LATIF

Ex 1. Prove that the homogeneous 2nd degree equation in x-y i.e. $ax^2+2hxy+by^2=0$ always represent a pair of st. lines through the origin.

Sol. Given equation is

$$ax^2+2hxy+by^2=0 \longrightarrow (1)$$

Let $b\neq 0$, dividing equation (1) by bx^2.

$$\therefore \left(\frac{y}{x}\right)^2+\frac{2h}{b}(y/x)+a/b=0$$

Putting $\frac{y}{x}=m$,

$$\therefore m^2+\frac{2h}{b}m+\frac{a}{b}=0 \longrightarrow (2)$$

Let m_1, m_2 are the roots of the quadratic equation (2).

$$\therefore m_1+m_2=-\frac{2h}{b} \longrightarrow (i) \quad \& \quad m_1m_2=\frac{a}{b} \longrightarrow (ii)$$

Eq^n. (2) can be written as

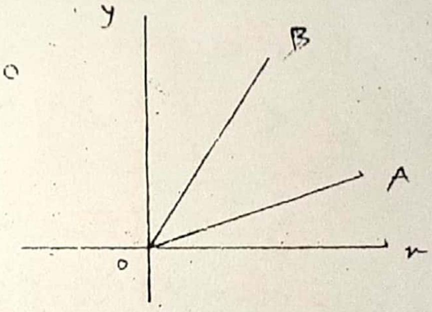
$$m^2-(m_1+m_2)m+m_1m_2=0$$

or, $(m-m_1)(m-m_2)=0$

or, $m=m_1$ & $m=m_2$

or, $\frac{y}{x}=m_1$ & $\frac{y}{x}=m_2$

or, $y=m_1x \longrightarrow (3)$ & $y=m_2x \longrightarrow (4)$



As equations (3) & (4) are two st. lines through the origin, so eq^n. (1) will always represent a pair of st. lines through the origin.

Col. If the st. lines (3) & (4) are real then the disc. of (2) i.e. $4\frac{h^2}{b^2}-4a/b=\frac{4}{b^2}(h^2-ab)>0$

i.e. $h^2-ab>0$.

For coincident st. lines $h^2-ab=0$.

For imaginary st. lines $h^2-ab<0$.

(2)

2. Prove that: a homogeneous equation of nth degree in x, y always represent n-st-lines through the origin.

Sol. We have a homogeneous eq^n. of nth degree in x, y is of the form,

$$y^n + a_1 x y^{n-1} + a_2 x^2 y^{n-2} + \dots + a_r x^r y^{n-r} + \dots + a_n x^n = 0 \longrightarrow (1)$$

Dividing the eq^n. by x^n.

$$\therefore \left(\frac{y}{x}\right)^n + a_1\left(\frac{y}{x}\right)^{n-1} + a_2\left(\frac{y}{x}\right)^{n-2} + \dots + a_r(y/x)^{n-r} + \dots + a_n = 0$$

& Putting $y/x = m$.

$$\text{i.e. } m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_r m^{n-r} + \dots + a_{n-1} m + a_n = 0 \longrightarrow (2)$$

Eq^n. (2) is an nth degree eq^n. in m so it has exactly n-roots (real, equal or imaginary).

Let the roots of eq^n. (2) are $m_1, m_2, m_3, \dots, m_r, \dots m_n$.

Eq^n. (2) be written as $[\Sigma m_1 = -a_1, \Sigma m_1 m_2 = a_2 \dots m_1 m_2 \dots m_n = (-1)^n a_n]$

$$(m - m_1)(m - m_2) \dots (m - m_r) \dots (m - m_n) = 0$$

$$\text{i.e. } \left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) \dots \left(\frac{y}{x} - m_r\right) \dots \left(\frac{y}{x} - m_n\right) = 0$$

$$\text{i.e. } (y - m_1 x)(y - m_2 x) \dots (y - m_r x) \dots (y - m_n x) = 0$$

i.e. The n-st. lines through the origin are

$$y - m_1 x = 0, \quad y - m_2 x = 0, \dots \quad y - m_r x = 0, \dots \quad y - m_n x = 0$$

$\therefore$ Eq^n. (1) will always represent n-st. lines through the origin.

