SPECTRUM ANALYSIS

OR

FOURIER ANALYSIS

By

Dr. A.N.M. Rezaul Karim
B.Sc. (Honors), M.Sc. in Mathematics (CU)
DCSA (BOU), PGD in ICT (BUET), Ph.D. (IU)

Associate Professor
Department of Computer Science & Engineering
International Islamic University Chittagong

Sunday, October 2, 2022

Signal: A signal is defined as any physical phenomenon that carries information and varies with time

What are the parameters of a signal?
The signal parameters – amplitude, frequency and phase

Depending on the nature of signal, it is categorized into several classes based on some criterion. Some of the classifications include continuous v/s discrete, **periodic v/s aperiodic,** energy v/s power, deterministic v/s random, stationary v/s non-stationary and so on**.**

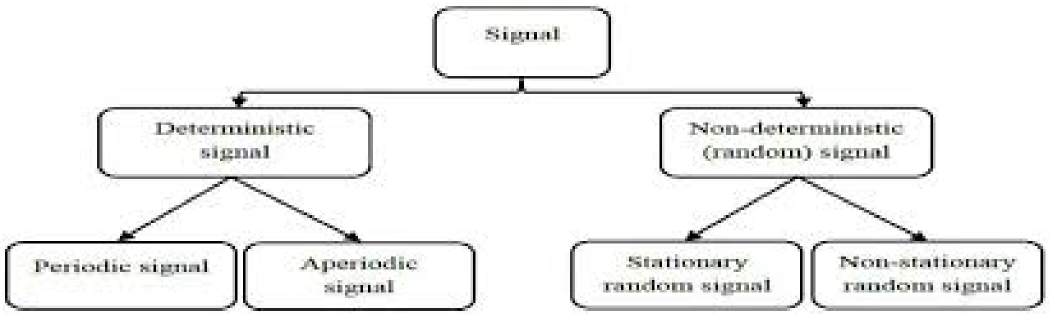


Figure 01:

The first natural division of all signals is into either stationary or non-stationary categories. Stationary signals are constant in their statistical parameters over time. If you look at a stationary signal for a few moments and then wait an hour and look at it again, it would look essentially the same, i.e. its overall level would be about the same and its amplitude distribution and standard deviation would be about the same. Rotating machinery generally produces stationary vibration signals.
A non-stationary signal is one whose frequency changes over time; e.g. human speech where frequencies vary over time depending on what words or syllables you are pronouncing

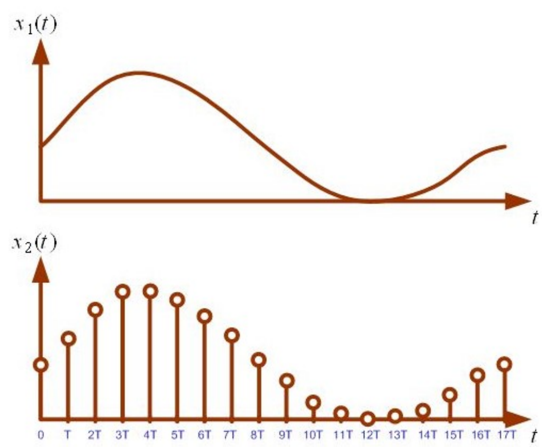


Figure 02: Continuous Vs Discrete Signals

Stationary signals are further divided into deterministic and random signals. Random signals are unpredictable in their frequency content and their amplitude level, but they still have relatively uniform statistical characteristics over time. Examples of random signals are rain falling on a roof, jet engine noise, turbulence in pump flow patterns and cavitations**.**

Stationary Signal: In stationary signal, the time interval is longer and the level is lower. Sine wave, triangular wave, square wave and so on are stationary in nature. All periodic signals are stationary signals. The time period for the stationary signal remains constant at all times.

Non-Stationary Signal: In non-stationary signal, the time interval is short and the level is more. The time period for a non-stationary signal varies with time and is not constant.

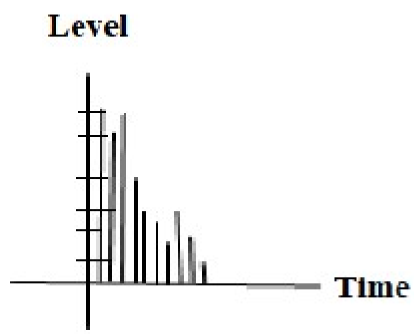


Fig: Non-Stationary Signal

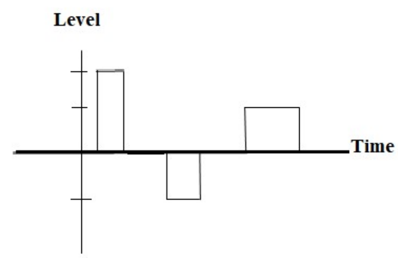


Fig: Stationary Signal

DC signal: In electronic circuits things happen. **Voltage/time**, *V*/*t*, graphs provide a useful method of describing the changes which take place.

The diagram below shows the *V*/*t* graph, which represents a DC signal
Voltage = **v,** Time = **t**

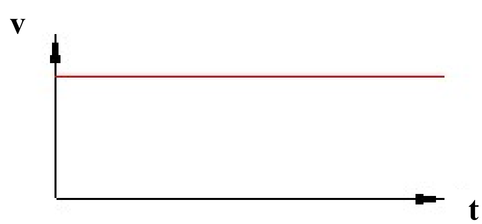


Figure 03

Direct current (DC) is produced by sources such as batteries, thermocouples, solar cells etc

Problem 01: Periodic Signal

What is periodic signal?
A signal which is repeating itself is a periodic signal

Periodic: Signal Pattern repeats over time

Example your voice is not periodic unless you say the same word over and over again the exact same way with the exact spacing!

Example **01**

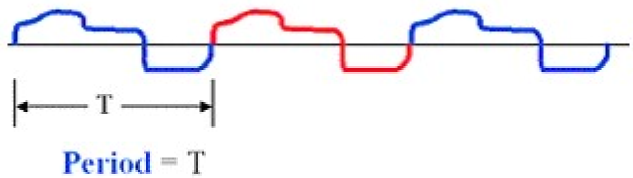


Figure 04: A periodic signal with period T

Example **02**

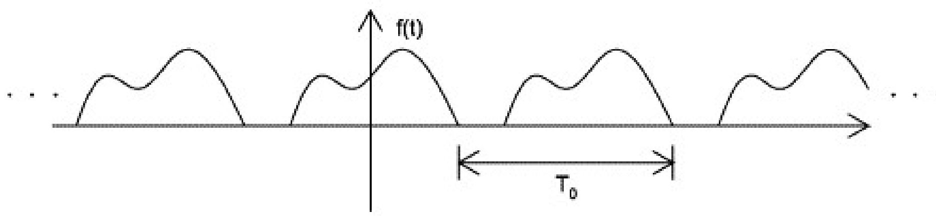


Figure 05: A periodic signal with period T_0

Example **03**

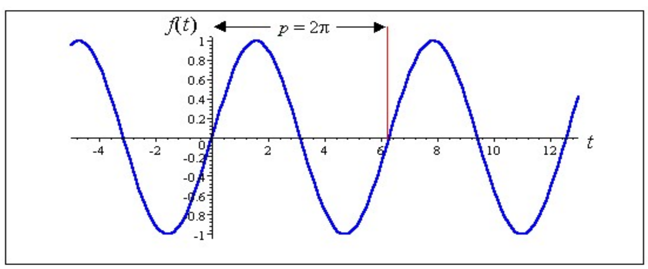


Figure 06: $\mathbf{f(t) = \sin t}$; Period $\mathbf{T = 2\pi}$

Example **04**

$\mathbf{f(t) = 3t;} \qquad \mathbf{-1 \le t < 1}.$

$\mathbf{f(t) = f(t+2)}$ [This indicates it is periodic with period 2.]

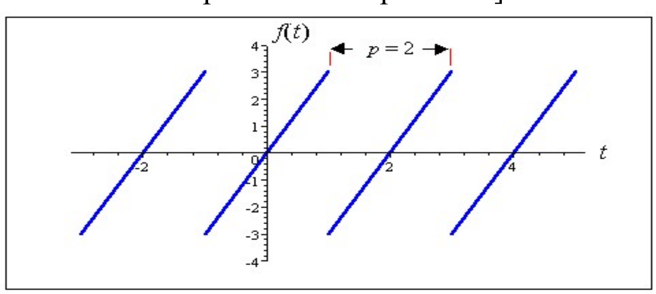


Figure 07: $\mathbf{f(t) = 3t;}$ Period $\mathbf{T = 2}$

Example **05**

$\mathbf{f(t) = t^2\ ;} \qquad \mathbf{0 \le t < 2}$

$\mathbf{f(t) = f(t+2)}$ [Indicating it is periodic with period 2.]

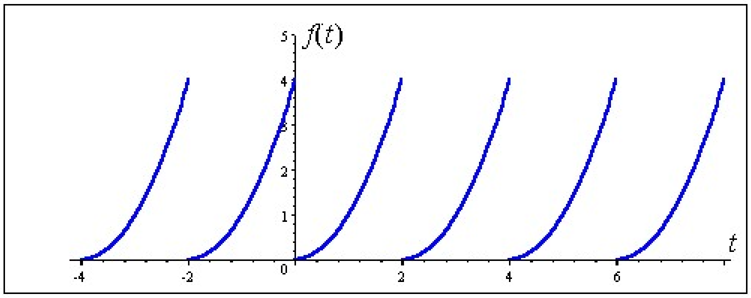


Figure 08: $\mathbf{f(t) = t^2}$ Period $\mathbf{T = 2}$

Example **06**

$\mathbf{f(t) = -3;} \qquad \mathbf{-1 \le t < 1}$

$\qquad \mathbf{= 3\,;} \qquad \mathbf{1 \le t < 3}$

$\mathbf{f(t) = f(t+4)}$ [The period is 4.]

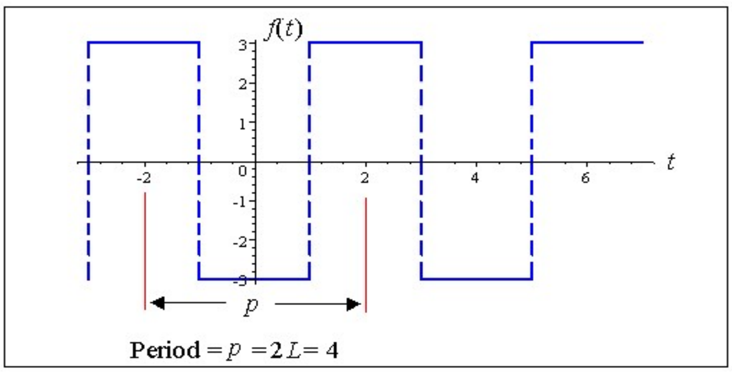


Figure 09; Period **T = 4**

Example **07**

$$f(t) = \begin{cases} -1 & \text{if} \quad 0 \le t < \frac{\pi}{2} \\ 1 & \text{if} \quad \frac{\pi}{2} \le t < \frac{3\pi}{2} \\ -1 & \quad \frac{3\pi}{2} \le t < 2\pi \end{cases}$$

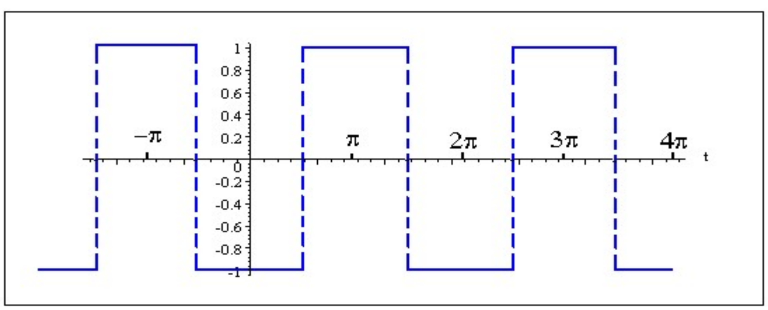


Figure 10; Period $T = 2\pi$

Example **08**

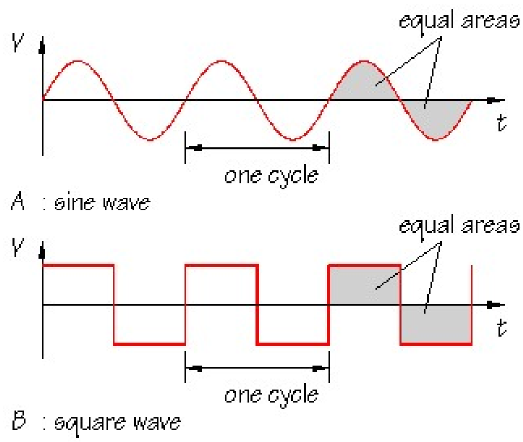


Figure 11: different shape of periodic signal

Periodic Signal Characteristics:

i. Amplitude (A): Signal Value, measured in volts
ii. Frequency (f): repletion rate, cycles per second or Hertz
iii. Period (T): Amount of time it takes for one repetition
iv. Phase (φ): Relative Position in time, measured in degrees

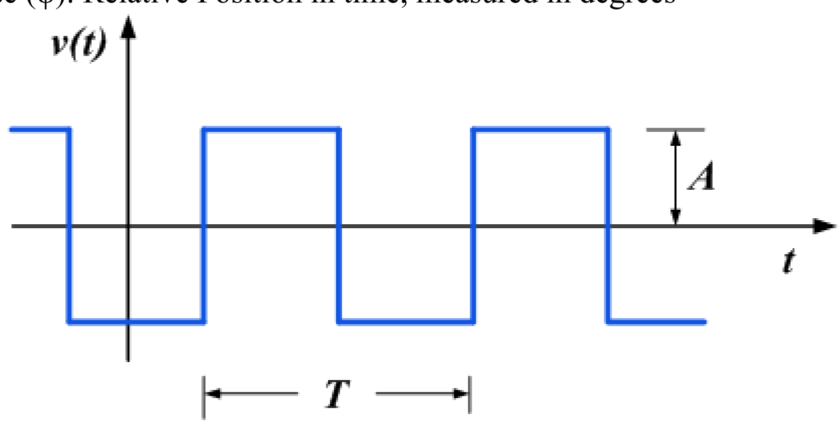


Figure 12

3. Any periodic function can be written as a sum of a symmetric and anti-symmetric function by writing

$$f(t) = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2}. \quad (2.9)$$

Problem 02:

Aperiodic Signal: Not Periodic

Example **09:**

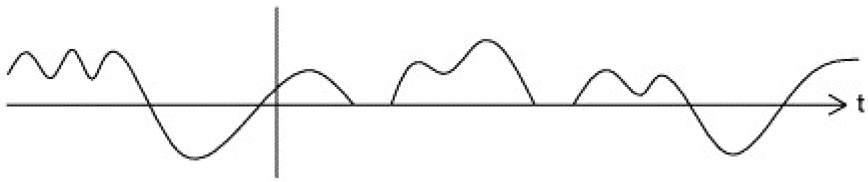


Figure 13: Non periodic signal

Cycles: A set of events or actions that happen again and again in the same order.

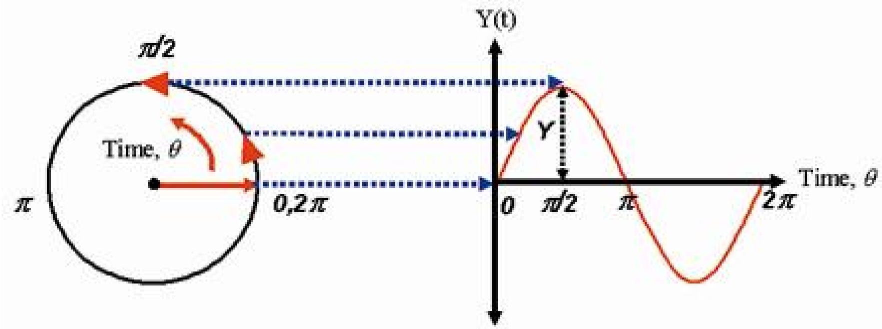


Figure 14

If we were to model the rotation of the engine graphically in one dimension, we could choose a point on the engine and watch it as a rotated. The motion that point produced over time as the engine rotated would look like a sinusoid. The horizontal axis would be time, and the vertical axis would be the vertical position of the point as a function of time.

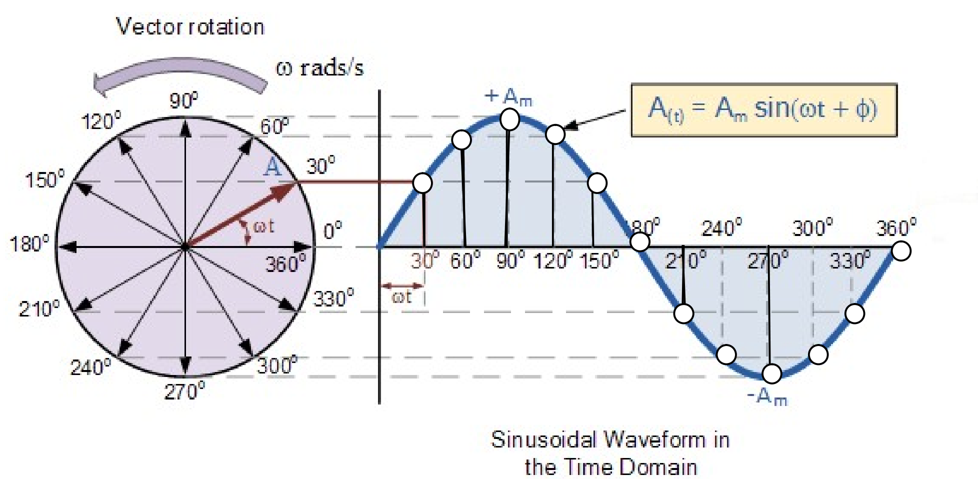


Figure 15

Frequency f: **<u>This is the number of cycles completed per second</u>**. The measurement unit for frequency is the **hertz**, **Hz**. 1 Hz = 1 cycle per second. $\boldsymbol{f = \frac{1}{T}}$

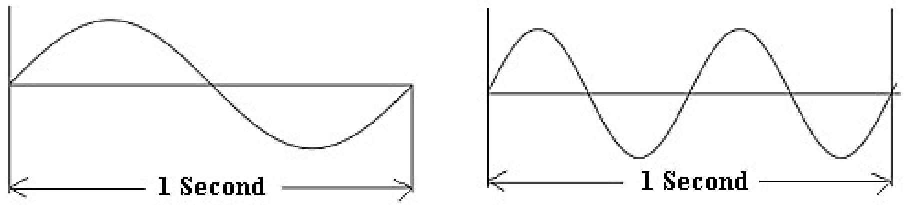


Fig: 1 Cycle/ Sec = 1 Hz   Fig: 2 Cycle/Sec = 2 Hz

Figure 16: Sine Wave with different frequencies

Period: T: The period is the time taken for one complete cycle of a repeating waveform

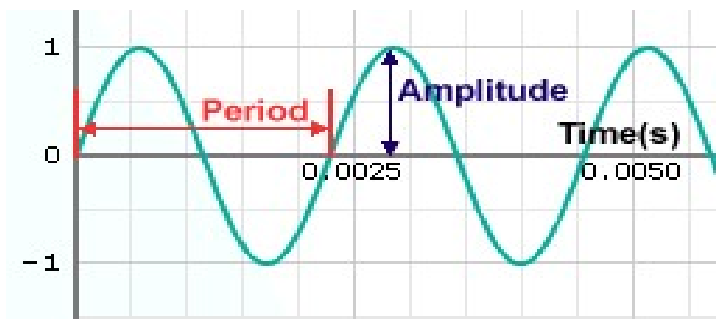


Figure 17

Problem 03: **The effect of frequency and phase**

We normally think of a sine wave as the result of a vector that rotates in a clockwise direction. However, a given sine wave can also be visualized as the sum of two vectors, one rotating clockwise, the other counterclockwise. The vectors are equal in length; each is half of the total. The clockwise rotating vector may be regarded as a positive frequency. The counterclockwise rotating vector is a negative frequency.

The expressions of a sinusoidal signal can be

$\mathbf{f(t) = A\sin(\omega t + \phi)}$ -------------------------------(i)

Here,

$\mathbf{A}$ = the Amplitude; ω = the frequency, ϕ = the phase shift and $\mathbf{t}$ is the time in seconds

In this function, $\mathbf{t}$ is a variable. The other quantities are in general fixed, and each of them influences the shape of the graph of this function as we change its three parameters called Note that ϕ (Relative Position in time), $\omega = 2\pi f$, $\mathbf{f}$ is the frequency in hertz (Hz), and $\mathbf{t}$ is time in seconds

Amplitude is the height of a wave which generates intensity. Example: the brightness of light, loudness of sound, power of electricity, etc.

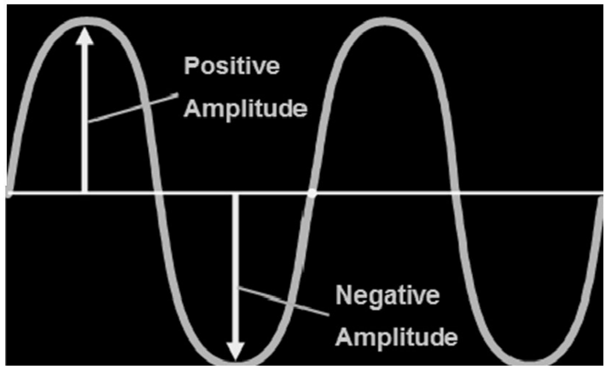


Figure 18

Example 10: $\mathbf{f(t) = A\sin(\omega t + \phi)}$

If $\mathbf{A = 1;\ \phi = 0}$, then draw the graph of $\mathbf{f(t) = ?}$

Then $\mathbf{f(t) = A\sin(\omega t + \phi)}$

$\mathbf{f(t) = 1.\sin(\omega t + 0)}$

$\Rightarrow \mathbf{f(t) = \sin\omega t}$

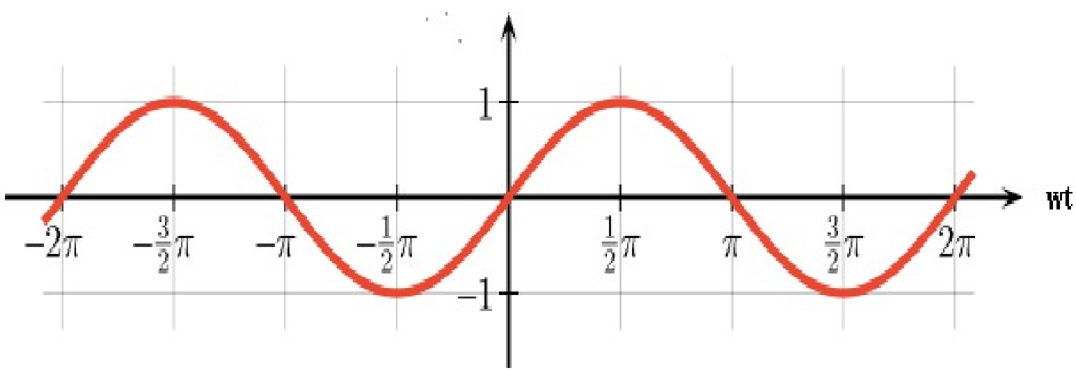


Figure 19: $f(t) = A * \sin\omega t$

Example 11: $\mathbf{f(t) = A\sin(\omega t + \phi)}$

If $\mathbf{A = 1;\ \phi = 90^0}$, then draw the graph of $\mathbf{f(t) = ?}$

$\mathbf{f(t) = A\sin(\omega t + \phi)}$

$\mathbf{f(t) = 1.\sin(\omega t + 90^0)}$

$\mathbf{f(t) = \sin(\omega t + 90^0)}$

$\mathbf{f(t) = \sin(90^0 + \omega t)}$

$\mathbf{f(t) = \sin(1.90^0 + \omega t)}$

[Odd Number Multiplication with 90]

$\mathbf{f(t) = \cos\omega t}$

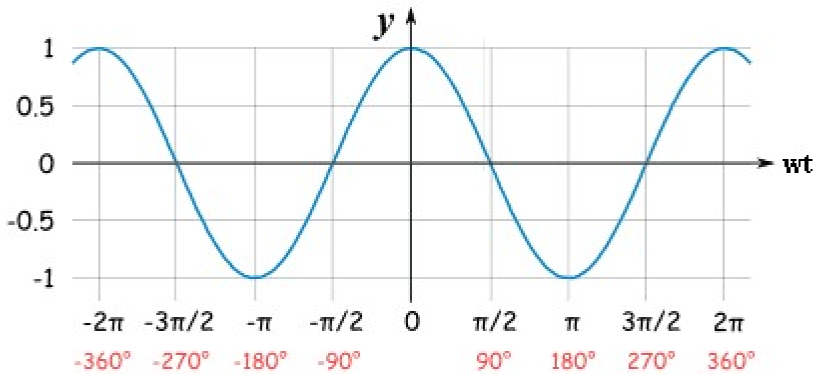


Figure 20: $f(t) = A * \cos \omega t$

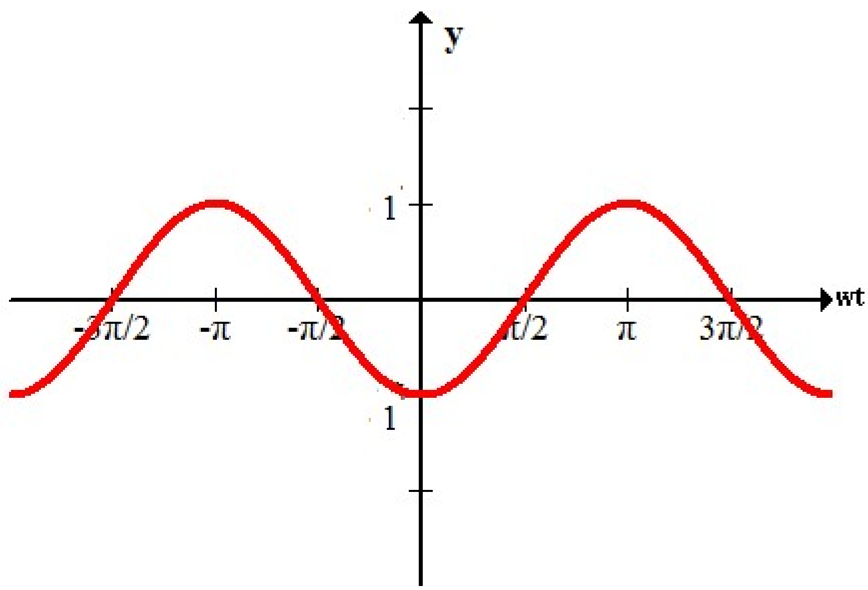


Figure 21: $\mathbf{f(t) = -A * \cos \omega t}$

$y = \sin\theta$

$\mathbf{y = \sin(\theta - \frac{\pi}{2})}$

$\mathbf{y = \sin\{-(\frac{\pi}{2} - \theta)\}}$

$\mathbf{y = -\sin(\frac{\pi}{2} - \theta)}$ $\mathbf{[\sin(-\theta) = -\sin\theta]}$

$\mathbf{y = -\sin(1.\frac{\pi}{2} - \theta)}$

$\mathbf{y = -\sin(1.90^0 - \theta)}$

$\mathbf{y = -\cos\theta}$

Example 12:

We have,

$$f(t) = A\sin(\omega t + \phi) \text{---(i)}$$

$$\Rightarrow f(t) = A[\sin\omega t\cos\phi + \cos\omega t\sin\phi]\ [\because \sin(x+y) = \sin x\cos y + \cos x\sin y]$$

$$\Rightarrow f(t) = A\sin\omega t\cos\phi + A\cos\omega t\sin\phi$$

$$\Rightarrow f(t) = A\cos\omega t\sin\phi + A\sin\omega t\cos\phi$$

$$\Rightarrow f(t) = A\sin\phi\cos\omega t + A\cos\phi\sin\omega t$$

$$\Rightarrow f(t) = a\cos\omega t + b\sin\omega t\ [\text{Let, } A\sin\phi = a,\ A\cos\phi = b] \text{--------------------(ii)}$$

***Problem 04: Phase difference or phase shift* between a Sine wave and a Cosine wave**

Sine and cosine signals of the same frequency have only a phase difference of $\pi/2$

Phase describes the position of the waveform relative to time zero.

