http://www.robots.ox.ac.uk/~sjrob/Teaching/Vectors/slides4.pdf

https://www.physicstutorials.org/exams-and-problem-solutions/dynamics-exams-and-problem-solutions/dynamics-exam1-and-problem-solutions/

Vector Analysis

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Contents

Introduction

Physical quantities can be divided into two main groups, scalar quantities and vector quantities

Scalar Quantities: A Physical Quantity which has magnitude only is called as a Scalar**.**

Example: Time, Temperature, Mass, Volume are examples of scalars.

That is, the measurement of years, months, weeks, days, hours, minutes, seconds, and even milliseconds, A temperature of 15°C, A mass of 0.2 kg, etc.

Vectors: A Physical Quantity which has both magnitude and direction is called as Vector Examples: velocity, displacement, acceleration, force etc.

Some Examples:

01. A speed of 10 km/h is a scalar quantity, but a velocity of 10 km/h **due north** is a vector quantity.

02. A temperature of 100^0c is a scalar quantity.

03. The weight of a 7 kg mass is a vector quantity. $[\,w = mg\,]$

Vector Notation: Typical notation to designate a vector $\vec{\mathbf{AB}}$ is a boldfaced character or a character with an arrow on it, or a character with a line under it (i.e, $\vec{\mathbf{AB}}$, $\underline{\mathbf{AB}}$, **AB**).

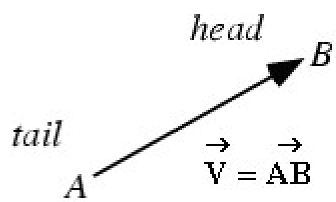
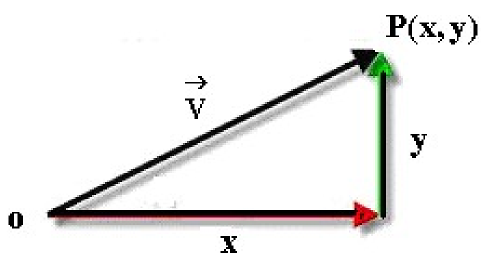


Figure: 01

The <u>Magnitude of a vector</u>: The magnitude of a vector $\vec{\mathbf{OP}}$ or $\vec{\mathbf{V}}$ **is <u>its length</u>** and is normally denoted by $|\vec{\mathbf{V}}|$ or V. Given a vector $\vec{\mathbf{V}}$ with tail at the origin O and head at $\mathbf{P(x,y)}$, what's its length?



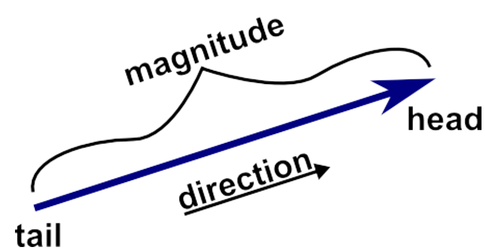


Figure: 02 Figure: 03

According to Pythagoras, the length of the hypotenuse OP is the square root of $\mathbf{x}^2 + \mathbf{y}^2$.

That is, **Magnitude of a vector** $\vec{\mathbf{V}}$ = **<u>its length</u>** = $\left|\vec{\mathbf{OP}}\right| = \left|\vec{\mathbf{V}}\right| = \sqrt{\mathbf{x^2 + y^2}}$

The **zero vectors** are **the vector with zero magnitude that is vector's length is zero.**

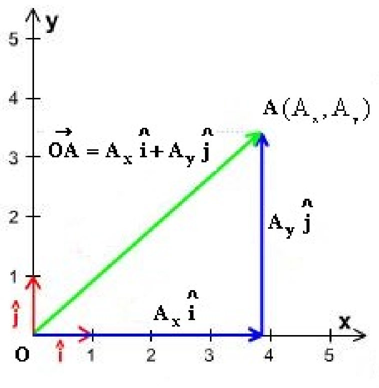


Figure # 04

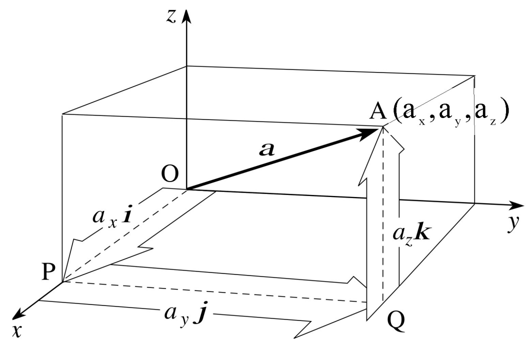


Figure # 05

From figure # 04

$$\vec{\mathbf{OA}} = \mathbf{A_x \hat{i} + A_y \hat{j}}$$

The length of the vector $\vec{\mathbf{OA}} = \sqrt{\mathbf{A_x^2 + A_y^2}}$

From figure # 05

$$\vec{\mathbf{OA}} = \vec{\mathbf{a}} = \mathbf{a_x \hat{i} + a_y \hat{j} + a_z \hat{k}}$$

The length of the vector $\vec{\mathbf{OA}} = \left|\vec{\mathbf{a}}\right| = \sqrt{\mathbf{a_x^2 + a_y^2 + a_z^2}}$

<u>Graphical Vector Addition</u>

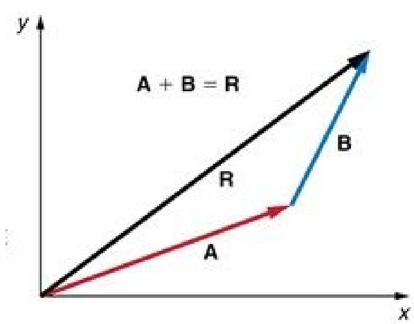


Figure 06

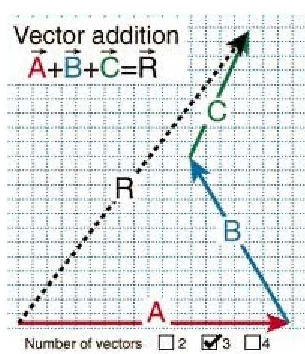


Figure 07

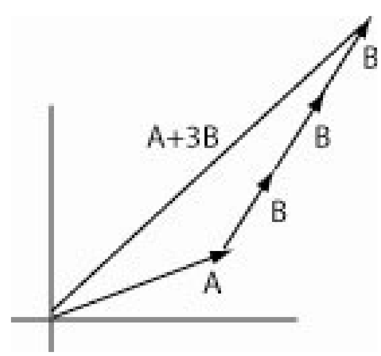


Figure: 08

Position vector:
In geometry, a position or position vector, also known as location vector or radius vector. The origin is (0, 0) and the position vector is basically just a straight line drawn from the origin to some other point.

The position vector is the <u>vector from the origin</u> of the coordinate system **O(0,0)** to the point $\mathbf{P(x, y)}$. It is shown as the vector op (Figure 09)

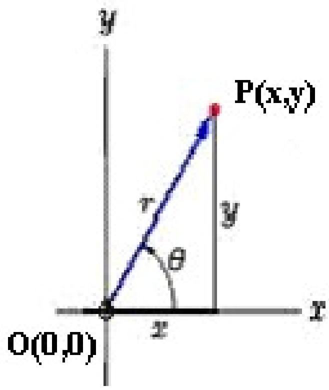


Figure: 09

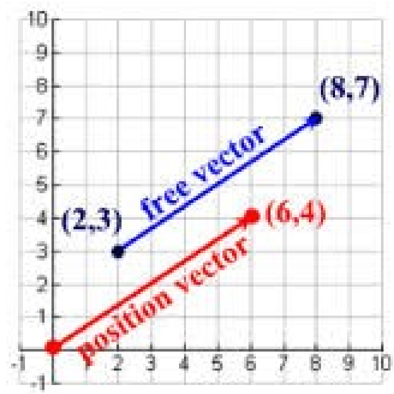


Figure: 10

Q# 01:
Draw the curves:

i) $\vec{G}(t) = t\,\hat{i} + 2t\,\hat{j}, \quad t \;\text{ in } [0,1]$

ii) $\vec{H}(t) = t^2\,\hat{i} + 2t^2\,\hat{j}, \quad t \;\text{ in } [0,1]$

iii) $\vec{J}(t) = (1 - \frac{t}{2})\,\hat{i} + (2 - t)\,\hat{j}, \quad t \;\text{ in } [0,2]$

iv) Compute $\int_C xy\,dx$, where C is the curve given by

$\vec{J}(t) = (1 - \frac{t}{2})\,\hat{i} + (2 - t)\,\hat{j}, \quad t \;\text{ in } [0,2]$

Answer Q # 01:

i) $\vec{G}(t) = t\,\hat{i} + 2t\,\hat{j}, \quad t \;\text{ in } [0,1]$

$\vec{G}(0) = 0\,\hat{i} + 0\,\hat{j},$

$\vec{G}(1) = 1\,\hat{i} + 2\,\hat{j},$

$\therefore \vec{OP} = \vec{G}(t) = t\,\hat{i} + 2t\,\hat{j}, \quad t \;\text{ in } [0,1]$ [Figure # 11]

ii) $\vec{H}(t) = t^2\,\hat{i} + 2t^2\,\hat{j}, \quad t \;\text{ in } [0,1]$

$\vec{H}(0) = 0\,\hat{i} + 0\,\hat{j},$

$\vec{H}(1) = 1\,\hat{i} + 2\,\hat{j},$

$\therefore \vec{OP} = \vec{H}(t) = t^2\,\hat{i} + 2t^2\,\hat{j}, \quad t \;\text{ in } [0,1]$ [Figure # 11]

iii) $\vec{\mathbf{J}}(\mathbf{t}) = (\mathbf{1} - \frac{\mathbf{t}}{\mathbf{2}})\hat{\mathbf{i}} + (\mathbf{2} - \mathbf{t})\hat{\mathbf{j}},\ \ \mathbf{t}\ \ \text{in } [0,2]$

$\vec{\mathbf{J}}(0) = 1\,\hat{\mathbf{i}} + 2\,\hat{\mathbf{j}},$

$\vec{\mathbf{J}}(2) = (1 - \frac{2}{2})\hat{\mathbf{i}} + (2 - 2)\hat{\mathbf{j}},$

$\vec{\mathbf{J}}(2) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}},$

$\therefore \vec{\mathbf{PO}} = \vec{\mathbf{J}}(\mathbf{t}) = (1 - \frac{t}{2})\hat{\mathbf{i}} + (2 - t)\hat{\mathbf{j}},\ \ t\ \ \text{in } [0,2]$ [Figure # 12]

iv) Compute $\int_C \mathbf{xy\ dx}$, where C is the curve given by

$\vec{\mathbf{J}}(\mathbf{t}) = (1 - \frac{t}{2})\hat{\mathbf{i}} + (2 - t)\hat{\mathbf{j}},\ \ t\ \ \text{in } [0,2]$

Here, $\mathbf{x = (1 - \frac{t}{2})}$ **and** $\mathbf{y = (2 - t)}$

$$\Rightarrow \frac{dx}{dt} = -\frac{1}{2}$$

$$\therefore \int_C xy\ dx = \int_0^2 (1 - \frac{t}{2})(2 - t)\frac{dx}{dt}dt = \int_0^2 (1 - \frac{t}{2})(2 - t)(-\frac{1}{2})dt = \int_0^2 (2 - t - t + \frac{t^2}{2})(-\frac{1}{2})dt$$

$$= -\frac{1}{2}\int_0^2 (2 - 2t + \frac{t^2}{2})dt = -\frac{1}{2}(2t - t^2 + \frac{t^3}{6})_0^2 = -\frac{1}{2}(4 - 4 + \frac{8}{6}) = -\frac{2}{3}\ \ \textbf{Answer}$$

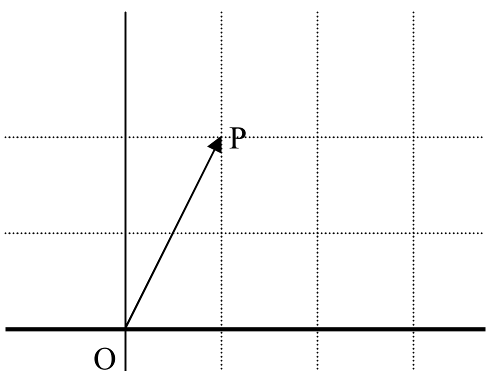


Figure # 11 for G (t) and H (t)

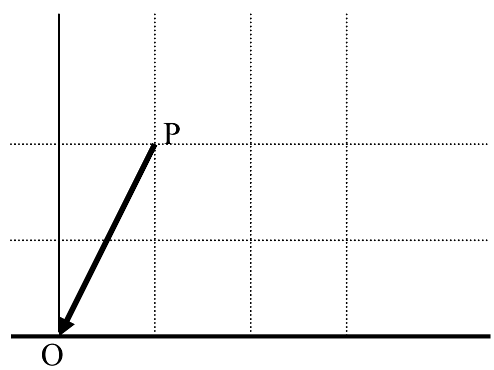


Figure # 12 for J (t)

Unit vectors:

A unit vector $\hat{\mathbf{e}}$ is a vector of unit length. A unit vector is sometimes denoted by replacing the arrow on a vector with a "^" on a boldfaced character (i.e, $\hat{\mathbf{e}}$). Therefore, $|\hat{\mathbf{e}}| = 1$

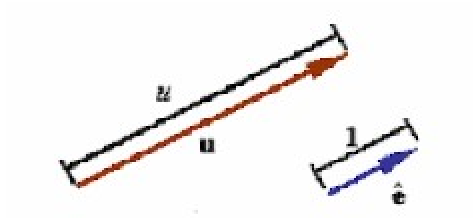


Figure # 13

Any vector can be made into a unit vector by dividing it by its length.

$$\hat{\mathbf{e}} = \frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|}$$

$$\Rightarrow \vec{u} = |\vec{u}| \times \hat{e}$$

Any vector $\vec{\mathbf{u}}$ can be fully represented by providing its magnitude and a unit vector along its direction. $\vec{\mathbf{u}} = |\vec{\mathbf{u}}| \times \hat{\mathbf{e}}$ [**That is, Any Vector = Length of this Vector × Unit Vector**]

Vector components:

For example, $\vec{\mathbf{u}} = \vec{\mathbf{u}}_1 + \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3$

Where, $\vec{\mathbf{u}}_1 = |\vec{\mathbf{u}}_1| \hat{\mathbf{e}}_1$ [∵ **Any Vector = Length of this Vector × Unit Vector**]

$$\vec{\mathbf{u}}_2 = |\vec{\mathbf{u}}_2| \hat{\mathbf{e}}_2$$

$$\vec{\mathbf{u}}_3 = |\vec{\mathbf{u}}_3| \hat{\mathbf{e}}_3$$

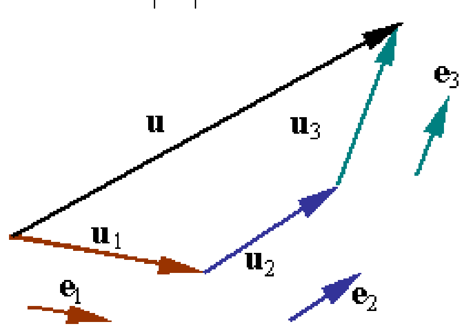


Figure # 14

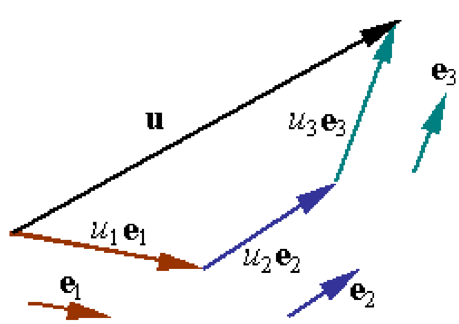


Figure # 15

The original vector $\vec{u}$ can now be written as

$$\therefore \vec{u} = \vec{u}_1 + \vec{u}_2 + \vec{u}_3$$

$$\therefore \vec{u} = \left|\vec{u}_1\right| \hat{e}_1 + \left|\vec{u}_2\right| \hat{e}_2 + \left|\vec{u}_3\right| \hat{e}_3$$

Vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are unit vectors and $\left|\vec{u}_1\right|, \left|\vec{u}_2\right|, \left|\vec{u}_3\right|$ are the length of the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ respectively.

As for Example: Here, $\hat{e}_1$ is a unit vector of $\vec{AB}$

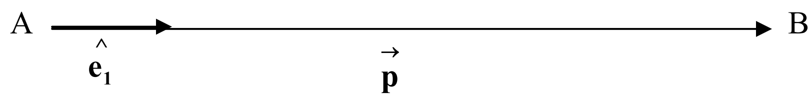


Figure 16

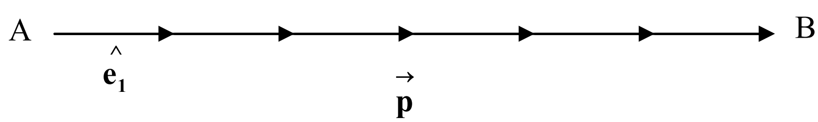


Figure 17

$\left|\vec{AB}\right| = 6$ *[From Figure 17]*

$\therefore \vec{AB} = \vec{P} = \left|\vec{AB}\right| \hat{e}_1 = 6\hat{e}_1$ **[Any Vector = Length of this Vector × Unit Vector]**

$$\therefore \text{Unit Vector } \hat{e}_1 = \frac{\vec{AB}}{\left|\vec{AB}\right|} = \frac{6\hat{e}_1}{6} = \hat{e}_1$$

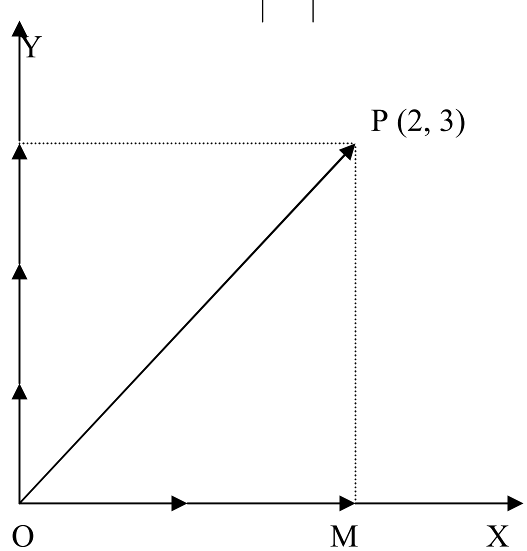


Figure # 18

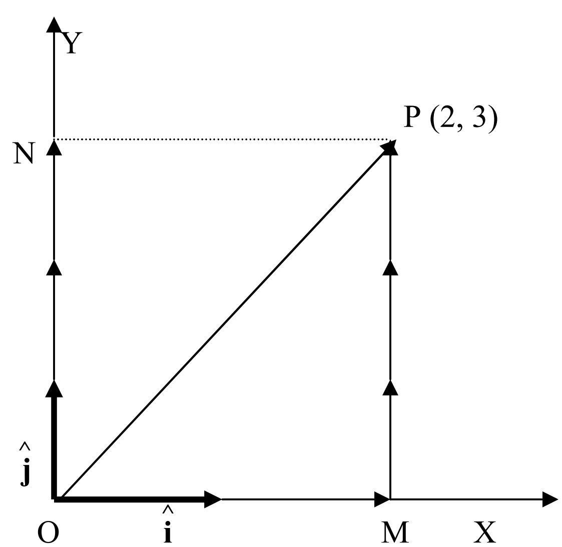


Figure # 19

From figure 18

$$\vec{\mathbf{OP}} = \begin{pmatrix} \mathbf{2} \\ \mathbf{3} \end{pmatrix}$$

From figure 19

$\vec{\mathbf{OM}} = \mathbf{2}\,\hat{\mathbf{i}}$ $\quad$ [$\vec{\mathbf{OM}} = \mathbf{2}\,\hat{\mathbf{i}}$ =**Length of this Vector** $\times$ **Unit Vector**]

