Fourier analysis

By

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Signal: Signal is a physical quantity that is measurable. Signals are functions of time**.** System is a physical entity that exists. Signal is produced from a system.

Depending on the nature of signal, it is categorized into several classes based on some criterion. Some of the classifications include continuous v/s discrete, periodic v/s aperiodic, energy v/s power, deterministic v/s random, stationary v/s non-stationary and so on**.**

The first natural division of all signals is into either stationary or non-stationary categories. Stationary signals are constant in their statistical parameters over time. If you look at a stationary signal for a few moments and then wait an hour and look at it again, it would look essentially the same, i.e. its overall level would be about the same and its amplitude distribution and standard deviation would be about the same. Rotating machinery generally produces stationary vibration signals.

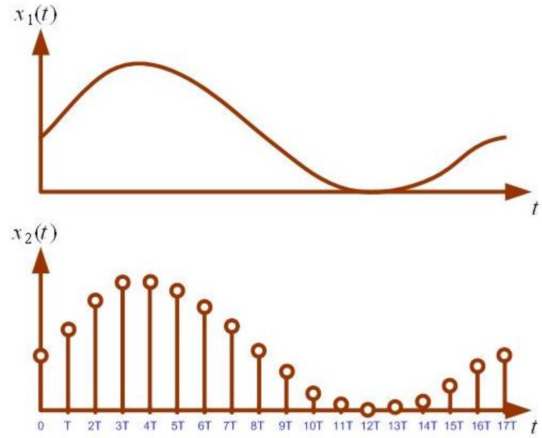


Figure 01: Continuous Vs Discrete Signals

Stationary signals are further divided into deterministic and random signals. Random signals are unpredictable in their frequency content and their amplitude level, but they still have relatively uniform statistical characteristics over time. Examples of random signals are rain falling on a roof, jet engine noise, turbulence in pump flow patterns and cavitations**.**

DC signal: In electronic circuits things happen. Voltage/time, *V*/*t*, graphs provide a useful method of describing the changes which take place.

The diagram below shows the *V*/*t* graph, which represents a DC signal
Voltage = **v,** Time = **t**

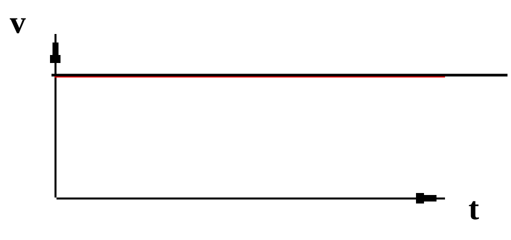


Figure 02

Direct current (DC) is produced by sources such as batteries, thermocouples, solar cells etc

Problem 01: Periodic Signal

What is periodic signal?
A signal which is repeating itself is a periodic signal

Periodic: Signal Pattern repeats over time
Example **01**

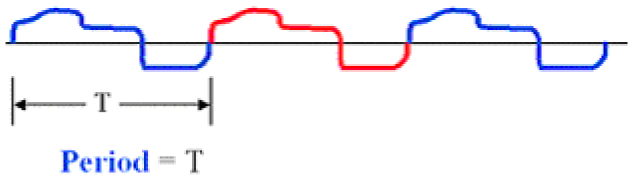


Figure 03: A periodic signal with period T

Example **02**

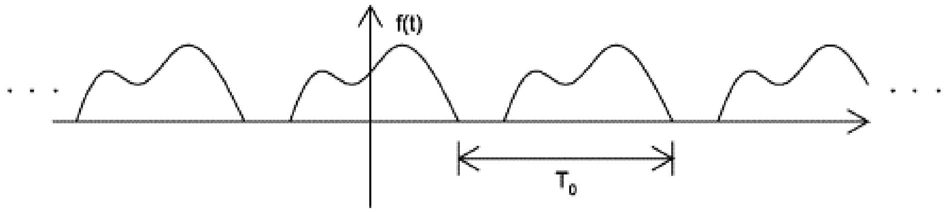


Figure 04: A periodic signal with period T_0

Example **03**

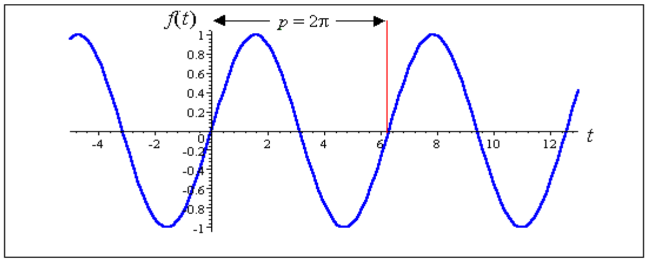


Figure 05: $\mathbf{f(t) = sint}$; Period $\mathbf{T = 2\pi}$

Example **04**

$\mathbf{f(t) = 3t;} \qquad \mathbf{-1 \le t < 1}.$

$\mathbf{f(t) = f(t+2)}$ [This indicates it is periodic with period 2.]

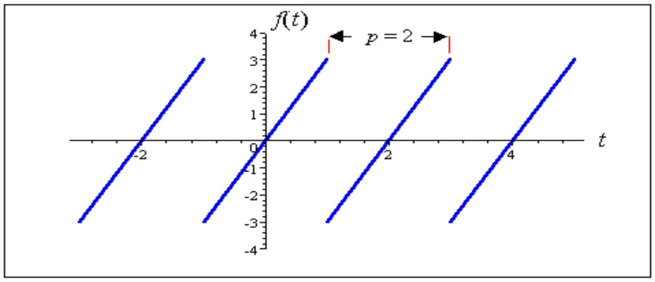


Figure 06: $\mathbf{f(t) = 3t;}$ Period $\mathbf{T = 2}$

Example **05**

$\mathbf{f(t) = t^2}$; $\qquad \mathbf{0 \le t < 2}$

$\mathbf{f(t) = f(t+2)}$ [Indicating it is periodic with period 2.]

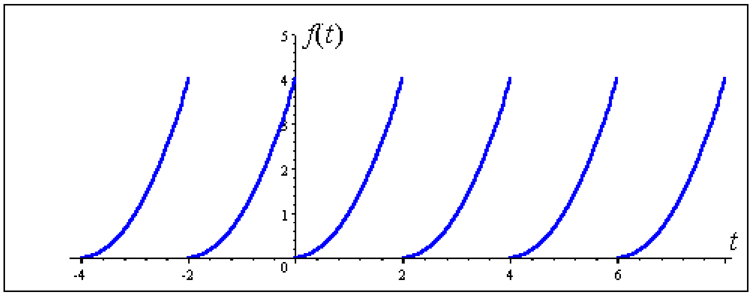


Figure 07: $\mathbf{f(t) = t^2}$ Period $\mathbf{T = 2}$

Example **06**

$\mathbf{f(t) = -3;} \qquad \mathbf{-1 \le t < 1}$

$\qquad \mathbf{= \ 3\,;} \qquad \mathbf{1 \le t < 3}$

$\mathbf{f(t) = f(t+4)}$ [The period is 4.]

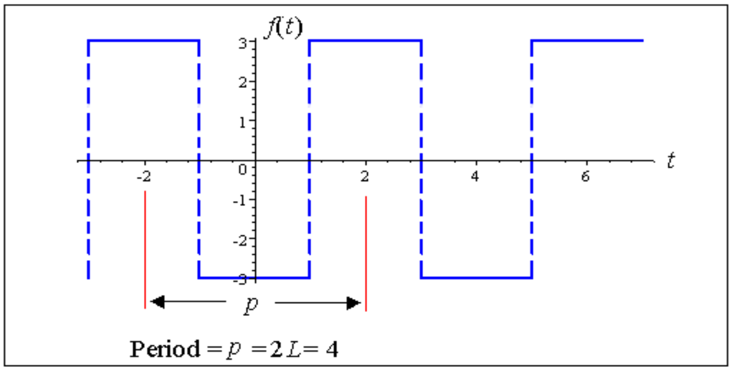


Figure 08; Period $\mathbf{T = 4}$

Example **07**

$$f(t) = \begin{cases} -1 & \text{if} \quad 0 \le t < \frac{\pi}{2} \\ 1 & \text{if} \quad \frac{\pi}{2} \le t < \frac{3\pi}{2} \\ -1 & \quad \frac{3\pi}{2} \le t < 2\pi \end{cases}$$

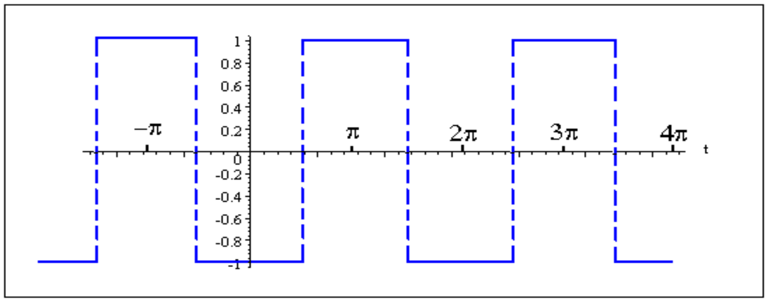


Figure 09; Period $\mathbf{T = 2\pi}$

Example **08**

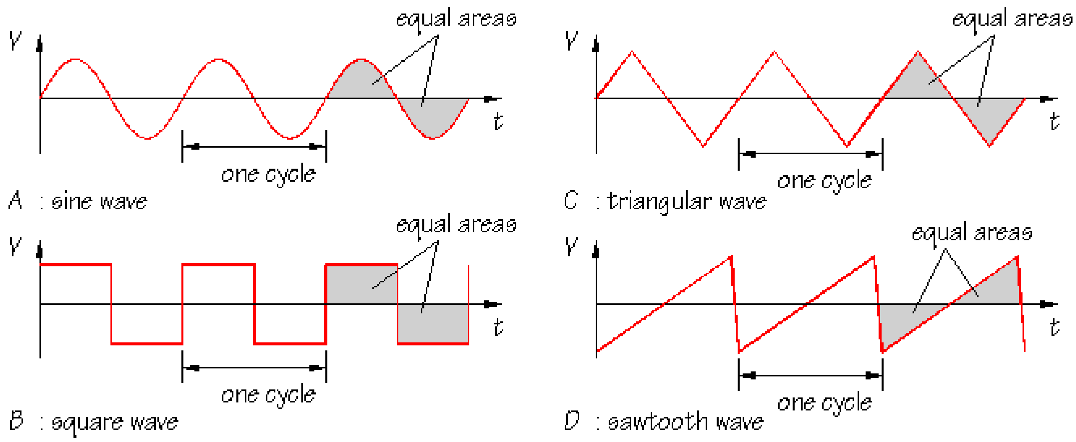
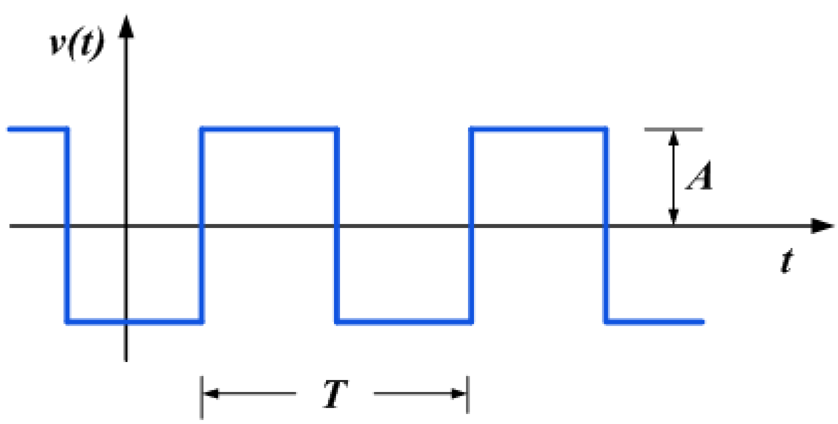


Figure 10: different shape of periodic signal

Periodic Signal Characteristics:

i. Amplitude (A): Signal Value, measured in volts
ii. Frequency (f): repletion rate, cycles per second or Hertz
iii. Period (T): Amount of time it takes for one repetition
iv. Phase (φ): Relative Position in time, measured in degrees



Problem 02:

Aperiodic Signal: Not Periodic

Example **09:**

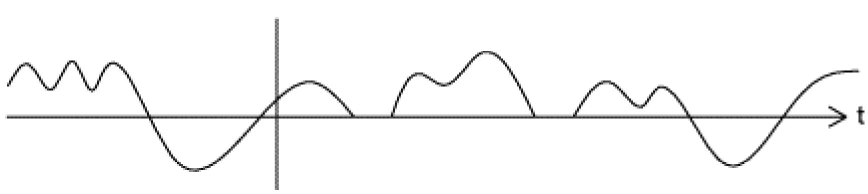


Figure 11: Non periodic signal

Cycles: A set of events or actions that happen again and again in the same order.

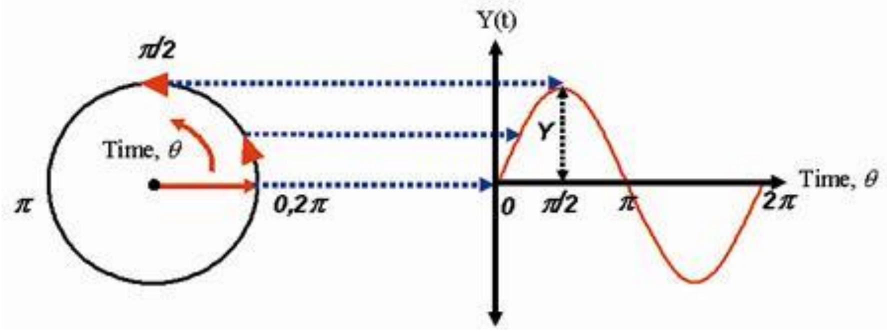


Figure 12

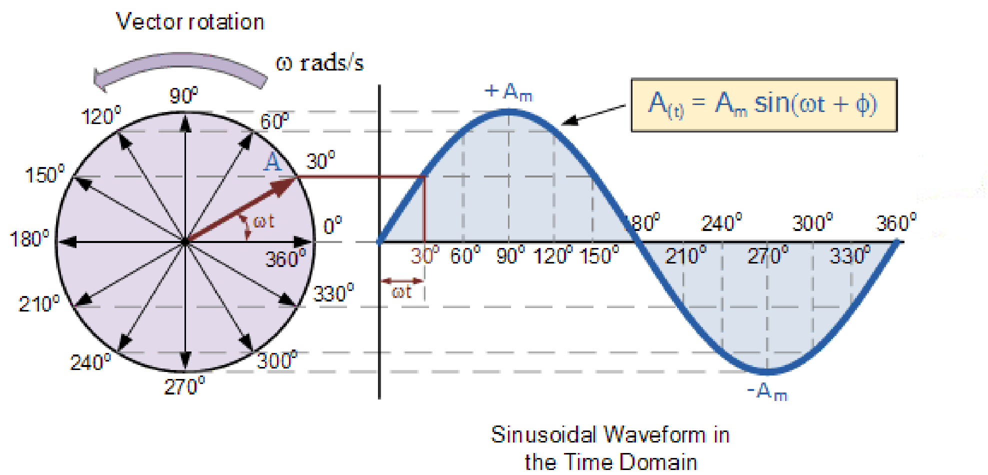


Figure 13

Frequency f: This is the number of cycles completed per second. The measurement unit for frequency is the **hertz**, **Hz**. 1 Hz = 1 cycle per second. $\boldsymbol{f = \frac{1}{T}}$

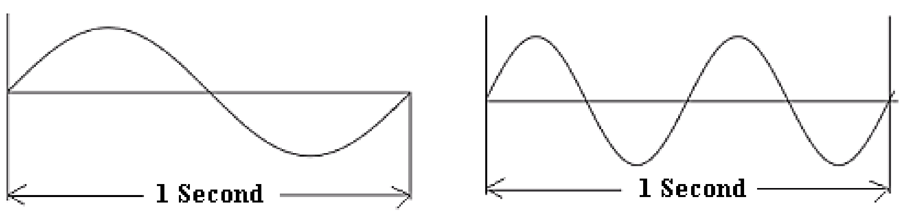


Fig: 1 Cycle/ Sec = 1 Hz

Fig: 2 Cycle/Sec = 2 Hz

Figure 14: Sine Wave with different frequencies

Period: **T**: The period is the time taken for one complete cycle of a repeating waveform

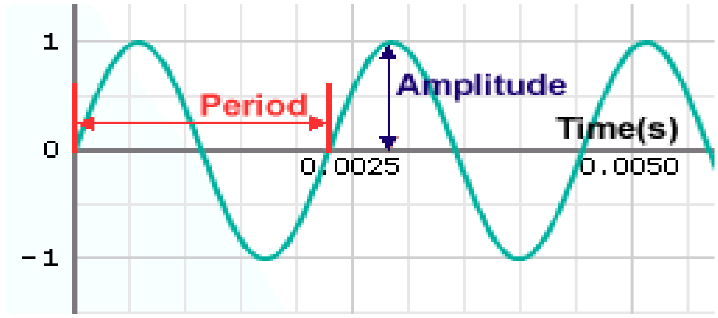


Figure 15

Problem 03: **The effect of frequency and phase**

We normally think of a sine wave as the result of a vector that rotates in a clockwise direction. However, a given sine wave can also be visualized as the sum of two vectors, one rotating clockwise, the other counterclockwise. The vectors are equal in length; each is half of the total. The clockwise rotating vector may be regarded as a positive frequency. The counterclockwise rotating vector is a negative frequency.

We would like to understand the behavior of the following function:

$\mathbf{f(t) = A\sin(\omega t + \phi)}$------------------------------(i)

Here,

$\mathbf{A}$ = the Amplitude; $\boldsymbol{\omega}$ = the frequency, $\boldsymbol{\phi}$ = the phase shift and $\mathbf{t}$ is the time in seconds

In this function, $\mathbf{t}$ is a variable. The other quantities are in general fixed, and each of them influences the shape of the graph of this function as we change its three parameters called Note that $\boldsymbol{\phi}$ (Relative Position in time), $\boldsymbol{\omega = 2\pi f}$, $\mathbf{f}$ is the frequency in hertz (Hz), and $\mathbf{t}$ is time in seconds

Amplitude is the height of a wave which generates intensity. Example: the brightness of light, loudness of sound, power of electricity, etc.

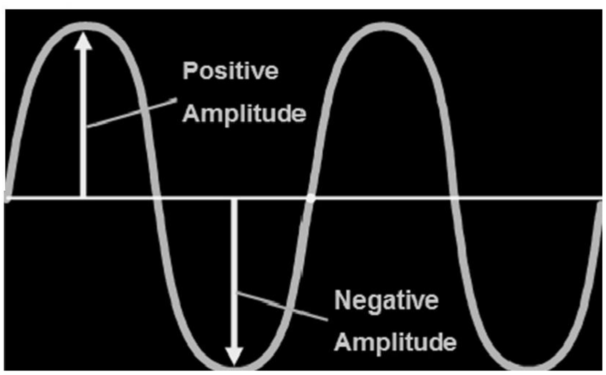


Figure 16

Example 10: $\mathbf{f(t) = A\sin(\omega t + \phi)}$

If $\mathbf{A = 1;\ \phi = 0}$, then draw the graph of $\mathbf{f(t) = ?}$

Then $\mathbf{f(t) = A\sin(\omega t + \phi)}$
$\mathbf{f(t) = 1.\sin(\omega t + 0)}$
$\mathbf{\Rightarrow f(t) = \sin\omega t}$

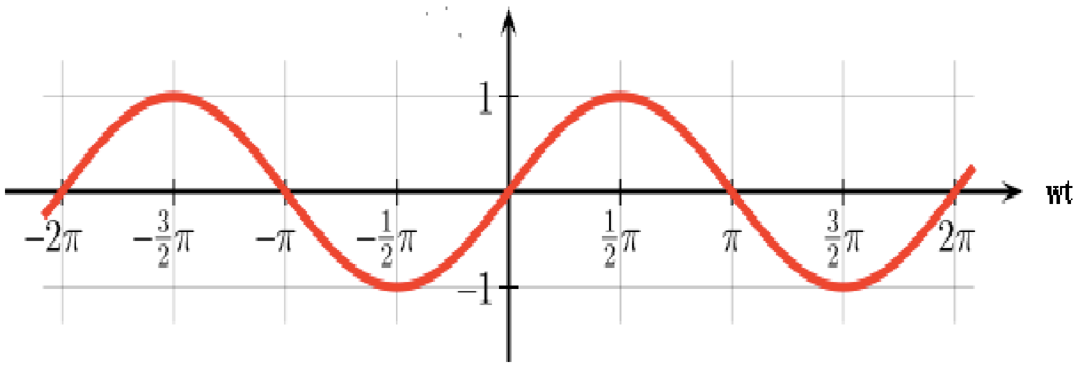


Figure 17: $f(t) = A * \sin\omega t$

Example 11: $\mathbf{f(t) = A\sin(\omega t + \phi)}$

If $\mathbf{A = 1;\ \phi = 90^0}$, then draw the graph of $\mathbf{f(t) = ?}$

$\mathbf{f(t) = A\sin(\omega t + \phi)}$

$\mathbf{f(t) = 1.\sin(\omega t + 90^0)}$

$\mathbf{f(t) = \sin(\omega t + 90^0)}$

$\mathbf{f(t) = \sin(90^0 + \omega t)}$

$\mathbf{f(t) = \sin(1.90^0 + \omega t)}$

[Odd Number Multiplication with 90]

$\mathbf{f(t) = \cos\omega t}$

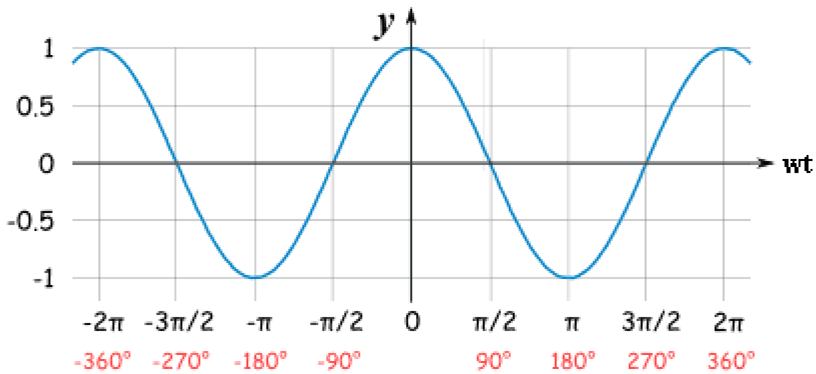


Figure 18: $f(t) = A * \cos\omega t$

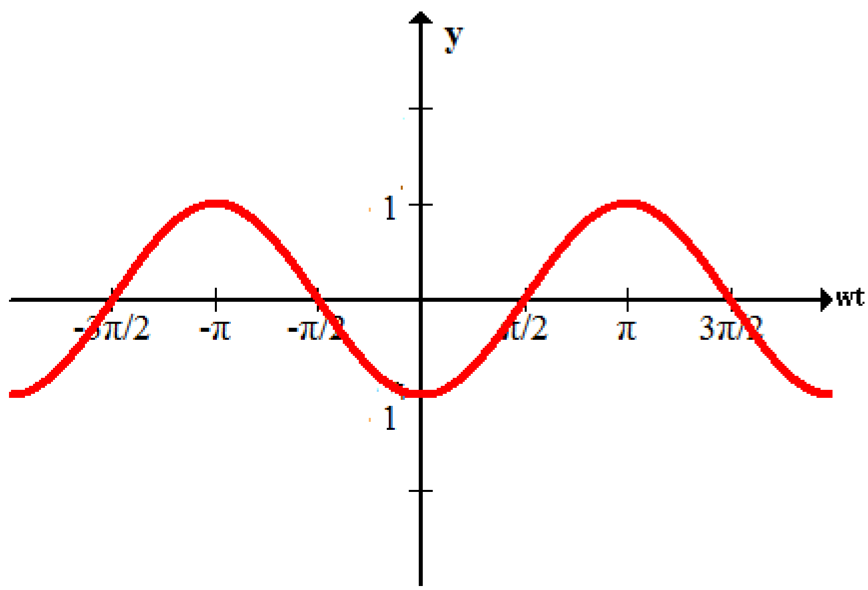


Figure 18: $f(t) = -A * \cos\omega t$

$$y = \sin(\theta - \frac{\pi}{2})$$

$$y = \sin\{-(\frac{\pi}{2} - \theta)\}$$

$$y = -\sin(\frac{\pi}{2} - \theta) \qquad [\sin(-\theta) = -\sin\theta]$$

$$y = -\sin(1.\frac{\pi}{2} - \theta)$$

$$y = -\sin(1.90^0 - \theta)$$

$$y = -\cos\theta$$

Example 12:

We have,

$f(t) = A\sin(\omega t + \phi)$---(i)

$\Rightarrow f(t) = A[\sin\omega t\cos\phi + \cos\omega t\sin\phi]$ $[\because \sin(x+y) = \sin x\cos y + \cos x\sin y]$

$\Rightarrow f(t) = A\sin\omega t\cos\phi + A\cos\omega t\sin\phi$

$\Rightarrow f(t) = A\cos\omega t\sin\phi + A\sin\omega t\cos\phi$

$\Rightarrow f(t) = A\sin\phi\cos\omega t + A\cos\phi\sin\omega t$

$\Rightarrow f(t) = a\cos\omega t + b\sin\omega t$ [Let, $A\sin\phi = a$, $A\cos\phi = b$] -------------------(ii)

***Problem 04: Phase difference or phase shift* between a Sine wave and a Cosine wave**

Sine and cosine signals of the same frequency have only a phase difference of $\pi/2$

Phase describes the position of the waveform relative to time zero.

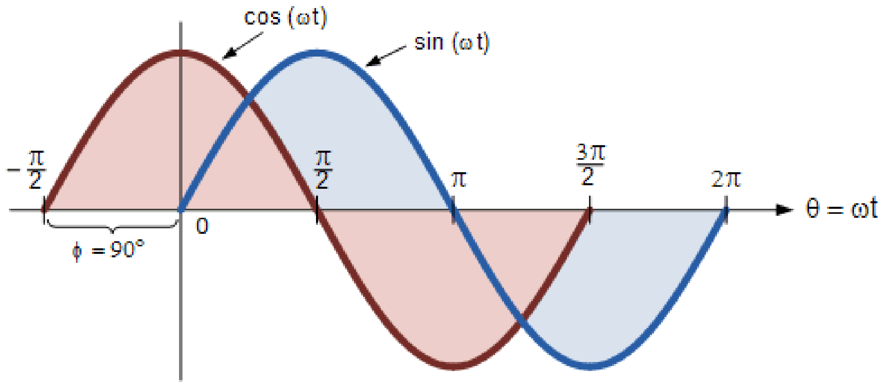


Figure 19: sine wave and cosine wave with Phase difference is 90^0

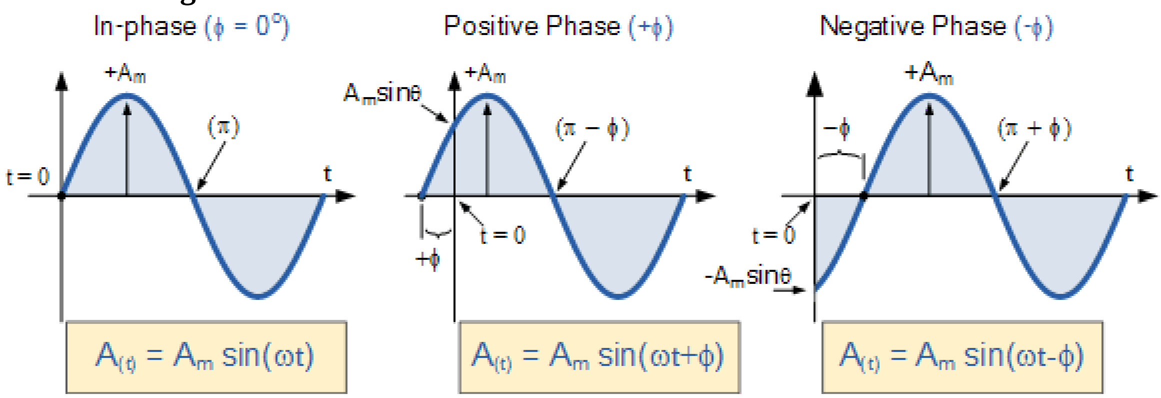


Figure 20

Problem 05: Even and Odd function

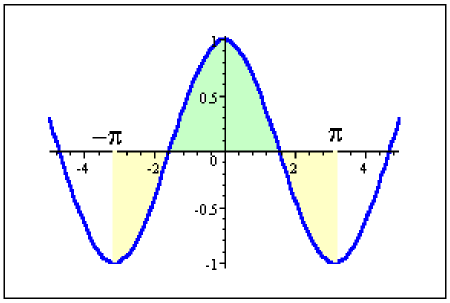


Figure 21: An even signal

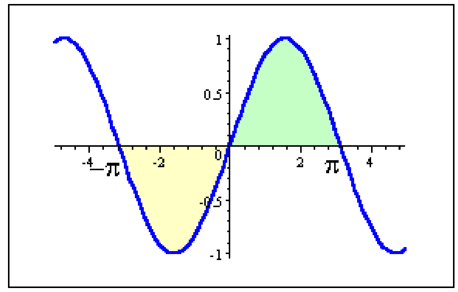


Figure 22: An odd signal

Example 13:

Absolute Value Function $\mathbf{f(x) = |x|}$

That is, $\mathbf{f(x) = -x \quad ; x < 0}$

$\mathbf{\quad\quad\quad\quad x \quad ; x \geq 0}$

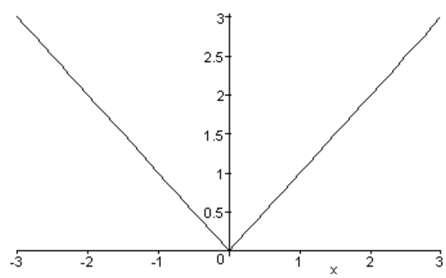


Figure 23

This is the absolute value function. It is really a split function defined in two pieces

Example 14:

$\mathbf{f(t) = -t \quad ; -\pi \leq t < 0}$

$\mathbf{\quad\quad t \quad ; 0 \leq t < \pi}$

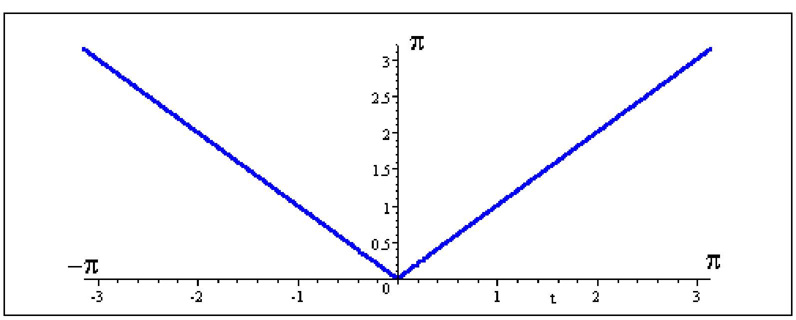


Figure 24

Example 15:

Making waves

Sine waves can be mixed with DC signals, or with other sine waves to produce new waveforms. Here is one example of a complex waveform**:**

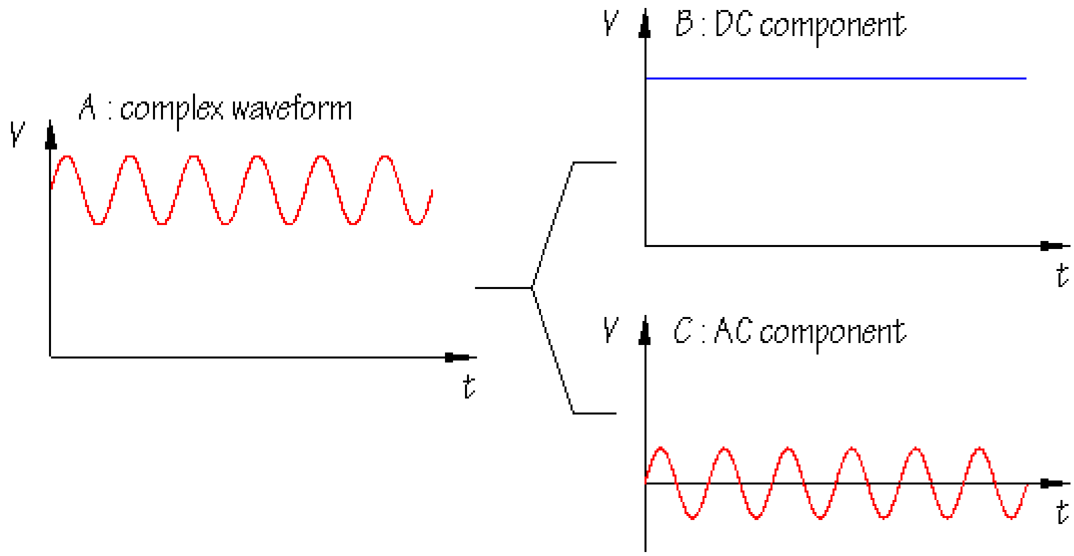


Figure 25

Problem 06: Sum of waves

- Complex wave forms can be reproduced with a sum of different amplitude, frequency sine waves
- Any waveform can be turned into a sum of different amplitude, frequency sine waves

Example 16:

The wave is the 'combination' or 'sum' of the two or more waves

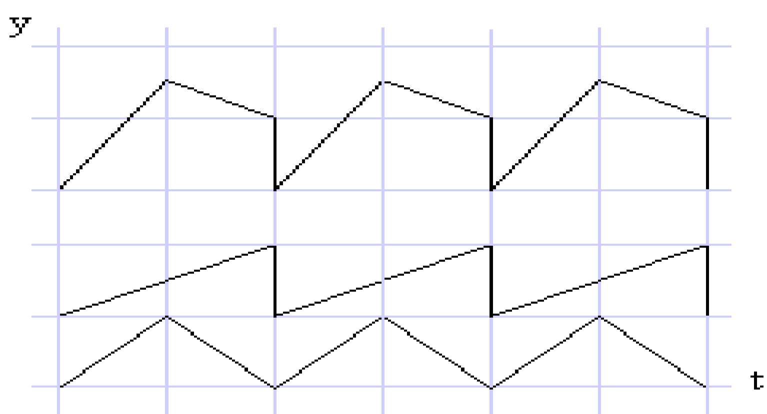


Figure 26

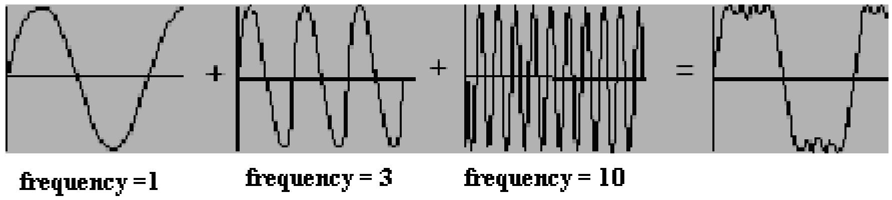


Figure 27

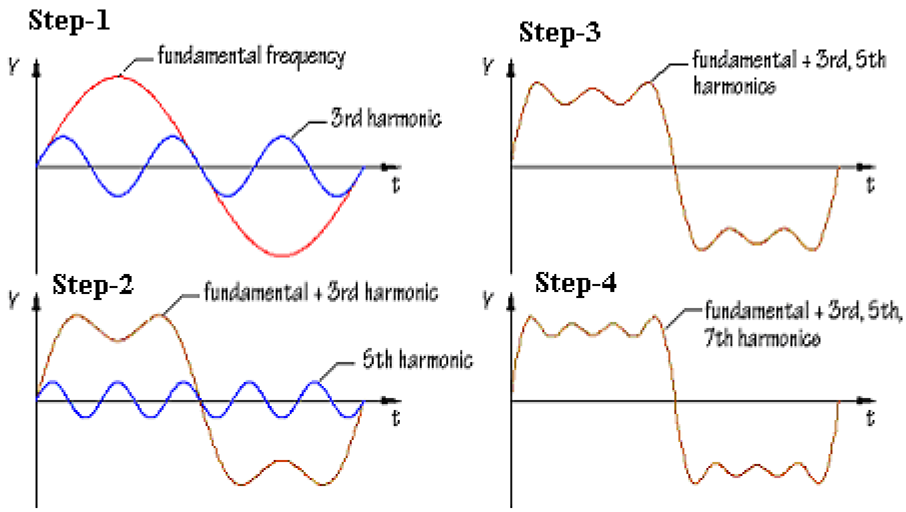


Figure 28: *The different amplitude, frequency of sine waves adds up to produce a complex wave*.

Harmonics: The angular frequencies of the Sinusoids above are all integer multiples of ω. They are called the *harmonics* of ω, which in turn is called the *fundamental*. In terms of pitch, the ω , 2ω ,.............harmonics . These frequencies are referred to as ***harmonics*** of the fundamental frequency

Harmonic analysis: The computation and study of Fourier series is known as harmonic analysis.

Example 17:

We will see functions like the following, which approximates a saw-tooth signal

$$\underbrace{f(t)}_{Complex\ wave} = \underbrace{1}_{DC\ value} \underbrace{+2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t + \ldots\ldots\ldots\ldots\ldots\ldots}_{AC\ value} \text{-----------(i)}$$

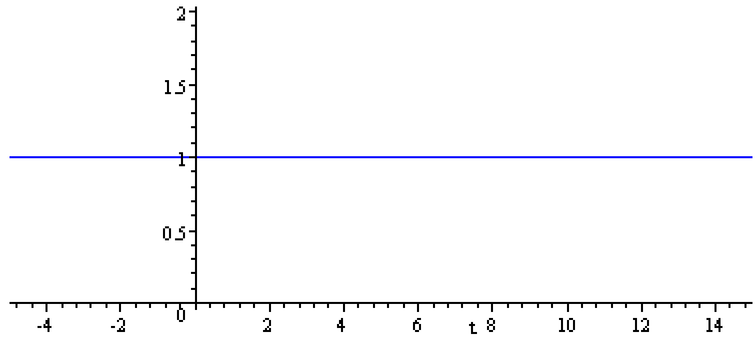


Figure 29: $f(t) = 1$

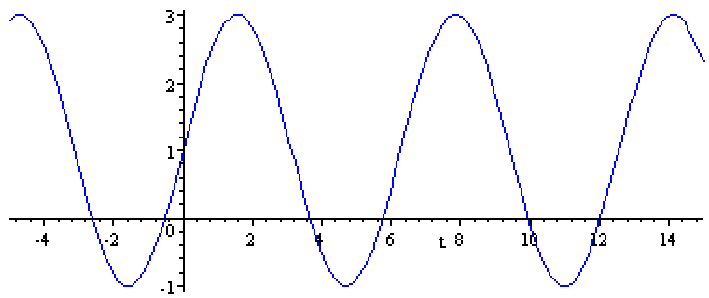


Figure 30: $f(t) = 1 + 2\sin t$

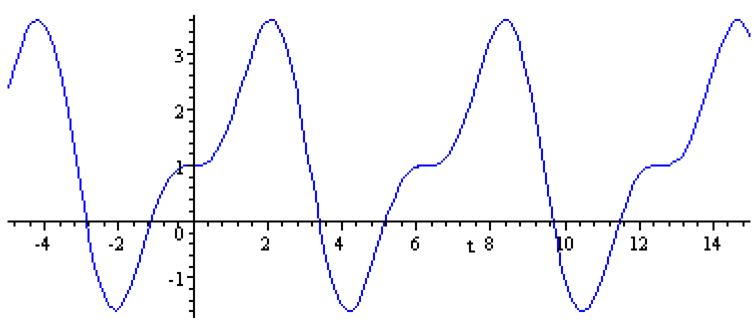


Figure 31: $f(t) = 1 + 2\sin t - \sin 2t$

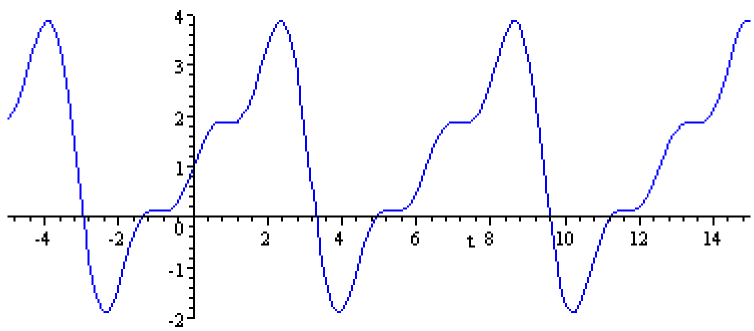


Figure 32: $f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t$

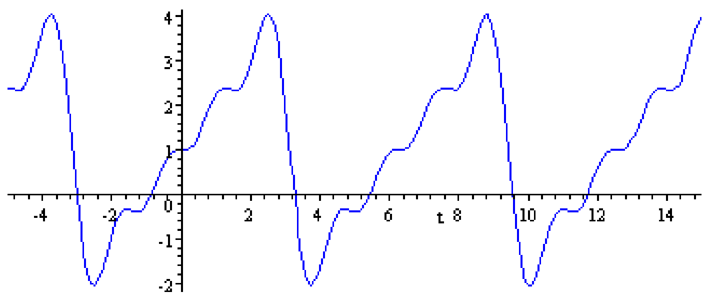


Figure 33: $f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t$

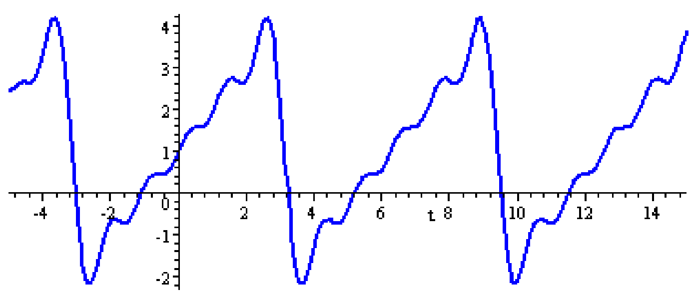


Figure 34: $\mathbf{f(t) = 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \frac{2}{5}\sin 5t}$

In this way, we say that the infinite Fourier series converge to the saw tooth curve. We can take any function of time and describe it as a sum of sine waves each with different amplitudes and frequencies

Example 18:

A sound can be represented by a mathematical function, with time as the free variable. When a function represents a sound, it is often referred to as a continuous signal.

Sounds as a sum of different amplitude signals each with a different frequency Here Sound is a complex wave.

Problem 07: Physical Significance of Fourier series

Any electromagnetic signal can be shown to consist of a collection of periodic analog signals (Sine waves) at different amplitudes, frequencies & phases.

Any composite/complex signal can be represented as a combination of simple sine waves with different frequencies, phases and amplitudes.

Speech of Fourier:

[GKwU Kg‡cø· I‡qe‡K fvO‡j wWwm I‡qf Ges Gwm I‡qf cvIqv hvq| Gwm I‡qf¸wj wWdv‡i›U wd«Kz‡qwÝ Ges wWdv‡i›U Gw¤cwjwPDW Gi mvBb I‡qf| wecixZµ‡g wWwm I‡qf Ges wWdv‡i›U wd«Kz‡qwÝ Ges wWdv‡i›U Gw¤cwjwPDW Gi A‡bK¸wj mvBb I‡qf †hvM Ki‡j GKwU Kg‡cø· I‡qe cvIqv hvq|]

One of the principles of Fourier analysis is that any imaginable waveform can be constructed out of a carefully chosen set of sine wave components, and conversely, any complex periodic signal can be broken down into a series of sine wave components for analysis. And most important, the described tasks are reciprocal operations — in the same way that integration and differentiation are reciprocal operations in Calculus, encoding and decoding signals are reciprocal operations in Fourier analysis.

Applications in signal processing

When processing signals, such as audio, radio waves, light waves, seismic waves, and even images, Fourier analysis can isolate individual components of a compound waveform, concentrating them for easier detection and/or removal.

Problem 08: Mathematical Expression of Fourier series

A Fourier (In 1822, Joseph Fourier, a French mathematician) series expresses any function as a sum of sine and cosine waves of different frequencies. For example, if you're talking about sound, you can take a wave of any shape you like and extract the frequency components of that wave by expressing it as a Fourier series.

Fourier series: Representation of Periodic Signals. A periodic signal can be described by Fourier decomposition as a Fourier series, i.e. as a sum of sinusoidal and co sinusoidal oscillations. By reversing this procedure a periodic signal can be generated by superimposing sinusoidal and co sinusoidal waves. ***The Fourier series tells you the amplitude and frequency of the sines and cosines that you should add up to recreate your original function***. The general function is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)) \text{---------------------------------------(i)}$$

$$\mathbf{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))} \text{-------------------------------------(ii)}$$

$$[let,\ \omega = \omega_0]$$

Where the variables are:

- $\frac{a_0}{2} \rightarrow$ The *average* (DC) value of the signal:
- The a_n holds the amplitudes of the cosine wave
- The b_n holds the amplitudes of the sine wave
- $n \rightarrow$ The *harmonic number*: 1=fundamental, 2=2nd harmonic, etc
- $a_n \rightarrow$ Peak value of the magnitude of the $n-th$ cosine harmonic
- $b_n \rightarrow$ Peak value of the magnitude of the $n-th$ sine harmonic
- $\omega_0 \rightarrow$ *Fundamental* frequency, $[\because \omega_0 = \frac{2\pi}{T}]$
- $T \rightarrow$ Period of $f(t)$

Putting $\omega_0 = \frac{2\pi}{T}$ in (ii),

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\frac{2\pi}{T}t) + b_n \sin(n\frac{2\pi}{T}t))\ [\because \omega_0 = \frac{2\pi}{T}] \text{-----------------------(iii)}$$

This is called Fourier series of a periodic function f (t).

Each periodic wave could be represented by a Fourier series. It's a summation of sinusoids with different amplitudes, frequencies and phases, ***Where*** a_0, a_n, b_n ***are called Fourier coefficients.***

Let the function $f(t)$ be periodic with period $T = 2L$ in (iii)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\frac{2\pi}{2L}t) + b_n \sin(n\frac{2\pi}{2L}t))\ \text{[Putting } T = 2L \text{ in (iii)]}$$

$$\boxed{\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{a_0}{2}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}t) + b_n \sin(\frac{n\pi}{L}t))}_{\text{AC value}} \text{-----------------------(iv)}}$$

This is called Fourier series of a periodic function $f(t)$ ***for period*** $T = 2L$; $\frac{a_0}{2}$ ***is the mean value, sometimes referred to as the dc level. For an electrical signal it represents DC component***

Example 19: Determine the value of Fourier coefficient a_0

Answer: From (i),

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)) \text{-------------------------(i)}$$

Integrate (i) on both sides from $-\pi$ **to** π i.e period = $T = 2\pi$

We have, $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

$$\int_{-\pi}^{\pi} f(t)dt = \int_{-\pi}^{\pi} [\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)]dt$$

$$= \frac{a_0}{2}\int_{-\pi}^{\pi} dt + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi}\cos(n\omega t)dt + b_n \int_{-\pi}^{\pi}\sin(n\omega t)dt$$

$$= \frac{a_0}{2}\left[t\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\left[a_n\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-\pi}^{\pi} - b_n\left[\frac{\cos(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}\right]$$

$[\because \int \sin mx\, dx = -\frac{1}{m}\cos mx \ \&\ \int \cos mx\, dx = \frac{1}{m}\sin mx\ ]$

$$= \frac{a_0}{2}\left[\pi - (-\pi)\right] + \sum_{n=1}^{\infty}\left[a_n\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-\pi}^{\pi} - b_n\left[\frac{\cos(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{a_0}{2}\left[\pi + \pi\right] + \sum_{n=1}^{\infty}\left[a_n\left[\frac{\sin(n\omega t)}{n\omega}\right]_{-\pi}^{\pi} - b_n\left[\frac{\cos(n\omega t)}{n\omega}\right]_{-\pi}^{\pi}\right]$$

$$= \frac{a_0}{2}2\pi + \sum_{n=1}^{\infty}[\frac{a_n}{n\omega}(\sin(n\omega\pi) - \sin(-n\omega\pi)) - \frac{b_n}{n\omega}(\cos(n\omega\pi) - \cos(-n\omega\pi))$$

$$= \pi a_0 + \sum_{n=1}^{\infty}[\frac{a_n}{n\omega}(\sin(n\omega\pi) + \sin(n\omega\pi)) - \frac{b_n}{n\omega}(\cos(\cancel{n}\omega\pi) - \cos(\cancel{n}\omega\pi))$$

$[\because \sin(-\theta) = -\sin\theta\ ,\ \cos(-\theta) = \cos\theta]$

$$= \pi a_0 + \sum_{n=1}^{\infty}[\frac{a_n}{n\omega}2\sin(n\omega\pi)] - 0 \text{ ------------------------(ii)}$$

When $n = 1$, ***and*** $\omega = 1$ **[Given]**

Then $\frac{a_n}{n\omega}2\sin(n\omega\pi) = \frac{a_1}{1}2\sin(\pi) = 0\ [\because \sin\pi = 0]$

n =2,

$\frac{a_n}{n\omega}2\sin(n\omega\pi) = \frac{a_2}{2}2\sin(2\pi) = 0\ [\because \sin 2\pi = 0]$

Similarly,

$\frac{a_n}{n\omega}2\sin(n\omega\pi) = 0$ For $n = 1,2,3,.................$

From (ii),

$$\therefore \int_{-\pi}^{\pi} f(t)dt = \pi a_0 + \sum_{n=1}^{\infty}[\frac{a_n}{n\omega}2\sin(n\omega\pi)] - 0$$

$$\therefore \int_{-\pi}^{\pi} f(t)dt = \pi a_0 + 0 \qquad [\because \frac{a_n}{n\omega}2\sin(n\omega\pi) = 0 \quad for \quad n = 1,2,3,..............]$$

$$\therefore \int_{-\pi}^{\pi} f(t)dt = \pi a_0$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

Example 20: Determine the value of Fourier coefficient a_n

Answer:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)) \text{------------------(i)}$$

Multiplying by **$\cos\omega mt$** [where m is a positive integer] Integrate on both sides from **$-\pi$ to π** i.e period = $T = 2\pi$

We have, $\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$--------------------------------------(ii)

$$\int_{-\pi}^{\pi} f(t)\cos(\omega mt)dt = \int_{-\pi}^{\pi}[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t)]\cos(\omega mt)dt$$

$$\int_{-\pi}^{\pi} f(t)\cos(mt)dt = \int_{-\pi}^{\pi}[\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(nt) + b_n \sin(nt)]\cos(mt)dt$$

$$= \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mt)dt + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\cos(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$= \frac{a_0}{2}\left[\frac{1}{m}\sin mt\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$= \frac{a_0}{2}\frac{1}{m}(\sin m\pi - \sin(-m\pi)) + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$= \frac{a_0}{2}\frac{1}{m}(\sin m\pi + \sin m\pi) + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$[\because \sin(-\theta) = -\sin\theta \;,\; \cos(-\theta) = \cos\theta]$$

$$= \frac{a_0}{2}\frac{1}{m}(2\sin m\pi) + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

Let m = 1, 2, 3, 4, ...then

$$2\sin m\pi = 2\sin\pi = 0 \qquad [\because \sin\pi = 0]$$

$$2\sin m\pi = 2\sin 2\pi = 0 \qquad [\because \sin 2\pi = 0]$$

....

....

$$2\sin m\pi = 2\sin 4\pi = 0 \qquad [\because \sin 4\pi = 0]$$

$$2\sin m\pi = 2\sin 5\pi = 0 \qquad [\because \sin 5\pi = 0]$$

$$= \frac{a_0}{2}\frac{1}{m}(0) + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt + b_n \int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$= 0 + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt + b_n \int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt] \text{----------------(iii)}$$

Now in the RHS of (iii)

$$a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt$$

$$= a_n \times \frac{1}{2}\int_{-\pi}^{\pi}2\cos(mt)\cos(nt)dt = a_n \times \frac{1}{2}\int_{-\pi}^{\pi}\{\cos(m+n)t + \cos(m-n)t\}dt$$

$$[\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$$

If $\mathbf{m \neq n}$

$$= a_n \times \frac{1}{2}\int_{-\pi}^{\pi}\{\cos(m+n)t + \cos(m-n)t\}dt$$

$$= a_n \times \frac{1}{2}\left[\frac{1}{m+n}[\sin(m+n)t\right]_{-\pi}^{\pi} + \frac{a_n}{2} \times \left[\frac{1}{m-n}[\sin(m-n)t\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\frac{a_n}{(m+n)}[\sin(m+n)\pi - \sin\{-(m+n)\pi] + \frac{a_n}{2} \times \frac{1}{(m-n)}[\sin(m-n)\pi - \sin\{-(m-n)\pi]$$

$$= \frac{1}{2}\frac{a_n}{(m+n)}[\sin(m+n)\pi + \sin\{(m+n)\pi] + \frac{a_n}{2} \times \frac{1}{(m-n)}[\sin(m-n)\pi + \sin\{(m-n)\pi]$$

$$[\because \sin(-\theta) = -\sin\theta\ ,\ \ \cos(-\theta) = \cos\theta]$$

$$= \frac{1}{2}\frac{a_n}{(m+n)}[2\sin(m+n)\pi] + \frac{a_n}{2} \times \frac{1}{(m-n)}[2\sin(m-n)\pi]$$

$$= \frac{a_n}{(m+n)}[\sin(m+n)\pi] + \frac{a_n}{(m-n)}[\sin(m-n)\pi]$$

Since $\mathbf{m \neq n}$, Let m = 2, n = 3

$$\sin(m+n)\pi = \sin(2+3)\pi = \sin 5\pi = 0$$

$$\& \sin(m-n)\pi = \sin(2-3)\pi = -\sin\pi = 0]$$

$$a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt = 0+0$$

$$a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt = 0 \text{------------------------------------(iv)}$$

And If m =n, from (iii),

$$a_n \int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt$$

$$= a_m \times \frac{1}{2}\int_{-\pi}^{\pi} 2\cos(mt)\cos(mt)dt = a_m \times \frac{1}{2}\int_{-\pi}^{\pi} 2\cos^2 mtdt = \frac{a_m}{2}\int_{-\pi}^{\pi}[1+\cos 2mt]dt$$

$$[\because 2\cos^2 x = 1+\cos 2x]$$

$$= \frac{a_m}{2}\left[t + \frac{1}{2m}\sin 2mt\right]_{-\pi}^{\pi}$$

$$= \frac{a_m}{2}[\pi-(-\pi)] + \frac{a_m}{2}\frac{1}{2m}[\sin 2m\pi - \sin(-2m\pi)]$$

$$= \frac{a_m}{2}[2\pi] + \frac{a_m}{2}\frac{1}{2m}[\sin 2m\pi + \sin 2m\pi] = a_m\pi + \frac{a_m}{2}\frac{1}{2m}[2\sin 2m\pi]$$

$$[\because \sin(-\theta) = -\sin\theta]$$

$$= a_m\pi + \frac{a_m}{2}\frac{1}{2m}\times 0 \qquad [\because 2\sin m\pi = 0, \text{ for } m = 1,2,3.........]$$

$$a_n \int_{-\pi}^{\pi} \cos(mt)\cos(nt)dt = a_m\pi \text{ --------------------------------(v)}$$

Again in the RHS of (iii),

$$b_n \int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = \frac{b_n}{2}\int_{-\pi}^{\pi} 2\sin(nt)\cos(mt)dt = \frac{b_n}{2}\int_{-\pi}^{\pi}\{\sin(n+m)t + \sin(n-m)t\}dt$$

$$[\because 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{b_n}{2}\left[\frac{-1}{(n+m)}[\cos(n+m)t\right]_{-\pi}^{\pi} + \frac{b_n}{2}\left[\frac{-1}{(n-m)}[\cos(n-m)t\right]_{-\pi}^{\pi}$$

$$= \frac{b_n}{2}\cdot\frac{-1}{(n+m)}\{\cos(n+m)\pi - \cos[-(n+m)\pi]\} + \frac{b_n}{2}\cdot\frac{-1}{(n-m)}\{\cos(n-m)\pi - \cos[-(n-m)\pi]\}$$

$$= \frac{b_n}{2}\cdot\frac{-1}{(n+m)}\{\cos(n\cancel{+}m)\pi - \cos[(n\cancel{+}m)\pi]\} + \frac{b_n}{2}\cdot\frac{-1}{(n-m)}\{\cos(n\cancel{-}m)\pi - \cos[(n\cancel{-}m)\pi]\}$$

$$[\because \sin(-\theta) = -\sin\theta\ ,\ \cos(-\theta) = \cos\theta]$$

If $\mathbf{m \neq n}$ or m = n

$$= \frac{b_n}{2}\cdot\frac{-1}{(n+m)}\{0\} + \frac{b_n}{2}\cdot\frac{-1}{(n-m)}\{0\}$$

= 0

$$\therefore b_n \int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = 0 \text{--------------------------------------(vi)}$$

Putting these values in (iii), we get,

When, $\mathbf{m \neq n}$,

$$\int_{-\pi}^{\pi} f(t)\cos(mt)dt = 0 + \sum_{n=1}^{\infty}[a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt + b_n \int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$\int_{-\pi}^{\pi} f(t)\cos(mt)dt = 0+0+0=0 \qquad \text{[From (iv) and (vi)]}$$

And when m =n,

Putting these values in (iii), we get,

$$\int_{-\pi}^{\pi} f(t)\cos(mt)dt = 0+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(mt)\cos(nt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt]$$

$$\int_{-\pi}^{\pi} f(t)\cos(mt)dt = 0+a_m\pi+0=a_m\pi \qquad \text{[From (v) and (vi)]}$$

$$\therefore a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)dt$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \qquad [m=n]$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

Example 21: Determine the value of Fourier coefficient b_n

We have,

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\omega t)+b_n\sin(n\omega t))\text{-----------------(i)}$$

Multiplying by $\sin\omega mt$ [where m is a positive integer] Integrate on both sides from $-\pi$ **to** π i.e period = $T=2\pi$

We have, $\omega=\frac{2\pi}{T}=\frac{2\pi}{2\pi}=1$--------------------------------------(ii)

$$\int_{-\pi}^{\pi} f(t)\sin(\omega mt)dt = \int_{-\pi}^{\pi}[\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(n\omega t)+b_n\sin(n\omega t)]\sin(\omega mt)dt$$

$$\int_{-\pi}^{\pi} f(t)\sin(mt)dt = \int_{-\pi}^{\pi}[\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos(nt)+b_n\sin(nt)]\sin(mt)dt$$

$$=\frac{a_0}{2}\int_{-\pi}^{\pi}\sin(mt)dt+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$=\frac{a_0}{2}\left[\frac{-1}{m}\cos mt\right]_{-\pi}^{\pi}+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$=\frac{a_0}{2}(\frac{-1}{m})(\cos m\pi-\cos(-m\pi))+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$=\frac{a_0}{2}(\frac{-1}{m})(\cancel{\cos m\pi}-\cancel{\cos(m\pi)})+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$[\because\cos(-\theta)=\cos\theta]$$

$$=\frac{a_0}{2}(\frac{-1}{m})(0)+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

$$0+\sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]\text{-------------(iii)}$$

Now in the RHS of (iii), we get,

$$a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt$$

$$=\frac{a_n}{2}\int_{-\pi}^{\pi}2\cos(nt)\sin(mt)dt=\frac{a_n}{2}\int_{-\pi}^{\pi}\{\sin(n+m)t-\sin(n-m)t\}dt$$

$$[\because 2\cos A\sin B=\sin(A+B)-\sin(A-B)]$$

$$=\frac{a_n}{2}\times\left[\frac{-1}{(n+m)}\cos[(n+m)t\right]_{-\pi}^{\pi}-\frac{a_n}{2}\times\left[\frac{-1}{(n-m)}\cos[(n-m)t\right]_{-\pi}^{\pi}$$

$$=\frac{a_n}{2}\times\frac{-1}{(n+m)}(\cos[(n+m)\pi-\cos\{-(n+m)\pi\})-\frac{a_n}{2}\times\frac{-1}{(n-m)}(\cos[(n-m)\pi-\cos[-(n-m)\pi])$$

$$=\frac{a_n}{2}\times\frac{-1}{(n+m)}(\cos[(\cancel{n+m})\pi-\cos\{(\cancel{n+m})\pi\})-\frac{a_n}{2}\times\frac{-1}{(n-m)}(\cos[(\cancel{n-m})\pi-\cos[(\cancel{n-m})\pi])$$

$$[\because\cos(-\theta)=\cos\theta]$$

$=0$ [$\mathbf{m\neq n}$ or m = n]---------------------------------(iv)

Again in the RHS of (iii) ,

$$b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt$$

$$=\frac{b_n}{2}\int_{-\pi}^{\pi}2\sin(nt)\sin(mt)dt=\frac{b_n}{2}\int_{-\pi}^{\pi}\{\cos(n-m)t-\cos(n+m)t\}dt$$

$$[\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)]$$

$$=\frac{b_n}{2}.\left[\frac{1}{(n-m)}[\sin(n-m)t\right]_{-\pi}^{\pi}-\frac{b_n}{2}.\left[\frac{1}{(n+m)}[\sin(n+m)t\right]_{-\pi}^{\pi}$$

$$=\frac{b_n}{2}.\frac{1}{(n-m)}\{\sin(n-m)\pi-\sin[-(n-m)\pi]\}-\frac{b_n}{2}.\frac{1}{(n+m)}\{\sin(n+m)\pi-\sin[-(n+m)\pi]\}$$

$$=\frac{b_n}{2}.\frac{1}{(n-m)}\{\sin(n-m)\pi+\sin[(n-m)\pi]\}-\frac{b_n}{2}.\frac{1}{(n+m)}\{\sin(n+m)\pi+\sin[(n+m)\pi]\}$$

$$[\because\sin(-\theta)=-\sin\theta]$$

$$=\frac{b_n}{2}.\frac{1}{(n-m)}\{2\sin(n-m)\pi\}-\frac{b_n}{2}.\frac{1}{(n+m)}\{2\sin(n+m)\pi\}\text{--------(v)}$$

From (v),

If $\mathbf{m\neq n}$

$$b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt=\frac{b1}{2}.\frac{1}{(1-2)}\{2\sin(1-2)\pi\}-\frac{b1}{2}.\frac{1}{(1+2)}\{2\sin(1+2)\pi\}$$

$$[\text{say } n=1,\ m=2]$$

$$=\frac{b1}{2}.\frac{1}{-1}\{2\sin(-\pi)-\frac{b1}{2}.\frac{1}{3}\{2\sin 3\pi\}$$

$$=\frac{b1}{2}.\frac{-1}{-1}\{2\sin(\pi)-\frac{b1}{2}.\frac{1}{3}\{2\sin 3\pi\}$$

= 0-0

= 0.----------------------(vi)

If m = n,

Now in the RHS of (iii) ,

$$b_n \int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt$$

$$= b_m \int_{-\pi}^{\pi} \sin(mt)\sin(mt)dt = \frac{b_m}{2}\int_{-\pi}^{\pi} 2\sin^2(mt)dt = \frac{b_m}{2}\int_{-\pi}^{\pi}(1-\cos mt)dt$$

$$[\because 2\sin^2 mt = 1-\cos mt]$$

$$= \frac{b_m}{2}\left[t - \frac{1}{m}\sin mt\right]_{-\pi}^{\pi} = \frac{b_m}{2}(\pi-(-\pi)) - \frac{b_m}{2}\frac{1}{m}(\sin m\pi - \sin(-m\pi))$$

$$= \frac{b_m}{2}(2\pi) - \frac{b_m}{2}\frac{1}{m}(\sin m\pi + \sin(m\pi)) = b_m\pi - \frac{b_m}{2}\frac{1}{m}(2\sin m\pi)\ [\because \sin(-\theta) = -\sin\theta]$$

$$= b_m\pi - 0 \qquad [\because \sin m\pi = 0 \text{ for } m = 1,2,3.........]$$

$$= b_m\pi \text{--}(vii)$$

Putting these values in (iii), we get,

When, $m \neq n$,

$$\int_{-\pi}^{\pi} f(t)\sin(mt)dt = 0 + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

= 0+0+0 [From (iv) and (vi)]

= 0.

