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Demoivre's theorem

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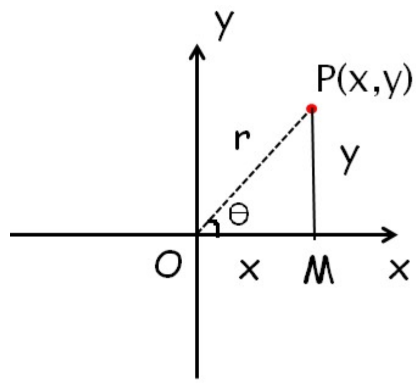

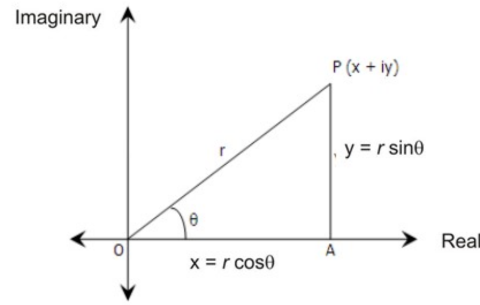

$$\frac{x}{r} = \cos\theta \qquad \text{and} \qquad \frac{y}{r} = \sin\theta$$

$$\Rightarrow x = r\cos\theta \qquad \Rightarrow y = r\sin\theta$$

$$\therefore (x + iy) = (r\cos\theta + i\,r\sin\theta)$$

$$\therefore (x + iy) = r(\cos\theta + i\sin\theta)$$

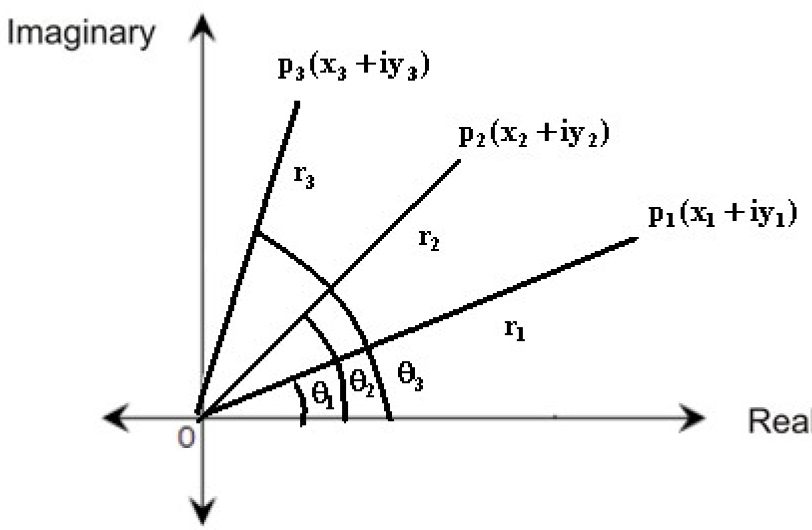

Figure 1

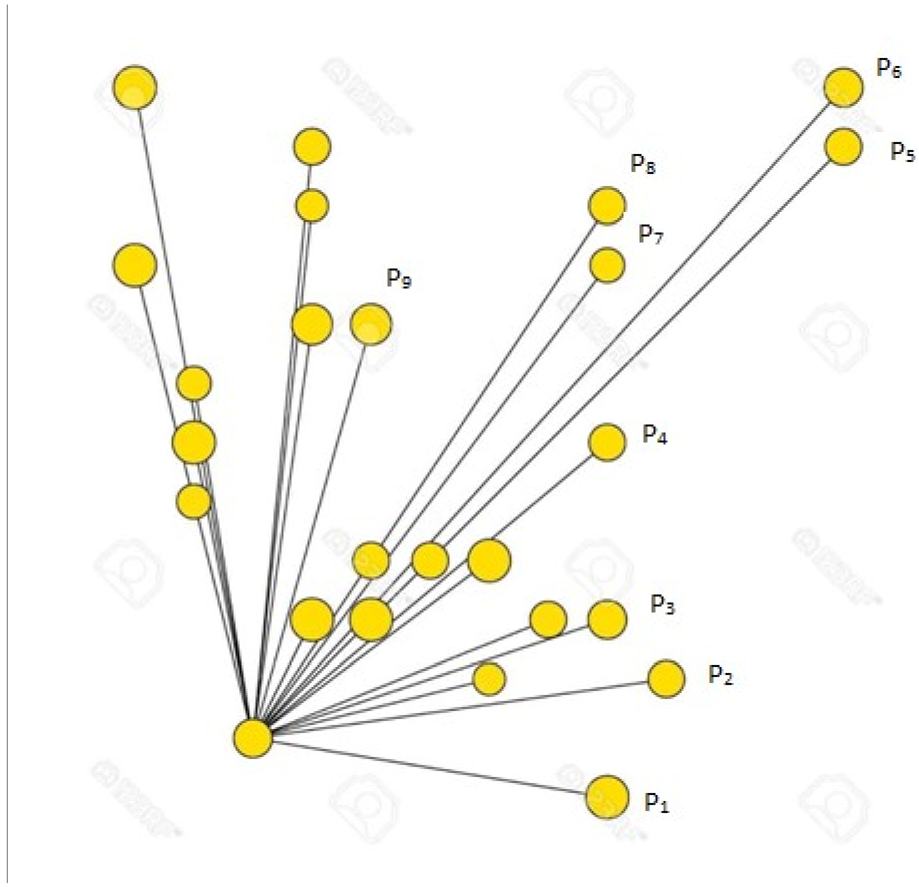

Figure 2

Let another complex number $\mathbf{p_1(x_1 + iy_1)}$

$$\therefore \mathbf{x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)}$$

$$\therefore \mathbf{x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\therefore \mathbf{x_3 + iy_3 = r_3(\cos\theta_3 + i\sin\theta_3)}$$

$$\therefore \mathbf{x_4 + iy_4 = r_4(\cos\theta_4 + i\sin\theta_4)}$$

$$\therefore \mathbf{x_n + iy_n = r_n(\cos\theta_n + i\sin\theta_n)}$$

State and Prove Demoivre's theorem

Statement: whatever be the value of n, positive or negative, integral or fractional,

$\cos n\theta + i\sin n\theta$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$

Proof:

Case 1: when n is a positive integer,

By actual multiplication $(x_1 + iy_1)(x_2 + iy_2) = r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1 r_2\{\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\}$

$= r_1 r_2\{\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\}$ $[i^2 = -1]$