$\vec{\mathbf{ON}} = \vec{\mathbf{MP}} = \mathbf{3}\,\hat{\mathbf{j}}$ $\quad$ [$\vec{\mathbf{ON}} = \vec{\mathbf{MP}} = \mathbf{3}\,\hat{\mathbf{j}}$ = **Length of this Vector** $\times$ **Unit Vector**]

From, ΔOMP,

$$\vec{\mathbf{OP}} = \vec{\mathbf{OM}} + \vec{\mathbf{MP}} = \mathbf{2}\,\hat{\mathbf{i}} + \mathbf{3}\,\hat{\mathbf{j}}$$

Here, $\left|\vec{\mathbf{OM}}\right| = \mathbf{2}$, $\left|\vec{\mathbf{MP}}\right| = \mathbf{3}$

$$\therefore \left|\vec{\mathbf{OP}}\right| = \sqrt{\mathbf{OM}^2 + \mathbf{MP}^2} = \sqrt{\mathbf{2}^2 + \mathbf{3}^2} = \sqrt{\mathbf{13}}$$

$$\therefore \text{Unit vector of } \vec{\mathbf{OM}} = \frac{\vec{\mathbf{OM}}}{\left|\vec{\mathbf{OM}}\right|} = \frac{\mathbf{2}\,\hat{\mathbf{i}}}{\mathbf{2}} = \hat{\mathbf{i}}$$

$$\therefore \text{Unit vector of } \vec{\mathbf{ON}} = \vec{\mathbf{MP}} = \frac{\vec{\mathbf{ON}}}{\left|\vec{\mathbf{ON}}\right|} = \frac{\mathbf{3}\,\hat{\mathbf{j}}}{\mathbf{3}} = \hat{\mathbf{j}}$$

$$\therefore \text{Unit vector of } \vec{\mathbf{OP}} = \hat{\mathbf{e}} = \frac{\vec{\mathbf{OP}}}{\left|\vec{\mathbf{OP}}\right|} = \frac{\mathbf{2}\,\hat{\mathbf{i}} + \mathbf{3}\,\hat{\mathbf{j}}}{\sqrt{\mathbf{13}}} = \frac{\mathbf{2}}{\sqrt{\mathbf{13}}}\,\hat{\mathbf{i}} + \frac{\mathbf{3}}{\sqrt{\mathbf{13}}}\,\hat{\mathbf{j}}$$

$$\therefore \text{Magnitude of Unit vector of } \vec{\mathbf{OP}} = \left|\hat{\mathbf{e}}\right| = \sqrt{\left(\frac{\mathbf{2}}{\sqrt{\mathbf{13}}}\right)^2 + \left(\frac{\mathbf{3}}{\sqrt{\mathbf{13}}}\right)^2} = \sqrt{\frac{\mathbf{4}}{\mathbf{13}} + \frac{\mathbf{9}}{\mathbf{13}}} = \sqrt{\frac{\mathbf{13}}{\mathbf{13}}} = \mathbf{1}$$

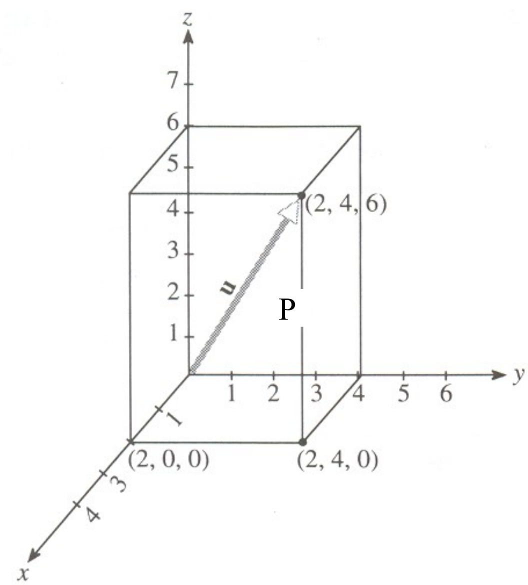


Figure 20

From figure 20

$\vec{\mathbf{OP}} == 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

$\therefore \left|\vec{\mathbf{OP}}\right| = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{4 + 16 + 36} = \sqrt{56}$

$\therefore$ Unit vector of $\vec{\mathbf{OP}} = \hat{\mathbf{e}} = \dfrac{\vec{\mathbf{OP}}}{\left|\vec{\mathbf{OP}}\right|} = \dfrac{2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 6\hat{\mathbf{k}}}{\sqrt{56}} = \dfrac{2}{\sqrt{56}}\hat{\mathbf{i}} + \dfrac{4}{\sqrt{56}}\hat{\mathbf{j}} + \dfrac{6}{\sqrt{56}}\hat{\mathbf{k}}$

$\therefore$ Magnitude of Unit vector of $\vec{\mathbf{OP}}$

$$= \left|\hat{\mathbf{e}}\right| = \sqrt{\left(\frac{2}{\sqrt{56}}\right)^2 + \left(\frac{4}{\sqrt{56}}\right)^2 + \left(\frac{6}{\sqrt{56}}\right)^2} = \sqrt{\frac{4}{56} + \frac{16}{56} + \frac{36}{56}} = \sqrt{\frac{56}{56}} = 1$$

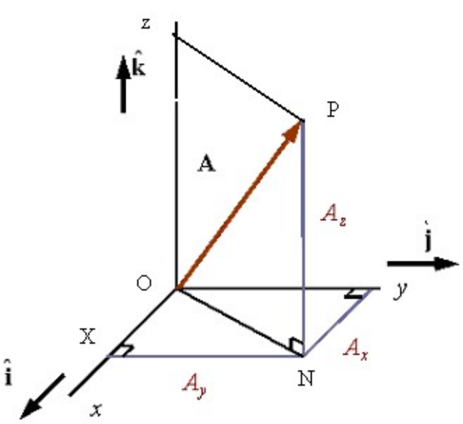


Figure 21

From figure 21

ΔONP,

$$\vec{OP} = \vec{A} = \vec{ON} + \vec{NP}$$
$$= (\vec{OX} + \vec{XN}) + \vec{NP}$$
$$= (\vec{OX} + \vec{OY}) + \vec{NP}$$
$$= (\vec{OX} + \vec{OY}) + \vec{OZ}$$
$$= A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

and

$$|\vec{A}| = |\vec{OP}| = \sqrt{|\vec{OP}|^2} = \sqrt{|\vec{ON}|^2 + |\vec{NP}|^2} = \sqrt{|\vec{OX}|^2 + |\vec{XN}|^2 + |\vec{OZ}|^2} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \text{ and } |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Q#02: Find the unit tangent *(slope, ঢাল, স্পর্শক, tan θ, m, rate of change, derivative/differentiation)* vector to the graph of $\vec{r}(t) = t^2 \hat{i} + t^3 \hat{j}$ at the point where t = 2

Answer # 02

Given, $\vec{r}(t) = t^2 \hat{i} + t^3 \hat{j}$ ---------------------------------------(i)

$\therefore \vec{r}(2) = 4\hat{i} + 8\hat{j}$ --------------------------------------(ii)

Here x = 4, y = 8

$\therefore \vec{OP} = \vec{r}(2) = 4\hat{i} + 8\hat{j}$ [See figure no 22]

The tangent vector will be drawn at (4, 8) which is $\vec{T}(2)$

From (i),

$$\vec{r}(t) = t^2 \hat{i} + t^3 \hat{j}$$

$$\frac{d\vec{r}}{dt} = \vec{r}'(t) = 2t\,\hat{i} + 3t^2\,\hat{j} \text{ --------------------------------------(iii)}$$

$$\therefore \vec{T}(t) = \vec{r}'(t) = 2t\,\hat{i} + 3t^2\,\hat{j} \text{ ----------------------------------(iv)}$$

$$\therefore \vec{T}(2) = \vec{r}'(2) = 2 \times 2\,\hat{i} + 3 \times 2^2\,\hat{j}$$

$\therefore$ Tangent vector: $\vec{PQ} = \vec{T}(2) = \vec{r}'(2) = 4\hat{i} + 12\hat{j}$ ------------(v)

$\therefore$ Unit tangent vector:

$$\frac{\vec{T}(2)}{\left|\vec{T}(2)\right|} = \frac{\vec{r}'(2)}{\left|\vec{r}'(2)\right|} = \frac{4\hat{i}+12\hat{j}}{\sqrt{4^2+12^2}} = \frac{4}{\sqrt{160}}\hat{i}+\frac{12}{\sqrt{160}}\hat{j} = \frac{4}{\sqrt{16\times 10}}\hat{i}+\frac{12}{\sqrt{16\times 10}}\hat{j}$$

$$= \frac{4}{4\sqrt{10}}\hat{i}+\frac{12}{4\sqrt{10}}\hat{j} = \frac{1}{\sqrt{10}}\hat{i}+\frac{3}{\sqrt{10}}\hat{j}$$

The Figure is following:

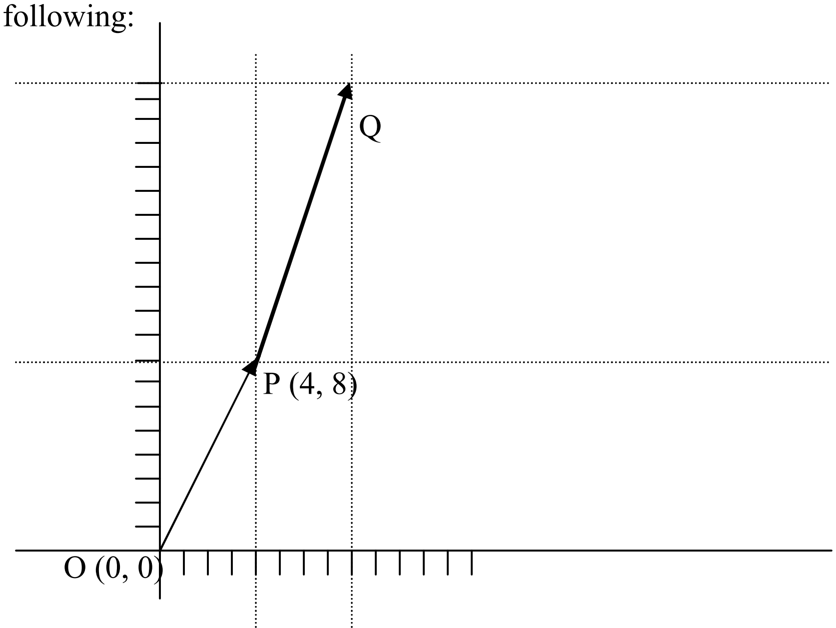


Figure: 22

When $t = 1,\ \vec{r}(1) = 1.\hat{i} + 1.\hat{j}$

When $t = 2,\ \vec{r}(2) = 4.\hat{i} + 8.\hat{j}$

When $t = 3,\ \vec{r}(3) = 9.\hat{i} + 27.\hat{j}$

From (iii),

$\vec{r}'(t) = 2t\hat{i} + 3t^2\hat{j}$

When $t = 2,\ \vec{r}'(2) = 2\times 2\hat{i} + 3\times 2^2\hat{j} = 4\hat{i} + 12\hat{j}$ Answer

Vector Components:

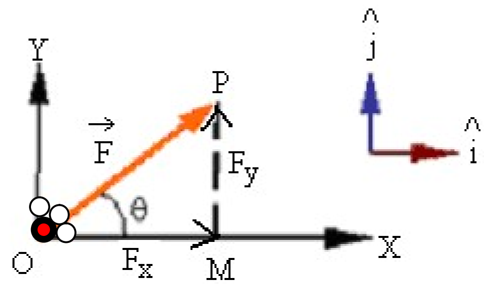


Figure 23

Here,

$\mathbf{OM = F_x, MP = F_y, OP = \left|\vec{F}\right|}$

[∵ **Any Vector = Length of this Vector × Unit Vector]**

$\mathbf{\overrightarrow{OM} = F_x\,\hat{i}}$ [$\overrightarrow{OM} = F_x\,\hat{i}$ =**Length of this Vector × Unit Vector**]

$\mathbf{\overrightarrow{MP} = F_y\,\hat{j}}$ [$\overrightarrow{MP} = F_y\,\hat{j}$ = **Length of this Vector × Unit Vector**]

From, ΔOMP,

$\overrightarrow{OP} = \vec{F} = \overrightarrow{OM} + \overrightarrow{MP} = F_x\,\hat{i} + F_y\,\hat{j}$ --------------------(i)

From, ΔOMP,

$$\mathbf{\frac{OM}{OP} = \cos\theta}$$

$\mathbf{OM = OP\cos\theta}$

$\mathbf{F_x = \left|\vec{F}\right|\cos\theta}$ ---(ii)

From, ΔOMP,

$$\mathbf{\frac{MP}{OP} = \sin\theta}$$

$\mathbf{MP = OP\sin\theta}$

$\mathbf{F_y = \left|\vec{F}\right|\sin\theta}$ ---(iii)

Projection: Imagine parallel rays of light shining vertically downwards on to the *x*-axis. The quantity $\left|\left|\vec{\mathbf{b}}\right|\mathbf{cos}\,\theta\right|$; which gives the size of the 'shadow' of vector $\vec{\mathbf{b}}$ on the *x*-axis, is often termed the **projection** of $\vec{\mathbf{b}}$ on to the *x*-axis

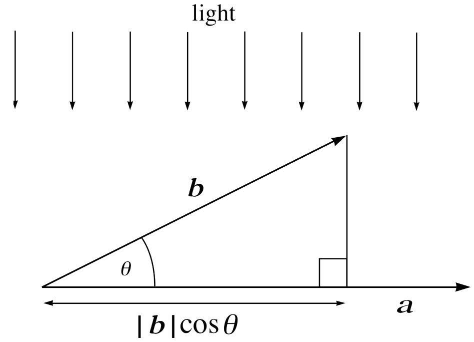


Figure: 24

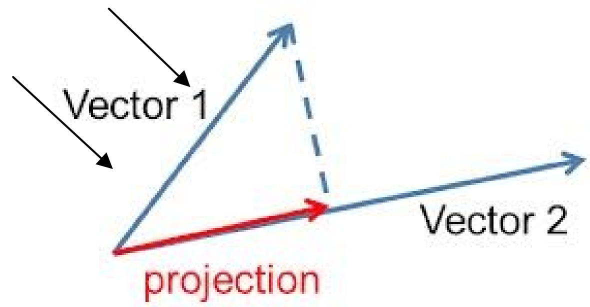


Figure: 25

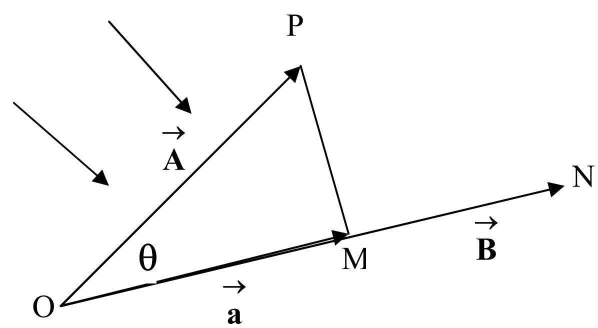


Figure: 26

Projection of $\vec{\mathbf{A}}$ on $\vec{\mathbf{B}}$ **is** $\vec{\mathbf{a}}$

Here, ΔOPM, $\dfrac{\mathbf{OM}}{\mathbf{OP}} = \mathbf{cos}\,\theta$

$$\therefore \frac{\left|\vec{\mathbf{a}}\right|}{\left|\vec{\mathbf{A}}\right|} = \mathbf{cos}\,\theta$$

Then, $\left|\vec{\mathbf{a}}\right| = \left|\vec{\mathbf{A}}\right|\mathbf{cos}\,\theta$

$\therefore$ Projection of $\vec{\mathbf{A}}$ on $\vec{\mathbf{B}}$ **is** : OM= $\left|\vec{\mathbf{a}}\right| = \left|\vec{\mathbf{A}}\right|\mathbf{cos}\,\theta$ --------------(i)

$$\therefore \frac{\left|\vec{\mathbf{a}}\right|}{\left|\vec{\mathbf{A}}\right|} = \cos\theta \text{ --(ii)}$$

Again,

$$\mathbf{OP} = \left|\vec{\mathbf{A}}\right|$$

$$\mathbf{ON} = \left|\vec{\mathbf{B}}\right|$$

Again,

$$\vec{A}.\vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|\cos\theta$$

$$\therefore \frac{\vec{A}.\vec{B}}{\left|\vec{B}\right|} = \left|\vec{A}\right|\cos\theta$$

$$\therefore \left|\vec{A}\right|\cos\theta = \frac{\vec{A}.\vec{B}}{\left|\vec{B}\right|} \qquad \text{-----------------------------(iii)}$$

From (i) & (iii),

$\therefore$ Projection of $\vec{\mathbf{A}}$ on $\vec{\mathbf{B}}$ **is** $= \dfrac{\vec{A}.\vec{B}}{\left|\vec{B}\right|}$

From (iii)

$$\cos\theta = \frac{\vec{A}.\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|} \text{ --(iv)}$$

From (ii) and (iii), we can write,

$$\therefore \frac{\left|\vec{\mathbf{a}}\right|}{\left|\vec{\mathbf{A}}\right|} = \frac{\vec{\mathbf{A}}.\vec{\mathbf{B}}}{\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|}$$

$$\therefore \left|\vec{\mathbf{a}}\right| = \frac{\vec{\mathbf{A}}.\vec{\mathbf{B}}}{\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|}\left|\vec{\mathbf{A}}\right|$$

The projection $\therefore \left|\vec{\mathbf{a}}\right| = \dfrac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{\left|\vec{\mathbf{B}}\right|}$ --(iv)

Again,

$\vec{\mathbf{a}} = \left|\vec{\mathbf{a}}\right| \times$ unit vector of $\vec{\mathbf{B}}$

$\vec{\mathbf{a}} = \left|\vec{\mathbf{a}}\right| \times \dfrac{\vec{\mathbf{B}}}{\left|\vec{\mathbf{B}}\right|}$ --(v)

Putting the value of $\left|\vec{\mathbf{a}}\right|$ from (iv) in (v)

$$\vec{\mathbf{a}} = \left|\vec{\mathbf{a}}\right| \times \frac{\vec{\mathbf{B}}}{\left|\vec{\mathbf{B}}\right|}$$

$$\vec{\mathbf{a}} = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{\left|\vec{\mathbf{B}}\right|} \times \frac{\vec{\mathbf{B}}}{\left|\vec{\mathbf{B}}\right|}$$

The projection $\vec{\mathbf{a}} = \dfrac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{\left|\vec{\mathbf{B}}\right|^{2}} \vec{\mathbf{B}}$

Three –dimensional Coordinate System

The point P(a, b, c) determines a rectangular box as the figure

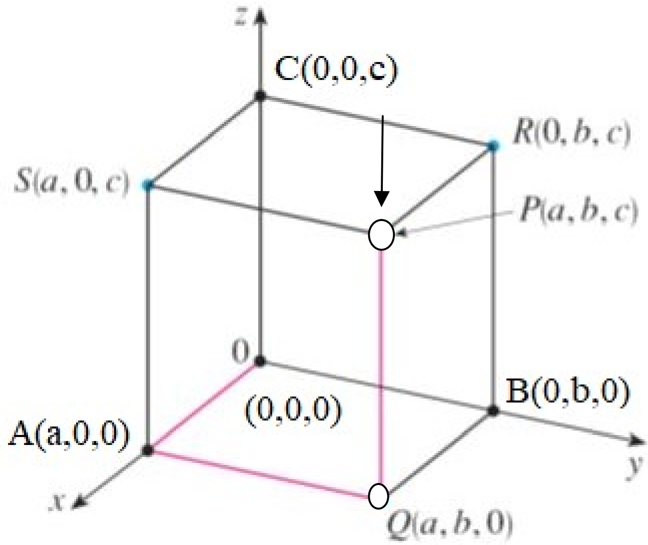


Figure: 27

If we drop a perpendicular from P to the xy-plane, we get a point Q with coordinates (a, b, 0) called the projection of P onto the xy-plane. Similarly, R(0, b, c) and S(a, 0, c) are the projections of P onto the yz-plane and xz-plane, respectively.