And When m = n,

$$\int_{-\pi}^{\pi} f(t)\sin(mt)dt = 0 + \sum_{n=1}^{\infty}[a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt + b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt]$$

= 0+ 0+ $b_m\pi$ [From (iv) and (vii)]

= $b_m\pi$

$$\therefore \int_{-\pi}^{\pi} f(t)\sin(mt)dt = b_m\pi$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

Hence the Fourier coefficients are:

$$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt}$$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt}$$

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt}$$

Remember to find Fourier series:

We have, $$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{a_0}{2}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}t) + b_n \sin(\frac{n\pi}{L}t))}_{\text{AC value}} \text{----------------------(i)}$$

When Period $T = 2\pi$

We find,

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \text{---(ii)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(n\omega t)dt$$

$$[\omega = 2\pi f = 2\pi\frac{1}{T} = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1]$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \text{------------------------------------(iii)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(n\omega t)dt$$

$$[\omega = 2\pi f = 2\pi\frac{1}{T} = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1]$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \text{---------------------------------------(iv)}$$

When Period $T = 2L$

Then,

$$\therefore a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt \text{--(v)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nwt)dt$$

$$[\text{Here, } \omega = \frac{2\pi}{T} = \frac{2\pi}{2L} = \frac{\pi}{L}]$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\frac{\pi}{L}t)dt \text{---------------------------------(vi)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nwt)dt$$

$$[w = 2\pi f = 2\pi\frac{1}{T} = \frac{2\pi}{T} = \frac{2\pi}{2L} = \frac{\pi}{L}]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(n\frac{\pi}{L}t)dt \text{---------------------------------(vii)}$$

Some Graphs

Figure 35. $y = f(x) = 3 \;;\; -4 \le x \le 4$

Figure 36. $y = f(x) = -3 \;;\; -4 \le x \le 4$

Figure 37. $y = f(x) = -3 \;;\; -4 \le x \le 0$

$\quad\quad = \;\; 3 \;;\quad 0 \le x \le 4$

Figure 38. $y = f(x) = 0 \;;\; -4 \le x \le 0$

$\quad\quad = 3 \;;\quad 0 \le x \le 4$

Figure 39. $y = f(x) = 0 \;; -3 \le x \le 0$

$\quad\quad = 1 \;;\quad 0 \le x \le 3$

Figure 40. $y = f(x) = x \;;\; -4 \le x \le 4$

Figure 41. $y = f(x) = -x \;;\; -4 \le x \le 4$

Figure 42. $y = f(x) = -x \;;\; -4 \le x \le 0$
$\quad\quad = \; x \;;\; \; 0 \le x \le 4$

Figure 43. $y = f(x) = \; x \;;\; -4 \le x \le 0$
$\quad\quad = -x \;;\; \; 0 \le x \le 4$

Figure 44. $y = f(x) = 0 \;;\; -4 \le x \le 0$
$\quad\quad = \; x \;;\; \; 0 \le x \le 4$

Figure 45. $y = f(x) = \; 0 \;;\; -4 \le x \le 0$
$\quad\quad = -x \;;\; \; 0 \le x \le 4$

Figure 46. $y = f(x) = x \ ; \ -4 \le x \le 0$

$= 0 \ ; \quad 0 \le x \le 4$

Figure 47. $y = f(x) = -x \ ; \ -4 \le x \le 0$

$= \ 0 \ ; \quad 0 \le x \le 4$

Figure 48. $y = f(x) = \sin x \ ; \ -\pi \le x \le \pi$

Figure 49. $y = f(x) = 0 \quad ; \ -\pi \le x \le 0$

$= \sin x \ ; \quad 0 \le x \le \pi$

Figure 50. $x = 4$

Figure 51. $x = -4$

Example 22

$$y = f(t) = 0\,; \quad -4 \le t \le 0$$
$$= 5\,; \qquad 0 \le t \le 4 \quad \text{-----------------------(i)}$$

$\mathbf{f(t) = f(t+8)}$ **Here, $\mathbf{T = 2L = 8}$** $\quad \mathbf{\therefore L = 4}$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

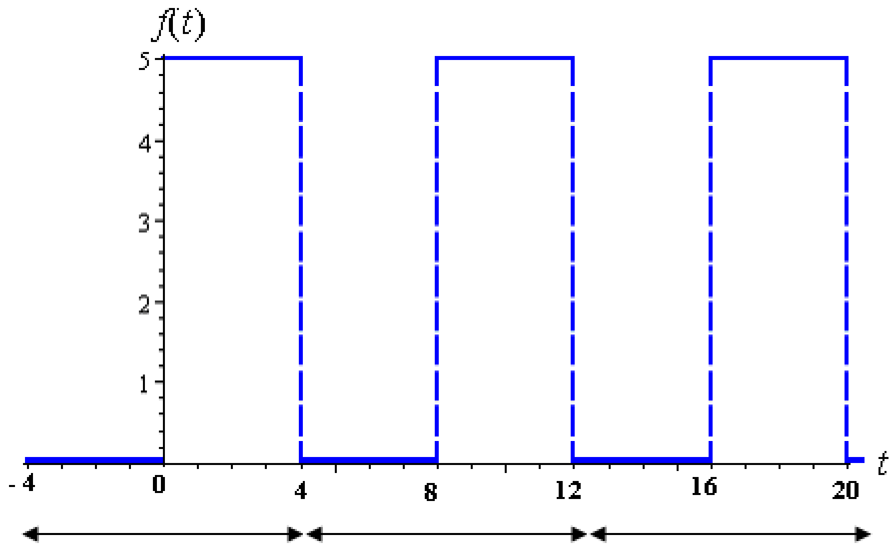


Figure 52: A periodic signal with period $\boldsymbol{T = 2L = 8}$

$$\mathbf{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt}$$

$$\mathbf{= \frac{1}{4}\int_{-4}^{4} f(t)dt}$$

$$\mathbf{= \frac{1}{4}\int_{-4}^{0} f(t)dt + \frac{1}{4}\int_{0}^{4} f(t)dt}$$

$$\mathbf{= \frac{1}{4}\int_{-4}^{0} 0.dt + \frac{1}{4}\int_{0}^{4} 5dt} \qquad \text{[From (i)]}$$

$$\mathbf{= 0 + \frac{1}{4}\left[5t\right]_0^4} \qquad \mathbf{[\int dt = t]}$$

$$\mathbf{= \frac{1}{4}\times\left[5\times 4\text{-}0\right]}$$

$$\mathbf{= \frac{20}{4} = 5}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$a_n = \frac{1}{4}\int_{-4}^{4} f(t)\cos\frac{n\pi t}{4}dt$$

$$a_n = \frac{1}{4}\int_{-4}^{0} f(t)\cos\frac{n\pi t}{4}dt + \frac{1}{4}\int_{0}^{4} f(t)\cos\frac{n\pi t}{4}dt$$

$$a_n = \frac{1}{4}\int_{-4}^{0} (0)\cos\frac{n\pi t}{4}dt + \frac{1}{4}\int_{0}^{4} (5)\cos\frac{n\pi t}{4}dt \qquad \text{[From (i)]}$$

$$a_n = 0 + \frac{1}{4}\times 5\int_{0}^{4} \cos\frac{n\pi t}{4}dt$$

$$= \frac{5}{4}\times\frac{1}{\frac{n\pi}{4}}\left[\sin\frac{n\pi t}{4}\right]_0^4 \qquad [\int \cos mx\,dx = \frac{1}{m}\sin mx]$$

$$= \frac{5}{4}\times\frac{4}{n\pi}\left[\sin\frac{n\pi\times 4}{4} - \sin\frac{n\pi\times 0}{4}\right]$$

$$= \frac{5}{n\pi}[\sin n\pi - \sin 0]$$

$$= \frac{5}{n\pi}\sin n\pi$$

$$= 0 \qquad [\because \sin\pi = \sin 2\pi = \sin 3\pi = ... = \sin n\pi = 0 \text{ and } \sin 0 = 0]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}dt$$

$$b_n = \frac{1}{4}\int_{-4}^{4} f(t)\sin\frac{n\pi t}{4}dt$$

$$b_n = \frac{1}{4}\int_{-4}^{0} f(t)\sin\frac{n\pi t}{4}dt + \frac{1}{4}\int_{0}^{4} f(t)\sin\frac{n\pi t}{4}dt$$

$$b_n = \frac{1}{4}\int_{-4}^{0} (0)\sin\frac{n\pi t}{4}dt + \frac{1}{4}\int_{0}^{4} (5)\sin\frac{n\pi t}{4}dt \qquad \text{[From (i)]}$$

$$b_n = 0 + \frac{1}{4}\times 5\int_{0}^{4} \sin\frac{n\pi t}{4}dt$$

$$b_n = \frac{5}{4}\int_{0}^{4} \sin\frac{n\pi t}{4}dt$$

$$= \frac{5}{4}\times\frac{1}{\frac{n\pi}{4}}\left[-\cos\frac{n\pi t}{4}\right]_0^4 = \frac{-5}{4}\times\frac{4}{n\pi}\left[\cos\frac{n\pi t}{4}\right]_0^4$$

$$[\int \cos mx\,dx = \frac{1}{m}\sin mx; \int \sin mx\,dx = \frac{-1}{m}\cos mx]$$

$$= \frac{-5}{n\pi}\left[\cos\frac{n\pi \times 4}{4} - \cos\frac{n\pi \times 0}{4}\right]$$

$$= \frac{-5}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-5}{n\pi}\left[\cos n\pi - 1\right]$$

The Fourier series for the above function:

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{5}{2} + \sum_{n=1}^{\infty}(0)\cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} -\frac{5}{n\pi}(\cos(n\pi) - 1)\sin\frac{n\pi t}{L}$$

$$= 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$$

$$\underbrace{f(t)}_{Complex\ wave} = \underbrace{2.5}_{DC\ value} + \underbrace{[-\frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}]}_{AC\ value} \textit{Answer}$$

Example 23:

$$\begin{aligned} y = f(t) &= -1\ ;\ -\pi < t < 0 \\ &= \ \ 1\ ;\ \ \ 0 < t < \pi \end{aligned} \text{----------------------(i)}$$

$f(t) = f(t + 2\pi)$ Here, $T = 2L = 2\pi$ $\therefore L = \pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

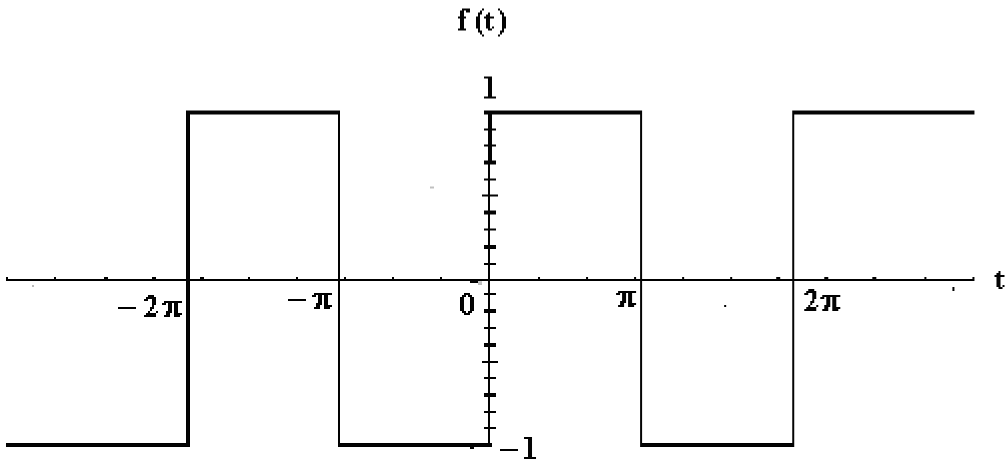


Figure 53: A periodic signal with period $T = 2L = 2\pi$

$$a_0 = \frac{1}{L}\int_{-\pi}^{\pi} f(t)dt$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)dt + \frac{1}{\pi}\int_{0}^{\pi} (1)dt \qquad \text{[From (i)]}$$

$$= \frac{-1}{\pi}[t]_{-\pi}^{0} + \frac{1}{\pi}[t]_{0}^{\pi}$$

$$= \frac{-1}{\pi}[0-(-\pi)] + \frac{1}{\pi}[\pi - 0]$$

$$= -\frac{1}{\pi}[\pi] + \frac{1}{\pi}[\pi]$$

$$= -1 + 1$$

$$= 0$$

$$a_n = \frac{1}{L}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{L}dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\cos\frac{n\pi t}{\pi} + \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cos nt dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\cos nt dt \qquad \text{[From (i)]}$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} \cos nt dt + \frac{1}{\pi}\int_{0}^{\pi} \cos nt dt$$

$$= -\frac{1}{\pi}\times\frac{1}{n}[\sin nt]_{-\pi}^{0} + \frac{1}{\pi}\times\frac{1}{n}[\sin nt]_{0}^{\pi} \qquad [\int \cos mx\, dx = \frac{1}{m}\sin mx]$$

$$= -\frac{1}{\pi n}[\sin 0 - \sin(-n\pi)] + \frac{1}{\pi n}[\sin n\pi - 0]$$

$$= -\frac{1}{\pi n}[0 - \sin(-n\pi)] + \frac{1}{\pi n}[\sin n\pi - 0]$$

$$= -\frac{1}{\pi n}[0 + \sin(n\pi)] + \frac{1}{\pi n}[\sin n\pi - 0] \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= -\frac{1}{\pi n}[0+0] + \frac{1}{\pi n}[0-0]\,[\therefore \sin n\pi = 0 \text{ for } n = 1,2,3\ldots\ldots\ldots\ldots]$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{L}dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin\frac{n\pi t}{\pi}dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin nt dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin nt dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-1)\sin nt dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\sin nt dt \qquad \text{[From (i)]}$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} \sin nt dt + \frac{1}{\pi}\int_{0}^{\pi} \sin nt dt$$

$$= -\frac{1}{\pi}\times\frac{1}{n}\left[-\cos nt\right]_{-\pi}^{0} + \frac{1}{\pi}\times\frac{1}{n}\left[-\cos nt\right]_{0}^{\pi} \; [\because \int \sin mx\, dx = \frac{-1}{m}\cos mx]$$

$$= +\frac{1}{\pi n}\left[\cos nt\right]_{-\pi}^{0} + \frac{1}{\pi n}\left[-\cos nt\right]_{0}^{\pi}$$

$$= \frac{1}{\pi n}\left[\cos 0 - \cos(-n\pi)\right] - \frac{1}{\pi n}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{1}{\pi n}\left[1 - \cos n\pi\right] - \frac{1}{\pi n}\left[\cos n\pi - 1\right] \qquad \left[\therefore \cos(-\theta) = \cos\theta\right]$$

$$= \frac{1}{\pi n} - \frac{1}{\pi n}\cos n\pi - \frac{1}{\pi n}\cos n\pi + \frac{1}{\pi n}$$

$$= \frac{2}{\pi n} - \frac{2}{\pi n}\cos n\pi$$

$$= \frac{2}{\pi n}\left(1 - \mathrm{conn}\pi\right)$$

The Fourier series for the above function:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{\pi} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{\pi}$$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$\therefore f(t) = 0 + \sum_{n=1}^{\infty} 0.\cos nt + \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - \cos n\pi)\sin nt$$

$$\underbrace{f(t)}_{\textit{Complex wave}} = \underbrace{0}_{\textit{DC value}} + \underbrace{\sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - \cos n\pi)\sin nt}_{\textit{AC value}}$$

Example 24:

$$y = f(t) = -t\ ; \quad -\pi \le t < 0$$
$$\quad = t\ ; \quad 0 \le t < \pi \quad \text{-----------------------(i)}$$

$f(t) = f(t + 2\pi)$ Here, $T = 2L = 2\pi \quad \therefore L = \pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

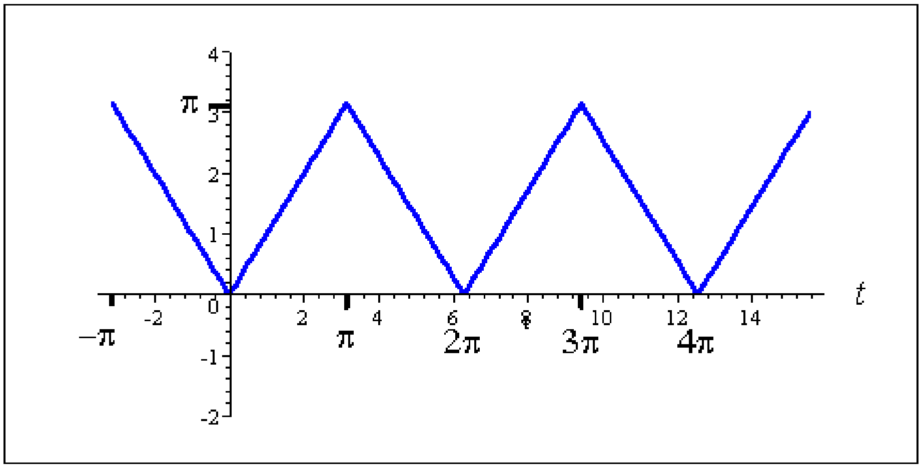


Figure 54: A periodic signal with period $T = 2L = 2\pi$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-t)dt + \frac{1}{\pi}\int_{0}^{\pi} tdt \qquad \text{[From (i)]}$$

$$= -\frac{1}{\pi}[\frac{t^2}{2}]_{-\pi}^{0} + \frac{1}{\pi}[\frac{t^2}{2}]_{0}^{\pi}$$

$$= -\frac{1}{\pi}[0 - \frac{(-\pi)^2}{2}] + \frac{1}{\pi}[\frac{\pi^2}{2} - 0]$$

$$= -\frac{1}{\pi}[-\frac{\pi^2}{2}] + \frac{1}{\pi}[\frac{\pi^2}{2}]$$

$$= \frac{1}{\pi}\times\frac{\pi^2}{2} + \frac{1}{\pi}\times\frac{\pi^2}{2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos(nt)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-t)\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)dt \qquad \text{[From (i)]}$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} t\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)dt \text{ -----------------(ii)}$$

Now, Let, $I = \int t\cos(nt)dt$

$$= t\int\cos(nt)dt - \int[\frac{d}{dt}(t)\int\cos(nt)dt]dt \; [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\frac{\sin(nt)}{n} - \int 1.\frac{\sin(nt)}{n}dt$$

$$= \frac{t}{n}\sin(nt) - \frac{1}{n}\int\sin(nt)dt$$

$$= \frac{t}{n}\sin(nt) - \frac{1}{n}\frac{(-\cos(nt))}{n}$$

$$= \frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt) \text{ -------------------------(iii)}$$

Putting the value of (iii) in (ii)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = -\frac{1}{\pi}\int_{-\pi}^{0} t\cos(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\cos(nt)dt$$

$$= -\frac{1}{\pi}\left[\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{t}{n}\sin(nt) + \frac{1}{n^2}\cos(nt)\right]_{0}^{\pi}$$

$$= -\frac{1}{\pi}[\frac{0}{n}\sin(0) + \frac{1}{n^2}\cos(0) - \frac{(-\pi)}{n}\sin(-n\pi) - \frac{1}{n^2}\cos(-n\pi)] + \frac{1}{\pi}[\frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\cos(n\pi)$$
$$- \frac{0}{n}\sin(0) - \frac{1}{n^2}\cos(0)]$$

$$= \frac{-1}{\pi}[0 + \frac{1}{n^2}\times 1 + \frac{\pi}{n}\sin(-n\pi) - \frac{1}{n^2}\cos(-n\pi)] + \frac{1}{\pi}[\frac{\pi}{n}\times 0 + \frac{1}{n^2}\cos(n\pi) - 0 - \frac{1}{n^2}\times 1]$$

$$= \frac{-1}{\pi}[\frac{1}{n^2} - \frac{\pi}{n}\sin(n\pi) - \frac{1}{n^2}\cos(n\pi)] + \frac{1}{\pi}[\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}]$$

$$[\sin(-\theta) = -\sin\theta \text{ ; } \cos(-\theta) = \cos\theta; \ \sin n\pi = 0 \text{ for } n = 1,2,3..........]$$

$$= \frac{-1}{\pi}[\frac{1}{n^2} - \frac{\pi}{n} \times 0 - \frac{1}{n^2}\cos(n\pi)] + \frac{1}{\pi}[\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}]$$

$$= \frac{-1}{\pi n^2} + \frac{1}{\pi n^2}\cos(n\pi) + \frac{1}{\pi n^2}\cos(n\pi) - \frac{1}{\pi n^2}$$

$$= \frac{2}{\pi n^2}\cos n\pi - \frac{2}{\pi n^2}$$

$$= \frac{2}{\pi n^2}(\cos n\pi - 1) \qquad [\text{Taking Common } \frac{2}{\pi n^2}]$$

$$b_n = \frac{1}{L}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{L}dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\frac{n\pi t}{\pi}dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin(nt)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-t)\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt$$

$$= -\frac{1}{\pi}\int_{-\pi}^{0} t\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt \text{ ---------------------(iv)}$$

Now $\int t\sin(nt)dt$

$$= t\int \sin(nt)dt - \int\{\frac{d}{dt}(t)\int \sin(nt)dt\}dt \ [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t(\frac{-\cos(nt)}{n}) - \int 1.\frac{-\cos(nt)}{n}dt$$

$$= -\frac{t}{n}\cos(nt) + \frac{1}{n}\int \cos(nt)dt$$

$$= \frac{-t}{n}\cos(nt) + \frac{1}{n}\frac{\sin(nt)}{n} \qquad [\int \cos mx\, dx = \frac{1}{m}\sin mx]$$

$$= -\frac{t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt) \text{ ------------------------------(v)}$$

Putting the value of (v) in (iv)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt = -\frac{1}{\pi}\int_{-\pi}^{0} t\sin(nt)dt + \frac{1}{\pi}\int_{0}^{\pi} t\sin(nt)dt$$

$$= -\frac{1}{\pi}\left[\frac{-t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{-t}{n}\cos(nt) + \frac{1}{n^2}\sin(nt)\right]_{0}^{\pi}$$

$$= -\frac{1}{\pi}[\frac{-0}{n}\cos(0) + \frac{1}{n^2}\sin(0) - \frac{-(-\pi)}{n}\cos(-n\pi) - \frac{1}{n^2}\sin(-n\pi)] + \frac{1}{\pi}[-\frac{\pi}{n}\cos(n\pi)$$

$$+ \frac{1}{n^2}\sin(n\pi) - \frac{-(0)}{n}\cos(0) - \frac{1}{n^2}\sin(0)]$$

$$= -\frac{1}{\pi}[0 + 0 - \frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi)] + \frac{1}{\pi}[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) + 0 - 0]$$

$$= -\frac{1}{\pi}[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\times 0] + \frac{1}{\pi}[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\times 0]$$

$$[\sin(-\theta) = -\sin\theta \; ; \; \cos(-\theta) = \cos\theta; \; \sin n\pi = 0 \text{ for } n = 1,2,3..........]$$

$$= \frac{\pi}{\pi n}\cancel{\cos}(n\pi) - \frac{\pi}{\pi n}\cancel{\cos}(n\pi)$$

$$= 0$$

The Fourier series for the above function is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi}{L})t + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi}{L})t$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}(\cos n\pi - 1)\cos(\frac{n\pi}{L})t + 0$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}(\cos n\pi - 1)\cos(\frac{n\pi}{\pi})t$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{\pi}{2}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty} \frac{2}{n^2\pi}(\cos n\pi - 1)\cos nt}_{\text{AC value}}$$

Example 25

$$y = f(t) = t \; ; \; -\pi \le t \le \pi \text{ ----------------------(i)}$$

Here, $T = 2L = 2\pi$ $\therefore L = \pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

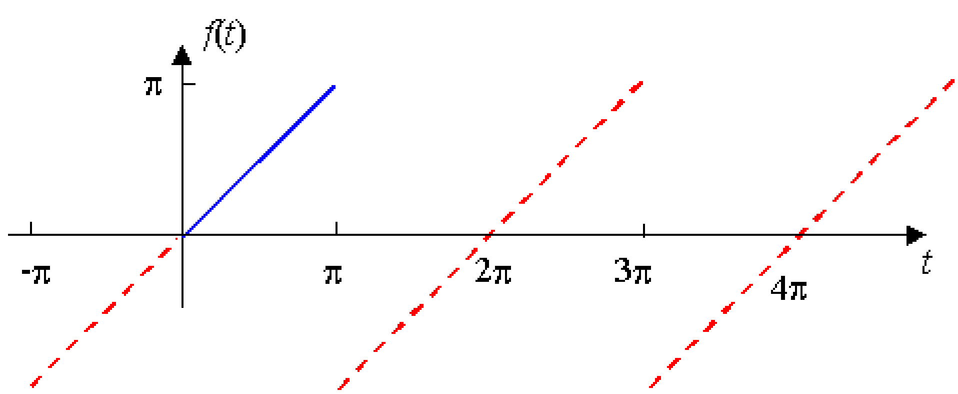


Figure 55: A periodic signal with period $T = 2L = 2\pi$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} tdt \qquad \text{[From (i)]}$$

$$= \frac{1}{\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{(-\pi)^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{L}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt dt \qquad \text{[From (i)]}$$

Now $\int t\cos nt dt$

$$= t\int \cos nt dt - \int\{\frac{d}{dt}(t)\int \cos nt dt\}dt \quad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= \frac{t}{n}\times \sin nt - \int 1.\frac{\sin nt}{n}dt$$

$$= \frac{t}{n}\sin nt - \frac{1}{n}.\frac{1}{n}.(-\cos nt)$$

$$= \frac{t}{n}\sin nt + \frac{1}{n^2}\cos(nt) \text{ --------------------------(ii)}$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt dt$$

$$= \frac{1}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{-\pi}^{\pi} \qquad \text{[From (ii)]}$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\sin n\pi+\frac{1}{n^2}\cos n\pi-\left\{\frac{-\pi}{n}\sin(-n\pi)+\frac{1}{n^2}\cos(-n\pi)\right\}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\sin n\pi+\frac{1}{n^2}\cos n\pi+\frac{\pi}{n}\sin(-n\pi)-\frac{1}{n^2}\cos(-n\pi)\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\sin n\pi+\frac{1}{n^2}\cos n\pi+\frac{\pi}{n}(-)\sin(n\pi)-\frac{1}{n^2}\cos(n\pi)\right]$$

$$[\because \sin(-\theta)=-\sin\theta\ ,\ \cos(-\theta)=\cos\theta]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\cancel{\sin}n\pi+\frac{1}{n^2}\cancel{\cos}n\pi-\frac{\pi}{n}\cancel{\sin}n\pi-\frac{1}{n^2}\cancel{\cos}n\pi\right]$$

$$=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t).\sin\frac{n\pi t}{L}dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin\frac{n\pi t}{\pi}dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}t\sin nt\,dt \qquad \text{[From (i)]}$$

Now, $\int t\sin nt\,dt$

$$=t\int \sin nt\,dt-\int\{\frac{d}{dt}(t)\int \sin nt\,dt\}dt \quad [\because \int uv\,dx=u\int v\,dx-\int\{\frac{d}{dx}(u)\int v\,dx\}dx]$$

$$=t.\frac{1}{n}.(-\cos nt)-\int 1.\frac{1}{n}(-\cos nt)dt$$

$$=-\frac{t}{n}\cos nt+\frac{1}{n}\int \cos nt\,dt$$

$$=-\frac{t}{n}\cos nt+\frac{1}{n^2}\sin nt \text{ ----------------------------(iii)} \quad [\int \cos mx\,dx=\frac{1}{m}\sin mx]$$

$$\therefore b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}t.\sin nt.dt$$

$$=\frac{1}{\pi}\left[-\frac{t}{n}\cos nt+\frac{1}{n^2}\sin nt\right]_{-\pi}^{\pi} \qquad \text{[From (iii)]}$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi-\left\{-\frac{-\pi}{n}\cos(-n\pi)+\frac{1}{n^2}\sin(-n\pi)\right\}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi-\left\{\frac{\pi}{n}\cos(-n\pi)+\frac{1}{n^2}\sin(-n\pi)\right\}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi-\frac{\pi}{n}\cos(-n\pi)-\frac{1}{n^2}\sin(-n\pi)\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi-\frac{\pi}{n}\cos(-n\pi)-\frac{1}{n^2}(-)\sin(n\pi)\right]$$

$$[\because \sin(-\theta)=-\sin\theta\ ;\cos(-\theta)=\cos\theta]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi-\frac{\pi}{n}\cos n\pi+\frac{1}{n^2}\sin n\pi\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\cos n\pi-\frac{\pi}{n}\cos n\pi\right] \qquad [\because \sin n\pi=0, \text{for } n=1,2,3...]$$

$$=-\frac{1}{\pi}\times\frac{2\pi}{n}\cos n\pi$$

$$=-\frac{2}{n}\cos n\pi$$

The Fourier series for the above function is:

$$\therefore f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nt+\sum_{n=1}^{\infty}b_n\sin nt$$

$$=0+0+\sum_{n=1}^{\infty}b_n\sin nt$$

$$=\sum_{n=1}^{\infty}-\frac{2}{n}\cos n\pi.\sin nt$$

$$\underbrace{f(t)}_{Complex\ wave}=\underbrace{0}_{DC\ value}+\underbrace{-2\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi.\sin nt}_{AC\ value}$$

Example 26

$$y=f(t)=\begin{cases}0 & ;\ -1\le t\le -\frac{1}{2}\\ \cos 3\pi t & ;\ -\frac{1}{2}\le t\le \frac{1}{2} \qquad \text{---------------------(i)}\\ 0 & ;\ \frac{1}{2}\le t\le 1\end{cases}$$

$f(t)=f(t+1)$ **Here** $T=2L=2\ \therefore L=1$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

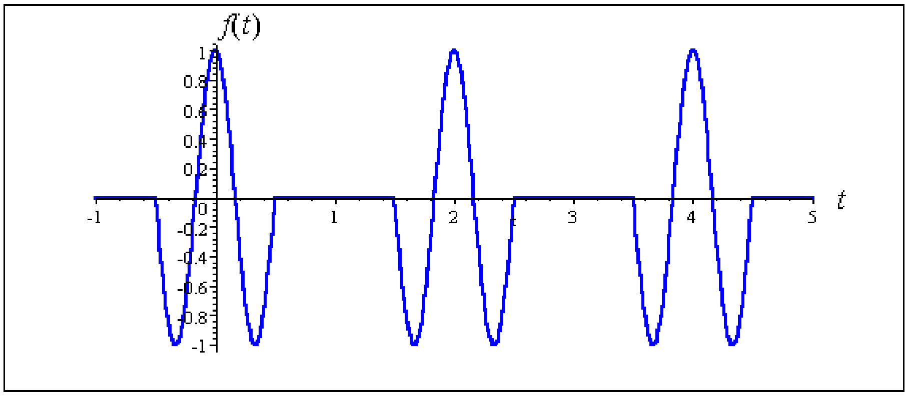


Figure 56: A periodic signal with period $T = 2L = 2$

$$\mathbf{a_0 = \frac{1}{L}\int_{L}^{L} f(t)dt}$$

$$= \frac{1}{1}\int_{-1}^{1} f(t)dt$$

$$= \int_{-1}^{\frac{-1}{2}} f(t)dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} f(t)dt + \int_{\frac{1}{2}}^{1} f(t)dt$$

$$= 0 + \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos 3\pi t dt + 0 \qquad \text{[From (i)]}$$

$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos 3\pi t dt$$

$$= \frac{1}{3\pi}\left[\sin 3\pi t\right]_{\frac{-1}{2}}^{\frac{1}{2}} \qquad [\int \cos mx\, dx = \frac{1}{m}\sin mx; \int \sin mx\, dx = \frac{-1}{m}\cos mx]$$

$$= \frac{1}{3\pi}\left[\sin 3\pi \times \frac{1}{2} - \sin(-\frac{3\pi}{2})\right]$$

$$= \frac{1}{3\pi}\left[\sin\frac{3\pi}{2} + \sin\frac{3\pi}{2}\right] \quad [\because \sin(-\theta) = -\sin\theta]$$

$$= \frac{1}{3\pi} \times 2\sin\frac{3\pi}{2}$$

$$= \frac{2}{3\pi}\sin\frac{3\pi}{2}$$

$$= \frac{2}{3\pi}(-1)\left[\because \sin\frac{3\pi}{2} = \sin 3\times 90 = \sin(3.90+0) = -\cos 0 = -1\right]$$

$$= \frac{-2}{3\pi}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$= \int_{-1}^{1} f(t)\cos\frac{n\pi t}{1}dt$$

$$= \int_{-1}^{1} f(t)\cos n\pi t dt$$

$$= \int_{-1}^{-\frac{1}{2}} f(t)\cos n\pi t dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\cos n\pi t dt + \int_{\frac{1}{2}}^{1} f(t)\cos n\pi t dt$$

$$= \int_{-1}^{-\frac{1}{2}} 0.\cos n\pi t dt + \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t)\cos n\pi t dt + \int_{\frac{1}{2}}^{1} 0.\cos n\pi t dt \qquad \text{[From (i)]}$$

$$= \int_{-1}^{\frac{-1}{2}} 0\cos n\pi t dt + \int_{\frac{-1}{2}}^{\frac{1}{2}} f(t)\cos n\pi t dt + \int_{\frac{1}{2}}^{1} 0\cos n\pi t dt$$

$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} f(t)\cos n\pi t dt$$

$$= \int_{\frac{-1}{2}}^{\frac{1}{2}} \cos 3\pi t \cos n\pi t dt \qquad \text{[From (i)]}$$

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} 2\cos 3\pi t \cos n\pi t dt$$

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \{\cos(3\pi t + n\pi t) + \cos(3\pi t - n\pi t)\}dt$$

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \cos(3\pi t + n\pi t)dt + \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \cos(3\pi t - n\pi t)dt$$

$$=\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}}\cos(3\pi+n\pi)tdt+\frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}}\cos(3\pi-n\pi)tdt$$

$$=\frac{1}{2}\frac{1}{(3\pi+n\pi)}\left[\sin(3\pi+n\pi)t\right]_{\frac{-1}{2}}^{\frac{1}{2}}+\frac{1}{2}\frac{1}{(3\pi-n\pi)}\left[\sin(3\pi-n\pi)t\right]_{\frac{-1}{2}}^{\frac{1}{2}}$$

$$=\frac{1}{2}\frac{1}{(3\pi+n\pi)}\left[\sin(3+n)\pi t\right]_{\frac{-1}{2}}^{\frac{1}{2}}+\frac{1}{2}\frac{1}{(3\pi-n\pi)}\left[\sin(3-n)\pi t\right]_{\frac{-1}{2}}^{\frac{1}{2}}$$

$$=\frac{1}{2}\frac{1}{(3\pi+n\pi)}\left[\sin(3+n)\frac{\pi}{2}-\sin(3+n)(-\frac{\pi}{2})\right]+\frac{1}{2(3\pi-n\pi)}\left[\sin(3-n)\frac{\pi}{2}-\sin(3-n)(-\frac{\pi}{2})\right]$$

$$=\frac{1}{2}\frac{1}{(3\pi+n\pi)}\left[\sin(3+n)\frac{\pi}{2}+\sin(3+n)(\frac{\pi}{2})\right]+\frac{1}{2(3\pi-n\pi)}\left[\sin(3-n)\frac{\pi}{2}+\sin(3-n)(\frac{\pi}{2})\right]$$

$[\because \sin(-\theta)=-\sin\theta]$

$$=\frac{1}{2}\frac{2}{(3\pi+n\pi)}\sin(3+n)\frac{\pi}{2}+\frac{1}{2}\times\frac{2}{(3\pi-n\pi)}\sin(3-n)\frac{\pi}{2}$$

$$=\frac{1}{(3+n)\pi}\sin(3+n)\frac{\pi}{2}+\frac{1}{(3-n)\pi}\sin(3-n)\frac{\pi}{2}$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi t}{L}dt$$

$$=\frac{1}{1}\int_{-1}^{1}f(t)\sin\frac{n\pi t}{1}dt$$

$$=\int_{-1}^{\frac{-1}{2}}f(t)\sin n\pi t dt+\int_{\frac{-1}{2}}^{\frac{1}{2}}f(t)\sin n\pi t dt+\int_{\frac{1}{2}}^{1}f(t)\sin n\pi t dt$$

$$=\int_{-1}^{\frac{-1}{2}}0.\sin n\pi t dt+\int_{\frac{-1}{2}}^{\frac{1}{2}}f(t)\sin n\pi t dt+\int_{\frac{1}{2}}^{1}0.\sin n\pi t dt$$ [From (i)]

$$=\int_{\frac{-1}{2}}^{\frac{1}{2}}f(t)\sin n\pi t dt$$

$$=\int_{\frac{-1}{2}}^{\frac{1}{2}}\cos 3\pi t\sin n\pi t dt$$ [From (i)]

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} 2\cos 3\pi t \sin n\pi t dt$$

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \{\sin(3\pi t + n\pi t) - \sin(3\pi t - n\pi t)\}dt \quad [\because 2\cos A \sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \sin(3\pi + n\pi)t dt - \frac{1}{2}\int_{\frac{-1}{2}}^{\frac{1}{2}} \sin(3\pi - n\pi)t dt$$

$$= \frac{1}{2}\frac{1}{(3\pi + n\pi)}\left[-\cos(3\pi + n\pi)t\right]_{\frac{-1}{2}}^{\frac{1}{2}} - \frac{1}{2}\frac{1}{(3\pi - n\pi)}\left[-\cos(3\pi - n\pi)t\right]_{\frac{-1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{2(3\pi + n\pi)}\left[-\cos(3\pi + n\pi)\frac{1}{2} + \cos(3\pi + n\pi)(-\frac{1}{2})\right] - \frac{1}{2(3\pi - n\pi)}\left[-\cos(3\pi - n\pi)\frac{1}{2} + \cos(3\pi - n\pi)(-\frac{1}{2})\right]$$

$$= \frac{1}{2(3\pi + n\pi)}\left[-\cos(3+n)\frac{\pi}{2} + \cos(3+n)(-\frac{\pi}{2})\right] - \frac{1}{2(3\pi - n\pi)}\left[-\cos(3-n)\frac{\pi}{2} + \cos(3-n)(-\frac{\pi}{2})\right]$$

$$= \frac{1}{2(3\pi + n\pi)}\left[-\cos(3+n)\frac{\pi}{2} + \cos(3+n)(\frac{\pi}{2})\right] - \frac{1}{2(3\pi - n\pi)}\left[-\cos(3-n)\frac{\pi}{2} + \cos(3-n)(\frac{\pi}{2})\right]$$

$$[\because \cos(-\theta) = \cos\theta]$$

$$= 0$$

The Fourier series for the above function is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{\frac{-2}{3\pi}}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{1} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{1}$$

$$= \frac{-1}{3\pi} + \sum_{n=1}^{\infty} a_n \cos n\pi t + \sum_{n=1}^{\infty} 0 \sin n\pi t$$

$$= \frac{-1}{3\pi} + \sum_{n=1}^{\infty}\{\frac{1}{(3\pi + n\pi)}\sin(3+n)\frac{\pi}{2} + \frac{1}{(3\pi - n\pi)}\sin(3-n)\frac{\pi}{2}\}\cos n\pi t$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{-1}{3\pi}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}\{\frac{1}{(3\pi + n\pi)}\sin(3+n)\frac{\pi}{2} + \frac{1}{(3\pi - n\pi)}\sin(3-n)\frac{\pi}{2}\}\cos n\pi t}_{\text{AC value}} \textit{ Answer}$$

Example 27

Given

$$y = f(t) = 1 - t^2 \text{ ------------------------(i)}$$

is to be represented by a Fourier series expression over the finite interval $0 < \mathbf{t} < 1$.

Here, $\mathbf{T = 2L = 1} \quad \therefore\ \mathbf{L = 1/2}$

Now $$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$= \frac{1}{\frac{1}{2}}\int_{0}^{2L} f(t)dt \qquad \left[\because \int_{-a}^{a} f(t)dt = \int_{0}^{2a} f(t)dt\right]$$

$$= 2\int_{0}^{2\times 1/2} f(t)dt$$

$$= 2\int_{0}^{1} f(t)dt$$

$$= 2\int_{0}^{1} (1-t^2)dt$$ [From (i)]

$$= 2\int_{0}^{1} (1)dt - 2\int_{0}^{1} t^2 dt$$

$$= 2\int_{0}^{1} dt - 2\int_{0}^{1} t^2 dt$$

$$= 2[t]_0^1 - 2[\frac{t^3}{3}]_0^1$$

$$= 2[1-0] - 2[\frac{1}{3} - 0]$$

$$= 2\times 1 - 2\times\frac{1}{3}$$

$$= 2 - \frac{2}{3}$$

$$= \frac{6-2}{3}$$

$$a_0 = \frac{4}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$a_n = \frac{1}{1/2}\int_{0}^{2L} f(t)\cos\frac{n\pi t}{1/2}dt$$

$$= 2\int_{0}^{2\times\frac{1}{2}} f(t)\cos(2n\pi t)dt$$

$$= 2\int_0^1 f(t)\cos(2n\pi t)dt$$

$$= 2\int_0^1 (1-t^2)\cos(2n\pi t)dt \qquad \text{[From (i)]}$$

$$= 2\int_0^1 \cos(2n\pi t)dt - 2\int_0^1 t^2\cos(2n\pi t)dt$$

Now,

$$\int \cos(2n\pi t)dt$$

$$= \frac{1}{2n\pi}\sin(2n\pi t) \ [\int \cos mx\,dx = \frac{1}{m}\sin mx; \int \sin mx\,dx = \frac{-1}{m}\cos mx]$$

And,

$$\int t^2\cos(2n\pi t)dt$$

$$= t^2\int \cos(2n\pi t)dt - \int\{(t^2)\frac{d}{dt}\int \cos(2n\pi t)dt\}.dt \ [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t^2\frac{1}{2n\pi}\sin(2n\pi t) - \int 2t\frac{1}{2n\pi}\sin(2n\pi t).dt$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) - \frac{1}{n\pi}\int t\sin(2n\pi t).dt$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) - \frac{1}{n\pi}\left[t\int \sin(2n\pi t)dt - \int\{\frac{d}{dt}(t)\int \sin(2n\pi t)dt\}.dt\right]$$

$$= \frac{t^2}{2n\pi}.\sin(2n\pi t) - \frac{1}{n\pi}\left[t\frac{1}{2n\pi}(-\cos 2n\pi t) - \int 1\frac{1}{2n\pi}(-\cos 2n\pi t)dt\right]$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) - \frac{1}{n\pi}\left[-\frac{t}{2n\pi}\cos 2n\pi t + \frac{1}{2n\pi}\int(\cos 2n\pi t)dt\right]$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) - \frac{1}{n\pi}\left[-\frac{t}{2n\pi}\cos 2n\pi t + \frac{1}{2n\pi}\frac{1}{2n\pi}\sin(2n\pi t)\right]$$

$$= \frac{t^2}{2n\pi}\sin(2n\pi t) + \frac{t}{2n^2\pi^2}\cos 2n\pi t - \frac{1}{4n^3\pi^3}\sin(2n\pi t) \text{--------------------(ii)}$$

$$a_n = 2\int_0^1 \cos(2n\pi t)dt - 2\int_0^1 t^2\cos(2n\pi t)dt$$

$$= 2\left[\frac{1}{2n\pi}\sin(2n\pi t)\right]_0^1 - 2\left[\frac{t^2}{2n\pi}\sin(2n\pi t) + \frac{t}{2n^2\pi^2}\cos 2n\pi t - \frac{1}{4n^3\pi^3}\sin(2n\pi t)\right]_0^1$$

[From (ii)]

$$= 2[\frac{1}{2n\pi}\sin(2n\pi) - 0] - 2[\frac{1}{2n\pi}\sin(2n\pi) - 0 + \frac{1}{2n^2\pi^2}\cos 2n\pi - 0 - \frac{1}{4n^3\pi^3}\sin(2n\pi) + 0]$$

$[\because \sin 2n\pi = 0$ For n=1, 2, 3……….]

$$= -\frac{1}{n^2\pi^2}\cos(2n\pi) + 0$$

$$= -\frac{1}{n^2\pi^2}\cos(2n\pi)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

$$= \frac{1}{L}\int_{0}^{2L} f(t)\sin\frac{n\pi t}{1/2}\,dt$$

$$= \frac{1}{1/2}\int_{0}^{2\times 1/2} f(t)\sin\frac{n\pi t}{1/2}\,dt$$

$$= 2\int_{0}^{1} f(t)\sin(2n\pi t)\,dt$$

$$= 2\int_{0}^{1} (1-t^2)\sin(2n\pi t)\,dt \qquad \text{[From (i)]}$$

$$= 2\int_{0}^{1} \sin(2n\pi t)\,dt - 2\int_{0}^{1} t^2\sin(2n\pi t)\,dt$$

Now, $\int \sin(2n\pi t)\,dt$

$$= -\frac{1}{2n\pi}\cos(2n\pi t)$$

And, $\int t^2 \sin(2n\pi t)\,dt$

$$= t^2\int \sin(2n\pi t)\,dt - \int\{\frac{d}{dt}(t^2)\int \sin(2n\pi t)\,dt\}.dt \quad [\because \int uv\,dx = u\int v\,dx - \int\{\frac{d}{dx}(u)\int v\,dx\}dx]$$

$$= t^2\,\frac{1}{2n\pi}\{-\cos(2n\pi t)\} - \int 2t\,\frac{1}{2n\pi}\{-\cos(2n\pi t)\}.dt$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\int t\cos(2n\pi t).dt$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\left[t\int \cos(2n\pi t).dt - \int\{\frac{d}{dt}(t)\int \cos(2n\pi t)dt\}.dt\right]$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\left[t\,\frac{1}{2n\pi}\sin(2n\pi t) - \int 1\,\frac{1}{2n\pi}\sin(2n\pi t)dt\right]$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\left[\frac{t}{2n\pi}\sin 2n\pi t - \frac{1}{2n\pi}\int \sin(2n\pi t)dt\right]$$

$$= -\frac{t^2}{2n\pi}\cos(2n\pi t) + \frac{1}{n\pi}\left[\frac{t}{2n\pi}\sin(2n\pi t) - \frac{1}{2n\pi}\,\frac{1}{2n\pi}.(-\cos 2n\pi t)\right]$$

$$= -\frac{t^2}{2n\pi}.\cos(2n\pi t) + \frac{t}{2n^2\pi^2}\sin 2n\pi t + \frac{1}{4n^3\pi^3}\cos(2n\pi t) \text{ ----------------(iii)}$$

$$b_n = 2\int_0^1 \sin(2n\pi t)dt - 2\int_0^1 t^2\sin(2n\pi t)dt$$

$$= -\frac{2}{2n\pi}\left[\cos(2n\pi t)\right]_0^1 - 2\left[-\frac{t^2}{2n\pi}.\cos(2n\pi t) + \frac{t}{2n^2\pi^2}\sin 2n\pi t + \frac{1}{4n^3\pi^3}\cos(2n\pi t)\right]_0^1$$

[From (iii)]

$$= -\frac{2}{2n\pi}[\cos(2n\pi) - 1] - 2[-\frac{1}{2n\pi}\cos(2n\pi) + \frac{1}{2n^2\pi^2}\sin 2n\pi + \frac{1}{4n^3\pi^3}\cos(2n\pi) + 0 - 0 - \frac{1}{4n^3\pi^3}]$$

$[\because \sin 2n\pi = 0$ For n=1, 2, 3, & $\cos 0 = 1]$

$$= -\frac{1}{n\pi}\cos(2n\pi) + \frac{1}{n\pi} + \frac{1}{n\pi}\cos(2n\pi) - \frac{1}{2n^3\pi^3}\cos(2n\pi) + \frac{1}{2n^3\pi^3}$$

$$= \frac{1}{n\pi} - \frac{1}{2n^3\pi^3}\cos(2n\pi) + \frac{1}{2n^3\pi^3}$$

$$= \frac{1}{n\pi} - \frac{1}{2n^3\pi^3}[\cos(2n\pi) - 1]$$

The Fourier series for the above function is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$= \frac{\frac{4}{3}}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{1/2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{1/2}\right)$$

$$= \frac{2}{3} + \sum_{n=1}^{\infty} a_n \cos(2n\pi t) + \sum_{n=1}^{\infty} b_n \sin(2n\pi t)$$

$$= \frac{2}{3} + \sum_{n=1}^{\infty}[-\frac{1}{n^2\pi^2}\cos(2n\pi)]\cos(2n\pi t) + \sum_{n=1}^{\infty}[\frac{1}{n\pi} - \frac{1}{2n^3\pi^3}\{\cos 2n\pi - 1\}]\sin(2n\pi t)$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{2}{3}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}[-\frac{1}{n^2\pi^2}\cos(2n\pi)]\cos(2n\pi t) + \sum_{n=1}^{\infty}[\frac{1}{n\pi} - \frac{1}{2n^3\pi^3}\{\cos 2n\pi - 1\}]\sin(2n\pi t)}_{\text{AC value}}$$

Answer

Example 28

$$y = f(t) = 3t \;; \quad 0 < t < 1$$
$$= 3 \;; \quad 1 < t < 2 \text{ --------------------(i)}$$

$f(t+2) = f(t)$ Here, $T = 2L = 2 \quad \therefore L = 1$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$= \frac{1}{L}\int_0^{2L} f(t)dt \qquad [\because \int_{-a}^{a} f(t)dt = \int_0^{2a} f(t)dt]$$

$$= \frac{1}{1}\int_0^2 f(t)dt$$

$$= \int_0^2 f(t)dt$$

$$= \int_0^1 f(t)dt + \int_1^2 f(t)dt$$

$$= \int_0^1 3tdt + \int_1^2 3dt$$ [From (i)]

$$= 3[\frac{t^2}{2}]_0^1 + [3t]_1^2$$

$$= \frac{3}{2}[1-0] + 3[2-1]$$

$$= \frac{3}{2}[1-0] + 3\times 1$$

$$= \frac{3}{2} + 3$$

$$= \frac{9}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$= \frac{1}{L}\int_0^{2L} f(t)\cos\frac{n\pi t}{L}dt$$

$$= \frac{1}{1}\int_0^2 f(t)\cos\frac{n\pi t}{1}dt$$

$$= \int_0^2 f(t)\cos n\pi t dt$$

$$= \int_0^1 f(t)\cos n\pi t dt + \int_1^2 f(t)\cos n\pi t dt$$

$$= \int_0^1 3t\cos n\pi t dt + \int_1^2 3\cos n\pi t dt$$ [From (i)]

Now, $\int 3t\cos n\pi t dt$

$$= 3t\int \cos n\pi t dt - \int\left[\frac{d}{dt}(3t)\int \cos n\pi t dt\right]dt \quad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= 3t\times\frac{1}{n\pi}\sin n\pi t - \int 3\times\frac{1}{n\pi}\sin n\pi t dt$$

$$= \frac{3t}{n\pi}\sin n\pi t - \frac{3}{n\pi}\int \sin n\pi t\,dt$$

$$= \frac{3t}{n\pi}\sin n\pi t - \frac{3}{n\pi}\times\frac{1}{n\pi}(-\cos n\pi t)\ [\int \cos mx\,dx = \frac{1}{m}\sin mx; \int \sin mx\,dx = \frac{-1}{m}\cos mx]$$

$$= \frac{3t}{n\pi}\sin n\pi t + \frac{3}{n^2\pi^2}\cos(n\pi t) \text{ ---------------------(ii)}$$

and $\int 3\cos n\pi t\,dt$

$$= 3\times\frac{1}{n\pi}\sin n\pi t\ [\int \cos mx\,dx = \frac{1}{m}\sin mx; \int \sin mx\,dx = \frac{-1}{m}\cos mx]$$

$$= \frac{3}{n\pi}\sin n\pi t \text{ -------------------------------(iii)}$$

$$\therefore a_n = \int_0^1 3t\cos n\pi t\,dt + \int_1^2 3\cos n\pi t\,dt$$

$$= \left[\frac{3t}{n\pi}\sin n\pi t + \frac{3}{n^2\pi^2}\cos(n\pi t)\right]_0^1 + \left[\frac{3}{n\pi}\sin n\pi t\right]_1^2 \quad \text{[From (ii) and From (iii)]}$$

$$= \left[\frac{3\times 1}{n\pi}\sin n\pi + \frac{3}{n^2\pi^2}\cos n\pi - \frac{3\times 0}{n\pi}\sin 0 - \frac{3}{n^2\pi^2}\cos 0\right] + \left[\frac{3}{n\pi}\sin 2n\pi - \frac{3}{n\pi}\sin n\pi\right]$$

$$= \frac{3}{n^2\pi^2}\cos n\pi - \frac{3}{n^2\pi^2}\times 1\ [\sin n\pi = 0 \text{ For n=1, 2, 3,...and } \cos 0 = 1, \sin 0 = 0]$$

$$= \frac{3}{n^2\pi^2}(\cos n\pi - 1)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

$$= \frac{1}{L}\int_0^{2L} f(t)\sin\frac{n\pi t}{L}\,dt$$

$$= \frac{1}{1}\int_0^2 f(t)\sin\frac{n\pi t}{1}\,dt$$

$$= \int_0^2 f(t)\sin n\pi t\,dt$$

$$= \int_0^1 f(t)\sin n\pi t\,dt + \int_1^2 f(t)\sin n\pi t\,dt$$

$$= \int_0^1 3t\sin n\pi t\,dt + \int_1^2 3\sin n\pi t\,dt \qquad \text{[From (i)]}$$

Now, $\int 3t\sin n\pi t\,dt$

$$= 3t\int \sin n\pi t dt - \int\left\{\frac{d}{dt}(3t)\int \sin n\pi t dt\right\}dt \quad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= 3t \times \frac{1}{n\pi}(-\cos n\pi t) - \int 3(-\cos n\pi t) \times \frac{1}{n\pi}dt$$

$$= -\frac{3t}{n\pi}\cos n\pi t + \frac{1}{n\pi}\int 3\cos n\pi t dt$$

$$= -\frac{3t}{n\pi}\cos n\pi t + \frac{3}{n\pi} \times \frac{1}{n\pi}\sin n\pi t$$

$$= -\frac{3t}{n\pi}\cos n\pi t + \frac{3}{n^2\pi^2}\sin n\pi t \text{ ----------------------------(iv)}$$