$= r_1 r_2\{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2\}$

$[\cos A\cos B - \sin A\sin B = \cos(A+B) \;\&\; \sin A\cos B + \cos A\sin B = \sin(A+B)]$

$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$

$= r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\therefore r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1 r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\therefore (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$ -----------------(i)

Now, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$= \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}(\cos\theta_3 + i\sin\theta_3)$

$= \cos(\theta_1 + \theta_2)\cos\theta_3 + i\sin(\theta_1 + \theta_2)\cos\theta_3 + i\cos(\theta_1 + \theta_2)\sin\theta_3 + i^2\sin(\theta_1 + \theta_2)\sin\theta_3$

$= \cos(\theta_1 + \theta_2)\cos\theta_3 + i\sin(\theta_1 + \theta_2)\cos\theta_3 + i\cos(\theta_1 + \theta_2)\sin\theta_3 - \sin(\theta_1 + \theta_2)\sin\theta_3$

$= \cos(\theta_1 + \theta_2)\cos\theta_3 - \sin(\theta_1 + \theta_2)\sin\theta_3 + i\{\sin(\theta_1 + \theta_2)\cos\theta_3 + \cos(\theta_1 + \theta_2)\sin\theta_3\}$

$= \cos(\theta_1 + \theta_2 + \theta_3) + i\{\sin(\theta_1 + \theta_2 + \theta_3\}$

So, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots\ldots\ldots\ldots\ldots\ldots(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots\ldots\ldots + \theta_n) + i\{\sin(\theta_1 + \theta_2 + \theta_3 + \theta_4\ldots\ldots\ldots\ldots + \theta_n\}$