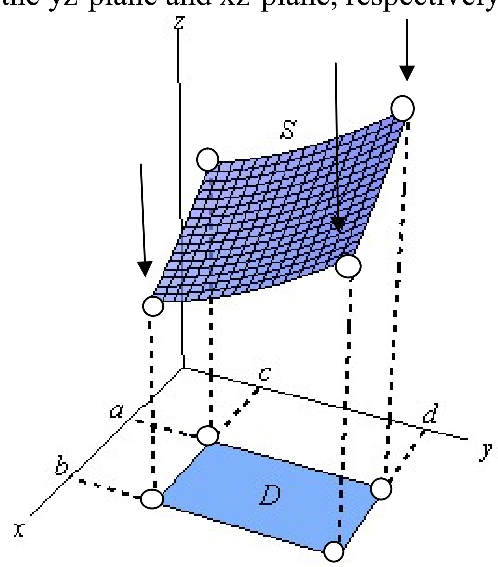


Figure: 28

Work done by dot products of two vectors:

If you have taken physics class, you have probably encountered the notion of work in mechanics.

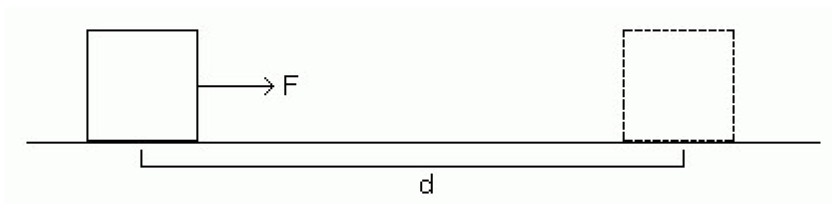


Figure: 29

If a constant force of F (in the direction of motion) is applied to move an object a distance d in a straight line, then the work exerted is

$$\mathbf{Work} = \mathbf{Force} \times \mathbf{distance}$$
$$\mathbf{W} = \mathbf{F} \times \mathbf{d}$$

The unit for force is N (newton) and the unit for distance is m (meter). The unit of work is joule=(newton)(meter).

Now suppose that the there is an angle theta between the direction in which the constant force is applied and the direction of motion.

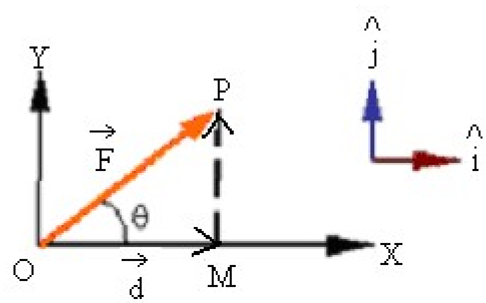


Figure: 30

In this case the work is given by:

Force Components in the direction of X-axis ×distance in the direction of Force components = $\left|\vec{\mathbf{F}}\right| \mathbf{cos}\,\theta \times \mathbf{OM} = \left|\vec{F}\right| \cos\theta \times \left|\vec{d}\right|$

Physical Significance of The scalar or dot product:

The man is pulling the block with a constant force **a** so that it moves along the horizontal ground. The **work done** in moving the block through a distance **b** is then given by the distance moved through multiplied by the magnitude of the component of the force in the direction of motion

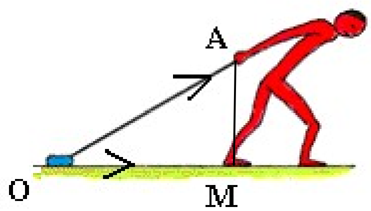


Figure 31

One important physical application of the scalar product is the calculation of work:

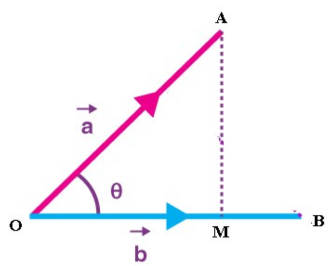


Figure 32

Here,

From Figure 30

From, $\Delta\mathbf{OMA},$

$$\frac{\mathbf{OM}}{\mathbf{OA}} = \cos\theta$$

$$\mathbf{OM} = \mathbf{OA}\cos\theta$$

$$\mathbf{OM} = \left|\vec{\mathbf{a}}\right|\cos\theta$$

$\therefore$ The force or vector components of the vector $\vec{\mathbf{a}}$ in the direction of OB is $\left|\vec{\mathbf{a}}\right|\cos\theta$

The scalar or dot product as

Work done = w = $\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\cos\theta \times \left|\vec{\mathbf{b}}\right|$

$$= \vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\cos\theta$$

That is, Force Components in the direction of OB × distance (OM) in the direction (OB) of Force or vector components $= \left|\vec{\mathbf{a}}\right| \cos\theta \times \left|\vec{\mathbf{b}}\right|$

The scalar or dot product as

Work done = w = $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right| \cos\theta \times \left|\vec{\mathbf{b}}\right|$

$\Rightarrow \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right| \cos\theta$

[দুটি Vector এর Dot Product মানে তাদের মধ্যে একটি Vector যদি Force Vector হয়, অপর Vector যদি displacement Vector হয় তাহলে Force Vector কর্তৃক সেখানে যে কাজ সংঘটিত হয় সেই কাজের পরিমানই হচ্ছে Dot Product, that is, Work done = w = $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right| \cos\theta$;

এখানে কাজের পরিমান : $\mathbf{w} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right| \cos\theta$]

Summery: ***From the physical interpretation of the dot product, the work done in moving an object a distance d by a force of magnitude F in the same direction(Figure 28) as the force is W =*** $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$

When a constant force $\vec{\mathbf{F}}$ ***is applied to a body acting at an angle*** θ ***to the direction of motion (Figure 28), then the work done by*** $\vec{\mathbf{F}}$ ***is defined to be W =*** $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = \left|\vec{\mathbf{F}}\right| \cos\theta \times \left|\vec{\mathbf{d}}\right| = \left|\vec{\mathbf{F}}\right|\left|\vec{\mathbf{d}}\right| \cos\theta$

Q # 03: A block of mass "m" moves from point A to B along a smooth plane surface under the action of force as shown in the figure. Find the work done.

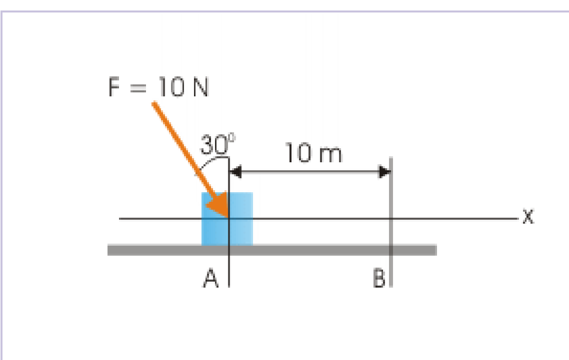


Figure 33

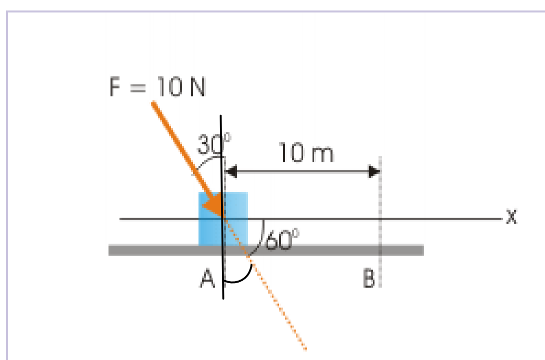


Figure 34

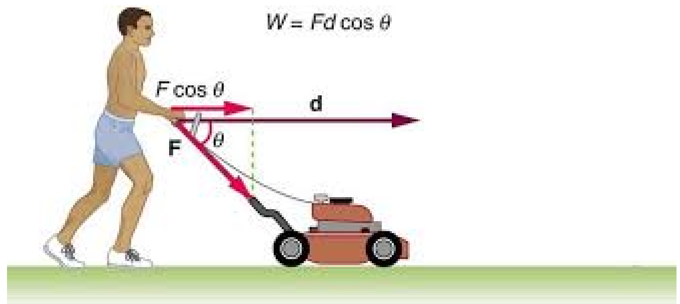


Figure 35

Answer: We have, Work done = w = $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = |\vec{\mathbf{F}}| \cos\theta \times |\vec{\mathbf{d}}| = |\vec{\mathbf{F}}||\vec{\mathbf{d}}| \cos\theta$

Here, $|\vec{\mathbf{F}}| = \mathbf{10N}$, $|\vec{\mathbf{d}}| = \mathbf{AB} = \mathbf{10}$ *meter* and $\theta = -60^0$

Work done = w = $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = |\vec{\mathbf{F}}| \cos\theta \times |\vec{\mathbf{d}}|$

Work done = w = $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = |\vec{\mathbf{F}}| \cos\theta \times |\vec{\mathbf{d}}| = 10 \times \cos(-60^0) \times 10$

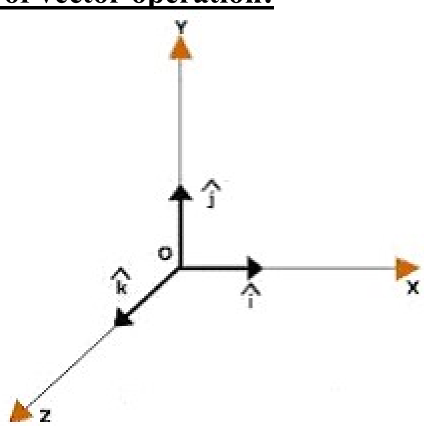
Work done = w = $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = |\vec{\mathbf{F}}| \cos\theta \times |\vec{\mathbf{d}}| = 10 \times \cos 60^0 \times 10$ $\quad [\because \cos(-\theta) = \cos\theta]$

Work done = w = $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = |\vec{\mathbf{F}}| \cos\theta \times |\vec{\mathbf{d}}| = 10 \times \frac{1}{2} \times 10$

Work done = w = $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = |\vec{\mathbf{F}}| \cos\theta \times |\vec{\mathbf{d}}| = \mathbf{10 \times 5}$

Work done = w = $\vec{\mathbf{F}} \cdot \vec{\mathbf{d}} = |\vec{\mathbf{F}}| \cos\theta \times |\vec{\mathbf{d}}| = \mathbf{50\ Joule}$

Laws of vector operation:



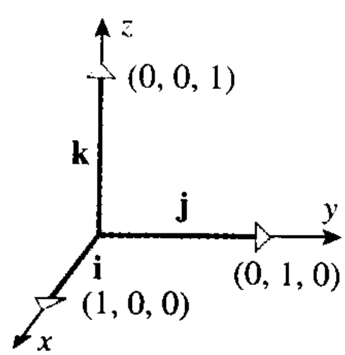


Figure: 36

We have, from figure 30

$$\vec{\mathbf{a}}\,.\,\vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\cos\theta$$

$$\therefore \hat{\mathbf{i}}\,.\,\hat{\mathbf{j}} = \left|\hat{\mathbf{i}}\right|\left|\hat{\mathbf{j}}\right|\cos\theta$$

$$\therefore \hat{\mathbf{i}}\,.\,\hat{\mathbf{j}} = \left|\hat{\mathbf{i}}\right|\left|\hat{\mathbf{j}}\right|\cos 90 = 1.1.0 = 0$$

Similarly,

$$\therefore \hat{\mathbf{k}}\,.\,\hat{\mathbf{i}} = \left|\hat{\mathbf{k}}\right|\left|\hat{\mathbf{i}}\right|\cos 90 = 1.1.0 = 0$$

Now,

$\therefore \hat{\mathbf{i}}\,.\,\hat{\mathbf{i}} = \left|\hat{\mathbf{i}}\right|\left|\hat{\mathbf{i}}\right|\cos 0 = 1.1.1 = 1$ [$\because$ The length or magnitude of unit vector is 1]

Similarly

$$\therefore \hat{\mathbf{j}}\,.\,\hat{\mathbf{j}} = \left|\hat{\mathbf{j}}\right|\left|\hat{\mathbf{j}}\right|\cos 0 = 1.1.1 = 1$$

$$\therefore \hat{\mathbf{k}}\,.\,\hat{\mathbf{k}} = \left|\hat{\mathbf{k}}\right|\left|\hat{\mathbf{k}}\right|\cos 0 = 1.1.1 = 1$$

Q # 04: **If** $\vec{\mathbf{A}} = \mathbf{A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}}$ and $\vec{\mathbf{B}} = \mathbf{B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}}$, Find $\vec{\mathbf{A}}\,.\,\vec{\mathbf{B}}$

Answer 04:

$$\vec{\mathbf{A}}\,.\,\vec{\mathbf{B}} = \mathbf{(A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k})\,.(B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k})}$$

$$= \mathbf{(A_x\,B_x + A_y\,B_y + A_z\,B_z)}$$

[$\because \mathbf{\hat{i}\,.\,\hat{i} = 1,\ \hat{j}\,.\,\hat{j} = 1,\ \hat{k}\,.\,\hat{k} = 1,\ \hat{i}\,.\,\hat{j} = 0\ ,\ \hat{i}\,.\,\hat{k} = 0\ ,\ \hat{j}\,.\,\hat{i} = 0\ ,\ \hat{j}\,.\,\hat{k} = 0\ ,\ \hat{k}\,.\,\hat{i} = 0\ ,\ \hat{k}\,.\,\hat{j} = 0}$]

Answer

Direction cosines:

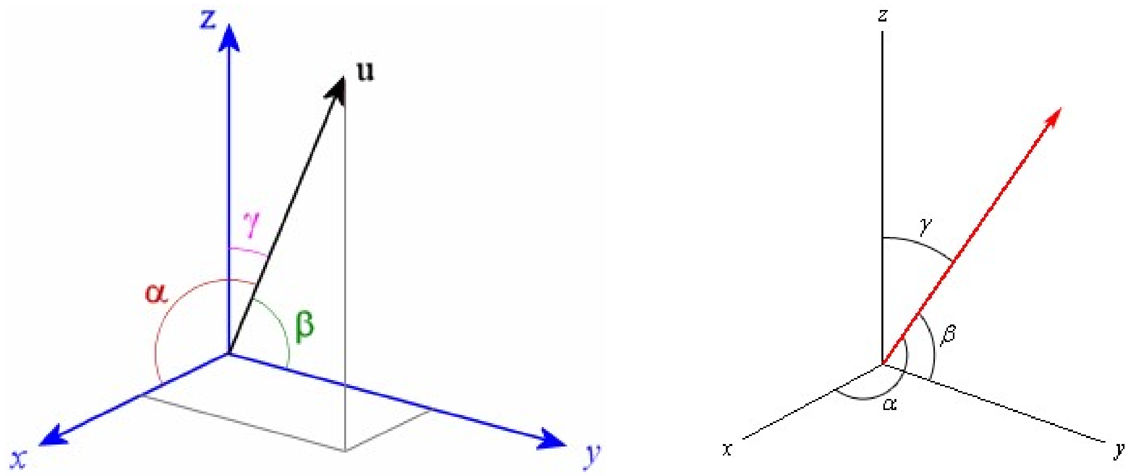


Figure # 37

Direction cosines are defined as

$\mathbf{l = \cos\alpha}$
$\mathbf{m = \cos\beta}$
$\mathbf{n = \cos\gamma}$

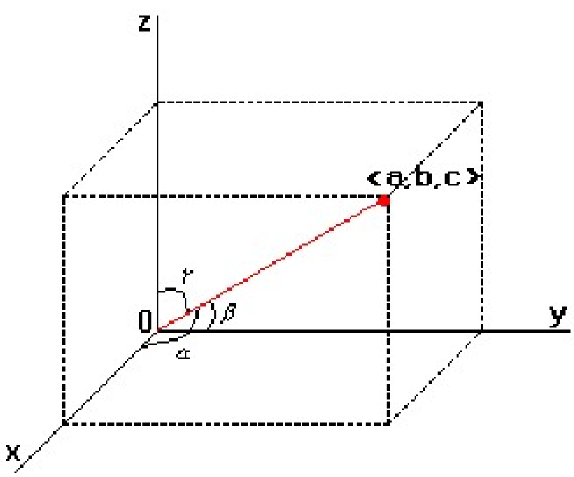


Figure # 38

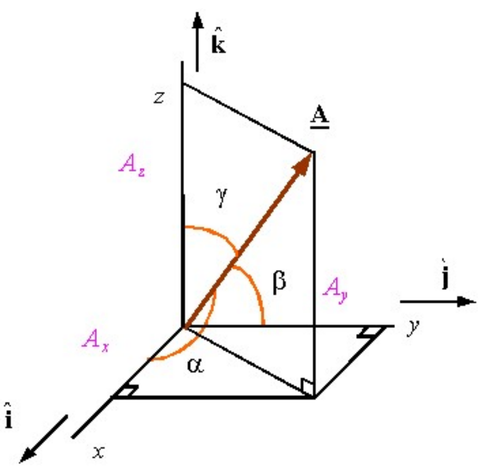


Figure # 39

Where the angles α, β and γ are the angles shown in the figure. As shown in the figure, the direction cosines represent the cosines of the angles made between the vector and the three coordinate directions.
The direction cosines can be calculated from the components of the vector and its magnitude through the relations

$$\mathbf{l = \cos\alpha = \frac{A_x}{A},\ m = \cos\beta = \frac{A_y}{A},\ n = \cos\gamma = \frac{A_z}{A}}$$ [*from figure 36*]