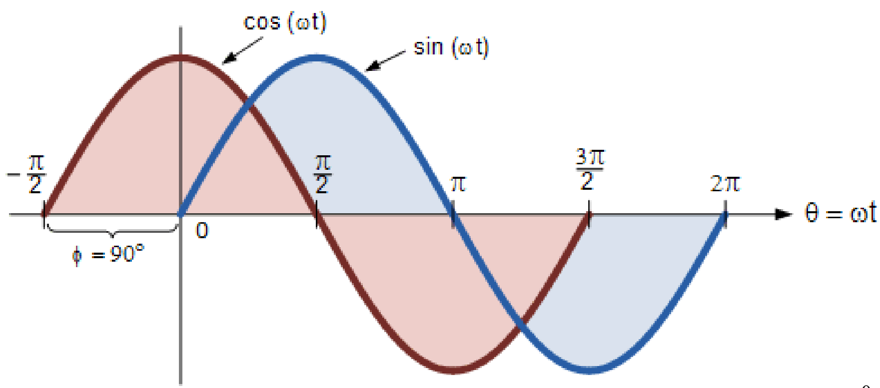


Figure 22: sine wave and cosine wave with Phase difference is 90^0

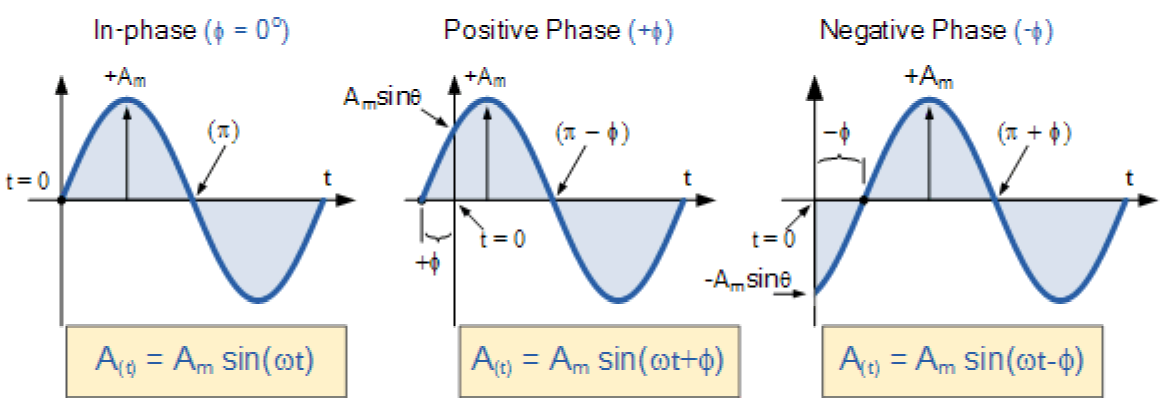


Figure 23

Problem 05: Even and Odd function

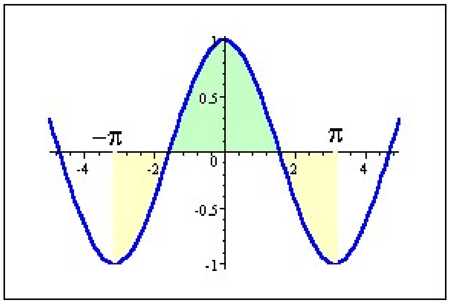


Figure 24: An even signal

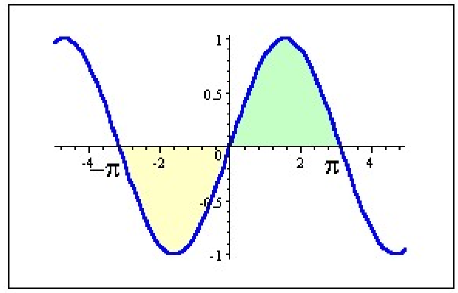


Figure 25: An odd signal

Example 13:

Absolute Value Function $\mathbf{f(x) = |x|}$

That is, $\mathbf{f(x) = -x \quad ; x < 0}$

$\mathbf{\qquad\qquad\quad x \quad ; x \geq 0}$

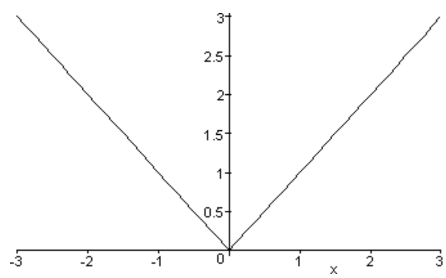


Figure 26

This is the absolute value function. It is really a split function defined in two pieces

Example 14:

$\mathbf{f(t) = -t \qquad ; -\pi \leq t < 0}$

$\mathbf{\qquad\quad t \qquad ; 0 \leq t < \pi}$

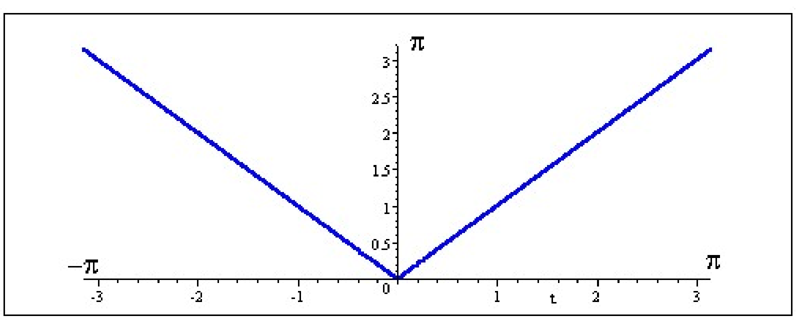


Figure 27

Example 15:

Making waves
Sine waves can be mixed with DC signals, or with other sine waves to produce new waveforms. Here is one example of a complex waveform**:**

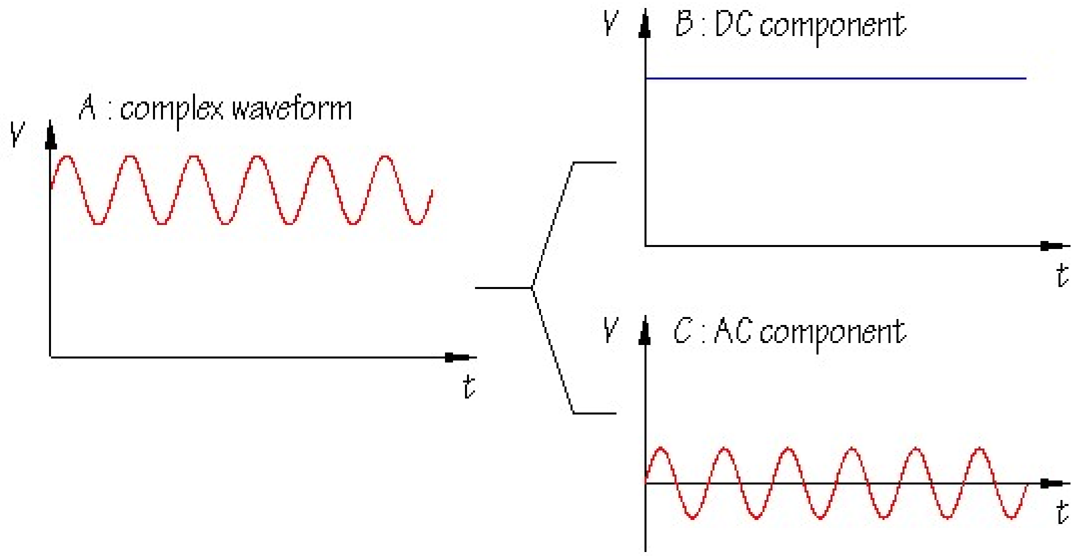


Figure 28

Problem 06: Sum of waves

- Complex wave forms can be reproduced with a sum of different amplitude, frequency sine waves
- Any waveform can be turned into a sum of different amplitude, frequency sine waves

Example 16:

The wave is the 'combination' or 'sum' of the two or more waves

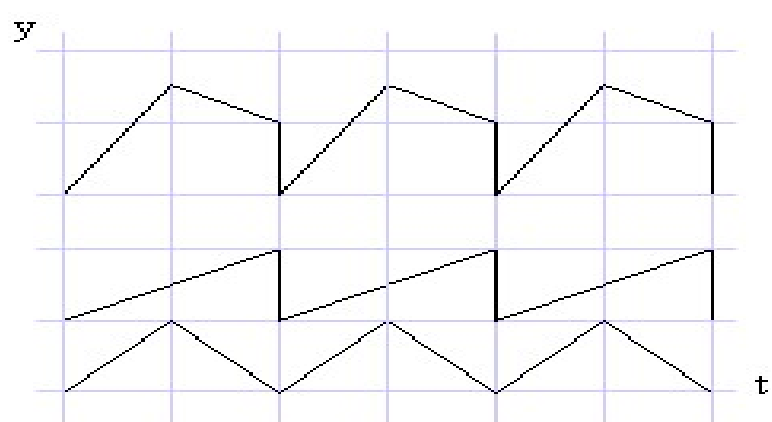


Figure 29

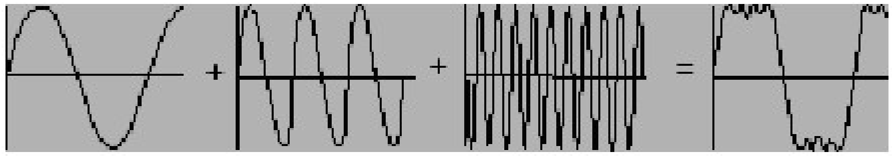


Figure 30

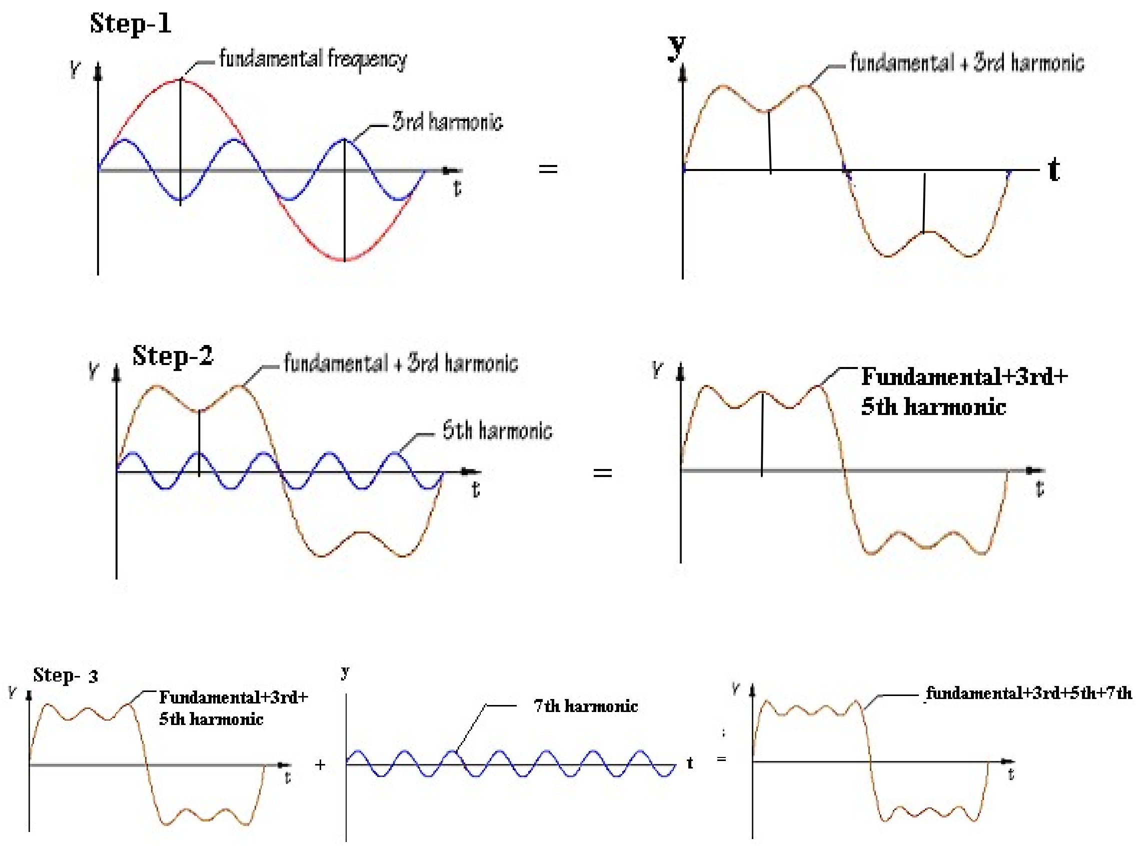


Figure 31: ***The different amplitude, different frequency of sine waves adds up to produce a complex wave.***

Complex Wave

A complex waveform may be treated as being composed of a number of sinusoid waveforms. These sinusoids are of various phases, frequencies and amplitudes. The description of the magnitude, phase and frequency of these various waves is known as the spectrum of the signal, by analogy with the spectrum of light.

Harmonics: The angular frequencies of the Sinusoids above are all integer multiples of ω. They are called the *harmonics* of ω, which in turn is called the *fundamental*. In terms of pitch, the ω , 2ω ,.............harmonics . These frequencies are referred to as ***harmonics*** of the fundamental frequency

Harmonic analysis: The computation and study of Fourier series is known as harmonic analysis.

Spectrum Analysis **or** ***Fourier analysis*** is the process of analyzing some time-domain waveform to find its spectrum.

Example 17:

A Fourier series takes a signal and decomposes it into a sum of sines and cosines of different frequencies. Also note that the Fourier representation is formally periodic, the beginning of the cycle will always equal the end.

We will see functions like the following, which approximates a saw-tooth signal

$$\underbrace{f(t)}_{Complex\ wave} = \underbrace{1}_{DC\ value} + \underbrace{2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t + \ldots\ldots\ldots\ldots}_{AC\ value} \text{ -----------(i)}$$

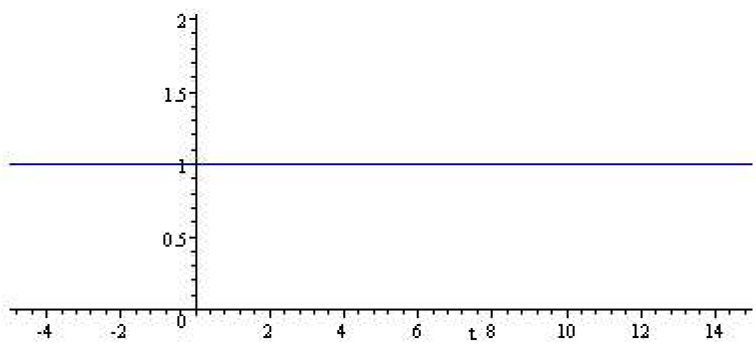


Figure 32: $f(t) = 1$

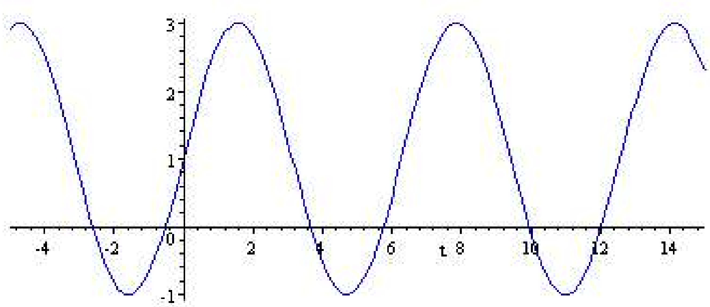


Figure 33: $f(t) = 1 + 2\sin t$

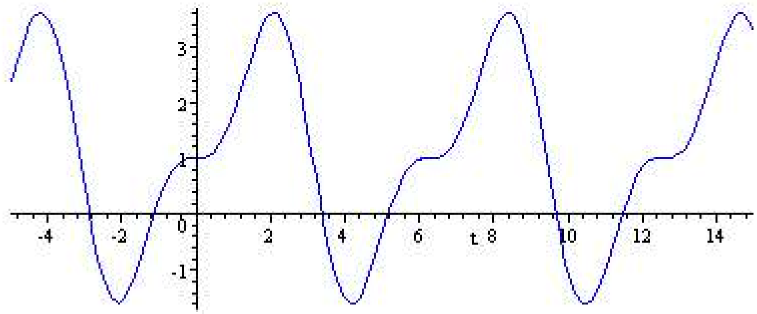


Figure 34: $f(t) = 1 + 2\sin t - \sin 2t$

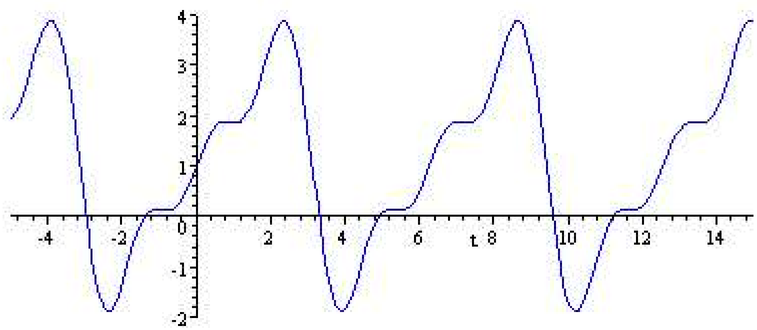


Figure 35: $f(t) = 1 + 2\sin t - \sin 2t + \dfrac{2}{3}\sin 3t$

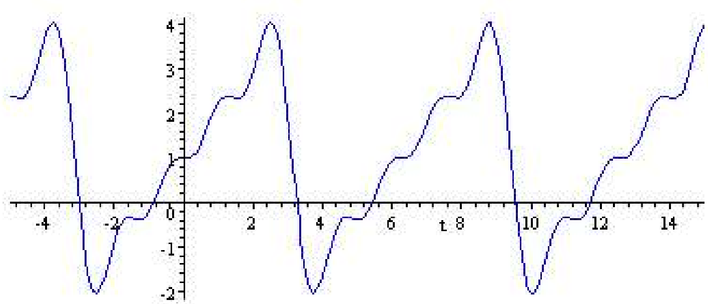


Figure 36: $f(t) = 1 + 2\sin t - \sin 2t + \dfrac{2}{3}\sin 3t - \dfrac{1}{2}\sin 4t$

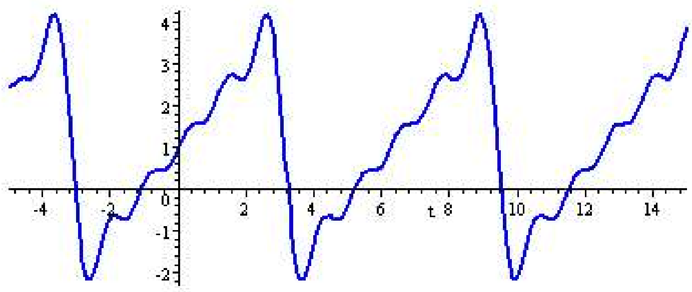


Figure 37: $\mathbf{f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t}$

In this way, we say that the infinite Fourier series converge to the saw tooth curve. We can take any function of time and describe it as a sum of sine waves each with different amplitudes and frequencies

Example 18:
A sound can be represented by a mathematical function, with time as the free variable. When a function represents a sound, it is often referred to as a continuous signal. Sounds as a sum of different amplitude signals each with a different frequency. However, everyday sounds are complex sounds, which are made up of many tonal frequency components.

<u>Problem 07: Physical Significance of Fourier series</u>

<u>Fourier series expresses a periodic function as a sum of sines and cosines of different frequencies.</u>

Any electromagnetic signal can be shown to consist of a collection of periodic analog signals (Sine waves) at different amplitudes, frequencies & phases.

Any composite/complex signal can be represented as a combination of simple sine waves with different frequencies, phases and amplitudes.

Speech of Fourier:

[একটি কমপ্লেক্স ওয়েবকে ভাঙলে ডিসি ওয়েভ এবং এসি ওয়েভ পাওয়া যায়। এসি ওয়েভগুলি ডিফারেন্ট ফ্রিকুয়েন্সি এবং ডিফারেন্ট এম্পিপলিচিউড এর সাইন ওয়েভ। বিপরীতক্রমে বলতে পারি ডিসি ওয়েভ এবং ডিফারেন্ট ফ্রিকুয়েন্সি এবং ডিফারেন্ট এম্পিপলিচিউড এর অনেকগুলি সাইন ওয়েভ যোগ করলে একটি কমপ্লেক্স ওয়েব পাওয়া যায়।]

One of the principles of Fourier analysis is that any imaginable waveform can be constructed out of a carefully chosen set of sine wave components, and conversely, any complex periodic signal can be broken down into a series of sine wave components for analysis. And most important, the described tasks are reciprocal operations — in the same way that integration and differentiation are reciprocal operations in Calculus, encoding and decoding signals are reciprocal operations in Fourier analysis.

Applications in signal processing

When processing signals, such as audio, radio waves, light waves, seismic waves, and even images, Fourier analysis can isolate individual components of a compound waveform, concentrating them for easier detection and/or removal.

Problem 08: Mathematical Expression of Fourier series

A Fourier (In 1822, Joseph Fourier, a French mathematician) series expresses any function as a sum of sine and cosine waves of different frequencies. For example, if you're talking about sound, you can take a wave of any shape you like and extract the frequency components of that wave by expressing it as a Fourier series.

Fourier series: Representation of Periodic Signals. A periodic signal can be described by Fourier decomposition as a Fourier series, i.e. as a sum of sinusoidal and co sinusoidal oscillations. By reversing this procedure a periodic signal can be generated by superimposing sinusoidal and co sinusoidal waves. ***The Fourier series tells you the amplitude and frequency of the sines and cosines that you should add up to recreate your original function***.

Mathematically, a periodic signal $f(t)$ may be represented by a Fourier Series as follows:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)) \text{---------------------------------------(i)}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)) \text{-------------------------------------(ii)}$$

$$[let, \omega = \omega_0]$$

Where the variables are:

- $\frac{a_0}{2} \rightarrow$ The *average* (DC) value of the signal:
- The a_n holds the amplitudes of the cosine wave
- The b_n holds the amplitudes of the sine wave
- $\mathbf{n} \rightarrow$ The *harmonic number*: 1=fundamental, 2=2nd harmonic, etc
- $a_n \rightarrow$ Peak value of the magnitude of the ***nth*** cosine harmonic
- $b_n \rightarrow$ Peak value of the magnitude of the ***nth*** sine harmonic
- $\omega_0 \rightarrow$ *Fundamental* frequency, $[\because \omega_0 = \frac{2\pi}{T}]$
- $\mathbf{T} \rightarrow$ This waveform repeats every T seconds where, T is the Period of $f(t)$

Putting $\omega_0 = \frac{2\pi}{T}$ in (ii),

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\frac{2\pi}{T}t) + b_n \sin(n\frac{2\pi}{T}t)) \; [\because \omega_0 = \frac{2\pi}{T}] \text{-----------------------(iii)}$$

This is called Fourier series of a periodic function f (t).

.

<u>Each periodic wave could be represented by a Fourier series</u>. It's a summation of sinusoids with different amplitudes, frequencies and phases, ***Where*** a_0, a_n, b_n ***are called Fourier coefficients.***

Let the function $f(t)$ be periodic with period $T = 2L$ in (iii)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(n\frac{2\pi}{2L}t\right) + b_n \sin\left(n\frac{2\pi}{2L}t\right)\right) \text{ [Putting } T = 2L \text{ in (iii)]}$$

$$\boxed{\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{a_0}{2}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi}{L}t\right) + b_n \sin\left(\frac{n\pi}{L}t\right)\right)}_{\text{AC value}} \text{----------------------(iv)}}$$

This is called Fourier series of a periodic function $f(t)$ ***for period*** $T = 2L$; $\frac{a_0}{2}$ ***is the mean value, sometimes referred to as the dc level. For an electrical signal it represents DC component***

Example

If you ever watched the blinking lights on a stereo equalizer then you have seen Fourier analysis at work. The lights represent whether the music contains lots of bass or treble. Jean Baptiste Joseph Fourier, a French Mathematician who once served as a scientific adviser to Napoleon, is credited with the discovery of the results that now bear his name

Applications of Fourier series

To recapitulate, Fourier series simplify the analysis of periodic, real valued functions. Specifically, it can break up a periodic function into an infinite series of sine and cosine waves. Consider the very common differential equation given by:

x00(t) + ax0(t) + b = f(t) (23)

This equation describes the motion of a damped harmonic oscillator that is driven by some function f(t). It can be used to model an extensive variety of physical phenomena, such as a driven mass on a spring, an analog circuit with a capacitor, resistor, and inductor, or a string vibrated at some frequency. There are two parts to the solution of equation (25). The first part is a transient that fades away (generally) fairly quickly. When the transient is gone, what remains is the steady state solution. This is what we will concern ourselves with.

If f(t) is a sinusoid, then the solution is also a sinusoid which is not very difficult to find. The problem is that the driver is generally not a simple sinusoid, but some other periodic function. In electronics, for example, a common driving voltage function is the square wave s(t), a periodic function (whose period we shall say is 2p) such that s(t) = 0 for -p = t < 0 and s(t) = 1 for 0 = t < p.

The physical property of oscillating systems that makes Fourier Analysis useful is the property of superposition in other words, suppose the driving force f1(t), along with some initial conditions, produces some steady state solution x1(t), and that another driving force, f2(t) produces the steady state solution x2(t). Then the driving force f3(t)= f1(t) + f2(t) produces the steady state response x3(t) = x1(t) + x2(t).

Then, since we can represent any period driving function as a Fourier series, and it is a simple matter to find the steady state solution to a sinusoidally driven oscillator, we can find the response to the arbitrary driving function

f(x) = a0 +(ancos(nx) + bnsin(nx)).

Example 19: Determine the value of Fourier coefficient a_0

Answer: From (i),

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\omega t)+b_n\sin(n\omega t))\text{ ------------------------(i)}$$

Integrate (i) on both sides from **- π to π** i.e period = **T = 2π**

We have, $\omega=\frac{2\pi}{T}=\frac{2\pi}{2\pi}=1$

$$\int_{-\pi}^{\pi}f(t)\,dt=\int_{-\pi}^{\pi}[\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\omega t)+b_n\sin(n\omega t)]dt$$

$$=\frac{a_0}{2}\int_{-\pi}^{\pi}dt+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(n\omega t)dt+b_n\int_{-\pi}^{\pi}\sin(n\omega t)dt$$

$$=\frac{a_0}{2}[t]_{-\pi}^{\pi}+\sum_{n=1}^{\infty}\left[a_n\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}-b_n\left[\frac{\cos(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}\right]$$

$$[\because\int\sin mx\,dx=-\frac{1}{m}\cos mx\;\&\;\int\cos mx\,dx=\frac{1}{m}\sin mx\,]$$

$$=\frac{a_0}{2}[\pi-(-\pi)]+\sum_{n=1}^{\infty}\left[a_n\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}-b_n\left[\frac{\cos(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}\right]$$

$$=\frac{a_0}{2}[\pi+\pi]+\sum_{n=1}^{\infty}\left[a_n\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}-b_n\left[\frac{\cos(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}\right]$$

$$=\frac{a_0}{2}2\pi+\sum_{n=1}^{\infty}[\frac{a_n}{n\omega}(\sin(n\omega\pi)-\sin(-n\omega\pi))-\frac{b_n}{n\omega}(\cos(n\omega\pi)-\cos(-n\omega\pi))$$

$$=\pi a_0+\sum_{n=1}^{\infty}[\frac{a_n}{n\omega}(\sin(n\omega\pi)+\sin(n\omega\pi))-\frac{b_n}{n\omega}(\cos(\cancel{n\omega}\pi)-\cos(\cancel{n\omega}\pi))$$

$$[\because \sin(-\theta) = -\sin\theta\ ,\ \cos(-\theta) = \cos\theta]$$

$$= \pi a_0 + \sum_{n=1}^{\infty} [\frac{a_n}{n\omega} 2\sin(n\omega\pi)] - 0 \text{------------------------(ii)}$$

When n* = 1, *and $\omega = 1$ **[Given]**

Then $\frac{a_n}{n\omega} 2\sin(n\omega\pi) = \frac{a_1}{1} 2\sin(\pi) = 0\ \ [\because \sin\pi = 0]$

n =2,

$$\frac{a_n}{n\omega} 2\sin(n\omega\pi) = \frac{a_2}{2} 2\sin(2\pi) = 0\ \ [\because \sin 2\pi = 0]$$

Similarly,

$\frac{a_n}{n\omega} 2\sin(n\omega\pi) = 0$ For $n = 1,2,3,.................$

From (ii),

$$\therefore \int_{-\pi}^{\pi} f(t)dt = \pi a_0 + \sum_{n=1}^{\infty} [\frac{a_n}{n\omega} 2\sin(n\omega\pi)] - 0$$

$$\therefore \int_{-\pi}^{\pi} f(t)dt = \pi a_0 + 0 \qquad [\because \frac{a_n}{n\omega} 2\sin(n\omega\pi) = 0\ \ for\ \ n = 1,2,3,..............]$$

$$\therefore \int_{-\pi}^{\pi} f(t)dt = \pi a_0$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

Example 20: Determine the value of Fourier coefficient a_n

Answer:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)) \text{------------------(i)}$$

Multiplying by $\cos\omega mt$ [where m is a positive integer] Integrate on both sides from $-\pi$ **to** π i.e period = $T = 2\pi$

We have, $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$ -------------------------------------(ii)

$$\int_{-\pi}^{\pi} f(t)\cos(\omega mt)dt = \int_{-\pi}^{\pi} [\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)]\cos(\omega mt)dt$$

$$\int_{-\pi}^{\pi} f(t)\cos(mt)dt = \int_{-\pi}^{\pi} [\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt)]\cos(mt)dt$$

$$=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mt)dt+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\cos(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$=\frac{a_0}{2}\left[\frac{1}{m}\sin mt\right]_{-\pi}^{\pi}+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$=\frac{a_0}{2}\frac{1}{m}(\sin m\pi-\sin(-m\pi))+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$=\frac{a_0}{2}\frac{1}{m}(\sin m\pi+\sin m\pi)+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$[\because \sin(-\theta)=-\sin\theta\ ,\ \cos(-\theta)=\cos\theta]$$

$$=\frac{a_0}{2}\frac{1}{m}(2\sin m\pi)+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

Let m = 1, 2, 3, 4, ...then

$2\sin m\pi = 2\sin\pi = 0 \qquad [\because \sin\pi = 0]$

$2\sin m\pi = 2\sin 2\pi = 0 \qquad [\because \sin 2\pi = 0]$

....