If $\theta_1 = \theta_2 = \theta_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \theta_n = \theta$

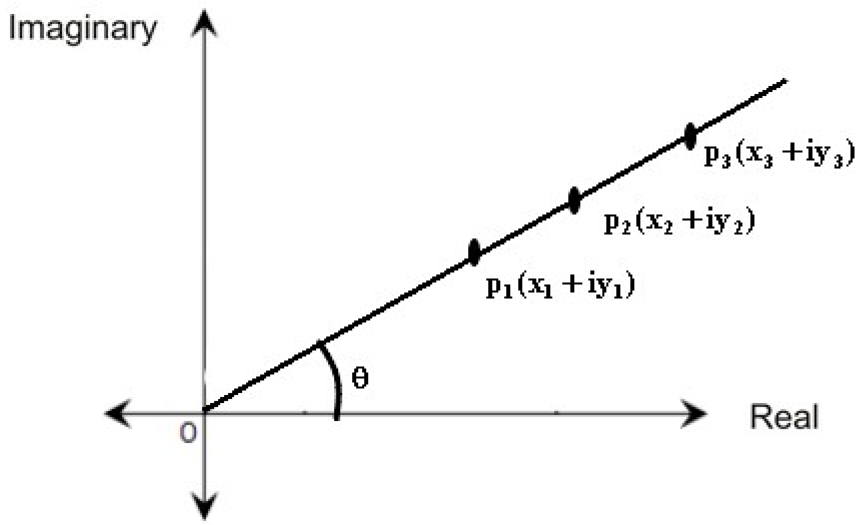

Figure 3

Then $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)......................(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) + i\{\sin(\theta_1 + \theta_2 + \theta_3 + \theta_3 .... + \theta_n\}$

$\Rightarrow (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)......................(\cos\theta + i\sin\theta)$

$= \cos(\theta + \theta + \theta + ... + \theta) + i\{\sin(\theta + \theta + \theta + \theta .... + \theta\}$

$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Case 2: when n is a negative integer

Let us suppose, $n = -m$, where m is a positive integer.

$\Rightarrow (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{1}{\cos m\theta + i\sin m\theta}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{(\cos m\theta - i\sin m\theta)}{\cos^2 m\theta - i^2\sin^2 m\theta}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{(\cos m\theta - i\sin m\theta)}{\cos^2 m\theta + \sin^2 m\theta} \qquad [i^2 = -1]$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{(\cos m\theta - i\sin m\theta)}{1} \qquad [\cos^2\theta + \sin^2\theta = 1]$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = (\cos m\theta - i\sin m\theta)$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos(-m\theta) + i\sin(-m\theta) \qquad [\cos(-\theta) = \cos\theta;\ \sin(-\theta) = -\sin\theta]$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad [\because n = -m]$$

Case 3: when n is a fraction, positive or negative

Let us suppose, $n = \frac{p}{q}$, where q is a positive integer and p is any integer, positive or negative.

From case 1:

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q = \cos q\frac{\theta}{q} + i\sin q\frac{\theta}{q} \qquad [(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$$

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q = \cos\theta + i\sin\theta$$

Taking the q-th roots on both sides,

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q = \cos\theta + i\sin\theta$$

$$\{(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q\}^{\frac{1}{q}} = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}) = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

So, $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$

Raising to the p-th power,

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}) = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

$$\{(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q})\}^p=\{(\cos\theta+i\sin\theta)^{\frac{1}{q}}\}^p$$

$$\{(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q})\}^p=\{(\cos\theta+i\sin\theta)^{\frac{p}{q}}\}$$

$$\{(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q})\}^p=\{(\cos\theta+i\sin\theta)^{\frac{p}{q}}\}$$

$$\Rightarrow \cos p\frac{\theta}{q}+i\sin p\frac{\theta}{q}=\{(\cos\theta+i\sin\theta)^{\frac{p}{q}}\}$$

$$\Rightarrow \cos\frac{p}{q}\theta+i\sin\frac{p}{q}\theta=\{(\cos\theta+i\sin\theta)^{\frac{p}{q}}\}$$

$$\Rightarrow (\cos\theta+i\sin\theta)^{\frac{p}{q}}=\cos\frac{p}{q}\theta+i\sin\frac{p}{q}\theta$$

$$\Rightarrow (\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta \qquad [n=\frac{p}{q}]$$