The three direction cosines are not independent and must satisfy the relation

$$\mathbf{l^2 + m^2 + n^2 = 1} \text{-----------------------------(i)}$$

This results form the fact that

$$\mathbf{l^2 + m^2 + n^2 = 1}$$

$$\Rightarrow \mathbf{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

$$\Rightarrow \left(\frac{\mathbf{A_x}}{\mathbf{A}}\right)^2 + \left(\frac{\mathbf{A_y}}{\mathbf{A}}\right)^2 + \left(\frac{\mathbf{A_z}}{\mathbf{A}}\right)^2 = \mathbf{1}$$

$$\Rightarrow \frac{\mathbf{A^2{}_x}}{\mathbf{A^2}} + \frac{\mathbf{A^2{}_y}}{\mathbf{A^2}} + \frac{\mathbf{A^2{}_z}}{\mathbf{A^2}} = \mathbf{1} \text{-----------------(ii)}$$

Since from figure 36

$$\vec{\mathbf{A}} = \mathbf{A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}}$$

$$\therefore \left|\vec{A}\right| = \sqrt{A^2{}_x + A^2{}_y + A^2{}_z}$$

A unit vector can be constructed along a vector using the direction cosines as its components along the *x*, *y*, and *z* directions. For example, the unit-vector $\hat{\mathbf{e}}$ along the vector $\vec{\mathbf{A}}$ is obtained from $\hat{\mathbf{e}} = \dfrac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|} = \dfrac{\mathbf{A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}}}{\left|\vec{\mathbf{A}}\right|}$

$$\Rightarrow \hat{\mathbf{e}} = \frac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|} = \frac{\mathbf{A_x\,\hat{i}}}{\left|\vec{\mathbf{A}}\right|} + \frac{\mathbf{A_y\,\hat{j}}}{\left|\vec{\mathbf{A}}\right|} + \frac{\mathbf{A_z\,\hat{k}}}{\left|\vec{\mathbf{A}}\right|}$$

$$\Rightarrow \hat{\mathbf{e}} = \frac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|} = \frac{\mathbf{A_x}}{\left|\vec{\mathbf{A}}\right|}\mathbf{\hat{i}} + \frac{\mathbf{A_y}}{\left|\vec{\mathbf{A}}\right|}\mathbf{\hat{j}} + \frac{\mathbf{A_z}}{\left|\vec{\mathbf{A}}\right|}\mathbf{\hat{k}}$$

$$\Rightarrow \hat{\mathbf{e}} = \frac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|} = \mathbf{\cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}} \text{-----------(iii)}$$

$$\Rightarrow \hat{\mathbf{e}} = \frac{\vec{\mathbf{A}}}{\left|\vec{\mathbf{A}}\right|} = \mathbf{l\,\hat{i} + m\,\hat{j} + n\,\hat{k}} \text{-------------------------(iv)}$$

Therefore, $\vec{\mathbf{A}} = \left|\vec{\mathbf{A}}\right| \times \hat{\mathbf{e}}$

$$\Rightarrow \vec{\mathbf{A}} = \left|\vec{\mathbf{A}}\right| \times \hat{\mathbf{e}}$$

$$\Rightarrow \vec{\mathbf{A}} = \left|\vec{\mathbf{A}}\right| \times (\mathbf{l}\,\hat{\mathbf{i}} + \mathbf{m}\,\hat{\mathbf{j}} + \mathbf{n}\,\hat{\mathbf{k}}) \qquad \text{[From (iv)]}$$

$$\Rightarrow \vec{\mathbf{A}} = \left|\vec{\mathbf{A}}\right| \times (\mathbf{cos}\,\alpha\,\hat{\mathbf{i}} + \mathbf{cos}\,\beta\,\hat{\mathbf{j}} + \mathbf{cos}\,\gamma\,\hat{\mathbf{k}}) \qquad \text{[From (iii)]}$$

Q # 05: How do you find the angle between a vector and the x-axis, y-axis, z-axis?

If a vector $\vec{OP} = \vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ makes an angle α with the x -axis, then

$$\cos\alpha = \frac{A_x}{\left|\vec{OP}\right|} = \frac{A_x}{\left|\vec{A}\right|} = \frac{A_x}{\sqrt{A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}}}$$

If a vector $\vec{OP} = \vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ makes an angle β with the y -axis, then

$$\cos\beta = \frac{A_y}{\left|\vec{OP}\right|} = \frac{A_y}{\left|\vec{A}\right|} = \frac{A_y}{\sqrt{A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}}}$$

If a vector $\vec{OP} = \vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ makes an angle γ with the z -axis, then

$$\cos\gamma = \frac{A_z}{\left|\vec{OP}\right|} = \frac{A_z}{\left|\vec{A}\right|} = \frac{A_z}{\sqrt{A_x^{\ 2} + A_y^{\ 2} + A_z^{\ 2}}}$$

As for an example $\vec{OP} = 2\,\hat{i} + 4\,\hat{j} + 6\,\hat{k}$ makes an angle α , β and γ with the x -axis, y-axis and z-axis respectively, then

$$\cos\alpha = \frac{2}{\left|\vec{OP}\right|} = \frac{2}{\sqrt{4+16+36}} = \frac{2}{\sqrt{56}}\,; \quad \alpha = \cos^{-1}(\frac{2}{\sqrt{56}})$$

$$\cos\beta = \frac{4}{\left|\vec{OP}\right|} = \frac{4}{\sqrt{4+16+36}} = \frac{4}{\sqrt{56}}\,; \quad \beta = \cos^{-1}(\frac{4}{\sqrt{56}})$$

$$\cos\gamma = \frac{6}{\left|\vec{OP}\right|} = \frac{6}{\sqrt{4+16+36}} = \frac{6}{\sqrt{56}}\,; \quad \gamma = \cos^{-1}(\frac{6}{\sqrt{56}})$$

Parallel Vectors:

When $\vec{A}$ **and** $\vec{B}$ are parallel to each other, their Dot Product is identical to the ordinary multiplication of their sizes, that is $\vec{A}.\vec{B} = AB$ since $\theta = 0^0$ and $\cos 0^0 = 1$.

Perpendicular Vectors:

When $\vec{A}$ and $\vec{B}$ are perpendicular to each other, their Dot Product is always Zero that is $\vec{A}.\vec{B} = 0$, since $\theta = 90^0$ and $\cos 90^0 = 0$

Q# 06: Determine whether $\vec{A} = 3\hat{i} + 5\hat{j} - 2\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} - 2\hat{k}$ are perpendicular

Answer:

Given, $\vec{A} = 3\hat{i} + 5\hat{j} - 2\hat{k}$ **and** $\vec{B} = 2\hat{i} - 2\hat{j} - 2\hat{k}$

$\therefore \vec{A}.\vec{B} = (3\hat{i} + 5\hat{j} - 2\hat{k}).(2\hat{i} - 2\hat{j} - 2\hat{k})$

$\therefore \vec{A}.\vec{B} = 3 \times 2 + 5 \times (-2) + (-2) \times (-2)$

$[\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0]$

$\therefore \vec{A}.\vec{B} = 6 - 10 + 4$

$\therefore \vec{A}.\vec{B} = 0$

Since $\vec{A}.\vec{B} = 0$ Then $\vec{A}$ and $\vec{B}$ are perpendicular to each other

Q# 07: Find the angle between $\vec{A} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{B} = 4\hat{i} + \hat{j} - 3\hat{k}$

Answer: **We have,**

$$\vec{A}.\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$\cos\theta = \frac{\vec{A}.\vec{B}}{|\vec{A}||\vec{B}|}$$

Given, $\vec{A} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{B} = 4\hat{i} + \hat{j} - 3\hat{k}$

$\therefore \vec{A}.\vec{B} = (2\hat{i} - 3\hat{j} + \hat{k}).(4\hat{i} + \hat{j} - 3\hat{k})$

$\therefore \vec{A}.\vec{B} = 2 \times 4 + (-3) \times 1 + 1 \times (-3)$

$[\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0]$

$\therefore \vec{A}.\vec{B} = 8 - 3 - 3$

$\therefore \vec{A}.\vec{B} = 8 - 6$

$\therefore \vec{A}.\vec{B} = 2$

Again, Given, $\vec{\mathbf{A}} = \mathbf{2\,\hat{i} - 3\,\hat{j} + \hat{k}}$ and $\vec{\mathbf{B}} = \mathbf{4\,\hat{i} + \hat{j} - 3\,\hat{k}}$

$$\therefore \left|\vec{\mathbf{A}}\right| = \sqrt{\mathbf{2^2 + (-3)^2 + 1^2}} = \sqrt{\mathbf{4+9+1}} = \sqrt{\mathbf{14}}$$

$$\textbf{and } \therefore \left|\vec{\mathbf{B}}\right| = \sqrt{\mathbf{4^2 + (1)^2 + (-3)^2}} = \sqrt{\mathbf{16+1+9}} = \sqrt{\mathbf{26}}$$

$$\cos\theta = \frac{\vec{\mathbf{A}}.\vec{\mathbf{B}}}{\left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|}$$

$$\cos\theta = \frac{\mathbf{2}}{\sqrt{\mathbf{14}} \times \sqrt{\mathbf{26}}}$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{2}}{\sqrt{\mathbf{14}} \times \sqrt{\mathbf{26}}}\right) \textbf{\textit{ Answer}}$$

Q# 08: A particle acted on by constant forces $\vec{\mathbf{F}}_1 = \mathbf{4\,\hat{i} + \hat{j} - 3\,\hat{k}}$ and $\vec{\mathbf{F}}_2 = \mathbf{3\,\hat{i} + \hat{j} - \hat{k}}$ (both measured in Newton), is displaced from the point $\mathbf{(1,2,3)}$ to the point $\mathbf{(5,4,1)}$ (measured in meters). Find the total work done by the forces.

Answer: Figure 30 shows the displacement of the particle and the forces acting on it. Although the forces are shown acting at the initial point A, they are assumed to act on the particle throughout the displacement from A to B. The resultant force is:

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 = \mathbf{4\,\hat{i} + \hat{j} - 3\,\hat{k} + 3\,\hat{i} + \hat{j} - \hat{k} = 7\,\hat{i} + 2\,\hat{j} - 4\,\hat{k}} \text{ ------------------------(i)}$$

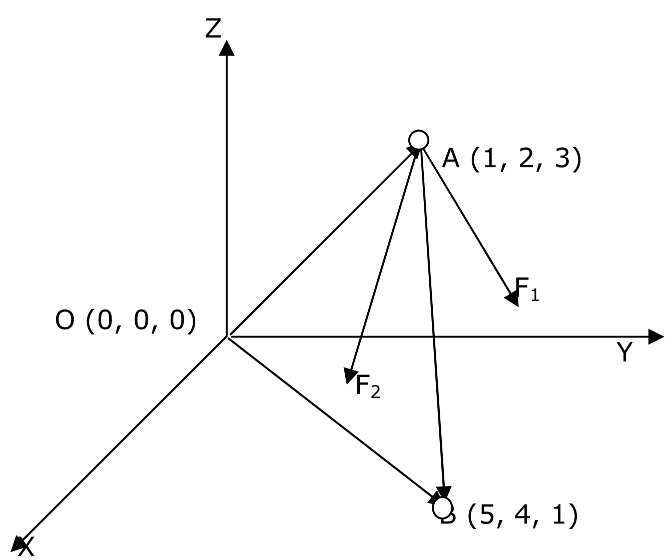


Figure 40

The displacement is the vector $\vec{d} = \vec{AB}$, **but,**

From ΔOAB,

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\vec{AB} = \vec{OB} - \vec{OA} = (5\hat{i} + 4\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\vec{d} = \vec{AB} = 4\hat{i} + 2\hat{j} - 2\hat{k} \text{ -------------------------------------(ii)}$$

The work done, W, is given by

$$\vec{F}.\vec{d} = (7\hat{i} + 2\hat{j} - 4\hat{k}).(4\hat{i} + 2\hat{j} - 2\hat{k}) = 7 \times 4 + 2 \times 2 + (-4) \times (-2) = 28 + 4 + 8 = 40 \text{ Joule}$$

Answer

Q# 09: A rope is attached to a 100-lb block (mass) on a ramp that is inclined at an angle of 30^0 with the ground (Figure no 38). How much force does the block exert against the ramp and how much force must be applied to the rope in a direction parallel to the ramp (slope) to prevent the block from sliding down the ramp? (Assume that the ramp is smooth, that is, exerts no frictional (ঘর্ষণ বল) forces)

Solution:

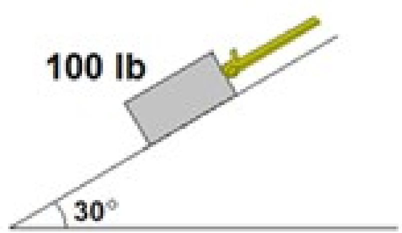


Figure 41

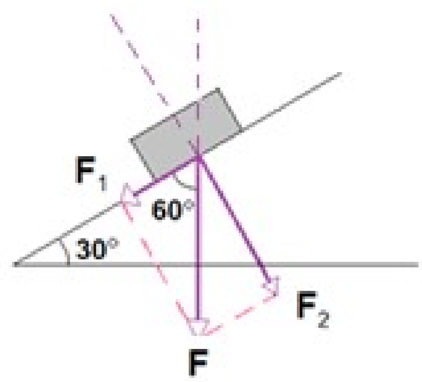


Figure 42

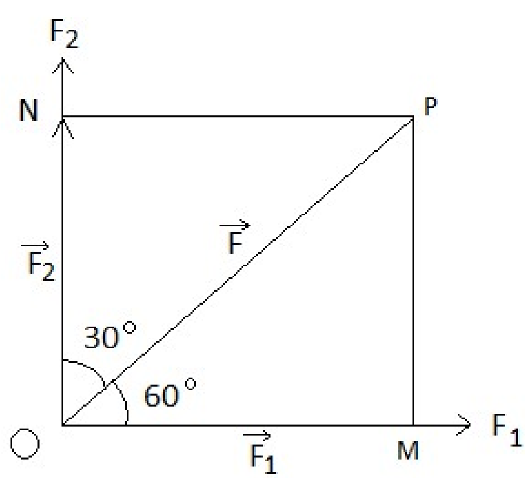


Figure 43

Let $\vec{\mathbf{F}}$ denote the downward force of gravity on the block. So $\left|\vec{\mathbf{F}}\right| = \mathbf{100}$ 1b and let

$\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ be the vector components of $\vec{\mathbf{F}}$ parallel and perpendicular to the ramp (Figure no 39).

From, ΔOMP,

$$\frac{\mathbf{OM}}{\mathbf{OP}} = \cos\theta$$

$$\mathbf{OM} = \mathbf{OP}\cos\theta$$

$$\left|\vec{F}_1\right| = \left|\vec{F}\right|\cos\theta \text{---(i)}$$

From, ΔOMP,

$$\frac{\mathbf{MP}}{\mathbf{OP}} = \sin\theta$$

$$\mathbf{MP} = \mathbf{OP}\sin\theta$$

$$\left|\vec{F}_2\right| = \left|\vec{F}\right|\sin\theta \text{---(ii)}$$

From (i),

$$\left|\vec{F}_1\right| = \left|\vec{F}\right|\cos\theta$$

$$\left|\vec{\mathbf{F}}_1\right| = \left|\vec{\mathbf{F}}\right|\cos 60^0 = 100 \times \frac{1}{2} = 50 \text{ 1b}$$

and from (ii),

$$\left|\vec{F}_2\right| = \left|\vec{F}\right|\sin\theta$$

$$\left|\vec{\mathbf{F}}_2\right| = \left|\vec{\mathbf{F}}\right|\sin 60^0 = 100 \times \frac{\sqrt{3}}{2} = 50\sqrt{3} \text{ 1b}$$

Thus the block exerts a force of approximately $\mathbf{50\sqrt{3}}$ -1b against the ramp, and it requires a force of 50-1b to prevent the block from sliding down the ramp.

Q# 10: A wagon is pulled horizontally by exerting a constant force of 10 1b on the handle at an angle of 60^0 with the horizontal. How much work is done in moving the wagon 50 ft?

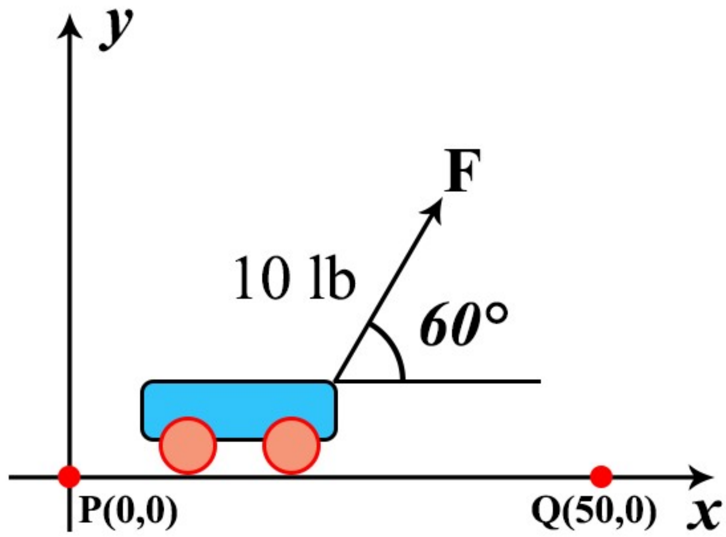


Figure 44

Answer: Introduce an xy-co-ordinate system so that the wagon moves from **P(0,0)** to **Q(50,0)** along the x-axis (Figure no 40). In the co-ordinate system $\overrightarrow{\mathbf{PQ}} = \mathbf{50}\,\hat{\mathbf{i}}$

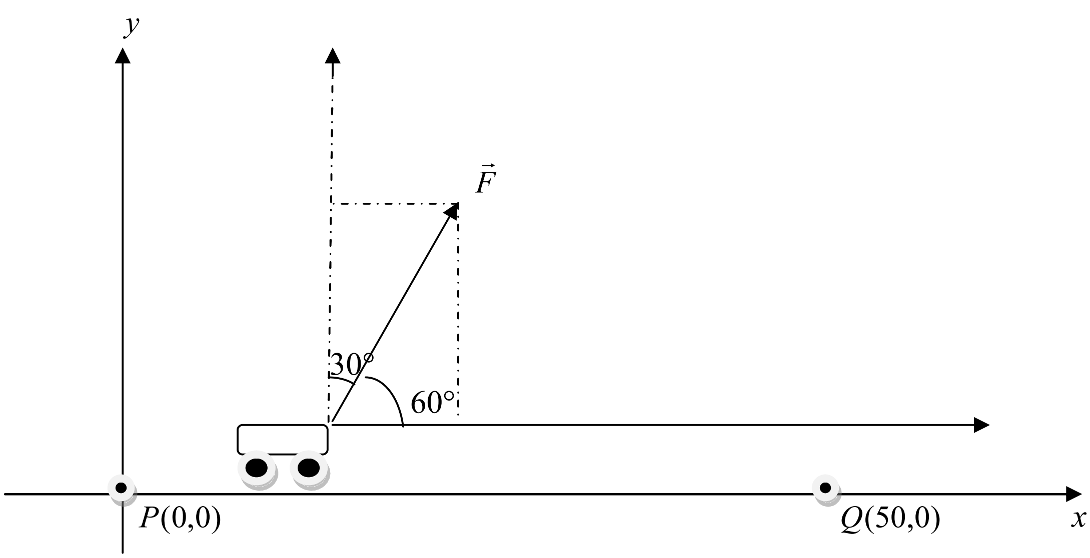


Figure 45

and $\overrightarrow{\mathbf{F}} = (\mathbf{10}\cos \mathbf{60^0})\,\hat{\mathbf{i}} + (\mathbf{10}\sin \mathbf{60^0})\,\hat{\mathbf{j}}$

$$\vec{F} = (10\times\frac{1}{2})\hat{i} + (10\times\frac{\sqrt{3}}{2})\hat{j}$$

$$\vec{F} = 5\hat{i} + 5\times\sqrt{3}\,\hat{j}$$

So, the work done is:

$$W = \vec{F}\,.\vec{PQ} = (5\hat{i} + 5\times\sqrt{3}\,\hat{j}).\,(50\,\hat{i})$$

$$W = \vec{F}\,.\vec{PQ} = (5\hat{i} + 5\times\sqrt{3}\,\hat{j}).\,(50\,\hat{i} + 0\,\hat{j})$$

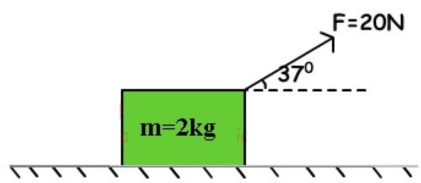
$W = \vec{F}\,.\vec{PQ} = \mathbf{250}$ 1b Answer

Or

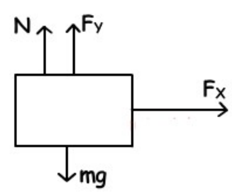
Component of Force $\vec{F}$ in the direction of X axis is, $\left|\vec{F}\right|\cos 60^0 = (10\times\frac{1}{2}) = 5$

Work done = force ×displacement = 5×50=250

Q-A box is pulled with 20N force. Mass of the box is 2kg and surface is frictionless. Find the acceleration of the box.



