$= -\frac{2}{r^4}\left(\vec{\nabla} \cdot \vec{r}\right) + \left\{\vec{\nabla}\left(\frac{-2}{r^4}\right)\right\} \cdot \vec{r} \qquad [\because \vec{\nabla}.(\phi \vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}]$$

$$= -\frac{2}{r^4}\left(\vec{\nabla} \cdot \vec{r}\right) - 2\left\{\vec{\nabla}(r^{-4})\right\} \cdot \vec{r}$$

$$= -\frac{6}{r^4} - 2\left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}(r^{-4})\right\} \cdot \vec{r} \qquad [\because \vec{\nabla} \cdot \vec{r} = 3]\ [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= -\frac{6}{r^4} - 2\left\{\frac{\vec{r}}{r}(-4)(r^{-4-1})\right\} \cdot \vec{r}$$

$$= -\frac{6}{r^4} - 2\left\{\frac{\vec{r}}{r}(-4)(r^{-5})\right\} \cdot \vec{r}$$

$$= -\frac{6}{r^4} - 2\left\{\vec{r}(-4)(r^{-5}r^{-1})\right\} \cdot \vec{r}$$

$$= -\frac{6}{r^4} + 8\left\{\vec{r}(r^{-6})\right\} \cdot \vec{r}$$

$$= -\frac{6}{r^4} + 8\frac{1}{r^6}\left(\vec{r} \cdot \vec{r}\right)$$

$$= -\frac{6}{r^4} + 8\frac{1}{r^6}r^2 \qquad [\because \vec{r}.\vec{r} = r^2]$$

$$= -\frac{6}{r^4} + \frac{8}{r^4}$$

$$= \frac{2}{r^4}.$$

Q# 62: Show that grad div $\vec{A} = -2r^{-3}\vec{r}$; Where, $\vec{A} = \frac{\vec{r}}{r}$

Answer: grad div $\vec{A}$

$$= \mathbf{grad}\,(\vec{\nabla}\,.\,\vec{A})$$

$$= \vec{\nabla}\,(\vec{\nabla}\,.\,\vec{A})$$

Now, $\mathbf{L.H.S = grad\quad div}\,\vec{A}$

$$= \mathbf{grad}\left(\vec{\nabla}\cdot\vec{A}\right)$$

$$= \vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)$$

$$= \vec{\nabla}\left(\vec{\nabla}\cdot\frac{\vec{r}}{r}\right) \qquad \left[\text{Given, } \vec{A} = \frac{\vec{r}}{r}\right]$$

$$= \vec{\nabla}\left[\vec{\nabla}\cdot\left(\frac{1}{r}\vec{r}\right)\right]$$

$$= \vec{\nabla}\left[\frac{1}{r}\left(\vec{\nabla}\cdot\vec{r}\right) + \left\{\vec{\nabla}\left(\frac{1}{r}\right)\right\}\cdot\vec{r}\right] \qquad [\because \vec{\nabla}.(\phi\vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}]$$

$$= \vec{\nabla}\left[\frac{3}{r} + \left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(r^{-1}\right)\right\}\cdot\vec{r}\right] \qquad [\because \vec{\nabla}\cdot\vec{r} = 3] \;\&\; [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \vec{\nabla}\left[\frac{3}{r} + \left\{\frac{\vec{r}}{r}(-1)(r^{-1-1})\right\}\cdot\vec{r}\right]$$

$$= \vec{\nabla}\left[\frac{3}{r} + \left\{\frac{\vec{r}}{r}(-1)(r^{-2})\right\}\cdot\vec{r}\right]$$

$$= \vec{\nabla}\left[\frac{3}{r} + \left\{\vec{r}(-1)(r^{-2}r^{-1})\right\}\cdot\vec{r}\right]$$

$$= \vec{\nabla}\left[\frac{3}{r} + \left\{\vec{r}(-1)(r^{-3})\right\}\cdot\vec{r}\right]$$