....

$2\sin m\pi = 2\sin 4\pi = 0 \qquad [\because \sin 4\pi = 0]$

$2\sin m\pi = 2\sin 5\pi = 0 \qquad [\because \sin 5\pi = 0]$

$$=\frac{a_0}{2}\frac{1}{m}(0)+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$=0+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]\text{----------------(iii)}$$

Now in the RHS of (iii)

$$a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt$$

$$=a_n\times\frac{1}{2}\int_{-\pi}^{\pi}2\cos(mt)\cos(nt)dt=a_n\times\frac{1}{2}\int_{-\pi}^{\pi}\{\cos(m+n)t+\cos(m-n)t\}dt$$

$$[\because 2\cos A\cos B=\cos(A+B)+\cos(A-B)]$$

If $m\neq n$

$$=a_n\times\frac{1}{2}\int_{-\pi}^{\pi}\{\cos(m+n)t+\cos(m-n)t\}dt$$

$$=a_n\times\frac{1}{2}\left[\frac{1}{m+n}[\sin(m+n)t\right]_{-\pi}^{\pi}+\frac{a_n}{2}\times\left[\frac{1}{m-n}[\sin(m-n)t\right]_{-\pi}^{\pi}$$

$$=\frac{1}{2}\frac{a_n}{(m+n)}[\sin(m+n)\pi-\sin\{-(m+n)\pi]+\frac{a_n}{2}\times\frac{1}{(m-n)}[\sin(m-n)\pi-\sin\{-(m-n)\pi]$$

$$=\frac{1}{2}\frac{a_n}{(m+n)}[\sin(m+n)\pi+\sin\{(m+n)\pi]+\frac{a_n}{2}\times\frac{1}{(m-n)}[\sin(m-n)\pi+\sin\{(m-n)\pi]$$

$$[\because \sin(-\theta)=-\sin\theta\ ,\ \ \cos(-\theta)=\cos\theta]$$

$$=\frac{1}{2}\frac{a_n}{(m+n)}[2\sin(m+n)\pi]+\frac{a_n}{2}\times\frac{1}{(m-n)}[2\sin(m-n)\pi]$$

$$=\frac{a_n}{(m+n)}[\sin(m+n)\pi]+\frac{a_n}{(m-n)}[\sin(m-n)\pi]$$

Since $\mathbf{m \neq n}$, Let m = 2, n = 3

$$\sin(m+n)\pi=\sin(2+3)\pi=\sin 5\pi=0$$
$$\&\sin(m-n)\pi=\sin(2-3)\pi=-\sin\pi=0]$$

$$a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt\ =0+0$$

$$a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt\ =0\text{------------------------------------}(iv)$$

And If m =n, from (iii),

$$a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt$$

$$=a_m\times\frac{1}{2}\int_{-\pi}^{\pi}2\cos(mt)\cos(mt)dt=\mathrm{a_m}\times\frac{1}{2}\int_{-\pi}^{\pi}2\cos^2\mathrm{mtdt}=\frac{a_m}{2}\int_{-\pi}^{\pi}[1+\cos 2mt]dt$$

$$[\because 2\cos^2 \mathrm{x}=1+\cos 2\mathrm{x}]$$

$$=\frac{\mathrm{a_m}}{2}\left[\mathrm{t}+\frac{1}{2\mathrm{m}}\sin 2\mathrm{mt}\right]_{-\pi}^{\pi}$$

$$=\frac{\mathrm{a_m}}{2}[\pi-(-\pi)]+\frac{\mathrm{a_m}}{2}\frac{1}{2\mathrm{m}}[\sin 2\mathrm{m}\pi-\sin(-2\mathrm{m}\pi)]$$

$$=\frac{\mathrm{a_m}}{2}[2\pi]+\frac{\mathrm{a_m}}{2}\frac{1}{2\mathrm{m}}[\sin 2\mathrm{m}\pi+\sin 2\mathrm{m}\pi]=a_m\pi+\frac{a_m}{2}\frac{1}{2m}[2\sin 2m\pi]$$

$$[\because \sin(-\theta)=-\sin\theta]$$

$$=a_m\pi+\frac{a_m}{2}\frac{1}{2m}\times 0 \qquad [\because 2\sin \mathrm{m}\pi=0,\ \text{for}\ \mathrm{m}=1,2,3.........]$$

$$a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt\ =a_m\pi\text{--------------------------------}(v)$$

Again in the RHS of (iii),

$$b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt=\frac{b_n}{2}\int_{-\pi}^{\pi}2\sin(nt)\cos(mt)dt=\frac{b_n}{2}\int_{-\pi}^{\pi}\{\sin(n+m)t+\sin(n-m)t\}dt$$

$$[\because 2\sin A\cos B = \sin(A+B)+\sin(A-B)]$$

$$= \frac{b_n}{2}\left[\frac{-1}{(n+m)}[\cos(n+m)t\right]_{-\pi}^{\pi} + \frac{b_n}{2}\left[\frac{-1}{(n-m)}[\cos(n-m)t\right]_{-\pi}^{\pi}$$

$$= \frac{b_n}{2}.\frac{-1}{(n+m)}\{\cos(n+m)\pi - \cos[-(n+m)\pi]\} + \frac{b_n}{2}.\frac{-1}{(n-m)}\{\cos(n-m)\pi - \cos[-(n-m)\pi]\}$$

$$= \frac{b_n}{2}.\frac{-1}{(n+m)}\{\cos(n\cancel{+}m)\pi - \cos[(n\cancel{+}m)\pi]\} + \frac{b_n}{2}.\frac{-1}{(n-m)}\{\cos(n\cancel{-}m)\pi - \cos[(n\cancel{-}m)\pi]\}$$

$$[\because \sin(-\theta) = -\sin\theta\ ,\ \cos(-\theta) = \cos\theta]$$

If $\mathbf{m \neq n}$ or m = n

$$= \frac{b_n}{2}.\frac{-1}{(n+m)}\{0\} + \frac{b_n}{2}.\frac{-1}{(n-m)}\{0\}$$

= 0

$$\therefore b_n \int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = 0 \text{-------------------------------------}(vi)$$

Putting these values in (iii), we get,

When, $\mathbf{m \neq n}$,

$$\int_{-\pi}^{\pi} f(t)\ \cos(mt)dt = 0 + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt\ + b_n \int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$\int_{-\pi}^{\pi} f(t)\ \cos(mt)dt = 0+0+0 = 0 \qquad \text{[From (iv) and (vi)]}$$

And when m =n,

Putting these values in (iii), we get,

$$\int_{-\pi}^{\pi} f(t)\ \cos(mt)dt = 0 + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt\ + b_n \int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$\int_{-\pi}^{\pi} f(t)\ \cos(mt)dt = 0 + a_m\pi + 0 = a_m\pi \qquad \text{[From (v) and (vi)]}$$

$$\therefore a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(mt)dt$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(nt)dt \qquad [m = n]$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(nt)dt$$

Example 21: Determine the value of Fourier coefficient b_n

We have,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\omega t) + b_n\sin(n\omega t)) \text{-----------------}(i)$$

Multiplying by $\mathbf{\sin\omega mt}$ [where m is a positive integer] Integrate on both sides from **- π to** π i.e period = $\mathbf{T = 2\pi}$

We have, $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$ --------------------------------------(ii)

$$\int_{-\pi}^{\pi} f(t)\sin(\omega mt)dt = \int_{-\pi}^{\pi} [\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\omega t) + b_n\sin(n\omega t)]\sin(\omega mt)dt$$

$$\int_{-\pi}^{\pi} f(t)\sin(mt)dt = \int_{-\pi}^{\pi} [\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nt) + b_n\sin(nt)]\sin(mt)dt$$

$$= \frac{a_0}{2}\int_{-\pi}^{\pi}\sin(mt)dt + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$= \frac{a_0}{2}\left[\frac{-1}{m}\cos mt\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$= \frac{a_0}{2}(\frac{-1}{m})(\cos m\pi - \cos(-m\pi) + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$= \frac{a_0}{2}(\frac{-1}{m})(\cancel{\cos m\pi - \cos(m\pi)}) + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$[\because \cos(-\theta) = \cos\theta]$$

$$= \frac{a_0}{2}(\frac{-1}{m})(0) + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$0 + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$ -------------(iii)

Now in the RHS of (iii), we get,

$$a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt$$

$$= \frac{a_n}{2}\int_{-\pi}^{\pi}2\cos(nt)\sin(mt)dt = \frac{a_n}{2}\int_{-\pi}^{\pi}\{\sin(n+m)t - \sin(n-m)t\}dt$$

$$[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{a_n}{2}\times\left[\frac{-1}{(n+m)}\cos[(n+m)t\right]_{-\pi}^{\pi} - \frac{a_n}{2}\times\left[\frac{-1}{(n-m)}\cos[(n-m)t\right]_{-\pi}^{\pi}$$

$$= \frac{a_n}{2}\times\frac{-1}{(n+m)}(\cos[(n+m)\pi - \cos\{-(n+m)\pi\}) - \frac{a_n}{2}\times\frac{-1}{(n-m)}(\cos[(n-m)\pi - \cos[-(n-m)\pi])$$

$$= \frac{a_n}{2}\times\frac{-1}{(n+m)}(\cos[(\cancel{n}+m)\pi - \cos\{(n\cancel{+}m)\pi\}) - \frac{a_n}{2}\times\frac{-1}{(n-m)}(\cos[(n\cancel{-}m)\pi - \cos[(n\cancel{-}m)\pi])$$

$$[\because \cos(-\theta) = \cos\theta]$$

$= 0$ [$\mathbf{m \neq n}$ or m = n]---------------------------------(iv)

Again in the RHS of (iii) ,

$$b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt$$

$$= \frac{b_n}{2}\int_{-\pi}^{\pi}2\sin(nt)\sin(mt)dt = \frac{b_n}{2}\int_{-\pi}^{\pi}\{\cos(n-m)t - \cos(n+m)t\}dt$$

$$[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$$

$$=\frac{b_n}{2}\cdot\left[\frac{1}{(n-m)}[\sin(n-m)t\right]_{-\pi}^{\pi}-\frac{b_n}{2}\cdot\left[\frac{1}{(n+m)}[\sin(n+m)t\right]_{-\pi}^{\pi}$$

$$=\frac{b_n}{2}\cdot\frac{1}{(n-m)}\{\sin(n-m)\pi-\sin[-(n-m)\pi]\}-\frac{b_n}{2}\cdot\frac{1}{(n+m)}\{\sin(n+m)\pi-\sin[-(n+m)\pi]\}$$

$$=\frac{b_n}{2}\cdot\frac{1}{(n-m)}\{\sin(n-m)\pi+\sin[(n-m)\pi]\}-\frac{b_n}{2}\cdot\frac{1}{(n+m)}\{\sin(n+m)\pi+\sin[(n+m)\pi]\}$$

$$[\because \sin(-\theta)=-\sin\theta]$$

$$=\frac{b_n}{2}\cdot\frac{1}{(n-m)}\{2\sin(n-m)\pi\}-\frac{b_n}{2}\cdot\frac{1}{(n+m)}\{2\sin(n+m)\pi\} \text{--------(v)}$$

From (v),

If $\mathbf{m \neq n}$

$$b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt=\frac{b1}{2}\cdot\frac{1}{(1-2)}\{2\sin(1-2)\pi\}-\frac{b1}{2}\cdot\frac{1}{(1+2)}\{2\sin(1+2)\pi\}$$

$$[\text{say } n=1,\ m=2]$$

$$=\frac{b1}{2}\cdot\frac{1}{-1}\{2\sin(-\pi)-\frac{b1}{2}\cdot\frac{1}{3}\{2\sin 3\pi\}$$

$$=\frac{b1}{2}\cdot\frac{-1}{-1}\{2\sin(\pi)-\frac{b1}{2}\cdot\frac{1}{3}\{2\sin 3\pi\}$$

= 0-0

= 0.----------------------(vi)

If m = n,

Now in the RHS of (iii) ,

$$b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt$$

$$=b_m\int_{-\pi}^{\pi}\sin(mt)\sin(mt)dt=\frac{b_m}{2}\int_{-\pi}^{\pi}2\sin^2(mt)dt=\frac{b_m}{2}\int_{-\pi}^{\pi}(1-\cos mt)dt$$

$$[\because 2\sin^2 mt=1-\cos mt]$$

$$=\frac{b_m}{2}\left[t-\frac{1}{m}\sin mt\right]_{-\pi}^{\pi}=\frac{b_m}{2}(\pi-(-\pi))-\frac{b_m}{2}\frac{1}{m}(\sin m\pi-\sin(-m\pi)$$

$$=\frac{b_m}{2}(2\pi)-\frac{b_m}{2}\frac{1}{m}(\sin m\pi+\sin(m\pi)=b_m\pi-\frac{b_m}{2}\frac{1}{m}(2\sin m\pi)\ [\because \sin(-\theta)=-\sin\theta]$$

$$=b_m\pi-0 \qquad [\because \sin m\pi=0 \text{ for } m=1,2,3.........]$$

$$=b_m\pi \text{--(vii)}$$

Putting these values in (iii), we get,

When, $\mathbf{m \neq n}$,

$$\int_{-\pi}^{\pi}f(t)\sin(mt)dt=0+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

= 0+0+0 [From (iv) and (vi)]

= 0.

And When m = n,

$$\int_{-\pi}^{\pi} f(t)\sin(mt)dt = 0 + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n \int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$= 0 + 0 + b_m\pi \qquad \text{[From (iv) and (vii)]}$$

$$= b_m\pi$$

$$\therefore \int_{-\pi}^{\pi} f(t)\sin(mt)dt = b_m\pi$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

Hence the Fourier coefficients are:

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt}$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt}$$

Remember to find Fourier series:

We have, $$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{a_0}{2}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}(a_n\cos(\frac{n\pi}{L}t) + b_n\sin(\frac{n\pi}{L}t))}_{\text{AC value}} \text{-----------------------(i)}$$

When Period $T = 2\pi = 2L$

We find,

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \text{ ---(ii)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\omega t)dt$$

$$[\omega = 2\pi f = 2\pi\frac{1}{T} = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1]$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \text{ ------------------------------------(iii)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\omega t)dt$$

$$[\omega = 2\pi f = 2\pi\frac{1}{T} = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1]$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \text{ ---------------------------------------(iv)}$$

When Period $T = 2L$
Then,

$$\therefore a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt \text{ ---(v)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nwt)dt$$

$$[\text{Here, } \omega = \frac{2\pi}{T} = \frac{2\pi}{2L} = \frac{\pi}{L}]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\frac{\pi}{L}t)dt \text{ ----------------------------------(vi)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\omega t)dt$$

$$[w = 2\pi f = 2\pi\frac{1}{T} = \frac{2\pi}{T} = \frac{2\pi}{2L} = \frac{\pi}{L}]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(n\frac{\pi}{L}t)dt \text{ ----------------------------------(vii)}$$

Some Graphs

Draw the graph of the following function

$y = 0.t + 1$

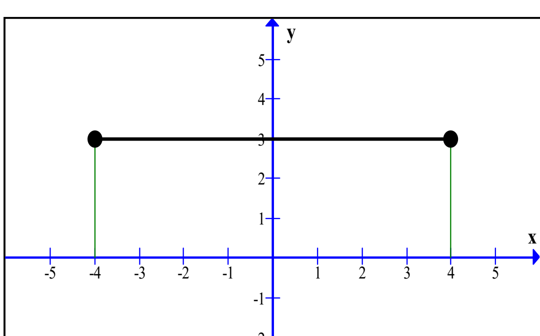
t	-2	-3	-1	0	1	2	3	-4	4
y	1	1	1	1	1	1	1	1	1

Draw the graph $y = t$;

t	-2	-3	-1	0	1	2	3	-4	4
y	-2	-3	-1	0	1	2	3	-4	4

Draw the graph $y = -t$

t	-2	-3	-1	0	1	2	3	-4	4
y	2	3	1	0	-1	-2	-3	4	-4



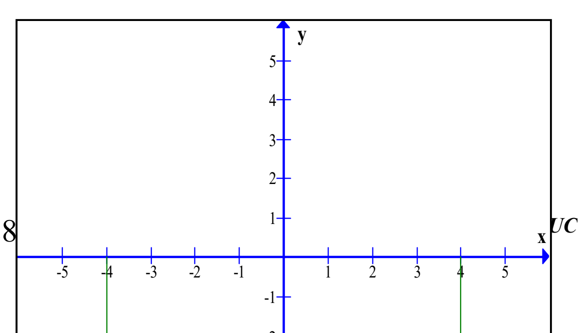
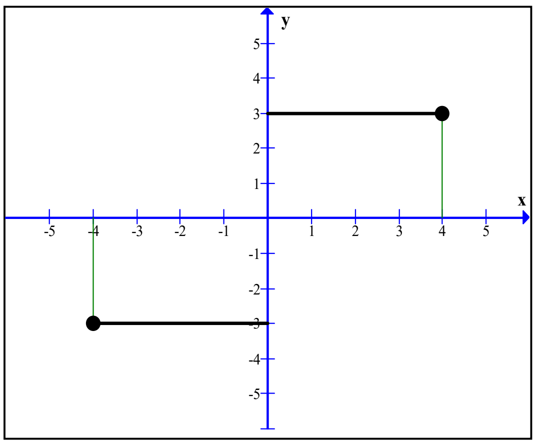


Figure 38. $y = f(x) = 3 \;;\; -4 \le x \le 4$

Figure 39. $y = f(x) = -3 \;;\; -4 \le x \le 4$



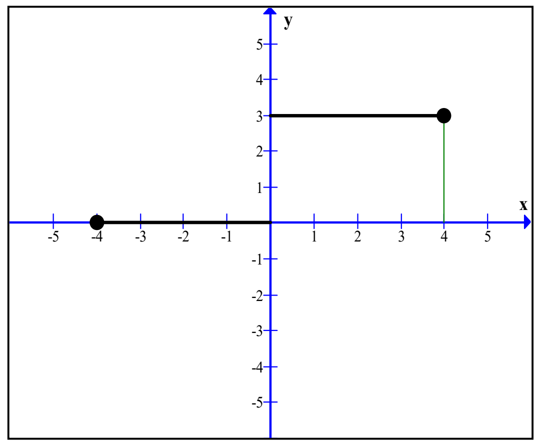


Figure 40. $y = f(x) = -3 \;;\; -4 \le x \le 0$
$\quad = \; 3 \;;\quad 0 \le x \le 4$

Figure 41. $y = f(x) = 0 \;;\; -4 \le x \le 0$
$\quad = 3 \;;\quad 0 \le x \le 4$

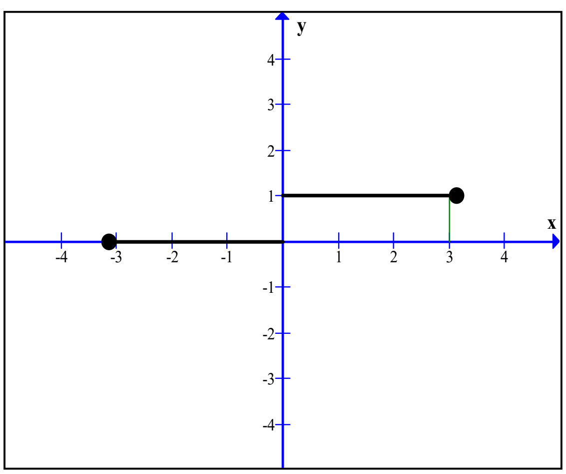
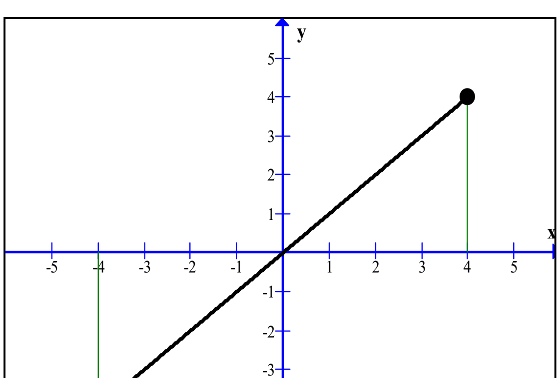


Figure 42. $y = f(x) = 0 \;;\; -3 \le x \le 0$
$\quad = 1 \;;\quad 0 \le x \le 3$



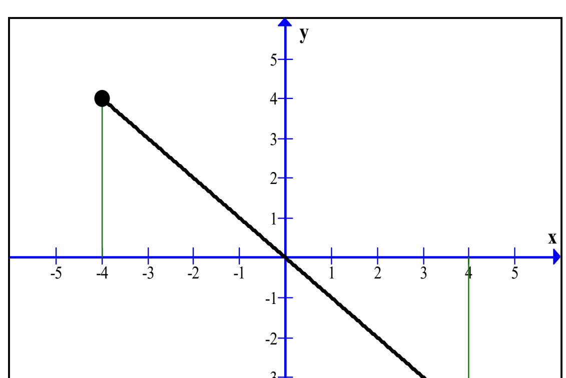
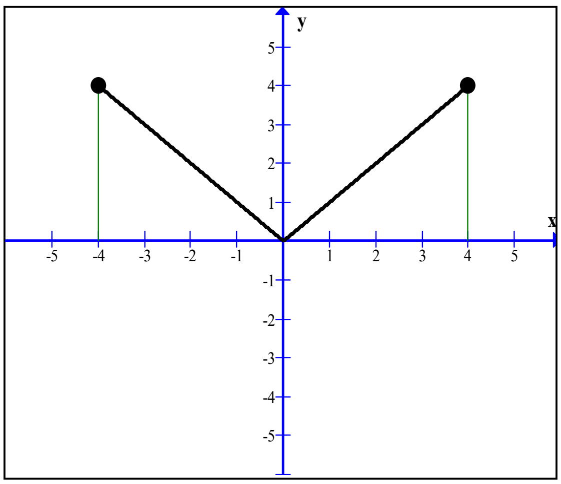


Figure 43. $y = f(x) = x \;\; ; \;\; -4 \le x \le 4$

Figure 44. $y = f(x) = -x \;\; ; \;\; -4 \le x \le 4$



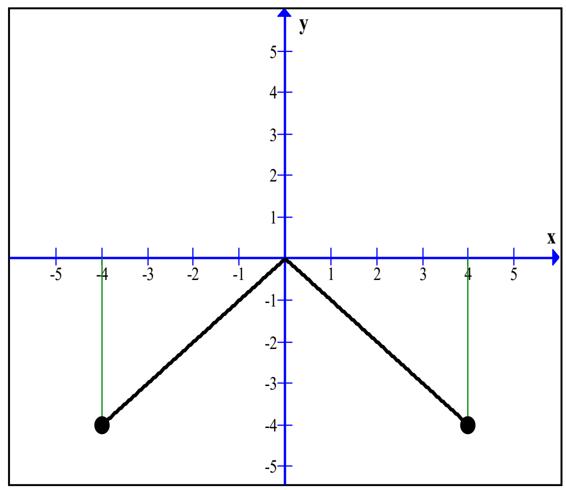
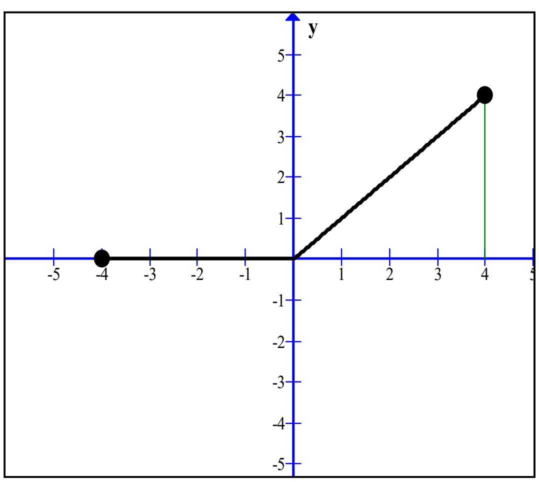


Figure 45. $y = f(x) = -x \;\; ; \;\; -4 \le x \le 0$

$= \;\; x \;\; ; \;\; 0 \le x \le 4$

Figure 46. $y = f(x) = \;\; x \;\; ; \;\; -4 \le x \le 0$

$= -x \;\; ; \;\; 0 \le x \le 4$



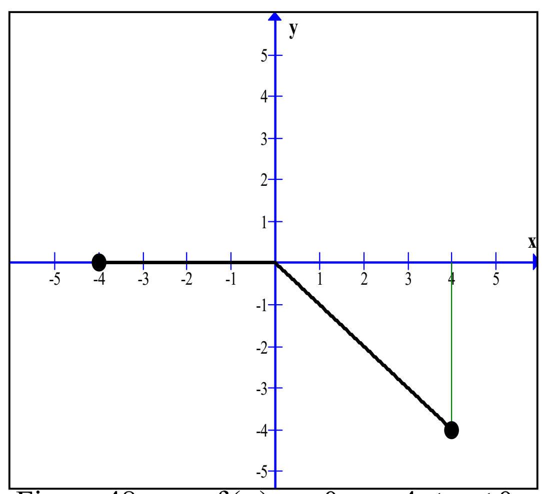
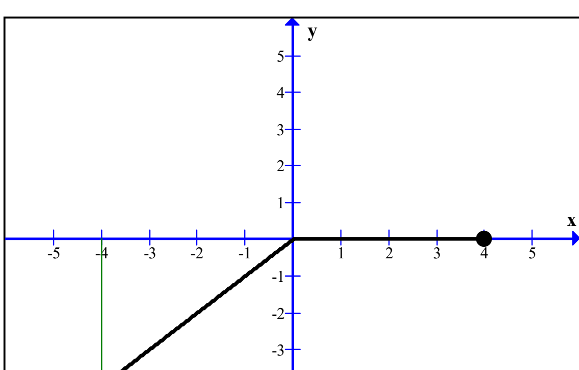