Find the angle between the st. lines represented by equation $ax^2 + 2hxy + by^2 = 0$.

Sol. Given equation is $ax^2 + 2hxy + by^2 = 0 \longrightarrow (1)$

Let the st. lines represented by eq^n. (1) are

$$y = m_1 x \longrightarrow (i) \quad \& \quad y = m_2 x \longrightarrow (ii)$$

where $m_1 + m_2 = -\dfrac{2h}{b} \longrightarrow (iii)$

& $m_1 m_2 = \dfrac{a}{b} \longrightarrow (iv)$.

Let θ be the angle between the st. lines (i) & (ii) Then we have $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 m_2}$

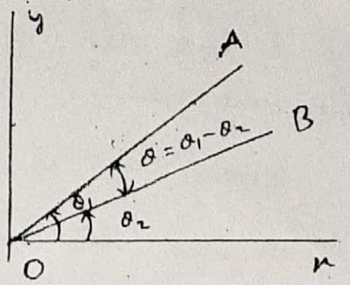
or, $\tan\theta = \dfrac{\sqrt{(m_1-m_2)^2}}{1+m_1m_2}$

$= \dfrac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2}$

$= \dfrac{\sqrt{(-2h/b)^2 - 4a/b}}{1+a/b}$

or, $\tan\theta = \dfrac{2\sqrt{h^2-ab}}{a+b}$ $\longrightarrow (2)$

or, $\theta = \tan^{-1}\left[\dfrac{2\sqrt{h^2-ab}}{a+b}\right]$



$m_1 = \tan\theta_1$,
$m_2 = \tan\theta_2$.

① Condition of perpendicularity:

If $\theta = 90°$ then the st. lines are perpendicular to each other..

$\therefore \tan\theta = \tan 90°$

or, $\dfrac{2\sqrt{h^2-ab}}{a+b} = \infty$

or, $a+b=0$.

Cor. As $a+b=0$ or, $b=-a$.

Putting this in eq^n. (1) then

$ax^2 + 2hxy - ay^2 = 0$

or; $x^2 + \frac{2h}{a}xy - y^2 = 0$

or, $x^2 + 2Kxy - y^2 = 0 \longrightarrow (3)$

Eq^n. (3) will represent a pair of perpendicular st. lines through the origin.

Condition of coincidence (parallelism):

If $\theta = 0$ or π then the st. lines are coincident.

$\therefore \tan\theta = \tan 0 \ (\tan\pi)$

or, $\dfrac{2\sqrt{h^2-ab}}{a+b} = 0$

or, $h^2 - ab = 0$.

Cor. Eq^n. (1) can be written as

$ax^2 + 2hxy + by^2 = 0$

or, $a^2x^2 + 2ahxy + aby^2 = 0$

or, $(ax)^2 + 2ax\cdot hy + (hy)^2 = 0$

or, $(ax+hy)^2 = 0$ or, $(ax+hy)(ax+hy) = 0$

i.e. the coincident lines are $ax+hy=0$, $ax+hy=0$.

(14) Find the bisector of angles between the st. lines represented by $ax^2+2hxy+by^2=0$.

Sol. Given eq^n is $ax^2+2hxy+by^2=0$. $\longrightarrow$ (1)

Let the st. lines represented by eq^n. (1) are

$m_1x-y=0 \longrightarrow$ (i) & $m_2x-y=0 \longrightarrow$ (ii)

where $m_1+m_2=-\frac{2h}{b} \longrightarrow$ (iii) & $m_1m_2=\frac{a}{b} \longrightarrow$ (iv)