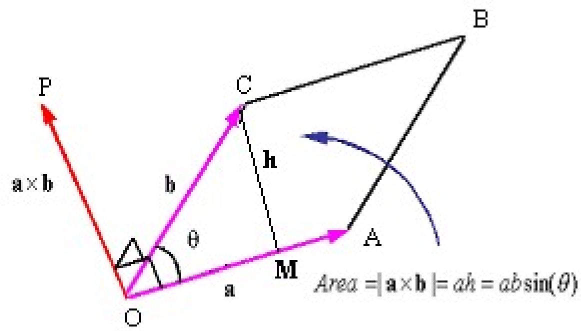
We show the forces acting on the box with following free body diagram.



X component of force gives acceleration to the box
F_x= Fcos37^o=20.0.8=16N
F_x= ma
16N = 2kga
a=8m/s²

The cross product:

The cross product of vectors $\vec{a}$ and $\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ and has a magnitude equal to the area of the parallelogram generated from $\vec{a}$ and $\vec{b}$. The direction of the cross product is given by the right-hand rule. The cross product is denoted by a "×" between the vectors



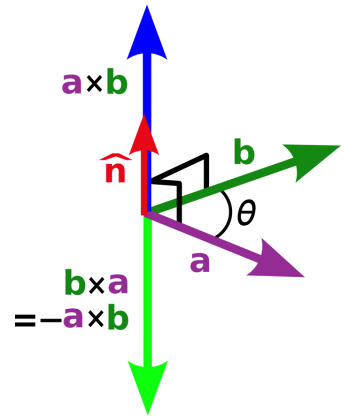


Figure # 46

Figure # 47

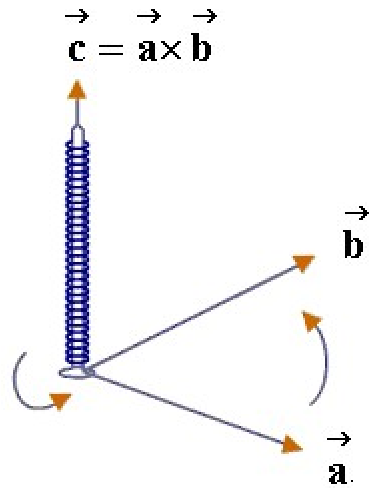


Figure # 48: *The direction of* $\vec{\mathbf{c}}$ *is that in which a right handed screw advances when turned from* $\vec{\mathbf{a}}$ *to* $\vec{\mathbf{b}}$

Area of Parallelogram OABC = base×height

Area of Parallelogram = $\mathbf{OA \times h}$ [from figure 41]

Area of Parallelogram = $\left|\vec{\mathbf{a}}\right| \times \mathbf{h}$ -------------------------------(i)

We have, From ΔOCM,

$$\frac{\text{CM}}{\text{OC}} = \sin\theta\,[\textit{Figure 43}]$$

$$\frac{\boldsymbol{h}}{\boldsymbol{OC}} = \mathbf{sin}\,\theta$$

$$\frac{\mathbf{h}}{\left|\vec{\mathbf{b}}\right|} = \mathbf{sin}\,\theta \;\; [\text{Figure 43}]$$

$\Rightarrow \mathbf{h} = \left|\vec{\mathbf{b}}\right| \sin\theta$

From (i),

$\therefore$ Area of Parallelogram $= \left|\vec{\mathbf{a}}\right| \mathbf{h} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right| \sin\theta$ $\qquad [\because \mathbf{h} = \left|\vec{\mathbf{b}}\right| \sin\theta]$

We have,

Any Vector = Length of this Vector × Unit Vector

$\therefore \vec{OP} =$ **the Length of this Vector** $\vec{OP} \times$ **Unit Vector of** $\vec{OP}$

$\therefore \vec{OP} = \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right| \sin\theta\, \hat{\eta}$ ------------------------------------(ii)

Here, $\left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right| \sin\theta$ is the magnitude (length) of the vector $\vec{\mathbf{OP}}$ or $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ and

$\hat{\eta}$ is the unit vector of $\vec{\mathbf{OP}}$ or $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$

where θ is the measure of the angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ $(0° \le \theta \le 180°)$ on the plane defined by the span of the vectors, and $\hat{\eta}$ is a unit vector perpendicular to both $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Order is important in the cross product. If the order of operations changes in a cross product the direction of the resulting vector is reversed. That is, $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = -\vec{\mathbf{b}} \times \vec{\mathbf{a}}$

[দুটি Vector $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ এর Cross Product করলে একটি নতুন Vector সৃষ্টি হবে । প্রদত্ত Vector ($\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$) দুটি যে সমতলে অবস্থিত নতুন Vector টি ঐ সমতলের উপর লম্ব হবে এবং নতুন Vector এর length হবে প্রদত্ত Vector ($\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$) কর্তৃক অংকিত সামন্তরিকের ক্ষেত্রফলের সমান। that is, $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right| \sin\theta\, \hat{\eta}$

এখানে নতুন Vector টির unit vector হচ্ছে $\hat{\eta}$ এবং $\left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right| \sin\theta$ হচ্ছে নতুন Vector এর length]

<u>Laws of vector operation:</u>

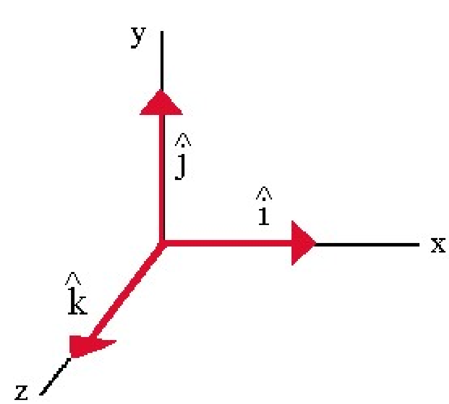


Figure # 49

We have,

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \sin\theta \, \hat{\eta}$$

From Figure 44,

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \sin\theta \, \hat{\eta} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}| \sin 90^0 \, \hat{\eta} = 1 \times 1 \times 1 \times \hat{\eta} = \hat{\eta}$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \text{ [say } \hat{\eta} = \hat{\mathbf{k}}] \text{ ----------------------(i)}$$

Similarly,

$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = |\hat{\mathbf{j}}||\hat{\mathbf{k}}| \sin\theta \, \hat{\mathbf{i}} = |\hat{\mathbf{j}}||\hat{\mathbf{k}}| \sin 90^0 \, \hat{\mathbf{i}} = 1 \times 1 \times 1 \times \hat{\mathbf{i}} = \hat{\mathbf{i}} \quad \text{[here } \hat{\eta} = \hat{\mathbf{i}}] \text{ -------------(ii)}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = |\hat{\mathbf{k}}||\hat{\mathbf{i}}| \sin\theta \, \hat{\mathbf{j}} = |\hat{\mathbf{k}}||\hat{\mathbf{i}}| \sin 90^0 \, \hat{\mathbf{j}} = 1 \times 1 \times 1 \times \hat{\mathbf{j}} = \hat{\mathbf{j}} \quad \text{[here } \hat{\eta} = \hat{\mathbf{j}}] \text{ -----------(iii)}$$

Again,

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = |\hat{\mathbf{j}}||\hat{\mathbf{i}}| \sin(-\theta) \, \hat{\eta} = -|\hat{\mathbf{j}}||\hat{\mathbf{i}}| \sin\theta \, \hat{\eta} = -|\hat{\mathbf{j}}||\hat{\mathbf{i}}| \sin 90^0 \, \hat{\eta} = -1 \times 1 \times 1 \times \hat{\eta} = -\hat{\eta}$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}} \text{ [say } \hat{\eta} = \hat{\mathbf{k}}] \text{ ---------------------(iv)}$$

Similarly,

$$\hat{\mathbf{i}} \times \hat{\mathbf{k}} = |\hat{\mathbf{i}}||\hat{\mathbf{k}}| \sin(-\theta) \, \hat{\eta} = -|\hat{\mathbf{i}}||\hat{\mathbf{k}}| \sin\theta \, \hat{\eta} = -|\hat{\mathbf{i}}||\hat{\mathbf{k}}| \sin 90^0 \, \hat{\eta} = -1 \times 1 \times 1 \times \hat{\eta} = -\hat{\eta}$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} \text{ [say } \hat{\eta} = \hat{\mathbf{j}}] \text{ ----------------------(v)}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = |\hat{\mathbf{k}}||\hat{\mathbf{j}}| \sin(-\theta) \, \hat{\eta} = -|\hat{\mathbf{k}}||\hat{\mathbf{j}}| \sin\theta \, \hat{\eta} = -|\hat{\mathbf{k}}||\hat{\mathbf{j}}| \sin 90^0 \, \hat{\eta} = -1 \times 1 \times 1 \times \hat{\eta} = -\hat{\eta}$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}} \text{ [say } \hat{\eta} = \hat{\mathbf{i}}] \text{ ---------------------(vi)}$$

Again,

$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}| \sin\theta \, \hat{\eta} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}| \sin 0^0 \, \hat{\eta} = 1 \times 1 \times 0 \times \hat{\eta} = 0 \text{ --------------------------(vii)}$$

$$\hat{\mathbf{j}} \times \hat{\mathbf{j}} = |\hat{\mathbf{j}}||\hat{\mathbf{j}}| \sin\theta \, \hat{\eta} = |\hat{\mathbf{j}}||\hat{\mathbf{j}}| \sin 0^0 \, \hat{\eta} = 1 \times 1 \times 0 \times \hat{\eta} = 0 \text{ -------------------------(viii)}$$

$$\hat{\mathbf{k}} \times \hat{\mathbf{k}} = |\hat{\mathbf{k}}||\hat{\mathbf{k}}| \sin\theta \, \hat{\eta} = |\hat{\mathbf{k}}||\hat{\mathbf{k}}| \sin 0^0 \, \hat{\eta} = 1 \times 1 \times 0 \times \hat{\eta} = 0 \text{ -----------------------(ix)}$$

Scalar triple product: $\vec{A}.(\vec{B}\times\vec{C})$ or $\vec{B}.(\vec{C}\times\vec{A})$ or $\vec{C}.(\vec{A}\times\vec{B})$ are known as a scalar triple product. It is symbolically denoted by $[ABC]$ or $[BCA]$ or $[CAB]$

We know, $(\vec{A}\times\vec{B}) = -(\vec{B}\times\vec{A})$

Hence, $\vec{A}.(\vec{B}\times\vec{C}) = -\vec{A}.(\vec{C}\times\vec{B})$

That is, $[ABC] = -[ACB]$

Q # 11: $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, **If** $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and Find $\vec{A}\times\vec{B}$

Answer: :

$$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) \times (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= A_xB_x(\hat{i}\times\hat{i}) + A_xB_y(\hat{i}\times\hat{j}) + A_xB_z(\hat{i}\times\hat{k}) + A_yB_x(\hat{j}\times\hat{i}) + A_yB_y(\hat{j}\times\hat{j}) + A_yB_z(\hat{j}\times\hat{k})$$
$$+ A_zB_x(\hat{k}\times\hat{i}) + A_zB_y(\hat{k}\times\hat{j}) + A_zB_z(\hat{k}\times\hat{k})$$

$$= A_xB_x\times 0 + A_xB_y(\hat{k}) + A_xB_z(-\hat{j}) + A_yB_x(-\hat{k}) + A_yB_y\times 0 + A_yB_z(\hat{i})$$
$$+ A_zB_x(\hat{j}) + A_zB_y(-\hat{i}) + A_zB_z\times 0$$

$$= A_xB_y\hat{k} - A_xB_z\hat{j} - A_yB_x\hat{k} + A_yB_z\hat{i} + A_zB_x\hat{j} - A_zB_y\hat{i}$$

$$= A_xB_y\hat{k} - A_xB_z\hat{j} - A_yB_x\hat{k} + A_yB_z\hat{i} + A_zB_x\hat{j} - A_zB_y\hat{i}$$

$$= A_yB_z\hat{i} - A_zB_y\hat{i} - A_xB_z\hat{j} + A_zB_x\hat{j} + A_xB_y\hat{k} - A_yB_x\hat{k}$$

$$\vec{A}\times\vec{B} = \hat{i}(A_yB_z - A_zB_y) - \hat{j}(A_xB_z - A_zB_x) + \hat{k}(A_xB_y - A_yB_x) \text{----------------------(i)}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \text{--(ii)}$$

Q # 12: Find a unit vector perpendicular to the vectors $\vec{a} = 3\hat{i} + \hat{j}$ and $\vec{b} = -\hat{i} + 2\hat{j} + 2\hat{k}$

Answer: A vector perpendicular to $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 0 \\ -1 & 2 & 2 \end{vmatrix} = \hat{i}(2-0) - \hat{j}(6-0) + \hat{k}(6+1) = 2\hat{i} - 6\hat{j} + 7\hat{k}$$

A unit vector, perpendicular to $\vec{a}$ and $\vec{b}$, in this direction is obtained by simply dividing $\vec{a}\times\vec{b}$ by its magnitude. Thus

$\hat{e} = \dfrac{2\hat{i} - 6\hat{j} + 7\hat{k}}{\sqrt{2^2 + (-6)^2 + 7^2}} = \dfrac{2\hat{i} - 6\hat{j} + 7\hat{k}}{\sqrt{89}}$ **is the required vector.**

$\therefore$ Magnitude of Unit vector of $\hat{e}$

$$= \left|\hat{e}\right| = \sqrt{\left(\frac{2}{\sqrt{89}}\right)^2 + \left(\frac{-6}{\sqrt{89}}\right)^2 + \left(\frac{7}{\sqrt{89}}\right)^2} = \sqrt{\frac{4}{89} + \frac{36}{89} + \frac{49}{89}} = \sqrt{\frac{89}{89}} = 1$$

Home Task:

Find a unit vector parallel to the resultant of vectors, $\vec{A} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{B} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\hat{e} = \frac{A + B}{|A + B|}$$

Q#13: Show that $\vec{\mathbf{A}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, $\vec{\mathbf{B}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\vec{\mathbf{C}} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ are coplanar.

Answer:

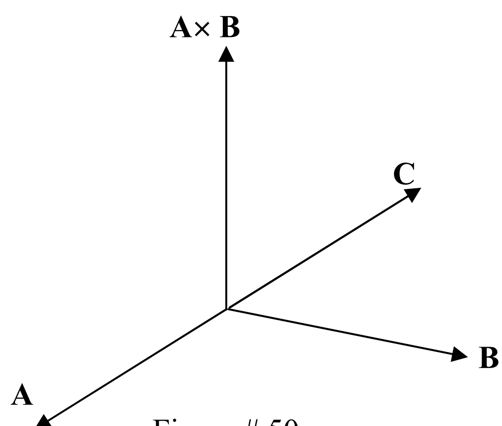


Figure # 50

If **A** is a third vector perpendicular to **(B×C),** then **A, B** and **C** are coplanar and **A. (B×C) =0**

Therefore, three vectors **A, B, C** are coplanar if **A. (B×C) = 0**

$$\mathbf{B\times C} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} == \hat{\mathbf{i}}(1-2) - \hat{\mathbf{j}}(-2-6) + \hat{\mathbf{k}}(2+3) = -\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

$\mathbf{A.\ (B\times C) =} (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}).(-\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$ = -1+16-15=0

Therefore A, B, C are coplanar.

Q#14: On which condition of λ the vectors $\hat{i}+2\hat{j}+3\hat{k},\ \lambda\hat{i}+4\hat{j}+7\hat{k},\ -3\hat{i}-2\hat{j}-5\hat{k}$ are collinear

Solution

We know that if the points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be collinear then

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ \lambda & 4 & 7 \\ -3 & -2 & -5 \end{vmatrix} = 0$$

$$\Rightarrow 1[4\times(-5) - 7\times(-2)] - 2[\lambda(-5) - (-3)\times 7] + 3[\lambda(-2) - 4(-3)] = 0$$

$$\Rightarrow 1[-20+14] - 2[-5\lambda + 21] + 3[-2\lambda + 12] = 0$$

$$\Rightarrow -6 + 10\lambda - 42 - 6\lambda + 36 = 0$$

$$\Rightarrow 4\lambda - 48 + 36 = 0$$

$$\Rightarrow 4\lambda - 12 = 0$$

$$\Rightarrow 4\lambda = 12$$

$$\Rightarrow \lambda = 3$$

Q# 15: **Figure no 46** shows a force $\vec{\mathbf{F}}$ of 100 N applied in the positive z-direction at the point **Q(1,1,1)** of a cube whose sides have a length of 1 m. assuming that the cube is free to rotate about the point **P(0,0,0)** (the origin), find the scalar moment of the force about **P** and describe the direction of rotation.