$$= \vec{\nabla}\left[\frac{3}{r} + \left\{(-1)(r^{-3})\right\}(\vec{r}\cdot\vec{r})\right]$$

$$= \vec{\nabla}\left[\frac{3}{r} - \frac{1}{r^3}r^2\right] \qquad [\because \vec{r}.\vec{r} = r^2]$$

$$= \vec{\nabla}\left[\frac{3}{r} - \frac{1}{r}\right]$$

$$= \vec{\nabla}\left(\frac{2}{r}\right)$$

$$= 2\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= 2\frac{\vec{r}}{r}\frac{\partial}{\partial r}(r^{-1})$$

$$= 2\frac{\vec{r}}{r}(-1)r^{-1-1}$$

$$= 2\frac{\vec{r}}{r}(-1)r^{-2}$$

$$= 2\vec{r}(-1)r^{-2}r^{-1}$$

$$= 2\vec{r}(-1)r^{-3}$$

$$= -2r^{-3}\vec{r}$$

Q# 63: i. Show that $\nabla^2 f(r) = f''(r) + \frac{2}{r}f'(r)$

$$\text{L.H.S.} = \nabla^2 f(r)$$

$$= \vec{\nabla}\cdot\vec{\nabla}f(r) \qquad [\because \vec{\nabla}.\vec{\nabla} = \nabla^2]$$

$$= \vec{\nabla}\cdot\left(\frac{\vec{r}}{r}\frac{\partial}{\partial r}f(r)\right) \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \vec{\nabla}\cdot\left(\frac{\vec{r}}{r}f'(r)\right) \qquad [\because \frac{\partial}{\partial r}f(r) = f'(r)]$$

$$= \vec{\nabla}\cdot\left[\frac{f'(r)}{r}\vec{r}\right]$$

$$= \frac{f'(r)}{r}(\vec{\nabla}\cdot\vec{r}) + \left\{\vec{\nabla}\left(\frac{f'(r)}{r}\right)\right\}\cdot\vec{r} \qquad [\because \vec{\nabla}.(\phi\vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}]$$

$$= 3\frac{f'(r)}{r} + \left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left\{\frac{f'(r)}{r}\right\}\right]\cdot\vec{r} \qquad [\because \vec{\nabla}\cdot\vec{r} = 3]$$

$$= 3\frac{f'(r)}{r} + \left[\frac{1}{r}\left\{\frac{rf''(r) - f'(r).1}{r^2}\right\}\right](\vec{r}\cdot\vec{r}) \qquad [\because \frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}]$$

$$= 3\frac{f'(r)}{r} + \left[\left\{\frac{rf''(r) - f'(r).1}{r^3}\right\}\right](\vec{r}\cdot\vec{r})$$

$$= 3\frac{f'(r)}{r} + \left[\left\{\frac{rf''(r) - f'(r).1}{r^3}\right\}\right]r^2 \qquad [\because \vec{r}.\vec{r} = r^2]$$

$$= 3\frac{f'(r)}{r} + \left[\left\{\frac{rf''(r) - f'(r).1}{r}\right\}\right]$$

$$= 3\frac{f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$= f''(r) + \frac{2}{r} f'(r)$$

ii. If $\nabla^2 f(r) = 0$

$$f''(r) + \frac{2}{r} f'(r) = 0 \qquad [\because \nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)]$$

$$\frac{\partial^2}{\partial r^2}(f(r)) + \frac{2}{r}\frac{\partial}{\partial r}(f(r)) = 0 \quad [\because \frac{\partial}{\partial r}(f(r)) = f'(r)] \,\&\, [\because \frac{\partial^2}{\partial r^2}(f(r)) = f''(r)]$$

$$\frac{\partial}{\partial r}\left\{\frac{\partial}{\partial r}(f(r))\right\} + \frac{2}{r}\frac{\partial}{\partial r}(f(r)) = 0$$