And, $\int 3\sin n\pi t dt$

$$= 3 \times \frac{1}{n\pi}(-\cos n\pi t)$$

$$= -\frac{3}{n\pi}\cos n\pi t \text{ ---------------------------------(v)}$$

$$\therefore b_n = \int_0^1 3t\sin n\pi t dt + \int_1^2 3\sin n\pi t dt$$

$$= \left[\frac{-3t}{n\pi}\cos n\pi t + \frac{3}{n^2\pi^2}\sin n\pi t\right]_0^1 + \left[\frac{-3}{n\pi}\cos n\pi t\right]_1^2 \quad \text{[From (iv) and From (v)]}$$

$$= [\frac{-3\times 1}{n\pi}\cos n\pi + \frac{3}{n^2\pi^2}\sin n\pi + \frac{3\times 0}{n\pi}\cos 0 - \frac{3}{n^2\pi^2}\sin 0] + [-\frac{3}{n\pi}\cos 2n\pi + \frac{3}{n\pi}\cos n\pi]$$

$$= -\frac{3}{n\pi}\cos n\pi - \frac{3}{n\pi}\cos 2n\pi + \frac{3}{n\pi}\cos n\pi$$

$$= -\frac{3}{n\pi}\cos 2n\pi$$

The Fourier series for the above function is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{1} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{1}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi t + \sum_{n=1}^{\infty} b_n \sin n\pi t$$

$$= \frac{9/2}{2} + \sum_{n=1}^{\infty}(\frac{3}{n^2\pi^2}(\cos n\pi - 1))\cos n\pi t + \sum_{n=1}^{\infty}(-\frac{3}{n\pi}\cos 2n\pi)\sin n\pi t$$

$$= \frac{9}{4} + \sum_{n=1}^{\infty}(\frac{3}{n^2\pi^2}(\cos n\pi - 1))\cos n\pi t + \sum_{n=1}^{\infty}(-\frac{3}{n\pi}\cos 2n\pi)\sin n\pi t$$

$$\underbrace{f(t)}_{Complex\ wave} = \underbrace{\frac{9}{4}}_{DC\ value} + \underbrace{\sum_{n=1}^{\infty}\left(\frac{3}{n^2\pi^2}(\cos n\pi - 1)\right)\cos n\pi t + \sum_{n=1}^{\infty}\left(-\frac{3}{n\pi}\cos 2n\pi\right)\sin n\pi t}_{AC\ value} \quad \textit{Answer}$$

Example 29

$$y = f(t) = t^2 + t \qquad ; -\pi < t < \pi \text{-----------------------(i)}$$

$$f(t) = f(t+2\pi) \qquad T = 2\pi = 2L \qquad \therefore L = \pi$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (t^2 + t)dt \qquad \text{[From (i)]}$$

$$= \frac{1}{\pi}[\frac{t^3}{3} + \frac{t^2}{2}]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} + [\frac{\pi^3}{3} + \frac{\pi^2}{2} - \frac{(-\pi)^3}{3} - \frac{(-\pi)^2}{2}]$$

$$= \frac{1}{\pi} + [\frac{\pi^3}{3} + \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2}]$$

$$= \frac{1}{\pi} + [\frac{2\pi^3}{3}]$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (t^2 + t)\cos nt dt \qquad \text{[From (i)]}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt dt + \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt dt$$

Now, $\int t^2 \cos nt dt$

$$= t^2\int \cos nt dt - \int\{\frac{d}{dt}(t^2)\int \cos nt dt\}dt \quad [\because \int uv dx = u\int v dx - \int\{\frac{d}{dx}(u)\int v dx\}dx]$$

$$= t^2\frac{1}{n}\sin nt - \int 2t\frac{1}{n}\sin nt dt$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\int t\sin nt\,dt$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\int \sin nt\,dt - \int\{\frac{d}{dt}(t)\int \sin nt\,dt\}dt]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\frac{1}{n}(-\cos nt) - \int 1.\frac{1}{n}(-\cos nt)dt]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\frac{1}{n}(-\cos nt) + \int 1.\frac{1}{n}(\cos nt)dt]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\frac{1}{n}(-\cos nt) + \frac{1}{n}\int \cos nt\,dt]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}[t\frac{1}{n}(-\cos nt) + \frac{1}{n}\frac{1}{n}\sin nt]$$

$$= \frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt \text{ ----------------------------(ii)}$$

And $\int t\cos nt\,dt$

$$= t\int \cos nt\,dt - \int\{\frac{d}{dr}(t)\int \cos nt\,dt\}dt \quad [\because \int uv\,dx = u\int v\,dx - \int\{\frac{d}{dx}(u)\int v\,dx\}dx]$$

$$= t\frac{1}{n}\sin nt - \int 1.\frac{1}{n}\sin nt\,dt$$

$$= \frac{t}{n}\sin nt - \frac{1}{n}\int \sin nt\,dt$$

$$= \frac{t}{n}\sin nt - \frac{1}{n}.\frac{1}{n}(-\cos nt) \quad [\int \cos mx\,dx = \frac{1}{m}\sin mx; \int \sin mx\,dx = \frac{-1}{m}\cos mx]$$

$$= \frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt \text{ ----------------------------------(iii)}$$

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos nt\,dt + \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{-\pi}^{\pi}$$

[From (ii) & (iii)]

$$= \frac{1}{\pi}[\frac{\pi^2}{n}\sin n\pi + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\sin n\pi - \frac{(-\pi)^2}{n}\sin(-n\pi) - \frac{2(-\pi)}{n^2}\cos(-n\pi) - (-\frac{2}{n^3})\sin(-n\pi)]$$

$$+ \frac{1}{\pi}[\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi - \frac{(-\pi)}{n}\sin(-n\pi) - \frac{1}{n^2}\cos(-n\pi)]$$

$$= \frac{1}{\pi}[\frac{\pi^2}{n}\sin n\pi + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\sin n\pi + \frac{\pi^2}{n}\sin n\pi + \frac{2\pi}{n^2}\cos n\pi - \frac{2}{n^3}\sin n\pi]$$

$$+\frac{1}{\pi}[\frac{\pi}{n}\sin n\pi+\frac{1}{n^2}\cos n\pi-\frac{\pi}{n}\sin n\pi-\frac{1}{n^2}\cos n\pi]$$

$$=\frac{1}{\pi}[0+\frac{2\pi}{n^2}\cos n\pi-0+0+\frac{2\pi}{n^2}\cos n\pi-0]+\frac{1}{\pi}[0+\frac{1}{n^2}\cos n\pi-0-\frac{1}{n^2}\cos n\pi]$$

$$[\sin(-\theta)=-\sin\theta\ ;\ \cos(-\theta)=\cos\theta;\ \sin n\pi=0 \text{ for } n=1,2,3..........]$$

$$=\frac{1}{\pi}[\frac{4\pi}{n^2}\cos n\pi]+\frac{1}{\pi}[\frac{1}{n^2}\cos n\pi-\frac{1}{n^2}\cos n\pi]$$

$$=\frac{4\pi}{\pi n^2}\cos n\pi+0$$

$$=\frac{4}{n^2}\cos n\pi$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi t}{L}dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin\frac{n\pi t}{\pi}dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\sin nt\,dt \qquad \text{[From (i)]}$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}t^2\sin nt\,dt+\frac{1}{\pi}\int_{-\pi}^{\pi}t\sin nt\,dt$$

Now $\int t^2\sin nt\,dt$

$$=t^2\int\sin nt\,dt-\int\{\frac{d}{dt}(t^2)\int\sin nt\,dt]dt\ [\because\int uv\,dx=u\int v\,dx-\int\{\frac{d}{dx}(u)\int v\,dx\}dx]$$

$$=t^2\int\sin nt\,dt-\int 2t\frac{1}{n}(-\cos nt)dt$$

$$=t^2\frac{1}{n}(-\cos nt)+\frac{2}{n}\int t\cos nt\,dt$$

$$=t^2\frac{1}{n}(-\cos nt)+\frac{2}{n}[t\int\cos nt\,dt-\int\{\frac{d}{dt}(t)\int\cos nt\,dt\}dt]$$

$$=t^2\frac{1}{n}(-\cos nt)+\frac{2}{n}[t.\frac{1}{n}\sin nt-\int 1.\frac{1}{n}\sin nt\,dt]$$

$$=-t^2\frac{1}{n}\cos nt+\frac{2}{n}[t.\frac{1}{n}\sin nt-\frac{1}{n}\int\sin nt\,dt]$$

$$=-t^2\frac{1}{n}\cos nt+\frac{2}{n}[t.\frac{1}{n}\sin nt-\frac{1}{n}\frac{1}{n}(-\cos nt)]$$

$$= -t^2\frac{1}{n}\cos nt + \frac{2}{n}[t.\frac{1}{n}\sin nt + \frac{1}{n^2}\cos nt]$$

$$= -t^2\frac{1}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt \text{ ---------------------(iv)}$$

$\int t\sin nt dt$

$$= t\int \sin nt dt - \int\{\frac{d}{dt}(t)\int \sin nt dt\}dt \;[\because \int uv dx = u\int v dx - \int\{\frac{d}{dx}(u)\int v dx\}dx]$$

$$= t\int \sin nt dt - \int 1.\frac{1}{n}(-\cos nt)dt$$

$$= t\frac{1}{n}(-\cos nt) + \frac{1}{n}\int \cos nt dt$$

$$= t\frac{1}{n}(-\cos nt) + \frac{1}{n}\frac{1}{n}\sin nt \;[\int \cos mx\, dx = \frac{1}{m}\sin mx; \int \sin mx\, dx = \frac{-1}{m}\cos mx]$$

$$= -t\frac{1}{n}\cos nt + \frac{1}{n^2}\sin nt \text{ -----------------------------(v)}$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\sin nt dt + \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt dt$$

$$= \frac{1}{\pi}\left[-t^2\frac{1}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt\right]_{-\pi}^{\pi} + \frac{1}{\pi}\left[-t\frac{1}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_{-\pi}^{\pi}$$

[From (iv) & (v)]

$$= \frac{1}{\pi}[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi - \frac{-(-\pi)^2}{n}\cos(-n\pi) - \frac{2(-\pi)}{n^2}\sin(-n\pi)$$

$$-\frac{2}{n^3}(\cos(-n\pi))] + \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi - \frac{-(-\pi)}{n}\cos(-n\pi) - \frac{1}{n^2}\sin(-n\pi)]$$

$$= \frac{1}{\pi}[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - \frac{2\pi}{n^2}\sin n\pi$$

$$-\frac{2}{n^3}\cos n\pi] + \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin n\pi - \frac{\pi}{n}\cos(-n\pi) + \frac{1}{n^2}\sin n\pi]$$

$$= \frac{1}{\pi}[-\frac{\pi^2}{n}\cos n\pi + 0 + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - 0 - \frac{2}{n^3}\cos n\pi] + \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi + 0$$

$$-\frac{\pi}{n}\cos n\pi + 0] \;[\because \sin(-\theta) = -\sin\theta\;,\;\; \cos(-\theta) = \cos\theta]$$

$$= \frac{1}{\pi}[-\frac{\pi^2}{n}\cos n\pi + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - \frac{2}{n^3}\cos n\pi] + \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi - \frac{\pi}{n}\cos n\pi]$$

$$= \frac{1}{\pi}[-\frac{\pi}{n}\cos n\pi - \frac{\pi}{n}\cos n\pi]$$

$$= -\frac{2}{n}\cos n\pi$$

The Fourier series for the above function is:

$$\therefore\ f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi t}{L}$$

$$\therefore f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi t}{\pi}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi t}{\pi}$$

$$\therefore f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nt+\sum_{n=1}^{\infty}b_n\sin nt$$

$$\therefore\ f(t)=\frac{2\pi^3/3}{2}+\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi\cos nt+\sum_{n=1}^{\infty}(-\frac{2}{n}\cos n\pi)\sin nt$$

$$\underbrace{f(t)}_{\text{Complex wave}}=\underbrace{\frac{2\pi^3/3}{2}}_{\text{DC value}}+\underbrace{\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi\cos nt+\sum_{n=1}^{\infty}(-\frac{2}{n}\cos n\pi)\sin nt}_{\text{AC value}}$$ Answer

Example 30:

$$\begin{aligned} y=f(t)&=t\ ; \quad 0\le t<\pi \\ &=\pi\ ; \quad \pi\le t<2\pi \end{aligned}$$ -----------------------(i)

Here, $T=2L=2\pi \qquad \therefore L=\pi$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

We have the Fourier series is:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi t}{L}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi t}{L}\right)$$ -------------------------(ii)

$$a_0=\frac{1}{L}\int_{-L}^{L}f(t)dt$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt \qquad [L=\pi]$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{0}^{2\pi}f(t)dt \qquad [\int_{-a}^{+a}f(x)dx=\int_{0}^{2a}f(x)dx;$$ **Method #19, Integral Calculus**]

$$\Rightarrow a_0=\frac{1}{\pi}\int_{0}^{\pi}f(t)dt+\frac{1}{\pi}\int_{\pi}^{2\pi}f(t)dt$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{0}^{\pi}t\,dt+\frac{1}{\pi}\int_{\pi}^{2\pi}\pi\,dt \qquad \text{[From (i)]}$$

$$\Rightarrow a_0=\frac{1}{\pi}\int_{0}^{\pi}t\,dt+\int_{\pi}^{2\pi}dt$$

$$\Rightarrow a_0=\frac{1}{\pi}\left[\frac{t^2}{2}\right]_0^{\pi}+[t]_{\pi}^{2\pi}$$

$$\Rightarrow a_0 = \frac{1}{2\pi}\left[\pi^2 - 0\right] + \left[2\pi - \pi\right]$$

$$\Rightarrow a_0 = \frac{1}{2\pi} \times \pi^2 + \pi$$

$$\Rightarrow a_0 = \frac{\pi}{2} + \pi$$

$$\Rightarrow a_0 = \frac{3\pi}{2}$$

We have,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\ \cos(n\frac{\pi}{L}t)dt$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \cos(n\frac{\pi}{\pi}t)dt \qquad [L = \pi]$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\ \text{cosntdt}$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(t)\ \text{cosntdt} \qquad [\int_{-a}^{+a} f(x)dx = \int_{0}^{2a} f(x)dx;\ \text{Method \#19, Integral Calculus}]$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{0}^{\pi} f(t)\text{cosntdt} + \frac{1}{\pi}\int_{\pi}^{2\pi} f(t)\text{cosntdt}$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{0}^{\pi} t \times \text{cosntdt} + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi \times \text{cosntdt}$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{0}^{\pi} t \times \text{cosntdt} + \int_{\pi}^{2\pi} \text{cosntdt} \qquad \text{-------------------------(iii)}$$

Now, Let

$$I = \int t \times \text{cosntdt}$$

$$= t\int \text{cosntdt} - \int\left[\frac{d}{dt}(t) \times \int \text{cosntdt}\right]dt \qquad [\int uvdx = u\int vdx - \int\left\{\frac{d}{dx}(u)\int vdx\right\}dx\ ]$$

$$= t \times \left(\frac{\text{sinnt}}{n}\right) - \int 1 \times \left(\frac{\text{sinnt}}{n}\right)dt$$

$$= \frac{t}{n}\text{sinnt} - \frac{1}{n} \times \left(\frac{-\text{cosnt}}{n}\right)$$

$$= \frac{t}{n}\text{sinnt} + \frac{1}{n^2}\text{cosnt} \qquad \text{-------------------------(iv)}$$

Putting Result of $\int t \times \text{cosnt}\, dt$ in (iii)

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} t \times \text{cosntdt} + \int_{\pi}^{2\pi} \text{cosntdt}$$

$$a_n = \frac{1}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi} + \left[\frac{\sin nt}{n}\right]_{\pi}^{2\pi}$$

$$a_n = \frac{1}{\pi}\left[\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi - \frac{0}{n}\sin(n\times 0) - \frac{1}{n^2}\cos(n\times 0)\right] + \frac{1}{n}\left[\sin(n\times 2\pi) - \sin n\pi\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{\pi}{n}\sin n\pi + \frac{1}{n^2}\cos n\pi - \frac{0}{n}\sin(n\times 0) - \frac{1}{n^2}\cos 0\right] + \frac{1}{n}\left[\sin 2n\pi - \sin n\pi\right]$$

$$a_n = \frac{1}{\pi}\left(\frac{\pi}{n}\times 0 + \frac{1}{n^2}\cos n\pi - 0 - \frac{1}{n^2}.1\right) + \frac{1}{n}\sin n\pi$$

[$\sin n\pi = 0$ For any integer values of n & $\cos 0 = 1$]

$$a_n = \frac{1}{\pi}\left(\frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right) + 0$$

$$a_n = \frac{1}{\pi n^2}(\cos n\pi - 1) \qquad \text{----------------------------------(v)}$$

Again, $b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(t)\,\sin(n\frac{\pi}{L}t)dt$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\sin(n\frac{\pi}{\pi}t)dt \qquad [L = \pi]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\sin nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(t)\sin nt\,dt \qquad [\int_{-a}^{+a} f(x)dx = \int_0^{2a} f(x)dx;\ \textbf{Method \#19, Integral Calculus}]$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} f(t)\sin nt\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} f(t)\sin nt\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} t\times\sin nt\,dt + \frac{1}{\pi}\int_{\pi}^{2\pi} \pi\times\sin nt\,dt$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} t\times\sin nt\,dt + \int_{\pi}^{2\pi}\sin nt\,dt \qquad \text{------------------------(vi)}$$

Now, Let

$$I = \int t\times\sin nt\,dt$$

$$= t\int\sin nt\,dt - \int\left\{\frac{d}{dt}(t)\int\sin nt\,dt\right\}dt \qquad [\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx\,]$$

$$= t\left[\frac{-\cos nt}{n}\right] - \int 1\times\left[\frac{-\cos nt}{n}\right]dt$$

$$= -\frac{t}{n}\cos nt + \frac{1}{n}\int\cos nt\,dt$$

$$= -\frac{t}{n}\cos nt + \frac{1}{n}\left[\frac{\sin nt}{n}\right]$$

$$= -\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt$$

$$= \frac{1}{n^2}\sin nt - \frac{t}{n}\cos nt \qquad \text{-----------------------------------(vii)}$$

Putting Result of $\int t \times \sin nt\,dt$ in (vi)

$$b_n = \frac{1}{\pi}\int_0^{\pi} t \times \sin nt\,dt + \int_{\pi}^{2\pi} \sin nt\,dt$$

$$b_n = \frac{1}{\pi}\left[\frac{1}{n^2}\sin nt - \frac{t}{n}\cos nt\right]_0^{\pi} + \left[\frac{-\cos nt}{n}\right]_{\pi}^{2\pi}$$

$$b_n = \frac{1}{\pi}\left(\frac{1}{n^2}\sin n\pi - \frac{\pi}{n}\cos n\pi - \frac{1}{n^2}\sin(n \times 0) - \frac{0}{n}\cos(n \times 0)\right) - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = \frac{1}{\pi}\left(\frac{1}{n^2} \times 0 - \frac{\pi}{n}\cos n\pi - \frac{1}{n^2}\sin 0 - \frac{0}{n}\cos 0\right) - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = \frac{1}{\pi}\left(0 - \frac{\pi}{n}\cos n\pi - 0 - 0\right) - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = \frac{1}{\pi}\left(-\frac{\pi}{n}\cos n\pi\right) - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = -\frac{1}{n}\cos n\pi - \frac{1}{n}(\cos 2n\pi - \cos n\pi)$$

$$b_n = -\frac{1}{n}\cos n\pi - \frac{1}{n}\cos 2n\pi + \frac{1}{n}\cos n\pi$$

$$b_n = -\frac{1}{n}\cos 2n\pi \qquad \text{-----------------------------------(viii)}$$

Putting the values of a_0, a_n, b_n in (ii)

The Fourier series for the above function is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$= \frac{\frac{3\pi}{2}}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}(\cos n\pi - 1)\cos\left(\frac{n\pi t}{\pi}\right) + \sum_{n=1}^{\infty} -\frac{1}{n}\cos 2n\pi \sin\left(\frac{n\pi t}{\pi}\right)$$

$$= \frac{3\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}(\cos n\pi - 1)\cos nt - \sum_{n=1}^{\infty} \frac{1}{n}\cos 2n\pi \sin nt$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{\frac{3\pi}{4}}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{\pi n^2}(\cos n\pi - 1)\cos nt - \sum_{n=1}^{\infty} \frac{1}{n}\cos 2n\pi \sin nt}_{\text{AC value}} \quad \textit{Answer}$$

Example 31:

If $y = f(t) = t^2$ over the integral $-\pi < t < \pi$ and has period 2π . Here, $T = 2L = 2\pi \therefore L = \pi$

a) **Sketch three cycles of $y = f(t)$ in the interval $-3\pi < t < 3\pi$**
b) **Find the Fourier series for the function.**

Answer:

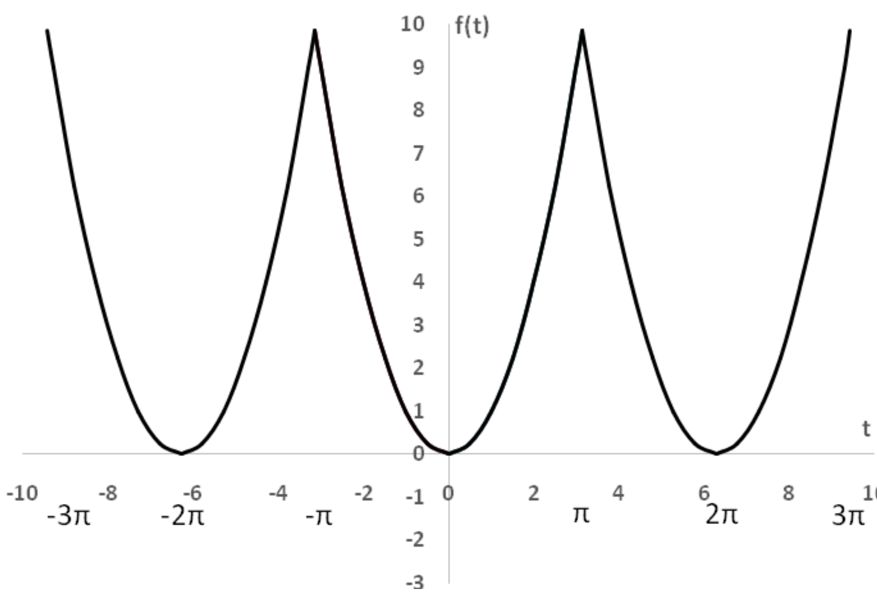


Figure 57: A periodic signal with period $\mathbf{T = 2L = 2\pi}$

Let $y = f(t) = t^2$ --------------- (i)

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt \qquad \text{[From (i)]}$$

$$= \frac{1}{\pi}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos\frac{n\pi t}{\pi}\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos nt\,dt \qquad \text{[From (i)]}$$

Now $\int t^2 \cos nt\,dt$

$$= t^2\int \cos nt\,dt - \int\{\frac{d}{dt}(t^2)\int \cos nt\,dt\}dt \; [\because \int uv\,dx = u\int v\,dx - \int\{\frac{d}{dx}(u)\int v\,dx\}dx]$$

$$= \frac{t^2}{n}\sin nt - \int\{2t.\frac{\sin nt}{n}\}dt$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\left[t\int \sin nt - \int\{\frac{d}{dt}(t)\int \sin nt\,dt\}dt\right]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\left[t.\frac{(-\cos nt)}{n} - \int\frac{(-\cos nt)}{n}dt\right]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\left[\frac{-t\cos nt}{n} + \int\frac{\cos nt}{n}dt\right]$$

$$= \frac{t^2}{n}\sin nt - \frac{2}{n}\left[\frac{-t\cos nt}{n} + \frac{\sin nt}{n^2}\right]$$

$$=\frac{t^2}{n}\sin nt+\frac{2t}{n^2}\cos nt-\frac{2}{n^3}\sin nt$$

$$\int t^2\cos nt\,dt=\frac{t^2}{n}\sin nt+\frac{2t}{n^2}\cos nt-\frac{2}{n^3}\sin nt \text{ --------------- (ii)}$$

$$\therefore a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}t^2\cos nt\,dt$$

$$=\frac{1}{\pi}\left[\frac{t^2}{n}\sin nt+\frac{2t}{n^2}\cos nt-\frac{2}{n^3}\sin nt\right]_{-\pi}^{\pi} \qquad \text{[From (ii)]}$$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{n}\sin n\pi+\frac{2\pi}{n^2}\cos n\pi-\frac{2}{n^3}\sin n\pi-\frac{(-\pi)^2}{n}\sin n(-\pi)-\frac{2(-\pi)}{n^2}\cos n(-\pi)+\frac{2}{n^3}\sin n(-\pi)\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{n}\sin n\pi+\frac{2\pi}{n^2}\cos n\pi-\frac{2}{n^3}\sin n\pi+\frac{\pi^2}{n}\sin n\pi+\frac{2\pi}{n^2}\cos n\pi-\frac{2}{n^3}\sin n\pi\right]$$

$$[\sin(-\theta)=-\sin\theta;\ \cos(-\theta)=\cos\theta]$$

$$=\frac{1}{\pi}\left[0+\frac{2\pi}{n^2}\cos n\pi-0+0+\frac{2\pi}{n^2}\cos n\pi-0\right]$$

$$[\sin n\pi=0; \text{For } n=1,2,3.........]$$

$$=\frac{1}{\pi}\left[\frac{2\pi}{n^2}\cos n\pi+\frac{2\pi}{n^2}\cos n\pi\right]$$

$$=\frac{1}{\pi}\left[\frac{4\pi}{n^2}\cos n\pi\right]$$

$$=\frac{4}{n^2}\cos n\pi$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi t}{L}\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin\frac{n\pi t}{\pi}\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}t^2\sin nt\,dt \qquad \text{[From (i)]}$$

Now $\int t^2\sin nt\,dt$

$$=t^2\int\sin nt\,dt-\int\{\frac{d}{dt}(t^2)\int\sin nt\,dt\}dt\ [\because\int uv\,dx=u\int v\,dx-\int\{\frac{d}{dx}(u)\int v\,dx\}dx]$$

$$= t^2 \int \sin nt dt - \int \{2t.\frac{1}{n}(-\cos nt)\} dt$$

$$= t^2 \frac{1}{n}(-\cos nt) + \frac{2}{n}\int t \cos nt dt$$

$$= t^2 \frac{1}{n}(-\cos nt) + \frac{2}{n}\left[t\int \cos nt - \int \{\frac{d}{dt}(t)\int \cos nt dt\} dt \right]$$

$$= -t^2 \frac{1}{n}\cos nt + \frac{2}{n}\left[t.\frac{1}{n}\sin nt - \int 1.\frac{1}{n}\sin nt dt \right]$$

$$= -t^2 \frac{1}{n}\cos nt + \frac{2}{n}\left[t.\frac{1}{n}\sin nt - \frac{1}{n}\int \sin nt dt \right]$$

$$= -t^2 \frac{1}{n}\cos nt + \frac{2}{n}\left[t.\frac{1}{n}\sin nt - \frac{1}{n}.\frac{1}{n}(-\cos nt) \right]$$

$$= -t^2 \frac{1}{n}\cos nt + \frac{2}{n}\left[t.\frac{1}{n}\sin nt + \frac{1}{n^2}\cos nt \right]$$

$$\int t^2 \sin nt dt = -t^2 \frac{1}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt \text{ ---------------(iii)}$$

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \sin nt dt$$

$$= \frac{1}{\pi}\left[-t^2 \frac{1}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt \right]_{-\pi}^{\pi} \qquad \text{[From (iii)]}$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi - \frac{-(-\pi)^2}{n}\cos n(-\pi) - \frac{2(-\pi)}{n^2}\sin n(-\pi) - \frac{2}{n^3}\cos n(-\pi) \right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2\pi}{n^2}\sin n\pi + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - \frac{2\pi}{n^2}\sin n\pi - \frac{2}{n^3}\cos n\pi \right]$$

$$[\sin(-\theta) = -\sin\theta;\ \cos(-\theta) = \cos\theta]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + 0 + \frac{2}{n^3}\cos n\pi + \frac{\pi^2}{n}\cos n\pi - 0 - \frac{2}{n^3}\cos n\pi \right]$$

$$[\sin n\pi = 0; \text{For } n = 1,2,3.........]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}\cancel{\cos n\pi} + \frac{2}{n^3}\cancel{\cos n\pi} + \frac{\pi^2}{n}\cancel{\cos n\pi} - \frac{2}{n^3}\cancel{\cos n\pi} \right]$$

$$= \frac{1}{\pi}[0]$$

$$= 0$$

The Fourier series for the above function is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$= \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi \cos nt + 0$$

$$= \frac{2\pi^2/3}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi \cos nt$$

$$\underbrace{f(t)}_{Complex\ wave} = \underbrace{\frac{2\pi^2/3}{2}}_{DC\ value} + \underbrace{\sum_{n=1}^{\infty} \frac{4}{n^2} \cos n\pi \cos nt}_{AC\ value}$$ Answer

Home Task

Find the Fourier series for the functions

01.

$$y = f(t) = 1\ ; \quad -\pi \le t < 0$$
$$= 0\ ; \quad 0 \le t < \pi$$

Here, $\mathbf{T = 2L = 2\pi \qquad \therefore L = \pi}$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function.

02.

$$y = f(t) = 0\ ; \quad -\pi \le t < 0$$
$$= t\ ; \quad 0 \le t < \pi$$

Here, $\mathbf{T = 2L = 2\pi \qquad \therefore L = \pi}$

a) Sketch the function for 3 cycles:
b) Find the Fourier series for the function

03.

$y = f(t) = \frac{t}{2}$ Over the interval $0 < t < 2\pi$ and has period 2π

Here, $\mathbf{T = 2L = 2\pi \qquad \therefore L = \pi}$

a) Sketch a graph of $y = f(t)$ in the interval $0 < t < 4\pi$
b) Find the Fourier series for the function

04.

$$y = f(t) = \pi - t\ ; \quad 0 < t < \pi$$
$$= 0\ ; \quad \pi \le t < 2\pi$$

Here, $\mathbf{T = 2L = 2\pi \qquad \therefore L = \pi}$

a) Sketch a graph of $y = f(t)$ in the interval $-2\pi < t < 2\pi$
b) Find the Fourier series for the function

Problem 09: Symmetry in Waveforms

Many periodic waveforms exhibit symmetry. The following three types of symmetry help to reduce tedious calculations in the analysis.

i. Even symmetry
ii. Odd symmetry
iii. Half-wave symmetry

Even Symmetry

A function $\mathbf{f(t)}$ exhibits even symmetry, when the region before the *y-axis* is the mirror image of the region after the *y-axis*. That is, $\mathbf{f(t) = f(-t)}$

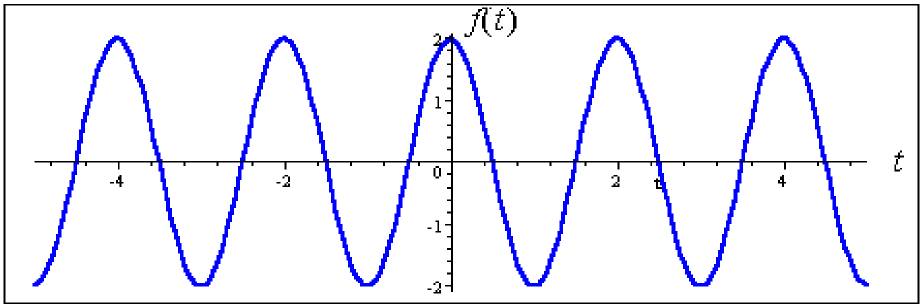


Figure 58

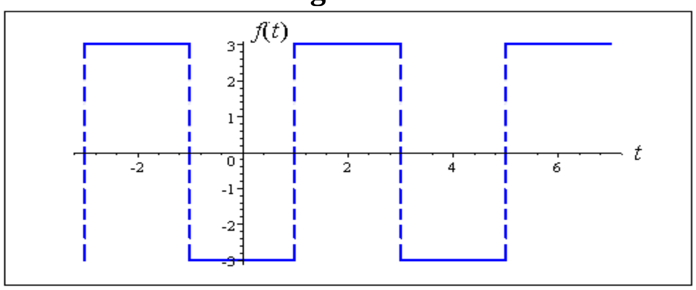


Figure 59: Even Square wave

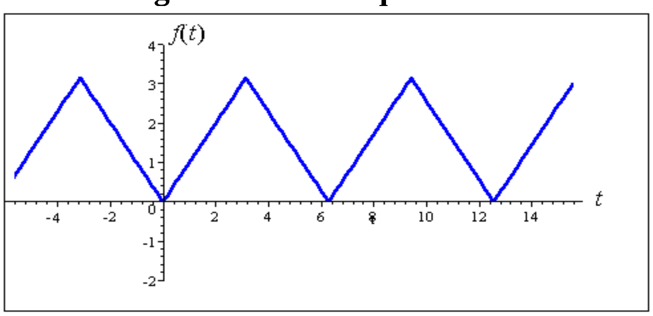


Figure 60: Triangular wave

$$f(t) = \begin{cases} t+\pi & \text{if } -\pi \le t < 0 \\ -t+\pi & \text{if } 0 \le t < \pi \end{cases}$$

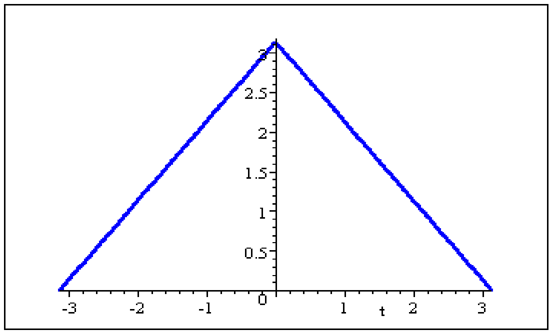


Figure 61: The graph that it is **even**

$$f(t) = \begin{cases} e^{t} & \text{if } -\pi \le t < 0 \\ e^{-t} & \text{if } 0 \le t < \pi \end{cases}$$

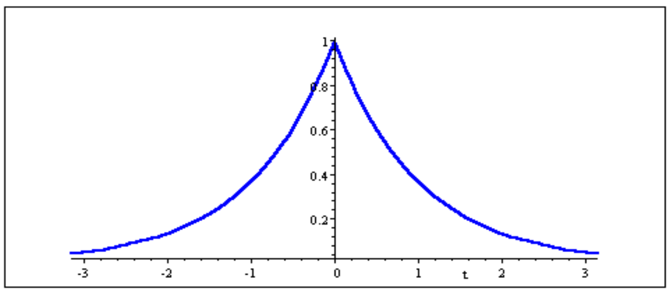


Figure 62

Odd Symmetry

A function $\mathbf{f(t)}$ exhibits even symmetry, when the region before the *y-axis* is the negative of the mirror image of the region after the *y-axis*. That is, $\mathbf{f(t) = -f(-t)}$

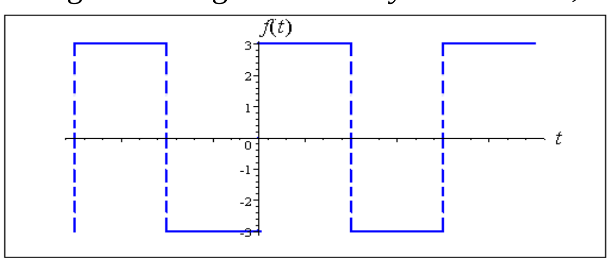


Figure 63: Odd Square wave

$$f(t) = \begin{cases} \left(t + \frac{\pi}{2}\right)^2 & \text{if } -\pi \le t < 0 \\ -\left(t - \frac{\pi}{2}\right)^2 & \text{if } 0 \le t < \pi \end{cases}$$

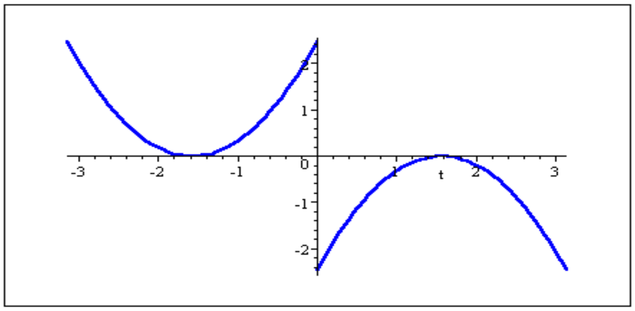


Figure 64: The graph that the function is **odd**

Half-wave Symmetry

A function $\mathbf{f(t)}$ exhibits half-wave symmetry, when one half of the waveform is exactly equal to the negative of the previous or the next half of the waveform. That is,

$$\mathbf{f(t) = (-)f(t - \frac{T}{2}) = (-)f(t + \frac{T}{2})}$$

Summary of Analysis of waveforms with symmetrical properties

i. With **even symmetry**, $\mathbf{b_n = 0}$ for all n, and $\mathbf{a_n}$ is twice the integral over half the cycle from zero time

ii. With **odd symmetry**, $\mathbf{a_n = 0}$ for all n, and $\mathbf{b_n}$ is twice the integral over half the cycle from zero time

iii. With **half-wave symmetry**, $\mathbf{a_n}$ and $\mathbf{b_n}$ are 0 for even n, and twice the integral over any half cycle for odd n

iv. If **half-wave symmetry** and either **even symmetry** or **odd symmetry** are present, then $\mathbf{a_n}$ and $\mathbf{b_n}$ are 0 for even n, and four times the integral over the quarter cycle for odd n for $\mathbf{a_n}$ or $\mathbf{b_n}$ respectively and zero for the remaining coefficient.

v. It is also to be noted that in any waveform, $\mathbf{\frac{a_0}{2}}$ corresponds to the mean value of the waveform and that sometimes a symmetrical property may be obtained by subtracting this value from the waveform

<u>Problem 10:</u>

Express $\boldsymbol{a\cos\theta \pm b\sin\theta}$ in the form $\boldsymbol{R\sin(\theta \pm \alpha)}$ that is $\boldsymbol{a\cos\theta \pm b\sin\theta = R\sin(\theta \pm \alpha)}$

Solution:

First we take the "plus" case, $\mathbf{(\theta + \alpha)}$ to make things easy.

Let, $\boldsymbol{a\cos\theta + b\sin\theta = R\sin(\theta + \alpha)}$ ------------------------------(i)

Here,

$\boldsymbol{a \rightarrow}$ **Amplitude of the cosine wave**

$\boldsymbol{b \rightarrow}$ **Amplitude of the sine wave**

$\boldsymbol{R \rightarrow}$ **Amplitude of the new signal**

$\boldsymbol{\alpha \rightarrow}$ **Phase Shift**

[$a\cos\theta + b\sin\theta = R\sin(\theta+\alpha)$ wjLv hvq| A_©vr $a\cos\theta$ & $b\sin\theta$ ‡hvM Ki‡j †h bZzb wmMb¨vj cvIqv hvq Zvi m‡e©Ÿv"P Amplitude n‡e R Avi GB bZzb wmMb¨vj sine n‡e hw` $\alpha = \pi$ or $0$ nq Ges GB bZzb wmMb¨vj cos n‡e hw` $\alpha = \frac{\pi}{2}$ nq]

Using the compound angle formula from before (Sine of the sum of angles),
$\sin(A+B) = \sin A\cos B + \cos A\sin B$---------------------------(ii)

We can expand the **RHS** of the above equation (i), as follows:

$R\sin(\theta+\alpha)$

$R\sin(\theta+\alpha) \equiv R(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$ $[\because \sin(A+B) = \sin A\cos B + \cos A\sin B]$

$R\sin(\theta+\alpha) \equiv R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$---------------------(iii)

From. (i),

So $a\cos\theta + b\sin\theta = R\sin(\theta+\alpha)$

$a\cos\theta + b\sin\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$ [From (iii)]

$a\cos\theta + b\sin\theta = R\sin\theta\cos\alpha + R\cos\theta\sin\alpha$

$a\cos\theta + b\sin\theta = \mathrm{R}\cos\theta\sin\alpha + \mathrm{R}\sin\theta\cos\alpha$------------------(iv)

Equating the coefficients of $\cos\theta$ and $\sin\theta$ from (iv), we have:

$a = R\sin\alpha$--------------------------(v)

$b = R\cos\alpha$--------------------------(vi)

From (v) and (vi),

$\frac{a}{b} = \frac{R\sin\alpha}{R\cos\alpha}$

$\Rightarrow \frac{a}{b} = \tan\alpha$

$\Rightarrow \tan\alpha = \frac{a}{b}$

$\Rightarrow \alpha = \tan^{-1}(\frac{a}{b})$-------------------(vii)

Again, from (v) and (vi), squaring and then adding

$a^2 + b^2 = R^2\sin^2\alpha + R^2\cos^2\alpha$

$\Rightarrow a^2 + b^2 = R^2(\sin^2\alpha + \cos^2\alpha)$

$\Rightarrow a^2 + b^2 = R^2.1$ $[\because \sin^2\alpha + \cos^2\alpha = 1]$

$\Rightarrow a^2 + b^2 = R^2$

$\Rightarrow R^2 = a^2 + b^2$

$\Rightarrow R = \sqrt{a^2 + b^2}$ -------------------(viii)

Similarly, for the minus case:

$a\cos\theta - b\sin\theta = R\sin(\theta-\alpha)$

We get,

$\Rightarrow R = \sqrt{a^2 + b^2}$ and $\Rightarrow \alpha = \tan^{-1}(\frac{a}{b})$

Here R is called the resultant amplitude and α is the phase difference.

[$a\cos\theta + b\sin\theta = R\sin(\theta + \alpha)$ wjLv hvq| A_©vr $a\cos\theta + b\sin\theta$ ‡hvM Ki‡j †h bZzb wmMb¨vj cvIqv hvq Zvi m‡e©Ÿv"P Amplitude n‡e R Avi GB bZzb wmMb¨vj sine n‡e hw` $\alpha = \pi$ *or* 0 nq Ges GB bZzb wmMb¨vj cos n‡e hw` $\alpha = \frac{\pi}{2}$ nq, †hLv‡b bZzb wmMb¨v‡ji Amplitude $R = \sqrt{a^2 + b^2}$]

Example 32:

Let, $x = \frac{1}{4}\cos 8t + \frac{1}{8}\sin 8t$ --(ix)

What is the amplitude of the resultant signal?

Answer:

We have, $a\cos\theta + b\sin\theta = R\sin(\theta + \alpha)$

So, in $x = \frac{1}{4}\cos 8t + \frac{1}{8}\sin 8t$

Here,

$a = \frac{1}{4}$ *and* $b = \frac{1}{8}$ *and* $\theta = 8$

(ix) bs mgxKiY n‡Z, cÖ_g wmMb¨v‡ji m‡ev"P© Amplitude $\frac{1}{4}$ Ges wØZxq wmMb¨v‡ji m‡ev"P© Amplitude $\frac{1}{8}$ | GB `ywU wmMb¨vj †hvM Ki‡j †h bZzb wmMb¨vj cvIqv hv‡e Zvi m‡e©v"P Amplitude hw` R nq Zvn‡j R Gi gvb †ei Ki‡Z cvwi ‡hLv‡b $R = \sqrt{(\frac{1}{4})^2 + (\frac{1}{8})^2}$ KviY $[\because \mathbf{R} = \sqrt{\mathbf{a}^2 + \mathbf{b}^2}]$

$$\Rightarrow \mathbf{R} = \sqrt{(\frac{1}{4})^2 + (\frac{1}{8})^2}$$

$$\Rightarrow \mathbf{R} = \sqrt{\frac{1}{16} + \frac{1}{64}}$$

$$\Rightarrow \mathbf{R} = \sqrt{\frac{4+1}{64}}$$

$$\Rightarrow \mathbf{R} = \sqrt{\frac{5}{64}}$$

$$\Rightarrow \mathbf{R} = \frac{\sqrt{5}}{8} \textit{ Answer}$$

Avi GB `ywU wmMb¨vj †hvM Ki‡j †mB bZzb wmMb¨vjwU sine Gi AvKvi aviY Ki‡Z cv‡i Avevi cosine Gi AvKvi aviY Ki‡Z cv‡i| Remember that sine wave and cosine wave with Phase difference is 90^0, Figure 19

Problem 11:

A signal can be viewed from two different standpoints:

- The time domain
- The frequency domain

The characteristics of electrical (and other) signals can be explored in two ways. The first is to examine their waveforms in the time domain, which can be done using an oscilloscope. The second is to examine their spectra in the frequency domain, which can be done using a *spectrum analyzer*. Both descriptions are useful, although frequency domain analysis is the most natural tool for many tasks in electrical circuit design, speech and music analysis, and for video signals. In fact, frequency domain analysis is of fundamental importance in all branches of engineering, from the design of bridges and roads to wireless Communications.

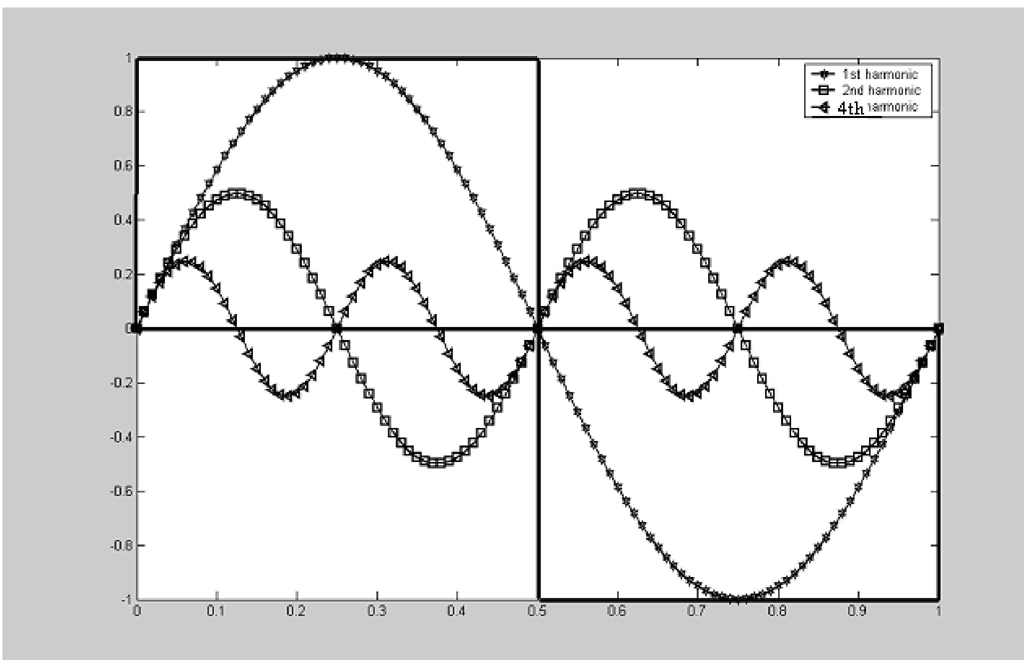


Figure 65: Time domain

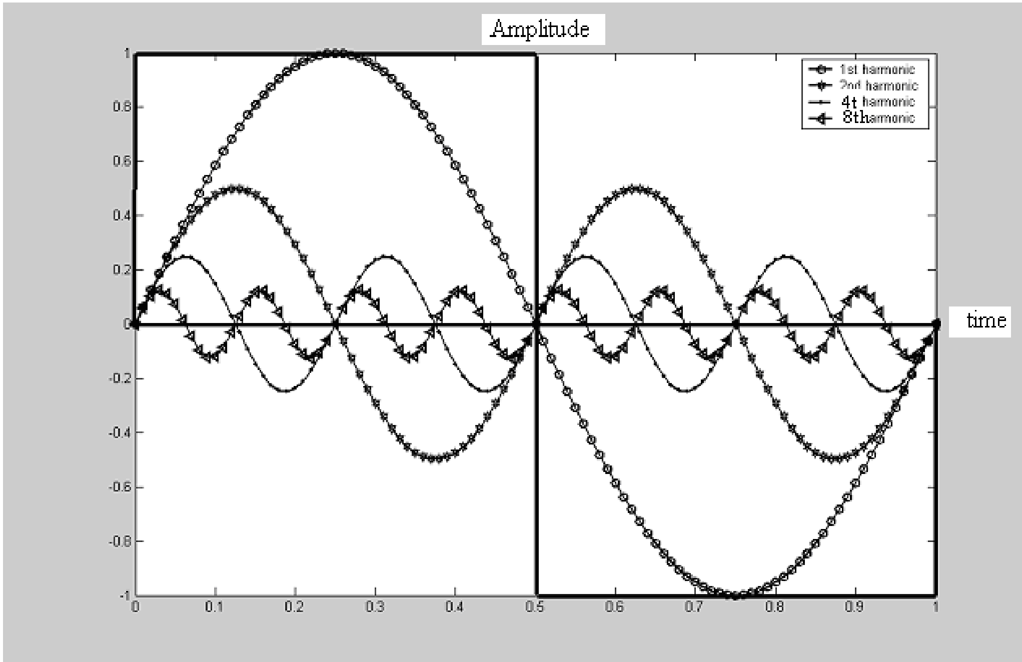


Figure 66: Time domain

<u>Problem 12:Frequency Spectrum</u>

Line spectrum or amplitude spectrum: A plot of amplitude against angular frequency is called the amplitude spectrum.

Phase spectrum: while that of phase against angular frequency is called the phase spectrum**.**

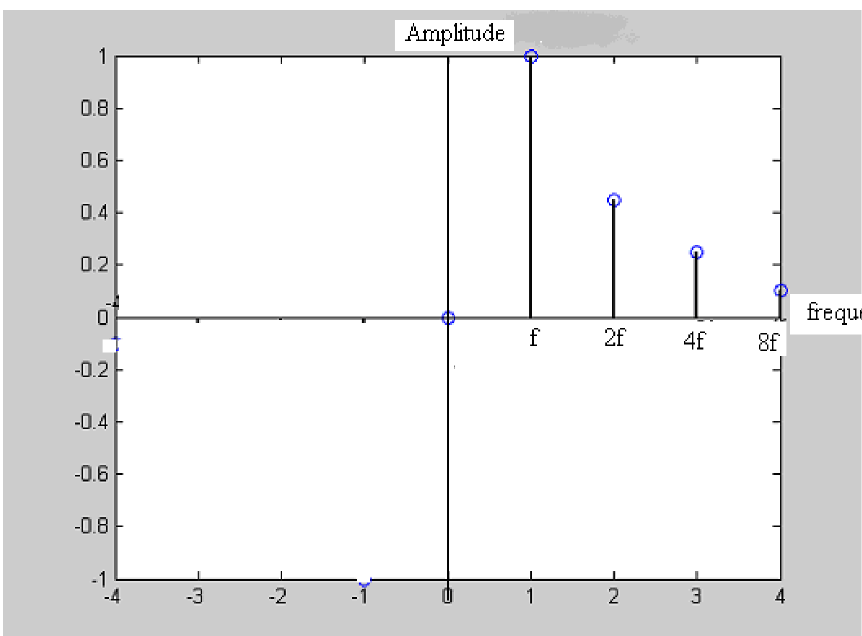


Figure 67: Frequency domain

Example 33:

A Fourier series is given below. List frequencies and plot the one-sided amplitude spectrum

$$\underbrace{x(t)}_{\text{Complex wave}} = \underbrace{12}_{\text{DC value}} \underbrace{+9\cos(2\pi\times 10t+\frac{\pi}{3})+6\cos(2\pi\times 20t-\frac{\pi}{6})+4\cos(2\pi\times 30t+\frac{\pi}{4})}_{\text{AC value}}$$

$$[\because \omega = 2\pi f]$$

Here DC value is 12; Amplitudes are 9, 6, and 4 with respect to frequencies 10, 20, 30 respectively where phases are $\frac{\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{4}$

The frequencies are 0 (dc), 10 Hz, 20 Hz, 30 Hz

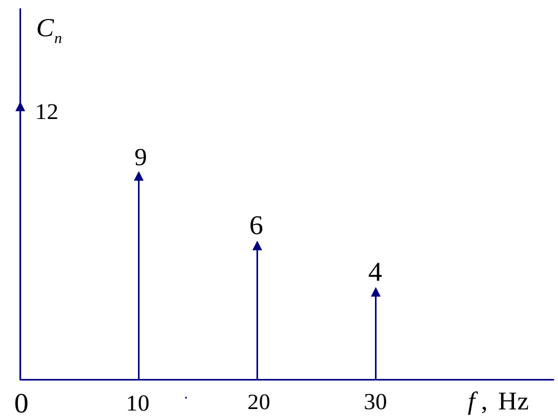


Figure 68

Spectrum Analysis

Spectrum analysis attempts to identify the relative quantities of different frequencies which are present in a given signal. Take, for example, a simple signal, $\mathbf{x(t)}$ made up of two pure sine-waves with frequencies 1 rad/second and 2 rad/second respectively:

$\mathbf{x(t) = 3\sin(1.t) + \sin(2t)}$--------------------------------(i)

There are exactly two `frequency' components present in this signal: 1 rad/second and 2 rad/second. The `amplitude' of the first is three times that of the second. We can represent this information in a plot of amplitude against frequency: a frequency amplitude spectrum

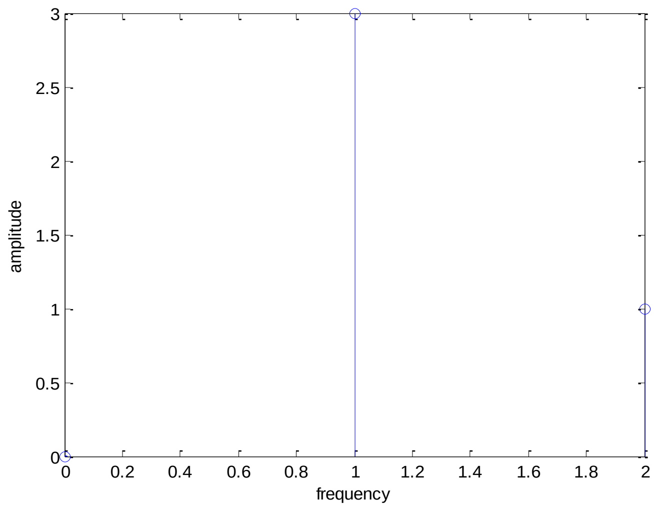


Figure 69

Problem 13: **Time domain / Frequency domain**

- Some signals are easier to visualize in the frequency domain
- Some signals are easier to visualize in the time domain
- Some signals are easier to define in the time domain (amount of information needed)
- Some signals are easier to define in the frequency domain (amount of information needed)

Example: speech recognition

Figure 70

Difficult to differentiate between different sounds in time domain
This time-domain plot shows a waveform that was once very familiar to electrical engineers: an amplitude modulated (AM) radio signal.

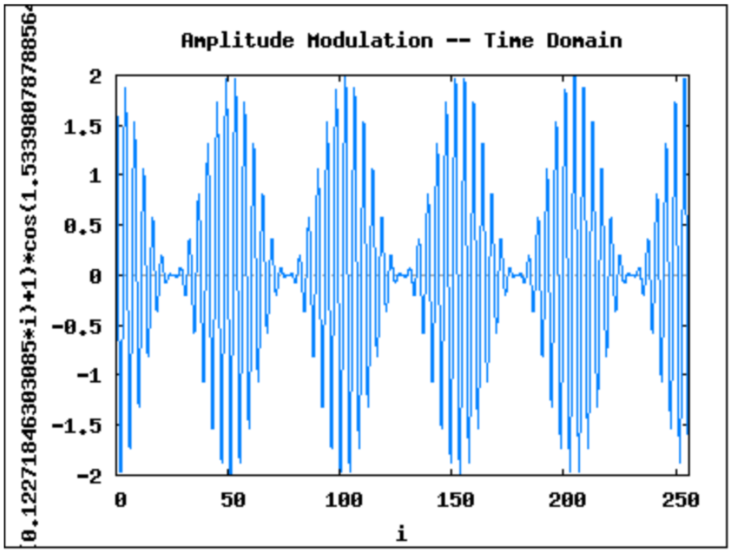


Figure 71

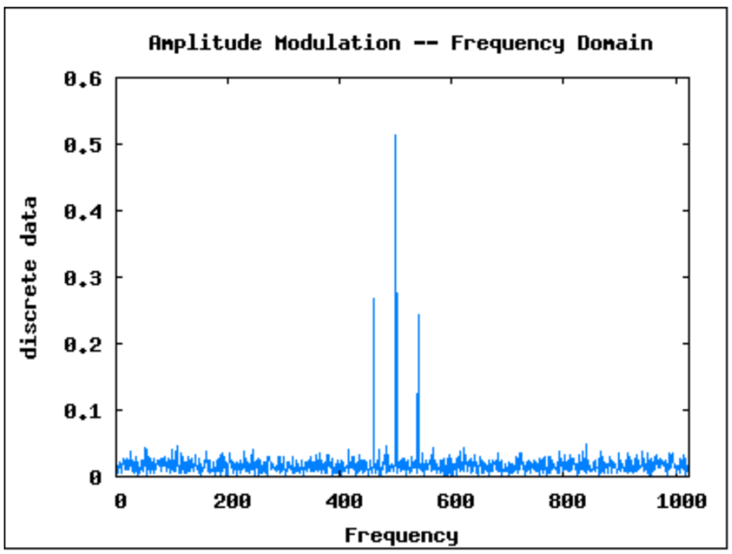


Figure 72

Here is the frequency-domain equivalent of the above time-domain plot

Problem 14: **Signal Spectrum**

Every signal has a frequency spectrum.

• The signal defines the spectrum

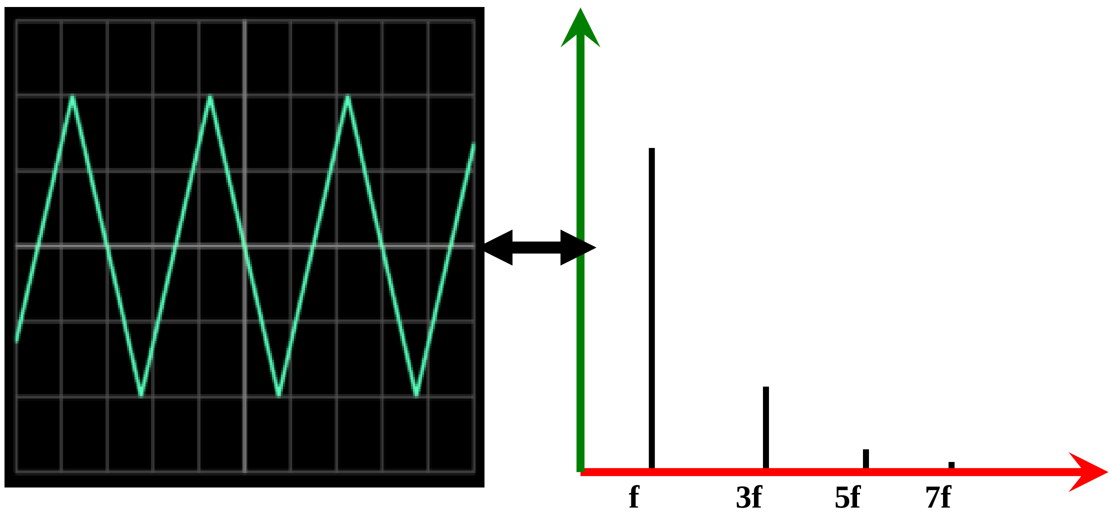
• The spectrum defines the signal

We can move back and forth between the time domain and the frequency domain without losing information

These sine terms are combined to create the time-varying waveform in the display. And if the opposite operation were to be performed on the result (something called a Fourier transform), these individual harmonic elements would reappear. It is important to reemphasize that waveform generation, and the Fourier transform, are reciprocal operations. You can use frequency components to generate a waveform in the time domain, then transform the result back to the frequency domain and recover what you started with. This reciprocal relationship is to Fourier analysis what the Fundamental Theorem of Calculus (the idea that integration and derivation are reciprocal operations) is to Calculus.

Waveform creation in the time domain, and harmonic analysis in the frequency domain, is reciprocal operations. One can take a list of harmonic components and use them to create a time-domain waveform, and then one can carry out a Fourier transform on the time-domain waveform to recapture the original harmonic components. It turns out that these two representations are equivalent and interchangeable.

Sound spectrum:



Time domain *frequency domain*

Figure 73

Spectrum Analysis* or *Fourier Analysis is the process of analyzing some time-domain waveform to find its spectrum. We also say that the time domain waveform is converted into a frequency spectrum by means of the *Fourier transform*. This process is reversible: using the *inverse Fourier transform* a spectrum may be converted back into a time-domain waveform.