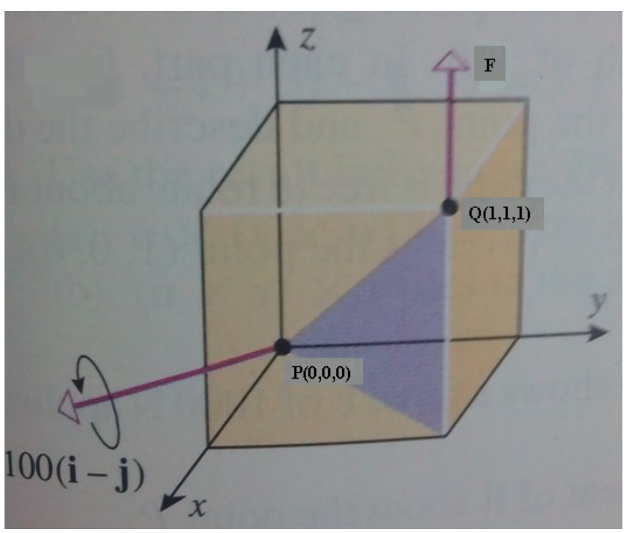


Figure 51

Answer: The force vector $\vec{\mathbf{F}} = \mathbf{0}.\hat{\mathbf{i}} + \mathbf{0}.\hat{\mathbf{j}} + \mathbf{100}\hat{\mathbf{k}}$ and the vector from $\mathbf{P}$ to $\mathbf{Q}$ is $\vec{\mathbf{PQ}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, so the vector moment of $\vec{\mathbf{F}}$ about $\mathbf{P}$ is

$$\vec{\mathbf{PQ}} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{100} \end{vmatrix} = \hat{\mathbf{i}}(\mathbf{100}-\mathbf{0}) - \hat{\mathbf{j}}(\mathbf{100}-\mathbf{0}) + \hat{\mathbf{k}}(\mathbf{0}-\mathbf{0})$$

$$= \mathbf{100}\,\hat{\mathbf{i}} - \mathbf{100}\,\hat{\mathbf{j}}$$

Thus the scalar moment of $\vec{\mathbf{F}}$ about $\mathbf{P}$ is

$$\left|\vec{\mathbf{PQ}} \times \vec{\mathbf{F}}\right| = \sqrt{(\mathbf{100})^2 + (-\mathbf{100})^2} = \sqrt{\mathbf{10000}+\mathbf{10000}} = \sqrt{\mathbf{20000}} = \sqrt{\mathbf{2}\times(\mathbf{100})^2} = \mathbf{100}\sqrt{\mathbf{2}}\ \text{N.m}$$

and the direction of rotation is counterclockwise looking along the vector $\mathbf{100}\,\hat{\mathbf{i}} - \mathbf{100}\,\hat{\mathbf{j}}$ $= \mathbf{100}(\hat{\mathbf{i}} - \hat{\mathbf{j}})$ towards its initial point **(Figure no 46)**

Q-16: If $\vec{a}.\vec{b} = \sqrt{3}$ and $\vec{a}\times\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$

Answer: We have, $\vec{\mathbf{a}}.\vec{\mathbf{b}} = \left|\vec{\mathbf{a}}\right|\left|\vec{\mathbf{b}}\right|\cos\theta$

Given, $\vec{a}.\vec{b} = \sqrt{3}$

$\therefore \left|\vec{a}\right|\left|\vec{b}\right|\cos\theta = \sqrt{3}$ ------------------------------------(i)

Again, we have, $\vec{a}\times\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\,\hat{\eta}$

Given, $\vec{a}\times\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$

$\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\,\hat{\eta} = \hat{i} + 2\hat{j} + 2\hat{k}$

$\hat{\eta}\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta = \hat{i} + 2\hat{j} + 2\hat{k}$ ---------------------------------------(ii)

We have, $\vec{a}\times\vec{b} = \left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\,\hat{\eta}$

Again, we can write, $\hat{\eta} = \dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta} = \dfrac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$ **---------------------------------------(iii)**

Given, $\vec{a}\times\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$

$$\therefore \left|\vec{a}\times\vec{b}\right| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$$

From (iii),

$$\hat{\eta} = \frac{\vec{a}\times\vec{b}}{\left|\vec{a}\times\vec{b}\right|}$$

$$\hat{\eta} = \frac{\hat{i}+2\hat{j}+2\hat{k}}{3}$$

Putting the value of $\hat{\eta}$ **in (ii), we get,**

$$\hat{\eta}\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta = \hat{i}+2\hat{j}+2\hat{k}$$

$$\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}\times\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta = \hat{i}+2\hat{j}+2\hat{k}$$

$$\frac{1}{3}\times\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta = 1$$

$$\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta = 3$$ **-----------------------------------(iv)**

(iv) $\div$(i)

$$\frac{\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta}{\left|\vec{a}\right|\left|\vec{b}\right|\cos\theta} = \frac{3}{\sqrt{3}}$$

$$\frac{\sin\theta}{\cos\theta} = \sqrt{3}$$

$$\tan\theta = \sqrt{3}$$

$$\tan\theta = \tan 60^0$$

$\therefore \theta = 60^0$ **Answer**

Q-17: Find all vectors $\vec{v}$ such that $(\hat{i}+2\hat{j}+\hat{k})\times\vec{v} = 3\hat{i}+\hat{j}-5\hat{k}$

Answer:

Given,

$$(\hat{i}+2\hat{j}+\hat{k})\times\vec{v} = 3\hat{i}+\hat{j}-5\hat{k} \text{---------------(i)}$$

Let $\vec{v} = x\hat{i}+y\hat{j}+z\hat{k}$

$$(\hat{i}+2\hat{j}+\hat{k})\times\vec{v}=(\hat{i}+2\hat{j}+\hat{k})\times x\hat{i}+y\hat{j}+z\hat{k}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 1\\ x & y & z\end{vmatrix}$$

$$\therefore (\hat{i}+2\hat{j}+\hat{k})\times\vec{v}=\hat{i}(2z-y)-\hat{j}(z-x)+\hat{k}(y-2x) \text{ ----------------------}\mathbf{(ii)}$$

Given,

$(\hat{i}+2\hat{j}+\hat{k})\times\vec{v}=3\hat{i}+\hat{j}-5\hat{k}$

From (ii), We can write,

$\therefore (\hat{i}+2\hat{j}+\hat{k})\times\vec{v}=\hat{i}(2z-y)-\hat{j}(z-x)+\hat{k}(y-2x)=3\hat{i}+\hat{j}-5\hat{k}$

$\Rightarrow \hat{i}(2z-y)-\hat{j}(z-x)+\hat{k}(y-2x)=3\hat{i}+\hat{j}-5\hat{k}$

Equating the coefficient of $\hat{i}$, $\hat{j}$, and $\hat{k}$ on both sides

$\therefore 2z-y=3$

$-(z-x)=1$

$y-2x=-5$

That is,

$$\begin{aligned}&\therefore 2z-y=3\\ \Rightarrow\quad & x-z=1\\ & y-2x=-5\end{aligned}$$

$$\begin{aligned}& x-z=1\\ \Rightarrow\quad & y-2x=-5\\ & 2z-y=3\end{aligned}$$

$$\begin{aligned}& x+0.y-z=1\\ \Rightarrow\quad & -2x+y+0.z=-5 \text{ ---(iii)}\\ & 0.x-y+2z=3\end{aligned}$$

We have,

$\mathbf{L_i \to -a_{i1}L_1 + a_{11}L_i}$

Here, $a_{11}=1,\ a_{12}=0,\ a_{13}=-1,\ a_{21}=-2,\ a_{22}=1,\ a_{23}=0,\ a_{31}=0,\ a_{32}=-1,\ a_{33}=2$

1st time:

$\mathbf{i=2,\ L_2 \to -a_{21}L_1 + a_{11}L_2}$

$=\ -(-2)(x+0.y-z-1)+1(-2x+y+0.z+5)$

$=2x-2z-2-2x+y+5$

$=y-2z+3$

$\mathbf{i = 3,\ L_3 \to -a_{31}L_1 + a_{11}L_3}$

$= \ -0(x+0.y-z-1)+1(0.x-y+2z-3)$

$= -y+2z-3$

Thus we obtain the following new system

$$x+0.y-z=1$$
$$y-2z=-3$$
$$-y+2z=3$$

2nd time:

$x\ +\ 0.y-z=1$

$$y-2z=-3 ------------- L_1$$
$$-y+2z=3 ------------- L_2$$

$$\begin{array}{l} \overset{a_{11}}{\uparrow} \quad \overset{a_{12}}{\uparrow} \\ 1\,y-2\,z=-3 \\ \overset{a_{21}}{\uparrow} \quad \overset{a_{22}}{\uparrow} \\ -1\,y+2\,z=3 \end{array}$$

We have,

$\mathbf{L_i \to -a_{i1}L_1 + a_{11}L_i}$

Here, $a_{11}=1,\ a_{12}=-2,\ a_{21}=-1,\ a_{22}=2$

$\mathbf{i = 2,\ L_2 \to -a_{21}L_1 + a_{11}L_2}$

$= \ -(-1)(y-2z+3)+1(-y+2z-3)$

$= y-2z+3-y+2z-3$

$= 0$

Thus we obtain the following new system

$$x+0.y-z=1$$
$$y-2z=-3$$
$$0=0$$

Thus we obtain the following new system

$$x+0.y-z=1$$
$$y-2z=-3$$

In echelon form, there are only two equations in three unknowns, then the system has a non-zero solution and in particular (3-2)=1 free variable, which is z. We obtain more than one solution of the system; hence we will get infinite number of vectors $\vec{v}$

Scalar Fields:

A scalar field is a map over some space of scalar values. That is, it is a map of values with no direction.

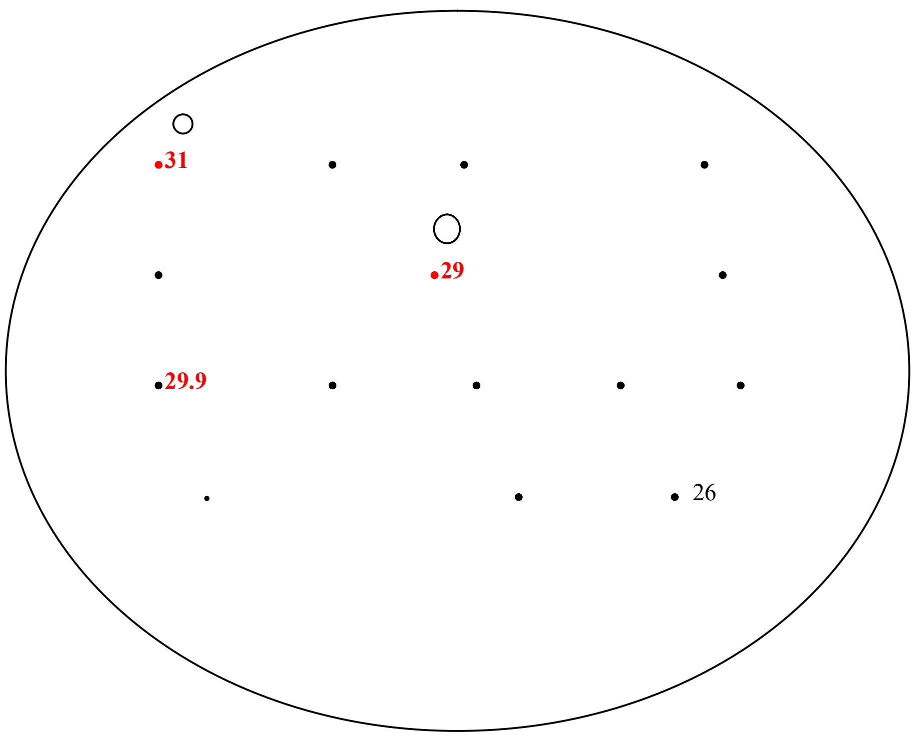


Figure 52

Examples:

1. **A simple example of a scalar field is a map of the temperature distribution in a room. The function that gives the temperature of any point in the room you are sitting is a scalar field**

- **Some parts of it, maybe near the door or windows, will probably be cooler, while other parts, maybe near a heater, will be warmer**. And in between these regions of course, there must be a continuous smooth change in temperature.
- This quantity "temperature", let's call it T, therefore, has various different values throughout that three-dimensional space that you're sitting in. Let's describe the position by the three Cartesian coordinates x, y and z.
- So at any given position $(\mathbf{x},\mathbf{y},\mathbf{z})$ the temperature T has a particular value, and if we change that position then T will probably change too. In other words T is a function of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ and we can write $\mathbf{T(x,y,z)}$. For example: $\mathbf{T(x,y,z)}$ can be used to represent the temperature at the point $(\mathbf{x},\mathbf{y},\mathbf{z})$
- This means that T is a **scalar field**.

As for example:

$$\mathbf{T(x,y,z) = x^2 + yz}$$
$$\mathbf{T(2,5,6) = 2^2 + 5\times 6 = 34^0}$$
$$\mathbf{T(4,2,8) = 4^2 + 2\times 8 = 32^0}$$
$$\mathbf{T(5,4,2) = 5^2 + 4\times 2 = 33^0}$$

....

....

- The temperature at that position just has a value, $\mathbf{34^0}$ degrees say, there is only one piece of information. There is no direction associated with that temperature.

2. To indicate the temperature distribution throughout space, or the air pressure
3. The temperature of a swimming pool is a scalar field: to each point we associate a scalar value of temperature.
4. In this course the most important example is the electromagnetic potential field.
5. A scalar valued function is a function that takes one or more values, but returns a single value. $\mathbf{f(x,y,z) = x^2 + 2yz^5}$ is an example of a scalar valued function.

তড়িৎ চুম্বকক্ষেত্র বিভব (Electromagnetic Potential Field):কোন একক ধনচার্জকে অসীম থেকে তড়িৎ ক্ষেত্রের কোন বিন্দুতে আনতে যে পরিমান কাজ করতে হয় তাকে ঐ ক্ষেত্রের ঐ বিন্দুর তড়িৎ ক্ষেত্র বিভব বলে। কাজ স্কেলার রাশি।

তড়িৎ চুম্বকক্ষেত্র (Electromagnetic Field):কোন একক ধনচার্জকে তড়িৎ ক্ষেত্রের কোন বিন্দুতে স্থাপন করলে সে যে বল অনুভব করে তাকে তড়িৎ চুম্বকক্ষেত্র বলে। যেমন একটি ধনচার্জের কাছে আর একটি ধন চার্জ রাখলে সে বিকর্ষন বল অনুভব করবে। তখন তার ডিরেকশন হবে বহি:মুখি। একটি ঋণচার্জের কাছে আর একটি ধন চার্জ রাখলে সে আকর্ষন বল অনুভব করবে। তখন তার ডিরেকশন হবে অন্ত:মুখি।

Newton’s Gravitational Field: একক ভরের কোন বস্তুকে পৃথিবী যে বলে আকর্ষন করে ।

সমুদ্রের পৃষ্ঠে, পাহাড়ের পাদদেশে, পাহাড়ের উপরে বিভিন্ন জায়গায় বাতাসের চাপ বিভিন্ন।

Vector space: Basically, a vector space is the set of all vectors that can be created by Linear combinations of a given set of vectors. If you take a vector and multiply it by any real number, and take another vector and multiply it by any real number, and then add them together, this new vector is a linear combination of the first two. So a vector space is all the possible linear combinations of the set of basis vectors. The basis vectors are said to "span" the vector space. You can find different sets of basis vectors that span the same vector space.

Vector fields:
A vector field can be considered a map of vectors over some space. . For example if one were to show wind vectors on a weather map; that would be a vector field. The electric field surrounding a charge is a vector field. A vector field in the plane, for instance, can be visualized as a collection of arrows with a given magnitude and direction each attached to a point in the plane

Examples:

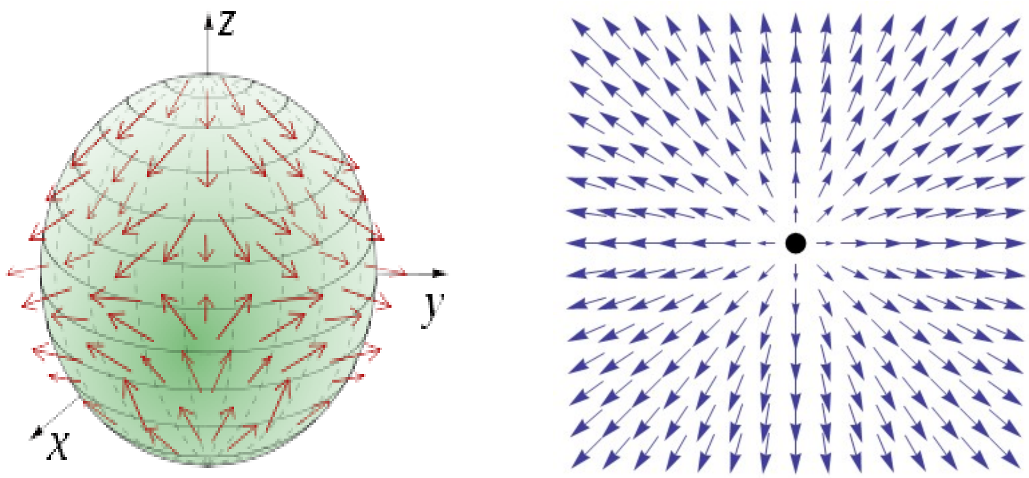


Figure # 53

1. Now imagine the air moving around in that room you're in. In some parts it will be moving quickly, above the heater maybe, or near an open window, or near your nose, while in other parts it will be moving slowly.
    - The quantity describing that air movement is "velocity", let's call it **v**. That quantity **v** also has a different value at different positions, so we can write $\mathbf{v(x,y,z)}$ and this quantity too is a field.
    - At any position $\mathbf{(x,y,z)}$ the air at that point is moving in a particular **direction**, with a particular **speed**.
2. **The water flow in the same pool is a vector field**
3. The speed and direction of a moving fluid throughout space, or the strength and direction of some force, such as the magnetic or gravitational force, as it changes from point to point.
4. Examples are movement of a fluid, or the force generated by a magnetic of gravitational field, or atmospheric models, where both the strength (speed) and the direction of winds are recorded.

5. Wind vectors on a weather map; that would be a vector field. The electric field surrounding a charge is a vector field
6. Examples of vector fields include the electromagnetic field and the Newtonian gravitational field.
7. Three vector fields are shown below. Which represents the electric field eminating from a positive point charge in the middle? (Note that vectors of similar magnitude are colored similarly in these plots)

কোন রুমে প্রতিটি ছাত্রের নাক যদি থ্রি ডাইমেনশনাল অক্ষের সাপেক্ষে এক একটি স্থানাংক (x,y,z) হয় তাহলে **<u>ছাত্রদের শ্বাস প্রশ্বাসের গতি এক একটি ভেক্টর হবে</u>**। এই সবগুলি ভেক্টর নিয়েই একটি ভেক্টর ফিল্ড হবে।

অতএব তা যদি ম্যথম্যটিক্যালি প্রকাশ করি তাহলে লিখব: $\vec{v}(x,y,z)$

Let $\vec{v}(x,y,z) = x^2y\,\hat{i} + yz\,\hat{j} + \hat{k}$

Visualize Scalar Field on a Surface: Surface is colored using the value of a scalar function defined on each vertex.