We have bisector of the angles between the st. lines (i) & (ii) are $\frac{m_1x-y}{\sqrt{m_1^2+1}}=\pm\frac{m_2x-y}{\sqrt{m_2^2+1}}$

or, $(m_2^2+1)(m_1x-y)^2=(m_1^2+1)(m_2x-y)^2$

or, $\{m_1^2(m_2^2+1)-m_2^2(m_1^2+1)\}x^2+(m_2^2+1-m_1^2-1)y^2$

$=\{2(m_2^2+1)m_1-2m_2(m_1^2+1)\}xy$

or, $(m_1^2-m_2^2)(x^2-y^2)=2\{(m_1-m_2)(1-m_1m_2)\}xy$

or, $(m_1+m_2)(x^2-y^2)=2(1-m_1m_2)xy \quad [m_1\neq m_2]$

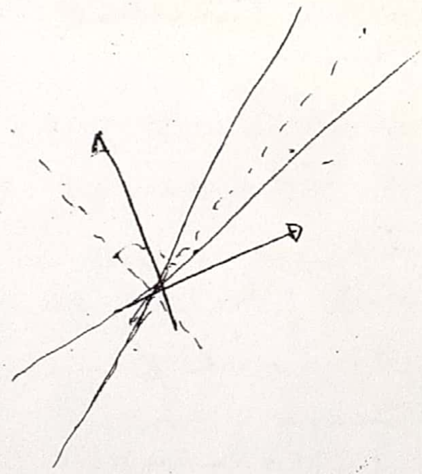
or, $-\frac{2h}{b}(x^2-y^2)=2(1-\frac{a}{b})xy$

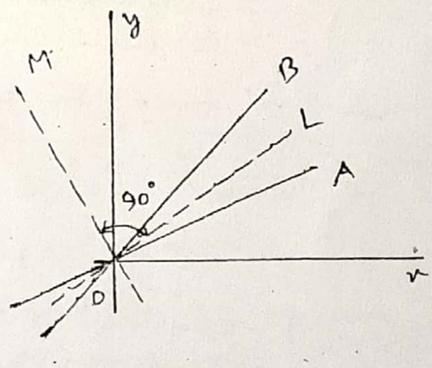
or, $h(x^2-y^2)=(a-b)xy$

or, $\frac{x^2-y^2}{a-b}=\frac{xy}{h}$, $\longrightarrow$ (2)

Eq^n. (2) represent the bisector of the angles bet^n the st. lines given by eq^n. (1).

We see that these st. lines given by (2) are at right angles; [as $a+b=h+(-h)=0$]





5. Find the condition that the general equation of 2nd degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ may represent a pair of st. lines.

Soln: Given eqn. is

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \longrightarrow (1)$$

1st method (Ordinary method):

Eqn. (1) can be written as

$$ax^2+2(hy+g)x+(by^2+2fy+c)=0$$

$$\text{or, } x = \frac{-2(hy+g) \pm \sqrt{4(hy+g)^2-4a(by^2+2fy+c)}}{2a}$$

$$\text{or, } ax+hy+g = \pm\sqrt{(h^2-ab)y^2+2(gh-af)y+g^2-ca} \longrightarrow (2)$$

If eqn. (1) will represent a pair of st. lines then the expression under radical sign in (2) must be a perfect square.

i.e. $\{(h^2-ab)y^2+2(gh-af)y+(g^2-ac)\}$ must be perfect square.

$\therefore$ The corresponding eqn.

$$(h^2-ab)y^2+2(gh-af)y+(g^2-ac)=0$$

has two equal roots.

$$\therefore 4(gh-af)^2-4(h^2-ab)(g^2-ac)=0$$

$$\text{or, } (gh-af)^2-(ab-h^2)(ca-g^2)=0$$

$$\text{or, } g^2h^2-2afgh+a^2f^2-a^2bc+abg^2+cah^2-g^2h^2=0$$

$$\text{or, } -a(abc+2fgh-af^2-bg^2-ch^2)=0 \quad [a\neq 0]$$

$$\text{or, } \Delta \equiv abc+2fgh-af^2-bg^2-ch^2=0. \longrightarrow (3)$$

This $\Delta=0$ is the required condition.

