Decomposition Theorem

Eigen Value Decomposition

① Define Eigen value decomposition. Find the Eigen Value decomposition for the matrix. $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ with justification.

Solution: Eigen Value decomposition: In linear algebra, eigen decomposition or sometimes spectral decomposition is the factorization of a matrix; into a canonical form, whereby the matrix is represented in terms of its eigen values and eigen vectors. It can be written as, $A = PDP^{-1}$.

Given that, $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

Now, $A - \lambda I = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}$

So, the characteristic equation,

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$

or, $(4-\lambda)(4-\lambda) - 1 = 0$

or, $16 - 4\lambda - 4\lambda + \lambda^2 - 1 = 0$

or, $\lambda^2 - 8\lambda + 15 = 0$

or, $\lambda^2 - 3\lambda - 5\lambda + 15 = 0$

or, $\lambda(\lambda - 3) - 5(\lambda - 3) = 0$

or, $(\lambda - 3)(\lambda - 5) = 0$

or, $\lambda = 3, 5$

Hence, the diagonalized matrix, $D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

Now, to find the eigen vector of matrix A.

For, $\lambda = 3$

Let, a non zero vector, $V_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

We have, $AV_1 = \lambda V_1$

or, $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 4x + y \\ x + 4y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$

or, $4x + y = 3x$
$x + 4y = 3y$

or, $x + y = 0$
$x + y = 0$

As the above two equations are same, so we can remove any one of them, we get,

$x + y = 0$

Here, number of equation is 1, but number of variable are 2. So $(2-1) = 1$ is free variable which is y.

Let, $y = 1$

$\therefore x = -1$

$\therefore V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Again, for $\lambda = 5$

Let, a non-zero vector, $V_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

We have, $AV_2 = \lambda V_2$

$$\text{or, } \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} 4x+y \\ x+4y \end{bmatrix} = \begin{bmatrix} 5x \\ 5y \end{bmatrix}$$

$$\text{or, } 4x+y = 5x$$
$$x+4y = 5y$$

$$\text{or, } -x+y=0$$
$$x-y=0$$

$$\text{or, } x-y=0$$
$$x-y=0$$

As the equations are identical, we can disregard one of them.

$$\therefore x-y=0$$

Now, number of equation is 1 but unknown variables are 2.

So, we have $(2-1)=1$ free variable, which is y.

Let, $y=1$

$\therefore x = 1$

$\therefore V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigen vector, $V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

We know that, $P = [V_1 \ V_2]$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Also, $P^{-1} = \frac{1}{-1-1} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$$= \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$\therefore$ $A = PDP^{-1}$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

(Ans)

Justification:

Here, $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

and, $PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3/2+0 & 3/2+0 \\ 0+5/2 & 0+5/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3/2 & 3/2 \\ 5/2 & 5/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2+5/2 & -3/2+5/2 \\ -3/2+5/2 & 3/2+5/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{2} & \frac{2}{2} \\ \frac{2}{2} & \frac{8}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

As, $A = PDP^{-1}$

so, justified.

② Find the eigen value decomposition for the matrix $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ with justification.

Solution: Given that, $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Now, $A - \lambda I = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$= \begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}$

So, the characteristic equation,

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)(2-\lambda) - 4 = 0$

or, $10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

or, $\lambda^2 - 7\lambda + 6 = 0$

or, $\lambda^2 - \lambda - 6\lambda + 6 = 0$

or, $\lambda(\lambda - 1) - 6(\lambda - 1) = 0$

or, $(\lambda - 1)(\lambda - 6) = 0$

$\therefore \lambda = 1; 6$

Hence, the diagonalized matrix, $D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$

Now, to find out the eigen vectors of matrix A

For $\lambda = 1$

Let, a non-zero vector, $V_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

We have, $AV_1 = \lambda V_1$

or, $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 5x+4y \\ x+2y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

or, $5x+4y = x$
$x+2y = y$

or, $4x+4y = 0$
$x+y = 0$

As the two equations are identical we can disregard one of them.

$\therefore x+y=0$

Now, member of equation is 1. but number of unknows variables are 2. so we have $(2-1)=1$ free variable which is y.

Let, $y=1$

$\therefore x=-1$

$\therefore V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Again, $\lambda=6$

Let a non zero vector, $V_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

We have, $AV_2 = \lambda V_2$

or, $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 5x+4y \\ x+2y \end{bmatrix} = \begin{bmatrix} 6x \\ 6y \end{bmatrix}$

or, $5x + 4y = 6x$
$x + 2y = 6y$.

or, $-x + 4y = 0$
$x - 4y = 0$

or, $x - 4y = 0$
$x - 4y = 0$

As the two equations are identical, we can disregard one of them.
$$x - 4y = 0$$

Now, number of equation is 1. but number of unknown variables are 2. so we have $(2-1) = 1$ free variable which is why.