Figure 47. $y = f(x) = 0 \;\; ; \;\; -4 \le x \le 0$

$= \;\; x \;\; ; \;\; 0 \le x \le 4$

Figure 48. $y = f(x) = \;\; 0 \;\; ; \;\; -4 \le x \le 0$

$= -x \;\; ; \;\; 0 \le x \le 4$



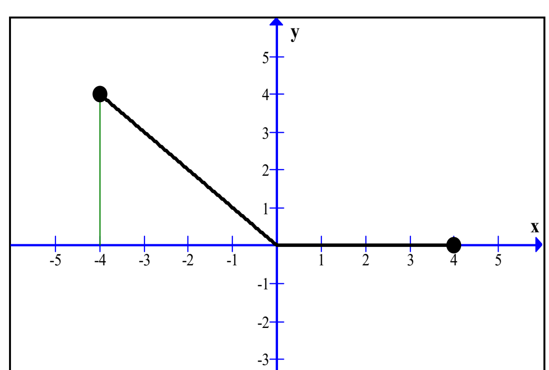
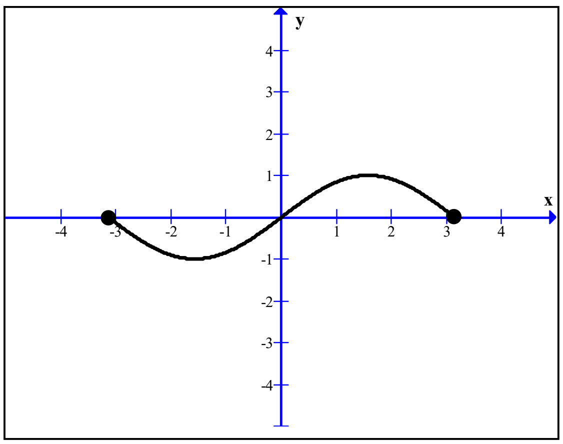


Figure 49. $y = f(x) = x \;;\; -4 \le x \le 0$

$= 0 \;;\quad 0 \le x \le 4$

Figure 50. $y = f(x) = -x \;;\; -4 \le x \le 0$

$= 0 \;;\quad 0 \le x \le 4$



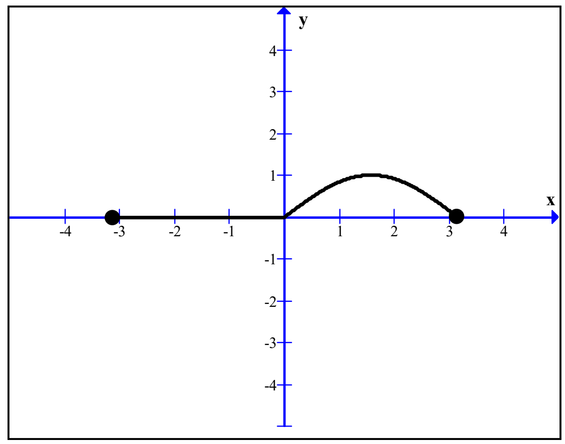
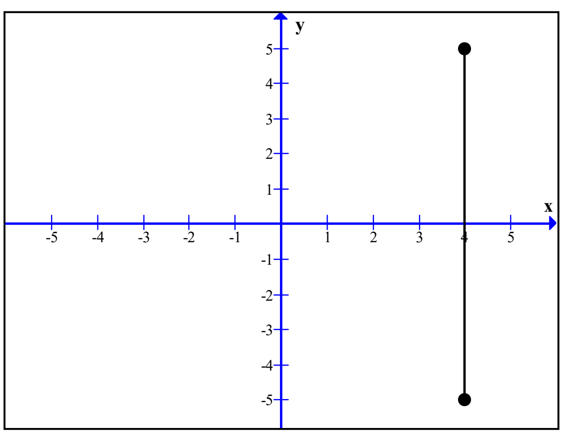


Figure 51. $y = f(x) = \sin x \;;\; -\pi \le x \le \pi$

Figure 52. $y = f(x) = 0 \quad ; \quad -\pi \le x \le 0$

$= \sin x \;;\quad 0 \le x \le \pi$



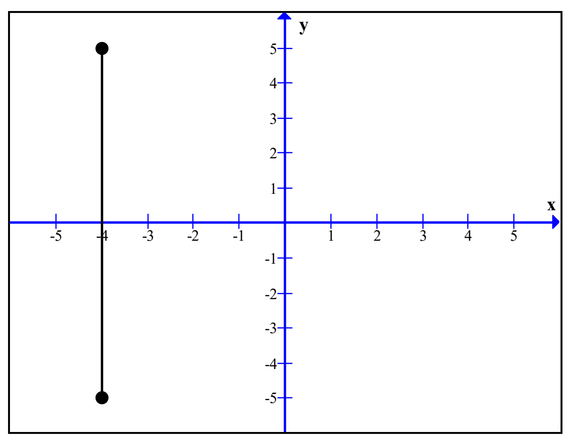


Figure 53. $x = 4$

Figure 54. $x = -4$

https://www.math24.net/fourier-series-definition-typical-examples/

Example 22

$$y = f(t) = 0\ ; \quad -4 \le t \le 0$$
$$y = 5\ ; \quad 0 \le t \le 4 \quad \text{-----------------------(i)}$$

f(t) = f(t + 8) **Here, T = 2L = 8** **∴ L = 4**

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

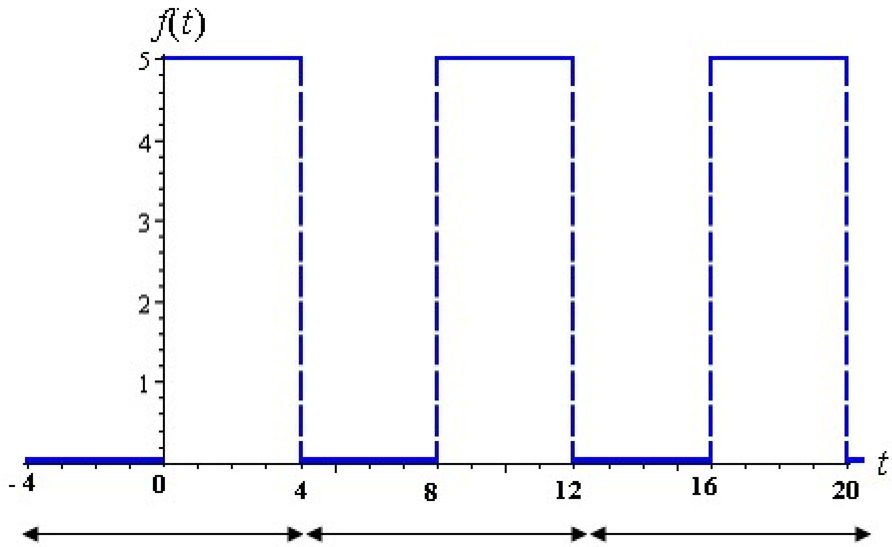


Figure 55: A periodic signal with period $\boldsymbol{T = 2L = 8}$

$$\mathbf{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt}$$
$$\mathbf{= \frac{1}{4}\int_{-4}^{4} f(t)dt}$$
$$\mathbf{= \frac{1}{4}\int_{-4}^{0} f(t)dt + \frac{1}{4}\int_{0}^{4} f(t)dt}$$
$$\mathbf{= \frac{1}{4}\int_{-4}^{0} 0.dt + \frac{1}{4}\int_{0}^{4} 5dt} \qquad \text{[From (i)]}$$
$$\mathbf{= 0 + \frac{1}{4}[5t]_0^4} \qquad \mathbf{[\int dt = t]}$$
$$\mathbf{= \frac{1}{4} \times [5 \times 4 - 0]}$$
$$\mathbf{= \frac{20}{4} = 5}$$
$$\mathbf{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt}$$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(t)\cos\frac{n\pi t}{4}dt$$

$$a_n = \frac{1}{4}\int_{-4}^{0} f(t)\cos\frac{n\pi t}{4}dt + \frac{1}{4}\int_{0}^{4} f(t)\cos\frac{n\pi t}{4}dt$$

$$a_n = \frac{1}{4}\int_{-4}^{0} (0)\cos\frac{n\pi t}{4}dt + \frac{1}{4}\int_{0}^{4} (5)\cos\frac{n\pi t}{4}dt \qquad \text{[From (i)]}$$

$$a_n = 0 + \frac{1}{4}\times 5\int_{0}^{4}\cos\frac{n\pi t}{4}dt$$

$$= \frac{5}{4}\times\frac{1}{\frac{n\pi}{4}}\left[\sin\frac{n\pi t}{4}\right]_0^4 \qquad [\int\cos mx\,dx = \frac{1}{m}\sin mx]$$

$$= \frac{5}{4}\times\frac{4}{n\pi}\left[\sin\frac{n\pi\times 4}{4} - \sin\frac{n\pi\times 0}{4}\right]$$

$$= \frac{5}{n\pi}\left[\sin n\pi - \sin 0\right]$$

$$= \frac{5}{n\pi}\sin n\pi$$

$$= 0 \qquad \left[\because \sin\ \pi = \sin 2\pi = \sin 3\pi = ... = \sin n\pi = 0 \text{ and } \sin 0 = 0\right]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt$$

$$b_n = \frac{1}{4}\int_{-4}^{4} f(t)\sin\frac{n\pi t}{4}dt$$

$$b_n = \frac{1}{4}\int_{-4}^{0} f(t)\sin\frac{n\pi t}{4}dt + \frac{1}{4}\int_{0}^{4} f(t)\sin\frac{n\pi t}{4}dt$$

$$b_n = \frac{1}{4}\int_{-4}^{0} (0)\sin\frac{n\pi t}{4}dt + \frac{1}{4}\int_{0}^{4} (5)\sin\frac{n\pi t}{4}dt \qquad \text{[From (i)]}$$

$$b_n = 0 + \frac{1}{4}\times 5\int_{0}^{4}\sin\frac{n\pi t}{4}dt$$

$$b_n = \frac{5}{4}\int_{0}^{4}\sin\frac{n\pi t}{4}dt$$

$$= \frac{5}{4}\times\frac{1}{\frac{n\pi}{4}}\left[-\cos\frac{n\pi t}{4}\right]_0^4 = \frac{-5}{4}\times\frac{4}{n\pi}\left[\cos\frac{n\pi t}{4}\right]_0^4$$

$$[\int\cos mx\,dx = \frac{1}{m}\sin mx; \int\sin mx\,dx = \frac{-1}{m}\cos mx]$$

$$= \frac{-5}{n\pi}\left[\cos\frac{n\pi\times 4}{4} - \cos\frac{n\pi\times 0}{4}\right]$$

$$= \frac{-5}{n\pi}[\cos n\pi - \cos 0]$$

$$= \frac{-5}{n\pi}[\cos n\pi - 1]$$

The Fourier series for the above function:

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{5}{2} + \sum_{n=1}^{\infty}(0)\cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} -\frac{5}{n\pi}(\cos(n\pi)-1)\sin\frac{n\pi t}{L}$$

$$= 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$$

$$\underbrace{f(t)}_{Complex\ wave} = \underbrace{2.5}_{DC\ value} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{AC\ value} \textit{Answer}$$

Example 23:

$$y = f(t) = -1\ ;\ -\pi < t < 0$$
$$= 1\ ;\quad 0 < t < \pi \qquad \text{----------------------(i)}$$

$f(t) = f(t + 2\pi)$ Here, $T = 2L = 2\pi$ $\therefore L = \pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

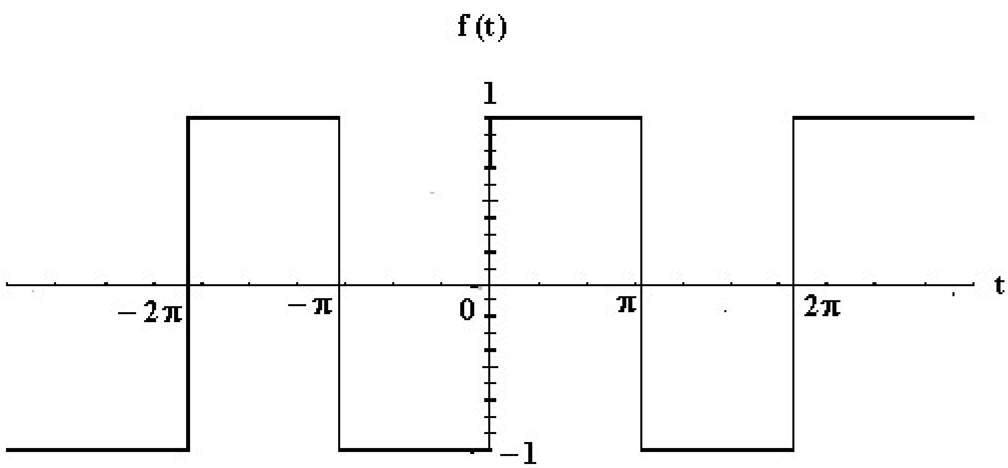


Figure 56: A periodic signal with period $T = 2L = 2\pi$

$$a_0 = \frac{1}{L}\int_{-\pi}^{\pi} f(t)dt$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)dt + \frac{1}{\pi}\int_{0}^{\pi} (1)dt \qquad \text{[From (i)]}$$

$$= \frac{-1}{\pi}[t]_{-\pi}^{0} + \frac{1}{\pi}[t]_{0}^{\pi}$$

$$= \frac{-1}{\pi}[0-(-\pi)] + \frac{1}{\pi}[\pi - 0]$$

$$= -\frac{1}{\pi}[\pi] + \frac{1}{\pi}[\pi]$$

$$= -1 + 1$$

$$= 0$$

$$a_n = \frac{1}{L}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{L}dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\cos\frac{n\pi t}{\pi} + \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cos ntdt + \frac{1}{\pi}\int_{0}^{\pi} (1)\cos ntdt \qquad \text{[From (i)]}$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} \cos ntdt + \frac{1}{\pi}\int_{0}^{\pi} \cos ntdt$$

$$= -\frac{1}{\pi}\times\frac{1}{n}[\sin nt]_{-\pi}^{0} + \frac{1}{\pi}\times\frac{1}{n}[\sin nt]_{0}^{\pi} \qquad [\int \cos mx\, dx = \frac{1}{m}\sin mx]$$

$$= -\frac{1}{\pi n}[\sin 0 - \sin(-n\pi)] + \frac{1}{\pi n}[\sin n\pi - 0]$$

$$= -\frac{1}{\pi n}[0 - \sin(-n\pi)] + \frac{1}{\pi n}[\sin n\pi - 0]$$

$$= -\frac{1}{\pi n}[0 + \sin(n\pi)] + \frac{1}{\pi n}[\sin n\pi - 0] \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= -\frac{1}{\pi n}[0+0] + \frac{1}{\pi n}[0-0]\ [\therefore \sin n\pi = 0 \text{ for } n = 1,2,3.............]$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{L}dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin\frac{n\pi t}{\pi}dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin ntdt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin ntdt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\sin ntdt + \frac{1}{\pi}\int_{0}^{\pi} (1)\sin ntdt \qquad \text{[From (i)]}$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} \sin ntdt + \frac{1}{\pi}\int_{0}^{\pi} \sin ntdt$$

$$= -\frac{1}{\pi}\times\frac{1}{n}\left[-\cos nt\right]_{-\pi}^{0} + \frac{1}{\pi}\times\frac{1}{n}\left[-\cos nt\right]_{0}^{\pi} \; [\because \int \sin mx\,dx = \frac{-1}{m}\cos mx]$$

$$= +\frac{1}{\pi n}\left[\cos nt\right]_{-\pi}^{0} + \frac{1}{\pi n}\left[-\cos nt\right]_{0}^{\pi}$$

$$= \frac{1}{\pi n}\left[\cos 0 - \cos(-n\pi)\right] - \frac{1}{\pi n}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{1}{\pi n}\left[1 - \cos n\pi\right] - \frac{1}{\pi n}\left[\cos n\pi - 1\right] \qquad \left[\therefore \cos(-\theta) = \cos\theta\right]$$

$$= \frac{1}{\pi n} - \frac{1}{\pi n}\cos n\pi - \frac{1}{\pi n}\cos n\pi + \frac{1}{\pi n}$$

$$= \frac{2}{\pi n} - \frac{2}{\pi n}\cos n\pi$$

$$= \frac{2}{\pi n}\left(1 - \text{conn}\pi\right)$$

The Fourier series for the above function:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{\pi} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{\pi}$$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$\therefore f(t) = 0 + \sum_{n=1}^{\infty} 0.\cos nt + \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - \cos n\pi)\sin nt$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{0}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - \cos n\pi)\sin nt}_{\text{AC value}}$$

Example 24:

$$y = f(t) = -t \; ; \quad -\pi \le t \le 0$$

$$y = t \; ; \quad 0 \le t \le \pi \text{ ----------------------(i)}$$

$\mathbf{f(t) = f(t + 2\pi)}$ Here, $\mathbf{T = 2L = 2\pi}$ $\therefore \mathbf{L = \pi}$

a) Sketch the function for 3 cycles:

b) Find the Fourier series for the function

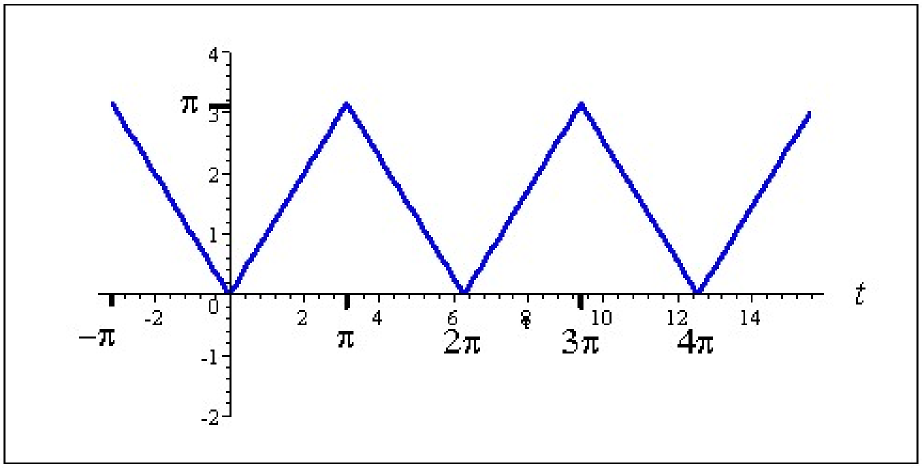


Figure 57: A periodic signal with period $\mathbf{T = 2L = 2\pi}$

$$\mathbf{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt}$$

$$\mathbf{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt}$$

$$\mathbf{= \frac{1}{\pi}\int_{-\pi}^{0} f(t)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)dt}$$

$$\mathbf{= \frac{1}{\pi}\int_{-\pi}^{0} (-t)dt + \frac{1}{\pi}\int_{0}^{\pi} tdt} \qquad \text{[From (i)]}$$

$$\mathbf{= -\frac{1}{\pi}[\frac{t^2}{2}]_{-\pi}^{0} + \frac{1}{\pi}[\frac{t^2}{2}]_{0}^{\pi}} \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + c \; ; n \ne -1]$$

$$\mathbf{= -\frac{1}{\pi}[0 - \frac{(-\pi)^2}{2}] + \frac{1}{\pi}[\frac{\pi^2}{2} - 0]}$$

$$\mathbf{= -\frac{1}{\pi}[-\frac{\pi^2}{2}] + \frac{1}{\pi}[\frac{\pi^2}{2}]}$$

$$\mathbf{= \frac{1}{\pi} \times \frac{\pi^2}{2} + \frac{1}{\pi} \times \frac{\pi^2}{2}}$$

$$\mathbf{= \frac{\pi}{2} + \frac{\pi}{2}}$$

$$= \frac{2\pi}{2}$$
$$= \pi$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos(nt)dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{0} (-t)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)dt \qquad \text{[From (i)]}$$
$$= -\frac{1}{\pi}\int_{-\pi}^{0} t\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)dt \text{------------------(ii)}$$

Now, Let, $\mathbf{I = \int t\cos(nt)dt}$

$$= t\int \cos(nt)dt - \int [\frac{d}{dt}(t)\int \cos(nt)dt]dt \quad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$
$$= t\frac{\sin(nt)}{n} - \int 1.\frac{\sin(nt)}{n}dt \qquad [\int \cos mx\, dx = \frac{1}{m}\sin mx]$$
$$= \frac{t}{n}\sin(nt) - \frac{1}{n}\int \sin(nt)dt$$
$$= \frac{t}{n}\sin(nt) - \frac{1}{n}\frac{(-\cos(nt))}{n} \qquad [\because \int \sin mx\, dx = \frac{-1}{m}\cos mx]$$
$$= \frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt) \text{--------------------------(iii)}$$

Putting the value of (iii) in (ii)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = -\frac{1}{\pi}\int_{-\pi}^{0} t\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)dt$$
$$= -\frac{1}{\pi}\left[\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt)\right]_{0}^{\pi}$$
$$= -\frac{1}{\pi}[\frac{0}{n}\sin(0) + \frac{1}{n^2}\cos(0) - \frac{(-\pi)}{n}\sin(-n\pi) - \frac{1}{n^2}\cos(-n\pi)] + \frac{1}{\pi}[\frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\cos(n\pi)$$
$$- \frac{0}{n}\sin(0) - \frac{1}{n^2}\cos(0)]$$
$$= \frac{-1}{\pi}[0 + \frac{1}{n^2}\times 1 + \frac{\pi}{n}\sin(-n\pi) - \frac{1}{n^2}\cos(-n\pi)] + \frac{1}{\pi}[\frac{\pi}{n}\times 0 + \frac{1}{n^2}\cos(n\pi) - 0 - \frac{1}{n^2}\times 1]$$

$$= \frac{-1}{\pi}[\frac{1}{n^2} - \frac{\pi}{n}\sin(n\pi) - \frac{1}{n^2}\cos(n\pi)] + \frac{1}{\pi}[\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}]$$

$[\sin(-\theta) = -\sin\theta \,;\, \cos(-\theta) = \cos\theta;\ \sin n\pi = 0 \text{ for } n = 1,2,3..........]$

$$= \frac{-1}{\pi}[\frac{1}{n^2} - \frac{\pi}{n}\times 0 - \frac{1}{n^2}\cos(n\pi)] + \frac{1}{\pi}[\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}]$$

$$= \frac{-1}{\pi n^2} + \frac{1}{\pi n^2}\cos(n\pi) + \frac{1}{\pi n^2}\cos(n\pi) - \frac{1}{\pi n^2}$$

$$= \frac{2}{\pi n^2}\cos n\pi - \frac{2}{\pi n^2}$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1) \qquad [\text{Taking Common } \frac{2}{\pi n^2}]$$

$$b_n = \frac{1}{L}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{L}dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{\pi}dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin(nt)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-t)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} t\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt \text{ ---------------------(iv)}$$

Now $\int t\sin(nt)dt$

$$= t\int \sin(nt)dt - \int\{\frac{d}{dt}(t)\int \sin(nt)dt\}dt \quad [\because \int uv dx = u\int v dx - \int\{\frac{d}{dx}(u)\int v dx\}dx]$$

$$= t(\frac{-\cos(nt)}{n}) - \int 1.\frac{-\cos(nt)}{n}dt$$

$$= -\frac{t}{n}\cos(nt) + \frac{1}{n}\int \cos(nt)dt$$

$$= \frac{-t}{n}\cos(nt) + \frac{1}{n}\frac{\sin(nt)}{n} \qquad [\int \cos mx\, dx = \frac{1}{m}\sin mx]$$

$$= -\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt) \text{ ------------------------------(v)}$$

Putting the value of (v) in (iv)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = -\frac{1}{\pi}\int_{-\pi}^{0} t\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt$$

$$=-\frac{1}{\pi}\left[\frac{-t}{n}\cos(nt)+\frac{1}{n^2}\sin(nt)\right]_{-\pi}^{0}+\frac{1}{\pi}\left[\frac{-t}{n}\cos(nt)+\frac{1}{n^2}\sin(nt)\right]_{0}^{\pi}$$

$$=-\frac{1}{\pi}[\frac{-0}{n}\cos(0)+\frac{1}{n^2}\sin(0)-\frac{-(-\pi)}{n}\cos(-n\pi)-\frac{1}{n^2}\sin(-n\pi)]+\frac{1}{\pi}[-\frac{\pi}{n}\cos(n\pi)$$

$$+\frac{1}{n^2}\sin(n\pi)-\frac{-(0)}{n}\cos(0)-\frac{1}{n^2}\sin(0)]$$

$$=-\frac{1}{\pi}[0+0-\frac{\pi}{n}\cos(n\pi)+\frac{1}{n^2}\sin(n\pi)]+\frac{1}{\pi}[-\frac{\pi}{n}\cos(n\pi)+\frac{1}{n^2}\sin(n\pi)+0-0]$$

$$=-\frac{1}{\pi}[-\frac{\pi}{n}\cos(n\pi)+\frac{1}{n^2}\times 0]+\frac{1}{\pi}[-\frac{\pi}{n}\cos(n\pi)+\frac{1}{n^2}\times 0]$$

$[\sin(-\theta)=-\sin\theta\ ;\ \cos(-\theta)=\cos\theta;\ \sin n\pi=0 \text{ for } n=1,2,3..........]$

$$=\frac{\pi}{\pi n}\cancel{\cos}(n\pi)-\frac{\pi}{\pi n}\cancel{\cos}(n\pi)$$

$$=0$$

The Fourier series for the above function is:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi}{L})t+\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi}{L})t$$

$$=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi}(\cos n\pi-1)\cos(\frac{n\pi}{L})t+0$$

$$=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{n^2\pi}(\cos n\pi-1)\cos(\frac{n\pi}{\pi})t$$

$$\underbrace{f(t)}_{Complex\ wave}=\underbrace{\frac{\pi}{2}}_{DC\ value}+\underbrace{\sum_{n=1}^{\infty}\frac{2}{n^2\pi}(\cos n\pi-1)\cos nt}_{AC\ value}$$

Example 25

$y=f(t)=t\ ;\ -\pi\le t\le\pi$ -----------------------(i)

Here, $T=2L=2\pi$ $\qquad\therefore L=\pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

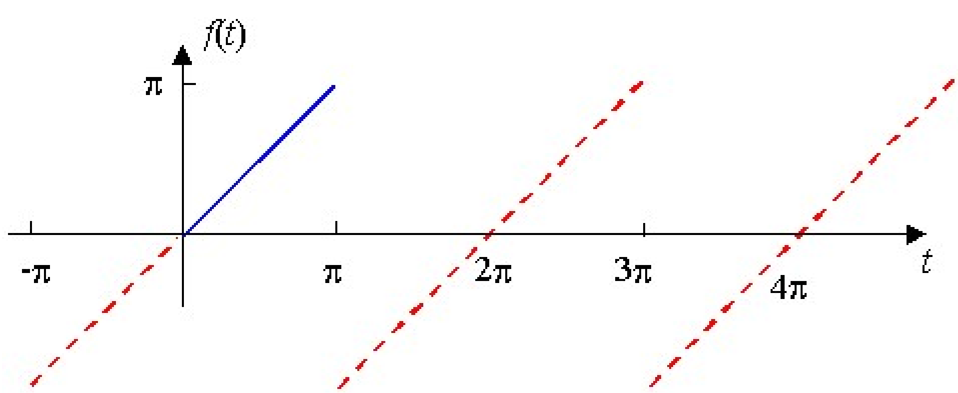


Figure 58: A periodic signal with period $\mathbf{T = 2L = 2\pi}$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} tdt \qquad \text{[From (i)]}$$

$$= \frac{1}{\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{L}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos ntdt \qquad \text{[From (i)]}$$

Now $\int t\cos ntdt$

$$= t\int \cos ntdt - \int\{\frac{d}{dt}(t)\int \cos ntdt\}dt \;\; [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= \frac{t}{n}\times \sin nt - \int 1.\frac{\sin nt}{n}dt$$

$$= \frac{t}{n}\sin nt - \frac{1}{n}.\frac{1}{n}.(-\cos nt)$$

$$= \frac{t}{n}\sin nt + \frac{1}{n^2}\cos(nt) \text{ --------------------------(ii)}$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos ntdt$$

$$= \frac{1}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{-\pi}^{\pi} \qquad \text{[From (ii)]}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} \sin n\pi + \frac{1}{n^2} \cos n\pi - \left\{\frac{-\pi}{n} \sin(-n\pi) + \frac{1}{n^2} \cos(-n\pi)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} \sin n\pi + \frac{1}{n^2} \cos n\pi + \frac{\pi}{n} \sin(-n\pi) - \frac{1}{n^2} \cos(-n\pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} \sin n\pi + \frac{1}{n^2} \cos n\pi + \frac{\pi}{n}(-) \sin(n\pi) - \frac{1}{n^2} \cos(n\pi)\right]$$