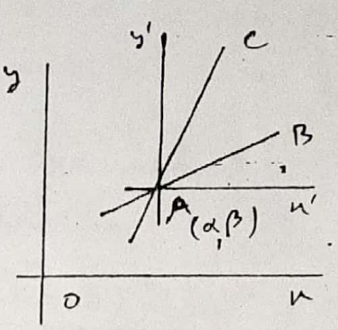
Δ can also be written as

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

2nd method (Shiftment of origin method):

Given eqⁿ. is $ax^2+2hxy+by^2+2gx+2fy+c=0.\longrightarrow(1)$

Let $A(\alpha,\beta)$ be the point of intersection of the st. lines (1). Shifting the origin to the point (α,β), i.e. replacing x by $(x+\alpha)$ & y by $(y+\beta)$ in (1).



$\therefore$ Eqⁿ.(1) becomes

$$a(x+\alpha)^2+2h(x+\alpha)(y+\beta)+b(y+\beta)^2+2g(x+\alpha)$$
$$+2f(y+\beta)+c=0$$

or, $$ax^2+2hxy+by^2+2(a\alpha+h\beta+g)x+2(h\alpha+b\beta+f)y$$
$$+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0.$$
$$\longrightarrow(2)$$

As eqⁿ.(2) passes through the new origin so it must be homogeneous 2nd degree in x & y.

$\therefore$ Co-effi. of x, y & constant term must be zero.

i.e. $$a\alpha+h\beta+g=0\longrightarrow(i)$$
$$h\alpha+b\beta+f=0\longrightarrow(ii)$$
& $$a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0$$
or, $$\alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+g\alpha+f\beta+c=0$$
or, $$\alpha\cdot 0+\beta\cdot 0+g\alpha+f\beta+c=0\ [\text{by (i) \& (ii)}]$$
or $$g\alpha+f\beta+c=0.\longrightarrow(iii)$$

Eliminating α,β from (i), (ii) & (iii), we get

$$\Delta\equiv\begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c\end{vmatrix}=0$$

i.e. $\Delta\equiv abc+2fgh-af^2-bg^2-ch^2=0$. Ans.

Col. Solving (i), & (ii) for α,β we will get

$$\alpha=\frac{hf-bg}{ab-h^2},\quad \beta=\frac{gh-af}{ab-h^2}.\quad\left[\alpha=\frac{G}{C},\ \beta=\frac{F}{C}\right]$$

i.e. The point of intersection of st. lines is at

$$\left(\frac{hf-bg}{ab-h^2},\ \frac{gh-af}{ab-h^2}\right).$$

3rd method (product of two determinant method):

Given equation is

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \longrightarrow (1)$$

Let the two st. lines given by eq^n. (1) are

$l_1x+m_1y+n_1=0 \rightarrow (i)$ & $l_2x+m_2y+n_2=0 \longrightarrow (ii)$

such that

$$(l_1x+m_1y+n_1)(l_2x+m_2y+n_2) = ax^2+2hxy+by^2+2gx+2fy+c$$

$\therefore$ $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, $m_1m_2=b$, $l_1n_2+l_2n_1=2g$

$m_1n_2+m_2n_1=2f$, $n_1n_2=c$.

It is required to eliminate $l_1, m_1, n_1, l_2, m_2, n_2$ from the above six relations.

We have the product of two determinants

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \times \begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} = 0$$

[Multiplying row×row method]

or, $$\begin{vmatrix} 2l_1l_2 & l_1m_2+l_2m_1 & l_1n_2+l_2n_1 \\ m_1l_2+m_2l_1 & 2m_1m_2 & m_1n_2+m_2n_1 \\ n_1l_2+n_2l_1 & n_1m_2+n_2m_1 & 2n_1n_2 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} 2l_1l_2 & l_1m_2+l_2m_1 & l_1n_2+l_2n_1 \\ l_1m_2+l_2m_1 & 2m_1m_2 & m_1n_2+m_2n_1 \\ l_1n_2+l_2n_1 & m_1n_2+m_2n_1 & 2n_1n_2 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$ or, $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\therefore$ $\Delta = abc+2fgh-af^2-bg^2-ch^2=0$

4th method (Calculus method):

Given eq^n is $ax^2+2hxy+by^2+2gx+2fy+c=0$. $\longrightarrow$ (1)

Let the st. lines represented by eq^n (1) are

$l_1x+m_1y+n_1=0$ & $l_2x+m_2y+n_2=0$.