$$\frac{\partial}{\partial r}(p) + \frac{2}{r} p = 0, \qquad [\because p = \frac{\partial}{\partial r}(f(r)) = f'(r)]$$

$$\frac{\partial p}{\partial r} \times \frac{\partial r}{p} + \frac{2p}{r} \times \frac{\partial r}{p} = 0 \qquad \text{[Multiplying by } \frac{\partial r}{p} \text{ on both sides]}$$

$$\frac{\partial p}{p} + \frac{2\partial r}{r} = 0$$

$$\int \frac{\partial p}{p} + \int \frac{2\partial r}{r} = \int 0 \qquad \text{[Integrating]}$$

$$\ln p + 2\ln r = \ln A$$

$$\ln p + \ln r^2 = \ln A$$

$$\ln pr^2 = \ln A \qquad [\because \ln ab = \ln a + \ln b]$$

$$pr^2 = A$$

$$p = Ar^{-2}$$

$$\frac{\partial f(r)}{\partial r} = Ar^{-2}$$

$$\int \frac{\partial f(r)}{\partial r}\partial r = \int Ar^{-2}\partial r$$

$$\int \frac{\partial}{\partial r}\{f(r)\}\partial r = \int Ar^{-2}\partial r$$

Integrating

$$f(r) = A\frac{r^{-2+1}}{-2+1} + B$$

$$f(r) = A\frac{r^{-1}}{-1} + B$$

$$f(r) = -A\frac{1}{r} + B$$

$$f(r) = \frac{-A}{r} + B$$

$$\mathbf{f}(\mathbf{r}) = \mathbf{B} + \frac{\mathbf{c}}{\mathbf{r}}, \qquad \text{Where, } \mathbf{c} = -\mathbf{A}$$

Q# 64: Show that $\nabla^2\left(\frac{1}{r}\right) = 0$

$$\text{L.H.S} = \nabla^2\left(\frac{1}{r}\right)$$

$$= \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{r}\right) \qquad [\because \vec{\nabla}.\vec{\nabla} = \nabla^2]$$

$$= \vec{\nabla}\cdot\left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right] \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \vec{\nabla}\cdot\left[\frac{\vec{r}}{r}\frac{\partial}{\partial r}(r^{-1})\right]$$

$$= \vec{\nabla}\cdot\left[\frac{\vec{r}}{r}(-1)r^{-1-1}\right]$$

$$= \vec{\nabla}\cdot\left[\frac{\vec{r}}{r}(-1)r^{-2}\right]$$

$$= \vec{\nabla}\cdot\left[\vec{r}(-1)r^{-2}r^{-1}\right]$$

$$= \vec{\nabla}\cdot\left[\vec{r}(-1)r^{-3}\right]$$

$$= \vec{\nabla}\cdot\left[-r^{-3}\vec{r}\right]$$

$$= \vec{\nabla}\cdot\left[-\frac{\vec{r}}{r^3}\right]$$

$$= \vec{\nabla}\cdot\left[-\vec{r}\,r^{-3}\right]$$

$$= \vec{\nabla}\cdot\left[-r^{-3}\vec{r}\right]$$

$$= -r^{-3}\left(\vec{\nabla}\cdot\vec{r}\right) - \left(\vec{\nabla}r^{-3}\right)\cdot\vec{r} \qquad [\because \vec{\nabla}.(\phi\vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}]$$

$$= -3r^{-3} - \left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}(r^{-3})\right\}\cdot\vec{r} \qquad [\because \vec{\nabla}\cdot\vec{r} = 3]\ \&\ [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= -\frac{3}{r^3} - \left\{\frac{1}{r}(-3)r^{-3-1}\right\}(\vec{r}\cdot\vec{r})$$