Q-01: If $x_r=\cos\frac{\pi}{3^r}+i\sin\frac{\pi}{3^r}$, prove that $x_1x_2x_3..................\inf=i$

Answer: Given,

$$x_r=\cos\frac{\pi}{3^r}+i\sin\frac{\pi}{3^r}$$

Putting r = 1, 2, 3, 4, …………..

$$x_1=\cos\frac{\pi}{3^1}+i\sin\frac{\pi}{3^1}$$

$$x_2=\cos\frac{\pi}{3^2}+i\sin\frac{\pi}{3^2}$$

$$x_3=\cos\frac{\pi}{3^3}+i\sin\frac{\pi}{3^3}$$

$$x_4=\cos\frac{\pi}{3^4}+i\sin\frac{\pi}{3^4}$$

…………………………………….

……………………………………

……………………………………..

$\therefore x_1 x_2 x_3$ ………………

$(\cos\frac{\pi}{3^1}+i\sin\frac{\pi}{3^1})\ (\cos\frac{\pi}{3^2}+i\sin\frac{\pi}{3^2})\ (\cos\frac{\pi}{3^3}+i\sin\frac{\pi}{3^3})\ (\cos\frac{\pi}{3^4}+i\sin\frac{\pi}{3^4})$…………

$=\cos(\frac{\pi}{3^1}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\frac{\pi}{3^4}+\ldots\ldots\ldots)+i\sin(\frac{\pi}{3^1}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\frac{\pi}{3^4}+\ldots\ldots\ldots)$

$=\cos\{\frac{\pi}{3^1}(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\ldots\ldots)\}+i\sin\{\frac{\pi}{3^1}(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\ldots\ldots)\}$

$[\because (1-x)^{-1}=1+x+x^2+x^3+x^4+\ldots\ldots\ldots\ldots]$

$=\cos\{\frac{\pi}{3^1}(1-\frac{1}{3})^{-1}\}+i\sin\{\frac{\pi}{3^1}(1-\frac{1}{3})^{-1}\}$

$=\cos\{\frac{\pi}{3^1}(\frac{3-1}{3})^{-1}\}+i\sin\{\frac{\pi}{3^1}(\frac{3-1}{3})^{-1}\}$

$=\cos\{\frac{\pi}{3^1}(\frac{2}{3})^{-1}\}+i\sin\{\frac{\pi}{3^1}(\frac{2}{3})^{-1}\}$

$=\cos\{\frac{\pi}{3^1}(\frac{1}{\frac{2}{3}})\}+i\sin\{\frac{\pi}{3^1}(\frac{1}{\frac{2}{3}})\}$

$=\cos\{\frac{\pi}{3^1}(\frac{3}{2})\}+i\sin\{\frac{\pi}{3^1}(\frac{3}{2})\}$

$=\cos\{\frac{\pi}{2}\}+i\sin\{\frac{\pi}{2}\}$

$=0+i.1$

$=i$

Q-02: If $(1+i\frac{x}{a})(1+i\frac{x}{b})(1+i\frac{x}{c})\ldots\ldots=A+iB$, Then prove that

(i) $(1+\frac{x^2}{a^2})(1+\frac{x^2}{b^2})(1+\frac{x^2}{c^2})\ldots\ldots\ldots\ldots=A^2+B^2$

(ii) $\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}+\ldots\ldots\ldots\ldots=\tan^{-1}\frac{B}{A}$

Answer:

Given, $(1+i\frac{x}{a})(1+i\frac{x}{b})(1+i\frac{x}{c})\ldots\ldots\ldots = A+iB$ ------------------------------------(i)

Let, $1 = r\cos\alpha$, $\frac{x}{a} = r\sin\alpha$ Again, Let, $1 = r\cos\beta$, $\frac{x}{b} = r\sin\beta$ Again, Let, $1 = r\cos\gamma$, $\frac{x}{c} = r\sin\gamma$