$\therefore F(x,y) \equiv ax^2+2hxy+by^2+2gx+2fy+c = (l_1x+m_1y+n_1)(l_2x+m_2y+n_2)$ $\longrightarrow$ (2)

Let (α,β) be the point of intersection of the lines given by eq^n(1).

$\therefore l_1\alpha+m_1\beta+n_1=0$ & $l_2\alpha+m_2\beta+n_2=0$. $\longrightarrow$ (i)

Taking the partial diff. of (2) w.r.to x & y,

or, $\frac{\partial F}{\partial x} = 2(ax+hy+g) = l_2(l_1x+m_1y+n_1)+l_1(l_2x+m_2y+n_2)$ $\longrightarrow$ (3)

& $\frac{\partial F}{\partial y} = 2(hx+by+f) = m_2(l_1x+m_1y+n_1)+m_1(l_2x+m_2y+n_2)$ $\longrightarrow$ (4)

Putting α,β for x,y in (3) & (4).

$\therefore 2(a\alpha+h\beta+g) = l_2(l_1\alpha+m_1\beta+n_1)+l_1(l_2\alpha+m_2\beta+n_2)$

& $2(h\alpha+b\beta+f) = m_2(l_1\alpha+m_1\beta+n_1)+m_1(l_2\alpha+m_2\beta+n_2)$

or, $a\alpha+h\beta+g=0$ $\longrightarrow$ (ii)

& $h\alpha+b\beta+f=0$. $\longrightarrow$ (iii)

Also we have α,β satisfies eq^n (1).

i.e. $a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0$

or, $\alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+g\alpha+f\beta+c=0$

or, $\alpha\cdot 0+\beta\cdot 0+g\alpha+f\beta+c=0$

or, $g\alpha+f\beta+c=0$ $\longrightarrow$ (iv)

Eliminating α,β from the relations (ii), (iii) & (iv).

$$\therefore \Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

i.e. $\Delta \equiv abc+2fgh-af^2-bg^2-ch^2=0$. Ans.

Cor^l: $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ the co-factors of a is $A = bc-f^2$

" " " b is $ca-g^2$

" " " c i $ab-h^2$

" " " h i $gf-ch$

" " " g is $G = hf-bg$

" " " f is $F = gh-af$

The point of intersection of the st. lines (1) is given by from (ii) & (iii)

$\alpha = \frac{hf-bg}{ab-h^2} = \frac{G}{C}$ & $\beta = \frac{gh-af}{ab-h^2} = \frac{F}{C}$.

6. Find the angle between the st. lines given by the general equation $ax^2+2hxy+by^2+2gx+2fy+c=0$.

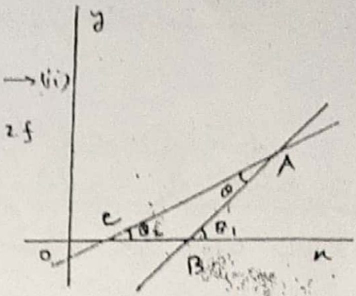
Sol. Given eq^n. is $ax^2+2hxy+by^2+2gx+2fy+c=0 \longrightarrow (1)$

Let the st. lines given by eq^n. (1) are

$l_1x+m_1y+n_1=0 \longrightarrow (i)$ & $l_2x+m_2y+n_2=0 \longrightarrow (ii)$

so that $l_1l_2=a$, $m_1m_2=b$, $n_1n_2=c$; $m_1n_2+m_2n_1=2f$

$n_1l_2+n_2l_1=2g$, $l_1m_2+l_2m_1=2h$.



Slope of the st. lines (i) & (ii) are

$m_1' = \tan\theta_1 = -\frac{l_1}{m_1}$

$m_2' = \tan\theta_2 = -l_2/m_2$.