Example 34:

A plot showing each of the harmonic amplitudes in the wave is called the line spectrum**.**
Plot the line spectrum (discrete frequency spectra) for the Fourier series:

$$f(t)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{1}{2n-1}\cos nt+\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\sin nt \text{ ---------------------------------------(i)}$$

This series has an interesting graph for the above function $\mathbf{f(t)}$

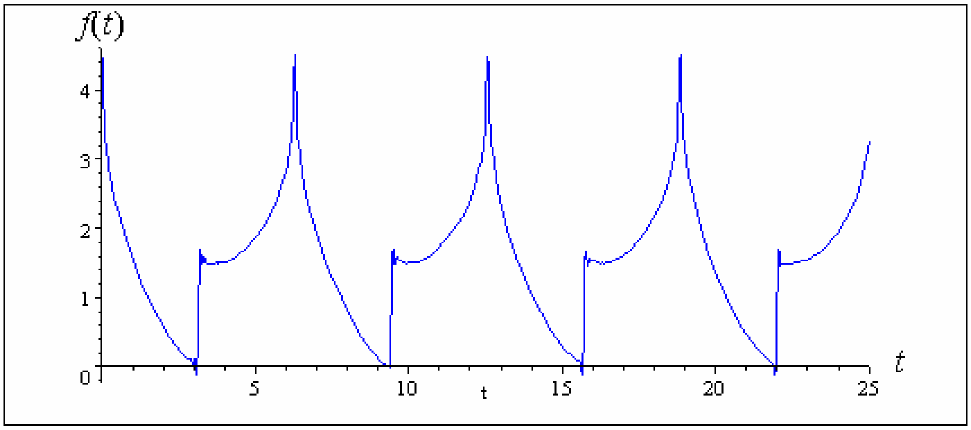


Figure 74

We have the Fourier series is $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t))$

We can see from the series (i) that

$$a_n = \frac{1}{2n-1} \qquad b_n = \frac{(-1)^n}{2n}$$

Now, using $\Rightarrow R = \sqrt{a^2 + b^2}$ [From (viii), page no 66]

Let $R_n = C_n = \sqrt{a_n^{\ 2} + b_n^{\ 2}}$

$a_n = \frac{1}{2n-1}$	$b_n = \frac{(-1)^n}{2n}$	$C_n = \sqrt{a_n^2 + b_n^2}$
$a_1 = 1$	$b_1 = -\frac{1}{2}$	$C_1 = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = 1.118$
$a_2 = \frac{1}{3}$	$b_2 = \frac{1}{4}$	$C_2 = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2} = 0.4167$
$a_3 = \frac{1}{5}$	$b_3 = -\frac{1}{6}$	$C_3 = \sqrt{\left(\frac{1}{5}\right)^2 + \left(-\frac{1}{6}\right)^2} = 0.260$
$a_4 = \frac{1}{7}$	$b_4 = \frac{1}{8}$	$C_4 = \sqrt{\left(\frac{1}{7}\right)^2 + \left(\frac{1}{8}\right)^2} = 0.190$

Here, from (i),

Here, $\mathbf{n\omega = n}$

For $n = 1$; **Fundamental Frequency = 1st Harmonic = $\omega = 1$**

For $n = 2$; **2nd Harmonic $= 2\omega = 2$**

For $n = 3$; **3rd Harmonic $= 3\omega = 3$**

For $n = 4$; **4th Harmonic $= 4\omega = 4$**

The resulting line spectrum is:

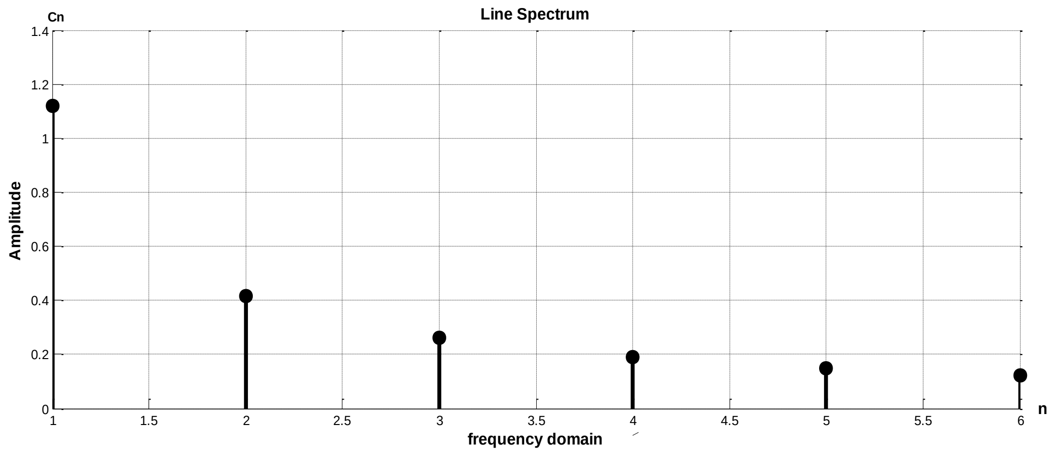


Figure 75: Line spectrum

Example 35:
Line spectrum: Example

$$s(t) = \frac{1}{2} + \frac{2}{\pi}\left(\cos(2\pi f_0 t) - \frac{1}{3}\cos(6\pi f_0 t) + \frac{1}{5}\cos(10\pi f_0 t) - \ldots\right)$$

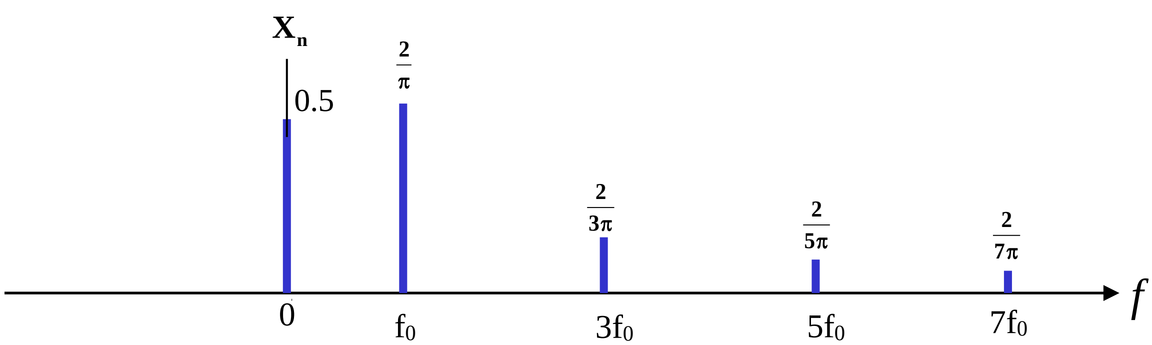


Figure 76

Where $X_n = \sqrt{a_n^2 + b_n^2}$

Home Task:

Plot the line spectrum (discrete frequency spectra) for the Fourier series:

$$f(t) = 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$$

Figure 74

We have the Fourier series is $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\omega t) + b_n \sin(n\omega t))$

Here, $a_n = 0$; $b_n = -\frac{5}{\pi}\frac{1}{n}(\cos n\pi - 1)$

Now, using $\Rightarrow R = \sqrt{a^2 + b^2}$

Let $R_n = C_n = \sqrt{a_n{}^2 + b_n{}^2}$

$a_n = 0$	$b_n = -\frac{5}{\pi}\frac{1}{n}(\cos n\pi - 1)$	$R_n = C_n = \sqrt{a_n{}^2 + b_n{}^2}$

Here, $n\omega = \frac{n\pi}{4}$

For n = 1; **Fundamental Frequency = 1st Harmonic =** $\omega = \frac{\pi}{4}$

For n = 2; **2nd Harmonic** $= 2\omega = \frac{2\pi}{4}$

For n = 3; **3rd Harmonic** $= 3\omega = \frac{3\pi}{4}$

For n = 4; **4th Harmonic** $= 4\omega = \frac{4\pi}{4}$

Problem 15: **Prism Analogy:**

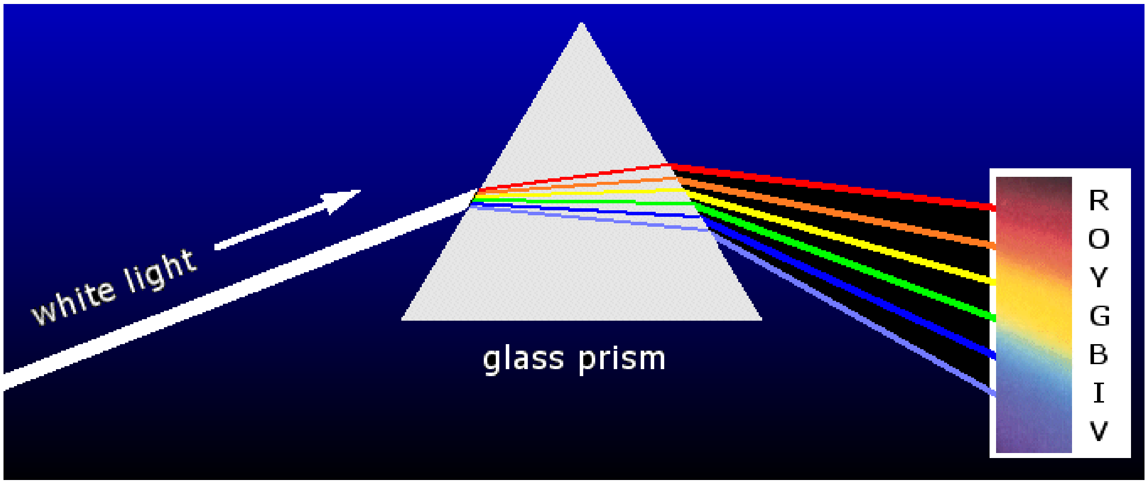


Figure 77

A prism which splits white light into a spectrum of colors. White light consists of all frequencies mixed together. The prism breaks them apart so we can see the separate frequencies

Signal g(t) → Fourier Transform → **Spectrum G (w)**

Figure 78

Problem 16: **How do we hear?**

In our inner ears, Cochlea consists of spiral of tissue with liquid and thousands of tiny hairs that gradually gets smaller. Each hair is connected to the nerve. The longer hairs resonate with lower frequencies, the shorter hairs resonate with higher frequencies. Thus the time-domain air pressure signal is transformed into frequency spectrum, which is then processed by the brain. ***Our ear is a Natural Fourier Transform Analyzer***

The cochlea transforms a time domain signal (the sound's waveform) into a frequency domain signal. The strength of the response in the auditory nerve fiber tuned to a particular frequency reflects the amplitude of the sound's waveform at that frequency. In other words, the auditory system takes a Fourier transform of the incoming signal, decomposing the sound into amplitudes as a function of frequency.

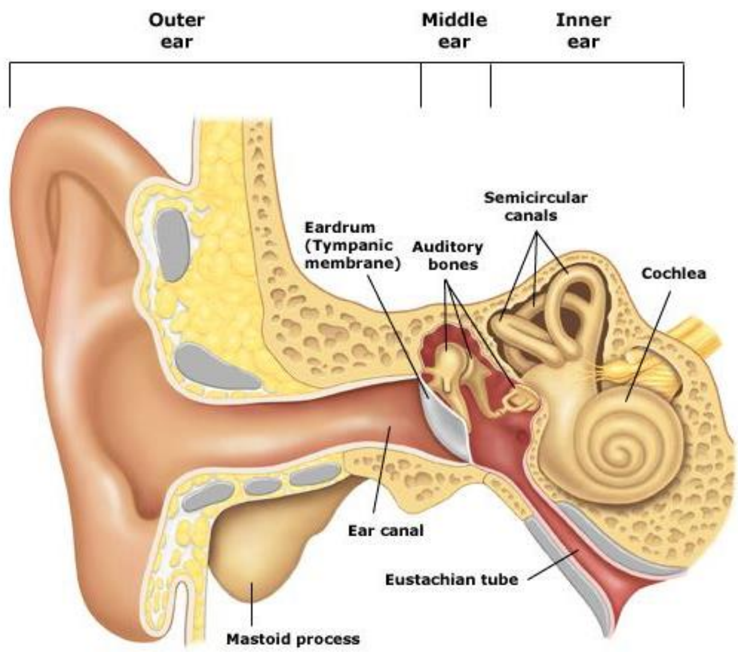


Figure 79

Sounds as a sum of different amplitude signals each with a different frequency (Figure 70). Here Sound is a complex wave.

In Fourier analysis a signal is decomposed into its constituent sinusoids, i.e. frequencies, the amplitudes of various frequencies form the so called frequency spectrum of the signal. In an inverse Fourier transform operation the signal can be synthesized by adding up its constituent frequencies. It turns out that many signals that we encounter in daily life such as speech, car engine noise, bird songs, music etc. have a periodic or quasi-periodic structure, and that the cochlea in the human hearing system performs a kind of harmonic analysis of the input audio signals. Therefore the concept of frequency is not a purely mathematical abstraction in that biological and physical systems have also evolved to make use of the frequency analysis concept.

Fourier analysis is by no means limited to these classic examples **—** ***it can analyze and process images, it can efficiently compress images and video streams, and it can assist in visual pattern recognition, where a complex pattern may be efficiently and concisely described using a set of Fourier terms***.

Problem 17: **Harmonic Analysis with Example:**

Harmonics: The angular frequencies of the Sinusoids above are all integer multiples of ω. They are called the *harmonics* of ω, which in turn is called the *fundamental*. In terms of pitch, the ω , 2ω ,.............harmonics. These frequencies are referred to as ***harmonics*** of the fundamental frequency

Example 36: Find Harmonic Analysis for the given Fourier series

$$f(t) = 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$$

We have, from Example 22, (Page no 29)

$$f(t) = 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$$ [Answer of Example 22, Page no 29]

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} - \underbrace{\frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi}{4}t}_{\text{AC value}} \text{-----------------------------(i)}$$

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2.5}_{\text{DC value}} - \underbrace{\frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(\cos n\pi - 1)}_{\text{Amplitude}} \sin \underbrace{\frac{n\pi}{4}}_{\text{Frequency}} t \text{ -------------------(ii)}$$

$$\textit{Here, } n\omega = \frac{n\pi}{4}$$

For n = 1; **Fundamental Frequency = 1st Harmonic =** $\omega = \frac{n\pi}{4} == \frac{1.\pi}{4} = \frac{\pi}{4}$

For n = 2; 2^{nd} **Harmonic** $= 2\omega = \frac{n\pi}{4} == \frac{2.\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

For n = 3; 3^{rd} **Harmonic** $= 3\omega = \frac{n\pi}{4} == \frac{3.\pi}{4} = \frac{3\pi}{4}$

For n = 4; 4^{th} **Harmonic** $= 4\omega = \frac{n\pi}{4} == \frac{4.\pi}{4} = \frac{4\pi}{4} = \pi$

Problem 18: Compact Trigonometric Fourier Series:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \text{-----------------------------------(i)}$$

The trigonometric Fourier series equation no (i) contains sine and cosine terms of the same frequency. We can combine the two terms in a single term of the same frequency using trigonometric identity:

$$a_n \cos(n\omega t) + b_n \sin(n\omega t) = C_n \cos(n\omega_0 t + \theta_n) \text{--------------------------------(ii)}$$

Where, $C_n = \sqrt{a_n{}^2 + b_n{}^2}$ --(iii)

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Problem 19: Derive Complex form of Fourier series

We have, the trigonometric form of Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) \text{----------------(iv)}$$

We have,

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + ----------------$$

Put $x = ix,$

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + --------$$

$$[i^2 = -1;\ i^3 = i^2.i = -i;\ i^4 = i^2.i^2 = (-1).(-1) = +1;\ i^5 = i^4.i = i]$$

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \frac{-ix^7}{7!} + --------$$

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ---- + (\frac{ix^1}{1!} - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + --------)$$

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ---- + i(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + --------)$$

$$[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + -----; \quad \sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ---------]$$

$$\therefore e^{ix} = \cos x + i \sin x \text{--------------------------------------(v)}$$

Similarly,

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + ----------------$$

Put $x = -ix,$

$$e^{-ix} = 1 + \frac{-ix^1}{1!} + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \frac{(-ix)^5}{5!} + \frac{(-ix)^6}{6!} + \frac{(-ix)^7}{7!} + ----$$

$$[(-i)^2 = -1; (-i)^3 = (-i)^2.(-i) = i; (-i)^4 = (-i)^2.(-i)^2 = (-1).(-1) = +1;$$

$$(-i)^5 = (-i)^4.(-i) = (+1).(-i) = -i]$$

$$e^{-ix} = 1 + \frac{-ix^1}{1!} + \frac{-x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{-ix^5}{5!} + \frac{-x^6}{6!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{ix^1}{1!} - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ---------- - \frac{ix^1}{1!} + \frac{ix^3}{3!} - \frac{ix^5}{5!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ---------- - i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ----\right)$$

$$[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + -----; \quad \sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ---------]$$

$$\therefore e^{-ix} = \cos x - i\sin x \quad \text{---(vi)}$$

Adding (v) and (vi),

$$e^{ix} = \cos x + i\sin x$$

$$\underline{e^{-ix} = \cos x - i\sin x}$$

$$e^{ix} + e^{-ix} = 2\cos x$$

$$\therefore \cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \text{ ---(vii)}$$

Again Subtracting (v) and (vi)

$$e^{ix} = \cos x + i\sin x$$

$$\underline{e^{-ix} = \cos x - i\sin x}$$

$$e^{ix} - e^{-ix} = 2i\sin x$$

$$\therefore \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \text{ ---(viii)}$$

From equation (vii)

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$$

Hence From equation (vii), we can write,

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$$\therefore \cos(n\omega t) = \frac{1}{2}(e^{in\omega t} + e^{-in\omega t})$$ ------------------------------(ix)

And from equation (vii)

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

Hence from equation (vii), we can write

$$\therefore \sin(n\omega t) = \frac{1}{2i}(e^{in\omega t} - e^{-in\omega t})$$ ---------------------------(x)

Putting the values of (ix) and (x) in (iv), we get,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \ \{\frac{1}{2}(e^{in\omega t} + e^{-in\omega t})\} + \sum_{n=1}^{\infty} b_n \ \{\frac{1}{2i}(e^{in\omega t} - e^{-in\omega t})\}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \ \{\frac{1}{2}(e^{in\omega t} + e^{-in\omega t})\} + b_n \ \{\frac{1}{2i}(e^{in\omega t} - e^{-in\omega t})\}]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \{\frac{1}{2}(a_n e^{in\omega t} + a_n e^{-in\omega t})\} + \{\frac{1}{2i}(b_n e^{in\omega t} - b_n e^{-in\omega t})\}\ ]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \frac{1}{2}(a_n e^{in\omega t} + a_n e^{-in\omega t}) + \frac{1}{2i}(b_n e^{in\omega t} - b_n e^{-in\omega t})\ ]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \frac{1}{2}(a_n e^{in\omega t}) + \frac{1}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{1}{2i}(-b_n e^{-in\omega t})\ ]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(-1)}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(-1)}{2i}(-b_n e^{-in\omega t})\ ]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(i^2)}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(i^2)}{2i}(-b_n e^{-in\omega t})\ ]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(i\ \ )}{2}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(i\ \ )}{2}(-b_n e^{-in\omega t})\ ]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \frac{1}{2}(a_n e^{in\omega t}) + \frac{-i}{2}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{-i}{2}(-b_n e^{-in\omega t})\ ]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \{\frac{1}{2}(a_n e^{in\omega t}) - \frac{i}{2}(b_n e^{in\omega t})\} + \{\frac{1}{2}(a_n e^{-in\omega t}) + \frac{i}{2}(b_n e^{-in\omega t})\}]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\ \frac{1}{2}(a_n - ib_n)e^{in\omega t}) + \frac{1}{2}(a_n + ib_n)e^{-in\omega t})\ ]$$ ----------------------(xi)

Let, $c_0 = \frac{a_0}{2}$ --(a)

$$c_n = \frac{1}{2}(a_n - ib_n)$$ ------------------------------------(b)

$$\mathbf{c^*_n = \frac{1}{2}(a_n + ib_n)}$$ --------------------------------(c)

Then from (viii), we get the series is:

$$\mathbf{f(t) = c_0 + \sum_{n=1}^{\infty}[\, c_n e^{in\omega t} + c^*_n\, e^{-in\omega t}]}$$

$$\mathbf{f(t) = c_0 + \sum_{n=1}^{\infty}[\, c_n e^{in\omega t} + c_{-n} e^{-in\omega t}]\ [Say\ c^*_n = c_{-n}]}$$

$$\mathbf{f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega t} + \sum_{n=1}^{\infty} c_{-n} e^{-in\omega t}}$$ -------(d)

$$\mathbf{f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega t} + \sum_{n=-1}^{-\infty} c_n e^{in\omega t}}$$ ---------(e)

$$\mathbf{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$$ -----------(f)

[We have, $\mathbf{c_n e^{in\omega t}}$

Put n = 0, then

$$\mathbf{c_n e^{in\omega t}}$$
$$= c_0 e^{i\times 0\times \omega t}$$
$$= c_0 e^0 = c_0.1 = c_0]$$

$$\mathbf{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$$ --------------------------------------***(xii) Which is referred to as the complex or exponential form of the Fourier Series expansion of the function f(t)***

Where

$\because \mathbf{e^{ix} = \cos x + i \sin x}$ [from (v)]

$\therefore \mathbf{e^{in\omega t} = \cos n\omega t + i \sin n\omega t}$ -------------------------------(xiii)

$$\mathbf{c_0 = \frac{a_0}{2} = \frac{1}{2}a_0 = \frac{1}{2}\frac{1}{L}\int_{-L}^{L} f(t)dt}$$

$$\mathbf{c_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt}$$

$$\mathbf{c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt}$$ **[Where Period T= 2L]----------------(xiv)**

and

$$\mathbf{c_n = \frac{1}{2}(a_n - ib_n)}$$ --(xv)

$$\mathbf{c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)}$$ --------------------------------(xvi)

We have,

$$\mathbf{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt$$

$$\therefore c_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt - i\frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\int_{-L}^{L}\{f(t)\cos(n\omega t) - if(t)\sin(nwt)\}dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\{\cos(n\omega t) - i\sin(nwt)\}dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\omega t}dt \text{ [Since } e^{-ix} = \cos x - i\sin x \text{ from (vi)]}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt \text{ [T = 2L] ----------------------(xvii)}$$

and

$$c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt + i\frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\left(\int_{-L}^{L} f(t)\cos(n\omega t)dt + i.\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\left(\int_{-L}^{L} f(t)\{\cos(n\omega t)dt + i\sin(nwt)dt\right)\}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)e^{in\omega t}dt \text{ [}\because e^{ix} = \cos x + i\sin x \text{ from (2)]}$$

$$\therefore c^*_n = c_{-n} = \frac{1}{2L}\int_{-L}^{L} f(t)e^{in\omega t}dt$$

$$\therefore c^*_n = c_{-n} = \frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega t}dt \text{ [Period T = 2L]-------(xviii)}$$

In summary, the complex form of the Fourier series expansion of a periodic function f (t), of period T, is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

Where,

$$c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c^*_n = c_{-n} = \frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega t}dt$$

Example 37: Find the complex form of the Fourier Series expansion of the periodic function $f(t)$ and find the trigonometric form of the Fourier Series expansion of the periodic function f(t) is given by:

$$f(t) = \cos\frac{1}{2}t \qquad ; -\pi < t < \pi \ [T = 2\pi]$$

Solution:

The complex form of the Fourier series expansion of a periodic function $f(t)$, of period T, is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \text{--}(A)$$

Where,

$$c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c^*_n = c_{-n} = \frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega t}dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in\frac{2\pi}{T}t}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in\frac{2\pi}{2\pi}t}dt \ \ [T = 2\pi]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\frac{1}{2}t\, e^{-int}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\{\frac{1}{2}(e^{i(\frac{1}{2}t)}+e^{-i(\frac{1}{2}t)}\}e^{-int}dt \ [\therefore \cos x = \frac{1}{2}(e^{ix}+e^{-ix}) \ ]$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\{(e^{i(\frac{1}{2}t)}+e^{-i(\frac{1}{2}t)}\}e^{-int}dt$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\{(e^{i(\frac{1}{2}t)}.e^{-int}+e^{-i(\frac{1}{2}t)}.e^{-int}\}dt$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\{(e^{i(\frac{1}{2}t-nt)}+e^{-i(\frac{1}{2}t+nt)}\}dt$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\{(e^{-i(n-\frac{1}{2})t}+e^{-i(n+\frac{1}{2})t}\}dt$$

$$= \frac{1}{4\pi}\left[\frac{e^{-i(n-\frac{1}{2})t}}{-i(n-\frac{1}{2})}+\frac{e^{-i(n+\frac{1}{2})t}}{-i(n+\frac{1}{2})}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\frac{2e^{-i\frac{(2n-1)t}{2}}}{-i(2n-1)}-\frac{2e^{-i\frac{(2n+1)t}{2}}}{i(2n+1)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-2e^{-int}.e^{\frac{it}{2}}}{i(2n-1)}-\frac{2e^{-int}.e^{-\frac{it}{2}}}{i(2n+1)}\right]_{-\pi}^{\pi}$$

$$= \frac{-2}{4\pi}\left[\frac{e^{-int}.e^{\frac{it}{2}}}{i(2n-1)}+\frac{e^{-int}.e^{-\frac{it}{2}}}{i(2n+1)}\right]_{-\pi}^{\pi}$$

$$= \frac{-2}{4\pi}\left[\frac{e^{-in\pi}.e^{\frac{i\pi}{2}}}{i(2n-1)}+\frac{e^{-in\pi}.e^{-\frac{i\pi}{2}}}{i(2n+1)}-\frac{e^{-in(-\pi)}.e^{\frac{-i\pi}{2}}}{i(2n-1)}-\frac{e^{-in(-\pi)}.e^{-\frac{i(-\pi)}{2}}}{i(2n+1)}\right]$$

$$= \frac{-2}{4\pi}\left[\frac{e^{-in\pi}.e^{\frac{i\pi}{2}}}{i(2n-1)}+\frac{e^{-in\pi}.e^{-\frac{i\pi}{2}}}{i(2n+1)}-\frac{e^{in\pi}.e^{\frac{-i\pi}{2}}}{i(2n-1)}-\frac{e^{in\pi}.e^{\frac{i\pi}{2}}}{i(2n+1)}\right] \text{--------------------(B)}$$

We have,

$e^{ix} = \cos x + i\sin x$

$\therefore e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i.1 = i$ ---------------------(*)

And, $e^{-ix} = \cos x - i\sin x$

$\therefore e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = 0 - i.1 = -i$ -------------------(*)

$$e^{in\pi} = \cos n\pi + i\sin n\pi = (-1)^n + i.0 = (-1)^n$$ ------------(*)

Since,

if n =1, $\cos n\pi = \cos\pi = -1$

if n =2, $\cos n\pi = \cos 2\pi = 1$

if n =3, $\cos n\pi = \cos 3\pi = -1$

Hence we can write,

$\cos n\pi = (-1)^n$ [n = 1, 2, 3, ...]

Again,

$$e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n - i.0 = (-1)^n$$ ------------(*)

Putting these values in (B),

$$c_n = \frac{-2}{4\pi}\left[\frac{e^{-in\pi}.e^{\frac{i\pi}{2}}}{i(2n-1)} + \frac{e^{-in\pi}.e^{-\frac{i\pi}{2}}}{i(2n+1)} - \frac{e^{in\pi}.e^{\frac{-i\pi}{2}}}{i(2n-1)} - \frac{e^{in\pi}.e^{\frac{i\pi}{2}}}{i(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{e^{-in\pi}.i}{i(2n-1)} + \frac{e^{-in\pi}.(-i)}{i(2n+1)} - \frac{e^{in\pi}.(-i)}{i(2n-1)} - \frac{e^{in\pi}.(i)}{i(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{e^{-in\pi}.i}{i(2n-1)} - \frac{e^{-in\pi}.(i)}{i(2n+1)} + \frac{e^{in\pi}.(i)}{i(2n-1)} - \frac{e^{in\pi}.(i)}{i(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{(-1)^n.i}{i(2n-1)} - \frac{(-1)^n.(i)}{i(2n+1)} + \frac{(-1)^n.(i)}{i(2n-1)} - \frac{(-1)^n.(i)}{i(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{(-1)^n}{(2n-1)} - \frac{(-1)^n}{(2n+1)} + \frac{(-1)^n}{(2n-1)} - \frac{(-1)^n}{(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{1}{(2n-1)} - \frac{1}{(2n+1)} + \frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right](-1)^n$$

$$c_n = \frac{-2}{4\pi}\left[\frac{2}{(2n-1)} - \frac{2}{(2n+1)}\right](-1)^n$$

$$c_n = \frac{-4}{4\pi}\left[\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right](-1)^n$$

$$c_n = \frac{1}{\pi}\left[\frac{1}{(2n+1)} - \frac{1}{(2n-1)}\right](-1)^n$$

$$c_n = \frac{1}{\pi}\left[\frac{2n-1-(2n+1)}{(2n+1)(2n-1)}\right](-1)^n$$

$$c_n = \frac{1}{\pi}\left[\frac{-2}{(2n+1)(2n-1)}\right](-1)^n$$

$$c_n = \frac{-2}{\pi}\left[\frac{1}{(4n^2-1))}\right](-1)^n$$

$$c_n = \frac{2}{\pi}\left[\frac{1}{(4n^2-1))}\right](-1)^n(-1)$$

$$c_n = \frac{2}{\pi}\left[\frac{1}{(4n^2-1))}\right](-1)^{n+1}$$

$\therefore$ From (A), the complex form of the Fourier series expansion of a periodic function f (t), of period T, is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

$$= \sum_{n=-\infty}^{\infty}\left\{\frac{2}{\pi}\left[\frac{1}{(4n^2-1))}\right](-1)^{n+1}\right\} e^{in\omega t}$$

$$= \sum_{n=-\infty}^{\infty}\left\{\frac{2}{\pi}\left[\frac{1}{(4n^2-1))}\right](-1)^{n+1}\right\} e^{int}\left[\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1\right]$$ Answer

Problem 20: Conversion of $f(t)$ from complex form to the trigonometric form

We have, the trigonometric form of the Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$[\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1]$$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$ ---------------------------------(B)

We have,

Let, $c_0 = \frac{a_0}{2}$

$\therefore a_0 = 2c_0$

We got,

$$c_n = \frac{2}{\pi}\left[\frac{1}{(4n^2-1))}\right](-1)^{n+1}$$

$$\therefore c_0 = \frac{2}{\pi}\left[\frac{1}{(4\times 0^2-1))}\right](-1)^{0+1} \text{ [putting } n = 0]$$

$$= \frac{2}{\pi}\left[\frac{1}{-1}\right](-1)^1]$$

$$= \frac{2}{\pi}$$

$$\therefore a_0 = 2c_0 = 2\times\frac{2}{\pi} = \frac{4}{\pi}$$

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$\therefore c_n + c_{-n} = \frac{1}{2}a_n + \frac{1}{2}a_n = a_n$$

$$\therefore a_n = c_n + c_{-n}$$

We have,

$$c_n = \frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1}$$

$$\therefore c_{-n} = \frac{2}{\pi}[\frac{1}{(4(-n)^2-1))}](-1)^{-n+1} \text{ [putting } n=-n]$$

$$\therefore a_n = c_n + c_{-n}$$

$$= \frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1} + \frac{2}{\pi}[\frac{1}{(4(-n)^2-1))}](-1)^{-n+1}$$

$$= \frac{2}{\pi}\frac{1}{(4n^2-1))}[(-1)^{n+1} + (-1)^{-n+1}]$$

$$= \frac{2}{\pi}\frac{1}{(4n^2-1))}[(-1)^{n+1} + (-1)^{-n+1}]$$

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$\therefore c_n - c_{-n} = -i\frac{1}{2}b_n - i\frac{1}{2}b_n = -i(\frac{1}{2}b_n + \frac{1}{2}b_n) = -ib_n$$

$$\therefore -ib_n = c_n - c_{-n}$$

$$\therefore b_n = -\frac{1}{i}(c_n - c_{-n}) = \frac{-1}{i}(c_n - c_{-n}) = \frac{i^2}{i}(c_n - c_{-n}) = i(c_n - c_{-n})$$

$$\therefore b_n = i(c_n - c_{-n})$$

$$= i\{\frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1} - \frac{2}{\pi}[\frac{1}{(4(-n)^2-1))}](-1)^{-n+1}\}$$

$$= \frac{2}{\pi}\frac{1}{(4n^2-1))}[(-1)^{n+1} - (-1)^{-n+1}]$$

$$= \frac{2}{\pi}\frac{1}{(4n^2-1))}[(-1)^{n+1} - (-1)^{-n+1}]$$

if n =1, $b_1 = 0$
if n =2, $b_2 = 0$
if n =3 , $b_3 = 0$

Since the given function is even function so the coefficient $\mathbf{b_n}$ will be zero
The trigonometric form of the Fourier series is:

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \text{ ---------------------------------(B)}$$

$$= \frac{4}{2\times\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{1}{(4n^2-1))} [(-1)^{n+1} + (-1)^{-n+1}] \cos nt + 0$$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{1}{(4n^2-1))} [(-1)^{n+1} + (-1)^{-n+1}] \cos nt \text{ Answer}$$

Example 38: Obtain the complex form of the Fourier series of the saw tooth function $\mathbf{f(t)}$ defined by $\mathbf{f(t) = \frac{2t}{T}(0 < t < 2T), \quad f(t+2T) = f(t)}$ [Period = 2T]

Answer:

We have, $\mathbf{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$ the complex or exponential form of the Fourier series.

Where, $$c_n = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-in\omega t} dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \frac{2t}{T} e^{-in\omega t} dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \frac{2t}{T} e^{-in\frac{2\pi}{T}t} dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \frac{2t}{T} e^{-in\frac{2\pi}{2T}t} dt$$

$$= \frac{1}{2T} \int_{0}^{2T} \frac{2t}{T} e^{-in\frac{\pi}{T}t} dt \quad [\because \int_{-a}^{a} f(x)dx = \int_{0}^{2a} f(x)dx] \text{ (Proved)}$$

$$= \frac{1}{2T^2} \int_{0}^{2T} 2t e^{-in\frac{\pi}{T}t} dt$$

$$= \frac{1}{T^2} \int_{0}^{2T} t e^{-in\frac{\pi}{T}t} dt$$

Now, $$\int t e^{-in\frac{\pi}{T}t} dt = t \int e^{-in\frac{\pi}{T}t} dt - \int \{\frac{d}{dt}(t) \int e^{-in\frac{\pi}{T}t} dt\}$$

$$= t \frac{e^{-in\frac{\pi}{T}t}}{-in\frac{\pi}{T}} - \int 1. \frac{e^{-in\frac{\pi}{T}t}}{-in\frac{\pi}{T}} dt$$

$$= t\frac{e^{-in\frac{\pi}{T}t}}{-in\frac{\pi}{T}} + \frac{T}{in\pi}\int e^{-in\frac{\pi}{T}t}dt$$

$$= Tt\,\frac{e^{-in\frac{\pi}{T}t}}{-in\pi} + \frac{T}{in\pi}.\frac{e^{-in\frac{\pi}{T}t}}{-in\frac{\pi}{T}}$$

$$= Tt\,\frac{e^{-in\frac{\pi}{T}t}}{-in\pi} - \frac{T^2}{(in\pi)^2}.\frac{e^{-in\frac{\pi}{T}t}}{1}$$

$$c_n = \frac{1}{T^2}\int_0^{2T} te^{-in\frac{\pi}{T}t}dt = \frac{1}{T^2}[Tt\,\frac{e^{-in\frac{\pi}{T}t}}{-in\pi} - \frac{T^2}{(in\pi)^2}.\frac{e^{-in\frac{\pi}{T}t}}{1}]_0^{2T}$$

$$= \frac{1}{T^2}[T.2T\frac{e^{-in\frac{\pi}{T}2T}}{-in\pi} - \frac{T^2}{(in\pi)^2}.\frac{e^{-in\frac{\pi}{T}2T}}{1} - 0 + \frac{T^2}{(in\pi)^2}.e^{-in\frac{\pi}{T}\times 0}]$$

$$= \frac{1}{T^2}[T.2T\frac{e^{-in2\pi}}{-in\pi} - \frac{T^2}{(in\pi)^2}.\frac{e^{-in2\pi}}{1} + \frac{T^2}{(in\pi)^2}\;]\;[e^0 = 1]$$

$$= \frac{1}{T^2}[2T^2\,\frac{e^{-in2\pi}}{-in\pi} + \frac{T^2}{(n\pi)^2}.\frac{e^{-in2\pi}}{1} - \frac{T^2}{(n\pi)^2}\;]\;[i^2 = -1]$$

We have,

$e^{-ix} = \cos x - i\sin x$

$e^{-i2n\pi} = \cos 2n\pi - i\sin 2n\pi$

$e^{-i2\pi} = \cos 2\pi - i\sin 2\pi = 1 + 0[\text{when } n = 1]$

$e^{-i4\pi} = \cos 4\pi - i\sin 4\pi = 1 + 0[\text{when } n = 2]$

$$c_n = \frac{1}{T^2}[2T^2\,\frac{e^{-in2\pi}}{-in\pi} + \frac{T^2}{(n\pi)^2}.\frac{e^{-in2\pi}}{1} - \frac{T^2}{(n\pi)^2}\;]\;[i^2 = -1]$$

$$c_n = \frac{1}{T^2}[2T^2\,\frac{1}{-in\pi} + \frac{T^2}{(n\pi)^2}.\frac{1}{1} - \frac{T^2}{(n\pi)^2}\;]$$

$$c_n = \frac{1}{T^2}[\frac{2T^2}{-in\pi} + \frac{T^2}{(n\pi)^2} - \frac{T^2}{(n\pi)^2}]$$

$$c_n = \frac{1}{T^2}[\frac{2T^2}{-in\pi}] = \frac{2}{-in\pi} = \frac{-2}{in\pi} = \frac{i^2 2}{in\pi} = \frac{i.2}{n\pi}\text{------------------------------(ii)}$$

Again,

$$c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt = \frac{1}{2T}\int_{-T}^{T}\frac{2t}{T}dt = \frac{1}{2T}\int_{0}^{2T}\frac{2t}{T}dt\,\frac{2}{2T^2}[\frac{t^2}{2}]_0^{2T} = \frac{1}{T^2}[\frac{4T^2}{2}] = 2$$

Hence,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} = c_0.e^0 + \sum_{n=-\infty}^{-1} c_n e^{in\omega t} + \sum_{n=1}^{\infty} c_n e^{in\omega t}$$

$$= c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{in\omega t}$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i2}{n\pi} e^{in\frac{2\pi}{T}t}$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i.2}{n\pi} e^{in\frac{2\pi}{2T}t}$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{n\pi}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})e^{in\frac{\pi}{T}t} \quad [\because \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i]$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{n\pi}(e^{i\frac{\pi}{2}})e^{in\frac{\pi}{T}t} \quad [\because e^{ix} = \cos x + i\sin x]$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{n\pi}(e^{i\frac{\pi}{2}+i\frac{n\pi t}{T}})$$

$$= 2 + \frac{2}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}(e^{i(\frac{\pi}{2}+\frac{n\pi t}{T}}) \text{--------(iii)-complex form of Fourier series}$$

Representation of Aperiodic Signals by Fourier Integral

Problem 21: Fourier transform

In the last chapter we discussed the spectral representation of periodic signals (Fourier Series). In this chapter we extend this spectral representation of aperiodic signals

Fourier Transform: ***Representation of Aperiodic Signals.*** The extension of a Fourier series for a non-periodic function is known as the Fourier transforms.
The Fourier series representation of periodic signals consists of harmonically related spectral lines spaced at the integer multiples of the fundamental frequency. The Fourier representation of aperiodic signals can be developed by regarding an aperiodic signal as a special case of a periodic signal with an infinite period. If the period of a signal is infinite, then the signal does not repeat itself and is aperiodic

When calculating the Fourier transform, rather than decomposing a signal in terms of sines and cosines, people often use complex exponentials. They can be a little easier to interpret, although they are mathematically equivalent. A complex exponential is defined as $\mathbf{Ae^{i\phi}}$ where $\mathbf{i^2 = -1}$ ($\mathbf{i}$ is the "imaginary" number), $\mathbf{A}$ is the amplitude, and $\boldsymbol{\phi}$ is the phase. A waveform can be decomposed in terms of complex exponentials rather than sines and cosines because of Euler's Theorem, which highlights the surprisingly close relationship between a complex exponential and sines/cosines.

Euler's Theorem

$$\mathbf{e^{i\phi} = \cos\phi + i\sin\phi}$$

The Fourier transform allows you to write any function $\mathbf{f(t)}$ as the integral (sum) across frequencies of complex exponentials of different amplitudes and phases $\mathbf{F(\omega)}$. $\mathbf{f(t)}$ is often called the "time domain" representation while $\mathbf{F(\omega)}$ is called the "frequency domain representation." The key thing to understand about Fourier transforms is that these two representations are different ways of expressing the same information

- An aperiodic/non-periodic signal, like an audio signal. Consider the train whistle
- Fourier transform compute the frequency spectrum
- The Fourier transform of a function produces a spectrum from which the original function can be reconstructed (aka *synthesized*) by an inverse transform. So it is reversible. In order to do that, it preserves not only the magnitude of each frequency component, but also its phase.
- The Fourier transform is an equation to calculate the frequency, amplitude and phase of each sine wave needed to make up any given signal.
- The **Fourier Transform** is a mathematical technique for doing a similar thing - resolving any time-domain function into a frequency spectrum.
- The Fourier Transform (FT) is a mathematical formula using integrals.
- The Discrete Fourier Transform (DFT) is a discrete numerical equivalent using sums instead of integrals.

- The Fast Fourier Transform (FFT) is just a computationally fast way to calculate the DFT

‡h‡nZz Fourier Transform n‡”Q Aperiodic/Non-Peridic Signal Gi Rb¨ †m‡nZz aperiodic/non periodic signal †K periodic signal Kivi Rb¨ a‡i wbe ∞ mgq c‡i G iKg GKwU SIGNAL REPEAT cvIqv hv‡e , Zvi gv‡b GLv‡b Period $\mathbf{T} \rightarrow \infty$

Problem 22: Angular Frequency

Angular frequency ω also referred to by the terms angular speed, radial frequency, and radian frequency. If we think of the wave as a rotating wheel, then this means that the wheel makes a full revolution the same number of times per second.

We also know that one full revolution of the wheel is 360 or 2π radians.

Consequently, if we multiply the frequency of the sound wave by 2π, we get the number of radians the wheel turns each second. This value is called the *angular frequency* or the *radian frequency*

$\boldsymbol{\omega = 2\pi f}$

If f = 1, that means one cycle occurs

If f =2, that means two cycle occurs

Angular Speed or Angular Frequency =ω; $\omega = \frac{2\pi}{t}$

$\theta = 2\pi , \therefore \omega = \frac{\theta}{t} \Rightarrow \theta = \omega t$

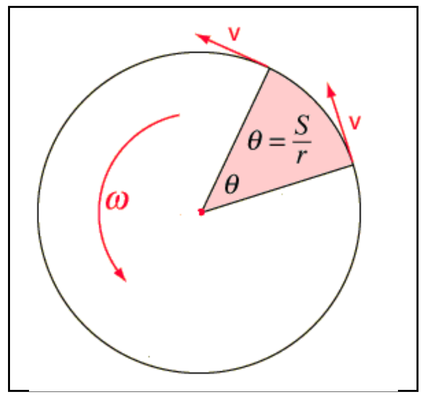


Figure 80

Problem 23:

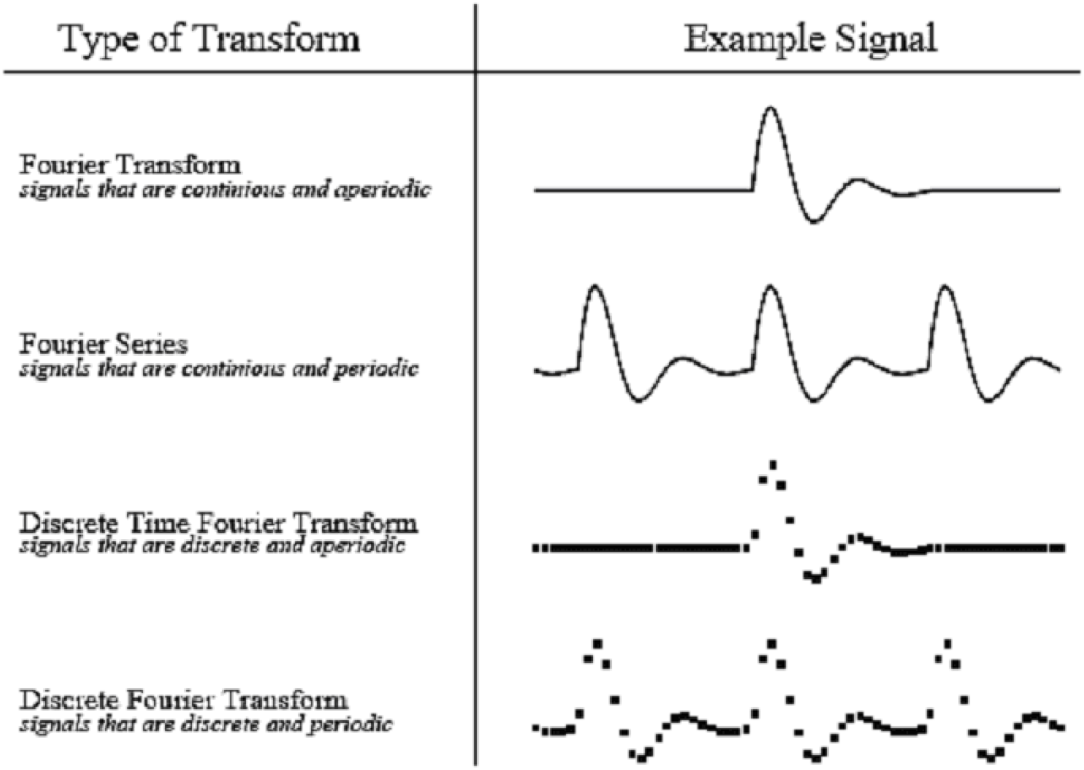


Figure 81

Revision of Previous lecture

Plot the line spectrum (discrete frequency spectra) for the Fourier series:

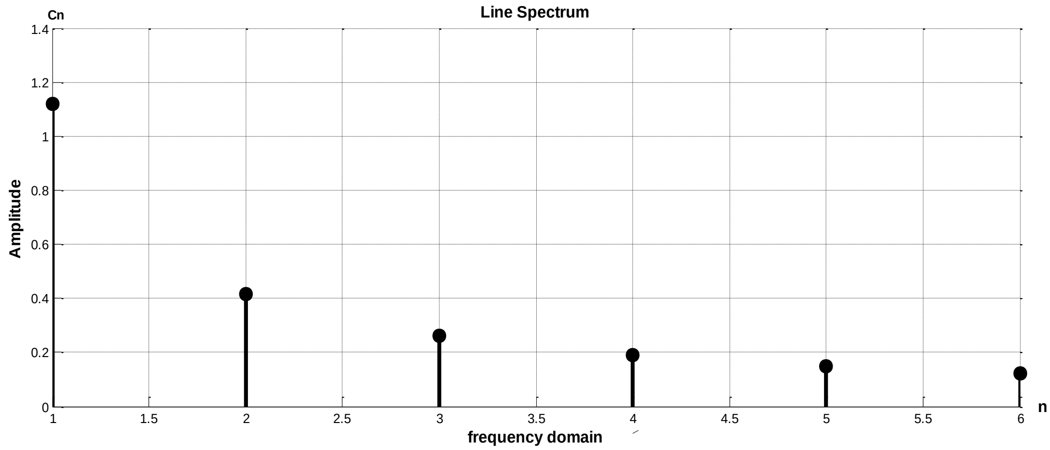
$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{2n-1}\cos n\omega t + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n}\sin n\omega t$$

$$a_n = \frac{1}{2n-1} \qquad b_n = \frac{(-1)^n}{2n}$$

Now, using $R = C_n = \sqrt{a_n{}^2 + b_n{}^2}$ for each term, we have:

$a_n = \frac{1}{2n-1}$	$b_n = \frac{(-1)^n}{2n}$	$C_n = \sqrt{a_n^2 + b_n^2}$
$a_1 = 1$	$b_1 = -\frac{1}{2}$	$C_1 = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = 1.118$
$a_2 = \frac{1}{3}$	$b_2 = \frac{1}{4}$	$C_2 = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2} = 0.4167$
$a_3 = \frac{1}{5}$	$b_3 = -\frac{1}{6}$	$C_3 = \sqrt{\left(\frac{1}{5}\right)^2 + \left(-\frac{1}{6}\right)^2} = 0.260$
$a_4 = \frac{1}{7}$	$b_4 = \frac{1}{8}$	$C_4 = \sqrt{\left(\frac{1}{7}\right)^2 + \left(\frac{1}{8}\right)^2} = 0.190$

The resulting line spectrum is:
Amplitude of different harmonics/frequencies



ω 2ω 3ω

Figure 82

Hence Fundamental Frequency/ Harmonic/First Harmonic $\omega_1 = \omega$ $[\omega = 2\pi f]$
Hence Second Harmonic $\omega_2 = 2\omega$ $[2\omega = 2\pi * 2f]$]
Hence Third Harmonic $\omega_3 = 3\omega$ $[3\omega = 2\pi * 3f]$
Hence Fourth Harmonic $\omega_4 = 4\omega$ $[4\omega = 2\pi * 4f]$

Problem 24: **Derive the equation of Fourier Transform from Fourier series**
Answer: we have the exponential form of Fourier series is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$ -----------------------------------(i) [see equation no ix, **page no 75**]

Where,

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$ -----------------------------(ii) [see equation no xiv, **page no 76**]

As period $T \to \infty$, ω tends to zero. [See explanation part why $T \to \infty$]
Since, We have,

$$\omega = \frac{2\pi}{T}$$

If $T \to \infty$, then $\omega = \frac{2\pi}{\infty} = 0$

Explanation:
An aperiodic signal may be looked at as a periodic signal with an infinite period.

That means an aperiodic signal (pulses) repeats after an infinite interval (say). That is $\mathbf{T \to \infty}$; then this aperiodic signal will be converted into periodic signal.

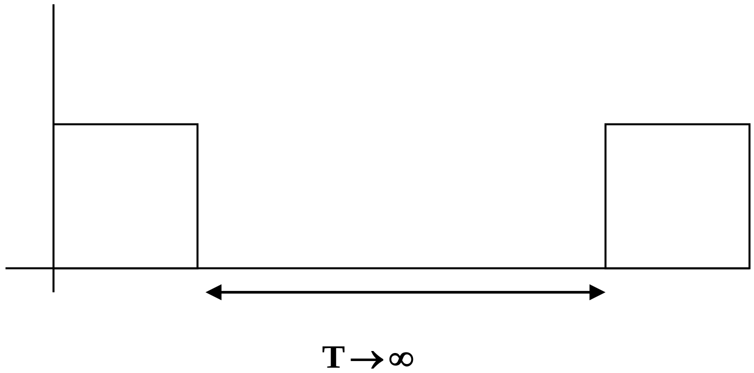


Figure 83

When $T \to \infty$ Then $\omega \to 0$,

We have,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\frac{1}{f}} = 2\pi f$$

$$\omega = \omega 1 = 2\pi f$$
$$2\omega = \omega 2 = 2\pi * 2f$$
$$3\omega = \omega 3 = 2\pi * 3f$$
$$4\omega = \omega 4 = 2\pi * 4f$$
$$5\omega = \omega 5 = 2\pi * 5f$$
$$6\omega = \omega 6 = 2\pi * 6f$$

When $T \to \infty$ Then $\omega \to 0$ means,

Say

$T \to \mathbf{1000000000000000000000000000000000000000000000000000010000}$

We have,

$$\omega = \frac{2\pi}{T}$$

$$\omega = \frac{2\pi}{1000000000000000000000000000000000000000000000000000010000}$$

1st harmonic say

$$\omega = \omega 1 \to 0.00000000000000000000000000000000000000000000000110000000000$$

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$$2\omega = \omega 2 \to 0.00000000000000000000000000000000000000000000002200000000000$$
$$3\omega = \omega 3 \to 0.00000000000000000000000000000000000000000000003300000000000$$
$$4\omega = \omega 4 \to 0.00000000000000000000000000000000000000000000004400000000000$$

So cÖ‡Z¨KwU harmonic Gi K¤úvsK LyeB Kg, Ges
$\omega 2-\omega 1,\ \omega 3-\omega 2,\ \omega 4-\omega 3,$ ....GKwU harmonic †_‡K Avi GKwU harmonic Gi g‡a¨ e¨eavb LyeB Kg, GK\`g jvMvjvwM| d‡j cÖ‡Z¨KwU harmonic Gi against G †h amplitude AvuKv n‡e Zv continuous spectrum n‡e|

Fourier Transform G $T \to \infty$, then $\omega = \frac{2\pi}{\infty} \to 0$ d‡j $\omega, 2\omega, 3\omega$ G‡\`i g‡a¨ cv\_©K¨ tends to zero d‡j Fourier Transform G line spectrum draw Ki‡j line spectrum ¸wj GK\`g KvQvKvwQ GKUvi mv‡_ Avi GKUv †j‡M _v‡K d‡j line spectrum ¸wj Avi Avjv\`v Kiv hvqbv †hgb GLv‡b Avjv\`v Kiv hvq A_v©r line spectrum ¸wj continuous nq|

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and n becomes meaningless as T approaches infinity. and the lines will become so dense that that the discrete frequency spectrum eventually (finally/ ultimately) approaches a smooth curve, resulting in a continuous spectrum
So, for a continuous spectrum, $n\omega \to \omega$ (n becomes meaningless as T approaches infinity) and Spacing between adjacent components $\omega \to \Delta\omega$

Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\Delta\omega}$

From (1),

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

$$f(t) = \sum_{\omega=-\infty}^{\infty} c_\omega e^{i\omega t} \text{ --------------------------------------(iii)} \qquad [n\omega \to \omega]$$

Where,

$$c_n = \frac{1}{T} \int_{-L}^{L} f(t) e^{-in\omega t} dt \text{ [From (ii)]}$$

$$\therefore c_\omega = \frac{1}{\frac{2\pi}{\Delta\omega}} \int_{-L}^{L} f(t) e^{-i\omega t} dt$$

$$\Rightarrow c_\omega = \frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(t) e^{-i\omega t} dt \text{ ----------------------------(iv)}$$

Hence from (iii), we get,

$$f(t) = \sum_{\omega=-\infty}^{\infty} c_\omega e^{i\omega t}$$

$$= \sum_{\omega=-\infty}^{\infty} c_\omega e^{i\omega t}$$

$$= \sum_{\omega=-\infty}^{\infty} [\frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(t) e^{-i\omega t} dt] e^{i\omega t}$$ [Putting the value of c_ω]

$$= \frac{1}{2\pi} [\sum_{\omega=-\infty}^{\infty} \int_{-L}^{L} f(t) e^{-i\omega t} dt] e^{i\omega t} \Delta\omega \text{----------(v)}$$

As $T \to \infty$, $\Delta\omega \to d\omega$ and $\sum \to \int$ [That is the discrete summation sign Σ should be replaced by the continuous summation integral $\int$]

Then equation (v) becomes,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt] e^{i\omega t} d\omega \text{----------------------(vi)}$$

Here, we have, $T = 2L$

$$\therefore L = \frac{T}{2}, \text{ If } T \to \infty, \text{then} L = \frac{\infty}{2} = \infty$$

Equation (vi) is form of the Fourier Integral.
The Fourier Integral (6) can be broken into a pair of relations:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [g(\omega)] e^{i\omega t} d\omega \qquad \text{-------------------------(vii)}$$

Where $g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ -------------------------(viii) ***is called the Fourier transform of f (t)***

The graph of $(\omega, |g(\omega)|)$ is called the amplitude spectrum of $f(t)$, ω is called the frequency of the spectrum

The equation (vii) and (viii) are called a Fourier transform pair; $g(\omega)$ ***is called the Fourier transform of f (t) and conversely, f (t) is called the inverse Fourier transform of*** $g(\omega)$.

The Fourier transform is an equation to calculate the frequency, amplitude and phase of each sine wave needed to make up any given signal f (t)
That is:

$f(t) = F^{-1}[g(\omega)]$ ---(ix)
$g(\omega) = F[f(t)]$ --(x) ***Answer***

The plot of $|g(\omega)|$ versus ω shows the relative frequency distribution of f (t).

Summary:

01. The Fourier transform transforms a function of t (in the time domain) into a function of ω (in the frequency domain), and the inverse Fourier transformation does the reverse, $g(\omega)$ is also called the spectrum function of f (t)

02. We call $\mathbf{g(\omega)}$ the direct Fourier Transform of f (t) and f (t) the inverse Fourier Transform of $\mathbf{g(\omega)}$. The Transform $\mathbf{g(\omega)}$ is the frequency –domain specification of f (t)

03. Fourier Transform: $$\mathbf{g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt}$$

04. Inverse Fourier Transform: $$\mathbf{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\ g(\omega)\ ] e^{i\omega t}\ d\omega}$$

Remember:

01. The Fourier Spectrum of a signal indicates the relative amplitudes and phases of the sinusoids that are required to synthesize that signal

02. ***A periodic signal Fourier Spectrum has finite amplitudes and exists at discrete frequencies (*** $\mathbf{\omega 1, \omega 2, \omega 3, \ldots\ldots\ldots\ldots}$ ***). Such a spectrum is easy to visualize, but the spectrum of a non periodic signal is not easy to visualize because it has a continuous spectrum that exists at every frequency (*** $\mathbf{\omega 1, \omega 2, \omega 3, \ldots\ldots\ldots\ldots}$ ***)***

03. ***Spectral representation***—the frequency representation of periodic and aperiodic signals indicates how their power or energy is allocated to different frequencies. Such a distribution over frequency is called the *spectrum of the signal*. For a periodic signal the spectrum is discrete, as its power is concentrated at frequencies multiples of a so-called *fundamental frequency*, directly related to the period of the signal. On the other hand, the spectrum of an aperiodic signal is a continuous function of frequency. The concept of spectrum is similar to the one used in optics for light, or in material science for metals, each indicating the distribution of power or energy over frequency.