Let, $y = 1$
$x = 4$

$\therefore V_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

We know that, $P = [V_1 \ V_2]$
$$= \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}$$

Also, $P^{-1} = \frac{1}{-1-4} \begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}$
$$= \frac{1}{-5} \begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \\ 1/5 & \frac{1}{5} \end{bmatrix}$$

$\therefore \quad A = PDP^{-1}$

$$= \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} -1/5 & 4/5 \\ 1/5 & 1/5 \end{bmatrix}$$

Justification:

Here $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

and $PDP^{-1} = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} -1/5 & 4/5 \\ 1/5 & 1/5 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{5}+0 & \frac{4}{5}+0 \\ 0+\frac{6}{5} & 0+\frac{6}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{5} & \frac{4}{5} \\ \frac{6}{5} & \frac{6}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1/5 + 24/5 & -4/5 + 24/5 \\ -1/5 + 6/5 & 4/5 + 6/5 \end{bmatrix}$$

$$= \begin{bmatrix} 25/5 & 20/5 \\ 5/5 & 10/5 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$= A$$

As, $A = PDP^{-1}$

so justified.

LU Decomposition:

LU Decomposition: If any linear system of equation can be performed as $AX = b$ with out row interchanges then A can be factored into the product of a linear triangular matrix L and an upper triangular matrix U. i.e. $A = LU$ which is known as LU decomposition or LU factorization.

① Solve the linear system of equations using LU decomposition.

$$U_1 + 2U_2 + 3U_3 = 5$$
$$2U_1 - 4U_2 + 6U_3 = 18$$
$$3U_1 - 9U_2 - 3U_3 = 6$$

Solution: Given that, the linear system,

$$U_1 + 2U_2 + 3U_3 = 5$$
$$2U_1 - 4U_2 + 6U_3 = 18$$
$$3U_1 - 9U_2 - 3U_3 = 6$$

which can be written as, $AU = b$ —①

where. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix}$, $U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$, $b = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$

Now, According to LU decomposition,

$A = LU$ —②

where,

$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$ and $U = \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$

From ②,

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix}$$

or,
$$\begin{bmatrix} l_{11} & l_{11}U_{12} & l_{11}U_{13} \\ l_{21} & l_{21}U_{12}+l_{22} & l_{21}U_{13}+l_{22}U_{23} \\ l_{31} & l_{31}U_{12}+l_{32} & l_{31}U_{13}+l_{32}U_{23}+l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 6 \\ 3 & -9 & -3 \end{bmatrix}$$

$\therefore l_{11} = 1$; $l_{11}U_{12} = 2$; $l_{11}U_{13} = 3$

or, $U_{12} = 2$; or, $U_{13} = 3$

$l_{21} = 2$; $l_{21}U_{12} + l_{22} = -4$; $l_{21}U_{13} + l_{22}U_{23} = 6$

or, $2.2 + l_{22} = -4$; or, $2.3 + (-8)U_{23} = 6$

or, $l_{22} = -8$; or, $6 - 8U_{23} = 6$

or, $-8U_{23} = 0$

$\therefore U_{23} = 0$

$l_{31} = 3$; $l_{31}U_{12} + l_{32} = -9$; $l_{31}U_{13} + l_{32}U_{23} + l_{33} = -3$

or, $3.2 + l_{32} = -9$; or, $3.3 + (-15).0 + l_{33} = -3$

or, $l_{32} = -15$; or, $l_{33} = -12$

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From equation ① and ② we get,

$$AU = b$$

$$LU_U = b \quad [A = LU] \quad —③$$

Therefore, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let, $Uu = V$ —④

where, $V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$

Now, from ③ and ④ we get,

$$LUu = b$$
$$LV = b$$

or, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -8 & 0 \\ 3 & -15 & -12 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$

or, $\begin{bmatrix} V_1 \\ 2V_1 - 8V_2 \\ 3V_1 - 15V_2 - 12V_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 18 \\ 6 \end{bmatrix}$

or, $V_1 = 5$

$2V_1 - 8V_2 = 18$

$3V_1 - 15V_2 - 12V_3 = 6$

$\therefore V_1 = 5$; $2V_1 - 8V_2 = 18$; $3V_1 - 15V_2 - 12V_3 = 6$

or, $2 \cdot 5 - 8V_2 = 18$

or, $-8V_2 = 18 - 10$

or, $V_2 = -1$

or, $3 \cdot 5 - 15(-1) - 12V_3 = 6$

or, $15 + 15 - 12V_3 = 6$

or, $-12V_3 = 6 - 30$

$\therefore V_3 = 2$

Now, from equation ④ we get,

$$Uu = V$$

or, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

or, $\begin{bmatrix} v_1+2v_2+3v_3 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$

