Oscillation

Simple Harmonic Motion:

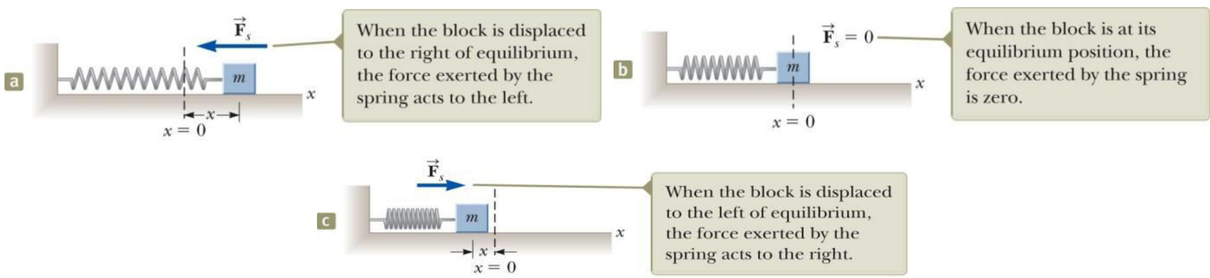
If the acceleration of a body be proportional to its displacement from its position of equilibrium, or any other fixed point in its path and be always directed towards it, the body is said to execute a simple harmonic motion.

Characteristics of simple harmonic motion:

Differential Equation of SHM:

Let us consider a body of mass m attached to an ideal spring of force constant K and free to move over a frictionless horizontal surface as an example of a simple harmonic motion. If the body moving back and forth about an equilibrium position through a potential that varies as

$$U_{(y)} = \frac{1}{2}\,ky^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(1)$$



Then the force acting on the particle is given by

$$F_{(y)} = -\frac{d}{dy}\left(U_y\right) = -\frac{d}{dy}\left(\frac{1}{2}\,ky^2\right)$$

$$\therefore\; F_{(y)} = -Ky \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(2)$$

From the Newton's second law of motion, we know that,

$$F = ma$$

$$\Rightarrow\; -Ky = m\,\frac{d^2y}{dt^2} \qquad \text{[From equation (2) F=-Ky]}$$

$$\Rightarrow m\,\frac{d^2y}{dt^2} + Ky = 0$$

$$\therefore \frac{d^2y}{dt^2} + \frac{K}{m}y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(3)$$

This is the equation of a SHM.

Let $\omega^2 = \frac{K}{m}$, then from equation (3), we have

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(4)$$

The general solution of equation (4) is

$$y = a\sin(\omega t + \alpha) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(5)$$

where, a is the amplitude.

Graphical Represenation of SHM:

Let P be a particle moving on the circumference of a circle of radius a. The foot of the perpendicular vibrates on the diameter YY'.

$$y = a \sin \omega t = a \sin 2\pi \frac{t}{T}$$

The displacement graph is a sine curve represented by ABCDE (Fig.-1).

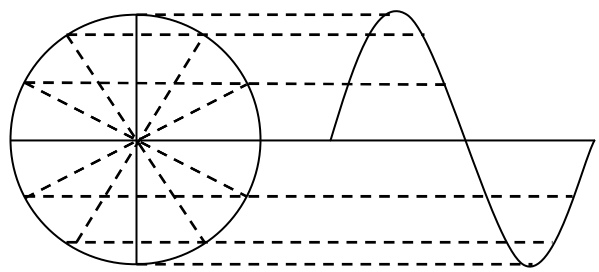


Fig.-1: displacement-time graph

The motion of the particle M is simple harmonic.

The velocity of a particle moving with simple harmonic motion is

$$v = \frac{dy}{dt} = +a\omega \cos \omega t$$

The velocity-time graph is shown in fig.-2. It is a cosine curve.

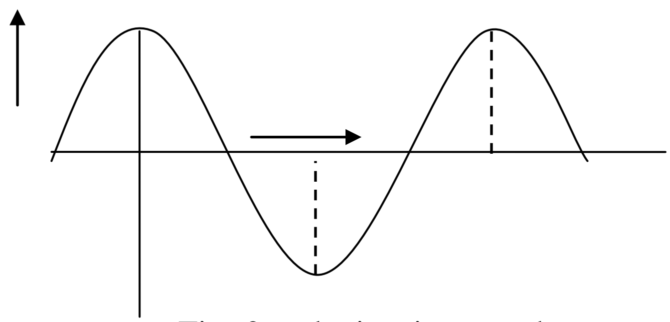


Fig.-2: velocity-time graph

The acceleration of a particle moving with simple harmonic motion is

$$\frac{d^2y}{dt^2} = -a\omega^2 \sin \omega t$$

The acceleration -time graph is shown in fig.-3. It is a negative sine curve.