$[\because \sin(-\theta) = -\sin\theta\ ,\ \cos(-\theta) = \cos\theta]$

$$= \frac{1}{\pi}\left[\frac{\pi}{n} \cancel{\sin} n\pi + \frac{1}{n^2} \cancel{\cos} n\pi - \frac{\pi}{n} \cancel{\sin} n\pi - \frac{1}{n^2} \cancel{\cos} n\pi\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t).\sin\frac{n\pi t}{L}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt dt \qquad \text{[From (i)]}$$

Now, $\int t\sin nt dt$

$$= t\int \sin nt dt - \int \{\frac{d}{dt}(t)\int \sin nt dt\}dt \quad [\because \int uv dx = u\int v dx - \int\{\frac{d}{dx}(u)\int v dx\}dx]$$

$$= t.\frac{1}{n}.(-\cos nt) - \int 1.\frac{1}{n}(-\cos nt)dt$$

$$= -\frac{t}{n}\cos nt + \frac{1}{n}\int \cos nt dt$$

$$= -\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt \text{ ----------------------------(iii) } [\int \cos mx\, dx = \frac{1}{m}\sin mx]$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t.\sin nt.dt$$

$$= \frac{1}{\pi}\left[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_{-\pi}^{\pi} \qquad \text{[From (iii)]}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n} \cos n\pi + \frac{1}{n^2} \sin n\pi - \left\{-\frac{-\pi}{n} \cos(-n\pi) + \frac{1}{n^2} \sin(-n\pi)\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n} \cos n\pi + \frac{1}{n^2} \sin n\pi - \left\{\frac{\pi}{n} \cos(-n\pi) + \frac{1}{n^2} \sin(-n\pi)\right\}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi-\frac{\pi}{n}\cos(-n\pi)-\frac{1}{n^2}\sin(-n\pi)\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi-\frac{\pi}{n}\cos(-n\pi)-\frac{1}{n^2}(-)\sin(n\pi)\right]$$

$$[\because \sin(-\theta)=-\sin\theta\ ;\cos(-\theta)=\cos\theta]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi-\frac{\pi}{n}\cos n\pi\right] \qquad [\because \sin n\pi=0, \text{for } n=1,2,3...]$$

$$=-\frac{1}{\pi}\times\frac{2\pi}{n}\cos n\pi$$

$$=-\frac{2}{n}\cos n\pi$$

The Fourier series for the above function is:

$$\therefore f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nt+\sum_{n=1}^{\infty}b_n\sin nt$$

$$=0+0+\sum_{n=1}^{\infty}b_n\sin nt$$

$$=\sum_{n=1}^{\infty}-\frac{2}{n}\cos n\pi.\sin nt$$

$$\underbrace{f(t)}_{Complex\ wave}=\underbrace{0}_{DC\ value}+\underbrace{-2\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi.\sin nt}_{AC\ value}$$

Example 26

$$\begin{aligned} y=f(t)=&\ 0 &;\quad -1\le t\le -\frac{1}{2}\\ =&\cos 3\pi t &;\quad -\frac{1}{2}\le t\le \frac{1}{2} \qquad \text{---------------------(i)}\\ =&\ 0 &;\quad \frac{1}{2}\le t\le 1\end{aligned}$$

$f(t)=f(t+1)$ **Here** $T=2L=2\ \therefore L=1$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

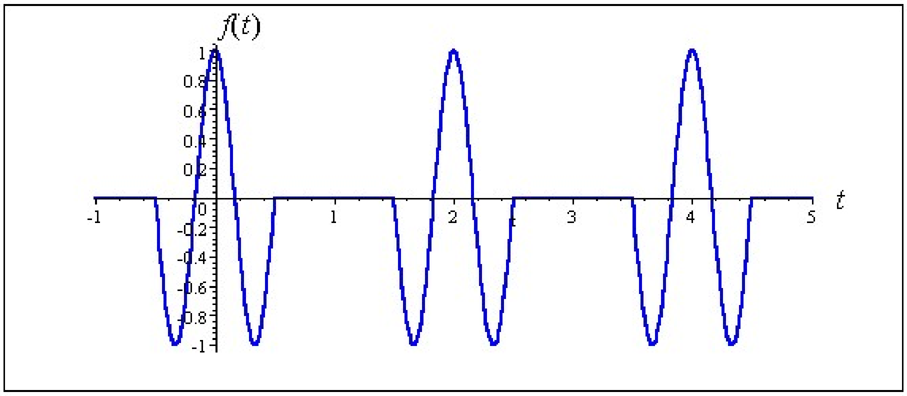


Figure 59: A periodic signal with period $T = 2L = 2$

$$a_0 = \frac{1}{L}\int_{L}^{L} f(t)dt$$

$$= \frac{1}{1}\int_{-1}^{1} f(t)dt$$

$$= \int_{-1}^{\frac{-1}{2}} f(t)dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} f(t)dt + \int_{\frac{1}{2}}^{1} f(t)dt$$

$$= 0 + \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos 3\pi t dt + 0 \qquad \text{[From (i)]}$$

$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos 3\pi t dt$$

$$= \frac{1}{3\pi}\left[\sin 3\pi t\right]_{\frac{-1}{2}}^{\frac{1}{2}} \qquad \left[\int \cos mx\, dx = \frac{1}{m}\sin mx; \int \sin mx\, dx = \frac{-1}{m}\cos mx\right]$$

$$= \frac{1}{3\pi}\left[\sin 3\pi \times \frac{1}{2} - \sin(-\frac{3\pi}{2})\right]$$

$$= \frac{1}{3\pi}\left[\sin\frac{3\pi}{2} + \sin\frac{3\pi}{2}\right] \quad \left[\because \sin(-\theta) = -\sin\theta\right]$$

$$= \frac{1}{3\pi} \times 2\sin\frac{3\pi}{2}$$

$$= \frac{2}{3\pi}\sin\frac{3\pi}{2}$$

$$=\frac{2}{3\pi}(-1)\left[\because \sin\frac{3\pi}{2}=\sin 3\times 90=\sin(3.90+0)=-\cos 0=-1\right]$$

$$=\frac{-2}{3\pi}$$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$=\int_{-1}^{1} f(t)\cos\frac{n\pi t}{1}dt$$

$$=\int_{-1}^{1} f(t)\cos n\pi t dt$$

$$=\int_{-1}^{-\frac{1}{2}} f(t)\cos n\pi t dt+\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\cos n\pi t dt+\int_{\frac{1}{2}}^{1} f(t)\cos n\pi t dt$$

$$=\int_{-1}^{-\frac{1}{2}} 0.\cos n\pi t dt+\int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\cos n\pi t dt+\int_{\frac{1}{2}}^{1} 0.\cos n\pi t dt \qquad \text{[From (i)]}$$

$$=\int_{-1}^{\frac{-1}{2}} 0\cos n\pi t dt+\int_{\frac{-1}{2}}^{\frac{1}{2}} f(t)\cos n\pi t dt+\int_{\frac{1}{2}}^{1} 0\cos n\pi t dt$$

$$=\int_{\frac{-1}{2}}^{\frac{1}{2}} f(t)\cos n\pi t dt$$

$$=\int_{\frac{-1}{2}}^{\frac{1}{2}} \cos 3\pi t\cos n\pi t dt \qquad \text{[From (i)]}$$

$$=\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} 2\cos 3\pi t\cos n\pi t dt$$

$$=\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \{\cos(3\pi t+n\pi t)+\cos(3\pi t-n\pi t)\}dt$$

$$=\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \cos(3\pi t+n\pi t)dt+\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \cos(3\pi t-n\pi t)dt$$

$$=\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}}\cos(3\pi+n\pi)tdt+\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}}\cos(3\pi-n\pi)tdt$$

$$=\frac{1}{2}\frac{1}{(3\pi+n\pi)}\left[\sin(3\pi+n\pi)t\right]_{\frac{-1}{2}}^{\frac{1}{2}}+\frac{1}{2}\frac{1}{(3\pi-n\pi)}\left[\sin(3\pi-n\pi)t\right]_{\frac{-1}{2}}^{\frac{1}{2}}$$

$$=\frac{1}{2}\frac{1}{(3\pi+n\pi)}\left[\sin(3+n)\pi t\right]_{\frac{-1}{2}}^{\frac{1}{2}}+\frac{1}{2}\frac{1}{(3\pi-n\pi)}\left[\sin(3-n)\pi t\right]_{\frac{-1}{2}}^{\frac{1}{2}}$$

$$=\frac{1}{2}\frac{1}{(3\pi+n\pi)}\left[\sin(3+n)\frac{\pi}{2}-\sin(3+n)(-\frac{\pi}{2})\right]+\frac{1}{2(3\pi-n\pi)}\left[\sin(3-n)\frac{\pi}{2}-\sin(3-n)(-\frac{\pi}{2})\right]$$

$$=\frac{1}{2}\frac{1}{(3\pi+n\pi)}\left[\sin(3+n)\frac{\pi}{2}+\sin(3+n)(\frac{\pi}{2})\right]+\frac{1}{2(3\pi-n\pi)}\left[\sin(3-n)\frac{\pi}{2}+\sin(3-n)(\frac{\pi}{2})\right]$$

$[\because \sin(-\theta)=-\sin\theta]$

$$=\frac{1}{2}\frac{2}{(3\pi+n\pi)}\sin(3+n)\frac{\pi}{2}+\frac{1}{2}\times\frac{2}{(3\pi-n\pi)}\sin(3-n)\frac{\pi}{2}$$

$$=\frac{1}{(3+n)\pi}\sin(3+n)\frac{\pi}{2}+\frac{1}{(3-n)\pi}\sin(3-n)\frac{\pi}{2}$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi t}{L}dt$$

$$=\frac{1}{1}\int_{-1}^{1}f(t)\sin\frac{n\pi t}{1}dt$$

$$=\int_{-1}^{\frac{-1}{2}}f(t)\sin n\pi tdt+\int_{\frac{-1}{2}}^{\frac{1}{2}}f(t)\sin n\pi tdt+\int_{\frac{1}{2}}^{1}f(t)\sin n\pi tdt$$

$$=\int_{-1}^{\frac{-1}{2}}0.\sin n\pi tdt+\int_{\frac{-1}{2}}^{\frac{1}{2}}f(t)\sin n\pi tdt+\int_{\frac{1}{2}}^{1}0.\sin n\pi tdt$$ [From (i)]

$$=\int_{\frac{-1}{2}}^{\frac{1}{2}}f(t)\sin n\pi tdt$$

$$=\int_{\frac{-1}{2}}^{\frac{1}{2}}\cos 3\pi t\sin n\pi tdt$$ [From (i)]

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} 2\cos 3\pi t \sin n\pi t dt$$

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \{\sin(3\pi t + n\pi t) - \sin(3\pi t - n\pi t)\}dt \ \left[\because 2\cos A \sin B = \sin(A+B) - \sin(A-B)\right]$$

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \sin(3\pi + n\pi)tdt - \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \sin(3\pi - n\pi)tdt$$

$$= \frac{1}{2}\frac{1}{(3\pi + n\pi)}\left[-\cos(3\pi + n\pi)t\right]_{\frac{-1}{2}}^{\frac{1}{2}} - \frac{1}{2}\frac{1}{(3\pi - n\pi)}\left[-\cos(3\pi - n\pi)t\right]_{\frac{-1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{2(3\pi + n\pi)}\left[-\cos(3\pi + n\pi)\frac{1}{2} + \cos(3\pi + n\pi)(-\frac{1}{2})\right] -$$
$$\frac{1}{2(3\pi - n\pi)}\left[-\cos(3\pi - n\pi)\frac{1}{2} + \cos(3\pi - n\pi)(-\frac{1}{2})\right]$$

$$= \frac{1}{2(3\pi + n\pi)}\left[-\cos(3+n)\frac{\pi}{2} + \cos(3+n)(-\frac{\pi}{2})\right] - \frac{1}{2(3\pi - n\pi)}\left[-\cos(3-n)\frac{\pi}{2} + \cos(3-n)(-\frac{\pi}{2})\right]$$

$$= \frac{1}{2(3\pi + n\pi)}\left[-\cos(3+n)\frac{\pi}{2} + \cos(3+n)(\frac{\pi}{2})\right] - \frac{1}{2(3\pi - n\pi)}\left[-\cos(3-n)\frac{\pi}{2} + \cos(3-n)(\frac{\pi}{2})\right]$$

$$[\because \cos(-\theta) = \cos\theta]$$

$= 0$

The Fourier series for the above function is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{\frac{-2}{3\pi}}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{1} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{1}$$

$$= \frac{-1}{3\pi} + \sum_{n=1}^{\infty} a_n \cos n\pi t + \sum_{n=1}^{\infty} 0 \sin n\pi t$$

$$= \frac{-1}{3\pi} + \sum_{n=1}^{\infty}\{\frac{1}{(3\pi + n\pi)}\sin(3+n)\frac{\pi}{2} + \frac{1}{(3\pi - n\pi)}\sin(3-n)\frac{\pi}{2}\}\cos n\pi t$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{-1}{3\pi}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}\{\frac{1}{(3\pi + n\pi)}\sin(3+n)\frac{\pi}{2} + \frac{1}{(3\pi - n\pi)}\sin(3-n)\frac{\pi}{2}\}\cos n\pi t}_{\text{AC value}} \ \textit{Answer}$$

Example 27

Given

$y = f(t) = 1 - t^2$ ------------------------(i) is to be represented by a Fourier series expression over the finite interval 0 < **t** < 1.

Here, $\mathbf{T = 2L = 1} \quad \therefore \mathbf{L = 1/2}$

Now
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$
$$= \frac{1}{\frac{1}{2}}\int_{0}^{2L} f(t)dt \qquad \left[\because \int_{-a}^{a} f(t)dt = \int_{0}^{2a} f(t)dt\right]$$
$$= 2\int_{0}^{2\times 1/2} f(t)dt$$
$$= 2\int_{0}^{1} f(t)dt$$
$$= 2\int_{0}^{1} (1-t^2)dt \qquad \text{[From (i)]}$$
$$= 2\int_{0}^{1} (1)dt - 2\int_{0}^{1} t^2 dt$$
$$= 2\int_{0}^{1} dt - 2\int_{0}^{1} t^2 dt$$
$$= 2[t]_0^1 - 2[\frac{t^3}{3}]_0^1$$
$$= 2[1-0] - 2[\frac{1}{3} - 0]$$
$$= 2\times 1 - 2\times\frac{1}{3}$$
$$= 2 - \frac{2}{3}$$
$$= \frac{6-2}{3}$$
$$a_0 = \frac{4}{3}$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$
$$a_n = \frac{1}{1/2}\int_{0}^{2L} f(t)\cos\frac{n\pi t}{1/2}dt$$
$$= 2\int_{0}^{2\times\frac{1}{2}} f(t)\cos(2n\pi t)dt$$

$$= 2\int_0^1 f(t)\cos(2n\pi t)dt$$

$$= 2\int_0^1 (1-t^2)\cos(2n\pi t)dt \qquad \text{[From (i)]}$$

$$= 2\int_0^1 \cos(2n\pi t)dt - 2\int_0^1 t^2\cos(2n\pi t)dt$$

Now,

$$\int \cos(2n\pi t)dt$$

$$= \frac{1}{2n\pi}\sin(2n\pi t) \quad [\int \cos mx\, dx = \frac{1}{m}\sin mx; \int \sin mx\, dx = \frac{-1}{m}\cos mx]$$

And,

$$\int t^2\cos(2n\pi t)dt$$

$$= t^2\int \cos(2n\pi t)dt - \int\{(t^2)\frac{d}{dt}\int \cos(2n\pi t)dt\}.dt \quad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t^2\frac{1}{2n\pi}\sin(2n\pi t) - \int 2t\frac{1}{2n\pi}\sin(2n\pi t).dt$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) - \frac{1}{n\pi}\int t\sin(2n\pi t).dt$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) - \frac{1}{n\pi}\left[t\int \sin(2n\pi t)dt - \int\{\frac{d}{dt}(t)\int \sin(2n\pi t)dt\}.dt\right]$$

$$= \frac{t^2}{2n\pi}.\sin(2n\pi t) - \frac{1}{n\pi}\left[t\frac{1}{2n\pi}(-\cos 2n\pi t) - \int 1\frac{1}{2n\pi}(-\cos 2n\pi t)dt\right]$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) - \frac{1}{n\pi}\left[-\frac{t}{2n\pi}\cos 2n\pi t + \frac{1}{2n\pi}\int(\cos 2n\pi t)dt\right]$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) - \frac{1}{n\pi}\left[-\frac{t}{2n\pi}\cos 2n\pi t + \frac{1}{2n\pi}\frac{1}{2n\pi}\sin(2n\pi t)\right]$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) + \frac{t}{2n^2\pi^2}\cos 2n\pi t - \frac{1}{4n^3\pi^3}\sin(2n\pi t) \text{--------------------(ii)}$$

$$a_n = 2\int_0^1 \cos(2n\pi t)dt - 2\int_0^1 t^2\cos(2n\pi t)dt$$

$$= 2\left[\frac{1}{2n\pi}\sin(2n\pi t)\right]_0^1 - 2\left[\frac{t^2}{2n\pi}\sin(2n\pi t) + \frac{t}{2n^2\pi^2}\cos 2n\pi t - \frac{1}{4n^3\pi^3}\sin(2n\pi t)\right]_0^1$$

[From (ii)]

$$= 2[\frac{1}{2n\pi}\sin(2n\pi) - 0] - 2[\frac{1}{2n\pi}\sin(2n\pi) - 0 + \frac{1}{2n^2\pi^2}\cos 2n\pi - 0 - \frac{1}{4n^3\pi^3}\sin(2n\pi) + 0]$$

$[\because \sin 2n\pi = 0$ For n=1, 2, 3……….]

$$= -\frac{1}{n^2\pi^2}\cos(2n\pi) + 0$$

$$= -\frac{1}{n^2\pi^2}\cos(2n\pi)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt$$

$$= \frac{1}{L}\int_{0}^{2L} f(t)\sin\frac{n\pi t}{1/2}dt$$

$$= \frac{1}{1/2}\int_{0}^{2\times 1/2} f(t)\sin\frac{n\pi t}{1/2}dt$$

$$= 2\int_0^1 f(t)\sin(2n\pi t)dt$$

$$= 2\int_0^1 (1-t^2)\sin(2n\pi t)dt \qquad \text{[From (i)]}$$

$$= 2\int_0^1 \sin(2n\pi t)dt - 2\int_0^1 t^2\sin(2n\pi t)dt$$

Now, $\int \sin(2n\pi t)dt$

$$= -\frac{1}{2n\pi}\cos(2n\pi t)$$

And, $\int t^2 \sin(2n\pi t)dt$

$$= t^2\int \sin(2n\pi t)dt - \int\{\frac{d}{dt}(t^2)\int \sin(2n\pi t)dt\}.dt \; [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t^2\frac{1}{2n\pi}\{-\cos(2n\pi t)\} - \int 2t\frac{1}{2n\pi}\{-\cos(2n\pi t)\}.dt$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\int t\cos(2n\pi t).dt$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\left[t\int \cos(2n\pi t).dt - \int\{\frac{d}{dt}(t)\int \cos(2n\pi t)dt\}.dt\right]$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\left[t\frac{1}{2n\pi}\sin(2n\pi t) - \int 1\frac{1}{2n\pi}\sin(2n\pi t)dt\right]$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\left[\frac{t}{2n\pi}\sin 2n\pi t - \frac{1}{2n\pi}\int \sin(2n\pi t)dt\right]$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\left[\frac{t}{2n\pi}\sin(2n\pi t) - \frac{1}{2n\pi}\frac{1}{2n\pi}.(-\cos 2n\pi t)\right]$$

$$= -\frac{t^2}{2n\pi}.\cos(2n\pi t) + \frac{t}{2n^2\pi^2}\sin 2n\pi t + \frac{1}{4n^3\pi^3}\cos(2n\pi t) \text{----------------(iii)}$$

$$b_n = 2\int_0^1 \sin(2n\pi t)dt - 2\int_0^1 t^2 \sin(2n\pi t)dt$$

$$= -\frac{2}{2n\pi}\left[\cos(2n\pi t)\right]_0^1 - 2\left[-\frac{t^2}{2n\pi}.\cos(2n\pi t) + \frac{t}{2n^2\pi^2}\sin 2n\pi t + \frac{1}{4n^3\pi^3}\cos(2n\pi t)\right]_0^1$$

[From (iii)]

$$= -\frac{2}{2n\pi}[\cos(2n\pi) - 1] - 2[-\frac{1}{2n\pi}\cos(2n\pi) + \frac{1}{2n^2\pi^2}\sin 2n\pi + \frac{1}{4n^3\pi^3}\cos(2n\pi) + 0 - 0 - \frac{1}{4n^3\pi^3}]$$

[$\because \sin 2n\pi = 0$ For n=1, 2, 3, & $\cos 0 = 1$]

$$= -\frac{1}{n\pi}\cos(2n\pi) + \frac{1}{n\pi} + \frac{1}{n\pi}\cos(2n\pi) - \frac{1}{2n^3\pi^3}\cos(2n\pi) + \frac{1}{2n^3\pi^3}$$

$$= \frac{1}{n\pi} - \frac{1}{2n^3\pi^3}\cos(2n\pi) + \frac{1}{2n^3\pi^3}$$

$$= \frac{1}{n\pi} - \frac{1}{2n^3\pi^3}[\cos(2n\pi) - 1]$$

The Fourier series for the above function is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$= \frac{\frac{4}{3}}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{1/2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{1/2}\right)$$

$$= \frac{2}{3} + \sum_{n=1}^{\infty} a_n \cos(2n\pi t) + \sum_{n=1}^{\infty} b_n \sin(2n\pi t)$$

$$= \frac{2}{3} + \sum_{n=1}^{\infty}[-\frac{1}{n^2\pi^2}\cos(2n\pi)]\cos(2n\pi t) + \sum_{n=1}^{\infty}[\frac{1}{n\pi} - \frac{1}{2n^3\pi^3}\{\cos 2n\pi - 1\}]\sin(2n\pi t)$$

$$\underbrace{f(t)}_{Complex\ wave} = \underbrace{\frac{2}{3}}_{DC\ value} + \underbrace{\sum_{n=1}^{\infty}[-\frac{1}{n^2\pi^2}\cos(2n\pi)]\cos(2n\pi t) + \sum_{n=1}^{\infty}[\frac{1}{n\pi} - \frac{1}{2n^3\pi^3}\{\cos 2n\pi - 1\}]\sin(2n\pi t)}_{AC\ value}$$

Answer

Example 28

$$y = f(t) = 3t \ ; \quad 0 < t < 1$$
$$= 3 \ ; \quad 1 < t < 2 \text{--------------------(i)}$$

$$f(t+2) = f(t) \text{ Here, } T = 2L = 2 \quad \therefore L = 1$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$= \frac{1}{L}\int_0^{2L} f(t)dt \qquad [\because \int_{-a}^{a} f(t)dt = \int_0^{2a} f(t)dt]$$

$$= \frac{1}{1}\int_0^{2} f(t)dt$$

$$= \int_0^{2} f(t)dt$$

$$= \int_0^{1} f(t)dt + \int_1^{2} f(t)dt$$

$$= \int_0^{1} 3tdt + \int_1^{2} 3dt \qquad \text{[From (i)]}$$

$$= 3[\frac{t^2}{2}]_0^1 + [3t]_1^2$$

$$= \frac{3}{2}[1-0] + 3[2-1]$$

$$= \frac{3}{2}[1-0] + 3\times 1$$

$$= \frac{3}{2} + 3$$

$$= \frac{9}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$= \frac{1}{L}\int_0^{2L} f(t)\cos\frac{n\pi t}{L}dt$$

$$= \frac{1}{1}\int_0^{2} f(t)\cos\frac{n\pi t}{1}dt$$

$$= \int_0^{2} f(t)\cos n\pi t dt$$

$$= \int_0^{1} f(t)\cos n\pi t dt + \int_1^{2} f(t)\cos n\pi t dt$$

$$= \int_0^{1} 3t\cos n\pi t dt + \int_1^{2} 3\cos n\pi t dt \qquad \text{[From (i)]}$$

Now, $\int 3t\cos n\pi t dt$

$$= 3t\int \cos n\pi t dt - \int\left[\frac{d}{dt}(3t)\int \cos n\pi t dt\right]dt \quad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= 3t \times \frac{1}{n\pi}\sin n\pi t - \int 3 \times \frac{1}{n\pi}\sin n\pi t dt$$

$$= \frac{3t}{n\pi}\sin n\pi t - \frac{3}{n\pi}\int \sin n\pi t dt$$

$$= \frac{3t}{n\pi}\sin n\pi t - \frac{3}{n\pi} \times \frac{1}{n\pi}(-\cos n\pi t) \; [\int \cos mx\, dx = \frac{1}{m}\sin mx; \int \sin mx\, dx = \frac{-1}{m}\cos mx]$$

$$= \frac{3t}{n\pi}\sin n\pi t + \frac{3}{n^2\pi^2}\cos(n\pi t) \text{ ----------------------(ii)}$$

and $\int 3\cos n\pi t dt$

$$= 3 \times \frac{1}{n\pi}\sin n\pi t \; [\int \cos mx\, dx = \frac{1}{m}\sin mx; \int \sin mx\, dx = \frac{-1}{m}\cos mx]$$

$$= \frac{3}{n\pi}\sin n\pi t \text{ --------------------------------(iii)}$$

$$\therefore a_n = \int_0^1 3t\cos n\pi t dt + \int_1^2 3\cos n\pi t dt$$

$$= \left[\frac{3t}{n\pi}\sin n\pi t + \frac{3}{n^2\pi^2}\cos(n\pi t)\right]_0^1 + \left[\frac{3}{n\pi}\sin n\pi t\right]_1^2 \quad \text{[From (ii) and From (iii)]}$$

$$= \left[\frac{3\times 1}{n\pi}\sin n\pi + \frac{3}{n^2\pi^2}\cos n\pi - \frac{3\times 0}{n\pi}\sin 0 - \frac{3}{n^2\pi^2}\cos 0\right] + \left[\frac{3}{n\pi}\sin 2n\pi - \frac{3}{n\pi}\sin n\pi\right]$$

$$= \frac{3}{n^2\pi^2}\cos n\pi - \frac{3}{n^2\pi^2} \times 1 \quad [\sin n\pi = 0 \text{ For n=1, 2, 3,...and } \cos 0 = 1, \sin 0 = 0]$$

$$= \frac{3}{n^2\pi^2}(\cos n\pi - 1)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L} dt$$

$$= \frac{1}{L}\int_0^{2L} f(t)\sin\frac{n\pi t}{L} dt$$

$$= \frac{1}{1}\int_0^{2} f(t)\sin\frac{n\pi t}{1} dt$$

$$= \int_0^2 f(t)\sin n\pi t dt$$

$$= \int_0^1 f(t)\sin n\pi t dt + \int_1^2 f(t)\sin n\pi t dt$$

$$= \int_0^1 3t\sin n\pi t dt + \int_1^2 3\sin n\pi t dt \quad \text{[From (i)]}$$

Now, $\int 3t\sin n\pi t dt$

$$= 3t\int \sin n\pi t dt - \int\left\{\frac{d}{dt}(3t)\int \sin n\pi t dt\right\}dt \ [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= 3t \times \frac{1}{n\pi}(-\cos n\pi t) - \int 3(-\cos n\pi t)\times\frac{1}{n\pi}dt$$

$$= -\frac{3t}{n\pi}\cos n\pi t + \frac{1}{n\pi}\int 3\cos n\pi t dt$$

$$= -\frac{3t}{n\pi}\cos n\pi t + \frac{3}{n\pi}\times\frac{1}{n\pi}\sin n\pi t$$

$$= -\frac{3t}{n\pi}\cos n\pi t + \frac{3}{n^2\pi^2}\sin n\pi t \text{ ----------------------------(iv)}$$