So, $\frac{\frac{x}{a}}{1} = \frac{r\sin\alpha}{r\cos\alpha}$

$\therefore \frac{x}{a} = \tan\alpha$

$\therefore \tan\alpha = \frac{x}{a}$

$\therefore \alpha = \tan^{-1}\frac{x}{a}$

So, $\frac{\frac{x}{b}}{1} = \frac{r\sin\beta}{r\cos\beta}$

$\therefore \frac{x}{b} = \tan\beta$

$\therefore \tan\beta = \frac{x}{b}$

$\therefore \beta = \tan^{-1}\frac{x}{b}$

So, $\frac{\frac{x}{c}}{1} = \frac{r\sin\gamma}{r\cos\gamma}$

$\therefore \frac{x}{c} = \tan\gamma$ etc. ,

$\therefore \tan\gamma = \frac{x}{c}$

$\therefore \gamma = \tan^{-1}\frac{x}{c}$

From (i), we get

Given, $(1+i\frac{x}{a})(1+i\frac{x}{b})(1+i\frac{x}{c})\ldots\ldots\ldots = A+iB$

$\Rightarrow (1+i\tan\alpha)(1+i\tan\beta)(1+i\tan\gamma)\ldots\ldots\ldots = A+iB$

$\Rightarrow (1+i\frac{\sin\alpha}{\cos\alpha})(1+i\frac{\sin\beta}{\cos\beta})(1+i\frac{\sin\gamma}{\cos\gamma})\ldots\ldots\ldots = A+iB$

$\Rightarrow (\frac{\cos\alpha+i\sin\alpha}{\cos\alpha})(\frac{\cos\beta+i\sin\beta}{\cos\beta})(\frac{\cos\gamma+i\sin\gamma}{\cos\gamma})\ldots\ldots\ldots = A+iB$

$\Rightarrow (\frac{1}{\cos\alpha})(\frac{1}{\cos\beta})(\frac{1}{\cos\gamma})\ldots\ldots\ldots.(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)\ldots\ldots\ldots = A+iB$

$\Rightarrow (\sec\alpha)(\sec\beta)(\sec\gamma)\ldots\ldots\ldots(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)\ldots\ldots\ldots = A+iB$

$\Rightarrow (\sec\alpha)(\sec\beta)(\sec\gamma)\ldots\ldots\ldots[\cos(\alpha+\beta+\gamma+\ldots\ldots\ldots)+i\sin(\alpha+\beta+\gamma+\ldots\ldots\ldots)] = A+iB$

Equating the real and imaginary part on both sides, we get,

$(\sec\alpha)(\sec\beta)(\sec\gamma)\ldots\ldots\ldots[\cos(\alpha+\beta+\gamma+\ldots\ldots\ldots)] = A$ --------------------(ii)

$(\sec\alpha)(\sec\beta)(\sec\gamma)\ldots\ldots\ldots[\sin(\alpha+\beta+\gamma+\ldots\ldots\ldots)] = B$ --------------------(iii)

Squaring (ii) and (iii), we get,

$$(\sec^2\alpha)(\sec^2\beta)(\sec^2\gamma)\ldots\ldots\ldots[\cos^2(\alpha+\beta+\gamma+\ldots\ldots\ldots)] = A^2 \text{ --------------------(iv)}$$

$$(\sec^2\alpha)(\sec^2\beta)(\sec^2\gamma)\ldots\ldots\ldots[\sin^2(\alpha+\beta+\gamma+\ldots\ldots\ldots)] = B^2 \text{ --------------------(v)}$$