04. ***Spectrum Analysis*** **or** ***Fourier analysis*** is the process of analyzing some time-domain waveform to find its spectrum. We also say that the time domain waveform is converted into a frequency spectrum by means of the *Fourier transform*. This process is reversible: using the *inverse Fourier transform* a spectrum may be converted back into a time-domain waveform**.**

Example 39: Consider the rectangular pulse figure 84 described as

$$\mathbf{f(t) = 1; \qquad -T \le t \le T}$$

$$\mathbf{= 0; \qquad |t| > T}$$

Find the Fourier Transform of $\mathbf{f(t)}$

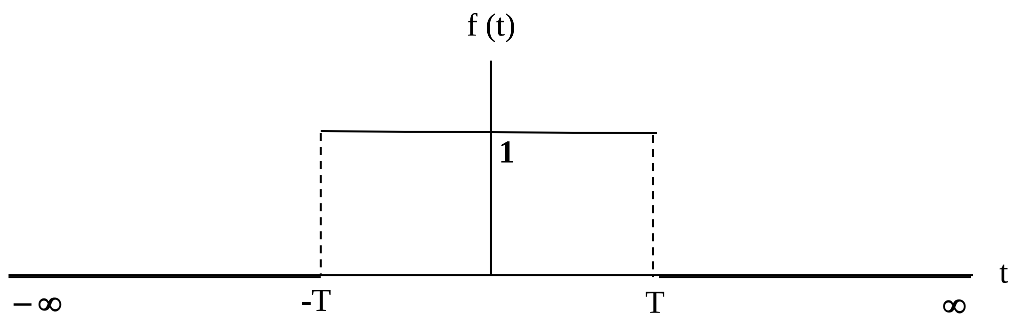

Figure 84: Rectangular Pulse

Answer:
We have,

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\, g(\omega)\,] e^{i\omega t}\, d\omega$$

Where $g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$$g(\omega) = \int_{-\infty}^{-T} f(t) e^{-i\omega t} dt + \int_{-T}^{T} f(t) e^{-i\omega t} dt + \int_{T}^{\infty} f(t) e^{-i\omega t} dt$$

$$g(\omega) = \int_{-\infty}^{-T} 0.e^{-i\omega t} dt + \int_{-T}^{T} 1.e^{-i\omega t} dt + \int_{T}^{\infty} 0.e^{-i\omega t} dt$$

$$g(\omega) = \int_{-T}^{T} 1.e^{-i\omega t} dt = [\frac{e^{-i\omega t}}{-i\omega}]_{-T}^{T} \qquad [\because \int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-1}{i\omega}(e^{-i\omega T} - e^{i\omega T}) = \frac{1}{i\omega}(e^{i\omega T} - e^{-i\omega T})$$

$$g(\omega) = \frac{2.1}{\omega}\{\frac{1}{2i}(e^{i\omega T} - e^{-i\omega T})\} = \frac{2}{\omega}\sin(\omega T) \text{------------------------(i)}$$

[$\because \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$**, page no, equation no]**

OR

[Since we have, $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$]

$$g(\omega) = \frac{2.1}{\omega}\{\frac{1}{2i}(e^{i\omega T} - e^{-i\omega T})\}$$

$$= \frac{2}{\omega} \sin(\omega T)$$

$$= \frac{2T}{\omega T} \sin(\omega T)$$

$$g(\omega) = 2T \frac{\sin(\omega T)}{\omega T}$$

Thus we write for all ω , $g(\omega) = 2T \frac{\sin(\omega T)}{\omega T}$ -----------------------------------(ii)

Both (i) & (ii) are correct

For $\omega = 0,$ the integral simplifies to 2T. It is straightforward to show using L' Hopital's rule that, from (i)

$$\lim_{\omega \to 0} \frac{2}{\omega} \sin(\omega T)$$

$$= \lim_{\omega \to 0} \frac{2\sin(\omega T)}{\omega} \left[\frac{0}{0}; \text{Indeterminate Form}\right]$$

$$= \lim_{\omega \to 0} \frac{2\cos(\omega T).T}{1} \text{ [Differentiate numerator and denominator with respect to } \omega \text{]}$$

$$= \frac{2\cos 0.T}{1}$$

$$= 2.1.T = 2T$$

$$\therefore \lim_{\omega \to 0} \frac{2}{\omega} \sin(\omega T) = 2T$$

Again, From (ii),

$$g(\omega) = 2T \frac{\sin(\omega T)}{\omega T}$$

$$\lim_{\omega \to 0} g(\omega) = \lim_{\omega \to 0} 2T \frac{\sin(\omega T)}{\omega T}$$

$$\lim_{\omega \to 0} g(\omega) = 2T \lim_{\omega \to 0} \frac{\sin(\omega T)}{\omega T}$$

$$\lim_{\omega \to 0} g(\omega) = 2T.1 \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

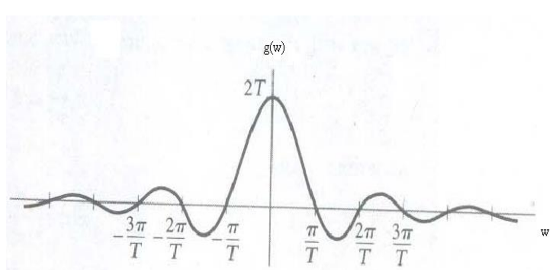

Figure 85: Fourier Transform of f (t)

Example 40: Find the Fourier Transform of $f(t) = e^{-at}u(t)$

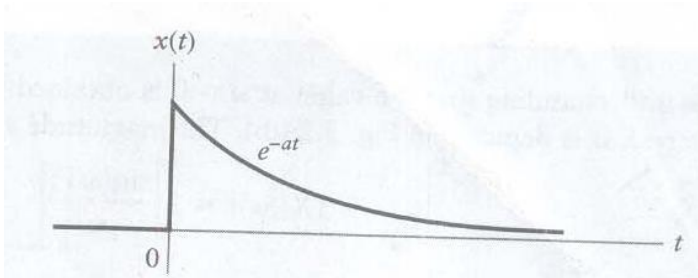

Figure 86: Exponential Signal

Answer: The Fourier Transform does not converge for $\mathbf{a \le 0}$, since f (t) is not absolutely integrable, as shown by:

i. **For a>0, we have,**

$$g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{0} e^{-at}u(t)e^{-i\omega t}dt + \int_{0}^{\infty} e^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{0} e^{-at}.0.e^{-i\omega t}dt + \int_{0}^{\infty} e^{-at}.1.e^{-i\omega t}dt \qquad \text{[By definition from unit step function]}$$

$$g(\omega) = 0 + \int_{0}^{\infty} e^{-at}.1.e^{-i\omega t}dt$$

$$g(\omega) = \int_{0}^{\infty} e^{-at}e^{-i\omega t}dt$$

$$g(\omega) = \int_{0}^{\infty} e^{-at}e^{-i\omega t}dt = \int_{0}^{\infty} e^{-(a+i\omega)t}dt \text{ ----------------------------------(i)}$$

$$= \frac{-1}{a+i\omega}\left[e^{-(a+i\omega)t}\right]_0^{\infty}$$

$$g(\omega) = \frac{-1}{a+i\omega}[e^{-(a+i\omega)\infty} - e^{-(a+i\omega)0}]$$

$$g(\omega) = \frac{-1}{a+i\omega}[e^{-\infty} - e^{-0}]$$

$$= \frac{-1}{a+i\omega}[\frac{1}{e^{\infty}} - \frac{1}{e^{0}}] = \frac{-1}{a+i\omega}\left[\frac{1}{\infty} - \frac{1}{1}\right] = \frac{-1}{a+i\omega}[0-1] = \frac{1}{a+i\omega} \quad [\because e^{\infty} = \infty]$$

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$$|g(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

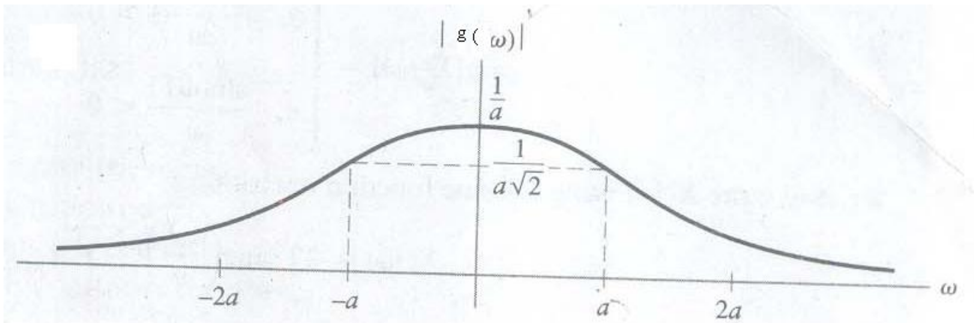

Figure 87: Fourier Transform of f (t) or Magnitude Spectrum

ii. for a < 0

$$g(\omega) = \int_0^{\infty} e^{-(a+i\omega)t}dt$$

$$g(\omega) = \int_0^{\infty} e^{-at-i\omega t}dt$$

$$g(\omega) = \int_0^{\infty} e^{at-i\omega t}dt \qquad [\text{for } a<0]$$

$$g(\omega) = \int_0^{\infty} e^{(a-i\omega)t}dt$$

$$g(\omega) = \frac{1}{a-i\omega}\left[e^{(a-i\omega)t}\right]_0^{\infty}$$

$$g(\omega) = \frac{1}{a-i\omega}\left[e^{(a-i\omega)\infty} - e^{(a-i\omega)0}\right]_0^{\infty}$$

$$g(\omega) = \frac{1}{a-i\omega}\left[e^{\infty} - e^{0}\right]$$

$$g(\omega) = \frac{1}{a-i\omega}\left[\infty - 1\right] = \infty$$

Example 41: Find Fourier Transform of

$$\begin{aligned} f(t) &= 1 \qquad ;0 \le t < 1 \\ &= -1 \qquad ;-1 \le t < 0 \\ &= 0 \qquad ;|t| > 1 \end{aligned}$$

We have $g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

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$$g(\omega)=\int_{-\infty}^{-1} f(t)e^{-i\omega t}dt+\int_{-1}^{0} f(t)e^{-i\omega t}dt+\int_{0}^{1} f(t)e^{-i\omega t}dt+\int_{1}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{-1} 0.e^{-i\omega t}dt+\int_{-1}^{0}(-1)e^{-i\omega t}dt+\int_{0}^{1}1.e^{-i\omega t}dt+\int_{1}^{\infty}0.e^{-i\omega t}dt$$

$$g(\omega)=-\int_{-1}^{0} e^{-i\omega t}dt+\int_{0}^{1} e^{-i\omega t}dt$$

$$g(\omega)=-\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{0}+\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{0}^{1} \qquad [\because \int e^{-mx}dx=\frac{e^{-mx}}{-m}]$$

$$g(\omega)=-\left[\frac{e^{-i\omega.0}}{-i\omega}-\frac{e^{-i\omega(-1)}}{-i\omega}\right]+\left[\frac{e^{-i\omega.1}}{-i\omega}-\frac{e^{-i\omega.0}}{-i\omega}\right]$$

$$g(\omega)=-\left[\frac{e^{-0}}{-i\omega}-\frac{e^{i\omega}}{-i\omega}\right]+\left[\frac{e^{-i\omega}}{-i\omega}-\frac{e^{-0}}{-i\omega}\right]$$

$$g(\omega)=-\left[\frac{\frac{1}{e^{0}}}{-i\omega}-\frac{e^{i\omega}}{-i\omega}\right]+\left[\frac{e^{-i\omega}}{-i\omega}-\frac{\frac{1}{e^{0}}}{-i\omega}\right]$$

$$g(\omega)=-\left[\frac{\frac{1}{1}}{-i\omega}-\frac{e^{i\omega}}{-i\omega}\right]+\left[\frac{e^{-i\omega}}{-i\omega}-\frac{\frac{1}{1}}{-i\omega}\right]$$

$$g(\omega)=-\left[\frac{1}{-i\omega}-\frac{e^{i\omega}}{-i\omega}\right]+\left[\frac{e^{-i\omega}}{-i\omega}-\frac{1}{-i\omega}\right]$$

$$g(\omega)=\left[\frac{1}{i\omega}-\frac{e^{i\omega}}{i\omega}\right]+\left[-\frac{e^{-i\omega}}{i\omega}+\frac{1}{i\omega}\right]$$

$$g(\omega)=\frac{1}{i\omega}+\frac{1}{i\omega}-\frac{e^{i\omega}}{i\omega}-\frac{e^{-i\omega}}{i\omega}$$

$$g(\omega)=\frac{2}{i\omega}-\frac{1}{i\omega}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{1}{i\omega}\frac{2}{2}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{2}{i\omega}\frac{1}{2}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{2}{i\omega}\cos\omega \qquad [\because \cos x=\frac{1}{2}(e^{ix}+e^{-ix})]$$

$$g(\omega)=\frac{2}{i\omega}(1-\cos\omega)$$ *Answer*

Example 42: Find Fourier Transform of $f(t) = e^{-|t|}$

Or

Find Fourier Transform of

$$f(t) = e^{-t} \qquad ;t > 0$$
$$= e^{t} \qquad ;t < 0$$

Answer:

Given

$$f(t) = e^{-t} \qquad ;t > 0$$
$$= e^{t} \qquad ;t < 0 \qquad \text{-------------------------(i)}$$

We have,

$$g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$g(\omega) = \int_{-\infty}^{0} f(t) e^{-i\omega t} dt + \int_{0}^{\infty} f(t) e^{-i\omega t} dt$$

$$g(\omega) = \int_{-\infty}^{0} e^{t}\, e^{-i\omega t} dt + \int_{0}^{\infty} e^{-t}\, e^{-i\omega t} dt \qquad \text{[Given equation no (i)]}$$

$$g(\omega) = \int_{-\infty}^{0} e^{t}\, e^{-i\omega t} dt + \int_{0}^{\infty} e^{-t}\, e^{-i\omega t} dt$$

$$g(\omega) = \int_{-\infty}^{0} e^{t - i\omega t} dt + \int_{0}^{\infty} e^{-t - i\omega t} dt$$

$$g(\omega) = \int_{-\infty}^{0} e^{(1-i\omega)t} dt + \int_{0}^{\infty} e^{-(1+i\omega)t} dt$$

$$g(\omega) = \left[\frac{e^{(1-i\omega)t}}{(1-i\omega)} \right]_{-\infty}^{0} + \left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)} \right]_{0}^{\infty}$$

$$g(\omega) = \left[\frac{e^{(1-i\omega).0}}{(1-i\omega)} - \frac{e^{(1-i\omega)(-\infty)}}{(1-i\omega)} \right] + \left[\frac{e^{-(1+i\omega)\infty}}{-(1+i\omega)} - \frac{e^{-(1+i\omega).0}}{-(1+i\omega)} \right]$$

$$g(\omega) = \left[\frac{e^{0}}{(1-i\omega)} - \frac{e^{-\infty}}{(1-i\omega)} \right] + \left[\frac{e^{-\infty}}{-(1+i\omega)} - \frac{e^{-0}}{-(1+i\omega)} \right]$$

$$g(\omega) = \left[\frac{1}{(1-i\omega)} - \frac{1}{e^{\infty}(1-i\omega)} \right] + \left[\frac{e^{-\infty}}{-(1+i\omega)} - \frac{e^{-0}}{-(1+i\omega)} \right]$$

$$g(\omega) = \frac{1}{(1-i\omega)} \left[1 - \frac{1}{e^{\infty}} \right] + \frac{-1}{1+i\omega} \left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}} \right]$$

$$g(\omega) = \frac{1}{(1-i\omega)} \left[1 - \frac{1}{\infty} \right] + \frac{-1}{1+i\omega} \left[\frac{1}{\infty} - \frac{1}{1} \right]$$

$$g(\omega)=\frac{1}{(1-i\omega)}[1-0]+\frac{-1}{1+i\omega}\left[\frac{1}{\infty}-\frac{1}{1}\right]$$

$$g(\omega)=\frac{1}{(1-i\omega)}+\frac{-1}{1+i\omega}[0-1]$$

$$g(\omega)=\frac{1}{(1-i\omega)}+\frac{1}{1+i\omega}$$

$$g(\omega)=\frac{1+i\omega+1-i\omega}{(1-i\omega)(1+i\omega)}$$

$$g(\omega)=\frac{2}{(1-i\omega)(1+i\omega)}$$

$$g(\omega)=\frac{2}{(1-i^2\omega^2)}$$

$$g(\omega)=\frac{2}{1+\omega^2} \qquad [i^2=-1]$$

Answer

Example 43: Find Fourier Transform for the given functions

$$\begin{aligned} f(t)&=e^{-2t} \qquad &;t\geq 0\\ f(t)&=0 \qquad &;t<0 \end{aligned}$$

Answer:

Given

$$\begin{aligned} f(t)&=e^{-2t} \qquad &;t\geq 0\\ f(t)&=0 \qquad &;t<0 \end{aligned} \qquad \text{--------------------------(i)}$$

We have,

$$g(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} f(t)e^{-i\omega t}dt+\int_{0}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} 0.e^{-i\omega t}dt+\int_{0}^{\infty} e^{-2t}\, e^{-i\omega t}dt \qquad \text{[Given equation no (i)]}$$

$$g(\omega)=\int_{0}^{\infty} e^{-2t}\, e^{-i\omega t}dt$$

$$g(\omega)=\int_{0}^{\infty} e^{-2t-i\omega t}dt$$

$$g(\omega)=\int_{0}^{\infty} e^{-(2+i\omega)t}dt$$

$$g(\omega)=\left[\frac{e^{-(2+i\omega)t}}{-(2+i\omega)}\right]_0^{\infty}$$

$$g(\omega)=\frac{-1}{2+i\omega}\left[e^{-\infty}-e^{-0}\right]$$

$$g(\omega)=\frac{-1}{2+i\omega}\left[\frac{1}{e^{\infty}}-\frac{1}{e^{0}}\right]$$

$$g(\omega)=\frac{-1}{2+i\omega}\left[\frac{1}{\infty}-\frac{1}{1}\right]$$

$$g(\omega)=\frac{-1}{2+i\omega}\left[0-1\right]$$

$$g(\omega)=\frac{1}{2+i\omega}$$

$$|\,g(\omega)\,|=\frac{1}{\sqrt{2^2+\omega^2}}$$

$$|\,g(\omega)\,|=\frac{1}{\sqrt{4+\omega^2}} \qquad \textit{Answer}$$

Example 44: Find Fourier Transform of

$$\begin{aligned} f(t) &= 1+\frac{t}{a} && ;-a<t<0 \\ &= 1-\frac{t}{a} && ;\ 0<t<a \\ &= 0 && ;\ \textit{otherwise} \end{aligned}$$

Answer:
We have,

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[g(\omega)]e^{i\omega t}d\omega$$

Where $g(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$g(\omega)=\int_{-\infty}^{-a} f(t)e^{-i\omega t}dt+\int_{-a}^{0} f(t)e^{-i\omega t}dt+\int_{0}^{a} f(t)e^{-i\omega t}dt+\int_{a}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{-a} 0.e^{-i\omega t}dt+\int_{-a}^{0}(1+\frac{t}{a}).e^{-i\omega t}dt+\int_{0}^{a}(1-\frac{t}{a}).e^{-i\omega t}dt+\int_{a}^{\infty} 0.e^{-i\omega t}dt$$

$$g(\omega)=\int_{-a}^{0}(1+\frac{t}{a}).e^{-i\omega t}dt+\int_{0}^{a}(1-\frac{t}{a}).e^{-i\omega t}dt$$

$$g(\omega)=\int_{-a}^{0}1.e^{-i\omega t}dt+\int_{-a}^{0}\frac{t}{a}.e^{-i\omega t}dt+\int_{0}^{a}1.e^{-i\omega t}dt+\int_{0}^{a}(-\frac{t}{a}).e^{-i\omega t}dt$$

$$g(\omega)=\int_{-a}^{0} e^{-i\omega t}dt+\frac{1}{a}\int_{-a}^{0} t.e^{-i\omega t}dt+\int_{0}^{a} e^{-i\omega t}dt-\frac{1}{a}\int_{0}^{a} t.e^{-i\omega t}dt$$

$$g(\omega)=\int_{-a}^{0} e^{-i\omega t}dt+\int_{0}^{a} e^{-i\omega t}dt+\frac{1}{a}\int_{-a}^{0} t.e^{-i\omega t}dt-\frac{1}{a}\int_{0}^{a} t.e^{-i\omega t}dt\text{..................................}(i)$$

Now,

$$\int t.e^{-i\omega t}dt$$

$$=t\int e^{-i\omega t}dt-\int\left\{\frac{d}{dt}(t)\int e^{-i\omega t}dt\right\}dt \qquad [\int \mathbf{uvdx}=\mathbf{u}\int \mathbf{vdx}-\int\{\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{u})\int \mathbf{vdx}\}\mathbf{dx}]$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]-\int\left\{1.\left[\frac{e^{-i\omega t}}{-i\omega}\right]\right\}dt$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]+\int\frac{e^{-i\omega t}}{i\omega}dt$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]+\frac{1}{i\omega}\int e^{-i\omega t}\,dt$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]+\frac{1}{i\omega}\left[\frac{e^{-i\omega t}}{-i\omega}\right]$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]-\frac{1}{i^2\omega^2}e^{-i\omega t}$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]-\frac{1}{-\omega^2}e^{-i\omega t} \qquad [i^2=-1]$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]+\frac{1}{\omega^2}e^{-i\omega t}$$

$$=\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t} \qquad \text{------------------------------------}(ii)$$

Putting the result of $\int t.e^{-i\omega t}dt$ in (i), we get

$$g(\omega)=\int_{-a}^{0} e^{-i\omega t}dt+\int_{0}^{a} e^{-i\omega t}dt+\frac{1}{a}\int_{-a}^{0} t.e^{-i\omega t}dt-\frac{1}{a}\int_{0}^{a} t.e^{-i\omega t}dt$$

$$g(\omega)=\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-a}^{0}+\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{0}^{a}+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\left[\frac{e^{-i\omega.0}}{-i\omega}-\frac{e^{-i\omega(-a)}}{-i\omega}\right]+\left[\frac{e^{-i\omega.a}}{-i\omega}-\frac{e^{-i\omega.0}}{-i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\left[\frac{e^{-0}}{-i\omega}-\frac{e^{+ai\omega}}{-i\omega}\right]+\left[\frac{e^{-i\omega.a}}{-i\omega}-\frac{e^{-0}}{-i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\left[\frac{1}{-i\omega}-\frac{e^{+ai\omega}}{-i\omega}\right]+\left[\frac{e^{-i\omega.a}}{-i\omega}-\frac{1}{-i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$[e^{-0}=\frac{1}{e^0}=\frac{1}{1}=1]$$

$$g(\omega)=\left[\frac{1}{-i\omega}+\frac{e^{+ai\omega}}{i\omega}\right]+\left[\frac{e^{-i\omega.a}}{-i\omega}+\frac{1}{i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\left[-\frac{1}{i\omega}+\frac{e^{+ai\omega}}{i\omega}\right]+\left[-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=-\frac{1}{i\omega}+\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{i\omega}+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{-i\omega}0.e^{-i\omega.0}+\frac{1}{\omega^2}e^{-i\omega.0}-\frac{1}{-i\omega}(-a)e^{-i\omega(-a)}-\frac{1}{\omega^2}e^{-i\omega(-a)}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[0+\frac{1}{\omega^2}e^{-0}-\frac{1}{-i\omega}(-a)e^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}.1+\frac{1}{-i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$[e^{-0}=\frac{1}{e^0}=\frac{1}{1}=1]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}ae^{-i\omega a}+\frac{1}{\omega^2}e^{-i\omega a}-\frac{1}{-i\omega}.0.e^{-i\omega.0}-\frac{1}{\omega^2}e^{-i\omega.0}\right]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}ae^{-i\omega a}+\frac{1}{\omega^2}e^{-i\omega a}-\frac{1}{\omega^2}e^{-0}\right]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}ae^{-i\omega a}+\frac{1}{\omega^2}e^{-i\omega a}-\frac{1}{\omega^2}.1\right]$$

$$[e^{-0}=\frac{1}{e^0}=\frac{1}{1}=1]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}ae^{-i\omega a}+\frac{1}{\omega^2}e^{-i\omega a}-\frac{1}{\omega^2}\right]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\left[\frac{1}{a\omega^2}-\frac{ae^{i\omega a}}{ai\omega}-\frac{e^{i\omega a}}{a\omega^2}\right]-\left[\frac{-ae^{-i\omega a}}{ai\omega}+\frac{e^{-i\omega a}}{a\omega^2}-\frac{1}{a\omega^2}\right]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a\omega^2}-\frac{ae^{i\omega a}}{ai\omega}-\frac{e^{i\omega a}}{a\omega^2}+\frac{ae^{-i\omega a}}{ai\omega}-\frac{e^{-i\omega a}}{a\omega^2}+\frac{1}{a\omega^2}$$

$$g(\omega)=\frac{1}{i\omega}(e^{+ai\omega}-e^{-i\omega a})+\frac{1}{a\omega^2}+\frac{1}{a\omega^2}+\frac{ae^{-i\omega a}}{ai\omega}-\frac{ae^{i\omega a}}{ai\omega}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{1}{i\omega}(e^{+ai\omega}-e^{-i\omega a})+\frac{2}{a\omega^2}+\frac{ae^{-i\omega a}}{ai\omega}-\frac{ae^{i\omega a}}{ai\omega}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{1}{i\omega}(e^{+ai\omega}-e^{-i\omega a})+\frac{2}{a\omega^2}+\frac{e^{-i\omega a}}{i\omega}-\frac{e^{i\omega a}}{i\omega}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega a}}{i\omega}+\frac{2}{a\omega^2}+\frac{e^{-i\omega a}}{i\omega}-\frac{e^{i\omega a}}{i\omega}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{2}{a\omega^2}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{1}{a\omega^2}(2-e^{i\omega a}-e^{-i\omega a})$$

Answer

Example 45: Find Fourier Transform of $\mathbf{f(t)=te^{-at}u(t)}$ for $\mathbf{a>0}$

Answer:

We have $g(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$\mathbf{g(\omega)=\int_{-\infty}^{\infty} te^{-at}u(t)e^{-i\omega t}dt}$$

$$\mathbf{g(\omega)=\int_{-\infty}^{0} te^{-at}u(t)e^{-i\omega t}dt+\int_{0}^{\infty} te^{-at}u(t)e^{-i\omega t}dt}$$

$$\mathbf{g(\omega)=\int_{-\infty}^{0} te^{-at}.0.e^{-i\omega t}dt+\int_{0}^{\infty} te^{-at}.1.e^{-i\omega t}dt}$$ *[As per the definition of unit function]*

$$\mathbf{g(\omega)=0+\int_{0}^{\infty} te^{-at}.1.e^{-i\omega t}dt}$$

$$g(\omega)=\int_0^\infty te^{-(a+i\omega)t}dt \quad ` \qquad ....(i)$$

Let $I_n=\int_0^\infty te^{-(a+i\omega)t}dt \qquad ....(ii)$

Now, $\int te^{-(a+i\omega)t}dt$

$$=t\int e^{-(a+i\omega)t}dt-\int\{\frac{d}{dt}(t)\int e^{-(a+i\omega)t}dt\}dt$$

$$=t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}-\int\left\{1.\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right\}dt$$

$$=t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}+\frac{1}{a+i\omega}\int e^{-(a+i\omega)t}dt$$

$$=t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}+\frac{1}{a+i\omega}.\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}$$

$$=t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}-\frac{1}{(a+i\omega)^2}.e^{-(a+i\omega)t} \quad ....(iii)$$

Putting the value of $\int t^n e^{-(a+i\omega)t}dt$ in (ii),

$$I_n=\int_0^\infty t^n e^{-(a+i\omega)t}dt$$

$$=\left[t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_0^\infty-\left[\frac{1}{(a+i\omega)^2}.e^{-(a+i\omega)t}\right]_0^\infty$$

$$=\left[t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_0^\infty-\frac{1}{(a+i\omega)^2}\left[e^{-(a+i\omega)t}\right]_0^\infty$$

$$=\left[\infty\frac{e^{-(a+i\omega).\infty}}{-(a+i\omega)}-0.\frac{e^{-(a+i\omega).0}}{-(a+i\omega)}\right]-\frac{1}{(a+i\omega)^2}\left[e^{-(a+i\omega)t}\right]_0^\infty$$

$$=\left[\infty.\frac{e^{-\infty}}{-(a+i\omega)}-0.\frac{e^{-0}}{-(a+i\omega)}\right]-\frac{1}{(a+i\omega)^2}\left[e^{-(a+i\omega).\infty}-e^{-(a+i\omega).0}\right]$$

$$=\left[\infty.\frac{1}{e^\infty}.\frac{1}{-(a+i\omega)}-0.\frac{1}{e^0}.\frac{1}{-(a+i\omega)}\right]-\frac{1}{(a+i\omega)^2}\left[e^{-\infty}-e^{-0}\right]$$

$$=\left[\infty.\frac{1}{\infty}.\frac{1}{-(a+i\omega)}-0.\frac{1}{1}.\frac{1}{-(a+i\omega)}\right]-\frac{1}{(a+i\omega)^2}\left[\frac{1}{e^\infty}-\frac{1}{e^0}\right]$$

$$=\left[\infty.0.\frac{1}{-(a+i\omega)}-0\right]-\frac{1}{(a+i\omega)^2}\left[\frac{1}{\infty}-\frac{1}{e^0}\right] \qquad [e^\infty=\infty]$$

$$=\left[\infty.0.\frac{1}{-(a+i\omega)}-0\right]-\frac{1}{(a+i\omega)^2}\left[0-\frac{1}{1}\right]$$

$$=\left[\infty.0.\frac{1}{-(a+i\omega)}-0\right]-\frac{1}{(a+i\omega)^2}\left[0-1\right]$$

$$=0+\frac{1}{(a+i\omega)^2}$$

$$=\frac{1}{(a+i\omega)^2}$$

$$\therefore g(\omega)=I_n=\int_0^{\infty} te^{-(a+i\omega)t}dt=\frac{1}{(a+i\omega)^2}$$ *Answer*

Example 46: Find Fourier Transform of $\mathbf{f(t)=t^n e^{-at}u(t)}$ for $\mathbf{a>0}$

Answer:

We have $g(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$g(\omega)=\int_{-\infty}^{\infty} t^n e^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} t^n e^{-at}u(t)e^{-i\omega t}dt+\int_0^{\infty} t^n e^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} t^n e^{-at}.0.e^{-i\omega t}dt+\int_0^{\infty} t^n e^{-at}.1.e^{-i\omega t}dt$$ *[As per the definition of unit function]*

$$g(\omega)=0+\int_0^{\infty} t^n e^{-at}.1.e^{-i\omega t}dt$$

$$g(\omega)=\int_0^{\infty} t^n e^{-(a+i\omega)t}dt \qquad \text{.......................(i)}$$

Let $I_n=\int_0^{\infty} t^n e^{-(a+i\omega)t}dt$ ....(ii)

$\therefore I_{n-1}=\int_0^{\infty} t^{n-1} e^{-(a+i\omega)t}dt$ ....(iii)

Now, $\int t^n e^{-(a+i\omega)t}dt$

$$=t^n\int e^{-(a+i\omega)t}dt-\int\{\frac{d}{dt}(t^n)\int e^{-(a+i\omega)t}dt\}dt$$

$$=t^n\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}-\int\left\{nt^{n-1}.\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right\}dt$$

$$= t^n \frac{e^{-(a+i\omega)t}}{-(a+i\omega)} + \frac{n}{a+i\omega}\int t^{n-1}e^{-(a+i\omega)t}dt$$

$$= t^n \frac{e^{-(a+i\omega)t}}{-(a+i\omega)} + \frac{n}{a+i\omega} I_{n-1} \text{....................(iv)} \qquad \text{[from (iii)]}$$

Putting the value of $\int t^n e^{-(a+i\omega)t}dt$ in (ii),

$$I_n = \int_0^\infty t^n e^{-(a+i\omega)t}dt$$

$$= \left[t^n \frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_0^\infty + \frac{n}{a+i\omega} I_{n-1}$$

$$= \left[\infty^n \frac{e^{-(a+i\omega).\infty}}{-(a+i\omega)} - 0^n \frac{e^{-(a+i\omega).0}}{-(a+i\omega)}\right] + \frac{n}{a+i\omega} I_{n-1}$$

$$= \left[\infty.\frac{e^{-\infty}}{-(a+i\omega)} - 0.\frac{e^{-0}}{-(a+i\omega)}\right] + \frac{n}{a+i\omega} I_{n-1}$$

$$= \left[\infty.\frac{1}{e^\infty}.\frac{1}{-(a+i\omega)} - 0.\frac{1}{e^0}.\frac{1}{-(a+i\omega)}\right] + \frac{n}{a+i\omega} I_{n-1}$$

$$= \left[\infty.\frac{1}{\infty}.\frac{1}{-(a+i\omega)} - 0.\frac{1}{1}.\frac{1}{-(a+i\omega)}\right] + \frac{n}{a+i\omega} I_{n-1}$$

$$= \left[\infty.0.\frac{1}{-(a+i\omega)} - 0\right] + \frac{n}{a+i\omega} I_{n-1}$$

$$= \left[\infty.0.\frac{1}{-(a+i\omega)} - 0\right] + \frac{n}{a+i\omega} I_{n-1}$$

$$= \left[\infty.0.\frac{1}{-(a+i\omega)} - 0\right] + \frac{n}{a+i\omega} I_{n-1}$$

$$= [0] + \frac{n}{a+i\omega} I_{n-1}$$

$$I_n = \frac{n}{a+i\omega} I_{n-1} \qquad \text{.................................(v)}$$

Put $\boldsymbol{n = n-1}$ in $\mathbf{(v)}$

$$\therefore I_{n-1} = \frac{n-1}{a+i\omega} I_{n-2} \quad \text{---}\mathbf{(vi)}$$

Again, Put $\boldsymbol{n = n-2}$ in $\mathbf{(v)}$

$$\therefore I_{n-2} = \frac{n-2}{a+i\omega} I_{n-3} \quad \text{---}\mathbf{(vii)}$$

Again Put, $\boldsymbol{n = n-3}$ in $\mathbf{(v)}$

$$\therefore I_{n-3} = \frac{n-3}{a+i\omega} I_{n-4} \text{ --- (viii)}$$

Put $n = 2$ in (v)

$$I_2 = \frac{2}{a+i\omega} I_{2-1}$$

$$\therefore I_2 = \frac{2}{a+i\omega} I_1 \text{ --- (ix)}$$

Put $n = 1$ in (v)

$$\therefore I_1 = \frac{1}{a+i\omega} I_{1-1}$$

$$= \frac{1}{a+i\omega} I_0 \text{ --- (x)}$$

Putting in values of $I_{n-1}, I_{n-2}, \ldots\ldots\ldots, I_2, I_1$ in (v)

$$\therefore I_n = \frac{n}{a+i\omega} I_{n-1}$$

$$= \frac{n}{a+i\omega} \cdot \frac{n-1}{a+i\omega} I_{n-2} \qquad [\text{form(vi)}]$$

$$= \frac{n}{a+i\omega} \cdot \frac{n-1}{a+i\omega} \cdot \frac{n-2}{a+i\omega} I_{n-3} \qquad [\text{form(vii)}]$$

$$= \frac{n}{a+i\omega} \cdot \frac{n-1}{a+i\omega} \cdot \frac{n-2}{a+i\omega} \cdot \frac{n-3}{a+i\omega} I_{n-4} \qquad [\text{form(viii)}]$$

$[\text{form(ix) and (x)}]$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots\ldots\ldots..2.1}{(a+i\omega)^n} I_0$$

$$\therefore I_n = \frac{n!}{(a+i\omega)^n} I_0 \text{ -------------------------------------- (xi)}$$

We have, $I_n = \int_0^\infty t^n e^{-(a+i\omega)t} dt$

Put $n = 0$

$$I_0 = \int_0^\infty t^0 e^{-(a+i\omega)t} dt$$

$$I_0 = \int_0^\infty 1.e^{-(a+i\omega)t} dt$$

$$I_0 = \int_0^\infty e^{-(a+i\omega)t}dt$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[e^{-(a+i\omega)t}\right]_0^\infty$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[e^{-(a+i\omega).\infty} - e^{-(a+i\omega).0}\right]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[e^{-\infty} - e^{-0}\right]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[\frac{1}{\infty} - \frac{1}{e^0}\right] \qquad [e^\infty = \infty]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[0 - \frac{1}{1}\right]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[0 - 1\right]$$

$$I_0 = \frac{1}{(a+i\omega)}$$

From (xi),

$$\therefore I_n = \frac{n!}{(a+i\omega)^n} I_0$$

$$\therefore I_n = \frac{n!}{(a+i\omega)^n} \cdot \frac{1}{(a+i\omega)}$$

$$\therefore I_n = \frac{n!}{(a+i\omega)^{n+1}}$$

$$\therefore g(\omega) = I_n = \int_0^\infty t^n e^{-(a+i\omega)t}dt = \frac{n!}{(a+i\omega)^{n+1}} \quad \textit{Answer}$$

Example 47: Find Fourier Transform of

$f(t) = 1 - t^2$ for $|t| < 1$

$= 0$ for $|t| > 1$

We have $g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$g(\omega) = \int_{-\infty}^{-1} f(t)e^{-i\omega t}dt + \int_{-1}^{1} f(t)e^{-i\omega t}dt + +\int_{1}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{-1}0.e^{-i\omega t}dt+\int_{-1}^{1}(1-t^2)e^{-i\omega t}dt+\int_{1}^{\infty}0.e^{-i\omega t}dt$$

$$g(\omega)=0+\int_{-1}^{1}(1-t^2)e^{-i\omega t}dt+0$$

$$g(\omega)=\int_{-1}^{1}e^{-i\omega t}dt-\int_{-1}^{1}t^2.e^{-i\omega t}dt$$

$$g(\omega)=\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{1}-\int_{-1}^{1}t^2.e^{-i\omega t}dt$$

$$g(\omega)=\left[\frac{e^{-i\omega}}{-i\omega}-\frac{e^{-i\omega(-1)}}{-i\omega}\right]-\int_{-1}^{1}t^2.e^{-i\omega t}dt$$

$$g(\omega)=\left[\frac{e^{-i\omega}}{-i\omega}-\frac{e^{i\omega}}{-i\omega}\right]-\int_{-1}^{1}t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{1}{-i\omega}\left[e^{-i\omega}-e^{i\omega}\right]-\int_{-1}^{1}t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{1}{i\omega}\left[e^{i\omega}-e^{-i\omega}\right]-\int_{-1}^{1}t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{1}{\omega}\frac{2}{2i}\left[e^{i\omega}-e^{-i\omega}\right]-\int_{-1}^{1}t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{2}{\omega}\frac{1}{2i}\left[e^{i\omega}-e^{-i\omega}\right]-\int_{-1}^{1}t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{2}{\omega}\sin\omega-\int_{-1}^{1}t^2.e^{-i\omega t}dt \quad \text{...............(i)} \qquad [\because \sin x=\frac{1}{2i}(e^{ix}-e^{-ix})]$$

Now, $\int t^2.e^{-i\omega t}dt$

$$=t^2\int e^{-i\omega t}dt-\int\{\frac{d}{dt}(t^2)\int e^{-i\omega t}dt\}dt$$

$$=t^2\frac{e^{-i\omega t}}{-i\omega}-\int 2t\frac{e^{-i\omega t}}{-i\omega}dt$$

$$=t^2\frac{e^{-i\omega t}}{-i\omega}+2\int t\frac{e^{-i\omega t}}{i\omega}dt$$

$$=t^2\frac{e^{-i\omega t}}{-i\omega}+2\frac{1}{i\omega}\int te^{-i\omega t}dt$$

$$=t^2\frac{e^{-i\omega t}}{-i\omega}+\frac{2}{i\omega}\int te^{-i\omega t}dt \quad \text{............................(ii)}$$

Now,

$$\int t.e^{-i\omega t}dt$$

$$= t\int e^{-i\omega t}dt - \int\left\{\frac{d}{dt}(t)\int e^{-i\omega t}dt\right\}dt \qquad [\int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] - \int\left\{1.\left[\frac{e^{-i\omega t}}{-i\omega}\right]\right\}dt$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] + \int\frac{e^{-i\omega t}}{i\omega}dt$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] + \frac{1}{i\omega}\int e^{-i\omega t}\,dt$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] + \frac{1}{i\omega}\left[\frac{e^{-i\omega t}}{-i\omega}\right]$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] - \frac{1}{i^2\omega^2}e^{-i\omega t}$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] - \frac{1}{-\omega^2}e^{-i\omega t} \qquad [i^2 = -1]$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] + \frac{1}{\omega^2}e^{-i\omega t}$$

$$= \frac{1}{-i\omega}te^{-i\omega t} + \frac{1}{\omega^2}e^{-i\omega t} \qquad \text{-----------------------------------(iii)}$$

Putting the value of $\int t.e^{-i\omega t}dt$

From (ii)

$$\int t^2.e^{-i\omega t}dt = t^2\frac{e^{-i\omega t}}{-i\omega} + \frac{2}{i\omega}\int te^{-i\omega t}dt$$

$$\int t^2.e^{-i\omega t}dt = t^2\frac{e^{-i\omega t}}{-i\omega} + \frac{2}{i\omega}\left[\frac{1}{-i\omega}te^{-i\omega t} + \frac{1}{\omega^2}e^{-i\omega t}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \left[t^2\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{1} + \frac{2}{i\omega}\left[\frac{1}{-i\omega}te^{-i\omega t} + \frac{1}{\omega^2}e^{-i\omega t}\right]_{-1}^{1}$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \left[1^2\frac{e^{-i\omega.1}}{-i\omega} - 1^2\frac{e^{-i\omega.(-1)}}{-i\omega}\right] +$$

$$\frac{2}{i\omega}\left[\frac{1}{-i\omega}1.e^{-i\omega.1} + \frac{1}{\omega^2}e^{-i\omega.1} - \frac{1}{-i\omega}(-1)e^{-i\omega(-1)} - \frac{1}{\omega^2}e^{-i\omega(-1)}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \left[-\frac{e^{-i\omega}}{i\omega} + \frac{e^{i\omega}}{i\omega}\right] + \frac{2}{i\omega}\left[\frac{1}{-i\omega}e^{-i\omega} + \frac{1}{\omega^2}e^{-i\omega} - \frac{1}{i\omega}e^{i\omega} - \frac{1}{\omega^2}e^{i\omega}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right] + \frac{2}{i\omega}\left[-\frac{1}{i\omega}e^{-i\omega} + \frac{1}{\omega^2}e^{-i\omega} - \frac{1}{i\omega}e^{i\omega} - \frac{1}{\omega^2}e^{i\omega}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right] + \frac{2}{i\omega}\left[-\frac{1}{i\omega}e^{i\omega} - \frac{1}{i\omega}e^{-i\omega} - \frac{1}{\omega^2}e^{i\omega} + \frac{1}{\omega^2}e^{-i\omega}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right] + \frac{2}{i\omega}.\frac{1}{i\omega}\left[-e^{i\omega} - e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{1}{\omega^2}\left[e^{i\omega} - e^{-i\omega}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{1}{i\omega}\left[e^{i\omega} + e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{1}{\omega^2}\left[e^{i\omega} - e^{-i\omega}\right]$$

$$[\therefore \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})\ ;\ \cos x = \frac{1}{2}(e^{ix} + e^{-ix})\ ]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{\omega}\frac{2}{2i}\left[e^{i\omega} - e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{1}{i\omega}\frac{2}{2}\left[e^{i\omega} + e^{-i\omega}\right] - \frac{2}{\omega}.\frac{1}{\omega^2}\frac{2}{2i}\left[e^{i\omega} - e^{-i\omega}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\frac{1}{2i}\left[e^{i\omega} - e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{2}{i\omega}\frac{1}{2}\left[e^{i\omega} + e^{-i\omega}\right] - \frac{2}{\omega}.\frac{2}{\omega^2}\frac{1}{2i}\left[e^{i\omega} - e^{-i\omega}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\sin\omega - \frac{2}{i\omega}.\frac{2}{i\omega}\cos\omega - \frac{2}{\omega}.\frac{2}{\omega^2}\sin\omega$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\sin\omega - \frac{4}{i^2\omega^2}\cos\omega - \frac{4}{\omega^3}\sin\omega$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\sin\omega - \frac{4}{-\omega^2}\cos\omega - \frac{4}{\omega^3}\sin\omega$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\sin\omega + \frac{4}{\omega^2}\cos\omega - \frac{4}{\omega^3}\sin\omega$$

Answer

Example 48: Find Fourier Transform of the unit impulse $\delta(t)$

Answer:

The Impulse function $\delta(t)$ is defined as

$$\delta(t) = 1 \qquad t = 0$$
$$= 0 \qquad \text{Otherwise}$$

Here, $f(t) = \delta(t)$

We have,

$$g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt$$

$$g(\omega) = \left[e^{-i\omega t}\right]_{t = location of\ \delta(t)\ function}$$

$$g(\omega) = \left[e^{-i\omega t}\right]_{t=0}$$

$$g(\omega) = \left[e^{-0}\right]$$

$$g(\omega) = \frac{1}{e^{0}}$$

$$g(\omega) = \frac{1}{1}$$

$$g(\omega) = 1$$

Summary:

01. Spectral analysis of periodic functions is achieved through the *Fourier series.* The three forms are:

- **i.** cosine-sine or trigonometric form: $\mathbf{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t + \sum_{n=1}^{\infty} b_n \sin(n\omega t)}$
- **ii.** amplitude-phase form $\mathbf{f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \phi_n)}$
- **iii.** Complex exponential form: $\mathbf{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$
- **iv.** These sine terms are combined to create the time-varying waveform in the display. And if the opposite operation were to be performed on the result (something called a Fourier transform), these individual harmonic elements would reappear. It is important to reemphasize that waveform generation, and the Fourier transform, are reciprocal operations. You can use frequency components to generate a waveform in the time domain, then transform the result back to the frequency domain and recover what you started with. This reciprocal relationship is to Fourier analysis what the Fundamental Theorem of Calculus (the idea that integration and derivation are reciprocal operations) is to Calculus

02. Why would you want to take a Fourier transform?

- **i.** Fourier analysis can be very useful for two main reasons**.** Many calculations are simpler in the frequency domain than the time domain. For example: filtering (convolving) becomes trivial in the frequency domain. We’ll talk about this next chapter

ii. Many neural processes can be described more effectively in the frequency domain. For example: The cochlea transforms a time domain signal (the sound's waveform) into a frequency domain signal. The strength of the response in the auditory nerve fiber tuned to a particular frequency reflects the amplitude of the sound's waveform at that frequency. In other words, the auditory system takes a Fourier transform of the incoming signal, decomposing the sound into amplitudes as a function of frequency.

iii. Many brain regions have oscillations of a particular frequency that can be easily characterized with Fourier analysis

03. Comparison of Fourier Series and Fourier Transform

i. The Fourier series is usually applied to a periodic function and the transform is usually applied to a non-periodic function

ii. The spectrum of a periodic function is a function of a discrete frequency variable. The spectrum of a non-periodic function is a function of a continuous frequency variable

04. What are the applications of Fourier analysis and Fourier Transform?

Application of Fourier analysis—the frequency representation of signals and systems is extremely important in signal processing and in communications. It explains filtering, modulation of messages in a communication system, the meaning of bandwidth, and how to design filters. Likewise, the frequency representation turns out to be essential in the sampling of analog signals—the bridge between analog and digital signal processing.

The applications of the Fourier transform include filtering. telecommunication, music processing, pitch modification, signal coding and signal synthesis feature extraction for pattern identification as in speech recognition, image processing, spectral analysis in astrophysics, radar signal processing. The Fourier transform is useful for extracting a signal from a noisy background.

Fourier methods have revolutionized fields of science and engineering, from radio astronomy to medical imaging, from seismology to spectroscopy. The wide application of Fourier methods is credited principally to the existence of the fast Fourier transform (FFT). The most direct application of the FFT are to the convolution or de convolution of data, correlation and autocorrelation, optimal filtering, power spectrum estimation, and the computational of Fourier integrals.

Example 49: Draw the graph of $\mathbf{y = f(t) = e^{5t}}$

Answer:

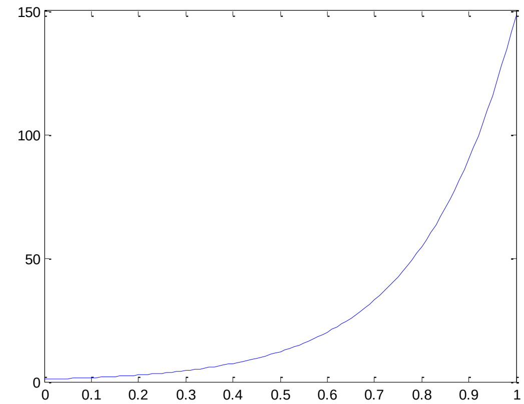


Figure 88

Example 50: Draw the graph of $\mathbf{y = f(t) = e^{-5t}}$

Answer:

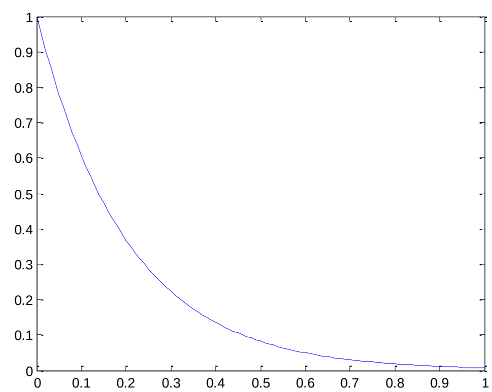


Figure 89

Example 51: Draw the graph of $\mathbf{y = f(t) = e^{-7t}}$

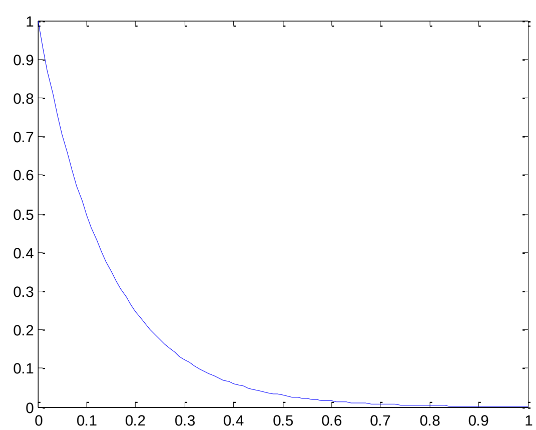


Figure 90

Example 52: Draw the graph of $\mathbf{y = f(t) = e^{5t} * e^{-7t}}$

Answer:

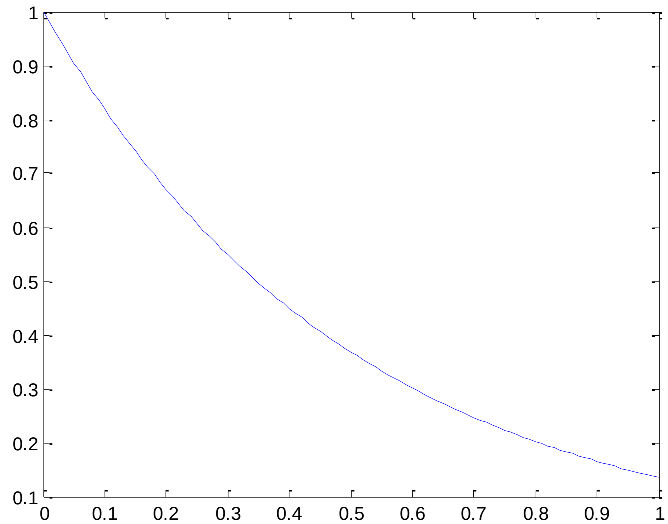


Figure 91

Example 53: Integrate the function of $\mathbf{y = f(t) = e^{5t}}$

Answer:

$$\int_0^\infty e^{5t} dt$$

$$= \left[\frac{e^{5t}}{5}\right]_0^\infty = \frac{1}{5}\left[e^\infty - e^0\right] = \frac{1}{5}[\infty - 1] = \infty$$

Example 54: Integrate the function of $\mathbf{y = f(t) = e^{-5t}}$

Answer:

$$\int_0^\infty e^{-5t} dt$$

$$= \left[\frac{e^{-5t}}{-5}\right]_0^\infty = -\frac{1}{5}\left[e^{-\infty} - e^{-0}\right] = -\frac{1}{5}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = -\frac{1}{5}\left[\frac{1}{\infty} - \frac{1}{1}\right] = -\frac{1}{5}[0 - 1] = \frac{1}{5}$$

$\mathbf{e^{5t}} \rightarrow$ This is unstable system. Unstable system ‡K absolute integrate Kiv hvqbv, gv‡b gvb cvIqv hvqbv| †m‡¶‡Î unstable system †K G †Rvi K‡i stable K‡i wb‡Z nq gv‡b $\mathbf{e^{-at}}$ Øviv ¸b Ki‡j signal Uv decrease n‡e, Unstable system G signal †K transform Kivi Rb¨ stable signal Øviv multiple K‡i wb‡Z nq| †hgb: $\mathbf{e^{5t} * e^{-7t}}$

Problem 25: Derive Laplace transform from Fourier transform

Answer:

We have

$$\mathbf{g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt} \text{ --------------------------------------(i)}$$

Let $\mathbf{g(i\omega) = \int_{0}^{\infty} f(t)e^{-i\omega t}dt}$ ------------------------------------(ii)

$\Rightarrow \mathbf{g(\sigma + i\omega) = \int_{-\infty}^{\infty} \{f(t)e^{-\sigma t}.\}e^{-i\omega t}dt}$ [Multiplying $\mathbf{by\ e^{-\sigma t}}$ to make stable,

Let $\mathbf{y = f(t) = e^{5t}}$]

$$\Rightarrow \mathbf{g(\sigma + i\omega) = \int_{-\infty}^{\infty} f(t)e^{-\sigma t - i\omega t}dt}$$

$$\Rightarrow \mathbf{g(\sigma + i\omega) = \int_{-\infty}^{\infty} f(t)e^{-(\sigma t + i\omega t)}dt}$$

$$\Rightarrow \mathbf{g(\sigma + i\omega) = \int_{-\infty}^{\infty} f(t)e^{-(\sigma + i\omega)t}dt}$$

$$\Rightarrow \mathbf{g(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt\ [let,\ \sigma + i\omega = s]} \text{ ----------------------------------(iii)}$$

Note that in Fourier Transform, $\mathbf{\sigma = 0}$, $\mathbf{\sigma} \rightarrow$ Initial Condition

Hence $\mathcal{L}\mathbf{(f(t)) = g(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt}$ is called Laplace Transform of $\mathbf{f(t)}$ ---------------(iv)

Example 55: Find Laplace Transform of $f(t) = 1$

Answer:

We have,

$$L(f(t)) = \int_0^{\infty} f(t)e^{-st}dt$$

Given,

$$f(t) = 1$$

$$\therefore\ L(1) = \int_0^{\infty} 1.e^{-st}dt$$

$$= \int_0^{\infty} e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty} \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$=-\frac{1}{s}\left[\frac{1}{e^{\infty}}-\frac{1}{e^{0}}\right]$$

$$=-\frac{1}{s}\left[\frac{1}{\infty}-\frac{1}{1}\right] \qquad [e^{\infty}=\infty]$$

$$=-\frac{1}{s}[0-1] \qquad \left[\frac{1}{\infty}=0\right]$$

$$=\frac{1}{s}$$

$$\therefore L(f(t))=L(1)=\frac{1}{s}$$

Example 56: Find Laplace Transform of $f(t)=a$

Answer:

We have,

$$L(f(t))=\int_0^{\infty} f(t)e^{-st}dt$$

Given,

$$f(t)=a$$

$$\therefore \quad L(a)=\int_0^{\infty} a.e^{-st}dt$$

$$=a\int_0^{\infty} e^{-st}dt$$

$$=a\left[\frac{e^{-st}}{-s}\right]_0^{\infty} \qquad [\int e^{-mx}dx=\frac{e^{-mx}}{-m}]$$

$$=-\frac{a}{s}\left[e^{-\infty}-e^{-0}\right]$$

$$=-\frac{a}{s}\left[\frac{1}{e^{\infty}}-\frac{1}{e^{0}}\right]$$

$$=-\frac{a}{s}\left[\frac{1}{\infty}-\frac{1}{1}\right] \qquad [e^{\infty}=\infty]$$

$$=-\frac{a}{s}[0-1] \qquad \left[\frac{1}{\infty}=0\right]$$

$$=\frac{a}{s}$$

$$\therefore L(f(t))=L(a)=\frac{a}{s} \text{ Answer}$$

Example 57: Find Laplace Transform of $f(t) = t$

Answer

We have,

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

Given,

$$f(t) = t$$

$$\therefore \; L(t) = \int_0^\infty t.e^{-st}dt \text{ ---(i)}$$

Now, $\int t e^{-st}dt$

$$= t\int e^{-st}dt - \int\left\{\frac{d}{dt}(t)\int e^{-st}dt\right\}dt \; [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}dt \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-t}{s}e^{-st} + \frac{1}{s}\int e^{-st}dt$$

$$= \frac{-t}{s}e^{-st} + \frac{1}{s}.\frac{e^{-st}}{-s}$$

$$= \frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

$\therefore$ From (i)

$$L(t) = \int_0^\infty t e^{-st}dt$$

$$= \left[\frac{-t}{s}e^{-st}\right]_0^\infty + \left[-\frac{1}{s^2}e^{-st}\right]_0^\infty$$

$$= 0 + \left[-\frac{1}{s^2}e^{-\infty} + \frac{1}{s^2}e^0\right]$$

$$= 0 + \left[-\frac{1}{s^2}\frac{1}{e^\infty} + \frac{1}{s^2}.1\right]$$

$$= 0 + \left[-\frac{1}{s^2}\frac{1}{\infty} + \frac{1}{s^2}.1\right] \qquad [e^\infty = \infty]$$

$$= 0 + \left[0 + \frac{1}{s^2}\right] \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \left[\frac{1}{s^2}\right]$$

$$= \frac{1}{s^2}$$

$$[\because e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0]$$

$$\therefore L(t) = = \frac{1}{s^2}$$

$$\therefore L(f(t)) = L(t) = \frac{1}{s^2} \textit{ Answer}$$

Details Example 57: Find Laplace Transform of $f(t) = t$

Answer

We have,

$$L(f(t)) = \int_0^{\infty} f(t)e^{-st}dt$$

Given,

$$f(t) = t$$

$$\therefore \ L(t) = \int_0^{\infty} t.e^{-st}dt \text{ ---(i)}$$

Now, $\int t e^{-st}dt$

$$= t\int e^{-st}dt - \int\left\{\frac{d}{dt}(t)\int e^{-st}dt\right\}dt \ [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}dt \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-t}{s}e^{-st} + \frac{1}{s}\int e^{-st}dt$$

$$= \frac{-t}{s}e^{-st} + \frac{1}{s}.\frac{e^{-st}}{-s}$$

$$= \frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

$\therefore$ From (i)

$$L(t) = \int_0^{\infty} t e^{-st}dt \ = \left[\frac{-t}{s}e^{-st}\right]_0^{\infty} + \left[-\frac{1}{s^2}e^{-st}\right]_0^{\infty} \text{..............................(ii)}$$

Now,

$$\left[\frac{-t}{s}e^{-st}\right]_0^{\infty}$$

$$= \lim_{t\to\infty}\left[\frac{-t}{s}e^{-st}\right] - \lim_{t\to 0}\left[\frac{-t}{s}e^{-st}\right]$$

$$= -\frac{1}{s}\lim_{t\to\infty}[t.e^{-st}] + \lim_{t\to 0}\frac{1}{s}[t.e^{-st}]$$

$$= -\frac{1}{s}\lim_{t\to\infty}\left[\frac{t}{e^{st}}\right] + \lim_{t\to 0}\frac{1}{s}\left[\frac{t}{e^{st}}\right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{e^{s\infty}}\right] + \frac{1}{s}\left[\frac{0}{e^{s.0}}\right]$$
$$= -\frac{1}{s}\left[\frac{\infty}{e^{\infty}}\right] + \frac{1}{s}\left[\frac{0}{e^{0}}\right]$$
$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right] + \frac{1}{s}\left[\frac{0}{1}\right]$$
$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right] + 0 \quad [Indeterminate\ form]$$

By L'Hospital's rule, Differentiate numerator and Denominator separately,

$$-\frac{1}{s}\lim_{t\to\infty}\left[\frac{t}{e^{st}}\right]$$
$$= \frac{-1}{s}\lim_{t\to\infty}\left[\frac{1}{se^{st}}\right]$$
$$= \frac{-1}{s}\left[\frac{1}{se^{s.\infty}}\right] \qquad \textit{[Differentiate numerator and Denominator separately]}$$
$$= \frac{-1}{s}\left[\frac{1}{se^{\infty}}\right]$$
$$= \frac{-1}{s}\left[\frac{1}{s.e^{\infty}}\right]$$
$$= \frac{-1}{s}\left[\frac{1}{s.\infty}\right]$$
$$= \frac{-1}{s}\left[\frac{1}{\infty}\right]$$
$$= \frac{-1}{s}[0]$$
$$= 0$$