And, $\int 3\sin n\pi t dt$

$$= 3\times\frac{1}{n\pi}(-\cos n\pi t)$$

$$= -\frac{3}{n\pi}\cos n\pi t \text{ ---------------------------------(v)}$$

$$\therefore b_n = \int_0^1 3t\sin n\pi t dt + \int_1^2 3\sin n\pi t dt$$

$$= \left[\frac{-3t}{n\pi}\cos n\pi t + \frac{3}{n^2\pi^2}\sin n\pi t\right]_0^1 + \left[\frac{-3}{n\pi}\cos n\pi t\right]_1^2 \quad \text{[From (iv) and From (v)]}$$

$$= [\frac{-3\times 1}{n\pi}\cos n\pi + \frac{3}{n^2\pi^2}\sin n\pi + \frac{3\times 0}{n\pi}\cos 0 - \frac{3}{n^2\pi^2}\sin 0] + [-\frac{3}{n\pi}\cos 2n\pi + \frac{3}{n\pi}\cos n\pi]$$

$$= -\frac{3}{n\pi}\cos n\pi - \frac{3}{n\pi}\cos 2n\pi + \frac{3}{n\pi}\cos n\pi$$

$$= -\frac{3}{n\pi}\cos 2n\pi$$

The Fourier series for the above function is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{1} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{1}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t + \sum_{n=1}^{\infty} b_n \sin n\pi t$$

$$= \frac{9/2}{2} + \sum_{n=1}^{\infty}(\frac{3}{n^2\pi^2}(\cos n\pi - 1))\cos n\pi t + \sum_{n=1}^{\infty}(-\frac{3}{n\pi}\cos 2n\pi)\sin n\pi t$$

$$= \frac{9}{4} + \sum_{n=1}^{\infty}(\frac{3}{n^2\pi^2}(\cos n\pi - 1))\cos n\pi t + \sum_{n=1}^{\infty}(-\frac{3}{n\pi}\cos 2n\pi)\sin n\pi t$$

$$\underbrace{f(t)}_{Complex\ wave} = \underbrace{\frac{9}{4}}_{DC\ value} + \underbrace{\sum_{n=1}^{\infty}\left(\frac{3}{n^2\pi^2}(\cos n\pi - 1)\right)\cos n\pi t + \sum_{n=1}^{\infty}\left(-\frac{3}{n\pi}\cos 2n\pi\right)\sin n\pi t}_{AC\ value} \quad \textit{Answer}$$

Example 29

$$\mathbf{y = f(t) = t^2 + t \qquad ; -\pi < t < \pi} \text{ -----------------------(i)}$$

Find the Fourier expression of f(t), Also

Deduce that $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + -------$

$$\mathbf{f(t) = f(t + 2\pi) \qquad T = 2\pi = 2L \qquad \therefore L = \pi}$$

$$\mathbf{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt}$$

$$\mathbf{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt}$$

$$\mathbf{= \frac{1}{\pi}\int_{-\pi}^{\pi} (t^2 + t)dt} \qquad \text{[From (i)]}$$

$$\mathbf{= \frac{1}{\pi}\left[\frac{t^3}{3} + \frac{t^2}{2}\right]_{-\pi}^{\pi}}$$

$$\mathbf{= \frac{1}{\pi} + \left[\frac{\pi^3}{3} + \frac{\pi^2}{2} - \frac{(-\pi)^3}{3} - \frac{(-\pi)^2}{2}\right]}$$

$$\mathbf{= \frac{1}{\pi} + \left[\frac{\pi^3}{3} + \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2}\right]}$$

$$\mathbf{= \frac{1}{\pi} + \left[\frac{2\pi^3}{3}\right]}$$

$$\mathbf{= \frac{2\pi^2}{3}}$$

$$\mathbf{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt}$$

$$\mathbf{= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt}$$

$$\mathbf{= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt}$$

$$\mathbf{= \frac{1}{\pi}\int_{-\pi}^{\pi} (t^2 + t)\cos ntdt} \qquad \text{[From (i)]}$$

$$\mathbf{= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos ntdt + \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos ntdt}$$

Now, $\mathbf{\int t^2 \cos ntdt}$

$$= t^2\int \cos ntdt - \int\{\frac{d}{dt}(t^2)\int \cos ntdt\}dt\ [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t^2\frac{1}{n}\sin nt - \int 2t\frac{1}{n}\sin ntdt$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\int t\sin ntdt$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\int \sin ntdt - \int\{\frac{d}{dt}(t)\int \sin ntdt\}dt]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\frac{1}{n}(-\cos nt) - \int 1.\frac{1}{n}(-\cos nt)dt]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\frac{1}{n}(-\cos nt) + \int 1.\frac{1}{n}(\cos nt)dt]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\frac{1}{n}(-\cos nt) + \frac{1}{n}\int \cos ntdt]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\frac{1}{n}(-\cos nt) + \frac{1}{n}\frac{1}{n}\sin nt]$$

$$= \frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt \text{ ----------------------------(ii)}$$

And $\int t\cos ntdt$

$$= t\int \cos ntdt - \int\{\frac{d}{dr}(t)\int \cos ntdt\}dt\ [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\frac{1}{n}\sin nt - \int 1.\frac{1}{n}\sin ntdt$$

$$= \frac{t}{n}\sin nt - \frac{1}{n}\int \sin ntdt$$

$$= \frac{t}{n}\sin nt - \frac{1}{n}.\frac{1}{n}(-\cos nt)\ [\int \cos mx\,dx = \frac{1}{m}\sin mx; \int \sin mx\,dx = \frac{-1}{m}\cos mx]$$

$$= \frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt \text{ -----------------------------------(iii)}$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos ntdt + \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos ntdt$$

$$= \frac{1}{\pi}\left[\frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{-\pi}^{\pi}$$

[From (ii) & (iii)]

$$=\frac{1}{\pi}[\frac{\pi^2}{n}\sin n\pi+\frac{2\pi}{n^2}\cos n\pi-\frac{2}{n^3}\sin n\pi-\frac{(-\pi)^2}{n}\sin(-n\pi)-\frac{2(-\pi)}{n^2}\cos(-n\pi)-(-\frac{2}{n^3})\sin(-n\pi)]$$

$$+\frac{1}{\pi}[\frac{\pi}{n}\sin n\pi+\frac{1}{n^2}\cos n\pi-\frac{(-\pi)}{n}\sin(-n\pi)-\frac{1}{n^2}\cos(-n\pi)]$$

$$=\frac{1}{\pi}[\frac{\pi^2}{n}\sin n\pi+\frac{2\pi}{n^2}\cos n\pi-\frac{2}{n^3}\sin n\pi+\frac{\pi^2}{n}\sin n\pi+\frac{2\pi}{n^2}\cos n\pi-\frac{2}{n^3}\sin n\pi]$$

$$+\frac{1}{\pi}[\frac{\pi}{n}\sin n\pi+\frac{1}{n^2}\cos n\pi-\frac{\pi}{n}\sin n\pi-\frac{1}{n^2}\cos n\pi]$$

$$=\frac{1}{\pi}[0+\frac{2\pi}{n^2}\cos n\pi-0+0+\frac{2\pi}{n^2}\cos n\pi-0]+\frac{1}{\pi}[0+\frac{1}{n^2}\cos n\pi-0-\frac{1}{n^2}\cos n\pi]$$

$[\sin(-\theta)=-\sin\theta\ ;\ \cos(-\theta)=\cos\theta;\ \sin n\pi=0\ \text{for}\ n=1,2,3..........]$

$$=\frac{1}{\pi}[\frac{4\pi}{n^2}\cos n\pi]+\frac{1}{\pi}[\frac{1}{n^2}\cos n\pi-\frac{1}{n^2}\cos n\pi]$$

$$=\frac{4\pi}{\pi n^2}\cos n\pi+0$$

$$=\frac{4}{n^2}\cos n\pi$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi t}{L}dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin\frac{n\pi t}{\pi}dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin ntdt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin ntdt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\sin ntdt \qquad \text{[From (i)]}$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}t^2\sin ntdt+\frac{1}{\pi}\int_{-\pi}^{\pi}t\sin ntdt$$

Now $\int t^2\sin ntdt$

$$=t^2\int\sin ntdt-\int\{\frac{d}{dt}(t^2)\int\sin ntdt]dt\ [\because\int uvdx=u\int vdx-\int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$=t^2\int\sin ntdt-\int 2t\frac{1}{n}(-\cos nt)dt$$

$$=t^2\frac{1}{n}(-\cos nt)+\frac{2}{n}\int t\cos ntdt$$

$$=t^2\frac{1}{n}(-\cos nt)+\frac{2}{n}[t\int\cos ntdt-\int\{\frac{d}{dt}(t)\int\cos ntdt\}dt]$$

$$= t^2 \frac{1}{n}(-\cos nt) + \frac{2}{n}[t.\frac{1}{n}\sin nt - \int 1.\frac{1}{n}\sin ntdt]$$

$$= -t^2 \frac{1}{n}\cos nt + \frac{2}{n}[t.\frac{1}{n}\sin nt - \frac{1}{n}\int \sin ntdt]$$

$$= -t^2 \frac{1}{n}\cos nt + \frac{2}{n}[t.\frac{1}{n}\sin nt - \frac{1}{n}\frac{1}{n}(-\cos nt)]$$

$$= -t^2 \frac{1}{n}\cos nt + \frac{2}{n}[t.\frac{1}{n}\sin nt + \frac{1}{n^2}\cos nt]$$

$$= -t^2 \frac{1}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt \text{ --------------------(iv)}$$

$$\int t \sin ntdt$$

$$= t\int \sin ntdt - \int \{\frac{d}{dt}(t)\int \sin ntdt]dt \; [\because \int uvdx = u\int vdx - \int \{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\int \sin ntdt - \int 1.\frac{1}{n}(-\cos nt)dt$$

$$= t\frac{1}{n}(-\cos nt) + \frac{1}{n}\int \cos ntdt$$

$$= t\frac{1}{n}(-\cos nt) + \frac{1}{n}\frac{1}{n}\sin nt \; [\int \cos mx\, dx = \frac{1}{m}\sin mx; \int \sin mx\, dx = \frac{-1}{m}\cos mx]$$

$$= -t\frac{1}{n}\cos nt + \frac{1}{n^2}\sin nt \text{ ----------------------------(v)}$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin ntdt + \frac{1}{\pi}\int_{-\pi}^{\pi} t \sin ntdt$$

$$= \frac{1}{\pi}\left[-t^2\frac{1}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[-t\frac{1}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_{-\pi}^{\pi}$$

[From (iv) & (v)]

$$= \frac{1}{\pi}[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi - \frac{-(-\pi)^2}{n}\cos(-n\pi) - \frac{2(-\pi)}{n^2}\sin(-n\pi)$$

$$- \frac{2}{n^3}(\cos(-n\pi)] + \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi - \frac{-(-\pi)}{n}\cos(-n\pi) - \frac{1}{n^2}\sin(-n\pi)]$$

$$= \frac{1}{\pi}[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - \frac{2\pi}{n^2}\sin n\pi$$

$$- \frac{2}{n^3}\cos n\pi] + \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi - \frac{\pi}{n}\cos(-n\pi) + \frac{1}{n^2}\sin n\pi]$$

$$= \frac{1}{\pi}[-\frac{\pi^2}{n}\cos n\pi + 0 + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - 0 - \frac{2}{n^3}\cos n\pi] + \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi + 0$$

$$- \frac{\pi}{n}\cos n\pi + 0] \; [\because \sin(-\theta) = -\sin\theta \;,\; \cos(-\theta) = \cos\theta]$$

$$=\frac{1}{\pi}[-\frac{\pi^2}{n}\cos n\pi+\frac{2}{n^3}\cos n\pi+\frac{\pi^2}{n}\cos n\pi-\frac{2}{n^3}\cos n\pi]+\frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi-\frac{\pi}{n}\cos n\pi]$$

$$=\frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi-\frac{\pi}{n}\cos n\pi]$$

$$=-\frac{2}{n}\cos n\pi$$

The Fourier series for the above function is:

$$\therefore f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi t}{L}$$

$$\therefore f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{\pi}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi t}{\pi}$$

$$\therefore f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nt+\sum_{n=1}^{\infty}b_n\sin nt$$

$$\therefore f(t)=\frac{2\pi^2/3}{2}+\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi\cos nt+\sum_{n=1}^{\infty}(-\frac{2}{n}\cos n\pi)\sin nt$$

$$\underbrace{f(t)}_{\text{Complex wave}}=\underbrace{\frac{2\pi^2/3}{2}}_{\text{DC value}}+\underbrace{\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi\cos nt+\sum_{n=1}^{\infty}(-\frac{2}{n}\cos n\pi)\sin nt}_{\text{AC value}}\text{ Answer}\qquad\text{--------------(vi)}$$

From (vi),

$$\underbrace{f(t)}_{\text{Complex wave}}=\underbrace{\frac{2\pi^2/3}{2}}_{\text{DC value}}+\underbrace{\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi\cos nt+\sum_{n=1}^{\infty}(-\frac{2}{n}\cos n\pi)\sin nt}_{\text{AC value}}$$

$$\underbrace{f(t)}_{\text{Complex wave}}=\underbrace{\frac{\pi^2}{3}}_{\text{DC value}}+\underbrace{\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi\cos nt+\sum_{n=1}^{\infty}(-\frac{2}{n}\cos n\pi)\sin nt}_{\text{AC value}}$$

$$\underbrace{f(t)}_{\text{Complex wave}}=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi\cos nt+\sum_{n=1}^{\infty}(-\frac{2}{n}\cos n\pi)\sin nt$$

$$f(t)=\frac{\pi^2}{3}+(\frac{4}{1^2}\cos 1\pi\cos t+\frac{4}{2^2}\cos 2\pi\cos 2t+\frac{4}{3^2}\cos 3\pi\cos 3t+---)+$$
$$\{(-\frac{2}{1}\cos\pi)\sin t+(-\frac{2}{2}\cos 2\pi)\sin 2t+(-\frac{2}{3}\cos 3\pi)\sin 3t+-------\}$$

$$f(t)=\frac{\pi^2}{3}+4(-\cos t+\frac{1}{2^2}\cos 2t-\frac{1}{3^2}\cos 3t+---)+\ 2(\sin t-\frac{1}{2}\sin 2t+\frac{1}{3}\sin 3t+--\}\text{ --(vii)}$$

Given

$$y=f(t)=t^2+t\qquad;-\pi<t<\pi$$

Therefore, from (vii),

$$t^2+t=\frac{\pi^2}{3}+4(-\cos t+\frac{1}{2^2}\cos 2t-\frac{1}{3^2}\cos 3t+--)+\ 2(\sin t-\frac{1}{2}\sin 2t+\frac{1}{3}\sin 3t+--\}\text{ -----(viii)}$$

Put $t=\pi$ in (viii)

$$t^2+t=\frac{\pi^2}{3}+4(-\cos t+\frac{1}{2^2}\cos 2t-\frac{1}{3^2}\cos 3t+--)+2(\sin t-\frac{1}{2}\sin 2t+\frac{1}{3}\sin 3t+--\}$$

$$\pi^2+\pi=\frac{\pi^2}{3}+4(-\cos\pi+\frac{1}{2^2}\cos 2\pi-\frac{1}{3^2}\cos 3\pi+--)+2(\sin\pi-\frac{1}{2}\sin 2\pi+\frac{1}{3}\sin 3\pi+--\}$$

$$\pi^2+\pi=\frac{\pi^2}{3}+4(-(-1)+\frac{1}{2^2}(1)-\frac{1}{3^2}(-1)+--)+2(0-\frac{1}{2}(0)+\frac{1}{3}(0)+--\}$$

$$\pi^2+\pi=\frac{\pi^2}{3}+4(1+\frac{1}{2^2}+\frac{1}{3^2}+--)+0$$

$$\pi^2+\pi=\frac{\pi^2}{3}+4(1+\frac{1}{2^2}+\frac{1}{3^2}+--) \text{----------------(ix)}$$

Put $t=-\pi$ in (viii)

$$(-\pi)^2+(-\pi)=\frac{\pi^2}{3}+4(-\cos(-\pi)+\frac{1}{2^2}\cos 2(-\pi)-\frac{1}{3^2}\cos 3(-\pi)+--)+$$

$$2(\sin(-\pi)-\frac{1}{2}\sin 2(-\pi)+\frac{1}{3}\sin 3(-\pi)+--\}$$

$$\Rightarrow \pi^2-\pi=\frac{\pi^2}{3}+4(-\cos(\pi)+\frac{1}{2^2}\cos 2(\pi)-\frac{1}{3^2}\cos 3(\pi)+--)+$$

$$-2(\sin\pi+\frac{1}{2}\sin 2(\pi)-\frac{1}{3}\sin 3\pi+--\}$$

$$\pi^2-\pi=\frac{\pi^2}{3}+4(1+\frac{1}{2^2}+\frac{1}{3^2}+--)+0$$

$$\pi^2-\pi=\frac{\pi^2}{3}+4(1+\frac{1}{2^2}+\frac{1}{3^2}+--) \text{------------(x)}$$

Adding (ix) and (x)

$$\pi^2+\pi+\pi^2-\pi=\frac{\pi^2}{3}+4(1+\frac{1}{2^2}+\frac{1}{3^2}+--)+\frac{\pi^2}{3}+4(1+\frac{1}{2^2}+\frac{1}{3^2}+--)$$

$$2\pi^2=\frac{2\pi^2}{3}+8(1+\frac{1}{2^2}+\frac{1}{3^2}+--)$$

$$2\pi^2-\frac{2\pi^2}{3}=8(1+\frac{1}{2^2}+\frac{1}{3^2}+--)$$

$$\frac{6\pi^2-2\pi^2}{3}=8(1+\frac{1}{2^2}+\frac{1}{3^2}+--)$$

$$\frac{4\pi^2}{3}=8(1+\frac{1}{2^2}+\frac{1}{3^2}+--)$$

$$\frac{4\pi^2}{24}=(1+\frac{1}{2^2}+\frac{1}{3^2}+--)$$

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+-- \qquad \text{Answer}$$

Example 30:

$$y = f(t) = t\,; \qquad 0 \le t < \pi$$
$$= \pi\,; \qquad \pi \le t < 2\pi \quad \text{------------------------(i)}$$

Here, $T = 2L = 2\pi \qquad \therefore L = \pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

We have the Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \qquad \text{-------------------------(ii)}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \qquad [L = \pi]$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(t)dt \qquad [\int_{-a}^{+a} f(x)dx = \int_{0}^{2a} f(x)dx\text{; Method \#19, Integral Calculus}]$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{0}^{\pi} f(t)dt + \frac{1}{\pi}\int_{\pi}^{2\pi} f(t)dt$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{0}^{\pi} tdt + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi dt \qquad \text{[From (i)]}$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{0}^{\pi} tdt + \int_{\pi}^{2\pi} dt$$

$$\Rightarrow a_0 = \frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi} + [t]_{\pi}^{2\pi}$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left[\pi^2 - 0\right] + \left[2\pi - \pi\right]$$

$$\Rightarrow a_0 = \frac{1}{2\pi} \times \pi^2 + \pi$$

$$\Rightarrow a_0 = \frac{\pi}{2} + \pi$$

$$\Rightarrow a_0 = \frac{3\pi}{2}$$

We have,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\ \cos(n\frac{\pi}{L}t)dt$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(n\frac{\pi}{\pi}t)dt \qquad [L = \pi]$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(t)\cos ntdt \qquad [\int_{-a}^{+a} f(x)dx = \int_{0}^{2a} f(x)dx\text{; Method \#19, Integral Calculus}]$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos ntdt + \frac{1}{\pi}\int_{\pi}^{2\pi} f(t)\cos ntdt$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{0}^{\pi} t\times\cos ntdt + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\times\cos ntdt$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{0}^{\pi} t\times\cos ntdt + \int_{\pi}^{2\pi}\cos ntdt \qquad \text{------------------------(iii)}$$

Now, Let

$$I = \int t\times\cos ntdt$$

$$= t\int\cos ntdt - \int\left[\frac{d}{dt}(t)\times\int\cos ntdt\right]dt \qquad [\int uvdx = u\int vdx - \int\left\{\frac{d}{dx}(u)\int vdx\right\}dx]$$

$$= t\times\left(\frac{\sin nt}{n}\right) - \int 1\times\left(\frac{\sin nt}{n}\right)dt$$

$$= \frac{t}{n}\sin nt - \frac{1}{n}\times\left(\frac{-\cos nt}{n}\right)$$

$$= \frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt \qquad \text{------------------------(iv)}$$

Putting Result of $\int t\times\cos nt\,dt$ in (iii)

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} t\times\cos ntdt + \int_{\pi}^{2\pi}\cos ntdt$$

$$a_n = \frac{1}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi} + \left[\frac{\sin nt}{n}\right]_{\pi}^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi - \frac{0}{n}\sin(n\times 0) - \frac{1}{n^2}\cos(n\times 0)\right] + \frac{1}{n}[\sin(n\times 2\pi) - \sin n\pi]$$

$$a_n = \frac{1}{\pi}\left[\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi - \frac{0}{n}\sin(n\times 0) - \frac{1}{n^2}\cos 0\right] + \frac{1}{n}[\sin 2n\pi - \sin n\pi]$$

$$a_n = \frac{1}{\pi}\left(\frac{\pi}{n}\times 0 + \frac{1}{n^2}\cos n\pi - 0 - \frac{1}{n^2}.1\right) + \frac{1}{n}\sin n\pi$$

[$\sin n\pi = 0$ For any integer values of n & $\cos 0 = 1$]

$$a_n = \frac{1}{\pi}\left(\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right) + 0$$

$$a_n = \frac{1}{\pi n^2}(\cos n\pi - 1) \qquad \text{---------------------------------(v)}$$

Again, $\mathbf{b_n = \frac{1}{L}\int_{-L}^{L} f(t)\ \sin(n\frac{\pi}{L}t)dt}$

$$\mathbf{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin(n\frac{\pi}{\pi}t)dt} \qquad \mathbf{[L = \pi]}$$

$$\mathbf{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \sin ntdt}$$

$$\mathbf{b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(t)\sin ntdt} \qquad [\mathbf{\int_{-a}^{+a} f(x)dx = \int_{0}^{2a} f(x)dx\ ;\ Method\ \#19,\ Integral\ Calculus}]$$

$$\mathbf{b_n = \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin ntdt + \frac{1}{\pi}\int_{\pi}^{2\pi} f(t)\sin ntdt}$$

$$\mathbf{b_n = \frac{1}{\pi}\int_{0}^{\pi} t\times\sin nt\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\times\sin ntdt}$$

$$\mathbf{b_n = \frac{1}{\pi}\int_{0}^{\pi} t\times\sin nt\,dt + \int_{\pi}^{2\pi} \sin ntdt} \qquad \text{------------------------(vi)}$$

Now, Let

$$\mathbf{I = \int t\times\sin ntdt}$$

$$\mathbf{= t\int \sin ntdt - \int\left\{\frac{d}{dt}(t)\int \sin ntdt\right\}dt} \qquad [\mathbf{\int uvdx = u\int vdx - \int\left\{\frac{d}{dx}(u)\int vdx\right\}dx}]$$

$$\mathbf{= t\left[\frac{-\cos nt}{n}\right] - \int 1\times\left[\frac{-\cos nt}{n}\right]dt}$$

$$\mathbf{= -\frac{t}{n}\cos nt + \frac{1}{n}\int \cos ntdt}$$

$$\mathbf{= -\frac{t}{n}\cos nt + \frac{1}{n}\left[\frac{\sin nt}{n}\right]}$$

$$\mathbf{= -\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt}$$

$$\mathbf{= \frac{1}{n^2}\sin nt - \frac{t}{n}\cos nt} \qquad \text{----------------------------------(vii)}$$

Putting Result of $\mathbf{\int t\times\sin ntdt}$ in (vi)

$$\mathbf{b_n = \frac{1}{\pi}\int_{0}^{\pi} t\times\sin nt\,dt + \int_{\pi}^{2\pi} \sin ntdt}$$

$$\mathbf{b_n = \frac{1}{\pi}\left[\frac{1}{n^2}\sin nt - \frac{t}{n}\cos nt\right]_0^{\pi} + \left[\frac{-\cos nt}{n}\right]_{\pi}^{2\pi}}$$

$$\mathbf{b_n = \frac{1}{\pi}\left(\frac{1}{n^2}\sin n\pi - \frac{\pi}{n}\cos n\pi - \frac{1}{n^2}\sin(n\times 0) - \frac{0}{n}\cos(n\times 0)\right) - \frac{1}{n}(\cos 2n\pi - \cos n\pi)}$$

$$b_n = \frac{1}{\pi}\left(\frac{1}{n^2}\times 0 - \frac{\pi}{n}\cos n\pi - \frac{1}{n^2}\sin 0 - \frac{0}{n}\cos 0\right) - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = \frac{1}{\pi}\left(0 - \frac{\pi}{n}\cos n\pi - 0 - 0\right) - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = \frac{1}{\pi}\left(-\frac{\pi}{n}\cos n\pi\right) - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = -\frac{1}{n}\cos n\pi - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = -\frac{1}{n}\cos n\pi - \frac{1}{n}\cos 2n\pi + \frac{1}{n}\cos n\pi$$

$$b_n = -\frac{1}{n}\cos 2n\pi \qquad \text{----------------------------------(viii)}$$

Putting the values of $\mathbf{a_0}$, $\mathbf{a_n}$, $\mathbf{b_n}$ in (ii)

The Fourier series for the above function is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$= \frac{\frac{3\pi}{2}}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}(\cos n\pi - 1)\cos\left(\frac{n\pi t}{\pi}\right) + \sum_{n=1}^{\infty} -\frac{1}{n}\cos 2n\pi \sin\left(\frac{n\pi t}{\pi}\right)$$

$$= \frac{3\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}(\cos n\pi - 1)\cos nt - \sum_{n=1}^{\infty} \frac{1}{n}\cos 2n\pi \sin nt$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{3\pi}{4}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{\pi n^2}(\cos n\pi - 1)\cos nt - \sum_{n=1}^{\infty} \frac{1}{n}\cos 2n\pi \sin nt}_{\text{AC value}} \quad \textit{Answer}$$

Example 31:

If $y = f(t) = t^2$ over the integral $-\pi < t < \pi$ and has period 2π. Here, $T = 2L = 2\pi \therefore L = \pi$

a) **Sketch three cycles of $y = f(t)$ in the interval $-3\pi < t < 3\pi$**

b) **Find the Fourier series for the function.**

c) *Hence, find the sum of the series* $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + --$

Answer:

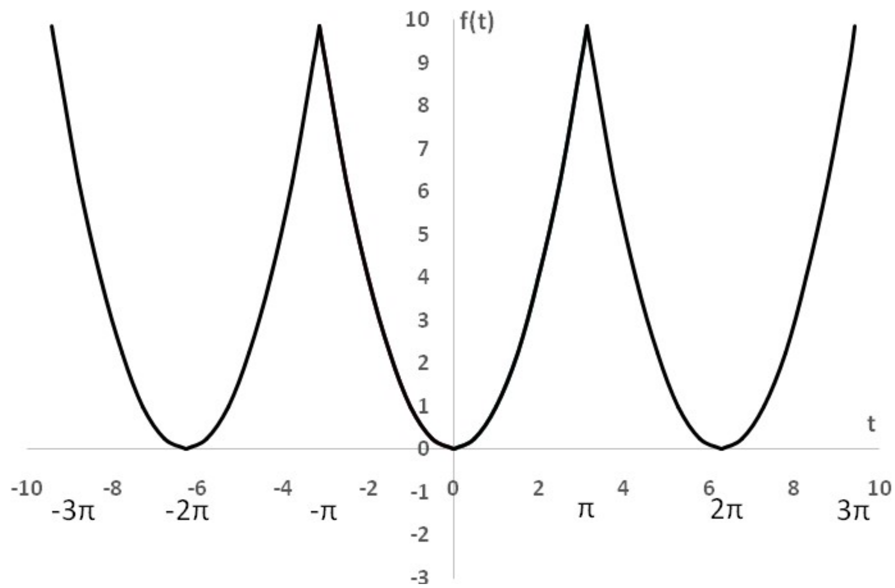


Figure 60: A periodic signal with period $\mathbf{T = 2L = 2\pi}$

Let $\boldsymbol{y = f(t) = t^2}$ --------------- (i)