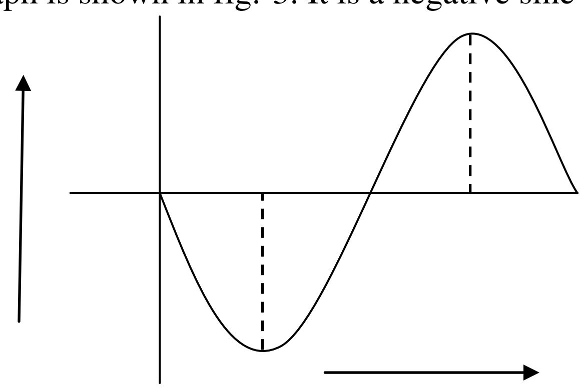


Fig.-3: Acceleration-time graph

Average Kinetic Energy of a Vibrating Particle:

The displacement of a vibrating particle is given by

$$y = a\sin(\omega t + \alpha)$$
$$v = \frac{dy}{dt} = a\omega\cos(\omega t + \alpha).$$

If m is the mass of the vibrating particle, the kinetic energy at any instant

$$= \frac{1}{2}mv^2 = \frac{1}{2}m.a^2\omega^2\cos^2(\omega t + \alpha).$$

The average kinetic energy of the Particle in one complete vibration

$$= \frac{1}{T}\int_0^T \frac{1}{2}ma^2\omega^2\cos^2(\omega t + \alpha)\,dt$$
$$= \frac{1}{T}\frac{ma^2\omega^2}{4}\int_0^T 2\cos^2(\omega t + \alpha)\,dt$$
$$= \frac{ma^2\omega^2}{4T}\int_0^T [1 + \cos 2\,(\omega t + \alpha)]\,dt$$
$$= \frac{ma^2\omega^2}{4T}\left[\int_0^T dt + \int_0^T \cos 2\,(\omega t + \alpha)dt\right]$$

But, $\int_0^T \cos 2\,(\omega t + \alpha)dt = 0$

$\therefore$ Average K.E. $= \frac{ma^2\omega^2}{4T}\,.T + 0$

$$= \frac{ma^2\omega^2}{4} = \frac{ma^2(4\pi^2n^2)}{4} = \pi^2ma^2n^2$$

where m is the mass of the vibrating particle, a is the amplitude of vibration and n is the frequency of vibration. Also, the average kinetic energy of a vibrating particle is directly proportional to the square of the amplitude.

Total Energy of a Vibrating Particle

$$y = a\sin(\omega t + \alpha)$$
$$\sin(\omega t + \alpha) = \frac{y}{a}$$
$$\cos(\omega t + \alpha) = \sqrt{1 - \frac{y^2}{a^2}} = \sqrt{\frac{a^2 - y^2}{a^2}}$$
$$= \frac{\sqrt{a^2 - y^2}}{a}$$

Velocity $v = a\omega\cos(\omega t + \alpha) = a\omega\frac{\sqrt{a^2-y^2}}{a} = \omega\sqrt{(a^2 - y^2)}$

$\therefore$ The kinetic energy of the Particle at the instant the displacement is y,

$$= \frac{1}{2}mv^2$$
$$= \frac{1}{2}m.\omega^2(a^2 - y^2)$$

Potential energy of the vibrating particle is the amount of work done in overcoming the force through a distance y.

Acceleration $= -\omega^2 y$

Force $= -m\omega^2 y$

(The –ve sign shows that the direction of the acceleration and force are opposite to the direction of the motion of the vibrating particle.)

$$\therefore \quad \text{P.E.} = \int_0^y m.\omega^2 y.dy$$

$$= m\omega^2.\frac{y^2}{2} = \frac{1}{2}\, m\omega^2 y^2.$$

Total energy of the particle at the instant the displacement is y

$$= \text{K.E.} + \text{P.E.}$$

$$= \frac{1}{2} m.\omega^2(a^2 - y^2) + \frac{1}{2}\, m\omega^2 y^2$$

$$= \frac{1}{2}\, m\omega^2.a^2$$

$$= \frac{1}{2}\, m(2\pi n)^2.a^2 = \mathbf{2\pi^2 m a^2 n^2}.$$

As the average kinetic energy of the vibrating particle$= 2\pi^2 ma^2n^2$, the average potential energy$= \pi^2 ma^2 n^2$. The total energy at any instant is a constant.