$$= -\frac{3}{r^3} - \left\{\frac{1}{r}(-3)r^{-4}\right\}(\vec{r}\cdot\vec{r})$$

$$= -\frac{3}{r^3} - \left\{(-3)r^{-4}r^{-1}\right\}(\vec{r}\cdot\vec{r})$$

$$= -\frac{3}{r^3} - \left\{(-3)r^{-4-1}\right\}(\vec{r}\cdot\vec{r})$$

$$= -\frac{3}{r^3} - \{(-3)r^{-5}\}(\vec{r}\cdot\vec{r})$$

$$= -\frac{3}{r^3} - \left\{\frac{-3}{r^5}\right\}(\vec{r}\cdot\vec{r})$$

$$= -\frac{3}{r^3} - \left\{\frac{-3}{r^5}\right\}r^2 \qquad [\because \vec{r}.\vec{r} = r^2]$$

$$= -\frac{3}{r^3} + \frac{3}{r^3} = 0$$

Q# 65: Find $\vec{\nabla}\phi$ if (a) $\phi = \ln|\vec{r}|$ (b) $\phi = \frac{1}{|\vec{r}|}$

Answer:

a) Given $\phi = \ln|\vec{r}|$

$$\vec{\nabla}\phi = \vec{\nabla}\ln|\vec{r}|$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}\frac{\partial}{\partial r}\ln|\vec{r}| \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}\frac{1}{r}$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r^2} \textit{ Answer}$$

b) Given, $\phi = \frac{1}{|\vec{r}|}$

$$\vec{\nabla}\phi = \vec{\nabla}\frac{1}{|\vec{r}|}$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}\frac{\partial}{\partial r}\frac{1}{|\vec{r}|} \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}\frac{\partial}{\partial r}|\vec{r}|^{-1}$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}(-1)\left|\vec{r}\right|^{-1-1} \qquad [\because \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\Rightarrow \vec{\nabla}\phi = \frac{\vec{r}}{r}(-1)\left|\vec{r}\right|^{-2}$$

$$\Rightarrow \vec{\nabla}\phi = -\frac{\vec{r}}{r}\left|\vec{r}\right|^{-2}$$

$$\Rightarrow \vec{\nabla}\phi = -\frac{\vec{r}}{r}\frac{1}{\left|\vec{r}\right|^{2}}$$

$$\Rightarrow \vec{\nabla}\phi = -\frac{\vec{r}}{r}\frac{1}{r^2}$$

$$\Rightarrow \vec{\nabla}\phi = -\frac{\vec{r}}{r^3} \quad \textit{Answer}$$

c) Prove that $\text{i.}\ \underline{\nabla}.[\frac{f(r)}{r}\vec{r}] = f'(r) + \frac{2}{r}f(r)$

$\text{ii}\ \underline{\nabla}.[\frac{f(r)}{r}\vec{r}] = \frac{1}{r^2}\frac{d}{dr}[r^2 f(r)]$

Answer: i. $\underline{\nabla}.[\frac{f(r)}{r}\vec{r}]$

$$= \frac{f(r)}{r}(\vec{\nabla}.\vec{r}) + \left\{\vec{\nabla}\frac{f(r)}{r}\right\}.\vec{r} \qquad [\because \vec{\nabla}.(\phi\vec{A}) = \phi(\vec{\nabla}.\vec{A}) + (\vec{\nabla}\phi).\vec{A}]$$

$$= \frac{3.f(r)}{r} + \left\{\frac{\vec{r}}{r}\frac{\partial}{\partial r}\left(\frac{f(r)}{r}\right)\right\}.\vec{r} \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \frac{3.f(r)}{r} + \left\{\frac{\vec{r}}{r}\left(\frac{r f'(r) - f(r).1}{r^2}\right)\right\}.\vec{r}$$

$$= \frac{3.f(r)}{r} + \left\{\left(\frac{r f'(r) - f(r).1}{r^3}\right)\right\}(\vec{r}.\vec{r})$$

$$= \frac{3.f(r)}{r} + \left\{\left(\frac{r f'(r) - f(r).1}{r^3}\right)\right\}r^2 \qquad [\because \vec{r}.\vec{r} = r^2]$$