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt \qquad \text{[From (i)]}$$

$$= \frac{1}{\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos ntdt \qquad \text{[From (i)]}$$

Now $\int t^2 \cos ntdt$

$$= t^2\int \cos ntdt - \int\{\frac{d}{dt}(t^2)\int \cos ntdt\}dt \quad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= \frac{t^2}{n}\sin nt - \int\{2t.\frac{\sin nt}{n}\}dt$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\left[t\int \sin nt - \int\{\frac{d}{dt}(t)\int \sin ntdt\}dt\right]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\left[t.\frac{(-\cos nt)}{n} - \int\frac{(-\cos nt)}{n}dt\right]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\left[\frac{-t\cos nt}{n} + \int\frac{\cos nt}{n}dt\right]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\left[\frac{-t\cos nt}{n} + \frac{\sin nt}{n^2}\right]$$

$$= \frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt$$

$$\int t^2 \cos ntdt = \frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt \text{ --------------- (ii)}$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos ntdt$$

$$= \frac{1}{\pi}\left[\frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_{-\pi}^{\pi} \qquad \text{[From (ii)]}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{n}\sin n\pi + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\sin n\pi - \frac{(-\pi)^2}{n}\sin n(-\pi) - \frac{2(-\pi)}{n^2}\cos n(-\pi) + \frac{2}{n^3}\sin n(-\pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{n}\sin n\pi + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\sin n\pi + \frac{\pi^2}{n}\sin n\pi + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\sin n\pi\right]$$

$$[\sin(-\theta) = -\sin\theta;\ \cos(-\theta) = \cos\theta]$$

$$= \frac{1}{\pi}\left[0 + \frac{2\pi}{n^2}\cos n\pi - 0 + 0 + \frac{2\pi}{n^2}\cos n\pi - 0\right]$$

$$[\sin n\pi = 0; \text{For } n = 1,2,3..........]$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi + \frac{2\pi}{n^2}\cos n\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\cos n\pi\right]$$

$$= \frac{4}{n^2}\cos n\pi$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\sin ntdt \qquad \text{[From (i)]}$$

Now $\int t^2 \sin ntdt$

$$= t^2\int \sin ntdt - \int\{\frac{d}{dt}(t^2)\int \sin ntdt\}dt \ [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t^2\int \sin ntdt - \int\{2t.\frac{1}{n}(-\cos nt)\}dt$$

$$= t^2\frac{1}{n}(-\cos nt) + \frac{2}{n}\int t\cos ntdt$$

$$= t^2\frac{1}{n}(-\cos nt) + \frac{2}{n}\left[t\int \cos nt - \int\{\frac{d}{dt}(t)\int \cos ntdt\}dt\right]$$

$$= -t^2\frac{1}{n}\cos nt + \frac{2}{n}\left[t.\frac{1}{n}\sin nt - \int 1.\frac{1}{n}\sin ntdt\right]$$

$$= -t^2\frac{1}{n}\cos nt + \frac{2}{n}\left[t.\frac{1}{n}\sin nt - \frac{1}{n}\int \sin ntdt\right]$$

$$= -t^2\frac{1}{n}\cos nt + \frac{2}{n}\left[t.\frac{1}{n}\sin nt - \frac{1}{n}.\frac{1}{n}(-\cos nt)\right]$$

$$= -t^2\frac{1}{n}\cos nt + \frac{2}{n}\left[t.\frac{1}{n}\sin nt + \frac{1}{n^2}\cos nt\right]$$

$$\int t^2 \sin ntdt = -t^2 \frac{1}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt \text{ ---------------(iii)}$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin ntdt$$

$$= \frac{1}{\pi}\left[-t^2 \frac{1}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt\right]_{-\pi}^{\pi} \qquad \text{[From (iii)]}$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi - \frac{-(-\pi)^2}{n}\cos n(-\pi) - \frac{2(-\pi)}{n^2}\sin n(-\pi) - \frac{2}{n^3}\cos n(-\pi)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - \frac{2\pi}{n^2}\sin n\pi - \frac{2}{n^3}\cos n\pi\right]$$

$$[\sin(-\theta) = -\sin\theta;\ \cos(-\theta) = \cos\theta]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + 0 + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - 0 - \frac{2}{n^3}\cos n\pi\right]$$

$$[\sin n\pi = 0; \text{For } n = 1,2,3.........]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cancel{\cos n\pi} + \frac{2}{n^3}\cancel{\cos n\pi} + \frac{\pi^2}{n}\cancel{\cos n\pi} - \frac{2}{n^3}\cancel{\cos n\pi}\right]$$

$$= \frac{1}{\pi}[0]$$

$$= 0$$

The Fourier series for the above function is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$= \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi \cos nt + 0$$

$$= \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi \cos nt$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{2\pi^2/3}{2}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi \cos nt}_{\text{AC value}} \qquad \text{Answer------------------(iv)}$$

$$f(t) = \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi \cos nt$$

$$f(t) = \frac{\pi^2}{3} + \left(\frac{4}{1^2}\cos\pi\cos t + \frac{4}{2^2}\cos 2\pi \cos 2t + \frac{4}{3^2}\cos 3\pi\cos 3t + \frac{4}{4^2}\cos 4\pi\cos 4t + -----\right)$$

$$f(t) = \frac{\pi^2}{3} + \left(-4\cos t + \frac{4}{2^2}\cos 2t - \frac{4}{3^2}\cos 3t + \frac{4}{4^2}\cos 4t + -----\right)$$

$$f(t)=\frac{\pi^2}{3}-4(\cos t-\frac{1}{2^2}\cos 2t+\frac{1}{3^2}\cos 3t-\frac{1}{4^2}\cos 4t+-----)\text{--------------(v)}$$

Given, $y = f(t) = t^2\,;-\pi < t < \pi$

$$t^2=\frac{\pi^2}{3}-4(\cos t-\frac{1}{2^2}\cos 2t+\frac{1}{3^2}\cos 3t-\frac{1}{4^2}\cos 4t+-----)\text{---------------------(vi)}$$

Put $t=0$ in (vi)

$$0^2=\frac{\pi^2}{3}-4(\cos 0-\frac{1}{2^2}\cos 2.0+\frac{1}{3^2}\cos 3.0-\frac{1}{4^2}\cos 4.0+-----)$$

$$0^2=\frac{\pi^2}{3}-4(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+-----)$$

$$-\frac{\pi^2}{3}=-4(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+-----)$$

$$\frac{\pi^2}{3}=4(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+-----)$$

$$\frac{\pi^2}{12}=(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+-----)$$

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+-----=\frac{\pi^2}{12}$$

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+-----=\frac{\pi^2}{12}\ \text{Answer}$$

Home Task

Find the Fourier series for the functions

01.

$y = f(t) = 1\,; \quad -\pi \le t < 0$

$\qquad\quad = 0\,; \qquad 0 \le t < \pi$

Here, $T = 2L = 2\pi \qquad \therefore L = \pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function.

02.

$y = f(t) = 0\,; \quad -\pi \le t < 0$

$\qquad\quad = t\,; \qquad 0 \le t < \pi$

Here, $T = 2L = 2\pi \qquad \therefore L = \pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

03.

$y = f(t) = \frac{t}{2}$ Over the interval $0 < t < 2\pi$ and has period 2π

Here, $T = 2L = 2\pi \qquad \therefore L = \pi$

a) Sketch a graph of $y = f(t)$ in the interval $\mathbf{0 < t < 4\pi}$
b) Find the Fourier series for the function

04.

$$y = f(t) = \pi - t\,; \qquad 0 < t < \pi$$
$$= 0\,; \qquad \pi \le t < 2\pi$$

Here, $\mathbf{T = 2L = 2\pi} \qquad \therefore \mathbf{L = \pi}$

a) Sketch a graph of $y = f(t)$ in the interval $\mathbf{-2\pi < t < 2\pi}$
b) Find the Fourier series for the function

https://www.youtube.com/watch?v=zcIvx6F-91I

Problem 09: Symmetry in Waveforms

Many periodic waveforms exhibit symmetry. The following three types of symmetry help to reduce tedious calculations in the analysis.

i. Even symmetry
ii. Odd symmetry
iii. Half-wave symmetry

Even Symmetry

A function $\mathbf{f(t)}$ exhibits even symmetry, when the region before the *y-axis* is the mirror image of the region after the *y-axis.* That is, $\mathbf{f(t) = f(-t)}$

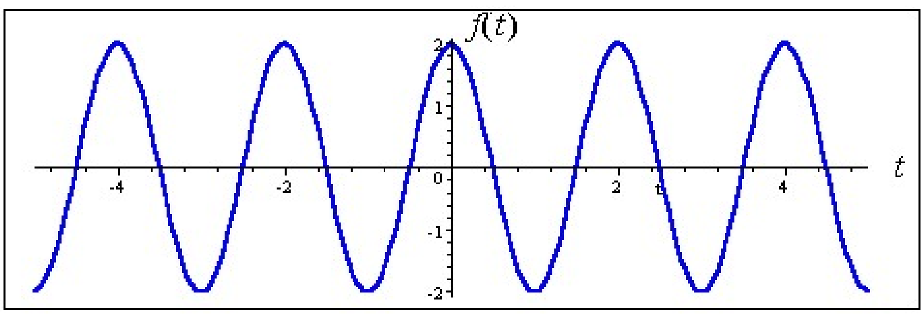


Figure 61

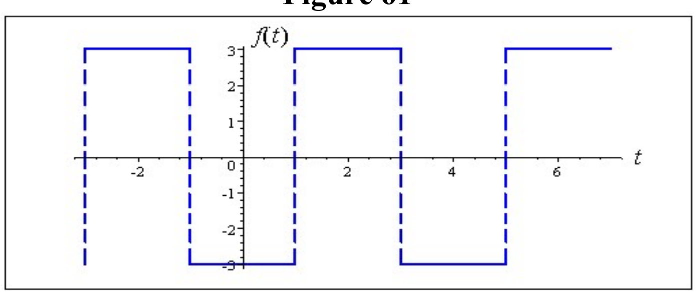


Figure 62: Even Square wave

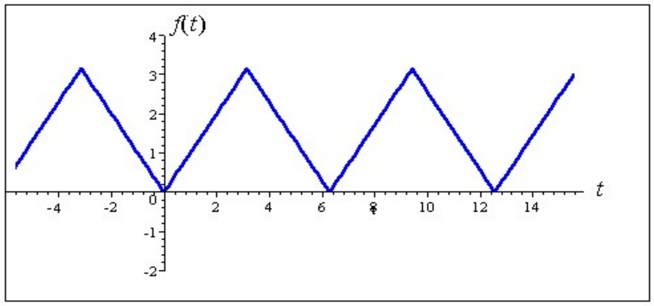


Figure 63: Triangular wave

$$f(t) = \begin{cases} t+\pi & \text{if } -\pi \le t < 0 \\ -t+\pi & \text{if } 0 \le t < \pi \end{cases}$$

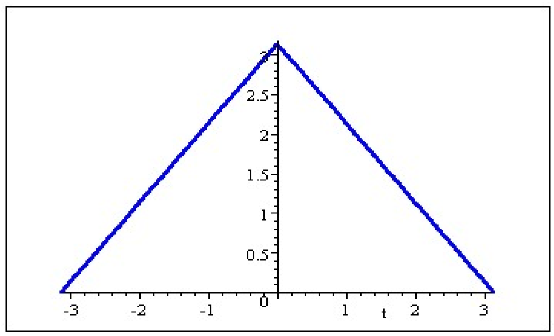


Figure 64: The graph that it is **even**

$$f(t) = \begin{cases} e^{t} & \text{if } -\pi \le t < 0 \\ e^{-t} & \text{if } 0 \le t < \pi \end{cases}$$

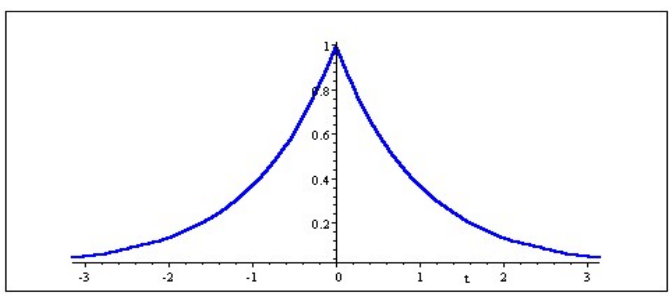


Figure 65

Odd Symmetry

A function $\mathbf{f(t)}$ exhibits even symmetry, when the region before the *y-axis* is the negative of the mirror image of the region after the *y-axis.* That is, $\mathbf{f(t) = -f(-t)}$

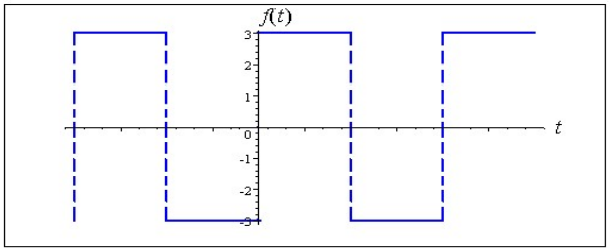


Figure 66: Odd Square wave

$$f(t) = \begin{cases} \left(t+\frac{\pi}{2}\right)^2 & \text{if } -\pi \le t < 0 \\ -\left(t-\frac{\pi}{2}\right)^2 & \text{if } 0 \le t < \pi \end{cases}$$

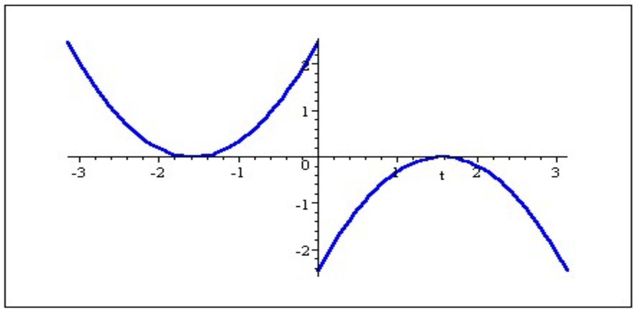


Figure 67: The graph that the function is **odd**

Half-wave Symmetry

A function $\mathbf{f(t)}$ exhibits half-wave symmetry, when one half of the waveform is exactly equal to the negative of the previous or the next half of the waveform. That is,

$$\mathbf{f(t) = (-)f(t - \frac{T}{2}) = (-)f(t + \frac{T}{2})}$$

Summary of Analysis of waveforms with symmetrical properties

i. With **even symmetry**, $\mathbf{b_n = 0}$ for all n, and $\mathbf{a_n}$ is twice the integral over half the cycle from zero time

ii. With **odd symmetry**, $\mathbf{a_n = 0}$ for all n, and $\mathbf{b_n}$ is twice the integral over half the cycle from zero time

iii. With **half-wave symmetry**, $\mathbf{a_n}$ and $\mathbf{b_n}$ are 0 for even n, and twice the integral over any half cycle for odd n

iv. If **half-wave symmetry** and either **even symmetry** or **odd symmetry** are present, then $\mathbf{a_n}$ and $\mathbf{b_n}$ are 0 for even n, and four times the integral over the quarter cycle for odd n for $\mathbf{a_n}$ or $\mathbf{b_n}$ respectively and zero for the remaining coefficient.

v. It is also to be noted that in any waveform, $\mathbf{\frac{a_0}{2}}$ corresponds to the mean value of the waveform and that sometimes a symmetrical property may be obtained by subtracting this value from the waveform

Problem 10:

Express $\boldsymbol{a\cos\theta \pm b\sin\theta}$ in the form $\boldsymbol{R\sin(\theta \pm \alpha)}$ that is $\boldsymbol{a\cos\theta \pm b\sin\theta = R\sin(\theta \pm \alpha)}$

Solution:

First we take the "plus" case, $(\theta+\alpha)$ to make things easy.

Let, $a\cos\theta + b\sin\theta = R\sin(\theta+\alpha)$ ------------------------------(i)

Here,

$a \rightarrow$ **Amplitude of the cosine wave**

$b \rightarrow$ **Amplitude of the sine wave**

$R \rightarrow$ **Amplitude of the new signal**

$\alpha \rightarrow$ **Phase Shift**

[$a\cos\theta + b\sin\theta = R\sin(\theta+\alpha)$ লিখা যায়। অর্থাৎ $a\cos\theta$ & $b\sin\theta$ যোগ করলে যে নতুন সিগন্যাল পাওয়া যায় তার সর্ব্বোচ্চ Amplitude হবে R আর এই নতুন সিগন্যাল sine হবে যদি $\alpha = \pi$ or 0 হয় এবং এই নতুন সিগন্যাল cos হবে যদি $\alpha = \frac{\pi}{2}$ হয়]

Using the compound angle formula from before (Sine of the sum of angles),

$\sin(A+B) = \sin A\cos B + \cos A\sin B$ ---------------------------(ii)

We can expand the **RHS** of the above equation (i), as follows:

$R\sin(\theta+\alpha)$

$R\sin(\theta+\alpha) \equiv R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$ $[\because \sin(A+B) = \sin A\cos B + \cos A\sin B]$

$R\sin(\theta+\alpha) \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$ ---------------------(iii)

From. (i),

So $a\cos\theta + b\sin\theta = R\sin(\theta+\alpha)$

$a\cos\theta + b\sin\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$ [From (iii)]

$a\cos\theta + b\sin\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$a\cos\theta + b\sin\theta = \mathrm{R}\cos\theta\sin\alpha + \mathrm{R}\sin\theta\cos\alpha$ ------------------(iv)

Equating the coefficients of $\cos\theta$ and $\sin\theta$ from (iv), we have:

$\mathrm{a} = \mathrm{R}\sin\alpha$ --------------------------(v)

$\mathbf{b} = \mathbf{R}\cos\alpha$ --------------------------(vi)

From (v) and (vi),

$\frac{a}{b} = \frac{R\sin\alpha}{R\cos\alpha}$

$\Rightarrow \frac{a}{b} = \tan\alpha$

$\Rightarrow \tan\alpha = \frac{a}{b}$

$\Rightarrow \alpha = \tan^{-1}(\frac{a}{b})$ -------------------(vii)

Again, from (v) and (vi), squaring and then adding

$a^2 + b^2 = R^2\sin^2\alpha + R^2\cos^2\alpha$

$\Rightarrow a^2 + b^2 = R^2(\sin^2\alpha + \cos^2\alpha)$

$\Rightarrow a^2 + b^2 = R^2.1$ $\qquad\qquad [\because \sin^2\alpha + \cos^2\alpha = 1]$

$\Rightarrow a^2 + b^2 = R^2$

$\Rightarrow R^2 = a^2 + b^2$

$\Rightarrow R = \sqrt{a^2 + b^2}$ ------------------(viii)

Similarly, for the minus case:
$a\cos\theta - b\sin\theta = R\sin(\theta - \alpha)$

We get,

$\Rightarrow R = \sqrt{a^2 + b^2}$ and $\Rightarrow \alpha = \tan^{-1}(\frac{a}{b})$

Here R is called the resultant amplitude and α is the phase difference.

[$a\cos\theta + b\sin\theta = R\sin(\theta + \alpha)$ লিখা যায়। অর্থাৎ $a\cos\theta + b\sin\theta$ যোগ করলে যে নতুন সিগন্যাল পাওয়া যায় তার সর্ব্বোচ্চ Amplitude হবে R আর এই নতুন সিগন্যাল sine হবে যদি $\alpha = \pi$ *or* 0 হয় এবং এই নতুন সিগন্যাল cos হবে যদি $\alpha = \frac{\pi}{2}$ হয়, যেখানে নতুন সিগন্যালের Amplitude $R = \sqrt{a^2 + b^2}$]

Example 32:

Let, $x = \frac{1}{4}\cos 8t + \frac{1}{8}\sin 8t$ ---(ix)

What is the amplitude of the resultant signal?

Answer:

We have, $a\cos\theta + b\sin\theta = R\sin(\theta + \alpha)$

So, in $x = \frac{1}{4}\cos 8t + \frac{1}{8}\sin 8t$

Here,

$a = \frac{1}{4}$ *and* $b = \frac{1}{8}$ *and* $\theta = 8$

(ix) নং সমীকরণ হতে, প্রথম সিগন্যালের সবোর্চ্চ Amplitude $\frac{1}{4}$ এবং দ্বিতীয় সিগন্যালের সবোর্চ্চ Amplitude $\frac{1}{8}$। এই দুটি সিগন্যাল যোগ করলে যে নতুন সিগন্যাল পাওয়া যাবে তার সর্বোচ্চ Amplitude যদি R হয় তাহলে R এর মান বের করতে পারি যেখানে $R = \sqrt{(\frac{1}{4})^2 + (\frac{1}{8})^2}$ কারণ $[\because \mathbf{R} = \sqrt{\mathbf{a}^2 + \mathbf{b}^2}]$

$$\Rightarrow R = \sqrt{(\frac{1}{4})^2 + (\frac{1}{8})^2}$$

$$\Rightarrow R = \sqrt{\frac{1}{16} + \frac{1}{64}}$$

$$\Rightarrow \mathbf{R} = \sqrt{\frac{\mathbf{4+1}}{\mathbf{64}}}$$

$$\Rightarrow \mathbf{R} = \sqrt{\frac{\mathbf{5}}{\mathbf{64}}}$$

$$\Rightarrow \mathbf{R} = \frac{\sqrt{\mathbf{5}}}{\mathbf{8}} = 0.2795\textit{\textbf{Answer}}$$

আর এই দুটি সিগন্যাল যোগ করলে সেই নতুন সিগন্যালটি sine এর আকার ধারণ করতে পারে আবার cosine এর আকার ধারণ করতে পারে। Remember that sine wave and cosine wave with Phase difference is 90^0, Figure 19

<u>Problem 11:</u>

A signal can be viewed from two different standpoints:

- The time domain
- The frequency domain

The frequency domain representation appears graphically as a series of Spikes occurring at the fundamental frequency (determined by the period of the original function) and its harmonics. The magnitudes of these spikes are the **Fourier coefficients**. This series of components are called the **wave spectrum.**

The characteristics of electrical (and other) signals can be explored in two ways. The first is to examine their waveforms in the time domain, which can be done using an oscilloscope. The second is to examine their spectra in the frequency domain, which can be done using a *spectrum analyzer*. Both descriptions are useful, although frequency domain analysis is the most natural tool for many tasks in electrical circuit design, speech and music analysis, and for video signals. In fact, frequency domain analysis is of fundamental importance in all branches of engineering, from the design of bridges and roads to wireless Communications.

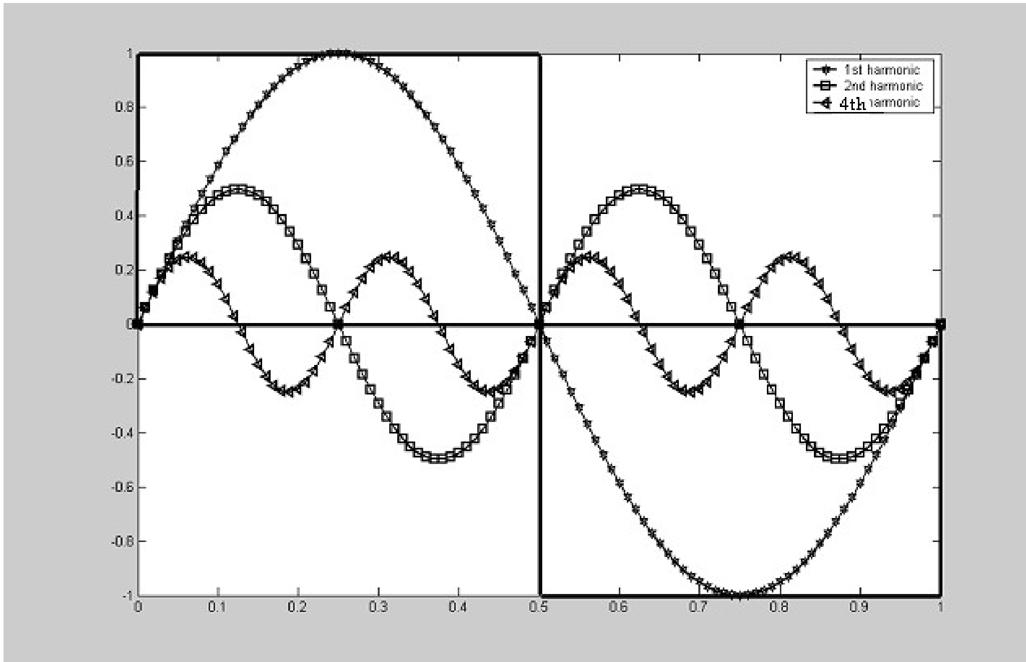


Figure 68: Time domain

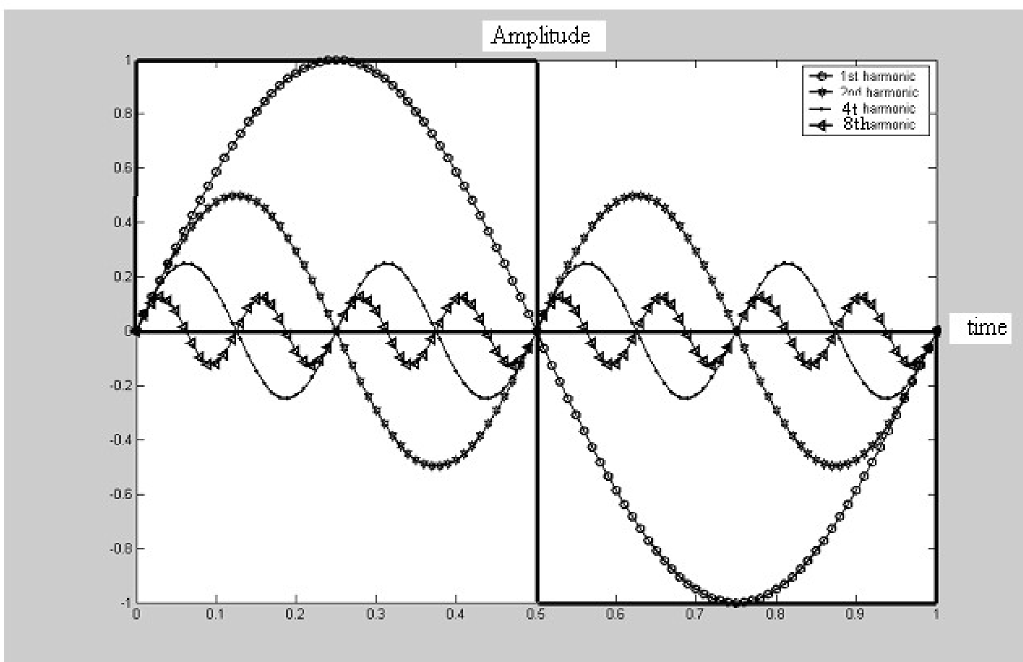


Figure 69: Time domain

Problem 12:Frequency Spectrum

The **Fourier series** is a method of expressing most periodic, time-domain functions in the frequency domain.