or, $v_1+2v_2+3v_3 = 5$

$v_2 = -1$

$v_3 = 2$

$\therefore v_2 = -1$; $v_3 = 2$; $v_1 + 2v_2 + 3v_3 = 5$

or, $v_1 + 2(-1) + 3.2 = 5$

$\therefore v_1 = 1$

$\therefore (v_1, v_2, v_3) = (1, -1, 2)$

(Ans)

② Solve the linear system using LU decomposition,
$x+y+z=6$, $x-y+z=2$ and $2x+y-z=1$.

<u>Solution:</u> Given that the linear system,

$$x+y+z=6$$
$$x-y+z=2$$
$$2x+y-z=1$$

which can be written as $AX=b$ — ①

where,

$$A=\begin{bmatrix}1&1&1\\1&-1&1\\2&1&-1\end{bmatrix},\quad X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } b=\begin{bmatrix}6\\2\\1\end{bmatrix}$$

Now, According to LU decomposition,

$$A=LU \quad — ②$$

where, $L=\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}$ and $U=\begin{bmatrix}1&U_{12}&U_{13}\\0&1&U_{23}\\0&0&1\end{bmatrix}$

From ②,

$$\begin{bmatrix}l_{11}&0&0\\l_{21}&l_{22}&0\\l_{31}&l_{32}&l_{33}\end{bmatrix}\begin{bmatrix}1&U_{12}&U_{13}\\0&1&U_{23}\\0&0&1\end{bmatrix}=\begin{bmatrix}1&1&1\\1&-1&1\\2&1&-1\end{bmatrix}$$

or,

$$\begin{bmatrix}l_{11}&l_{11}U_{12}&l_{11}U_{13}\\l_{21}&l_{21}U_{12}+l_{22}&l_{21}U_{13}+l_{22}U_{23}\\l_{31}&l_{31}U_{12}+l_{32}&l_{31}U_{13}+l_{32}U_{23}+l_{33}\end{bmatrix}=\begin{bmatrix}1&1&1\\1&-1&1\\2&1&-1\end{bmatrix}$$

$\therefore l_{11}=1$; $l_{11}U_{12}=1$; $l_{11}U_{13}=1$
or, $U_{12}=1$; or, $U_{13}=1$

$\therefore l_{21} = 1$; $l_{21}U_{12} + l_{22} = -1$

or, $1 + l_{22} = -1$

or, $l_{22} = -2$

; $l_{21}U_{13} + l_{22}U_{23} = 1$

or, $1 + (-2)U_{23} = -1$

or, $U_{23} = 0$

$\therefore l_{31} = 2$; $l_{31}U_{12} + l_{32} = 1$

or, $2 + l_{32} = 1$

or, $l_{32} = -1$

; $l_{31}U_{13} + l_{32}U_{23} + l_{33} = -1$

or, $2 + (-1).0 + l_{33} = -1$

or, $l_{33} = -3$

Therefore,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & -1 & -3 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From ① and ② we get,

$$AX = b$$

$$A = LU$$

$$\therefore LUX = b \quad \text{— ③}$$

Let, $UX = Y$. — ④

Now, from ③ and ④ we get,

$$LUX = b$$

$$LY = b$$

or, $$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

or, $$\begin{bmatrix} y_1 \\ y_1 - 2y_2 \\ 2y_1 - y_2 - 3y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore y_1 = 6$; $y_1 - 2y_2 = 1$; $2y_1 - y_2 - 3y_3 = 2$

or, $6 - 2y_2 = 1$ or, $2 \cdot 6 - 2 - 3y_3 = 2$

or, $y_2 = 2$ or, $y_3 = 3$

Now, from equation (4) we get,

$$UX = Y$$

$$\text{or, } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x+y+z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore y = 2$; $z = 3$; $x + y + z = 6$

or, $x + 2 + 3 = 6$

$\therefore x = 1$

$\therefore (x, y, z) = (1, 2, 3)$

(Ans)

Singular Value Decomposition

Singular Value decomposition: The singular value decomposition of a matrix A is the factorization of A into the product of three matrices $A = UDV^T$ where the column of U and V are orthogonal matrices and D is diagonal matrices with positive real entries.

Example-43: compute singular value Decomposition (SVD) for the given matrix $\begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$ with justification.