Adding (iv) & (v)

$$(\sec^2\alpha\sec^2\beta\sec^2\gamma\ldots.)\cos^2(\alpha+\beta+\gamma+\ldots.) + (\sec^2\alpha\sec^2\beta\sec^2\gamma\ldots)\sin^2(\alpha+\beta+\gamma+\ldots) = A^2+B^2$$

$$\Rightarrow (\sec^2\alpha\sec^2\beta\sec^2\gamma\ldots\ldots\ldots\ldots)\{\cos^2(\alpha+\beta+\gamma+\ldots\ldots)+\sin^2(\alpha+\beta+\gamma+\ldots.)\} = A^2+B^2$$

$$\Rightarrow (\sec^2\alpha\sec^2\beta\sec^2\gamma\ldots\ldots\ldots\ldots)1 = A^2+B^2$$

$$\Rightarrow \sec^2\alpha\sec^2\beta\sec^2\gamma\ldots\ldots\ldots\ldots = A^2+B^2$$

$$\Rightarrow (1+\tan^2\alpha)(1+\tan^2\beta)(1+\tan^2\gamma)\ldots\ldots\ldots\ldots = A^2+B^2$$

$$\therefore (1+\frac{x^2}{a^2})(1+\frac{x^2}{b^2})(1+\frac{x^2}{c^2})\ldots\ldots\ldots\ldots = A^2+B^2 \text{ proved (i)}$$

Now, (iii) $\div$ (ii)

$$\frac{(\sec\alpha)(\sec\beta)(\sec\gamma)\ldots\ldots\ldots[\sin(\alpha+\beta+\gamma+\ldots\ldots\ldots)]}{(\sec\alpha)(\sec\beta)(\sec\gamma)\ldots\ldots\ldots[\cos(\alpha+\beta+\gamma+\ldots\ldots\ldots)]} = \frac{B}{A}$$

$$\Rightarrow \frac{[\sin(\alpha+\beta+\gamma+\ldots\ldots\ldots)]}{[\cos(\alpha+\beta+\gamma+\ldots\ldots\ldots)]} = \frac{B}{A}$$

$$\Rightarrow \tan(\alpha+\beta+\gamma+\ldots\ldots\ldots) = \frac{B}{A}$$

$$\Rightarrow \alpha+\beta+\gamma+\ldots\ldots\ldots = \tan^{-1}\frac{B}{A}$$

$$\therefore (\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}+\ldots\ldots\ldots\ldots) = \tan^{-1}\frac{B}{A} \text{ proved (ii)}$$

Q-03: Using Demoivre's theorem, solve the equation $x^6+x^5+x^4+x^3+x^2+x^1+1=0$

Answer: We have, $x^6+x^5+x^4+x^3+x^2+x^1+1=0$

Multiplying the given equation by (x-1), we get,

$$(x-1)(x^6+x^5+x^4+x^3+x^2+x^1+1)=0$$

$$\Rightarrow x^7-1=0$$

$$\Rightarrow x^7=1$$

cos(4.pi+0)

=cos(8.90 +0)

=cos0

=1

$$\Rightarrow x^7 = 1$$

$$\Rightarrow x = (1)^{1/7}$$

$$\Rightarrow x = (\cos 0 + i \sin 0)^{1/7}$$

$$\Rightarrow x = \{\cos(2n\pi + 0) + i \sin(2n\pi + 0)\}^{1/7}$$

$$\Rightarrow x = \cos\frac{2n\pi}{7} + i \sin\frac{2n\pi}{7} \qquad [(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta]$$

Putting n =0, 1, 2, 3, 4, 5 and 6, we get roots of equation as

$$\Rightarrow x = \cos 0 + i \sin 0$$

$$\Rightarrow x = \cos\frac{2\pi}{7} + i \sin\frac{2\pi}{7}$$

$$\Rightarrow x = \cos\frac{4\pi}{7} + i \sin\frac{4\pi}{7}$$

$$\Rightarrow x = \cos\frac{6\pi}{7} + i \sin\frac{6\pi}{7}$$

$$\Rightarrow x = \cos\frac{8\pi}{7} + i \sin\frac{8\pi}{7}$$

$$\Rightarrow x = \cos\frac{10\pi}{7} + i \sin\frac{10\pi}{7}$$

$$\Rightarrow x = \cos\frac{12\pi}{7} + i \sin\frac{12\pi}{7}$$

Q-04: Using Demoivre's theorem, find the values of

$\mathbf{i.}(\sqrt{3}+\mathbf{i.1})^{1/5}$

$\mathbf{ii.}(8\mathbf{i})^{1/3}$

$\mathbf{iii.}(32)^{1/5}$

Answer:

$\mathbf{i.}(\sqrt{3}+\mathbf{i.1})^{1/5}$

Let, $\sqrt{3}=r\cos\theta$ --------------------(i)

$1=r\sin\theta$ ----------------------(ii)

Squaring (i) & (ii),

$3=r^2\cos^2\theta$ -----------------------(iii)

$1^2=r^2\sin^2\theta$ ----------------------(iv)

Adding (iii) & (iv),

$3+1=r^2\cos^2\theta+r^2\sin^2\theta$

$4=r^2(\cos^2\theta+\sin^2\theta)$

$4=r^2.1$

$r=2$ ----------------------(v)

(ii) ÷ (i),

$$\frac{1}{\sqrt{3}}=\frac{r\sin\theta}{r\cos\theta}$$

$$\Rightarrow\frac{1}{\sqrt{3}}=\tan\theta$$

$$\Rightarrow\tan\frac{\pi}{6}=\tan\theta$$

$$\Rightarrow\tan\theta=\tan\frac{\pi}{6}$$

$$\Rightarrow\theta=\frac{\pi}{6}$$ ----------------------(vi)

Given

$$(\sqrt{3}+i.1)^{1/5}$$

$$=(r\cos\theta+i.r\sin\theta)^{1/5}$$

$$=\{r(\cos\theta+i\sin\theta)\}^{1/5}$$

$$=\{2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})\}^{1/5}$$

$$=\{2^{1/5}(\cos(2n\pi+\frac{\pi}{6})+i\sin(2n\pi+\frac{\pi}{6})\}^{1/5}$$

$$=\{2^{1/5}\{\cos(\frac{12n\pi+\pi}{6})+i\sin(\frac{12n\pi+\pi}{6})\}^{1/5}$$

$$=\{2^{1/5}\{\cos(\frac{12n\pi+\pi}{30})+i\sin(\frac{12n\pi+\pi}{30})\}\ [(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta]$$

Putting n = 0, 1 , 2 , 3, 4

II) $(8i)^{1/3}$

$$8i=8(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$$

$$(8i)^{1/3}=\{8(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})\}^{\frac{1}{3}}$$

$$(8i)^{1/3}=8^{1/3}\{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\}^{\frac{1}{3}}$$

$$(8i)^{1/3}=(2^3)^{1/3}\{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\}^{\frac{1}{3}}$$

$$(8i)^{1/3}=2\{\cos(2n\pi+\frac{\pi}{2})+i\sin(2n\pi+\frac{\pi}{2})\}^{\frac{1}{3}}$$

$$(8i)^{1/3}=2\{\cos(\frac{4n\pi+\pi}{2})+i\sin(\frac{4n\pi+\pi}{2})\}^{\frac{1}{3}}\ [(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta]$$

$$(8i)^{1/3}=2\{\cos\frac{1}{3}(\frac{4n\pi+\pi}{2})+i\sin\frac{1}{3}(\frac{4n\pi+\pi}{2})\}$$

$$(8i)^{1/3}=2\{\cos(\frac{4n\pi+\pi}{6})+i\sin(\frac{4n\pi+\pi}{6})\}$$

Putting n = 0, 1 , 2

$$(8i)^{1/3}=2\{\cos(\frac{4.0\pi+\pi}{6})+i\sin(\frac{4.0\pi+\pi}{6})\}=2\{\cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})\}$$

$$(8i)^{1/3} = 2\{\cos(\frac{4.1.\pi + \pi}{6}) + i\sin(\frac{4.1.\pi + \pi}{6})\} = 2\{\cos(\frac{5\pi}{6}) + i\sin(\frac{5\pi}{6})\}$$
$$(8i)^{1/3} = 2\{\cos(\frac{4.2.\pi + \pi}{6}) + i\sin(\frac{4.2.\pi + \pi}{6})\} = 2\{\cos(\frac{9\pi}{6}) + i\sin(\frac{9\pi}{6})\}$$