Line spectrum or **amplitude spectrum: A plot of amplitude against angular frequency is called the amplitude spectrum.**
Phase spectrum: while that of phase against angular frequency is called the phase spectrum**.**

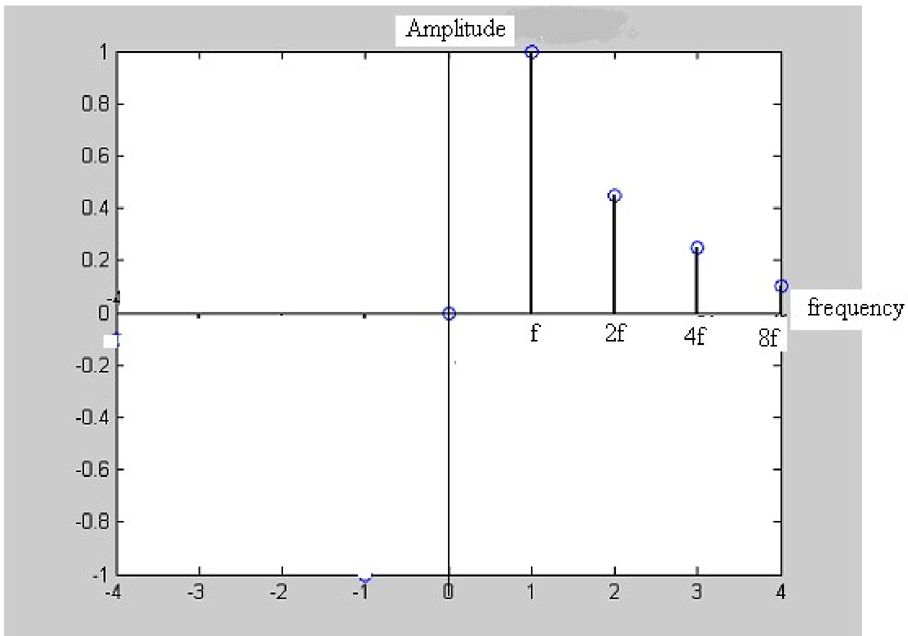


Figure 70: Frequency domain

Example 33:
A Fourier series is given below. List frequencies and plot the one-sided amplitude spectrum

$$\underbrace{x(t)}_{Complex\ wave} = \underbrace{12}_{DC\ value} \underbrace{+9\cos(2\pi\times 10t+\frac{\pi}{3})+6\cos(2\pi\times 20t-\frac{\pi}{6})+4\cos(2\pi\times 30t+\frac{\pi}{4})}_{AC\ value}$$

$[\because \omega = 2\pi f]$

Here DC value is 12; Amplitudes are 9, 6, and 4 with respect to frequencies 10, 20, 30 respectively where phases are $\frac{\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{4}$

The frequencies are 0 (dc), 10 Hz, 20 Hz, 30 Hz

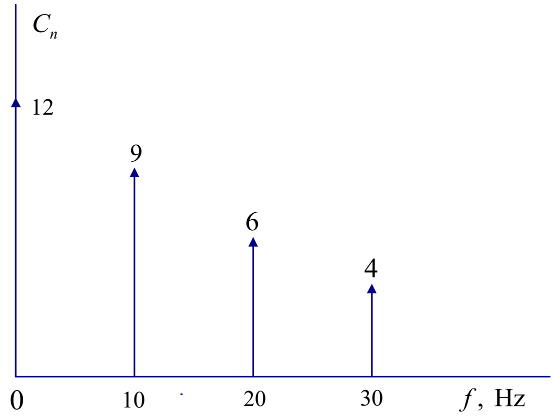


Figure 71

Spectrum Analysis

Spectrum analysis attempts to identify the relative quantities of different frequencies which are present in a given signal. Take, for example, a simple signal, $\mathbf{x(t)}$ made up of two pure sine-waves with frequencies 1 rad/second and 2 rad/second respectively:

$\mathbf{x(t) = 3\sin(1.t) + \sin(2t)}$ ---------------------------------(i)

There are exactly two `frequency' components present in this signal: 1 rad/second and 2 rad/second. The `amplitude' of the first is three times that of the second. We can represent this information in a plot of amplitude against frequency: a frequency amplitude spectrum

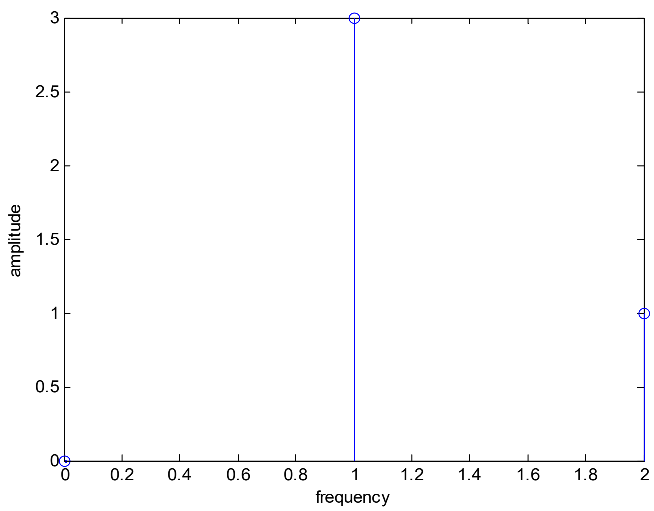


Figure 72

Problem 13: **Time domain / Frequency domain**

- Some signals are easier to visualize in the frequency domain
- Some signals are easier to visualize in the time domain
- Some signals are easier to define in the time domain (amount of information needed)
- Some signals are easier to define in the frequency domain (amount of information needed)

Example: speech recognition

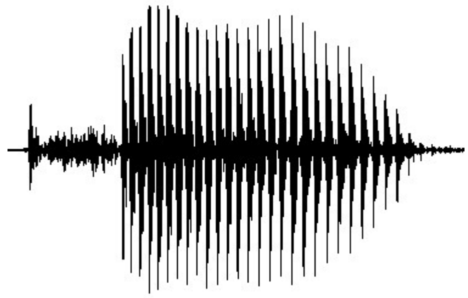


Figure 73

Difficult to differentiate between different sounds in time domain
This time-domain plot shows a waveform that was once very familiar to electrical engineers: an amplitude modulated (AM) radio signal.

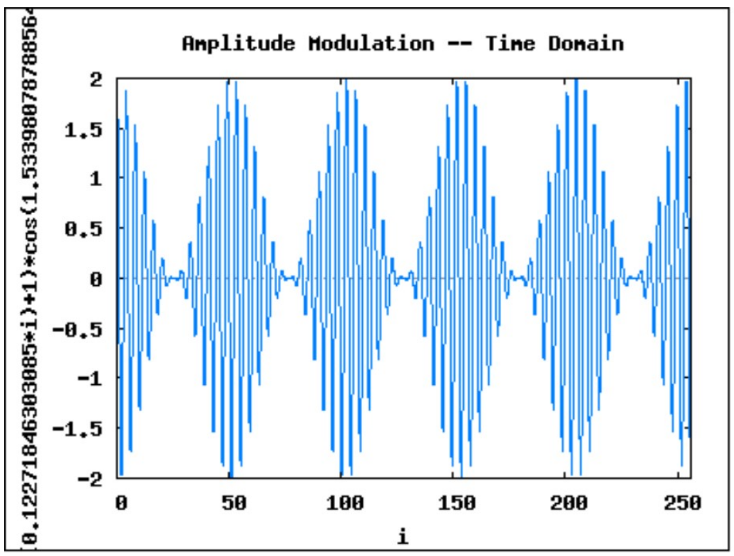


Figure 74

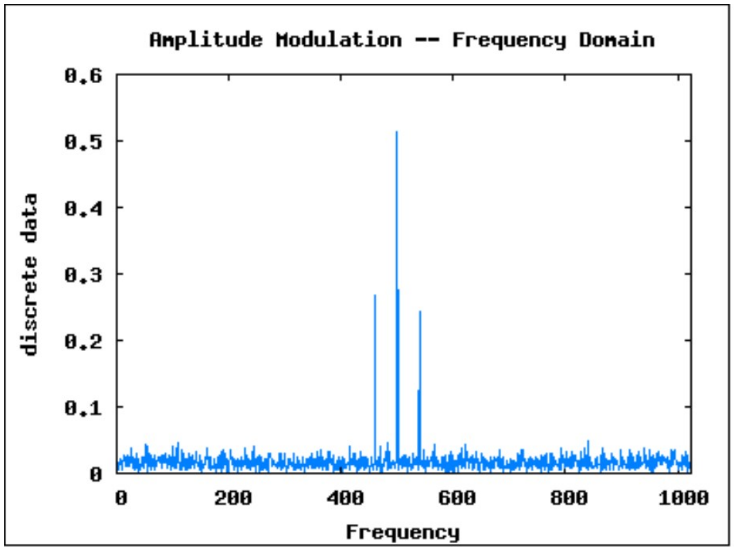


Figure 75

Here is the frequency-domain equivalent of the above time-domain plot

Problem 14: **Signal Spectrum**

Every signal has a frequency spectrum.

• The signal defines the spectrum

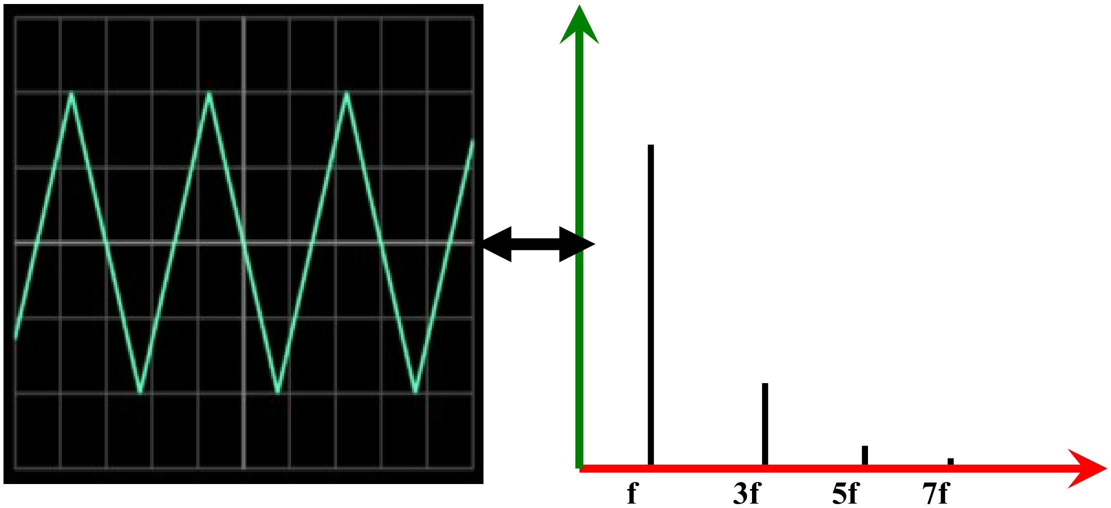
• The spectrum defines the signal

We can move back and forth between the time domain and the frequency domain without losing information

These sine terms are combined to create the time-varying waveform in the display. And if the opposite operation were to be performed on the result (something called a Fourier transform), these individual harmonic elements would reappear. It is important to reemphasize that waveform generation, and the Fourier transform, are reciprocal operations. You can use frequency components to generate a waveform in the time domain, then transform the result back to the frequency domain and recover what you started with. This reciprocal relationship is to Fourier analysis what the Fundamental Theorem of Calculus (the idea that integration and derivation are reciprocal operations) is to Calculus.

Waveform creation in the time domain, and harmonic analysis in the frequency domain, is reciprocal operations. One can take a list of harmonic components and use them to create a time-domain waveform, and then one can carry out a Fourier transform on the time-domain waveform to recapture the original harmonic components. It turns out that these two representations are equivalent and interchangeable.

Sound spectrum:



Time domain *frequency domain*

Figure 76

Spectrum Analysis **or** ***Fourier Analysis*** is the process of analyzing some time-domain waveform to find its spectrum. We also say that the time domain waveform is converted into a frequency spectrum by means of the *Fourier transform*. This process is reversible: using the *inverse Fourier transform* a spectrum may be converted back into a time-domain waveform.

Example 34:

A plot showing each of the harmonic amplitudes in the wave is called the line spectrum**.**

<u>Plot the line spectrum</u> (discrete frequency spectra) for the Fourier series:

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{2n-1} \cos nt + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \sin nt \text{ ---------------------------------------(i)}$$

This series has an interesting graph for the above function $\mathbf{f(t)}$

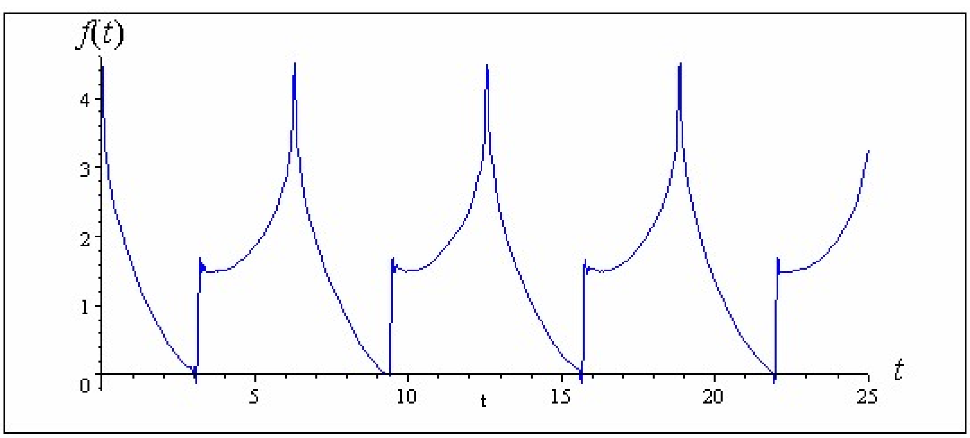


Figure 77

We have the Fourier series is $\boldsymbol{f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\omega t)+b_n\sin(n\omega t))}$

We can see from the series (i) that

$a_n = \frac{1}{2n-1} \qquad b_n = \frac{(-1)^n}{2n}$

Now, using $\Rightarrow \boldsymbol{R=\sqrt{a^2+b^2}}$ [From (viii), page no 66]

Let $\boldsymbol{R_n = C_n = \sqrt{a_n{}^2+b_n{}^2}}$

$a_n = \frac{1}{2n-1}$	$b_n = \frac{(-1)^n}{2n}$	$C_n = \sqrt{a_n^2+b_n^2}$
$a_1 = 1$	$b_1 = -\frac{1}{2}$	$C_1 = \sqrt{1^2+\left(-\frac{1}{2}\right)^2} = 1.118$
$a_2 = \frac{1}{3}$	$b_2 = \frac{1}{4}$	$C_2 = \sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{4}\right)^2} = 0.4167$
$a_3 = \frac{1}{5}$	$b_3 = -\frac{1}{6}$	$C_3 = \sqrt{\left(\frac{1}{5}\right)^2+\left(-\frac{1}{6}\right)^2} = 0.260$
$a_4 = \frac{1}{7}$	$b_4 = \frac{1}{8}$	$C_4 = \sqrt{\left(\frac{1}{7}\right)^2+\left(\frac{1}{8}\right)^2} = 0.190$

Here, from (i),

Here, $\mathbf{n\omega = n}$

For n = 1; **Fundamental Frequency = 1st Harmonic =** $\omega = 1$

For n = 2; **2nd Harmonic** $= 2\omega = 2$

For n = 3; **3rd Harmonic** $= 3\omega = 3$

For n = 4; **4th Harmonic** $= 4\omega = 4$

The resulting line spectrum is:

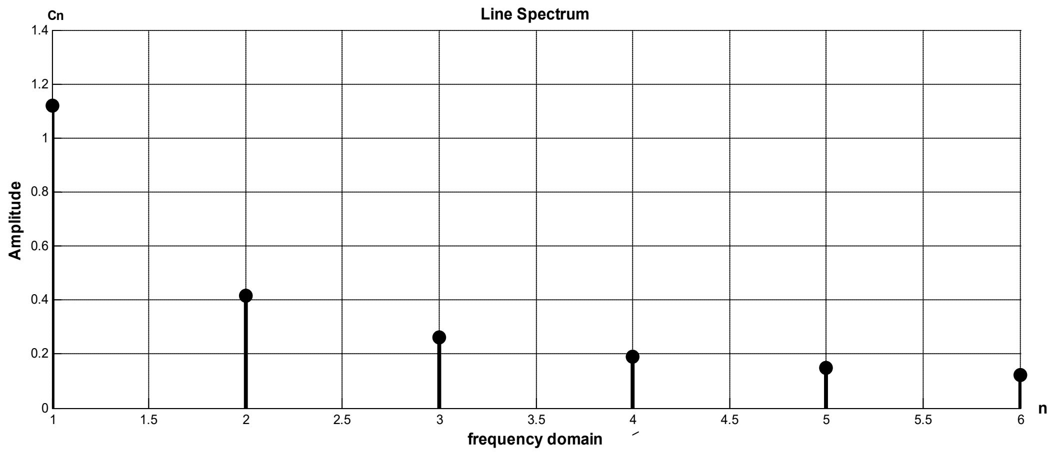


Figure 78: Line spectrum

Example 35:
Line spectrum: Example

$$\mathbf{s(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t) + \frac{1}{5}\cos(10\pi f_0 t) - \ldots\right)}$$

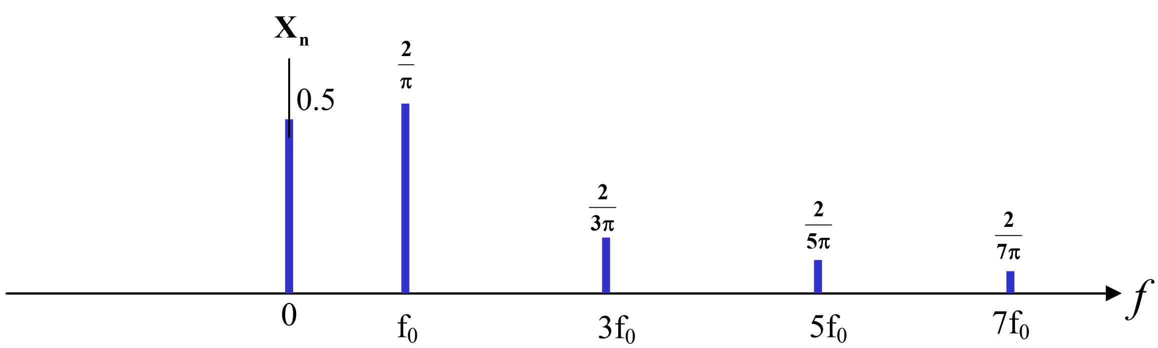


Figure 79

Where $\mathbf{X_n = \sqrt{a_n^2 + b_n^2}}$

Home Task:

Plot the line spectrum (discrete frequency spectra) for the Fourier series:

$$f(t) = 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$$

Figure 74

We have the Fourier series is $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t))$

Here, $a_n = 0$; $b_n = -\frac{5}{\pi}\frac{1}{n}(\cos n\pi - 1)$

Now, using $\Rightarrow R = \sqrt{a^2 + b^2}$

Let $R_n = C_n = \sqrt{a_n^2 + b_n^2}$

$a_n = 0$	$b_n = -\frac{5}{\pi}\frac{1}{n}(\cos n\pi - 1)$	$R_n = C_n = \sqrt{a_n^2 + b_n^2}$
$a_1 = 0$	$b_1 = -\frac{5}{\pi}\frac{1}{1}(\cos \pi - 1)$ $b_1 = -\frac{5}{\pi}\frac{1}{1}(-1 - 1)$ $b_1 = \frac{10}{\pi}$	$R_1 = C_1 = \sqrt{a_1^2 + b_1^2}$ $R_1 = C_1 = \sqrt{0^2 + (\frac{10}{\pi})^2}$ $R_1 = C_1 = \frac{10}{\pi}$ $R_1 = C_1 = \frac{10}{3.14} = 3.18$

Here, $n\omega = \frac{n\pi}{4}$

For n = 1; Fundamental Frequency = 1st Harmonic = $\omega = \frac{\pi}{4}$ **=0.785**

For n = 2; 2nd Harmonic $= 2\omega = \frac{2\pi}{4}$ **=1.570**

For n = 3; 3rd Harmonic $= 3\omega = \frac{3\pi}{4}$ **=2.356**

For n = 4; 4th Harmonic $= 4\omega = \frac{4\pi}{4}$ **=3.1416**

Problem 15: **Prism Analogy:**

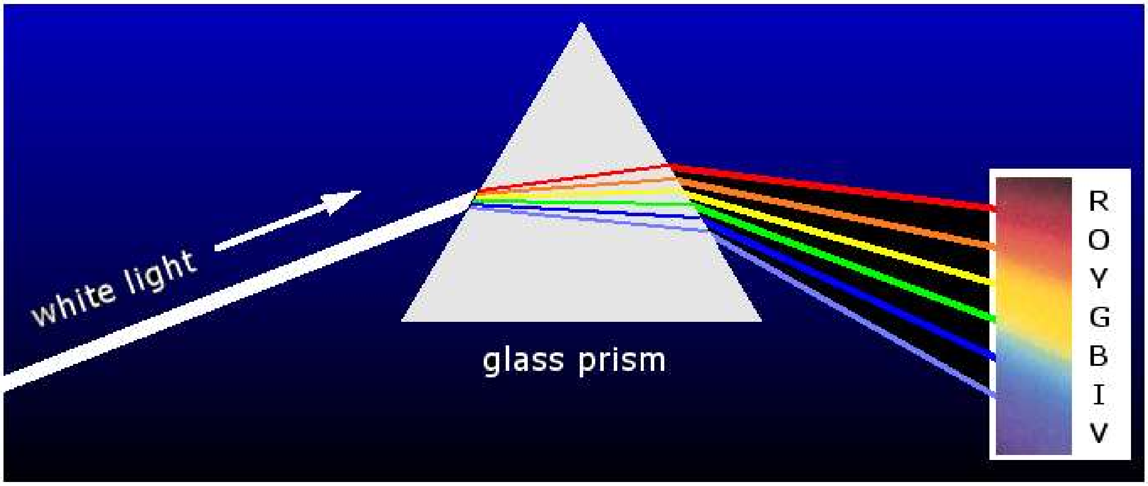


Figure 80

A prism which splits white light into a spectrum of colors. White light consists of all frequencies mixed together. The prism breaks them apart so we can see the separate frequencies

Signal g(t) → Fourier Transform → **Spectrum G (w)**

Figure 81

Problem 16: **How do we hear?**

The pinna catches sound waves (complex wave) and channels them down the external auditory canal, where they hit the tympanic membrane and make it vibrate **.** ***In our inner ears, Cochlea consists of spiral of tissue filled with liquid and thousands of tiny hairs, which gradually get smaller*** **from *the* outside of *the spiral* to *the inside*. *Each hair is connected to the nerve. The longer hairs resonate with lower frequencies, the shorter hairs resonate with higher frequencies*.**

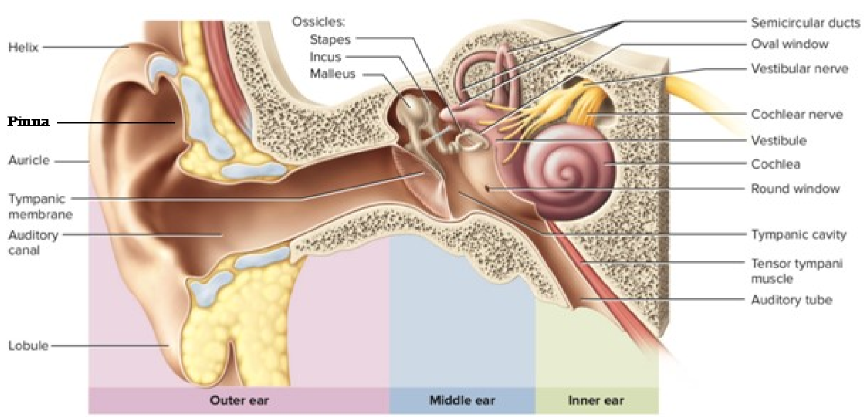


Figure 82

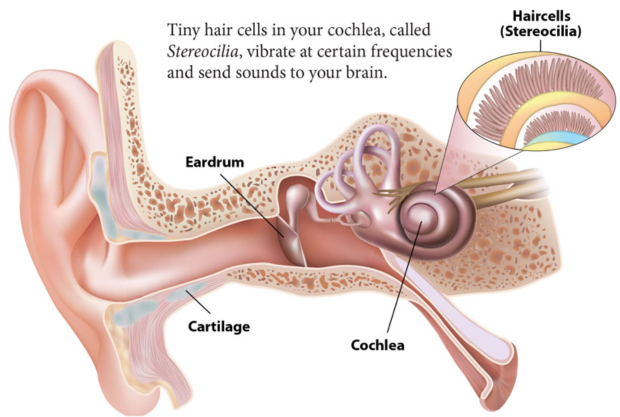
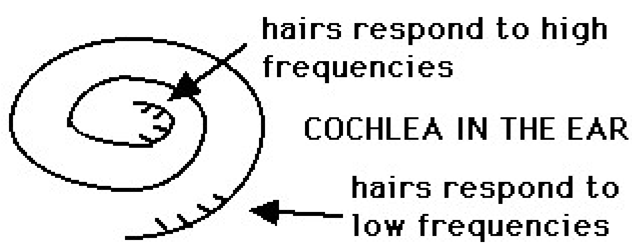


Figure 83



Sound enters the outer ear and travels through the ear canal.

i. The sound waves press against the tympanic membrane, or "ear drum," causing it to vibrate.

ii. Vibrations of the ear drum push against a series of three small bones: the malleus, incus, and stapes, moving them back and forth.

iii. The movement of the stapes against the cochlea transmits pressure waves into the fluid within the cochlea, causing the fluid to vibrate.

iv. The fluid vibrations cause tiny hair cells located within the cochlea to move gently back and forth. As the hair cells move, they release chemical signals that stimulate nerve fibers near the cochlea.

v. The nerve fibers transmit the signals to the auditory nerve and on to the brain.

***Sounds as a sum of different amplitude signals each with a different frequency* (Figure 70). *Here Sound is a complex wave*.** A sound is characterized by its frequency and intensity.

Thus the time-domain air pressure signal is transformed into frequency spectrum, which is then processed by the brain. ***Our ear is a Natural Fourier Transform Analyzer***

The cochlea transforms a time domain signal (the sound's waveform) into a frequency domain signal. The strength of the response in the auditory nerve fiber tuned to a particular frequency reflects the amplitude of the sound's waveform at that frequency. In other words, the auditory system takes a Fourier transform of the incoming signal, decomposing the sound into amplitudes as a function of frequency

In Fourier analysis a signal is decomposed into its constituent sinusoids, i.e. frequencies, the amplitudes of various frequencies form the so called frequency spectrum of the signal. In an inverse Fourier transform operation the signal can be synthesized by adding up its constituent frequencies. It turns out that many signals that we encounter in daily life such as speech, car engine noise, bird songs, music etc. have a periodic or quasi-periodic structure, and that the cochlea in the human hearing system performs a kind of harmonic analysis of the input audio signals. Therefore the concept of frequency is not a purely mathematical abstraction in that biological and physical systems have also evolved to make use of the frequency analysis concept.

Fourier analysis is by no means limited to these classic examples — ***it can analyze and process images, it can efficiently compress images and video streams, and it can assist in visual pattern recognition, where a complex pattern may be efficiently and concisely described using a set of Fourier terms***.

Problem 17: **Harmonic Analysis with Example:**

Harmonics: The angular frequencies of the Sinusoids above are all integer multiples of ω. They are called the *harmonics* of ω, which in turn is called the *fundamental*. In terms of pitch, the ω , 2ω ,.............harmonics. These frequencies are referred to as ***harmonics*** of the fundamental frequency

Example 36: Find Harmonic Analysis for the given Fourier series

$$\mathbf{f(t) = 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}}$$

We have, from Example 22, (Page no 33)

$$\mathbf{f(t) = 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}} \quad \text{[Answer of Example 22, Page no 33]}$$

$$\underbrace{\mathbf{f(t)}}_{\text{Complex wave}} = \underbrace{\mathbf{2.5}}_{\text{DC value}} \underbrace{- \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi}{4}t}_{\text{AC value}} \text{-----------------------------(i)}$$

$$\underbrace{\mathbf{f(t)}}_{\text{Complex wave}} = \underbrace{\mathbf{2.5}}_{\text{DC value}} - \underbrace{\frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)}_{\text{Amplitude}} \sin \underbrace{\frac{n\pi}{4}}_{\text{Frequency}} t \text{ -------------------(ii)}$$

We have the Fourier series is $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(n\omega t) + b_n\sin(n\omega t))$

Here, $n\omega = \frac{n\pi}{4}$

For n = 1; **Fundamental Frequency = 1st Harmonic =** $\omega = \frac{n\pi}{4} == \frac{1.\pi}{4} = \frac{\pi}{4}$

For n = 2; **2nd Harmonic** $= 2\omega = \frac{n\pi}{4} == \frac{2.\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

For n = 3; **3rd Harmonic** $= 3\omega = \frac{n\pi}{4} == \frac{3.\pi}{4} = \frac{3\pi}{4}$

For n = 4; **4th Harmonic** $= 4\omega = \frac{n\pi}{4} == \frac{4.\pi}{4} = \frac{4\pi}{4} = \pi$