∴ From (ii)

$$L(t) = \int_0^\infty t\,e^{-st}dt$$
$$= \left[\frac{-t}{s}e^{-st}\right]_0^\infty + \left[-\frac{1}{s^2}e^{-st}\right]_0^\infty$$
$$= \left[\frac{-t}{s}e^{-st}\right]_0^\infty + \left[-\frac{1}{s^2}e^{-st}\right]_0^\infty$$
$$= 0 + \left[-\frac{1}{s^2}e^{-\infty} + \frac{1}{s^2}e^{0}\right]$$
$$= 0 + \left[-\frac{1}{s^2}\frac{1}{e^{\infty}} + \frac{1}{s^2}.1\right]$$
$$= 0 + \left[-\frac{1}{s^2}\frac{1}{\infty} + \frac{1}{s^2}.1\right] \qquad [e^{\infty} = \infty]$$
$$= 0 + \left[0 + \frac{1}{s^2}\right] \qquad \left[\frac{1}{\infty} = 0\right]$$
$$= \left[\frac{1}{s^2}\right]$$

$$= \frac{1}{s^2}$$

$$[\because e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0]$$

$$\therefore L(t) = = \frac{1}{s^2}$$

$$\therefore L(f(t)) = L(t) = \frac{1}{s^2} \text{ Answer}$$

Example 58: Find Laplace Transform of $f(t) = e^{at}$

Answer:

We have,

$$L(f(t)) = \int_0^{\infty} f(t) e^{-st} dt$$

Given,

$$f(t) = e^{at}$$

$$\therefore \ L(e^{at}) = \int_0^{\infty} e^{at}.e^{-st} dt$$

$$= \int_0^{\infty} e^{at-st} dt$$

$$= \int_0^{\infty} e^{-st+at} dt$$

$$= \int_0^{\infty} e^{-(st-at)} dt$$

$$= \int_0^{\infty} e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{1}{-(s-a)} \left[e^{-\infty} - e^{-0} \right]$$

$$= -\frac{1}{(s-a)} \left[\frac{1}{e^{\infty}} - \frac{1}{e^0} \right]$$

$$= -\frac{1}{(s-a)} \left[\frac{1}{\infty} - \frac{1}{1} \right] \qquad [e^{\infty} = \infty]$$

$$= -\frac{1}{(s-a)} [0 - 1] \qquad \left[\frac{1}{\infty} = 0 \right]$$

$$= \frac{1}{s-a} \; Answer$$

$$\therefore L(f(t)) = L(e^{at}) = \frac{1}{s-a} \; Answer$$

Example 59: Find Laplace Transform of $f(t) = t^n$

We have,

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

Given,

$$f(t) = t^n$$

$$\therefore \; L(t^n) = \int_0^\infty t^n e^{-st}dt$$

Let, $I_n = L(t^n) = \int_0^\infty t^n e^{-st}dt$ ---------------------------------(i)

Now, $\int t^n e^{-st}dt$

$$= t^n \int e^{-st}dt - \int \left\{ \frac{d}{dt}(t^n) \int e^{-st}dt \right\} dt \; [\because \int uv dx = u\int v dx - \int \{\frac{d}{dx}(u)\int v dx\} dx]$$

$$= t^n \frac{e^{-st}}{-s} - \int n t^{n-1} \frac{e^{-st}}{-s} dt \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-t^n}{s}e^{-st} + \frac{n}{s}\int t^{n-1}e^{-st}dt \text{ ----------------------------(ii)}$$

Since $I_n = L(t^n) = \int_0^\infty t^n e^{-st}dt$

$$\therefore I_{n-1} = \int_0^\infty t^{n-1}e^{-st}dt \text{ --(iii)}$$

$\therefore$ From (i)

$$I_n = L(t^n) = \int_0^\infty t^n e^{-st}dt$$

$$= \left[\frac{-t^n}{s} e^{-st} \right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \; [\text{From(ii)}: \int t^n e^{-st}dt = \frac{-t^n}{s}e^{-st} + \frac{n}{s}\int t^{n-1}e^{-st}dt \;]$$

$$= 0 + \frac{n}{s} I_{n-1} \qquad [\text{From (iii)}: I_{n-1} = \int_0^\infty t^{n-1}e^{-st}dt \;]$$

$$[\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0 \;\text{ and }\; 0^n = 0]$$

$$\therefore I_n = \frac{n}{s} I_{n-1} \text{ -- } (iv)$$

Put $n = n-1$ in (iv)

$$\therefore I_{n-1} = \frac{n-1}{s} I_{n-2} \text{ --- } (v)$$

Again, Put $n = n-2$ in (iv)

$$\therefore I_{n-2} = \frac{n-2}{s} I_{n-3} \text{ --- } (vi)$$

Again Put, $n = n-3$ in (iv)

$$\therefore I_{n-3} = \frac{n-3}{s} I_{n-4} \text{ -- } (vii)$$

Put $n = 2$ in (iv)

$$I_2 = \frac{2}{s} I_{2-1}$$

$$\therefore I_2 = \frac{2}{s} I_1 \text{ -- } (viii)$$

Put $n = 1$ in (iv)

$$\therefore I_1 = \frac{1}{s} I_{1-1}$$

$$= \frac{1}{s} I_0 \text{ --- } (ix)$$

Putting in values of $I_{n-1}, I_{n-2,}..........., I_2, I_1$ in (iv)

$$\therefore I_n = \frac{n}{s} I_{n-1}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} I_{n-2} \qquad [form(v)]$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} I_{n-3} \qquad [form(vi)]$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} I_{n-4} \quad [form(vii)]$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdot \frac{n-4}{s} \ldots\ldots\ldots\ldots\ldots\ldots \frac{2}{s} \cdot \frac{1}{s} I_0 \quad [form(viii) \text{ and } (ix)]$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)....................2.1}{s^n} I_0$$

$$\therefore I_n = \frac{n!}{s^n} I_0 \text{--} (x)$$

We have, $I_n = \int_0^\infty t^n e^{-st} dt$

Put $n = 0$

$$I_0 = \int_0^\infty t^0 e^{-st} dt$$

$$I_0 = \int_0^\infty 1.e^{-st} dt$$

$$I_0 = \int_0^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= -\frac{1}{s} \left[e^{-\infty} - e^{-0} \right]$$

$$= -\frac{1}{s} \left[\frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= -\frac{1}{s} \left[\frac{1}{\infty} - \frac{1}{1} \right]$$

$$= -\frac{1}{s} [0 - 1]$$

$$I_0 = \frac{1}{s} \text{--} (xi)$$

From (x)

$$I_n = \frac{n!}{s^n} I_0$$

$$\mathrm{I_n} = \frac{\mathrm{n!}}{\mathrm{s^n}} \cdot \frac{1}{\mathrm{s}} \qquad \text{[From} (xi) \text{]}$$

$$I_n = \frac{n!}{s^{n+1}} \text{ ---(xii)}$$

$$\therefore L(f(t)) = L(t^n) = \frac{n!}{s^{n+1}} \ \textit{Answer}$$

As for example,

if $n = 2$

$$\therefore L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}}$$

That is, $I_2 = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$ [From (xii)]

$if \quad \mathbf{n = 4}$

$$\therefore L(f(t)) = L(t^4) = \frac{4!}{s^{4+1}}$$

That is,, $\mathbf{I_4} = \frac{4!}{s^{4+1}} = \frac{4!}{s^5}$ [From (xii)]

$if \quad n = 7$

$$\therefore L(f(t)) = L(t^7) = \frac{7!}{s^{7+1}}$$

That is,, $\mathbf{I_7} = \frac{7!}{s^{7+1}} = \frac{7!}{s^8}$ *Answer*

Details Example 59: Find Laplace Transform of $f(t) = t^n$

We have,

$$L(f(t)) = \int_0^\infty f(t) e^{-st} dt$$

Given,

$$f(t) = t^n$$

$$\therefore \quad L(t^n) = \int_0^\infty t^n e^{-st} dt$$

Let, $I_n = L(t^n) = \int_0^\infty t^n e^{-st} dt$ --------------------------------(i)

Now, $\int t^n e^{-st} dt$

$$= t^n \int e^{-st} dt - \int \left\{ \frac{d}{dt}(t^n) \int e^{-st} dt \right\} dt \quad [\because \int uv dx = u \int v dx - \int \{\frac{d}{dx}(u) \int v dx\} dx]$$

$$= t^n \frac{e^{-st}}{-s} - \int n t^{n-1} \frac{e^{-st}}{-s} dt \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-t^n}{s} e^{-st} + \frac{n}{s} \int t^{n-1} e^{-st} dt \text{ ----------------------------} (ii)$$

Since $\mathbf{I_n} = L(t^n) = \int_0^\infty t^n e^{-st} dt$

$$\therefore I_{n-1} = \int_0^\infty t^{n-1} e^{-st} dt \text{ --} (iii)$$

$\therefore$ From (i)

$$I_n = L(t^n) = \int_0^\infty t^n e^{-st} dt$$

$$=\left[\frac{-t^n}{s}e^{-st}\right]_0^\infty+\frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(iv)$$

[since From (ii): $\int t^n e^{-st}dt=\frac{-t^n}{s}e^{-st}+\frac{n}{s}\int t^{n-1}e^{-st}dt$]

Here,

$$\left[-\frac{t^n}{s}e^{-st}\right]_0^\infty$$

$$=-\frac{1}{s}[t^n.e^{-st}]_0^\infty$$

$$=-\frac{1}{s}\lim_{t\to\infty}[t^n.e^{-st}]-(\frac{-1}{s})\lim_{t\to 0}[t^n.e^{-st}]$$

$$=-\frac{1}{s}\lim_{t\to\infty}[t^n.e^{-st}]+\frac{1}{s}\lim_{t\to 0}[t^n.e^{-st}]$$

$$=-\frac{1}{s}\lim_{t\to\infty}\left[\frac{t^n}{e^{st}}\right]+\frac{1}{s}\lim_{t\to 0}\left[\frac{t^n}{e^{st}}\right]$$

$$=-\frac{1}{s}\left[\frac{\infty^n}{e^{s.\infty}}\right]+\frac{1}{s}\left[\frac{0^n}{e^{s.0}}\right]$$

$$=-\frac{1}{s}\left[\frac{\infty}{e^\infty}\right]+\frac{1}{s}\left[\frac{0}{e^0}\right]$$

$$=-\frac{1}{s}\left[\frac{\infty}{\infty}\right]+\frac{1}{s}\left[\frac{0}{1}\right]$$

$$=-\frac{1}{s}\left[\frac{\infty}{\infty}\right]+\frac{1}{s}\times 0$$

$$=-\frac{1}{s}\left[\frac{\infty}{\infty}\right]$$

It is an indeterminate form.

Now, Applying L'Hospital's rule in $-\frac{1}{s}\lim_{t\to\infty}\left[\frac{t^n}{e^{st}}\right]$

Differentiate numerator and denominator separately we get**,**

$$=-\frac{1}{s}\lim_{t\to\infty}\left[\frac{n.t^{n-1}}{s\ e^{st}}\right]$$

$$=-\frac{1}{s}\left[\frac{n.\infty^{n-1}}{s.e^{s.\infty}}\right]$$

$$=-\frac{1}{s}\left[\frac{n\infty}{s.e^\infty}\right]$$

$$=-\frac{1}{s}\left[\frac{\infty}{s.\infty}\right]$$

$$=-\frac{1}{s}\left[\frac{\infty}{\infty}\right]$$

It is an indeterminate form

Again, Differentiate numerator and denominator separately we get,

$$-\frac{1}{s}\lim_{t\to\infty}\left[\frac{n.t^{n-1}}{s.e^{st}}\right]$$

$$=-\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)t^{n-2}}{s.se^{st}}$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)t^{n-2}}{s^2 e^{st}}$$
$$= -\frac{1}{s}\left[\frac{n(n-1).\infty^{n-2}}{s^2 e^{s.\infty}}\right]$$
$$= -\frac{1}{s}\left[\frac{n(n-1).\infty}{s^2 e^{\infty}}\right]$$
$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right] \qquad \text{[Indeterminate Form]}$$

…………………………………………………………

…………………………………………………………

n- times derivative in,

$$-\frac{1}{s}\lim_{t\to\infty}\left[\frac{t^n}{e^{st}}\right]$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\{n-(n-1)\}t^{n-n}}{s^n e^{st}}$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..1.t^0}{s^n e^{st}}$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)(n-2)(n-3)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..1}{s^n e^{st}}$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n!}{s^n e^{st}}$$
$$=-\frac{1}{s}\left[\frac{n!}{s^n e^{s.\infty}}\right]$$
$$=-\frac{1}{s}\left[\frac{n!}{s^n e^{\infty}}\right]$$
$$=-\frac{1}{s}\left[\frac{n!}{s^n.\infty}\right]$$
$$=-\frac{1}{s}\left[\frac{n!}{\infty}\right]$$
$$= -\frac{1}{s}\times 0$$
$$=0$$

$\therefore$ From (iv)

$$I_n = L(t^n) = \int_0^\infty t^n e^{-st}dt$$

$$= \left[\frac{-t^n}{s}e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$= 0 + \frac{n}{s}I_{n-1} \qquad \text{[From (iii): } I_{n-1} = \int_0^\infty t^{n-1}e^{-st}dt \text{]}$$

$$[\because e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0 \text{ and } 0^n = 0]$$

$$\therefore I_n = \frac{n}{s} I_{n-1} \text{ -- (v)}$$

Put $n = n-1$ in $(\mathbf{v})$

$$\therefore I_{n-1} = \frac{n-1}{s} I_{n-2} \text{ -- (vi)}$$

Again, Put $n = n-2$ in $(\mathbf{v})$

$$\therefore I_{n-2} = \frac{n-2}{s} I_{n-3} \text{ --- (vii)}$$

Again Put, $n = n-3$ in $(\mathbf{v})$

$$\therefore I_{n-3} = \frac{n-3}{s} I_{n-4} \text{ -- (viii)}$$

Put $n = 2$ in $(\mathbf{v})$

$$I_2 = \frac{2}{s} I_{2-1}$$

$$\therefore I_2 = \frac{2}{s} I_1 \text{ -- (ix)}$$

Put $n = 1$ in $(\mathbf{v})$

$$\therefore I_1 = \frac{1}{s} I_{1-1}$$

$$= \frac{1}{s} I_0 \text{ --- (x)}$$

Putting in values of $I_{n-1}, I_{n-2}, \ldots\ldots\ldots, I_2, I_1$ in $(\mathbf{v})$

$$\therefore I_n = \frac{n}{s} I_{n-1}$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} I_{n-2} \qquad [\text{form(vi)}]$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} I_{n-3} \qquad [\text{form(vii)}]$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} I_{n-4} \quad [\text{form(viii)}]$$

$$= \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdot \frac{n-4}{s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{2}{s} \cdot \frac{1}{s} I_0 \quad [\text{form(ix) and (x)}]$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)..................2.1}{s^n} I_0$$

$$\therefore I_n = \frac{n!}{s^n} I_0 \text{---} (xi)$$

We have, $I_n = \int_0^\infty t^n e^{-st} dt$

Put $n = 0$

$$I_0 = \int_0^\infty t^0 e^{-st} dt$$

$$I_0 = \int_0^\infty 1.e^{-st} dt$$

$$I_0 = \int_0^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= -\frac{1}{s} \left[e^{-\infty} - e^{-0} \right]$$

$$= -\frac{1}{s} \left[\frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= -\frac{1}{s} \left[\frac{1}{\infty} - \frac{1}{1} \right]$$

$$= -\frac{1}{s} [0 - 1]$$

$$I_0 = \frac{1}{s} \text{---} (xii)$$

From **(xi)**

$$I_n = \frac{n!}{s^n} I_0$$

$$\mathrm{I_n} = \frac{\mathrm{n!}}{\mathrm{s^n}} . \frac{1}{\mathrm{s}} \qquad \text{[From (xii)]}$$

$$I_n = \frac{n!}{s^{n+1}} \text{---} (xiii)$$

$$\therefore L(f(t)) = L(t^n) = \frac{n!}{s^{n+1}} \textit{ Answer}$$

As for example,

if $n = 2$

$$\therefore L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}}$$

That is, $I_2 = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$ [From (xii)]

$if \quad \mathbf{n = 4}$

$$\therefore L(f(t)) = L(t^4) = \frac{4!}{s^{4+1}}$$

That is,, $\mathbf{I_4} = \frac{4!}{s^{4+1}} = \frac{4!}{s^5}$ [From (xii)]

$if \quad n = 7$

$$\therefore L(f(t)) = L(t^7) = \frac{7!}{s^{7+1}}$$

That is,, $\mathbf{I_7} = \frac{7!}{s^{7+1}} = \frac{7!}{s^8}$ *Answer*

Example 60: Find Laplace Transform of $f(t) = \sin at$

Answer:

We have,

$$L(f(t)) = \int_0^\infty f(t) e^{-st} dt \text{ ------------------------------------(i)}$$

Given,

$$f(t) = \sin at$$

$$L(\sin at) = \int_0^\infty \sin at \; e^{-st} dt \text{ --------------------------------(ii)}$$

We have, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore e^{iat} = \cos at + i \sin at$$

Here, $\cos at$ is the real $(\Re)$ part of e^{iat} and $\sin at$ is the imaginary (I) part of e^{iat}.

That is, $\cos at = \Re(e^{iat})$ and $\sin at = I(e^{iat})$

[$\Re(e^{iat})$ Means real part of e^{iat} and $I(e^{iat})$ means imaginary part of e^{iat}.]

We have, $L(f(t)) = \int_0^\infty f(t) e^{-st} dt$

$$\Rightarrow L(f(t)) = L(\sin at) = L\left\{I(e^{iat})\right\}$$

From (ii),

$$L(\sin at) = \int_0^\infty \sin at \; e^{-st} dt$$

$$= \int_0^\infty I\, e^{iat} . e^{-st} dt \qquad [\because \sin at = I(e^{iat})]$$

$$= I\int_0^\infty e^{iat}.e^{-st}dt$$ ----------------------------(iii)

$$= I\int_0^\infty e^{(ia-s)t}dt$$

$$= I\int_0^\infty e^{-(s-ia)t}dt$$

$$= I\left\{\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty\right\} \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= I\left\{\frac{e^{-(s-ia)\infty}}{-(s-ia)} - \frac{e^{-(s-ia)\cdot 0}}{-(s-ia)}\right\}$$

$$= I\left\{0 - \frac{e^{-0}}{-(s-ia)}\right\} \quad [\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0]$$

$$= I\left[\frac{1}{s-ia}\right] \qquad [\because e^{-0} = \frac{1}{e^0} = \frac{1}{1} = 1]$$

$$= I\left\{\frac{s+ia}{(s+ia)(s-ia)}\right\}$$ [Multiplying by $s+ia$ on numerator and denominator]

$$= I\left\{\frac{s+ia}{s^2 - i^2a^2}\right\}$$

$$= I\left\{\frac{s+ia}{s^2+a^2}\right\} \; [i^2 = -1]$$

$$= I\left\{\frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}\right\}$$ --------------------------(iv)

Taking that imaginary part from (iv)

$\therefore L(f(t)) = L\ (\sin at) = \dfrac{a}{s^2+a^2}$ *Answer*

Example 61: Find Laplace Transform of $f(t) = \cos at$

Answer:

We have,

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$ ----------------------------(i)

Given,

$$f(t) = \cos at$$

$$L(\cos at) = \int_0^\infty \cos at\ e^{-st}dt$$ --------------------(ii)

We have, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore e^{iat} = \cos at + i\sin at$$

Here, $\cos at$ is the real part of e^{iat} and $\sin at$ is the imaginary part of e^{iat}.

That is, $\cos at = \Re(e^{iat})$ and $\sin at = I(e^{iat})$

[$\Re(e^{iat})$ Means real part of e^{iat} and $I(e^{iat})$ means imaginary part of e^{iat}.]

We have, $L(f(t)) = \int_0^\infty f(t)e^{-st}dt$

$$\Rightarrow L(f(t)) = L(\cos at) = L\{\Re(e^{iat})\}$$

From (ii),

$$L(\cos at) = \int_0^\infty \cos at\ e^{-st}dt$$

$$= \int_0^\infty \Re e^{iat}.e^{-st}dt \qquad [\because \cos at = \Re(e^{iat})]$$

$$= \Re\int_0^\infty e^{iat}.e^{-st}dt \qquad \text{---------------------------(iii)}$$

$$= \Re\int_0^\infty e^{(ia-s)t}dt$$

$$= \Re\int_0^\infty e^{-(s-ia)t}dt$$

$$= \Re\left\{\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty\right\} \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= \Re\left\{\frac{e^{-(s-ia)\infty}}{-(s-ia)} - \frac{e^{-(s-ia)\cdot 0}}{-(s-ia)}\right\}$$

$$= \Re\left\{0 - \frac{e^{-0}}{-(s-ia)}\right\} \quad [\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0]$$

$$= \Re\left[\frac{1}{s-ia}\right] \qquad [\because e^{-0} = \frac{1}{e^0} = \frac{1}{1} = 1]$$

$$= \Re\left\{\frac{s+ia}{(s+ia)(s-ia)}\right\}$$ [Multiplying by $s+ia$ on numerator and denominator]

$$= \Re\left\{\frac{s+ia}{s^2 - i^2a^2}\right\}$$

$$= \Re\left\{\frac{s+ia}{s^2+a^2}\right\} \ [i^2 = -1]$$

$$= \Re\left\{\frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}\right\} \text{--------------------------(iv)}$$

Taking that real part from (iv)

$$\therefore L(f(t)) = L\,(\cos at) = \frac{s}{s^2+a^2} \quad \textit{Answer}$$

As for example
We get,

$$L\,(\sin at) = \frac{a}{s^2+a^2}$$

$$\therefore L\,(\sin 5t) = \frac{5}{s^2+5^2} \quad \textit{Answer}$$

and

$$L\,(\cos at) = \frac{s}{s^2+a^2}$$

$$L\,(\cos 3t) = \frac{s}{s^2+3^2} \quad \textit{Answer}$$

Example 62: Find Laplace Transform of $f(t) = \sinh at$

Given $f(t) = \sinh at$

We have, $f(t) = \sinh at = \frac{1}{2}(e^{at} - e^{-at})$

$$\therefore \mathcal{L}(\sinh at) = \int_0^\infty \sinh at . e^{-st} dt$$

$$= \int_o^\infty \frac{1}{2}(e^{at} - e^{-at}) e^{-st} dt$$

$$= \frac{1}{2}\int_o^\infty (e^{at} - e^{-at}) e^{-st} dt$$

$$= \frac{1}{2}\int_o^\infty (e^{at-st} - e^{-(at+st)}) dt$$

$$= \frac{1}{2}\int_o^\infty \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\} dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_o^\infty \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s-a)} - \frac{e^{-o}}{-(s-a)} - \frac{e^{-\infty}}{-(s+a)} + \frac{e^{-o}}{-(s+a)}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{e^\infty}.\frac{1}{(s-a)} + \frac{1}{e^0}.\frac{1}{(s-a)} + \frac{1}{e^\infty}.\frac{1}{(s+a)} - \frac{1}{e^0}.\frac{1}{(s+a)}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{\infty}.\frac{1}{(s-a)} + \frac{1}{1}.\frac{1}{(s-a)} + \frac{1}{\infty}.\frac{1}{(s+a)} - \frac{1}{1}.\frac{1}{(s+a)}\right]$$

$$=\frac{1}{2}\left[-0.\frac{1}{(s-a)}+\frac{1}{(s-a)}+0.\frac{1}{(s+a)}-\frac{1}{(s+a)}\right] [\because e^{\infty}=\infty, \frac{1}{\infty}=0, e^{0}=1]$$

$$=\frac{1}{2}\left[\frac{1}{(s-a)}-\frac{1}{(s+a)}\right]$$

$$=\frac{1}{2}\left[\frac{s+a-s+a}{(s^2-a^2)}\right]$$

$$=\frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$$\therefore \mathcal{L}(\text{sinhat})=\left[\frac{a}{s^2-a^2}\right]$$

$$\therefore L(f(t))=\mathcal{L}(\text{sinhat})=\frac{a}{s^2-a^2} \quad \mathcal{A}nswer$$

Example 63: Find Laplace Transform of $f(t)=\text{coshat}$

Given $f(t)=\text{coshat}$

We have, $f(t)=\text{coshat}=\frac{1}{2}(e^{at}+e^{-at})$

$$\therefore \mathcal{L}(\text{coshat})=\int_{0}^{\infty}\text{coshat}.e^{-st}dt$$

$$=\int_{o}^{\infty}\frac{1}{2}(e^{at}+e^{-at})e^{-st}dt$$

$$=\frac{1}{2}\int_{o}^{\infty}(e^{at}+e^{-at})e^{-st}dt$$

$$=\frac{1}{2}\int_{o}^{\infty}(e^{at-st}+e^{-(at+st)})dt$$

$$=\frac{1}{2}\int_{o}^{\infty}\left\{e^{-(s-a)t}+e^{-(s+a)t}\right\}dt$$

$$=\frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)}+\frac{e^{-(s+a)t}}{-(s+a)}\right]_{o}^{\infty} \qquad [\int e^{-mx}dx=\frac{e^{-mx}}{-m}]$$

$$=\frac{1}{2}\left[\frac{e^{-\infty}}{-(s-a)}-\frac{e^{-o}}{-(s-a)}+\frac{e^{-\infty}}{-(s+a)}-\frac{e^{-o}}{-(s+a)}\right]$$

$$=\frac{1}{2}\left[-\frac{1}{e^{\infty}}.\frac{1}{(s-a)}+\frac{1}{e^{0}}.\frac{1}{(s-a)}-\frac{1}{e^{\infty}}.\frac{1}{(s+a)}+\frac{1}{e^{0}}.\frac{1}{(s+a)}\right]$$

$$=\frac{1}{2}\left[-\frac{1}{\infty}.\frac{1}{(s-a)}+\frac{1}{1}.\frac{1}{(s-a)}-\frac{1}{\infty}.\frac{1}{(s+a)}+\frac{1}{1}.\frac{1}{(s+a)}\right]$$

$$=\frac{1}{2}\left[-0.\frac{1}{(s-a)}+\frac{1}{(s-a)}-0.\frac{1}{(s+a)}+\frac{1}{(s+a)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{(s-a)}+\frac{1}{(s+a)}\right]$$

$$=\frac{1}{2}\left[\frac{s+a+s-a}{(s^2-a^2)}\right]$$

$$=\frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

$$\therefore \mathcal{L}(\text{coshat})=\left[\frac{s}{s^2-a^2}\right]$$

$$\therefore L(f(t))=\mathcal{L}(\text{coshat})=\frac{s}{s^2-a^2}\quad \mathcal{A}nswer$$

Problem 26:

The First Shift Theorem

We have seen that a Laplace transform of $f(t)$ is a function of s only, i.e.

$L\{f(t)\}=f(s)$

The first shift theorem states that,

If $L\{f(t)\}=f(s)$ ---------------------------(i)

Then $L\{e^{-at}f(t)\}=f(s+a)$ ---------------------------(ii)

Example 64: Find $L\{e^{-at}t\}=?$

Answer:

Here, $f(t)=t$

The first shift theorem states that,

If $L\{f(t)\}=f(s)$ ---------------------------(i)

Then $L\{e^{-at}f(t)\}=f(s+a)$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t))=L(t)=\frac{1}{s^2}$ **[Here $f(s)=\frac{1}{s^2}$]**

If $f(s)=\frac{1}{s^2}$

$$\therefore f(s+a)=\frac{1}{(s+a)^2}$$

Hence, according to equation no (ii), we can write

$$L\{e^{-at}t\}=f(s+a)=\frac{1}{(s+a)^2}\quad \textit{Answer}$$

Example 65: Find $L\{e^{-3t}t\}=?$

Answer:

Here, $f(t)=t$

The first shift theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $L\{e^{-3t}f(t)\} = f(s+3)$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(t) = \frac{1}{s^2}$ **[Here $f(s) = \frac{1}{s^2}$]**

If $f(s) = \frac{1}{s^2}$

$\therefore f(s+3) = \frac{1}{(s+3)^2}$

Hence, according to equation no (ii), we can write

$L\{e^{-3t}t\} = f(s+3) = \frac{1}{(s+3)^2}$ *Answer*

Example 66: Find $L\{e^{-at}e^{4t}\} = ?$

Here, $f(t) = e^{4t}$

The first shift theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $L\{e^{-at}f(t)\} = f(s+a)$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(e^{4t}) = \frac{1}{(s-4)}$ **[Here**

$f(s) = \frac{1}{(s-4)}$**]**

If $f(s) = \frac{1}{(s-4)}$

$\therefore f(s+a) = \frac{1}{(s+a-4)}$

Hence, according to equation no (ii), we can write

$L\{e^{-at}e^{4t}\} = f(s+a) = \frac{1}{(s+a-4)}$ *Answer*

Example 67: Find $L\{e^{-3t}e^{4t}\} = ?$

Here, $f(t) = e^{4t}$

The first shift theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $L\{e^{-3t}f(t)\} = f(s+3)$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(e^{4t}) = \frac{1}{(s-4)}$ **[Here**

$f(s) = \frac{1}{(s-4)}$**]**

If $\mathbf{f(s) = \frac{1}{(s-4)}}$

$\therefore \mathbf{f(s+3) = \frac{1}{(s+3-4)}}$

Hence, according to equation no (ii), we can write

$\mathbf{L\{e^{-3t}e^{4t}\} = f(s+3) = \frac{1}{(s+3-4)} = \frac{1}{(s-1)}}$ *Answer*

Example 68: Find $\mathbf{L\{e^{-3t} * e^{5t}\} = ?}$

Here, $\mathbf{f(t) = e^{5t}}$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $\mathbf{L\{e^{-3t}f(t)\} = f(s+3)}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(e^{5t}) = \frac{1}{(s-5)}}$ **[Here $\mathbf{f(s) = \frac{1}{(s-5)}}$]**

If $\mathbf{f(s) = \frac{1}{(s-5)}}$

$\therefore \mathbf{f(s+3) = \frac{1}{(s+3-5)}}$

Hence, according to equation no (ii), we can write

$\mathbf{L\{e^{-3t} * e^{5t}\} = f(s+3) = \frac{1}{(s+3-5)} = \frac{1}{(s-2)}}$ *Answer*

Example 69: Find $\mathbf{L\{e^{-4t}t^2\} = ?}$

Answer:

We have,

If $\mathbf{f(t) = t^n}$

Then $\mathbf{L(f(t)) = L(t^n) = \frac{n!}{s^{n+1}}}$

For $\mathbf{n = 2}$;

If $\mathbf{f(t) = t^2}$

Then $\mathbf{L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}}$

We are to find $\mathbf{L\{e^{-4t}t^2\} = ?}$

Here, $\mathbf{f(t) = t^2}$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $\mathbf{L\{e^{-4t}t^2\} = f(s+4)}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}}$ **[Here** $\mathbf{f(s) = \frac{2!}{s^3}}$**]**

If $\mathbf{f(s) = \frac{2!}{s^3}}$**]**

$\mathbf{\therefore f(s+4) = \frac{2!}{(s+4)^3}}$

Hence, according to equation no (ii), we can write

$\mathbf{L\{e^{-4t} * t^2\} = f(s+4) = \frac{2!}{(s+4)^3}}$ *Answer*

Example 70: Find $\mathbf{L\{e^{-5t} * t^2\} = ?}$

Answer:

Here, $\mathbf{f(t) = t^2}$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $\mathbf{L\{e^{-5t} t^2\} = f(s+5)}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}}$ **[Here** $\mathbf{f(s) = \frac{2!}{s^3}}$**]**

If $\mathbf{f(s) = \frac{2!}{s^3}}$**]**

$\mathbf{\therefore f(s+5) = \frac{2!}{(s+5)^3}}$

Hence, according to equation no (ii), we can write

$\mathbf{L\{e^{-5t} * t^2\} = f(s+5) = \frac{2!}{(s+5)^3}}$ *Answer*

Problem 27:

Multiplication Theorem: Laplace Transform of $\boldsymbol{t.\, f(t)}$ (Multiplication by t)

Statement: This theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then, $\mathbf{L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}}$ ---------------------------(ii)

Also, $\boldsymbol{L(t^n.\, f(t)) = (-1)^n \frac{d^n}{ds^n}\left\{f(s)\right\}}$---------------------------(iii)

Proof:

We have, $\boldsymbol{L(f(t)) = f(s) = \int_0^\infty f(t) e^{-st} dt}$ ----------------------(iv)

Differentiating (iv) with respect to $\boldsymbol{s}$, we get

$$\boldsymbol{L(f(t)) = f(s) = \int_0^\infty f(t) e^{-st} dt}$$

$$\frac{d}{ds}\{f(s)\} = \frac{d}{ds}\left[\int_0^\infty f(t)e^{-st}dt\right]$$

$$\frac{d}{ds}\{f(s)\} = \int_0^\infty \frac{d}{ds}\left[f(t)e^{-st}dt\right]$$

$$\frac{d}{ds}\{f(s)\} = \int_0^\infty \frac{d}{ds}\left[e^{-st}\right]f(t)dt$$

$$\frac{d}{ds}\{f(s)\} = \int_0^\infty \left[e^{-st}\right].\frac{d}{ds}\left[-st\right]f(t)dt \quad [\because \frac{d}{dx}(e^{mx}) = e^{mx}.\frac{d}{dx}(mx) = e^{mx}.(m) = me^{mx}]$$

$$\frac{d}{ds}\{f(s)\} = \int_0^\infty \left[e^{-st}\right].(-t)\ f(t)dt$$

$$\frac{d}{ds}\{f(s)\} = -\int_0^\infty t\ f(t)e^{-st}dt$$

$$\frac{d}{ds}\{f(s)\} = -L(t.f(t)) \qquad [\because L(f(t)) = \int_0^\infty f(t)e^{-st}dt]$$

$$L(t.f(t)) = -\frac{d}{ds}\{f(s)\} \qquad \text{(Proved equation no ii)}$$

Similarly,

$$L(t^2.f(t)) = (-1)^2\frac{d^2}{ds^2}\{f(s)\}$$

$$L(t^3.f(t)) = (-1)^3\frac{d^3}{ds^3}\{f(s)\}$$

$$L(t^4.f(t)) = (-1)^4\frac{d^4}{ds^4}\{f(s)\}$$

$$L(t^n.f(t)) = (-1)^n\frac{d^n}{ds^n}\{f(s)\} \qquad \text{(Proved equation no iii)}$$

Example 71: Find $\mathbf{L\{t.t\} = ?}$

Answer:

Here, $\mathbf{f(t) = t}$

This theorem states that

If $\mathbf{L\{f(t)\} = f(s)}$ ----------------------------(i)

Then, $\mathbf{L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}}$ ----------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(t) = \frac{1}{s^2}}$ $\quad$ **[Here** $\mathbf{f(s) = \frac{1}{s^2}}$**]**

Hence, according to equation no (ii), we can write

$$L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}$$

$$L\{t.t\} = -\frac{d}{ds}\{f(s)\} \qquad [f(t) = t]$$

$$L\{t.t\} = -\frac{d}{ds}\{f(s)\}$$

$$L\{t.t\} = -\frac{d}{ds}(\frac{1}{s^2}) \qquad [f(s) = \frac{1}{s^2}]$$

$$L\{t.t\} = -\frac{d}{ds}(s^{-2})$$

$$L\{t.t\} = -(-2)s^{-2-1}$$

$$L\{t.t\} = -(-2)s^{-3}$$

$$L\{t.t\} = 2s^{-3}$$

$$L\{t.t\} = \frac{2}{s^3}$$

That is, $L\{t^2\} = \frac{2}{s^3}$ *(Answer)*

Example 72: Find $L\{t.e^{4t}\} = ?$

Here, $f(t) = e^{4t}$

This theorem states that

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then, $L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(e^{4t}) = \frac{1}{(s-4)}$ [Here $f(s) = \frac{1}{(s-4)}$]

Hence, according to equation no (ii), we can write

$$L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}$$

$$L\{t.e^{4t}\} = -\frac{d}{ds}\{f(s)\} \qquad [f(t) = e^{4t}]$$

$$L\{t.e^{4t}\} = -\frac{d}{ds}\{\frac{1}{(s-4)}\} \qquad [f(s) = \frac{1}{(s-4)}]$$

$$L\{t.e^{4t}\} = -\frac{d}{ds}\{(s-4)^{-1}\}$$

$$L\{t.e^{4t}\} = -(-1)(s-4)^{-1-1}\frac{d}{ds}(s-4)$$

$$L\{t.e^{4t}\} = (s-4)^{-2}(1-0)$$

$$L\{t.e^{4t}\} = (s-4)^{-2}.1$$

$$L\{t.e^{4t}\} = \frac{1}{(s-4)^2}$$ *Answer*

Example 73: Find Laplace Transform of $t^2 \sin 2t$

Here, $f(t) = \sin 2t$

This theorem states that

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then, $L\{t^2 . f(t)\} = (-1)^2 \dfrac{d^2}{ds^2}\{f(s)\}$ --------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(\sin 2t) = \dfrac{2}{s^2+2^2} = \dfrac{2}{s^2+4}$

[Here $f(s) = \dfrac{2}{s^2+4}$]

Hence, according to equation no (ii), we can write

$$L\{t^2 . f(t)\} = (-1)^2 \frac{d^2}{ds^2}\{f(s)\}$$

$$L\{t^2 .\sin 2t\} = (-1)^2 \frac{d^2}{ds^2}(f(s)) \qquad [\ f(t) = \sin 2t\ ]$$

$$L\{t^2 .\sin 2t\} = (-1)^2 \frac{d^2}{ds^2}\{\frac{2}{s^2+4}\} \qquad [\ f(s) = \frac{2}{s^2+4}]$$

$$L\{t^2 .\sin 2t\} = +\frac{d}{ds}\left[\frac{d}{ds}\{\frac{2}{s^2+4}\}\right]$$

$$L\{t^2 .\sin 2t\} = \frac{d}{ds}\left[\frac{d}{ds}\{2(s^2+4)^{-1}\}\right]$$

$$L\{t^2 .\sin 2t\} = 2\frac{d}{ds}\left[(-1)(s^2+4)^{-1-1} . \frac{d}{ds}(s^2+4)\right]$$

$$L\{t^2 .\sin 2t\} = 2\frac{d}{ds}\left[(-1)(s^2+4)^{-2} .(2s)\right]$$

$$L\{t^2 .\sin 2t\} = -4\frac{d}{ds}\left[(s^2+4)^{-2} .s\right]$$

$$L\{t^2 .\sin 2t\} = -4\left[s\frac{d}{ds}\left[(s^2+4)^{-2}\right] + \left[(s^2+4)^{-2}\right]\frac{d}{ds}(s)\right]$$

$$L\{t^2 .\sin 2t\} = -4\left[s\left[(-2)(s^2+4)^{-2-1} . \frac{d}{ds}(s^2+4)\right] + \left[(s^2+4)^{-2}\right].1\right]$$

$$L\{t^2 .\sin 2t\} = -4\left[s\left[(-2)(s^2+4)^{-3} .(2s)\right] + \left[(s^2+4)^{-2}\right]\right]$$

$$L\{t^2 .\sin 2t\} = -4\left[s\left[\frac{-4s}{(s^2+4)^3}\right] + \left[\frac{1}{(s^2+4)^2}\right]\right]$$

$$L\{t^2.\sin 2t\} = \left[\left[\frac{16s^2}{(s^2+4)^3}\right] - \left[\frac{4}{(s^2+4)^2}\right]\right]$$

$$L\{t^2.\sin 2t\} = \frac{16s^2}{(s^2+4)^3} - \frac{4}{(s^2+4)^2}$$

$$L\{t^2.\sin 2t\} = \frac{16s^2 - 4(s^2+4)}{(s^2+4)^3}$$

$$L\{t^2.\sin 2t\} = \frac{16s^2 - 4s^2 - 16}{(s^2+4)^3}$$

$$L\{t^2.\sin 2t\} = \frac{12s^2 - 16}{(s^2+4)^3} \text{ Answer}$$

Example 74: Find the Laplace Transform $L\{t^2 \cos 3t\}$

Answer:

Here, $f(t) = \cos 3t$

This theorem states that,

If $L\{f(t)\} = f(s)$ --------------- (i)

Then $L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\{f(s)\}$ --------------- (ii)

We have, according to equation no (i), $L\{f(t)\} = L(\cos 3t) = \frac{s}{s^2+3^2} = \frac{s}{s^2+9}$

[Here $f(s) = \frac{s}{s^2+a^2}$]

Hence, according to equation no (ii), we can write,

$$L\{t^2.f(t)\} = (-1)^2 \frac{d^2}{ds^2}\{f(s)\}$$

$$L\{t^2.\cos 3t\} = (-1)^2 \frac{d^2}{ds^2}\{f(s)\} \qquad [\, f(t) = \cos 3t \,]$$

$$L\{t^2.\cos 3t\} = (-1)^2 \frac{d^2}{ds^2}\{\frac{s}{s^2+9}\}$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[\frac{d}{ds}\{\frac{s}{s^2+9}\}\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[\frac{d}{ds}\{s(s^2+9)^{-1}\}\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[s\frac{d}{ds}\{(s^2+9)^{-1}\} + (s^2+9)^{-1}\frac{d}{ds}s\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[s\{(-1)(s^2+9)^{-1-1}.\frac{d}{ds}(s^2+9)\} + (s^2+9)^{-1}.1\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[-s(s^2+9)^{-2}.(2s) + (s^2+9)^{-1}\right]$$
$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[-2s^2(s^2+9)^{-2} + (s^2+9)^{-1}\right]$$
$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[-2s^2(s^2+9)^{-2}\right] + \frac{d}{ds}\left[(s^2+9)^{-1}\right]$$
$$L\{t^2.\cos 3t\} = -2\frac{d}{ds}\left[s^2(s^2+9)^{-2}\right] + \frac{d}{ds}\left[(s^2+9)^{-1}\right]$$
$$L\{t^2.\cos 3t\} = -2\left[s^2\frac{d}{ds}\left[(s^2+9)^{-2}\right] + (s^2+9)^{-2}\frac{d}{ds}s^2\right] + \frac{d}{ds}\left[(s^2+9)^{-1}\right]$$
$$L\{t^2.\cos 3t\} = -2\left[s^2.(-2).(s^2+9)^{-2-1}\frac{d}{ds}(s^2+9) + (s^2+9)^{-2}.(2s)\right] + (-1)(s^2+9)^{-1-1}\frac{d}{ds}\left[(s^2+9)\right]$$

$$L\{t^2.\cos 3t\} = -2\left[s^2.(-2).(s^2+9)^{-3}.(2s) + (s^2+9)^{-2}.(2s)\right] + (-1)(s^2+9)^{-2}.(2s)$$
$$L\{t^2.\cos 3t\} = -2\left[-4s^3(s^2+9)^{-3} + 2s(s^2+9)^{-2}\right] - 2s(s^2+9)^{-2}$$
$$L\{t^2.\cos 3t\} = 8s^3(s^2+9)^{-3} - 4s(s^2+9)^{-2} - 2s(s^2+9)^{-2}$$
$$L\{t^2.\cos 3t\} = 8s^3(s^2+9)^{-3} - 6s(s^2+9)^{-2}$$
$$L\{t^2.\cos 3t\} = \frac{8s^3}{(s^2+9)^3} - \frac{6s}{(s^2+9)^2}$$
$$L\{t^2.\cos 3t\} = \frac{8s^3 - 6s(s^2+9)}{(s^2+9)^3}$$
$$L\{t^2.\cos 3t\} = \frac{8s^3 - 6s^3 - 54s}{(s^2+9)^3}$$
$$L\{t^2.\cos 3t\} = \frac{2s^3 - 54s}{(s^2+9)^3} \quad \textbf{Answer}$$

Problem 28: Division Theorem: Laplace transform of $\frac{1}{t}f(t)$ (Division by t)

That is $L(\frac{1}{t}f(t)) = ?$

Division theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty f(s)\,ds$ ---------------------------(ii)

Proof: We have $L(f(t)) = \int_0^\infty f(t)e^{-st}dt = f(s)$ ---------------------------(iii)

Integrating (iii) with respect to s

$$\int_s^\infty f(s)ds = \int_s^\infty \left[\int_0^\infty f(t)e^{-st}dt\right]ds$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[\int_s^\infty f(t)e^{-st}ds\right]dt \qquad \text{-----------------------------(iv)}$$

Now

$$\int f(s)ds = \int \left[\int f(t)e^{-st}ds\right]dt$$

$$\int f(s)ds = \int \left[f(t)\int e^{-st}ds - \int \left\{\frac{d}{ds}(f(t))\int e^{-st}ds\right\}ds\right]dt$$

$$[\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$\int f(s)ds = \int \left[f(t)\left\{\frac{e^{-st}}{-t}\right\} - \int 0.\frac{e^{-st}}{-t}ds\right]dt$$

$$\int f(s)ds = \int \left[f(t)\left\{\frac{e^{-st}}{-t}\right\} - 0\right]dt$$

$$\int f(s)ds = \int \left[f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt \qquad \text{-----------------------------(v)}$$

Putting the values of (v) in (iv), we can write

$$\int_s^\infty f(s)ds = \int_0^\infty \left[\int_s^\infty f(t)e^{-st}ds\right]dt$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]_s^\infty dt \qquad \text{[From v]}$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[f(t)\left\{\frac{e^{-\infty t}}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[f(t)\left\{\frac{e^{-\infty}}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[f(t)\left\{\frac{1}{e^\infty}.\frac{1}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[f(t)\left\{\frac{1}{\infty}.\frac{1}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[f(t)\left\{0.\frac{1}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[f(t)\{0\} + f(t)\left\{\frac{e^{-st}}{t}\right\}\right]dt$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[f(t)\left\{\frac{e^{-st}}{t}\right\}\right]dt$$

$$\int_s^\infty f(s)ds = \int_0^\infty \left[\frac{1}{t} f(t)e^{-st}\right]dt$$

$$\int_s^\infty f(s)ds = L\left[\frac{1}{t} f(t)\right] \qquad [\because L(f(t)) = \int_0^\infty f(t)e^{-st}dt = f(s)]$$

(Proved)

That is

$$L\left[\frac{1}{t} f(t)\right] = \int_s^\infty f(s)ds$$

Example 75: Find Laplace Transform of $\{\frac{\sin 2t}{t}\}$

Answer:

Here, $f(t) = \sin 2t$

Division theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $L\left[\frac{1}{t} f(t)\right] = \int_s^\infty f(s)ds$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(\sin 2t) = \frac{2}{s^2+4}$

[Here $f(s) = \frac{2}{s^2+4}$]

Hence, according to equation no (ii), we can write

$$L\left[\frac{1}{t} f(t)\right] = \int_s^\infty f(s)ds$$

$$L\left[\frac{1}{t} f(t)\right] = L\left[\frac{1}{t}\sin 2t\right] = \int_s^\infty f(s)ds$$

$$L\left[\frac{1}{t} f(t)\right] = L\left[\frac{1}{t}\sin 2t\right] = \int_s^\infty \frac{2}{s^2+4}ds$$

$$L\left[\frac{1}{t} f(t)\right] = L\left[\frac{1}{t}\sin 2t\right] = \int_s^\infty \frac{2}{s^2+2^2}ds$$

$$L\left[\frac{1}{t} f(t)\right] = L\left[\frac{1}{t}\sin 2t\right] = 2\int_s^\infty \frac{1}{s^2+2^2}ds$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^{\infty}$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\tan^{-1}\frac{\infty}{2}-\tan^{-1}\frac{s}{2}\right]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\tan^{-1}\infty-\tan^{-1}\frac{s}{2}\right]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\tan^{-1}\tan\frac{\pi}{2}-\tan^{-1}\frac{s}{2}\right] \qquad [\because \tan\frac{\pi}{2}=\infty]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\frac{\pi}{2}-\tan^{-1}\frac{s}{2}\right]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=\left[\frac{\pi}{2}-\tan^{-1}\frac{s}{2}\right]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=\left[\frac{\pi}{2}-\tan^{-1}\frac{s}{2}\right] \quad \textit{Answer}$$

Example 76: Prove that $L(f''(t))=s^2L\{f(t)\}-s\,f(0)-f'(0)$

We have

$$L(f(t))=\int_0^{\infty}f(t)e^{-st}dt \qquad \text{---(i)}$$

$$\therefore L(f'(t))=\int_0^{\infty}f'(t)e^{-st}dt \qquad [f(t)=f'(t)] \text{-------------------------(ii)}$$

Now, $\int f'(t)e^{-st}dt$

$$\int f'(t)e^{-st}dt=e^{-st}\int f'(t)dt-\int\{\frac{d}{dt}(e^{-st})\int f'(t)dt\}dt$$

$$[\because \int uvdx=u\int vdx-\int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$\int f'(t)e^{-st}dt=e^{-st}\int\frac{d}{dt}(f(t))dt-\int\{\frac{d}{dt}(e^{-st})\int\frac{d}{dt}(f(t))dt\}dt \qquad [\because \frac{d}{dt}(f(t))=f'(t)]$$

$$\int f'(t)e^{-st}dt=e^{-st}f(t)-\int\{\frac{d}{dt}(e^{-st})f(t)\}dt$$

$$\int f'(t)e^{-st}dt=e^{-st}f(t)-\int -s(e^{-st})f(t)dt$$

$$\int f'(t)e^{-st}dt=e^{-st}f(t)+\int s(e^{-st})f(t)dt$$

$$\int f'(t)e^{-st}dt=e^{-st}f(t)+s\int e^{-st}f(t)dt$$

From (ii),

$$\therefore L(f'(t)) = \int_0^\infty f'(t)e^{-st}dt = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = \left[e^{-s\times\infty} f(\infty) - e^{-s\times 0} f(0)\right] + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = \left[e^{-\infty} f(\infty) - e^{0} f(0)\right] + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{e^\infty} f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{e^\infty} f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{\infty} f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = \left[0.f(\infty) - f(0)\right] + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = -f(0) + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = -f(0) + sL\{f(t)\} \qquad [\because L(f(t)) = \int_0^\infty f(t)e^{-st}dt]$$

$$\therefore L(f'(t)) = -f(0) + sL\{f(t)\}$$

$$\therefore L(f'(t)) = sL\{f(t)\} - f(0) \qquad \text{---------------------------------------(iii)}$$

Now replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in (iii), we get

$$\therefore L(f'(t)) = sL\{f(t)\} - f(0)$$

$$\therefore L(f''(t)) = sL\{f'(t)\} - f'(0) \qquad \text{--(iv)}$$

Putting the value of $L(f'(t))$ from (iii) in (iv), we get

$$\therefore L(f''(t)) = sL\{f'(t)\} - f'(0)$$

$$\therefore L(f''(t)) = s\left[sL\{f(t)\} - f(0)\right] - f'(0) \qquad [L(f'(t)) = sL\{f(t)\} - f(0)]$$

$$\therefore L(f''(t)) = s^2L\{f(t)\} - s\,f(0) - f'(0) \qquad \text{--(v)}$$

$$\therefore L(f''(t)) = s^2L\{f(t)\} - s\,f(0) - f'(0) \qquad \text{(Proved)}$$

Similarly

$$\therefore L(f'''(t)) = s^3L\{f(t)\} - s^2\,f(0) - s\,f'(0) - f''(0)$$

$$\therefore L(f^{iv}(t)) = s^4L\{f(t)\} - s^3\,f(0) - s^2\,f'(0) - s\,f''(0) - f'''(0)$$

Summary

Fourier Series: Analyze Periodic Signal
Fourier Transform: Analyze Aperiodic Signal
Laplaces Transform: Analyze Unstable Signal

Problem 29: Unit Step Function

The unit function, also called Heaviside's unit function, $\mathbf{u(t)}$ is defined as

$$\mathbf{u(t) = 1 \qquad t \geq 0}$$

$$\mathbf{= 0 \qquad t < 0}$$

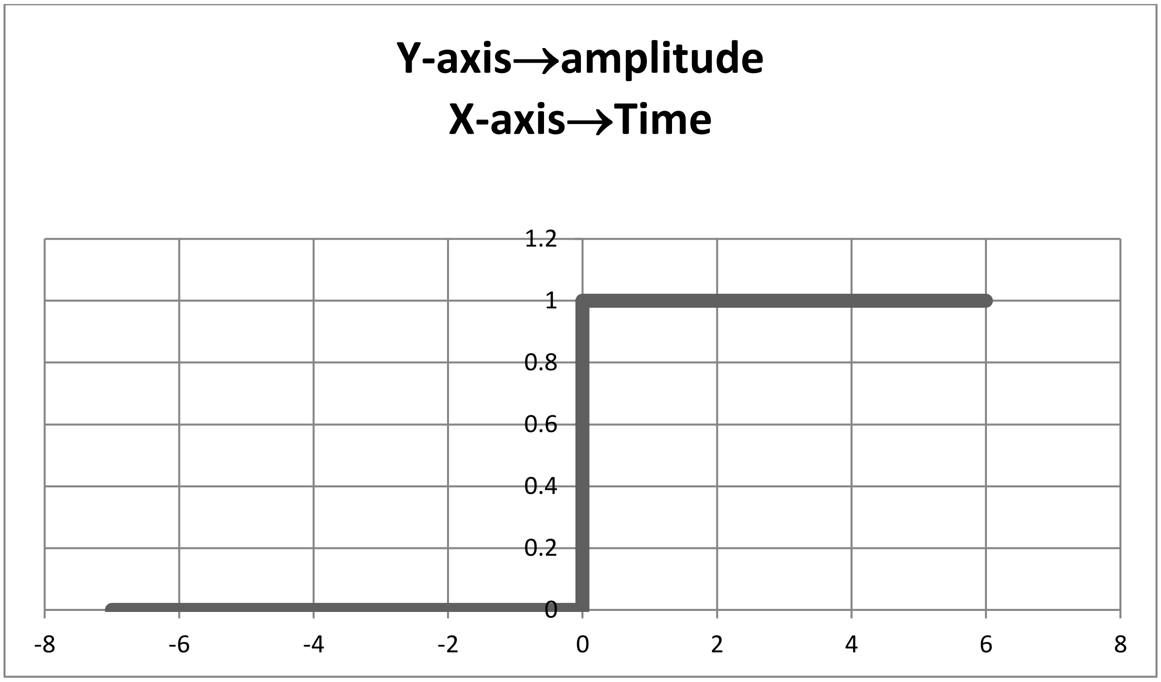


Figure 92

Example 77:

2u(t)

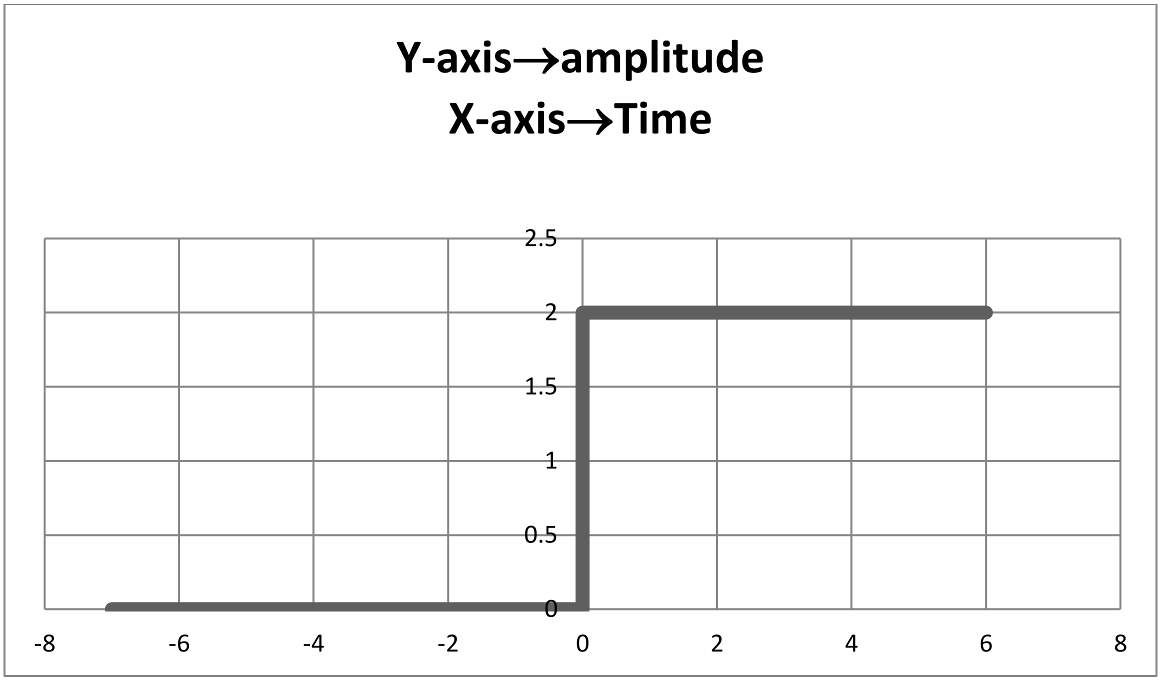


Figure 93

Example 78:

u(t-2)

Here,

$$t-2=0$$

$$\therefore t=2$$

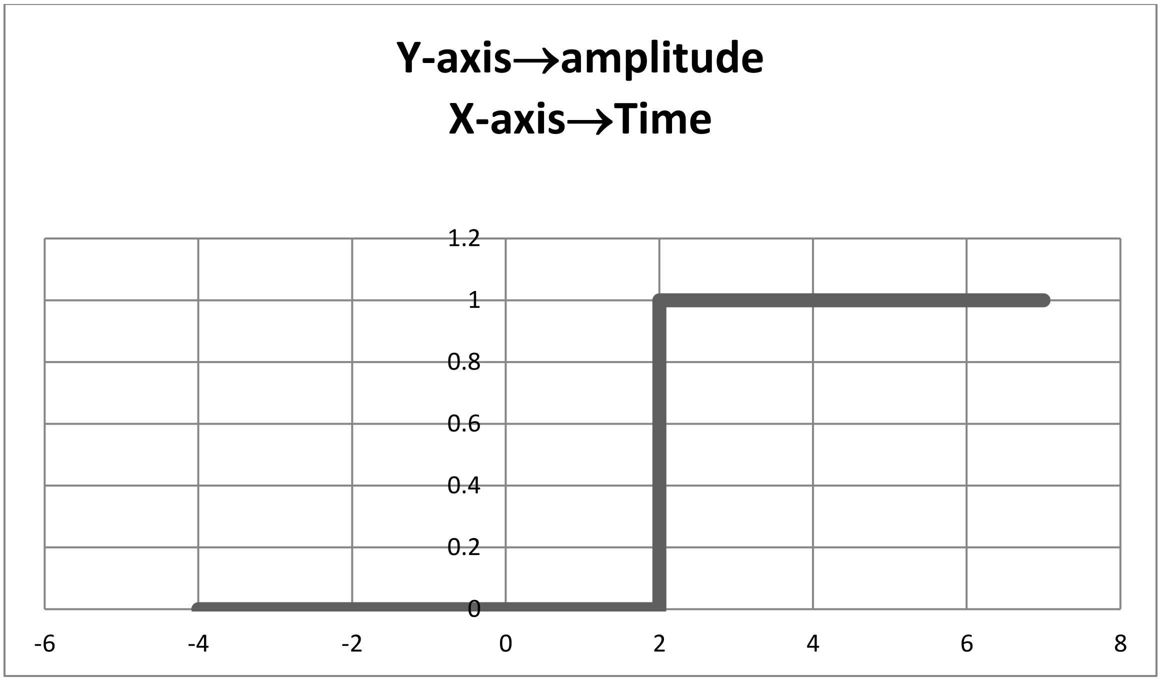


Figure 94

Example 79:

u(t-1)