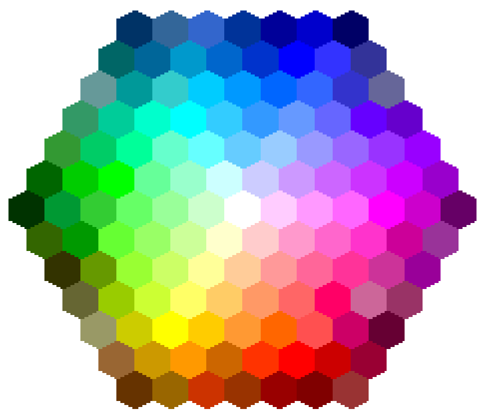


Figure # 54

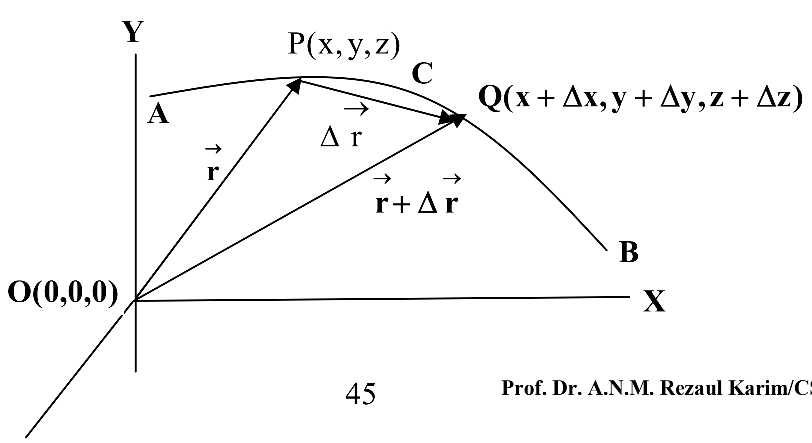
Visualize Vector Field on a Surface:

1. **<u>Imagine what happens when you throw a stone into the water</u>**
2. **<u>Imagine what happens when you throw a stone to the honeycomb</u>**

<u>Differentiation of Vectors:</u>

In many practical problems, we often deal with vectors that change with time, e.g.

Velocity, acceleration, etc.



Z

Figure # 55

We consider a position vector $\vec{OP} = \vec{r}$ **,** which is drawn from O to P then $\vec{OP}$ moves from P to Q . Then $\Delta \vec{r}$ **is** small increment from P to Q .So, $\vec{OQ} = \vec{r} + \Delta \vec{r}$ **is a new vector drawn** from O to Q. Assume that $\vec{r}$ is a vector function of x, y, z and depends on a scalar variable t.

From figure,

$$\vec{OP} = \vec{r}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k} \text{ -------------------------------(i)}$$

$$\therefore \Delta \vec{r} = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k} \text{ ---------------------------------------(ii)}$$

$$\vec{OQ} = (x + \Delta x)\hat{i} + (y + \Delta y)\hat{j} + (z + \Delta z)\hat{k}$$

$$\vec{OQ} = (x\hat{i} + y\hat{j} + z\hat{k}) + (\Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k})$$

$$\vec{OQ} = \vec{r} + \Delta \vec{r} \text{ --(iii)}$$

From

Δ **OPQ,**

$$\vec{OP} + \vec{PQ} = \vec{OQ}$$

$$\vec{r} + \Delta \vec{r} = \vec{r} + \Delta \vec{r}$$

$$\Delta \vec{r} = \vec{r} + \Delta \vec{r} - \vec{r} \text{ ---(iv)}$$

Δt সময়ে ভেক্টরের অবস্থানের পরিবর্তন $\Delta \vec{r}$
$\therefore$ একক .... $\dfrac{\Delta \vec{r}}{\Delta t}$

Then $\dfrac{\Delta \vec{r}}{\Delta t}$ is the average rate of change of $\vec{r}$ with respect to time t.

i.e. $\dfrac{\Delta \vec{r}}{\Delta t} = \dfrac{(\vec{r} + \Delta \vec{r}) - \vec{r}}{\Delta t}$ -------------------------------(v)

When $Q \to P$ then PQ will be tangent

So, then $\Delta \mathbf{t} \to \mathbf{0}$, then $\Delta \vec{\mathbf{r}} \to \mathbf{d}\vec{\mathbf{r}}$

[Note: $\mathbf{d}\vec{\mathbf{r}}$ ***is a tangent vector to any point to the curve]***

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{r}}}{\Delta \mathbf{t}} = \lim_{\Delta t \to 0} \frac{(\vec{\mathbf{r}} + \Delta \vec{\mathbf{r}}) - \vec{\mathbf{r}}}{\Delta \mathbf{t}}$$

$$\vec{\mathbf{v}} = \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{r}}}{\Delta \mathbf{t}} = \lim_{\Delta t \to 0} \frac{(\vec{\mathbf{r}} + \Delta \vec{\mathbf{r}}) - \vec{\mathbf{r}}}{\Delta \mathbf{t}} \text{ -----------------(vi)}$$

$\dfrac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}$ represents the velocity $\vec{\mathbf{v}}$

This is the derivative of $\vec{\mathbf{r}}$ with respect to the scalar variable t.

Again,

$$\vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t}$$

$$\vec{\mathbf{a}} = \frac{\mathbf{d}\vec{\mathbf{v}}}{\mathbf{dt}} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}\right) = \frac{\mathbf{d}^2\vec{\mathbf{r}}}{\mathbf{dt}^2} \text{ -------------------------------------(vii)} \qquad [\because \vec{v} = \frac{d\vec{r}}{dt}]$$

$\dfrac{\mathbf{d}\vec{\mathbf{v}}}{\mathbf{dt}} = \dfrac{\mathbf{d}^2\vec{\mathbf{r}}}{\mathbf{dt}^2}$ represents the acceleration $\vec{\mathbf{a}}$ along the curve.

Again,

$$\vec{\mathbf{OP}} = \vec{\mathbf{r}}(\mathbf{x,y,z}) = \mathbf{x}\hat{\mathbf{i}} + \mathbf{y}\hat{\mathbf{j}} + \mathbf{z}\hat{\mathbf{k}}$$

$$\therefore \vec{\mathbf{v}} = \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}} = \frac{\mathbf{dx}}{\mathbf{dt}}\hat{\mathbf{i}} + \frac{\mathbf{dy}}{\mathbf{dt}}\hat{\mathbf{j}} + \frac{\mathbf{dz}}{\mathbf{dt}}\hat{\mathbf{k}} \text{ --------------------------------(viii)}$$

Remark: If $\vec{\mathbf{r}}(\mathbf{t})$ is the position function of a particle moving along a curve in 2-space (2-dimensional space) or 3-space, then the instantaneous velocity , instantaneous acceleration and instantaneous speed of the particle at time t are defined by

Velocity: $\vec{\mathbf{v}}(\mathbf{t}) = \dfrac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}$

Acceleration: $\vec{\mathbf{a}} = \dfrac{\mathbf{d}\vec{\mathbf{v}}}{\mathbf{dt}} = \dfrac{\mathbf{d}}{\mathbf{dt}}\left(\dfrac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}}\right) = \dfrac{\mathbf{d}^2\vec{\mathbf{r}}}{\mathbf{dt}^2}$

Speed: $\left|\vec{\mathbf{v}}(\mathbf{t})\right| = \dfrac{\mathbf{ds}}{\mathbf{dt}}$

Theorem:

If C is the graph in 2-space or 3-space of a smooth vector-valued function $\vec{\mathbf{r}}(\mathbf{t})$, then its arc length L from $\mathbf{t} = \mathbf{a}$ to $\mathbf{t} = \mathbf{b}$ is

$$L = \int_b^b \left| \frac{d\vec{r}}{dt} \right| dt$$

Displacement Vector and Distance Traveled:
If a particle travels along a curve C in 2-space or 3-space, the displacement of the particle over the time interval $t_1 \le t \le t_2$ is commonly denoted by $\Delta \vec{r}$ and is defined by

$$\Delta \vec{r} = \vec{r}(t_2) - \vec{r}(t_1) \qquad \text{--------------------------(i)}$$

The displacement vector, which describes the change in position of the particle during the time interval, can be obtained by integrating the velocity function from t_1 to t_2.

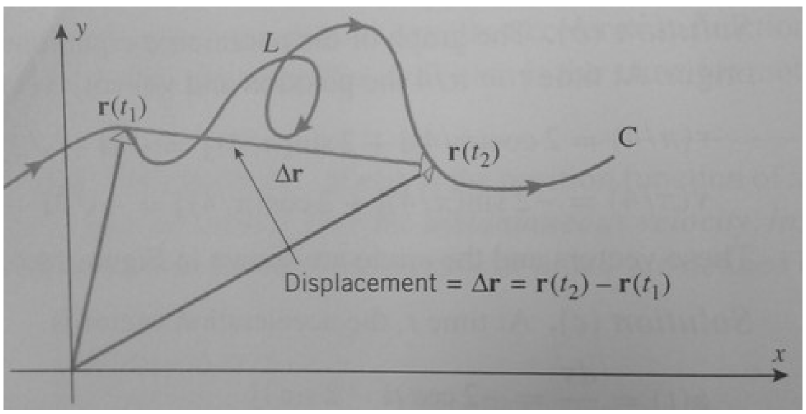


Figure 56

$$\Delta \vec{r} = \int_{t_1}^{t_2} \vec{v}(t)\,dt = \int_{t_1}^{t_2} \frac{dr}{dt}\,dt = \left[\vec{r}(t)\right]_{t_1}^{t_2} = \vec{r}(t_2) - \vec{r}(t_1) \qquad \text{--------------(ii)}$$

The distance travelled by the particle over the time interval $t_1 \le t \le t_2$ is:

$$s = \int_{t_1}^{t_2} \left| \frac{dr}{dt} \right| dt = \int_{t_1}^{t_2} \left| \vec{v}(t) \right| dt \qquad \text{-------------------------(iii)}$$

Q#18: A particle moves along a curve whose parametric equations are $x = e^{-t}, y = 2\cos 3t, z = 2\sin 3t,$ Where t is the time.

(a) Determine its velocity and acceleration at any time
(b) Find the magnitudes of the velocity and acceleration at t = 0.

Answer: The position vector $\vec{r}$ of the particle is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow \vec{r} = e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

Then the velocity is $\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(e^{-t})\hat{i} + \frac{d}{dt}(2\cos 3t)\hat{j} + \frac{d}{dt}(2\sin 3t)\hat{k}$

$$\frac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\hat{j} + 6\cos 3t\hat{k} \qquad [\frac{d}{dx}(e^{mx}) = me^{mx}]$$

and the acceleration is: $\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(\frac{d\vec{r}}{dt}) = \frac{d}{dt}(-e^{-t}\hat{i} - 6\sin 3t\hat{j} + 6\cos 3t\hat{k})$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\hat{i} - 18\cos 3t\hat{j} - 18\sin 3t\hat{k}$$

b) At $t = 0$,

Then the velocity is $\vec{V} = \frac{d\vec{r}}{dt} = -e^{-t}\hat{i} - 6\sin 3t\hat{j} + 6\cos 3t\hat{k}$

$$\vec{V} = \frac{d\vec{r}}{dt} = -e^{-0}\hat{i} - 6\sin 3\times 0\hat{j} + 6\cos 3\times 0\hat{k} \qquad [t = 0]$$

$$\vec{V} = \frac{d\vec{r}}{dt} = -e^{-0}\hat{i} - 6\sin 0 + 6\cos 0\hat{k}$$

$$\vec{V} = \frac{d\vec{r}}{dt} = -\hat{i} + 6\hat{k} \quad [e^{-0} = \frac{1}{e^0} = 1;\ \sin 0 = 0;\ \cos 0 = 1]$$

and the acceleration is:

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\hat{i} - 18\cos 3t\hat{j} - 18\sin 3t\hat{k}$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = e^{-0}\hat{i} - 18\cos 3\times 0\hat{j} - 18\sin 3\times 0\hat{k} \qquad [t = 0]$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = \hat{i} - 18\hat{j} \quad [e^{-0} = \frac{1}{e^0} = 1;\ \sin 0 = 0;\ \cos 0 = 1]$$

The Magnitude of the velocity is $|\vec{V}| = \sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

The Magnitude of the acceleration is $|\vec{a}|: \sqrt{(1)^2 + (-18)^2} = \sqrt{325}$ Answer

Q#19: A particle moves along a curve whose parametric equations are $x = 2t^2,\ y = t^2 - 4t,\ z = 3t - 5,$ where t is time. Find the component of the velocity at time t = 1 in the direction $\vec{a} = \hat{i} - 3\hat{j} + 2\hat{k}$

Answer:

The position vector $\vec{\mathbf{r}}$ of the particle is $\vec{\mathbf{r}} = \mathbf{x}\,\hat{\mathbf{i}} + \mathbf{y}\,\hat{\mathbf{j}} + \mathbf{z}\,\hat{\mathbf{k}}$

$$\Rightarrow \vec{r} = 2t^2\,\hat{i} + (t^2 - 4t)\,\hat{j} + (3t - 5)\,\hat{k}$$

Then the velocity is $\vec{\mathbf{V}} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}[2t^2\,\hat{i} + (t^2 - 4t)\,\hat{j} + (3t - 5)\,\hat{k}]$

$$\vec{\mathbf{V}} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[2t^2\,\hat{i}] + \frac{d}{dt}[(t^2 - 4t)\,\hat{j}] + \frac{d}{dt}[(3t - 5)\,\hat{k}]$$

$$\vec{\mathbf{V}} = \frac{d\vec{r}}{dt} = 4t\,\hat{i} + (2t - 4)\,\hat{j} + 3\,\hat{k} \text{ -------------------------(i)}$$

The Velocity at t = 1;

$$\vec{\mathbf{V}} = \frac{d\vec{r}}{dt} = 4t\,\hat{i} + (2t - 4)\,\hat{j} + 3\,\hat{k}$$

$$\vec{\mathbf{V}} = \frac{d\vec{r}}{dt} = 4.1\,\hat{i} + (2.1 - 4)\,\hat{j} + 3\,\hat{k}$$

$$\vec{\mathbf{V}} = \frac{d\vec{r}}{dt} = 4\,\hat{i} - 2\,\hat{j} + 3\,\hat{k} \qquad \text{---------------------------(ii)}$$

The unit vector of $\vec{a}$ is $\hat{e} = \dfrac{\hat{a}}{\left|\hat{a}\right|} = \dfrac{\hat{i} - 3\,\hat{j} + 2\,\hat{k}}{\sqrt{1^2 + (-3)^2 + 2^2}} = \dfrac{\hat{i} - 3\,\hat{j} + 2\,\hat{k}}{\sqrt{1 + 9 + 4}} = \dfrac{\hat{i} - 3\,\hat{j} + 2\,\hat{k}}{\sqrt{14}}$

$$= \frac{\hat{i}}{\sqrt{14}} - \frac{3\,\hat{j}}{\sqrt{14}} + \frac{2\,\hat{k}}{\sqrt{14}} \text{ ----------------------------------(iii)}$$

The component of the velocity in the given direction $\vec{a} = \hat{i} - 3\,\hat{j} + 2\,\hat{k}$ is $\vec{V}.\hat{e}$, where $\hat{e}$ is a unit vector in the direction of a.

$$\vec{\mathbf{V}}.\hat{e} = (4\,\hat{i} - 2\,\hat{j} + 3\,\hat{k})\ .(\frac{\hat{i}}{\sqrt{14}} - \frac{3\,\hat{j}}{\sqrt{14}} + \frac{2\,\hat{k}}{\sqrt{14}})$$

$$\vec{\mathbf{V}}.\hat{e} = (\frac{4}{\sqrt{14}} + \frac{6}{\sqrt{14}} + \frac{6}{\sqrt{14}})$$

$\vec{\mathbf{V}}.\hat{e} = \dfrac{16}{\sqrt{14}}$ Answer

Q#20: A particle moves so that its position vector is given by $\vec{\mathbf{r}} = \cos\omega t\,\hat{\mathbf{i}} + \sin\omega t\,\hat{\mathbf{j}}$, where ω is a constant. Show that (a) the velocity $\vec{\mathbf{V}}$ of the particle is perpendicular to $\vec{\mathbf{r}}$, (b) The acceleration $\vec{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin (c) $\vec{\mathbf{r}} \times \vec{\mathbf{V}}$ = a constant vector

Answer: Given, $\vec{r} = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}$

a) Then the velocity is $\vec{V} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(\cos\omega t)\,\hat{i} + \dfrac{d}{dt}(\sin\omega t)\,\hat{j}$

$$\vec{V} = \frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j}$$

Op = r
Op = -r

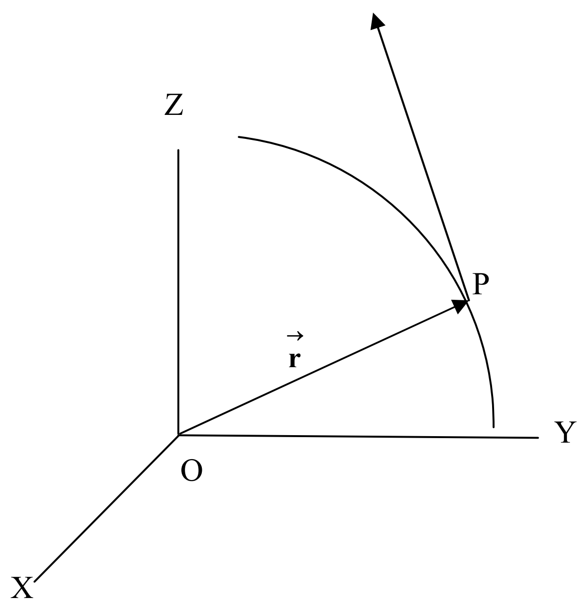


Figure # 57

Then $\vec{r}\,.\vec{V} = (\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}).(-\omega\sin\omega t\,\hat{i} + \omega\cos\omega t\,\hat{j})$

Then $\vec{r}\,.\vec{V} = (\cos\omega t)(-\omega\sin\omega t) + (\sin\omega t)(\omega\cos\omega t)$

Then $\vec{r}\,.\vec{V} = -\,\omega\sin\omega t\cos\omega t + \omega\sin\omega t\cos\omega t$

Then $\vec{r}\,.\vec{V} = 0$

Hence $\vec{r}$ and $\vec{V}$ are perpendicular.

b) The acceleration is: $\vec{a} = \dfrac{d\vec{V}}{dt} = \dfrac{d}{dt}(-\omega \sin \omega t\,\hat{i} + \omega \cos \omega t\,\hat{j})$

The acceleration is: $\vec{a} = \dfrac{d\vec{V}}{dt} = -\omega^2 \cos \omega t\,\hat{i} - \omega^2 \sin \omega t\,\hat{j}$

The acceleration is: $\vec{a} = \dfrac{d\vec{V}}{dt} = -\omega^2(\cos \omega t\,\hat{i} + \sin \omega t\,\hat{j}) = -\omega^2\,\vec{r}$

$$\vec{a} = -\omega^2\,\vec{r} \text{ --(i)}$$

$$\vec{a} \propto \vec{r}$$

$$|\vec{a}| \propto |\vec{r}| \text{ --(ii)}$$

From equation (1), the acceleration is opposite to the direction of $\vec{r}$. i.e. it is directed toward the origin, Its magnitude is proportional to $|\vec{r}|$ which is the distance from the origin.