Let θ be the angle between the st. lines (i) & (ii).

$$\therefore \tan\theta = \frac{m_1'-m_2'}{1+m_1'm_2'} = \frac{-l_1/m_1 + l_2/m_2}{1+\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}}$$

$$= \frac{l_2m_1-l_1m_2}{l_1l_2+m_1m_2} = \frac{\sqrt{(l_1m_2+l_2m_1)^2-4l_1l_2m_1m_2}}{a+b}$$

$$= \frac{\sqrt{4h^2-4ab}}{a+b}$$

$$= \frac{2\sqrt{h^2-ab}}{a+b}.$$

$$\text{or, } \theta = \tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right).$$

($\because$ The st. lines are parallel to those given by homogeneous eq^{n}s. above)

Condition of parallelism:

If the st. lines given by (1) are parallel then $\theta=0$

$\therefore \tan\theta = \tan 0 = 0$

or, $\frac{2\sqrt{h^2-ab}}{a+b} = 0$ or, $h^2-ab=0$.

Also we have $\Delta \equiv abc+2fgh-af^2-bg^2-ch^2=0$

or, $c(ab-h^2)+2fgh-af^2-bg^2=0$

or, $c\cdot 0+2fg\sqrt{ab}-af^2-bg^2=0$

or, $(\sqrt{a}f)^2-2\sqrt{a}f\cdot\sqrt{b}g+(\sqrt{b}g)^2=0$

or, $(\sqrt{a}f-\sqrt{b}g)^2=0$

or, $\sqrt{a}f=\sqrt{b}g$

or, $g/f = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{a}\sqrt{a}}{\sqrt{ab}} = \frac{a}{\sqrt{h^2}} = \frac{a}{h}$

or, $\frac{a}{h}=\frac{g}{f}$ & $h^2=ab$ gives $\frac{a}{h}=\frac{h}{b}$.

$$\therefore \frac{a}{h}=\frac{h}{b}=\frac{g}{f}$$

i.e. $a:h = h:b = g:f$.

Condition of perpendicularity :

If the st. lines (i) & (ii) are perpendicular to each other then

$l_1l_2 + m_1m_2 = 0$ or, $a+b=0$.

Also here $\theta = 90^\circ$ so that $\tan\theta = \tan 90^\circ = \infty$

or, $\dfrac{2\sqrt{h^2-ab}}{a+b} = \infty$

$\therefore$ $a+b=0$.

Condition of coincidence :

If the st. lines (i) & (ii) are coincident then

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$$

or, $\dfrac{l_1}{l_2} = \dfrac{m_1}{\cancel{m_2}}$ or, $l_1m_2 - l_2m_1 = 0$ or, $(l_1m_2 - l_2m_1)^2 = 0$

or, $(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2 = 0$

or, $4h^2 - 4ab = 0$

or, $h^2 - ab = 0$.

& $\dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$ or, $m_1n_2 - m_2n_1 = 0$ or, $(m_1n_2 - m_2n_1)^2 = 0$

or, $(m_1n_2 + m_2n_1)^2 - 4m_1m_2n_1n_2 = 0$

or, $4f^2 - 4bc = 0$

or, $f^2 - bc = 0$.

& $\dfrac{l_1}{l_2} = \dfrac{n_1}{n_2}$ or, $n_1l_2 - n_2l_1 = 0$ or, $(n_1l_2 - n_2l_1)^2 = 0$

or, $(n_1l_2 + n_2l_1)^2 - 4n_1n_2l_1l_2 = 0$

or, $4g^2 - 4ca = 0$

or, $g^2 - ca = 0$.

$\therefore$ condition of coincidence are

$h^2 - ab = 0$, $f^2 - bc = 0$, $g^2 - ca = 0$

$\therefore$ Eq^n (1) can be written as

$ax^2 + 2\sqrt{ab}\,xy + by^2 + 2\sqrt{ca}\,x + 2\sqrt{bc}\,y + c = 0$

or, $(\sqrt{a}x + \sqrt{b}y + \sqrt{c})^2 = 0$

or, $\sqrt{a}x + \sqrt{b}y + \sqrt{c} = 0$ & $\sqrt{a}x + \sqrt{b}y + \sqrt{c} = 0$ etc

7. Find the bisector of the angles between the st. lines $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Sol. Given eqn. is $ax^2+2hxy+by^2+2gx+2fy+c=0 \longrightarrow (1)$

Let $A(\alpha,\beta)$ be the point of intersection of the st. lines (1).
Shifting the origin to the point (α,β) by replacing x,y by $(x+\alpha)$, $(y+\beta)$ in (1).