$$= \frac{3.f(r)}{r} + \left\{\left(\frac{r f'(r) - f(r).1}{r}\right)\right\}$$

$$= \frac{3.f(r) + r f'(r) - f(r).1}{r}$$

$$= \frac{3.f(r) + r f'(r) - f(r)}{r}$$

$$= \frac{2f(r) + r f'(r)}{r}$$

$$= \frac{2f(r)}{r} + f'(r)$$

$$= f'(r) + \frac{2f(r)}{r}$$

$$= f'(r) + \frac{2f(r)}{r} \qquad \text{(Proved)}$$

$$= \frac{1}{r^2}\left[r^2 f'(r) + 2r f(r)\right]$$

$$= \frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right] \qquad \text{(Proved)}$$

Q# 66: Show that: $\vec{\nabla} r^n = nr^{n-2}\vec{r}$

Answer: L.H.S.

$$\vec{\nabla} r^n \qquad [\because \vec{\nabla} = \frac{\vec{r}}{r}\frac{\partial}{\partial r}]$$

$$= \frac{\vec{r}}{r}\frac{\partial}{\partial r} r^n$$

$$= \frac{\vec{r}}{r} nr^{n-1}$$

$$= \vec{r}.nr^{n-1}.r^{-1}$$

$$= \vec{r}.nr^{n-2}$$

$$= nr^{n-2}\vec{r} \quad \text{(Proved)}$$

Q# 67: **Evaluate** $\vec{\nabla}.(\vec{A} \times \vec{r})$ if $\vec{\nabla} \times \vec{A} = 0$

Answer:

$$\vec{\nabla}.(\vec{A} \times \vec{r}) = \vec{r}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{r}) \qquad [\because \vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B})]$$

$$= \vec{r}.0 - \vec{A}.(\vec{\nabla} \times \vec{r})$$

$$= \vec{r}.0 - \vec{A}.\left\{\vec{\nabla} \times (x\hat{i} + y\hat{j} + z\hat{k})\right\}$$

$$= 0 - \vec{A}.\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x & y & z \end{vmatrix}$$

$$= \vec{A}.\hat{i}[\frac{\delta}{\delta y}(z) - \frac{\delta}{\delta z}(y)] - \hat{j}[\frac{\delta}{\delta x}(z) - \frac{\delta}{\delta z}(x)] + \hat{k}[\frac{\delta}{\delta x}(y) - \frac{\delta}{\delta y}(x)]$$

$$= \vec{A}.\hat{i}[0-0] - \hat{j}[0-0] + \hat{k}[0-0]$$

$= 0$ ***Answer***

Q# 68: If $\vec{v} = \vec{\omega} \times \vec{r}$, Prove $\vec{\omega} = \frac{1}{2}\text{curl } \vec{v}$, Where $\vec{\omega}$ is a constant vector.

Answer:

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$

Then $\text{curl } \vec{v} = \vec{\nabla} \times \vec{v}$

$$\text{curl } \vec{v} = \vec{\nabla} \times (\vec{\omega} \times \vec{r})$$

$$\text{curl } \vec{v} = (\vec{r}.\vec{\nabla})\vec{\omega} - \vec{r}(\vec{\nabla}.\vec{\omega}) - (\vec{\omega}.\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\text{curl } \vec{v} = (x\hat{i} + y\hat{j} + z\hat{k}).(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) - \vec{r}(\vec{\nabla}.\vec{\omega}) - (\vec{\omega}.\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\text{curl } \vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) - \vec{r}(\vec{\nabla}.\vec{\omega}) - (\vec{\omega}.\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\text{curl } \vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) - \vec{r}(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}).(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$
$$- (\vec{\omega}.\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\text{curl } \vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) - \vec{r}(\frac{\delta}{\delta x}\omega_1 + \frac{\delta}{\delta y}\omega_2 + \frac{\delta}{\delta z}\omega_3)$$
$$- (\vec{\omega}.\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\text{curl } \vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) - \vec{r}(0+0+0) - (\vec{\omega}.\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\text{curl } \vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) - 0 - (\vec{\omega}.\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\text{curl } \vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}) - (\vec{\omega}.\vec{\nabla})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\mathbf{curl}\,\vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) - (\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}).(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k})\vec{r}$$