Solution: Let, $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$

Now, $A^T = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}$

So, $AA^T = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2\times2+(-2)(-2) & 2+(-2) \\ 2+1(-2) & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

Now, $AA^T - \lambda I = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 8-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix}$$

So the characteristics equation of matrix AA^T is given by.

$$|AA^T - \lambda I| = 0$$

or, $\begin{vmatrix} 8-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$

or, $16 - 8\lambda - 2\lambda + \lambda^2 = 0$

or, $8(2-\lambda) - \lambda(2-\lambda) = 0$

or, $(2-\lambda)(8-\lambda) = 0$

or, $\lambda = 8, 2$

So, $D = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$

$\therefore S = \sqrt{D} = \begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

Let, a non-zero vector $U_1 = \begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore AA^T U_1 = \lambda U_1$

or, $\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 8 \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 8x \\ 2y \end{bmatrix} = \begin{bmatrix} 8x \\ 8y \end{bmatrix}$

or, $8x = 8x$

$2y = 8y$

or, $0 = 0$

$y = 0$

Let, $x = -1$

$\therefore U_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ [when $\lambda = 8$]

$|U_1| = \sqrt{(-1)^2 + (0)^2}$

$= 1$

$\therefore U_1$ is a unit vector.

Again, a non zero vector $U_2 = \begin{bmatrix} x \\ y \end{bmatrix}$.

$\therefore AA^T U_2 = \lambda U_2$

or, $\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 8x \\ 2y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

or, $8x = 2x$
$2y = 2y$

or, $x = 0$
$0 = 0$

Let, $y = 1$

$\therefore U_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ [when $\lambda = 2$].

$|U_2| = \sqrt{(0)^2 + (1)^2}$

$= 1$

$\therefore U_2$ is a unit vector.

$\therefore U = [U_1 \quad U_2]$

$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

Now, $A^TA = \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$

Now, $A^TA - \lambda I = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{bmatrix}$

So the characteristics equation of matrix A^TA is given by,

$|A^TA - \lambda I| = 0$

or, $\begin{vmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix} = 0$

or, $25 - 5\lambda - 5\lambda + \lambda^2 - 9 = 0$

or, $\lambda^2 - 10\lambda + 16 = 0$

or, $\lambda^2 - 2\lambda - 8\lambda + 16 = 0$

or, $\lambda(\lambda-2) - 8(\lambda-2) = 0$

or, $(\lambda-2)(\lambda-8) = 0$

or, $\lambda = 2, 8$

Now, For $\lambda = 8$.

Let, a non-zero vector, $V_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ where

$A^TAV_1 = \lambda V_1$

or, $\begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 8 \begin{bmatrix} x \\ y \end{bmatrix}$

or, $\begin{bmatrix} 5x - 3y \\ -3x + 5y \end{bmatrix} = \begin{bmatrix} 8x \\ 8y \end{bmatrix}$

$$or,\ 5x - 3y = 8x$$
$$-3x + 5y = 8y$$

$$or,\ 3x + 3y = 0$$
$$3x + 3y = 0$$

$$or,\ x + y = 0$$
$$x + y = 0$$

As the above two equations are same. so we can remove any one of them, we get,

$$x + y = 0$$

Here, number of equation 1 but number of Variable are 2. so $(2-1) = 1$ is free variable which is y.

Let, $y = 1$
$x = -1$

$\therefore\ V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\qquad \therefore\ |V_1| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2} \neq 1$

It is not a unit vector.

$\therefore$ unit vector, $V_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Again, for $\lambda = 2$.

Let a non-zero vector, $V_2 = \begin{bmatrix} x \\ y \end{bmatrix}$

$$A^TA\ V_2 = \lambda V_2$$

$$or,\ \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$or,\ \begin{bmatrix} 5x - 3y \\ -3x + 5y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

or, $5x - 3y = 2x$

$-3x + 5y = 2y$

or, $3x - 3y = 0$

$3x - 3y = 0$

or, $x - y = 0$

$x - y = 0$

As the two equations are same. so we can remove any one of them, we get,

$$x - y = 0$$

Here number of equation is 1. but number of variable are 2. so, we ~~we~~ have $(2-1) = 1$ free variable which is y.

Let, $y = 1$

$x = 1$

$\therefore V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore |V_2| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} \neq 1$

$\therefore$ It is not an unit vector.

$\therefore$ unit vector, $V_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\therefore V = [V_1 \ \ V_2]$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\therefore V^T = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

So, the required relation,

$$A = USV^T$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Justification:

Here, $A = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$

and, $USV^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} -2\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}$$

As, $A = USV^T$ so justified.