$\therefore\ a(x+\alpha)^2+2h(x+\alpha)(y+\beta)+b(y+\beta)^2+2g(x+\alpha)+2f(y+\beta)+c=0$

$$ax^2+2hxy+by^2+2(a\alpha+h\beta+g)x+2(h\alpha+b\beta+f)y+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \longrightarrow (2)$$

Since eqn. (2) passes through the new origin so it reduces to a homogeneous eqn.

$$ax^2+2hxy+by^2=0 \longrightarrow (3)$$

where $a\alpha+h\beta+g=0$, $h\alpha+b\beta+f=0$, $a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0$.

We have eqn. of the bisector of the angles betn the st. lines (3) are

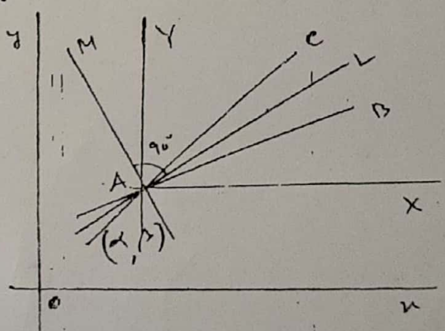
$$\frac{x^2-y^2}{a-b}=\frac{xy}{h}. \longrightarrow (4)$$

Eqn. (4) are w.r.to the new origin.
Eqn. (4) w.r.to old origin becomes after replacing x & y by $(x-\alpha)$ & $(y-\beta)$ in (4).

$\therefore$ The bisectors w.r.to old origin becomes

$$\frac{(x-\alpha)^2-(y-\beta)^2}{a-b}=\frac{(x-\alpha)(y-\beta)}{h}. \longrightarrow (5)$$

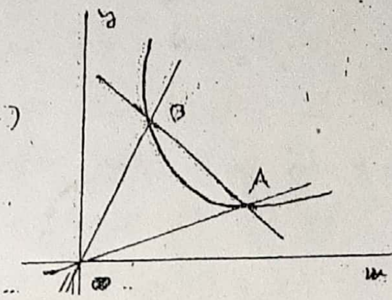
Eqn. (5) are the bisector of the angles betn the st. lines given by eqn. (1).



where $\alpha=\dfrac{hf-bg}{ab-h^2}$

$\beta=\dfrac{gh-af}{ab-h^2}$.

8. Straight lines joining the origin to the point of intersection of a curve and a st. line.



Sol. Let the eq^n. of curve be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \longrightarrow (1)$$

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$,

and the st. line be

$$lx + my + n = 0 \longrightarrow (2)$$

or $\frac{lx+my}{-n} = 1$

Let the st. line (2) meet the curve (1) at A and B, joining OA & OB. The combined eq^n. of these two st. lines is a hom. 2nd degree equation.

i.e. The pair of st. lines joining the origin to the intersection of a curve and a st. line is a homogeneous 2nd degree in x & y.

Making eq^n. (1) homogeneous (2nd degree) with the help of (2).

$$\therefore \quad ax^2 + 2hxy + by^2 + 2(gx + fy)\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$$

$$\text{or, } \left(a - \frac{2gl}{n} + c\frac{l^2}{n^2}\right)x^2 + 2\left(h - \frac{gm}{n} - \frac{fl}{n} + \frac{clm}{n^2}\right)xy + \left(b - \frac{2fm}{n} + \frac{cm^2}{n^2}\right)y^2 = 0$$

i.e. $\quad Ax^2 + 2Hxy + By^2 = 0. \longrightarrow (3)$

Eq^n. (3) are the required pair of st. lines through the origin.