$$+\vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\mathbf{curl}\,\vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) - (\omega_1\frac{\delta}{\delta x} + \omega_2\frac{\delta}{\delta y} + \omega_3\frac{\delta}{\delta z})\vec{r} + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\mathbf{curl}\,\vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) - (\omega_1\frac{\delta}{\delta x} + \omega_2\frac{\delta}{\delta y} + \omega_3\frac{\delta}{\delta z})(x\hat{i} + y\hat{j} + z\hat{k}) + \vec{\omega}(\vec{\nabla}.\vec{r})$$

$$\mathbf{curl}\,\vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) - (\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) + \vec{\omega}3 \quad [\because(\vec{\nabla}.\vec{r}) = 3]$$

$$\mathbf{curl}\,\vec{v} = (x\frac{\delta}{\delta x} + y\frac{\delta}{\delta y} + z\frac{\delta}{\delta z})(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) - \vec{\omega} + 3\vec{\omega}$$

$$\mathbf{curl}\,\vec{v} = (x.0 + y.0 + z.0) + 2\vec{\omega}$$

$$\mathbf{curl}\,\vec{v} = 0 + 2\vec{\omega}$$

$$\mathbf{curl}\,\vec{v} = 2\vec{\omega}$$

Therefore, $\vec{\omega} = \frac{1}{2}\mathbf{curl}\,\vec{v}$ ***(Proved)***

Q# 69: Show that $\phi(x,y,z)$ is any solution of Laplace's equation. Then $\nabla\phi$ is a vector which both solenoidal and irrotational.

Answer:

We have, A solenoidal vector field satisfies $\vec{\nabla}.\vec{B} = 0$

A vector field $\vec{\nabla}$ is said to be *irrotational* if its curl is zero. That is, if $\vec{\nabla}\times\vec{v} = 0$.
A conservative vector field is also **irrotational.**

Since $\phi(x,y,z)$ satisfies the Laplace's equation hence, $\nabla^2\phi = 0$ or $\nabla.\nabla\phi = 0$

Therefore, $\vec{\nabla}\phi$ is solenoidal.

and also $\mathbf{curl}\,\vec{v} = \vec{\nabla}\times(\vec{\nabla}\phi) = (\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z})\times(\hat{i}\frac{\delta}{\delta x} + \hat{j}\frac{\delta}{\delta y} + \hat{k}\frac{\delta}{\delta z})\phi$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{\delta\phi}{\delta x} & \frac{\delta\phi}{\delta y} & \frac{\delta\phi}{\delta z} \end{vmatrix}$$

$$= \hat{i}(\frac{\delta^2\phi}{\delta y\delta z} - \frac{\delta^2\phi}{\delta z\delta y}) - \hat{j}(\frac{\delta^2\phi}{\delta x\delta z} - \frac{\delta^2\phi}{\delta z\delta x}) + \hat{k}(\frac{\delta^2\phi}{\delta x\delta y} - \frac{\delta^2\phi}{\delta y\delta x})$$

$$= \hat{\mathbf{i}} \times 0 - \hat{\mathbf{j}} \times 0 + \hat{\mathbf{k}} \times 0$$
$$= \mathbf{0}$$

Hence $\vec{\nabla}\phi$ is also irrotational. (Proved)

Q# 70: If $\vec{\mathbf{A}}$ **and** $\vec{\mathbf{B}}$ are irrotational then prove that $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is solenoidal.