Here,

$$t-1=0$$

$$\therefore t=1$$

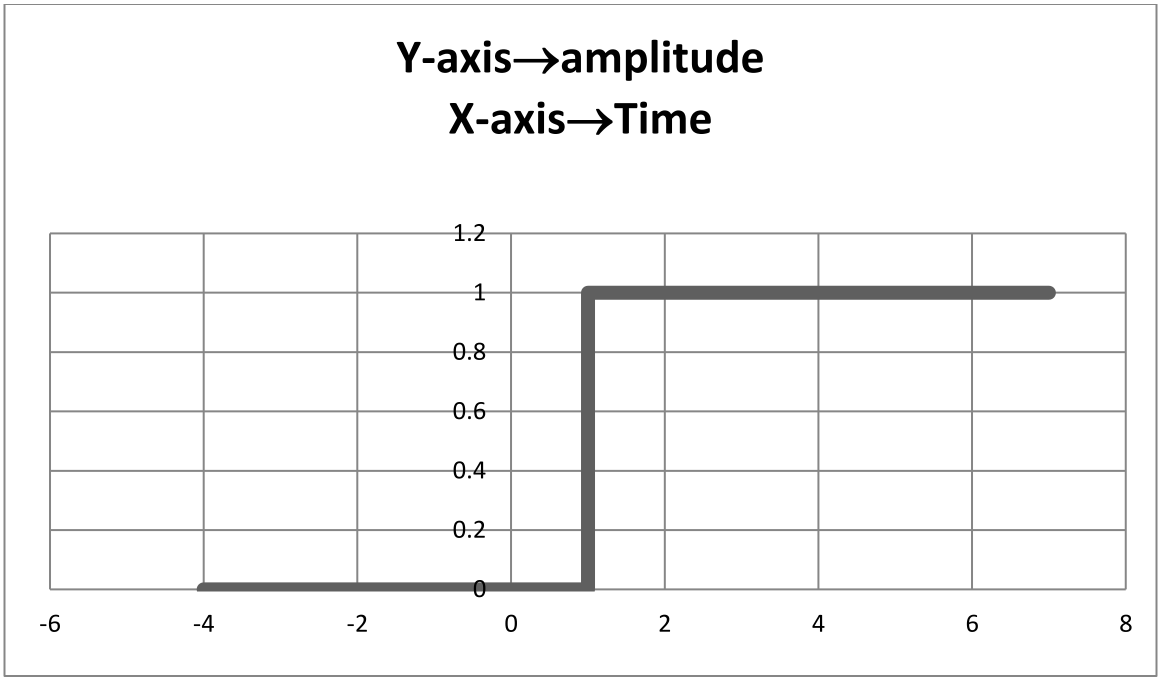


Figure 95

Example 80:

u(t+1)

Here,

$$t+1=0$$
$$\therefore t=-1$$

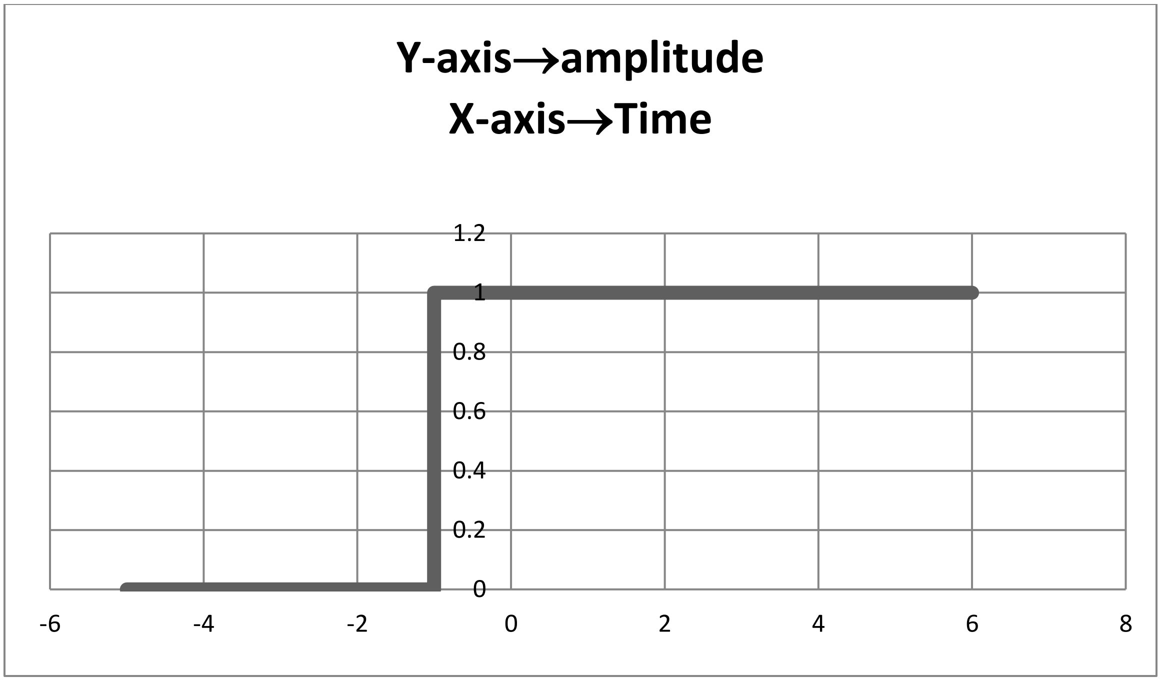


Figure 96

Example 81:

u(t+2)

Here,

$$t + 2 = 0$$

$$\therefore t = -2$$

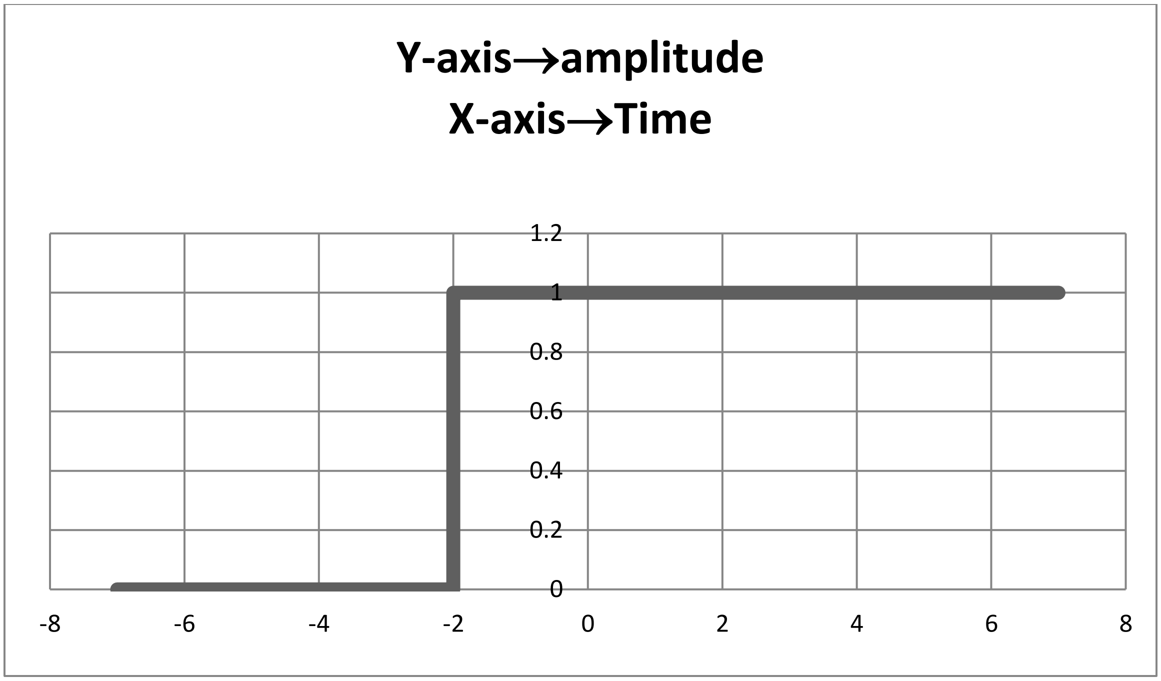


Figure 97

Example 82:

-u(t)

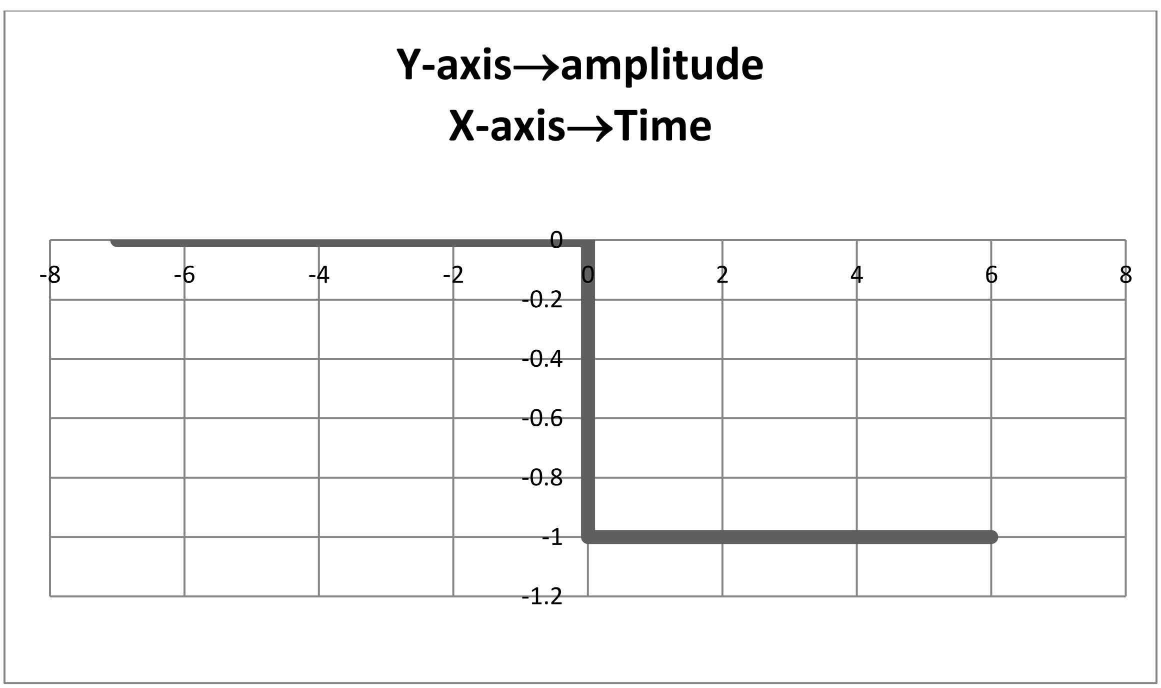


Figure 98

Example 83:

-2u(t)

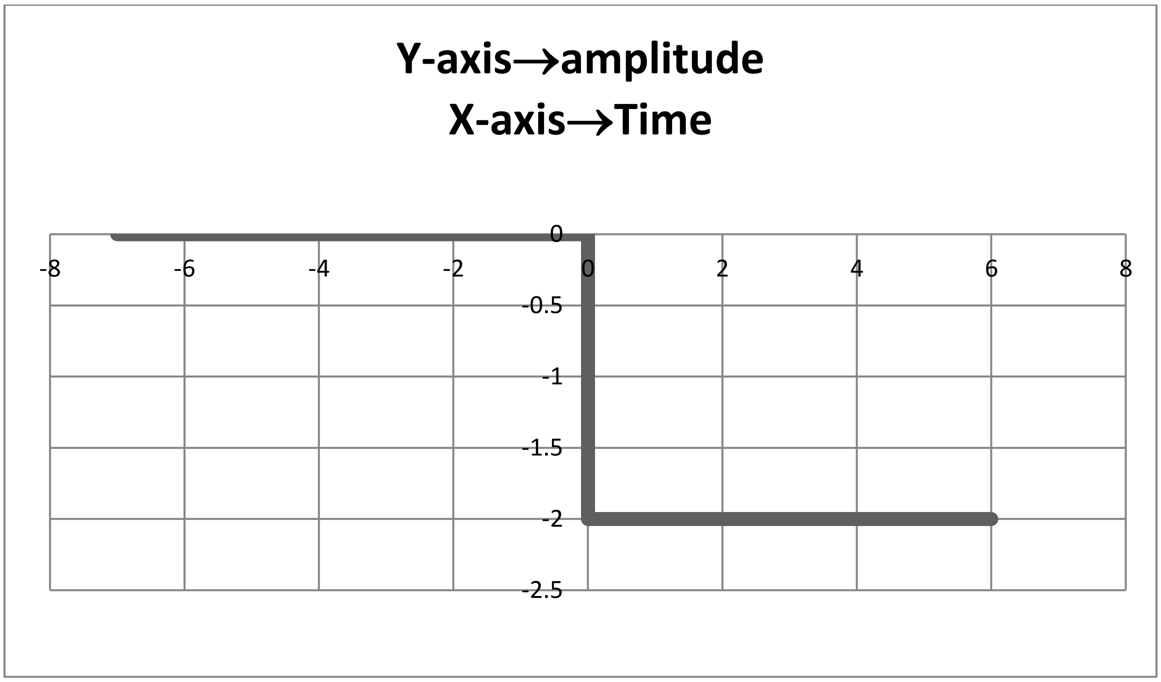


Figure 99

Example 84:

-2u(t-1)

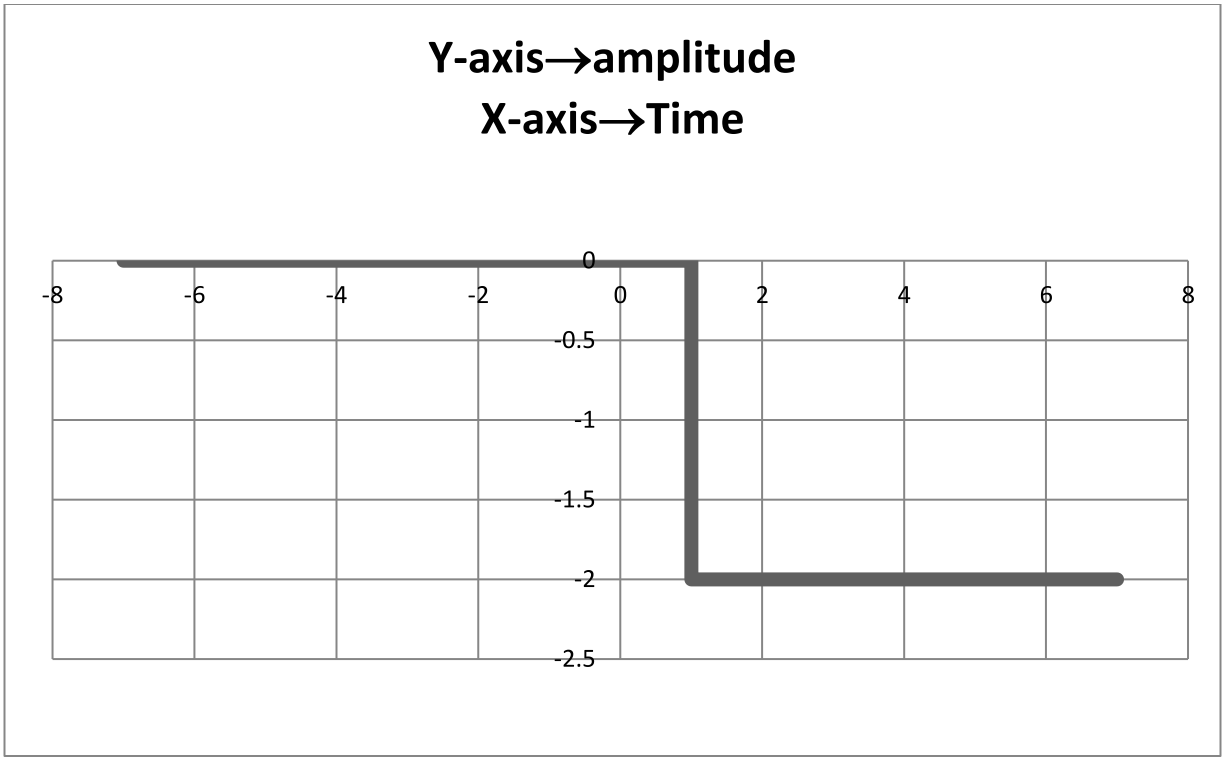


Figure 100

Example 85:

u(t)

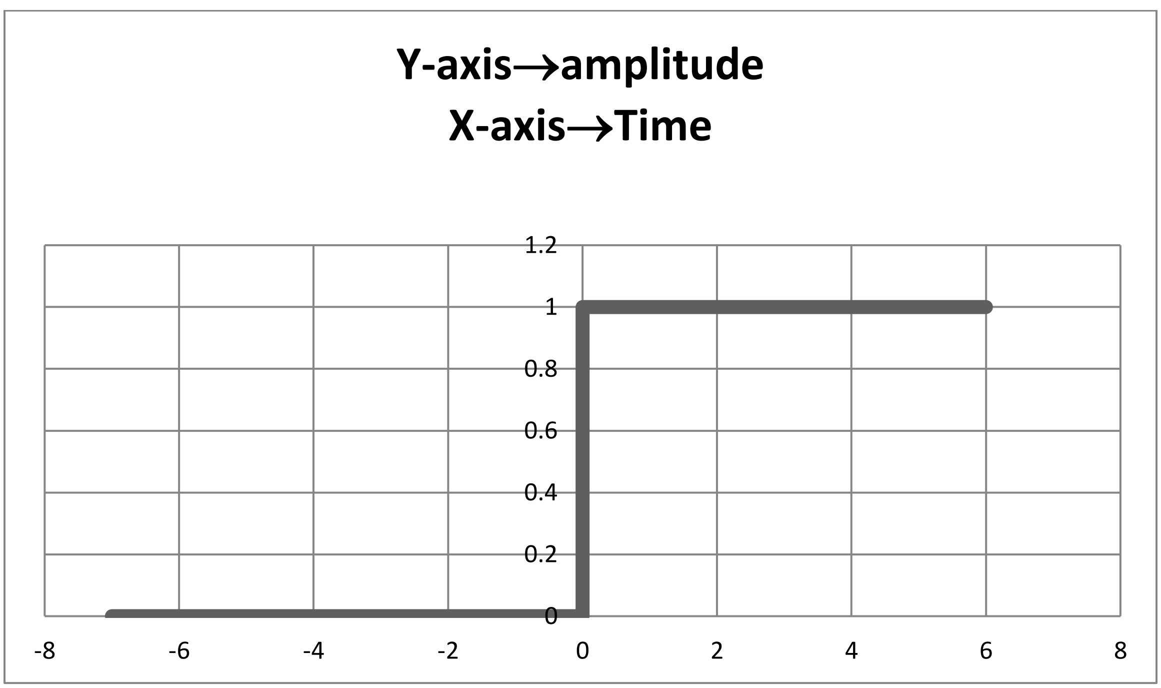


Figure 101

Example 86:

-u(t-2)

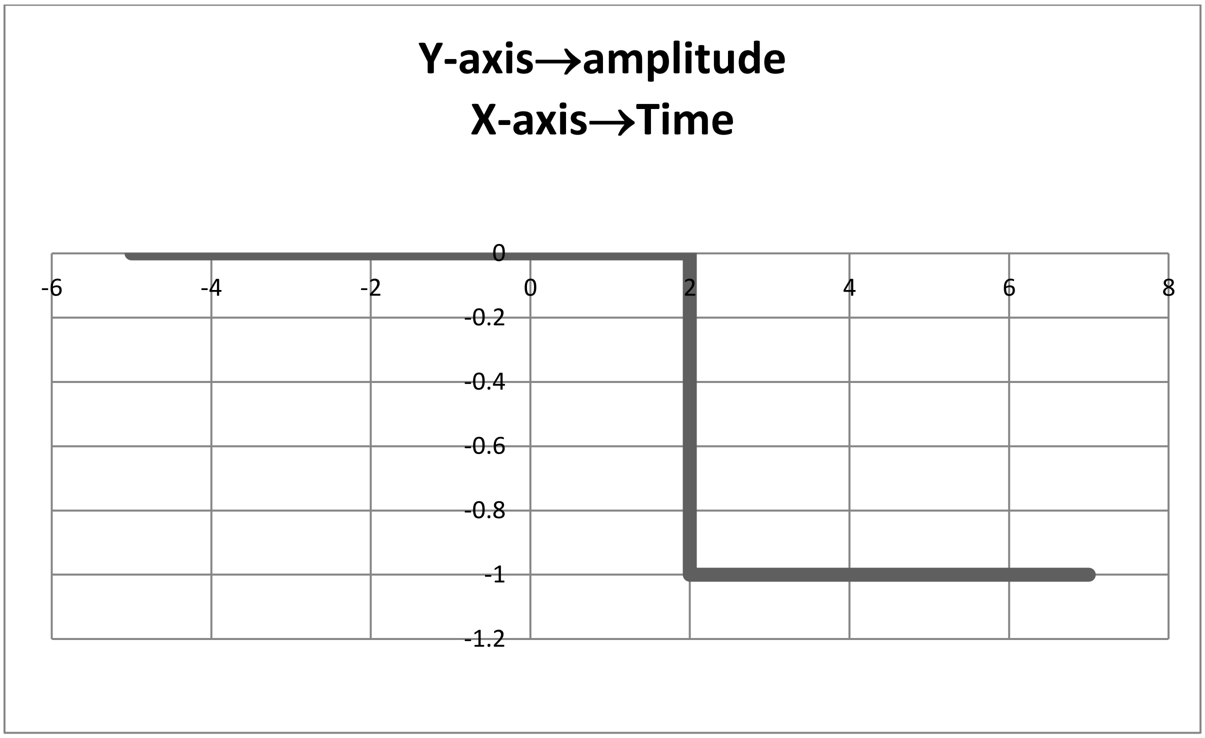


Figure 102

Example 87:

u(t)-u(t-2)

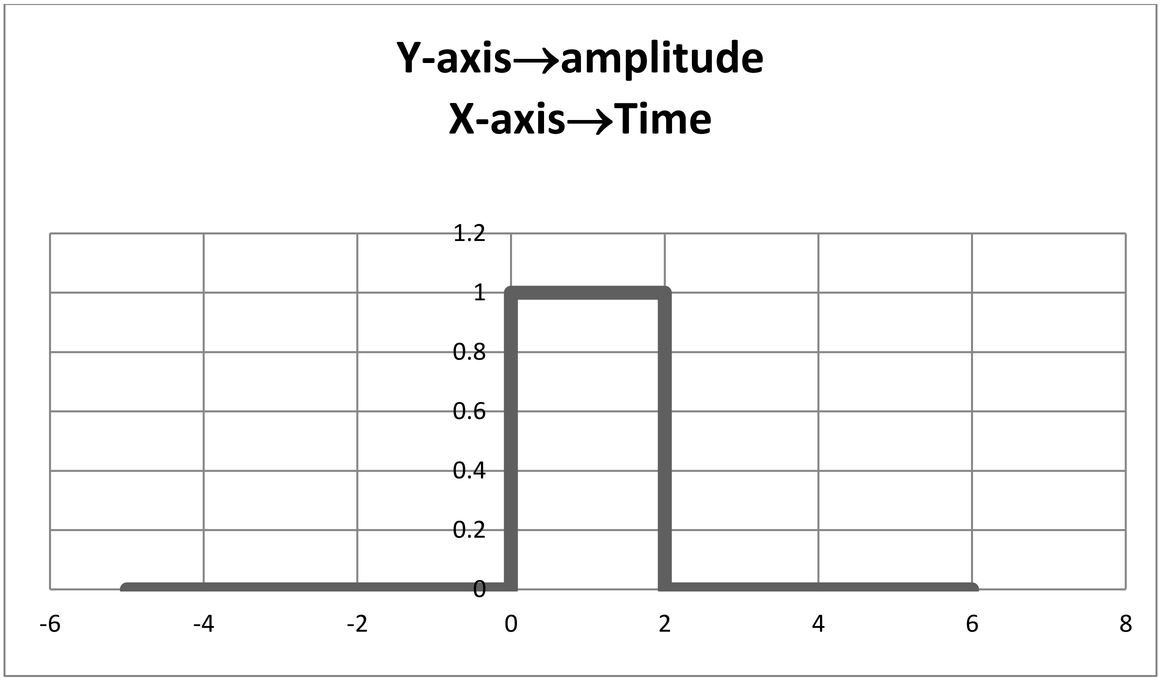


Figure 103

Example 88:

u(t-pi/2)

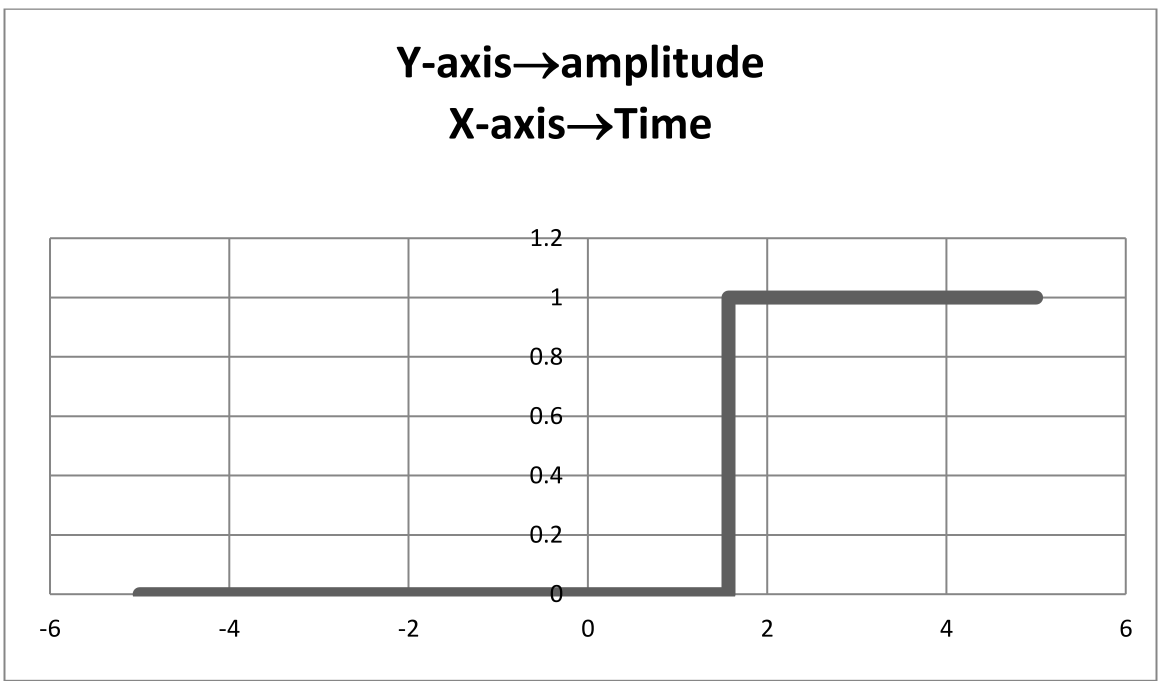


Figure 104

Problem 30: Ramp Function

The ramp function $\mathbf{r(t)}$ is defined as

$$\mathbf{r(t) = t \qquad t \geq 0}$$
$$\mathbf{= 0 \qquad t < 0}$$

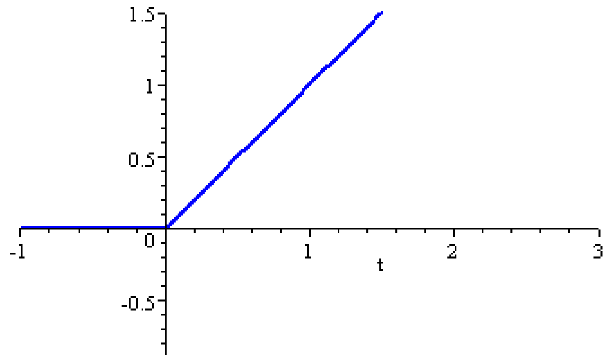


Figure 105

Example 89:

Given that, $\boldsymbol{x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)}$

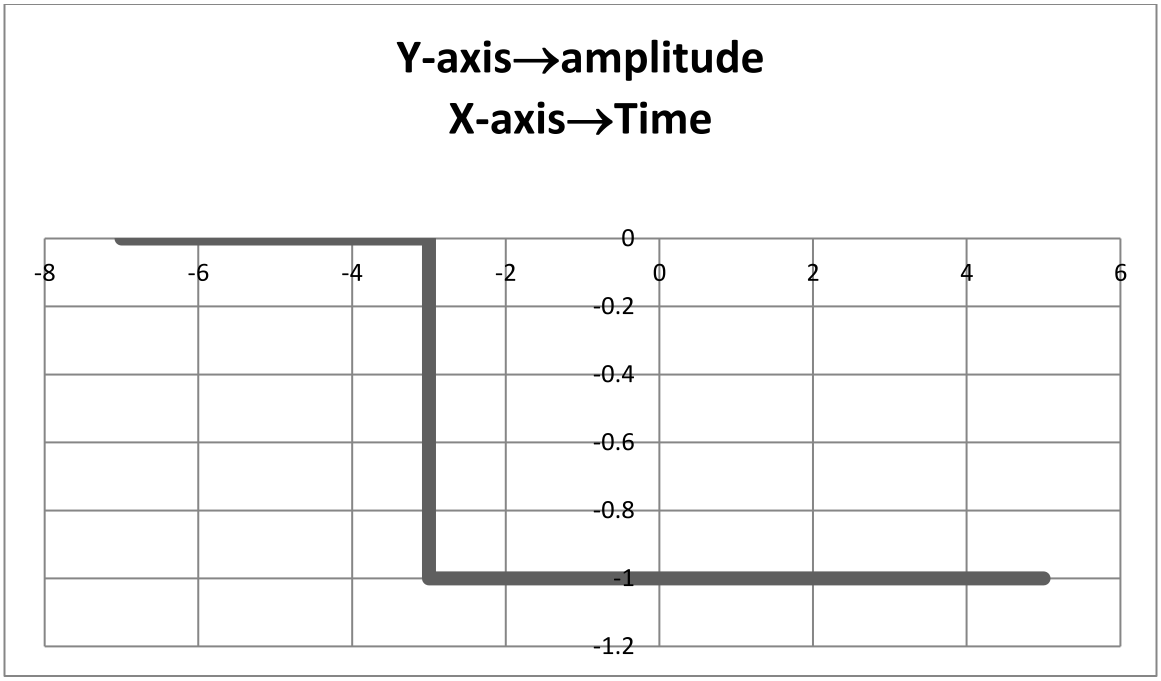
Answer:

01. $\boldsymbol{-u(t+3) =>}$

So,

$-u(t+3) = -1\ ;\ t \geq -3;$ $\qquad$ $here, t + 3 = 0$

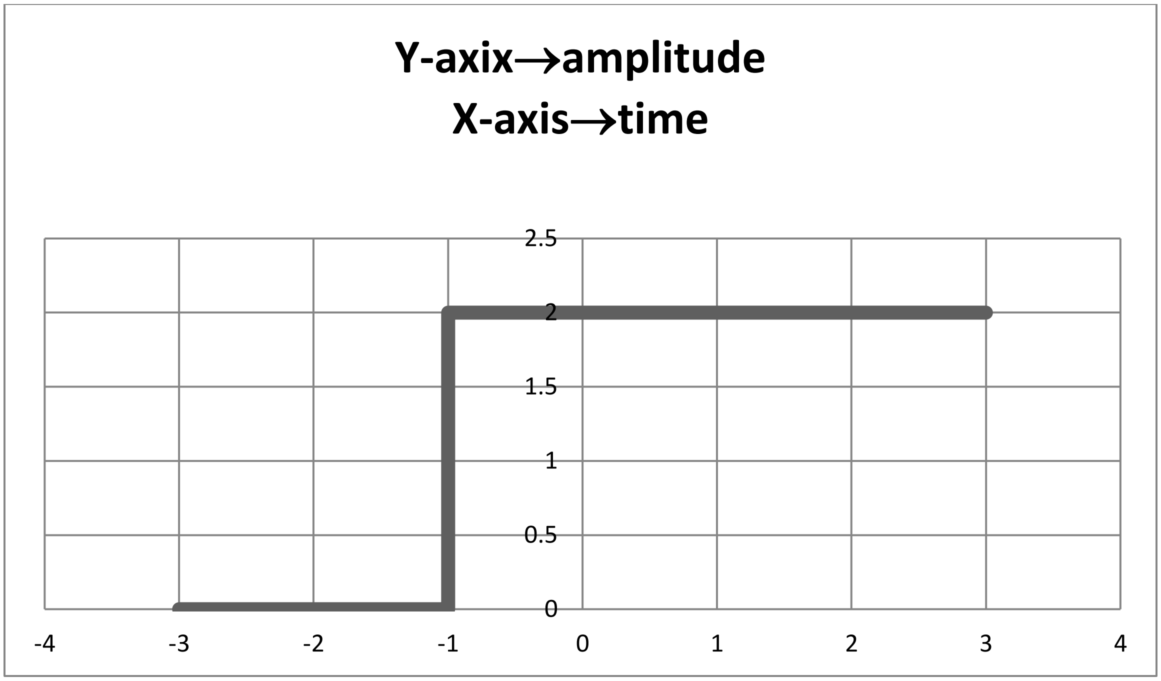
$= 0;\ \ t < -3$ $\qquad$ $\therefore t = -3$



02. $\mathbf{2u(t+1)}$

$2u(t+1) = 2; t \geq -1$ $\qquad$ $here, t + 1 = 0$

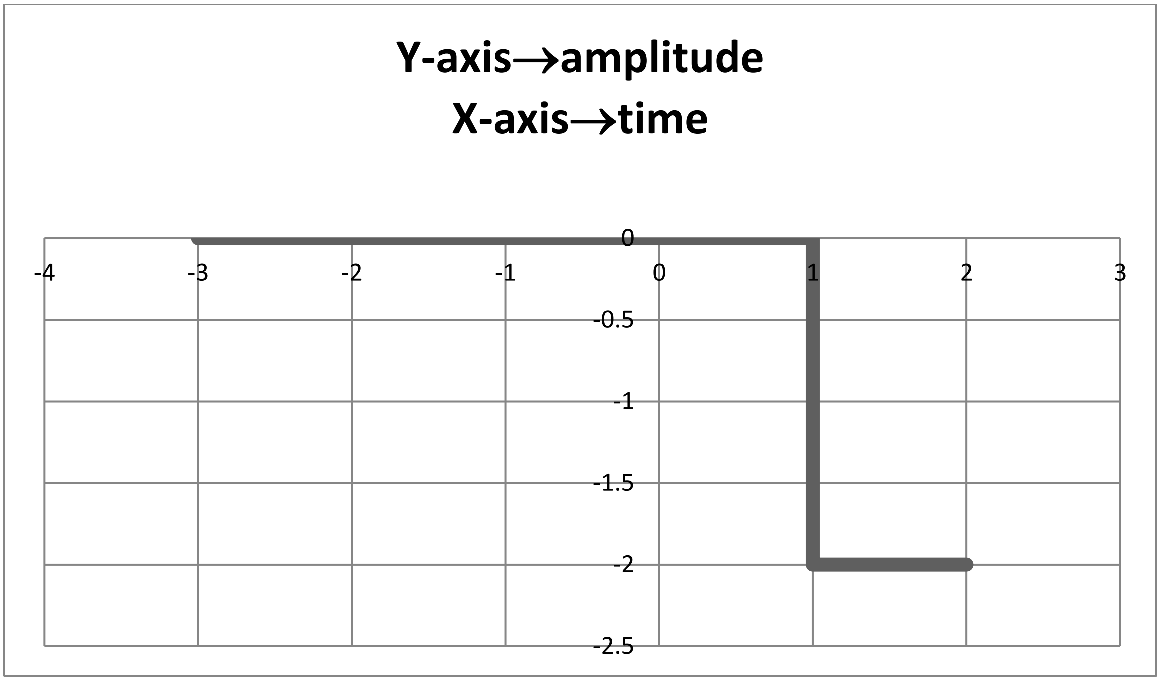
$\qquad = 0; t < -1$ $\qquad$ $t = -1$



03. $\boldsymbol{-2u\ (t-1)}$

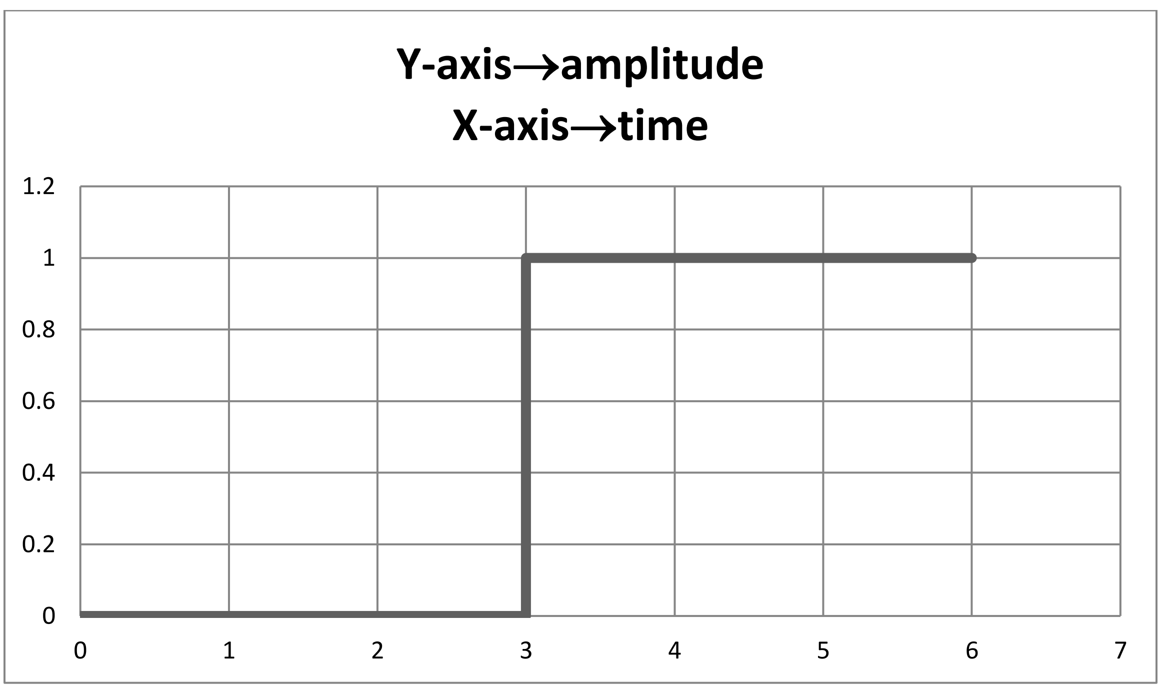
$\therefore\ -2u(t-1) = -2; t \geq 1$ $\qquad$ $here, t-1=0$

$\qquad = 0;\ t < 1$ $\qquad$ $t = 1$

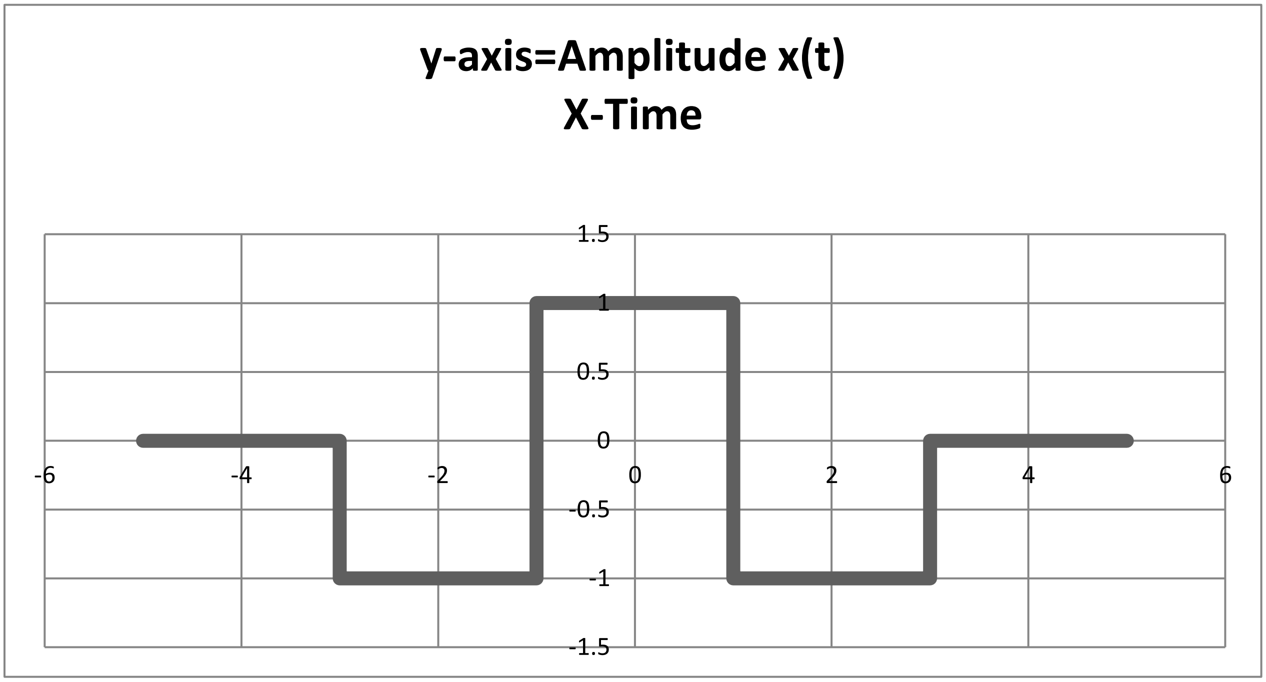


04. $\boldsymbol{u(t-3)} =>$ $\qquad here, t - 3 = 0$

$t = 3$



$$\boldsymbol{x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)}$$



Example 90:

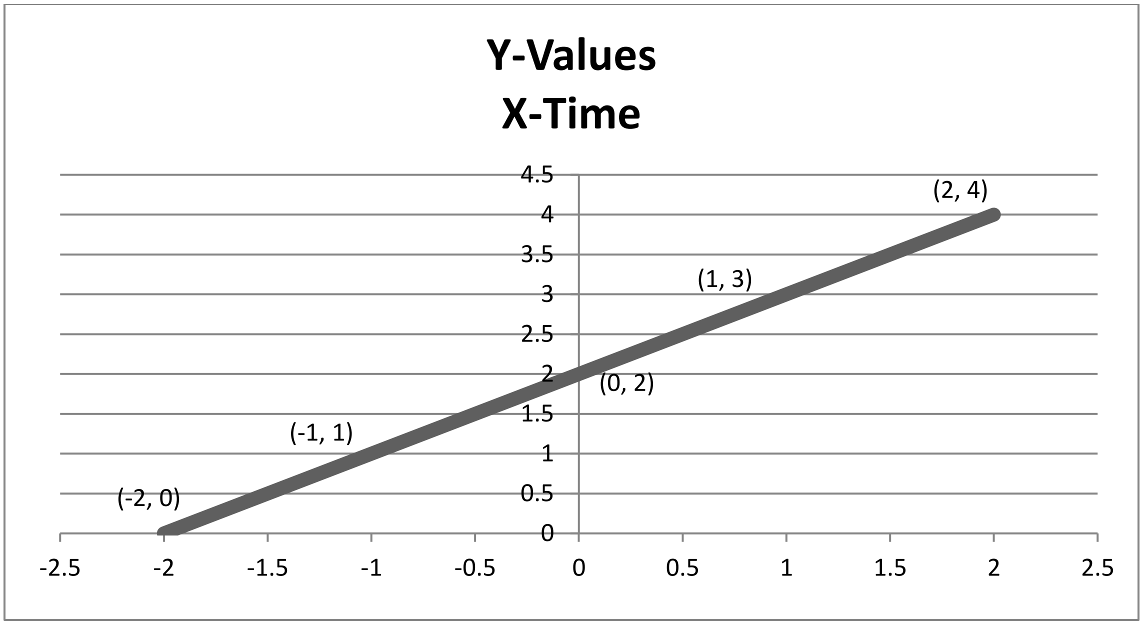
$$\boldsymbol{x(t) = r(t+2) - r(t+1) - r(t-1) + r(t-2)}$$

<u>*Solve:*</u>

$r(t+2) = t+2;\ t \geq -2$ $\qquad$ $Here,\ t+2=0$

$\qquad = 0;\ \ t < -2$ $\qquad$ $\therefore t = -2$

t	-2	-1	0	1	2	3
$r(t+2) = t+2$	0	1	2	3	4	5

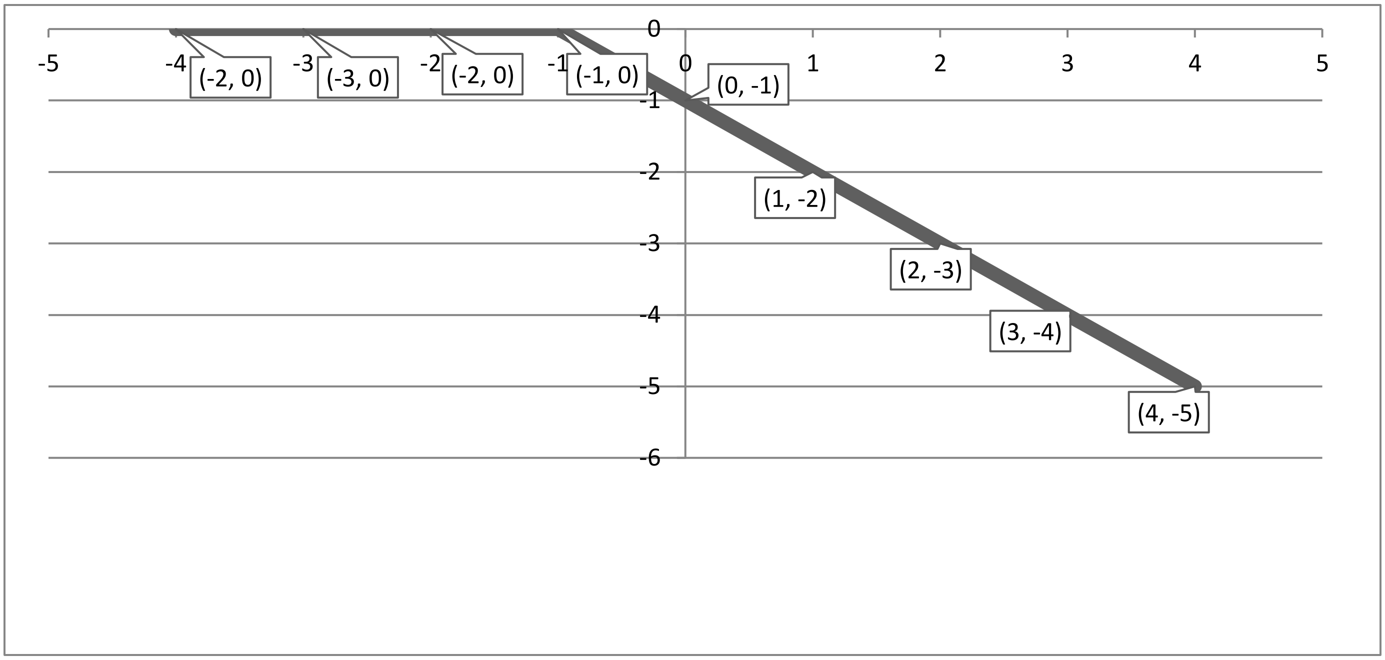


Again,

$-r(t+1) = -(t+1); t \geq -1$ $\qquad$ $Here, t+1=0$

$\qquad = 0; \;\; t < -1$ $\qquad$ $\therefore t = -1$

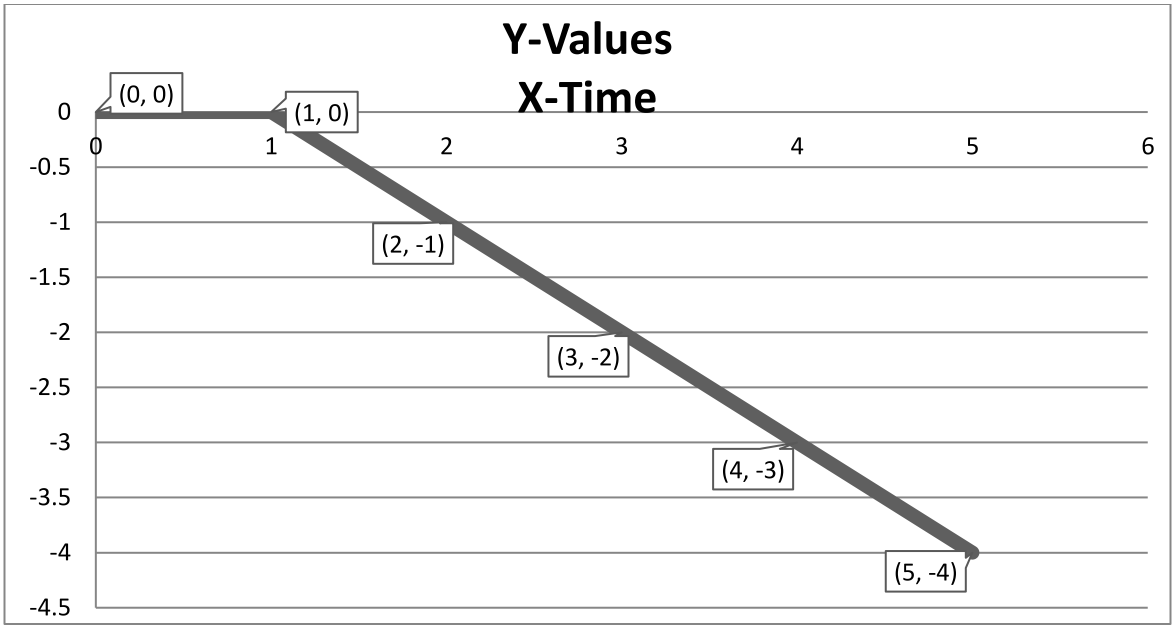
t	-1	0	1	2	3	4
$-r(t+1) = -(t+1)$	0	-1	-2	-3	-4	-5



$$\therefore -r(t-1) = -(t-1) \quad ; t \geq 1 \qquad\qquad Here,\ t-1=0$$

$$= 0 \quad ; t < 1 \qquad\qquad \therefore t = 1$$

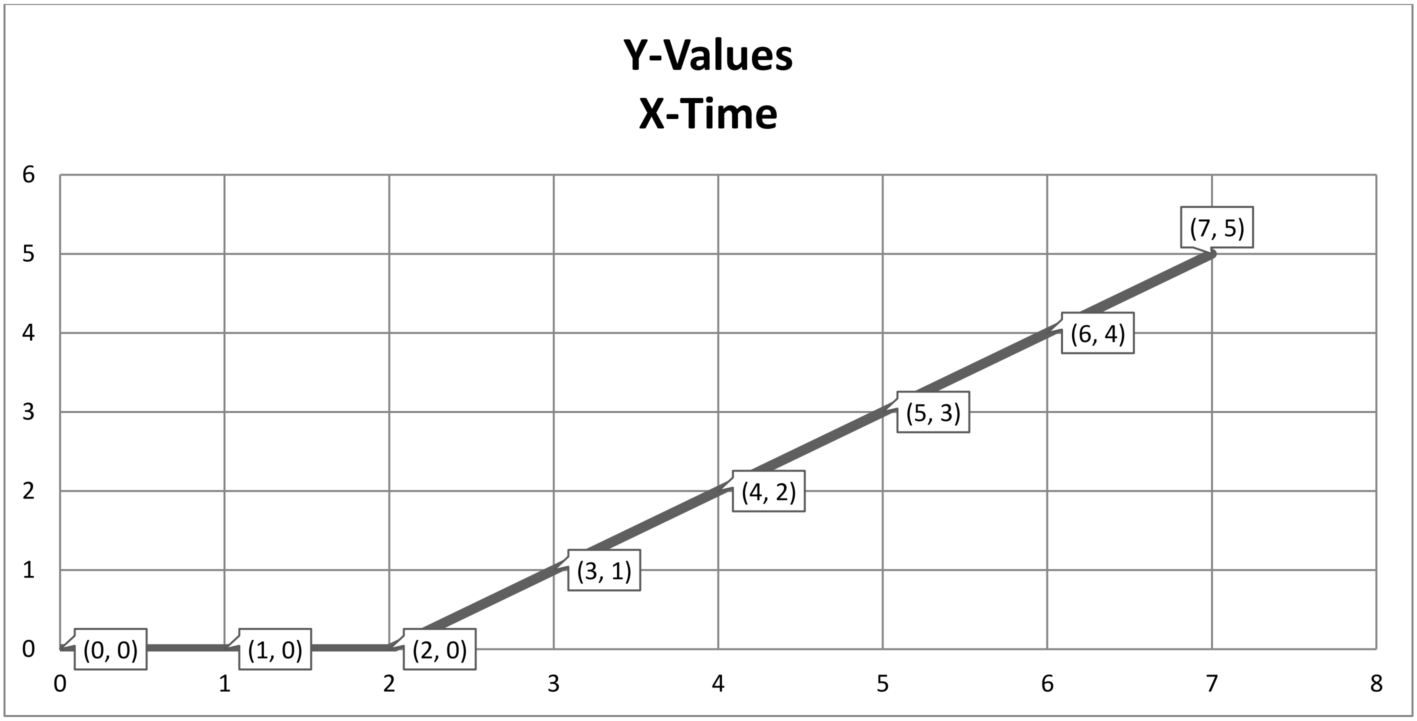
t	1	2	3	4	5
$-r(t-1) = -(t-1)$	0	-1	-2	-3	-4



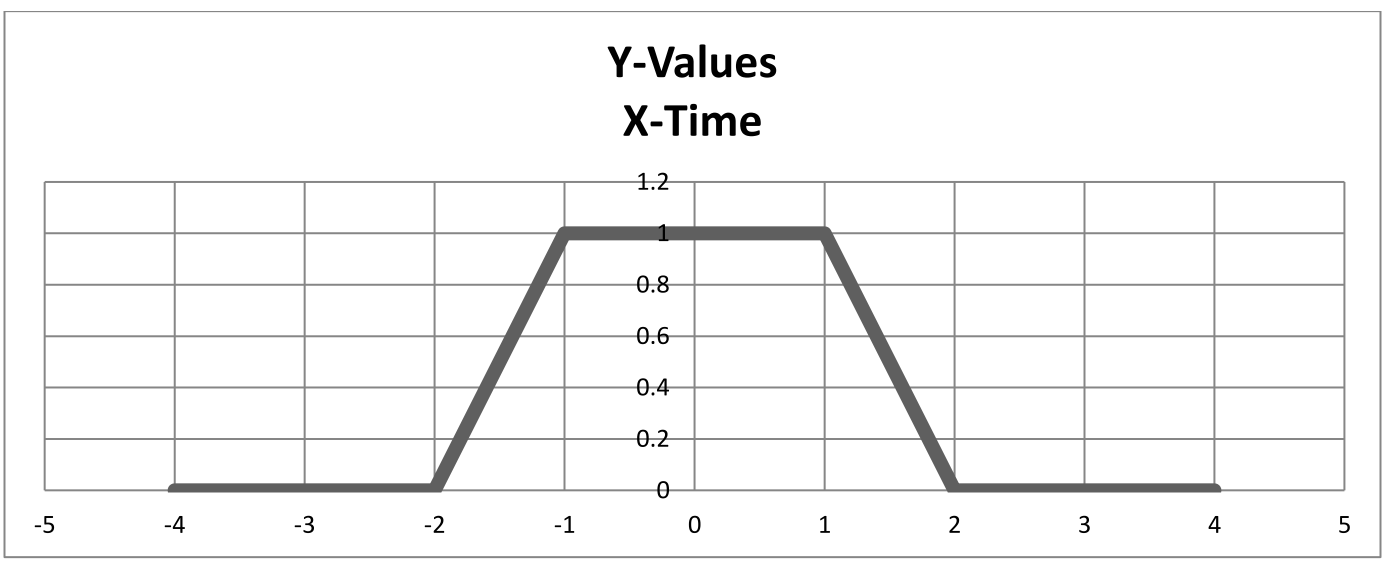
$$r(t-2) = (t-2) \quad ; t \geq 2$$
$$= 0 \quad ; t < 2$$

$Here, t - 2 = 0$
$\therefore t = 2$

t	2	3	4	5	6	7
$r(t-2) = t-2$	0	1	2	3	4	5



$$\therefore x(t) = r(t+2) - r(t+1) - r(t-1) + r(t-2)$$



<u>Problem 31: Impulse Function</u>

The Impulse function $\delta(t)$ is defined as

$$\delta(t) = 1 \qquad t = 0$$
$$= 0 \qquad \text{Otherwise}$$

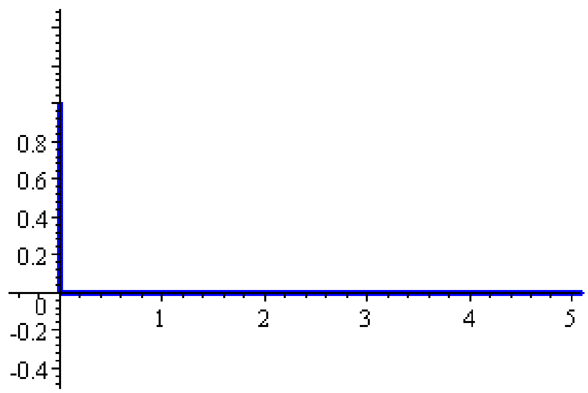
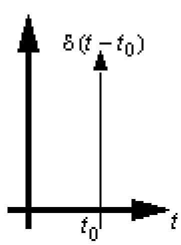


Figure 106



Delta (Impulse) Function

Example 91: Find $L[u(t-a)] = \dfrac{e^{-as}}{s}$

Proof: We have

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

Here, $f(t) = u(t-a)$

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

$$L(u(t-a)) = \int_0^\infty u(t-a)e^{-st}dt$$

$$L(u(t-a)) = \int_0^a 0.e^{-st}dt + \int_a^\infty 1.e^{-st}dt$$

[The unit function $u(t-a)$ is defined as

$$\begin{aligned} u(t) &= 1 \qquad t \geq a \\ &= 0 \qquad t < a \end{aligned}$$

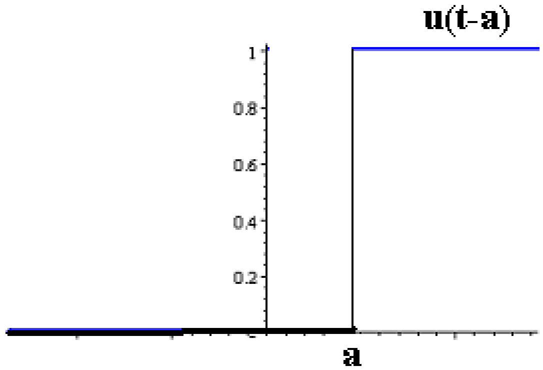


Figure 107

$$L(u(t-a)) = \int_0^a 0.e^{-st}dt + \int_a^\infty 1.e^{-st}dt$$

$$= 0 + \int_a^\infty 1.e^{-st}dt$$

$$= \int_a^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= -\frac{1}{s}\left[e^{-s\times\infty} - e^{-s\times a}\right]$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-as}\right]$$

$$= -\frac{1}{s}\left[\frac{1}{e^{\infty}} - e^{-as}\right]$$

$$= -\frac{1}{s}\left[\frac{1}{\infty} - e^{-as}\right]$$

$$= -\frac{1}{s}\left[0 - e^{-as}\right]$$

$$= +\frac{1}{s}\left[e^{-as}\right]$$

$$= \frac{e^{-as}}{s}$$

$$\therefore L(u(t-a)) = \frac{e^{-as}}{s} \text{ (Proved)}$$

$$\therefore L(u(t-2)) = \frac{e^{-2s}}{s}$$

Example 92: Express the following function in terms of unit step functions and find its Laplace transform:

$$f(t) = \begin{cases} 8\,; & t < 2 \\ 6\,; & t > 2 \end{cases}$$

Solution:

We have

$$L(u(t-a)) = \frac{e^{-as}}{s}$$

$$\therefore L(u(t-2)) = \frac{e^{-2s}}{s} -------------(i)$$

Given,

$$f(t) = \begin{cases} 8\,; & t < 2 \\ 6\,; & t > 2 \end{cases}$$

$$f(t) = \begin{cases} 8+0; & t < 2 \\ 8-2\,; & t > 2 \end{cases}$$

$$f(t) = 8 + \begin{cases} 0 \;; & t < 2 \\ -2 \;; & t > 2 \end{cases}$$

$$f(t) = 8 + (-2)\begin{cases} 0 \;; & t < 2 \\ 1 \;; & t > 2 \end{cases}$$

$$f(t) = 8 + (-2)\begin{cases} 1 \;; & t > 2 \\ 0 \;; & t < 2 \end{cases}$$

$$f(t) = 8 + (-2)u(t-2)$$

$$f(t) = 8 - 2u(t-2)$$

$$L\{f(t)\} = L\{8 - 2u(t-2)\}$$

$$L\{f(t)\} = L\{8\} - 2Lu(t-2)\}$$

$$L\{f(t)\} = 8L\{1\} - 2Lu(t-2)\}$$

$$L\{f(t)\} = 8 \times \frac{1}{s} - 2 \times \frac{e^{-2s}}{s}$$

$$[\because L(1) = \frac{1}{S}(\textit{ from example} 55)] \;\&\; L[u(t-2)] = \frac{e^{-2s}}{s}(\textit{ from example } 89)]$$

Problem 32:

Product of $\mathbf{u(t)}$ vs. Shifting the Function Along the *t*-axis

1. $\mathbf{f(t).u(t)}$

The $\mathbf{f(t)}$ part begins at $\mathbf{t=0}$

2. $\mathbf{f(t).u(t-a)}$

The $\mathbf{f(t)}$ part begins at $\mathbf{t=a}$

3. $\mathbf{f(t-a).u(t)}$

The $\mathbf{f(t)}$ part has been **shifted** to the right by *a* units and begins at $\mathbf{t=0}$

4. $\mathbf{f(t-a).u(t-a)}$

The $\mathbf{f(t)}$ part has been **shifted** to the right by *a* units and begins at $\mathbf{t=a}$

Example 93: $\mathbf{g(t)=sint.u(t)}$

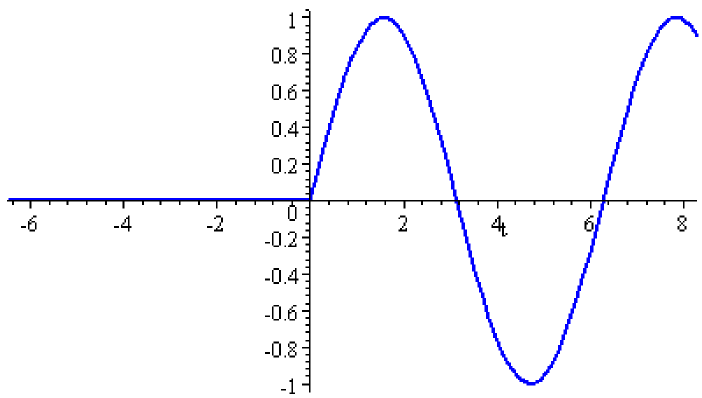


Figure 108: the $\mathbf{sint}$ part starts at $\mathbf{t=0}$

Example 94: $\mathbf{g(t)=sint.u(t-0.7)}$

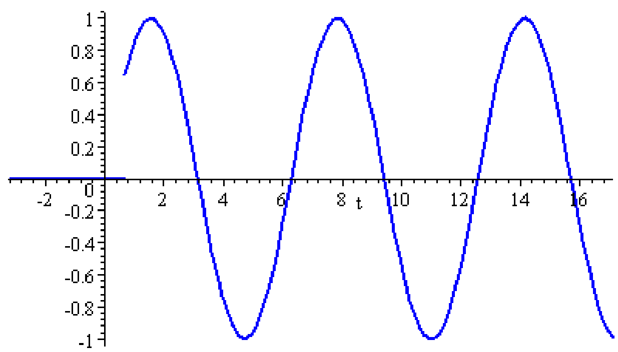


Figure 109: the $\mathbf{sint}$ part starts at $\mathbf{t=0.7}$

Example 95: $\mathbf{g(t)=sin(t-0.7).u(t)}$

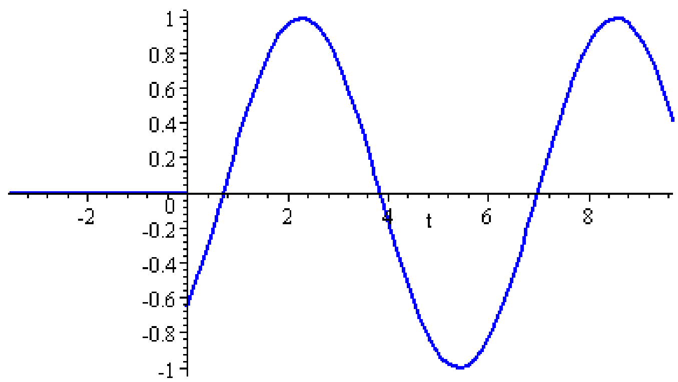


Figure 110: the $\mathbf{sint}$ part has been shifted 0.7 units to the right, and it starts at $\mathbf{t = 0}$

Example 96: $\mathbf{g(t) = sin(t - 0.7).u(t - 0.7)}$

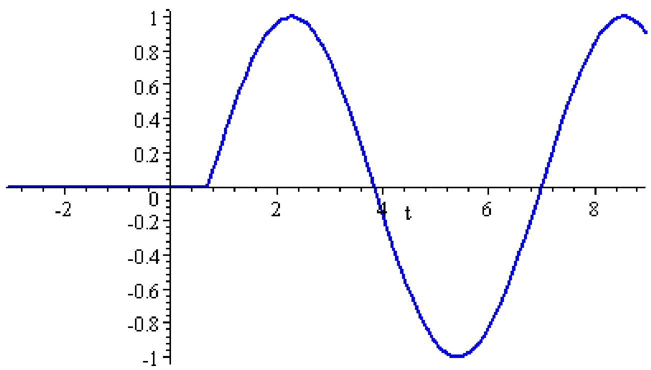


Figure 111: the $\mathbf{sint}$ part has been shifted 0.7 units to the right, and it starts at $\mathbf{t = 0.7}$

Example 97: If $\mathbf{f(t) = sint}$ then the graph of $\mathbf{g(t) = sint.u(t - 2\pi)}$ is

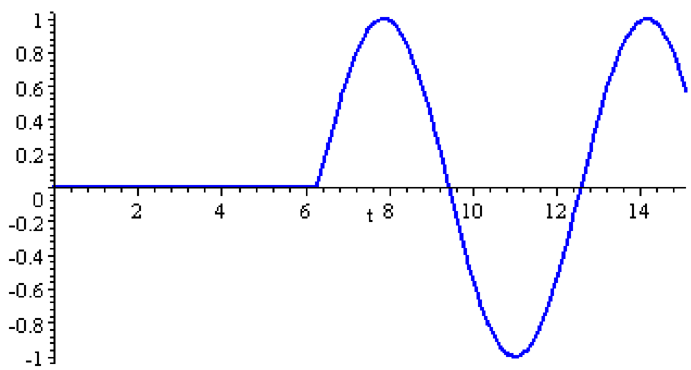


Figure 112: The $\mathbf{sint}$ portion starts at $\mathbf{t = 2\pi}$ because we have multiplied $\mathbf{sint}$ by $\mathbf{u(t - 2\pi)}$

Example 98: If $\mathbf{f(t) = 10e^{-2t}}$, then the graph of $\mathbf{g(t) = 10e^{-2t}.u(t - 5)}$ is

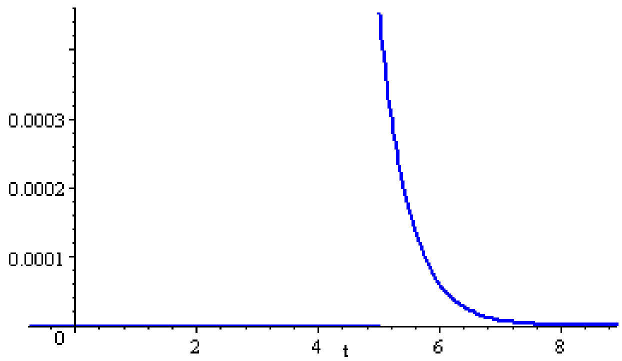


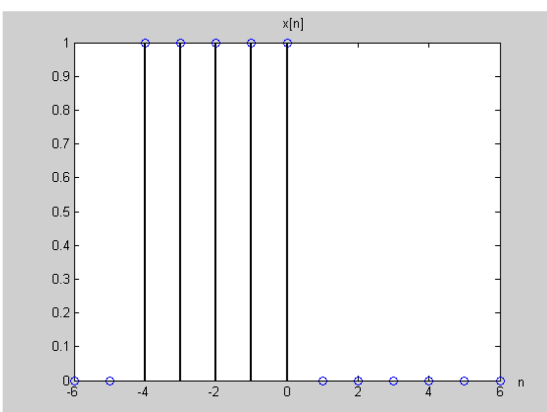
Figure 113: The portion $\mathbf{10e^{-2t}}$ starts at $\mathbf{t=5}$

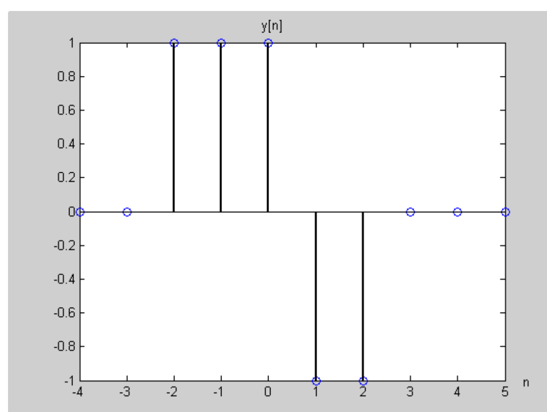
Home Task:

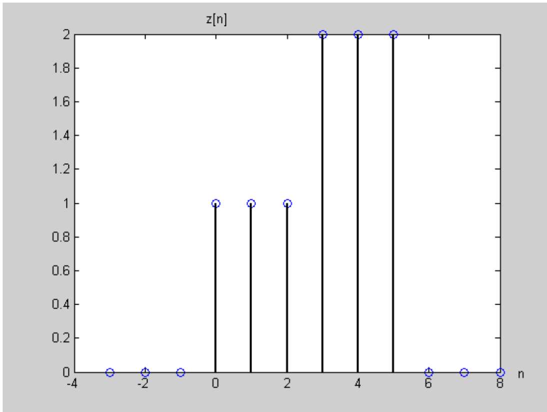
Draw the graph of $\mathbf{f(t)=4u(t)-8u(t-1)+4u(t-2)}$

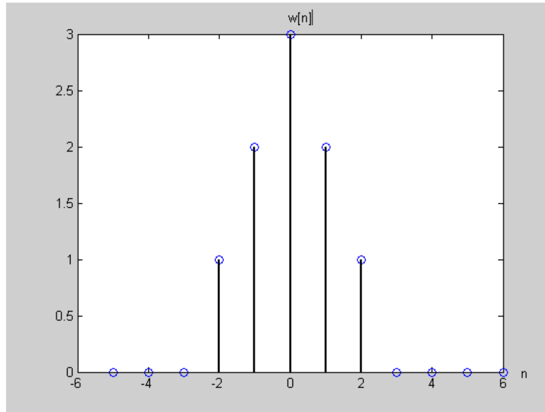
1. Consider the discrete-time signals depicted in the following figures. Evaluate the convolution sums indicated below:

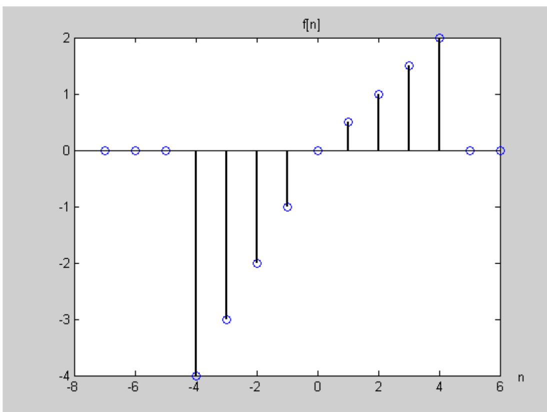
1. m[n] = x[n]*z[n]
2. m[n] = x[n]*y[n]
3. m[n] = x[n]*f[n]
4. m[n] = x[n]*g[n]
5. m[n] = y[n]*z[n]
6. m[n] = y[n]*g[n]
7. m[n] = y[n]*w[n]
8. m[n] = y[n]*f[n]
9. m[n] = z[n]*g[n]
10. m[n] = w[n]*g[n]
11. m[n] = f[n]*g[n]

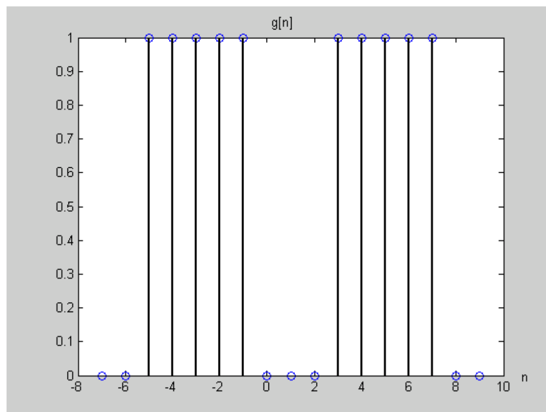






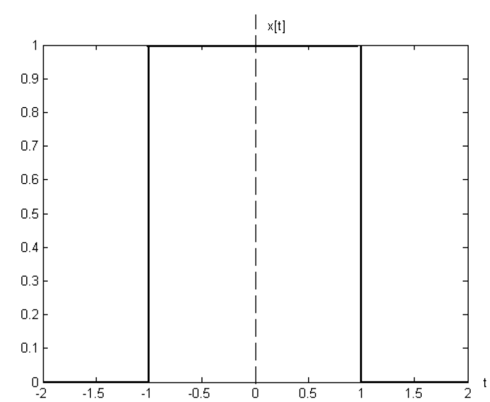


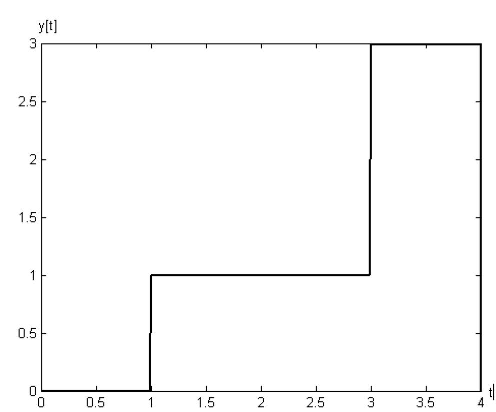


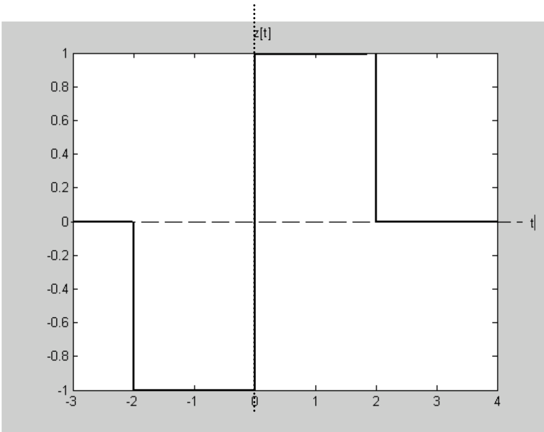


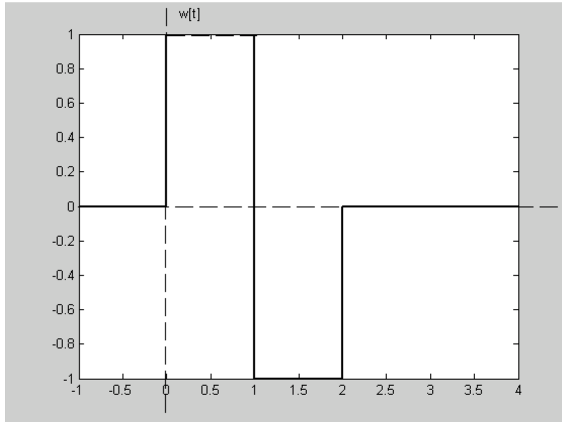
2. Consider the Continuous-time signals depicted in the following figures. Evaluate the convolution Integrals indicated below:

i) m(t)=x(t)*y(t)
ii) m(t)=x(t)*z(t)
iii) m(t)=y(t)*z(t)
iv) m(t)=y(t)*w(t)









3. Sketch the waveforms of the following signals:

a) x(t) = u(t)-u(t-2)

b) x(t) = u(t+1)-2u(t)+u(t-1)

c) $\boldsymbol{x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)}$

d) $\mathbf{x(t) = 4u(t-1) - 8u(t-4) + 4u(t-6)}$

e) $\mathbf{x(t) = r(t-1) + u(t) - u(t-1) - 2r(t-2) + r(t-3)}$

f) $\mathbf{x(t) = r(t+1) - r(t) + r(t-2)}$

g) $\mathbf{x(t) = r(t+2) - r(t+1) - r(t-1) + r(t-2)}$

<u>Prblem 33: Inverese Laplaces transform</u>

Example 99:

$If\ L[f(t)] = F(S), Then\ L^{-1}[F(S)] = f(t)$

$Where\ L^{-1}\ is\ called\ the\ inverse\ Laplace\ transform\ operator.$

$(1) L^{-1}\left(\frac{1}{S}\right) = 1\left[\because (1) L[f(t)) = L(1) = \frac{1}{s}\right]$

$(2) L^{-1}\left(\frac{a}{s}\right) = a\left[\because L(f(t)) = L(a) = \frac{a}{s}\right]$

$(3) L^{-1}\left(\frac{1}{s^2}\right) = t\left[\because L(f(t)) = L(t) = \frac{1}{s^2}\right]$

$(4) L^{-1}\left(\frac{1}{s-a}\right) = e^{at}\left[\because L(f(t)) = L(e^{at}) = \frac{1}{s-a}\right]$

$(5) L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}\left[\because\ L(f(t)) = L(e^{2t}) = \frac{1}{s-2}\right]$

$(6) L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at[\because\ L(f(t)) = L(\sin at) = \frac{a}{s^2+a^2}]$

$(7) L^{-1}\left(\frac{2}{S^2+2^2}\right) = \sin 2t[\because\ L(f(t)) = L(\sin 2t) = \frac{2}{s^2+2^2}]$

$(8) L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at[\because\ L(f(t)) = L(\cos at) = \frac{s}{s^2+a^2}]$

$(9) L^{-1}\left(\frac{s}{s^2+2^2}\right) = \cos 2t [\therefore\ L\big(f(t)\big) = L(\cos 2t) = \frac{s}{s^2+2^2}]$

$(10) L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sin hat [\therefore\ L\big(f(t)\big) = L(\sin hat) = \frac{a}{s^2-a^2}]$

$(11) L^{-1}\left(\frac{2}{S^2-2^2}\right) = \sin h2t [\therefore\ L\big(f(t)\big) = L(\sin h2t) = \frac{2}{s^2-2^2}]$

$(12) L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cos hat [\therefore\ L\big(f(t)\big) = L(\cos hat) = \frac{s}{s^2-a^2}]$

$(13) L^{-1}\left(\frac{s}{s^2-2^2}\right) = \cos h2t [\therefore\ L\big(f(t)\big) = L(\cos h2t) = \frac{s}{s^2-2^2}]$

Example 100 Find the inverse Laplace transforms of the following:

(i) $\frac{1}{s-2}$, (ii) $\frac{s}{s^2-16}$, (iii) $\frac{5}{s^2+25}$, (iv) $\frac{1}{(s+3)^2-4}$, (v) $\frac{1}{2s-7}$

Answers:

(i) Given $f(s) = \frac{1}{s-2}$

We have, $\therefore L(f(t)) = L(e^{at}) = \frac{1}{s-a}$ **[Example 58]**

$$\therefore e^{at} = L^{-1}\left(\frac{1}{s-a}\right)$$

$$\therefore e^{2t} = L^{-1}\left(\frac{1}{s-2}\right)$$

$$\therefore L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$$ Answer

(ii) $L^{-1}\left(\frac{s}{s^2-16}\right) = ?$

We have, $\therefore L(f(t)) = \mathrm{L}(\mathbf{coshat}) = \frac{\mathbf{s}}{\mathbf{s^2-a^2}}$ **[Example 63]**

$$\therefore \cosh at = L^{-1}\left(\frac{s}{s^2-a^2}\right)$$

$$\therefore \cosh 4t = L^{-1}\left(\frac{s}{s^2-4^2}\right)$$

$$\therefore L^{-1}\left(\frac{s}{s^2-4^2}\right) = \cosh 4t$$ Answer

(iii) We have, $\therefore L(f(t)) = L\ (\sin at) = \frac{a}{s^2+a^2}$ **[Example 60]**

$$\therefore \sin at = L^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$\therefore \sin 5t = L^{-1}\left(\frac{5}{s^2+5^2}\right)$$

$$\therefore L^{-1}\left(\frac{5}{s^2+5^2}\right) = \sin 5t$$

$$\therefore L^{-1}\left(\frac{5}{s^2+5^2}\right) = \sin 5t \qquad \textit{Answer}$$

Example 101:

Find inverse Laplace transform of: $\frac{s+4}{s(s-1)(s-2)}$

Solution:

Let,

$$\frac{s+4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (i)$$

Multiplying by $s(s-1)(s-2)$ in both sides

$$\Rightarrow \frac{s+4}{s(s-1)(s-2)} \times s(s-1)(s-2) = A\frac{s(s-1)(s-2)}{s} + B\frac{s(s-1)(s-2)}{(s-1)} + C\frac{s(s-1)(s-2)}{(s-2)}$$

$$\Rightarrow s+4 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1) \quad \ldots\ldots\ldots\ldots \ (ii)$$

Put $s=0$ in equation (ii),

$\Rightarrow 0+4 = A(0-1)(0-2) + B\times 0(0-2) + C\times 0(0-1)$

$\Rightarrow 4 = 2A$

$\therefore A = 2$

Put $s-1=0$, i.e. $s=1$ in equation (ii),

$\Rightarrow 1+4 = A(1-1)(1-2) + B\times 1(1-2) + C\times 1(1-1)$

$\Rightarrow 5 = 0 - B + 0$

$\therefore B = -5$

Put $s-2=0$, i.e. $s=2$ in equation (ii),

$\Rightarrow 2+4 = A(2-1)(2-2) + B\times 2(2-2) + C\times 2(2-1)$

$\Rightarrow 6 = 0 + 0 + C(4-2)$

$\Rightarrow 6 = 0 + 0 + 2C$

$\therefore C = 3$

Putting the value of A, B, C in equation (i), we get,

$$\frac{s+4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$$

$$\frac{s+4}{s(s-1)(s-2)}=\frac{2}{s}+\frac{-5}{(s-1)}+\frac{3}{(s-2)}$$

$$\therefore\ L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right)=L^{-1}\left(\frac{2}{s}\right)+L^{-1}\left(\frac{-5}{s-1}\right)+L^{-1}\left(\frac{3}{s-2}\right)$$

$$L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right)=2L^{-1}\left(\frac{1}{s}\right)-5L^{-1}\left(\frac{1}{s-1}\right)+3L^{-1}\left(\frac{1}{s-2}\right)--------(iii)$$

Since

01. We have $\therefore L(f(t))=L(1)=\dfrac{1}{s}$ [Example 55]

$$\therefore 1=L^{-1}(\frac{1}{s})$$

$$\therefore L^{-1}(\frac{1}{s})=1$$

02. We have, $\therefore L(f(t))=L(e^{at})=\dfrac{1}{s-a}$ [Example 58]

$$\therefore e^{at}=L^{-1}(\frac{1}{s-a})$$

$$\therefore e^{t}=L^{-1}(\frac{1}{s-1})$$

$$\therefore L^{-1}(\frac{1}{s-1})=e^{t}$$

03. $\therefore L(f(t))=L(e^{at})=\dfrac{1}{s-a}$ [Example 58]

$$\therefore e^{at}=L^{-1}(\frac{1}{s-a})$$

$$\therefore e^{2t}=L^{-1}(\frac{1}{s-2})$$

$\therefore L^{-1}(\dfrac{1}{s-2})=e^{2t}$ Answer

Putting these values in (iii), we get

$$L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right)=2L^{-1}\left(\frac{1}{s}\right)-5L^{-1}\left(\frac{1}{s-1}\right)+3L^{-1}\left(\frac{1}{s-2}\right)$$

$=2.1-5e^{t}+3e^{2t}$ Answer

Convolution Sum

The following steps are to be taken

i. Folding
ii. Shifting
iii. Multiplication
iv. Summation

1st times:

i. Folding
ii. Multiplication
iii. Summation

2nd times and more

i. Shifting
ii. Multiplication
iii. Summation

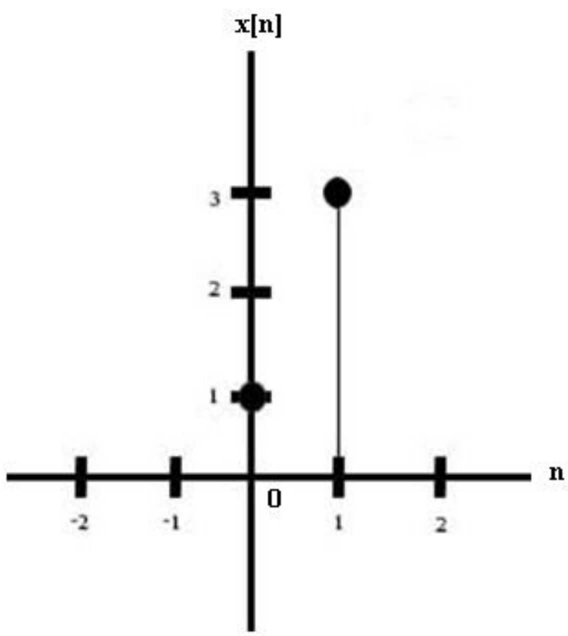
Example 102:

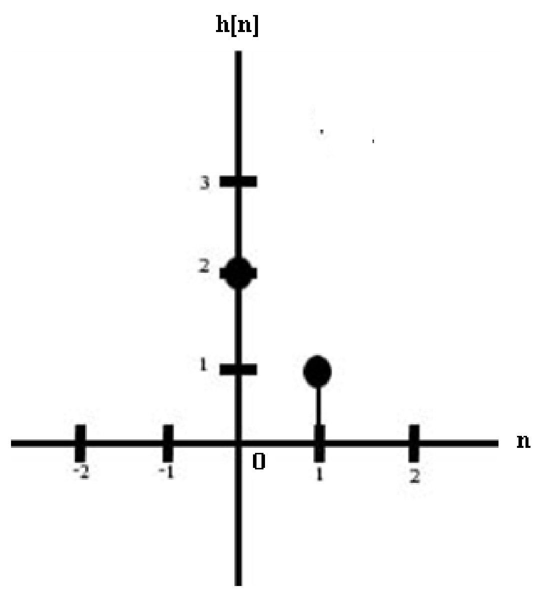
Evaluate the convolution sums of y [n] = x [n]*h [n]

Where,

$$\mathbf{x[n] = 1, \ n = 0 \quad 3, \ n = 1} \quad \text{and} \quad \mathbf{h[n] = 2, \ n = 0 \quad 1, \ n = 1}$$; ***n represents the time index***

Solution:





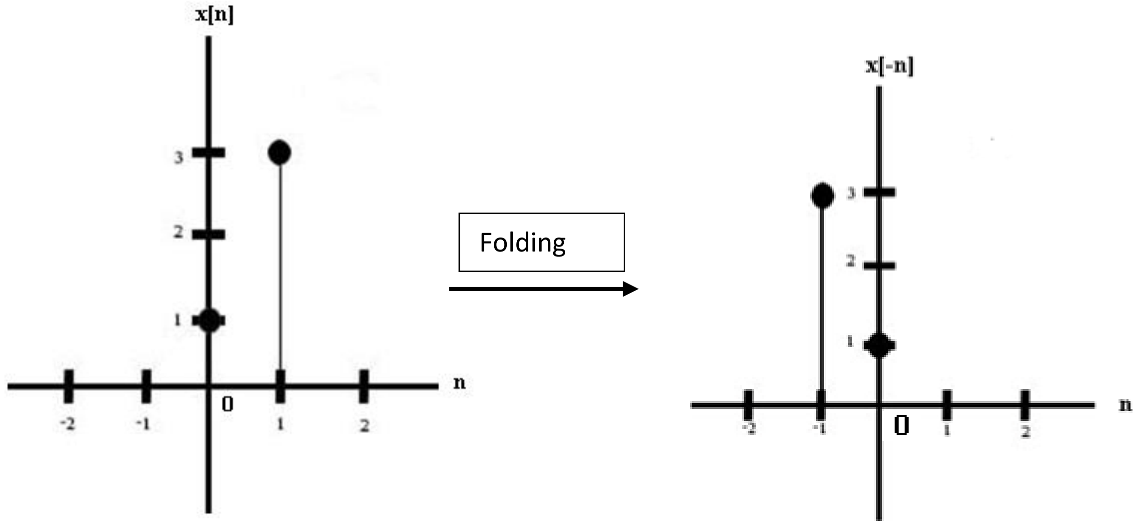
1st time:

(i). Folding:

$$x[n] = x[-n]$$

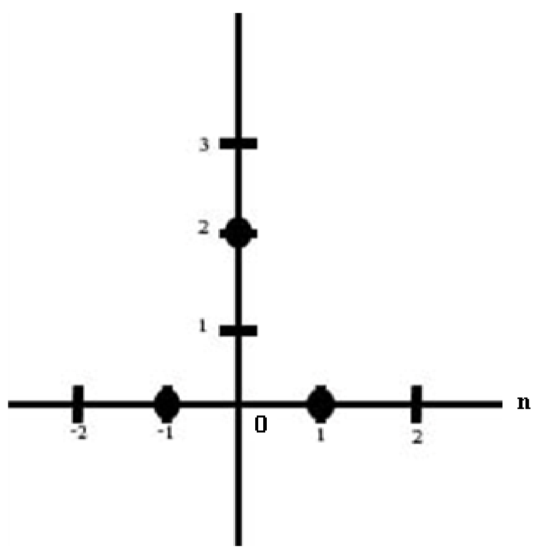
i.e. $x[0] = x[0]$

$x[1] = x[-1]$



(ii). Multiplication:

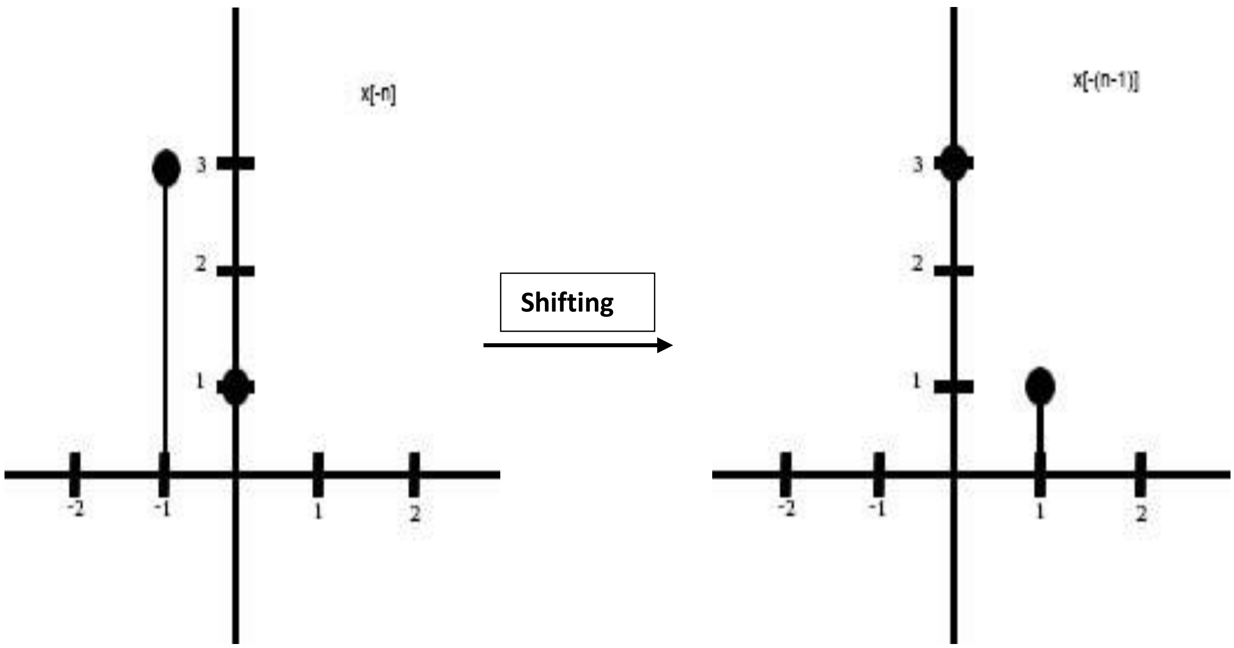
$$x[-n] * h[n]$$



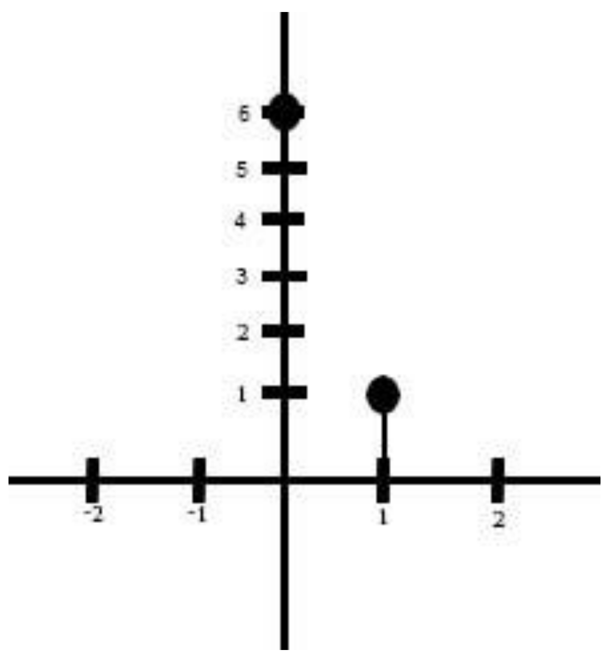
(iii). Summation: y[0] = 0 + 2 + 0 = 2

2nd time:

(i). Shifting:



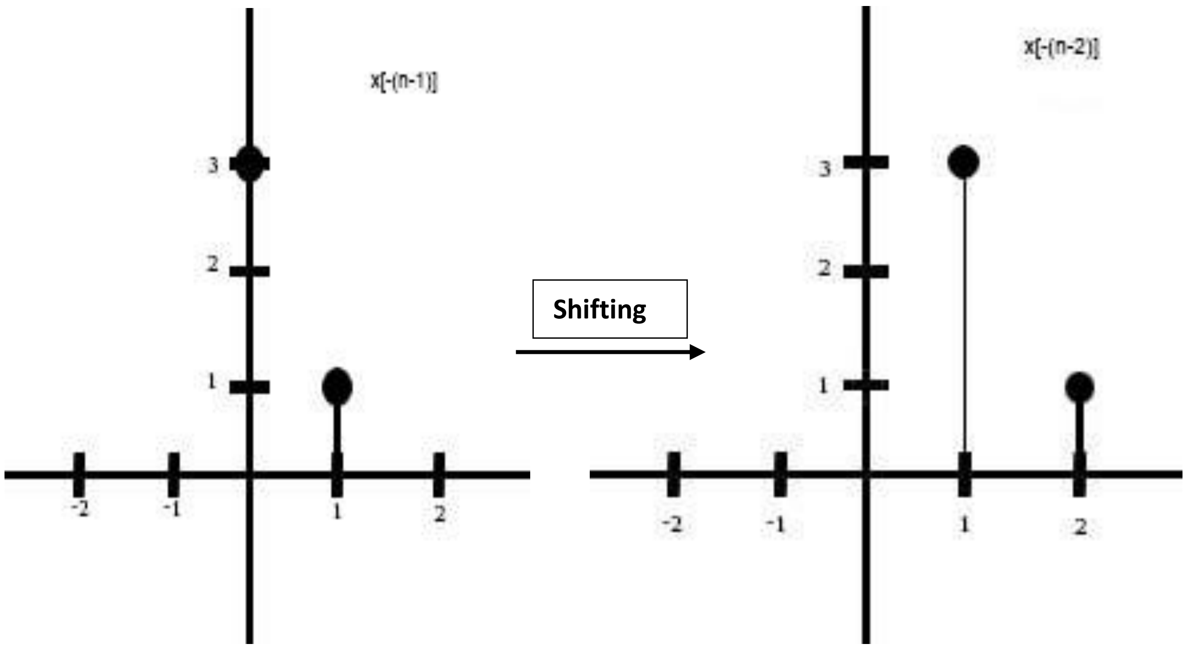
(ii) Multiplication: x[-(n-1)] * h[n]



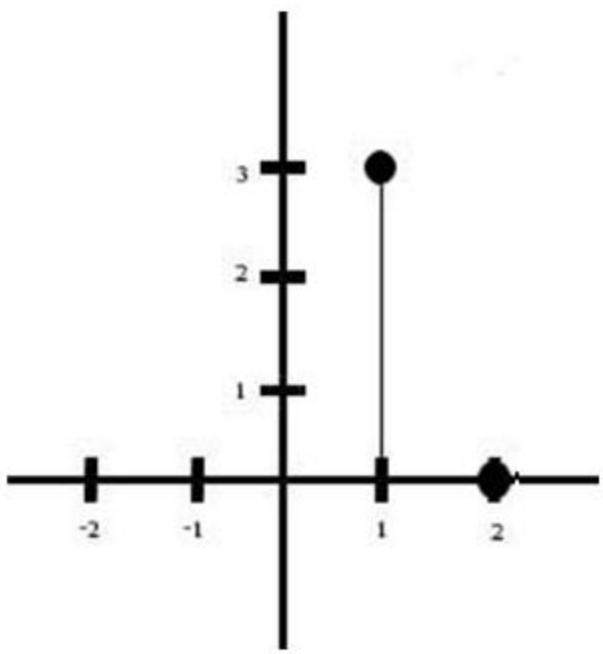
(iii). Summation: y[1] = 0 + 6 + 1 + 0 = 7

3rd time:

(i). Shifting:



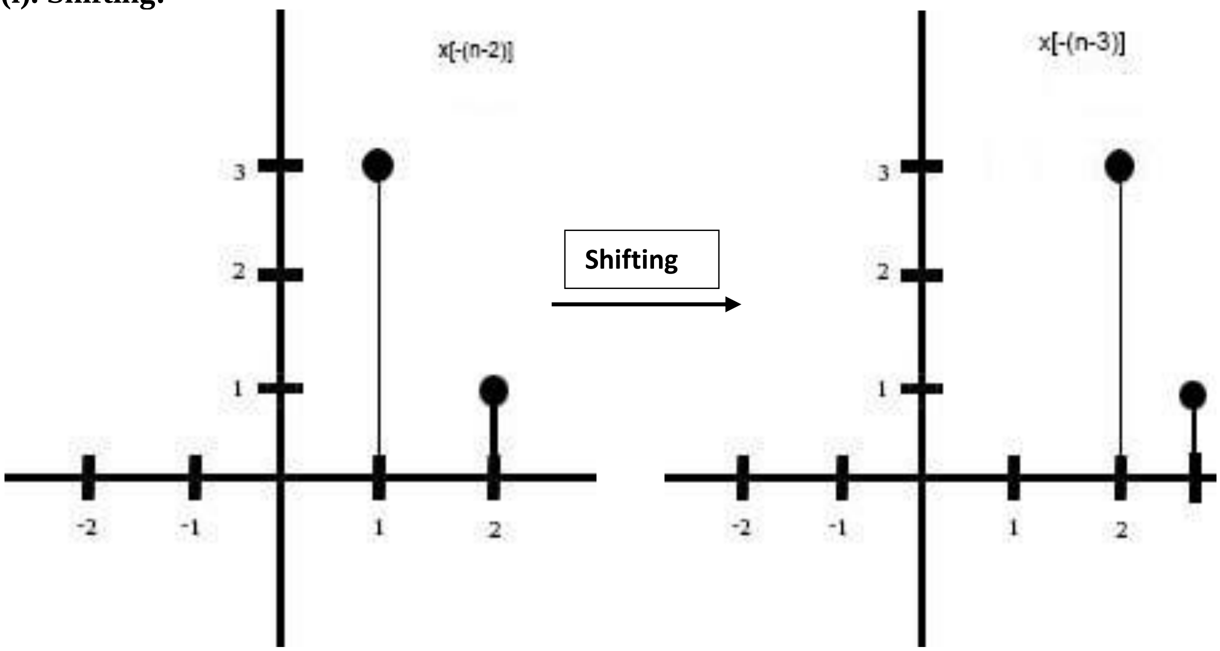
(ii). Multiplication: x[-(n-2)] * h[n]



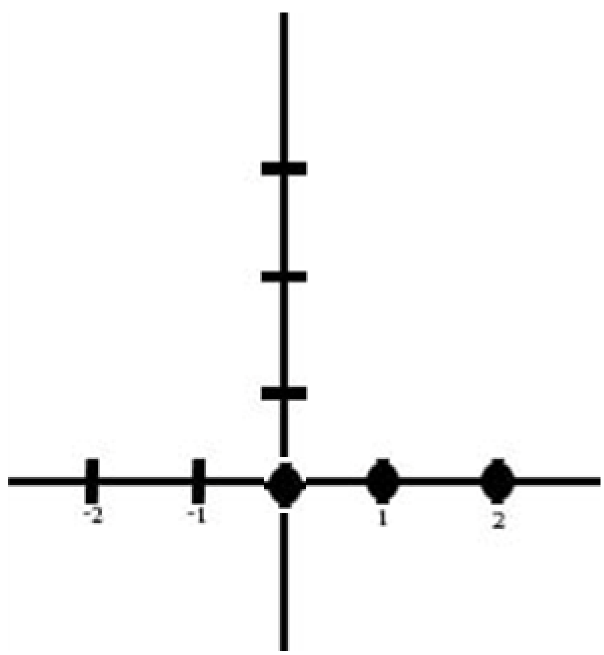
(iii). Summation: y [2] = 0 + 0 + 3 + 0 = 3

4th time:

(i). Shifting:



(ii). Multiplication: x[-(n-3)] * h[n]



iii) Summation: y[3] = 0+0+0+0=0
Finally we get,
∴ y[0] = 2
   y[1] = 7
   y[2] = 3

∴ Convolution Sum: y[n] = [2 7 3]

Application to Differential Equations

Q-103: Solve $Y'' + Y = t$ $\quad Y(0) = 1$ $\quad Y'(0) = -2$

Solution

Let $Y = f(t)$

$Y' = f'(t)$

$Y'' = f''(t)$

Given,

$Y'' + Y = t$

That is $\dfrac{d^2Y}{dt^2} + Y = t$

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$Y'' + Y = t$

$L\{Y''\} + L\{Y\} = L\{t\}$

We have,

$L(f''(t)) = s^2 L\{f(t)\} - s\,f(0) - f'(0)$

$L\{Y''\} + L\{Y\} = L\{t\}$

$s^2 L\{Y\} - sf(0) - f'(0) + L\{Y\} = L\{t\}$

$s^2 y - sf(0) - f'(0) + y = L\{t\}$ $\qquad [\text{let}, L\{Y\} = y]$

$s^2 y - sf(0) - f'(0) + y = \dfrac{1}{s^2}$ $\qquad [\text{let}, L\{t\} = \dfrac{1}{s^2}]$

$s^2 y - s.1 - (-2) + y = \dfrac{1}{s^2}$ $\qquad [\text{Given}, Y(0) = f(0) = 1 \quad Y'(0) = f'(0) = -2]$

$s^2 y + y - s.1 + 2 = \dfrac{1}{s^2}$

$s^2 y + y - s.1 + 2 - \dfrac{1}{s^2} = 0$

$y(s^2 + 1) - s + 2 - \dfrac{1}{s^2} = 0$

$y(s^2 + 1) = s - 2 + \dfrac{1}{s^2}$

$y = \dfrac{s-2}{s^2+1} + \dfrac{1}{s^2(s^2+1)}$

$y = \dfrac{s}{s^2+1} - \dfrac{2}{s^2+1} + \dfrac{1}{s^2} - \dfrac{1}{(s^2+1)}$

$y = \dfrac{s}{s^2+1} + \dfrac{1}{s^2} - \dfrac{3}{s^2+1}$

$\therefore L\{Y\} = y = \dfrac{s}{s^2+1} + \dfrac{1}{s^2} - \dfrac{3}{s^2+1}$

$$\therefore Y = L^{-1}(y) = L^{-1}[\frac{s}{s^2+1} + \frac{1}{s^2} - \frac{3}{s^2+1}]$$

$$\therefore Y = L^{-1}(y) = L^{-1}[\frac{s}{s^2+1}] + L^{-1}[\frac{1}{s^2}] - 3L^{-1}[\frac{1}{s^2+1}]$$

$\therefore Y = L^{-1}(y) = \cos t + t - 3\sin t$ Answer

Proof:

$\therefore Y = \cos t + t - 3\sin t$

$\therefore Y' = -\sin t + 1 - 3\cos t$

$\therefore Y'' = -\cos t + 0 + 3\sin t$

$\therefore Y'' + Y = -\cos t + 0 + 3\sin t + \cos t + t - 3\sin t$

$\therefore Y'' + Y = t$

Again,

$\therefore Y = \cos t + t - 3\sin t$

$\therefore Y(0) = \cos 0 + 0 - 3\sin 0$

$\therefore Y(0) = 1$

Again

$\therefore Y' = -\sin t + 1 - 3\cos t$

$\therefore Y'(0) = -\sin 0 + 1 - 3\cos 0$

$\therefore Y'(0) = 0 + 1 - 3.1$

$\therefore Y'(0) = -2$

Q-104: Solve $Y'' + 4Y = 12t \quad Y(0) = 0 \quad Y'(0) = 7$

Solution

Given,

$Y'' + 4Y = 12t$

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$Y'' + 4Y = 12t$

$L\{Y''\} + 4L\{Y\} = 12L\{t\}$

We have,

$L(f''(t)) = s^2 L\{f(t)\} - s\, f(0) - f'(0)$

$L\{Y''\} + 4L\{Y\} = 12L\{t\}$

$s^2 L\{Y\} - sf(0) - f'(0) + 4L\{Y\} = 12L\{t\}$

$s^2 y - sf(0) - f'(0) + 4y = 12L\{t\}$ $\quad$ $[\text{let}, L\{Y\} = y]$

$s^2 y - sf(0) - f'(0) + 4y = 12\frac{1}{s^2}$ $\quad$ $[\text{let}, L\{t\} = \frac{1}{s^2}]$

$$s^2y - s.0 - 7 + 4y = 12\frac{1}{s^2}$$ [Given, $Y(0) = 0$ $Y^{'}(0) = 7$]

$$s^2y - 7 + 4y = \frac{12}{s^2}$$

$$s^2y + 4y - 7 - \frac{12}{s^2} = 0$$

$$y(s^2 + 4) - 7 - \frac{12}{s^2} = 0$$

$$y(s^2 + 4) = 7 + \frac{12}{s^2}$$

$$y = \frac{7}{(s^2 + 4)} + \frac{12}{s^2(s^2 + 4)}$$

$$y = \frac{7}{(s^2 + 4)} + 3 \times \frac{4}{s^2(s^2 + 4)}$$

$$y = \frac{7}{(s^2 + 4)} + 3 \times [\frac{1}{s^2} - \frac{1}{(s^2 + 4)}]$$

$$y = \frac{7}{(s^2 + 4)} + [\frac{3}{s^2} - \frac{3}{(s^2 + 4)}]$$

$$\therefore L\{Y\} = y = \frac{7}{(s^2 + 4)} + [\frac{3}{s^2} - \frac{3}{(s^2 + 4)}]$$

$$\therefore Y = L^{-1}(y) = L^{-1}[\frac{7}{(s^2 + 4)} + [\frac{3}{s^2} - \frac{3}{(s^2 + 4)}]]$$

$$\therefore Y = L^{-1}(y) = L^{-1}[\frac{7}{(s^2 + 4)}] + L^{-1}[\frac{3}{s^2} - \frac{3}{(s^2 + 4)}]$$

$$\therefore Y = L^{-1}(y) = 7L^{-1}[\frac{1}{(s^2 + 4)}] + 3L^{-1}[\frac{1}{s^2} - \frac{1}{(s^2 + 4)}]$$

$$\therefore Y = L^{-1}(y) = 7L^{-1}[\frac{1}{(s^2 + 4)}] + 3L^{-1}[\frac{1}{s^2}] - 3L^{-1}[\frac{1}{(s^2 + 4)}]$$

$$\therefore Y = L^{-1}(y) = 4L^{-1}[\frac{1}{(s^2 + 4)}] + 3L^{-1}[\frac{1}{s^2}]$$

$$\therefore Y = L^{-1}(y) = 4\frac{1}{2}L^{-1}[\frac{2}{(s^2 + 4)}] + 3L^{-1}[\frac{1}{s^2}]$$

$$\therefore Y = L^{-1}(y) = 4\frac{1}{2}\sin 2t + 3t$$

$$\therefore Y = L^{-1}(y) = 2\sin 2t + 3t$$

Q-105: Solve $Y^{''} - 3Y^{'} + 2Y = 4e^{2t}$ $Y(0) = -3$ $Y^{'}(0) = 5$

Solution

$Y = f(t)$

Given,

$$Y'' - 3Y' + 2Y = 4e^{2t}$$

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$$Y'' - 3Y' + 2Y = 4e^{2t}$$

$$L\{Y''\} - 3L\{Y'\} + 2L\{Y\} = 4L\{e^{2t}\}$$

We have,

$$L(f''(t)) = s^2 L\{f(t)\} - s\,f(0) - f'(0)$$

And

$$\therefore L(f'(t)) = sL\{f(t)\} - f(0)$$

$$\therefore L\{Y''\} - 3L\{Y'\} + 2L\{Y\} = 4L\{e^{2t}\}$$

$$s^2 L\{f(t)\} - sf(0) - f'(0) - 3[sL\{f(t)\} - f(0)] + 2y = 4\frac{1}{s-2} \quad [\text{let}, L\{Y\} = y \ \& \ L(e^{at}) = \frac{1}{s-a}]$$

$$s^2 L\{Y\} - sf(0) - f'(0) - 3[sL\{Y\} - f(0)] + 2y = 4\frac{1}{s-2} \quad [Y = f(t)]$$

$$s^2 y - sf(0) - f'(0) - 3[sy - f(0)] + 2y = 4\frac{1}{s-2} \quad [\text{let}, L\{Y\} = y]$$

$$s^2 y - sf(0) - f'(0) - 3sy + 3f(0) + 2y = 4\frac{1}{s-2}$$

$$s^2 y - s\{-3\} - 5 - 3sy + 3(-3) + 2y = 4\frac{1}{s-2}$$

$$s^2 y + 3s - 5 - 3sy - 9 + 2y = 4\frac{1}{s-2}$$

$$s^2 y + 3s - 3sy - 14 + 2y = 4\frac{1}{s-2}$$

$$s^2 y - 3sy + 2y + 3s - 14 = 4\frac{1}{s-2}$$

$$s^2 y - 3sy + 2y = -3s + 14 + 4\frac{1}{s-2}$$

$$y(s^2 - 3s + 2) = -3s + 14 + 4\frac{1}{s-2}$$

$$y(s^2 - 2s - s + 2) = -3s + 14 + 4\frac{1}{s-2}$$

$$y\{s(s-2) - 1(s-2) = -3s + 14 + 4\frac{1}{s-2}$$

$$y(s-1)(s-2) = -3s + 14 + 4\frac{1}{s-2}$$

$$y = -3\frac{s}{(s-1)(s-2)} + 14\frac{1}{(s-1)(s-2)} + 4\frac{1}{(s-2)}\frac{1}{(s-1)(s-2)}$$

$$y=\frac{-3s}{(s-1)(s-2)}+\frac{14}{(s-1)(s-2)}+\frac{4}{(s-1)(s-2)^2}$$
$$y=\frac{-3s+14}{(s-1)(s-2)}+\frac{4}{(s-1)(s-2)^2}$$
$$y=\frac{(-3s+14)(s-2)+4}{(s-1)(s-2)^2}$$
$$y=\frac{-3s^2+6s+14s-28+4}{(s-1)(s-2)^2}$$
$$y=\frac{-3s^2+6s+14s-24}{(s-1)(s-2)^2}$$

Applying partial fraction

Let,

$$y=\frac{-3s^2+6s+14s-24}{(s-1)(s-2)^2}=\frac{A}{(s-1)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}$$

……………………………

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$$y=\frac{-3s^2+6s+14s-24}{(s-1)(s-2)^2}=\frac{-7}{(s-1)}+\frac{4}{(s-2)}+\frac{4}{(s-2)^2}$$
$$\therefore L\{Y\}=y=\frac{-3s^2+6s+14s-24}{(s-1)(s-2)^2}=\frac{-7}{(s-1)}+\frac{4}{(s-2)}+\frac{4}{(s-2)^2}$$
$$\therefore Y=L^{-1}(y)=-7\frac{1}{2}L^{-1}[\frac{1}{(s-1)}]+4L^{-1}[\frac{1}{(s-2)}]+4L^{-1}[\frac{4}{(s-2)^2}]$$
$$\therefore Y=L^{-1}(y)=-7e^{t}+4e^{2t}+4te^{2t}$$

Q-106: Solve $Y''+9Y=\cos 2t$ $\qquad Y(0)=1$ $\qquad Y(\frac{\pi}{2})=-1$

Solution

$$Y=f(t)$$

Given,

$$Y''+9Y=\cos 2t$$

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$$Y''+9Y=\cos 2t$$
$$L\{Y''\}+L\{9Y\}=L\{\cos 2t\}$$

We have,

$$L(f''(t))=s^2L\{f(t)\}-s\,f(0)-f'(0)$$
$$\therefore L\{Y''\}+L\{9Y\}=L\{\cos 2t\}$$
$$\therefore L\{Y''\}+9L\{Y\}=L\{\cos 2t\}$$

$$s^2L\{f(t)\} - s\,f(0) - f^{'}(0) + 9y = \frac{s}{s^2+4} \qquad [\mathrm{let, L\{Y\} = y}]$$

$$s^2L\{Y\} - s\,f(0) - f^{'}(0) + 9y = \frac{s}{s^2+4}$$

$$s^2y - s\,f(0) - f^{'}(0) + 9y = \frac{s}{s^2+4} \qquad [\mathrm{let, L\{Y\} = y}]$$

$$s^2y - s.1 - c + 9y = \frac{s}{s^2+4} \qquad [let, f^{'}(0) = c]$$

$$y(s^2+9) - s.1 - c = \frac{s}{s^2+4}$$

$$y(s^2+9) = s.1 + c + \frac{s}{s^2+4}$$

$$y = \frac{s}{(s^2+9)} + \frac{c}{(s^2+9)} + \frac{s}{(s^2+4)(s^2+9)}$$

$$y = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{(s^2+9)} + \frac{c}{(s^2+9)}$$

……………………………………………

………………………………………

Applying partial fraction

$$y = \frac{s}{5(s^2+4)} - \frac{s}{5(s^2+9)} + \frac{s}{(s^2+9)} + \frac{c}{(s^2+9)}$$

$$y = \frac{s}{5(s^2+4)} + \frac{s}{(s^2+9)} - \frac{s}{5(s^2+9)} + \frac{c}{(s^2+9)}$$

$$y = \frac{s}{5(s^2+4)} + \frac{5s-s}{(s^2+9)} + \frac{c}{(s^2+9)}$$

$$y = \frac{s}{5(s^2+4)} + \frac{4s}{5(s^2+9)} + \frac{c}{(s^2+9)}$$

$$\therefore L\{Y\} = y = \frac{s}{5(s^2+4)} + \frac{4s}{5(s^2+9)} + \frac{c}{(s^2+9)}$$

$$\therefore Y = L^{-1}(y) = L^{-1}\{\frac{s}{5(s^2+4)}\} + L^{-1}\{\frac{4s}{5(s^2+9)}\} + L^{-1}\{\frac{c}{(s^2+9)}\}$$

$$\therefore Y = L^{-1}(y) = L^{-1}\{\frac{s}{5(s^2+4)}\} + L^{-1}\{\frac{4s}{5(s^2+9)}\} + L^{-1}\{\frac{c}{(s^2+9)}\}$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}L^{-1}\{\frac{s}{(s^2+4)}\} + \frac{4}{5}L^{-1}\{\frac{s}{(s^2+9)}\} + \frac{c}{3}L^{-1}\{\frac{3}{(s^2+9)}\}$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}L^{-1}\{\frac{s}{(s^2+2^2)}\} + \frac{4}{5}L^{-1}\{\frac{s}{(s^2+3^2)}\} + \frac{c}{3}L^{-1}\{\frac{3}{(s^2+3^2)}\}$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t \;\; ....(i)$$

Given,

$$\therefore Y(\frac{\pi}{2}) = -1$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t$$

$$\therefore Y(\frac{\pi}{2}) = \frac{1}{5}\cos 2(\frac{\pi}{2}) + \frac{4}{5}\cos 3(\frac{\pi}{2}) + \frac{c}{3}\sin 3(\frac{\pi}{2})$$

$$\therefore Y(\frac{\pi}{2}) = \frac{1}{5}\cos \pi + \frac{4}{5}\cos 3(\frac{\pi}{2}) + \frac{c}{3}\sin(\frac{3\pi}{2})$$

$$\therefore Y(\frac{\pi}{2}) = \frac{1}{5}(-1) + \frac{4}{5}\cos(\frac{3\pi}{2}) + \frac{c}{3}\sin(\frac{3\pi}{2})$$

$$\therefore Y(\frac{\pi}{2}) = \frac{1}{5}(-1) + \frac{4}{5}\times 0 + \frac{c}{3}(-1)$$

$$\therefore Y(\frac{\pi}{2}) = -\frac{1}{5} - \frac{c}{3}$$

$$-1 = \frac{1}{5}(-1) + \frac{c}{3}(-1) \qquad [\because Y(\frac{\pi}{2}) = -1]$$

$$-1 = -\frac{1}{5} - \frac{c}{3}$$

$$\frac{c}{3} = -\frac{1}{15} + 1$$

$$\frac{c}{3} = \frac{-1+15}{15}$$

$$\frac{c}{3} = \frac{14}{15}$$

$$c = \frac{14}{5}$$

Putting the value of c in (i)

$$\therefore Y = L^{-1}(y) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t \;\; ....(i)$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{14}{15}\sin 3t$$

Answer