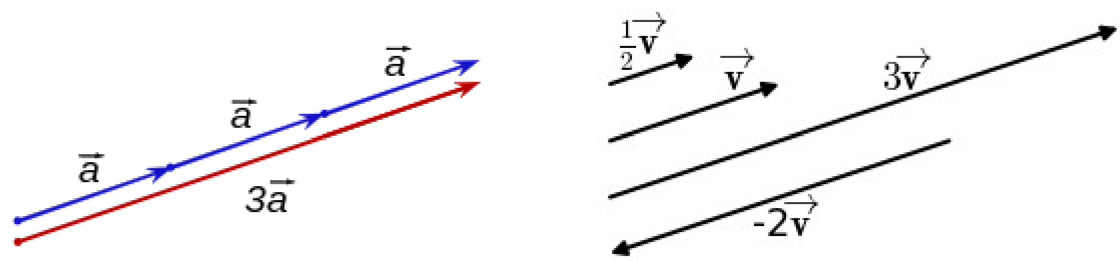


Figure # 58

c) Here $\vec{r} = \cos \omega t\,\hat{i} + \sin \omega t\,\hat{j}$ and $\vec{V} = -\omega \sin \omega t\,\hat{i} + \omega \cos \omega t\,\hat{j}$

$$\vec{r} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos \omega t & \sin \omega t & 0 \\ -\omega \sin \omega t & \omega \cos \omega t & 0 \end{vmatrix}$$

$$= \hat{i}(\sin \omega t \times 0 - 0 \times \omega \cos \omega t) - \hat{j}(\cos \omega t \times 0 - (-\omega \sin \omega t) \times 0 + \hat{k}(\cos \omega t \times \omega \cos \omega t$$
$$- (-\omega \sin \omega t) \times \sin \omega t)$$

$$= \hat{i} \times 0 - \hat{j} \times 0 + \hat{k}(\omega \cos^2 \omega t + \omega \sin^2 \omega t)$$

$$= \hat{k}\,\omega(\cos^2 \omega t + \sin^2 \omega t)$$

$= \hat{k}\,\omega.1 \qquad\qquad [\because \cos^2\omega t + \sin^2\omega t = 1]$

$= \hat{k}\,\omega$

Q#21: A particle moves along a circular path in such a way that its x- and y-coordinates at time t are $\mathbf{x = 2\cos t}$, $\mathbf{y = 2\sin t}$

a) Find the instantaneous velocity and speed of the particle at time t.
b) Sketch the path of the particle and show the position and velocity vectors at time $\mathbf{t = \frac{\pi}{4}}$ with the velocity vector drawn so that its initial point is at the top of the position vector
c) Show that at each instant the acceleration vector is perpendicular to the velocity vector

Answer:
a)
Let the position vector at any time t is:

$$\vec{OP} = \vec{r}(t) = x\hat{i} + y\hat{j}$$

At time t, the position vector is:

$$\vec{OP} = \vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} \text{ [Given, } x = 2\cos t,\ y = 2\sin t\text{]} \quad \text{--------------------(i)}$$

So the instantaneous velocity is:

$$\vec{V}(t) = \frac{d\vec{r}}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j} \quad \text{---------------------------------(ii)}$$

So the instantaneous speed is:

$$\left|\vec{V}(t)\right| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$$

<u>Answer (b):</u>

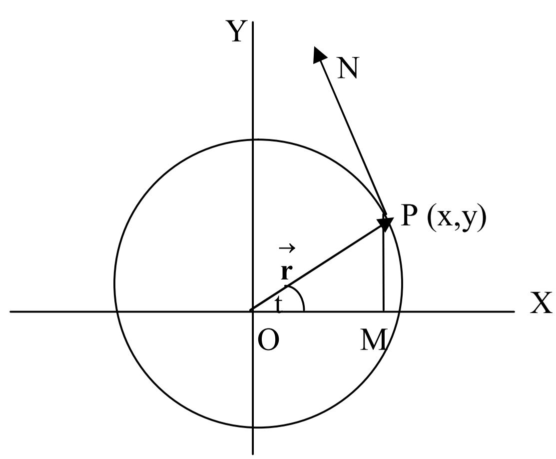


Figure #: 59

$\vec{OP} = \vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$

Here, $\mathbf{OM = x = 2\cos t}$ and $\mathbf{PM = y = 2\sin t}$

$\angle POM = t$

$\frac{PM}{OP} = \sin t$

$\frac{2\sin t}{OP} = \sin t$ [Given $\mathbf{PM = y = 2\sin t}$]

$OP\sin t = 2\sin t$

$OP = 2$ ---------------------------------(iii)

Similarly,

$\frac{OM}{OP} = \cos t$

$\frac{2\cos t}{OP} = \cos t$ [$\mathbf{OM = x = 2\cos t}$]

$OP\cos t = 2\cos t$

$OP = 2$ ----------------------------------(iv)

Hence the radius of the circle is OP = 2.

At time $t = \frac{\pi}{4}$, the position and velocity vector of the particles are:

$\vec{OP} = \vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j}$ [From (i)]

$\vec{r}(\frac{\pi}{4}) = 2\cos\frac{\pi}{4}\hat{i} + 2\sin\frac{\pi}{4}\hat{j}$

$\vec{r}(\frac{\pi}{4}) = 2\frac{1}{\sqrt{2}}\hat{i} + 2\frac{1}{\sqrt{2}}\hat{j}$

$\vec{OP} = \vec{r}(\frac{\pi}{4}) = \sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$ When $t = \frac{\pi}{4}$.

From (ii),

$\vec{V}(t) = \frac{d\vec{r}}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$

$\vec{V}(\frac{\pi}{4}) = -2\sin\frac{\pi}{4}\hat{i} + 2\cos\frac{\pi}{4}\hat{j}$

$\vec{PN} = \vec{V}(\frac{\pi}{4}) = -2\frac{1}{\sqrt{2}}\hat{i} + 2\frac{1}{\sqrt{2}}\hat{j} = \vec{V}(\frac{\pi}{4}) = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j}$

Answer (c):

We have,

$\vec{V}(t) = \frac{d\vec{r}}{dt} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$ [From (ii)]

At time t, the acceleration vector is:

$$\therefore \vec{a}(t) = \frac{d\vec{v}}{dt} = -2\cos t\,\hat{i} - 2\sin t\,\hat{j} \text{ ------------------------(v)}$$

Test: From (ii) & (v),

$$\vec{V}(t).\vec{a}(t) = (-2\sin t\,\hat{i} + 2\cos t\,\hat{j}).\ (-2\cos t\,\hat{i} - 2\sin t\,\hat{j}) = 4\sin t\cos t - 4\sin t\cos t = 0$$

Since the dot product of Velocity Vector (ii) and acceleration Vector (v) is Zero, Hence acceleration vector is perpendicular to the velocity vector.

(Proved)

Q# 22: A particle moves through 3-space in such a way that its velocity is $\vec{v}(t) = \hat{i} + t\,\hat{j} + t^2\,\hat{k}$. Find the co-ordinates of the particle at time $t = 1$ given that the particle is at the point $(-1,2,4)$ at time $t = 0$

Answer: We have,

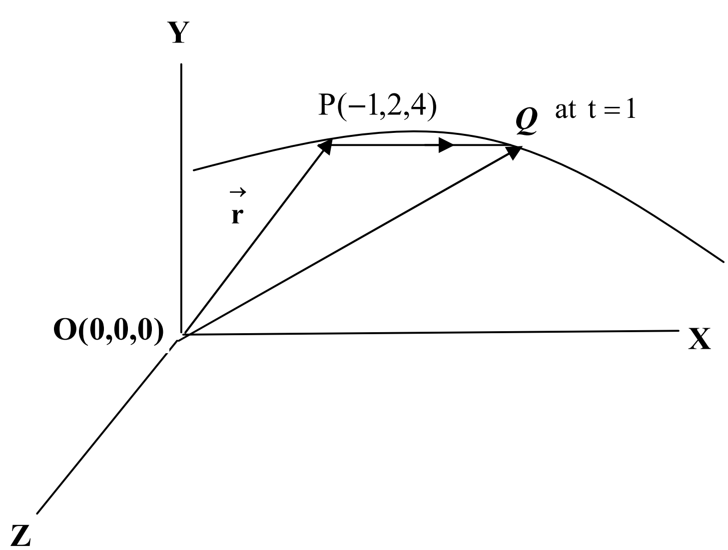


Figure 60

$$\vec{v}(t) = \frac{d\vec{r}}{dt} \quad \text{Where } \vec{r} \text{ is a position vector.}$$

Given, $\vec{v}(t) = \hat{i} + t\,\hat{j} + t^2\,\hat{k}$

$$\therefore \vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \hat{i} + t\,\hat{j} + t^2\,\hat{k}$$

$$\frac{d\vec{r}(t)}{dt} = \hat{i} + t\,\hat{j} + t^2\,\hat{k}$$

$$d\vec{r}(t) = (\hat{i} + t\,\hat{j} + t^2\,\hat{k})dt \text{ ---------------------------(i)}$$

Integrate (i) both sides, we get,

$$\int d\vec{r}(t) = \int (\hat{i} + t\,\hat{j} + t^2\,\hat{k})dt$$

$$\int d\vec{r}(t) = \int \hat{i}\,dt + \int t\,\hat{j}\,dt + \int t^2\,\hat{k}\,dt$$

$$\vec{r}(t) = t\,\hat{i} + \frac{t^2}{2}\hat{j} + \frac{t^3}{3}\hat{k} + C \qquad \text{------------------------(ii)}$$

Where C is a vector constant of integration. Since the coordinates of the particle at time $t = 0$ are $(-1,2,4)$, the position vector at time $t = 0$ is

We have the position vector

$$\vec{r}(t) = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$

$$\vec{r}(0) = (-1)\,\hat{i} + 2\,\hat{j} + 4\,\hat{k} \qquad [\text{at time } t = 0\text{, the position vector is at } (-1,2,4)]$$

$$\vec{r}(0) = (-1)\,\hat{i} + 2\,\hat{j} + 4\,\hat{k} \qquad \text{------------------------(iii)}$$

Again, putting $t = 0$ in (ii), we get,

$$\vec{r}(t) = t\,\hat{i} + \frac{t^2}{2}\hat{j} + \frac{t^3}{3}\hat{k} + C$$

$$\vec{r}(0) = 0.\,\hat{i} + \frac{0^2}{2}\hat{j} + \frac{0^3}{3}\hat{k} + C$$

$$\vec{r}(0) = 0 + 0 + 0 + C$$

$$\vec{r}(0) = C \qquad \text{------------------------(iv)}$$

Comparing (iii) and (iv), we get,

$$\vec{r}(0) = (-1)\,\hat{i} + 2\,\hat{j} + 4\,\hat{k} = C$$

$$C = -\,\hat{i} + 2\,\hat{j} + 4\,\hat{k} \qquad \text{-------------------------(v)}$$

Putting the value of C in (ii), we get,

$$\vec{r}(t) = t\,\hat{i} + \frac{t^2}{2}\hat{j} + \frac{t^3}{3}\hat{k} + C$$

$$\vec{r}(t) = t\,\hat{i} + \frac{t^2}{2}\hat{j} + \frac{t^3}{3}\hat{k} - \hat{i} + 2\,\hat{j} + 4\,\hat{k}$$

$$\vec{r}(t) = (t-1)\,\hat{i} + (\frac{t^2}{2} + 2)\,\hat{j} + (\frac{t^3}{3} + 4)\,\hat{k} \qquad \text{-------------------(vi)}$$

Thus, at time $t = 1$, the position vector of the particle is

From (vi),

$$\vec{r}(t) = (t-1)\,\hat{i} + (\frac{t^2}{2} + 2)\,\hat{j} + (\frac{t^3}{3} + 4)\,\hat{k}$$

$$\vec{r}(1) = (1-1)\,\hat{i} + (\frac{1^2}{2} + 2)\,\hat{j} + (\frac{1^3}{3} + 4)\,\hat{k}$$

$$\vec{r}(1) = 0\,\hat{i} + \frac{5}{2}\hat{j} + \frac{13}{3}\hat{k}.$$

So, the coordinates of the particle at time $\mathbf{t = 1}$ is $\mathbf{(0, \frac{5}{2}, \frac{13}{3})}$ the Answer

Q# 23: Suppose that a particle moves along a circular helix (**figure 56**) in 3-space so that its position vector at time t is $\vec{\mathbf{r}}(\mathbf{t}) = (\mathbf{4\cos\pi t})\,\hat{\mathbf{i}} + (\mathbf{4\sin\pi t})\,\hat{\mathbf{j}} + \mathbf{t}\,\hat{\mathbf{k}}$. Find the distance traveled and the displacement of the particle during the time interval $\mathbf{1 \le t \le 5}$

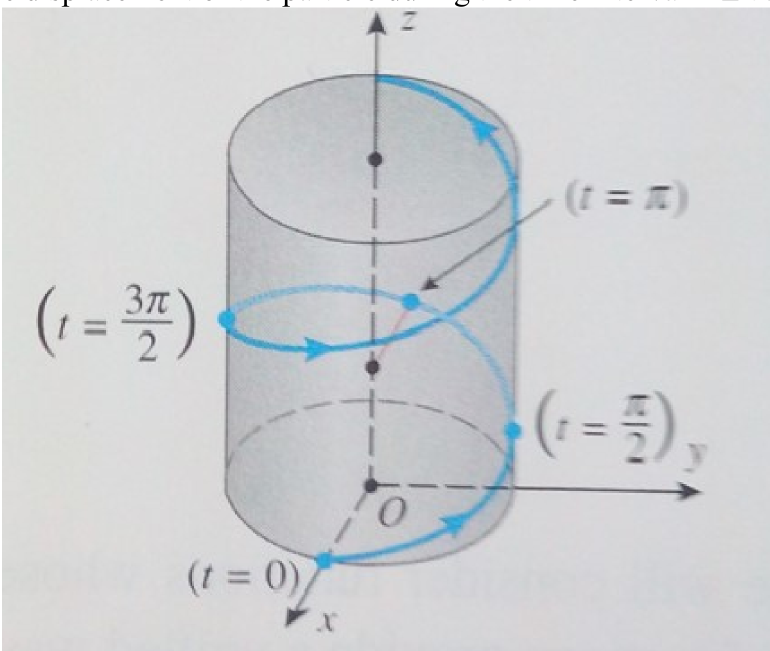


Figure 61

Answer: Given,

$$\vec{\mathbf{r}}(\mathbf{t}) = (\mathbf{4\cos\pi t})\,\hat{\mathbf{i}} + (\mathbf{4\sin\pi t})\,\hat{\mathbf{j}} + \mathbf{t}\,\hat{\mathbf{k}} \qquad \text{------------------(i)}$$

$$\therefore \vec{\mathbf{v}}(\mathbf{t}) = \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}} = \frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{4\cos\pi t}\,\hat{\mathbf{i}} + \mathbf{4\sin\pi t}\,\hat{\mathbf{j}} + \mathbf{t}\,\hat{\mathbf{k}})$$

$$\therefore \vec{\mathbf{v}}(\mathbf{t}) = \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}} = -\mathbf{4\sin\pi t}.\frac{\mathbf{d}}{\mathbf{dt}}(\boldsymbol{\pi}\mathbf{t})\,\hat{\mathbf{i}} + \mathbf{4\cos\pi t}.\frac{\mathbf{d}}{\mathbf{dt}}(\boldsymbol{\pi}\mathbf{t})\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\therefore \vec{\mathbf{v}}(\mathbf{t}) = \frac{\mathbf{d}\vec{\mathbf{r}}}{\mathbf{dt}} = -\mathbf{4\sin\pi t}.(\boldsymbol{\pi})\,\hat{\mathbf{i}} + \mathbf{4\cos\pi t}.(\boldsymbol{\pi})\,\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\therefore \vec{v}(t) = \frac{d\vec{r}}{dt} = -4\pi \sin \pi t\, \hat{i} + 4\pi \cos \pi t\, \hat{j} + \hat{k} \qquad \text{-------------------(ii)}$$

$$\therefore \left|\vec{v}(t)\right| = \sqrt{(-4\pi \sin \pi t)^2 + (4\pi \cos \pi t)^2 + 1^2}$$

$$\therefore \left|\vec{v}(t)\right| = \sqrt{16\pi^2 \sin^2 \pi t + 16\pi^2 \cos^2 \pi t + 1^2}$$

$$\therefore \left|\vec{v}(t)\right| = \sqrt{16\pi^2 (\sin^2 \pi t + \cos^2 \pi t) + 1}$$

$$\therefore \left|\vec{v}(t)\right| = \sqrt{16\pi^2 .1 + 1} \qquad [\sin^2 t + \cos^2 t = 1]$$

$$\therefore \left|\vec{v}(t)\right| = \sqrt{16\pi^2 + 1} \qquad \text{---------------------(iii)}$$

The distance travelled by the particle from time $t = 1$ **to** $t = 5$ **is:**

$$s = \int_1^5 \left|\frac{dr}{dt}\right| dt = \int_1^5 \left|\vec{v}(t)\right| dt$$

$$s = \int_1^5 \left|\frac{dr}{dt}\right| dt = \int_1^5 \sqrt{16\pi^2 + 1}\, dt$$

$$s = \int_1^5 \left|\frac{dr}{dt}\right| dt = \sqrt{16\pi^2 + 1}[t]_1^5$$

$$s = \int_1^5 \left|\frac{dr}{dt}\right| dt = \sqrt{16\pi^2 + 1}[5 - 1]$$

$$s = \int_1^5 \left|\frac{dr}{dt}\right| dt = \sqrt{16\pi^2 + 1}[4]$$

$$s = \int_1^5 \left|\frac{dr}{dt}\right| dt = 4\sqrt{16\pi^2 + 1}$$

Again,
From (i)

$$\vec{r}(t) = (4\cos \pi t)\, \hat{i} + (4\sin \pi t)\, \hat{j} + t\, \hat{k}$$

$$\therefore \vec{r}(5) = (4\cos 5\pi)\, \hat{i} + (4\sin 5\pi)\, \hat{j} + 5\hat{k}$$

$$\therefore \vec{r}(1) = (4\cos \pi)\, \hat{i} + (4\sin \pi)\, \hat{j} + \hat{k}$$

Moreover, the displacement over the time interval is:

$$\Delta\vec{r} = \vec{r}(5) - \vec{r}(1)$$

$$\Delta\vec{r} = (4\cos 5\pi)\, \hat{i} + (4\sin 5\pi)\, \hat{j} + 5\hat{k} - (4\cos \pi\, \hat{i} + 4\sin \pi\, \hat{j} + \hat{k})$$

$$\vec{\nabla r} = 4(-1)\hat{\mathbf{i}} + 4.0.\hat{\mathbf{j}} + 5\hat{\mathbf{k}} - [4.(-1)\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}} + \hat{\mathbf{k}}]$$

$$\vec{\nabla r} = -4\hat{\mathbf{i}} + 5\hat{\mathbf{k}} + 4\hat{\mathbf{i}} - \hat{\mathbf{k}}$$

$$\vec{\nabla r} = 4\hat{\mathbf{k}}$$

Which tells us that the change in the position of the particle over the time interval was 4 units straight up. *Answer*