iii) $(32)^{1/5}$

$$(32)^{1/5} = \{32(1)\}^{1/5}$$

$$(32)^{1/5} = 32^{1/5}(\cos 0 + i\sin 0)^{1/5}$$

$$(32)^{1/5} = (2^5)^{1/5}(\cos 0 + i\sin 0)^{1/5}$$

$$(32)^{1/5} = 2\{\cos(2n\pi + 0) + i\sin(2n\pi + 0)\}^{1/5} \ [(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$$

$$(32)^{1/5} = 2\{\cos(\frac{2n\pi}{5}) + i\sin(\frac{2n\pi}{5})\}$$

Putting n = 0, 1 , 2 , 3, 4
$$(32)^{1/5} = 2\{\cos(\frac{2.0.\pi}{5}) + i\sin(\frac{2.0.\pi}{5})\} = 2(\cos 0 + i\sin 0) = 2(1+0) = 2$$

$$(32)^{1/5} = 2\{\cos(\frac{2.1.\pi}{5}) + i\sin(\frac{2.1.\pi}{5})\} = 2\{\cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})\}$$

$$(32)^{1/5} = 2\{\cos(\frac{2.2.\pi}{5}) + i\sin(\frac{2.2.\pi}{5})\} = 2\{\cos(\frac{4\pi}{5}) + i\sin(\frac{4\pi}{5})\}$$

$$(32)^{1/5} = 2\{\cos(\frac{2.3.\pi}{5}) + i\sin(\frac{2.3.\pi}{5})\} = 2\{\cos(\frac{6\pi}{5}) + i\sin(\frac{6\pi}{5})\}$$

$$(32)^{1/5} = 2\{\cos(\frac{2.4.\pi}{5}) + i\sin(\frac{2.4.\pi}{5})\} = 2\{\cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5})\}$$

Q-5: Using Demoivres theorem find the quadratic equation whose roots are the nth power of the roots of the equation, $x^2 - 2x\cos\theta + 1 = 0$

Answer:

The given equation is $x^2 - 2x\cos\theta + 1 = 0$

$$x = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4.1.1}}{2.1}$$
$$x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$
$$x = \frac{2\cos\theta \pm \sqrt{-4 + 4\cos^2\theta}}{2}$$
$$x = \frac{2\cos\theta \pm \sqrt{-4(1-\cos^2\theta)}}{2}$$
$$x = \frac{2\cos\theta \pm \sqrt{-4\sin^2\theta}}{2}$$
$$x = \frac{2\cos\theta \pm \sqrt{4i^2\sin^2\theta}}{2}$$
$$x = \frac{2\cos\theta \pm 2i\sqrt{\sin^2\theta}}{2}$$
$$x = \frac{2\cos\theta \pm 2i\sin\theta}{2}$$
$$x = \frac{2(\cos\theta \pm i\sin\theta)}{2}$$
$$x = \cos\theta \pm i\sin\theta$$

Let α and β are the roots of the equation $x^2 - 2x\cos\theta + 1 = 0$

$\therefore \alpha = \cos\theta + i\sin\theta$ and $\beta = \cos\theta - i\sin\theta$

We have to form a new equation whose roots are α^n and β^n

We know any equation is $x^2 - (\text{sum of the roots})x + \text{product of the roots} = 0$

$$x^2 - (\alpha^n + \beta^n)x + \alpha^n\beta^n = 0$$
$$x^2 - [(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n]x + (\cos\theta + i\sin\theta)^n(\cos\theta - i\sin\theta)^n = 0$$
$$[\because (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]\ [\because (\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta]$$
$$x^2 - [(\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)]x + (\cos n\theta + i\sin n\theta)(\cos n\theta - i\sin n\theta) = 0$$
$$x^2 - [\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta]x + [\cos^2 n\theta - (i)^2\sin^2 n\theta] = 0$$
$$x^2 - [\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta]x + [\cos^2 n\theta + \sin^2 n\theta] = 0$$
$$x^2 - [2\cos n\theta]x + 1 = 0$$
$$x^2 - 2x\cos n\theta + 1 = 0 \text{ Answer.}$$