Free Vibrations:

When the bob of a simple pendulum (in vacuum) is displaced from its mean position and left, it executes simple harmonic motion. The time period of oscillation depends only on the length of the pendulum and the acceleration due to gravity at the place. The pendulum will continue to oscillate with the same time period and amplitude for any length of time. In such cass there is no loss of energy by friction or otherwise. In all similar cases, the vibrations will be undamped free vibrations. The amplitude of swing remains constant.

Damped Oscillation:

In actual practice, when the pendulum vibrates in air medium, there are frictional forces and consequently energy is dissipated in each vibration. The amplitude of swing decreases continuously with time and finally the oscillations die out. Such vibrations are called free damped vibrations.

In a word, in the presence of frictional or viscous forces, the amplitude of vibration decreases continuously with time and finally the oscillations die out. Such vibrations are called free damped vibrations.

Forced Oscillation:

The time period of a body executing simple harmonic motion depends on the dimensions of the body and its elastic properties. The vibrations of such a body die out with time due to dissipation of energy. If some external periodic force is constantly applied on the body, the body continues to oscillate under the influence of such external forces. Such vibrations of the body are called forced vibrations.

Stationary/standing waves:

Stationary/standing waves.
Let us consider two progressive waves A and B with same amplitude, same velocity and same wavelength.
The wave A is moving with velocity V along the direction of X-axis i.e., from left to right.The wave B is moving with same velocity along the opposite direction i.e., from right to left.Then, the equation of these waves will be,

$$y_1 = a\, sin\frac{2\pi}{\lambda}(vt - x) \ldots\ldots\ldots\ldots\ldots\ldots \;(1)$$

$$y_2 = a\, sin\frac{2\pi}{\lambda}(vt + x) \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where, a, λ *and* v are the amplitude, wavelength and velocity of the waves respectively.
Due to the superposition of the two waves, the resultant displacement of the particle can be written as

$$y = y_1 + y_2$$
$$y = \; a\, sin\frac{2\pi}{\lambda}(vt - x) + a\, sin\frac{2\pi}{\lambda}(vt + x)$$
$$y = a[sin\frac{2\pi}{\lambda}(vt + x) + sin\frac{2\pi}{\lambda}(vt - x)]$$
$$= 2a\, sin\frac{2\pi}{\lambda}.\frac{2vt}{2}.cos\frac{2\pi}{\lambda}.\frac{2x}{2}$$
$$= [2a\, cos\frac{2\pi x}{\lambda}.sin\frac{2\pi vt}{2}$$
$$\therefore y = A \sin\frac{2\pi vt}{2} \ldots\ldots\ldots\ldots.(3),$$

where, $A = 2a\, cos\frac{2\pi x}{\lambda}$
Equation (3) is the required expression for standing waves.

Nodes: From the expression of standing waves, the amplitude is,

$$A = 2a\,cos\frac{2\pi x}{\lambda}$$

This amplitude depends on the position, x of the particle. Therefore, for different positions of different particle, the values of A will be different .At points, where A=0 i.e., amplitude will be zero, then nodes will be created at that point.

Now, the condition for $2a\,cos\frac{2\pi x}{\lambda} = 0, will\ be$

$$cos\frac{2\pi x}{\lambda} = 0$$

$$\Rightarrow \frac{2\pi}{\lambda}x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots\ldots etc.$$

$$\Rightarrow x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots\ldots. etc.$$

These points are the nodes.The distance between two successive nodes will be = $\left(\frac{3\lambda}{4} - \frac{\lambda}{4}\right) = \frac{\lambda}{2}$

Therefore, the nodes are equidistant and seperated by $\frac{\lambda}{2}$.

Antinodes: At which points, the resultant amplitude, A is maximum i.e., A=±2a, antinodes will be produced at that point.

Now, the condition for $2a\,cos\frac{2\pi x}{\lambda} = \pm 2\text{a}, will\ be$

$$cos\frac{2\pi x}{\lambda} = \pm 1$$

$$\Rightarrow \frac{2\pi}{\lambda}x = 0, \pi, 2\pi \ldots\ldots etc.$$

$$\Rightarrow x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2} \ldots\ldots. \frac{n\lambda}{2}. (n = 0,1,2,3, \ldots.)$$

These points are the antinodes.The distance between two successive antinodes will be = $\left(\frac{2\lambda}{2} - \frac{\lambda}{2}\right) = \frac{\lambda}{2}$.

Therefore, the antinodes are equidistant and seperated by $\frac{\lambda}{2}$.