Answer: Since $\vec{\mathbf{A}}$ **and** $\vec{\mathbf{B}}$ are irrotational, hence $\vec{\nabla} \times \vec{\mathbf{A}} = \mathbf{0}$ **and** $\vec{\nabla} \times \vec{\mathbf{B}} = \mathbf{0}$

and if $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is solenoidal then $\vec{\nabla} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \mathbf{0}$

L.H.S. $\vec{\nabla} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}})$

$$= \vec{\mathbf{B}} \cdot (\vec{\nabla} \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\vec{\nabla} \times \vec{\mathbf{B}}) \qquad [\because \vec{\nabla} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \vec{\mathbf{B}} \cdot (\vec{\nabla} \times \vec{\mathbf{A}}) - \vec{\mathbf{A}} \cdot (\vec{\nabla} \times \vec{\mathbf{B}})]$$
$$= \vec{\mathbf{B}} \cdot \mathbf{0} - \vec{\mathbf{A}} \cdot \mathbf{0}$$
$$= \mathbf{0} \ \textit{(Proved)}$$

Hence $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is solenoidal. (Proved)

Q# 71: Prove that $(A \times B).(B \times C) \times (C \times A) = [ABC]^2$

Solution:
L.H.S

$$(A \times B).(B \times C) \times (C \times A)$$
$$let, B \times C = X$$
$$\therefore (A \times B).(B \times C) \times (C \times A)$$
$$= (A \times B).(X) \times (C \times A)$$
$$= (A \times B).[(X.A)C - (X.C)A]$$

[From Q # 43, $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \cdot \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \cdot \vec{\mathbf{b}})\vec{\mathbf{c}}$]

$$\therefore (A \times B).(B \times C) \times (C \times A) = (A \times B).[(B \times C.A)C - (B \times C.C)A] - - - -(i)$$

$[\because B \times C = X]$

Now, $\vec{B} \times \vec{C}$

$$= (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) \times (c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$
$$= \hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)$$

$$\therefore B \times C.C$$

$= [\hat{i}(b_2c_3 - b_3c_2) - \hat{j}(b_1c_3 - b_3c_1) + \hat{k}(b_1c_2 - b_2c_1)].\ (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$= c_1(b_2c_3 - b_3c_2) - c_2(b_1c_3 - b_3c_1) + c_3(b_1c_2 - b_2c_1)]\ [\therefore \hat{i}.\hat{i} = 1,\ \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1]$

$= c_1b_2c_3 - c_1b_3c_2 - c_2b_1c_3 + c_2b_3c_1 + c_3b_1c_2 - c_3b_2c_1$

$\therefore B \times C.C = 0$ --------------------------------------(ii)

From (i)

$$\begin{aligned}\therefore (A \times B).(B \times C) \times (C \times A) &= (A \times B).[(B \times C.A)C - (B \times C.C)A] \\ &= (A \times B).[(B \times C.A)C - 0] \quad [B \times C.C = 0; \textit{from}\,(ii)] \\ &= (A \times B).[(B \times C.A)C] \\ &= [A \times B.C][B \times C.A] \\ &= [ABC][ABC]\end{aligned}$$

We have, Scalar triple product: $\vec{A}.(\vec{B}\times\vec{C})$ or $\vec{B}.(\vec{C}\times\vec{A})$ or $\vec{C}.(\vec{A}\times\vec{B})$ are known as a scalar triple product. It is symbolically denoted by $[\mathbf{ABC}]$ or $[\mathbf{BCA}]$ or $[\mathbf{CAB}]$

$= [ABC]^2$

(Proved)

Home task

Find the values of constants $a,\ b$ and c so that,
$\vec{F} = (x + 2y + az)\hat{\imath} + (bx - 3y - z)\hat{\jmath} + (4x + cy + 2z)\hat{k}$ is irrotational.

If $\emptyset(x, y, z) = xy^2z$ and $\vec{A} = xz\hat{\imath} - xy^2\hat{\jmath} + yz^2\hat{k}$, then find $\frac{\partial^3}{\partial x^2\,\partial z}(\emptyset\vec{A})$ at the point $(2, -1, 1)$