Chapter 3: Vector Integration

Categories of curves

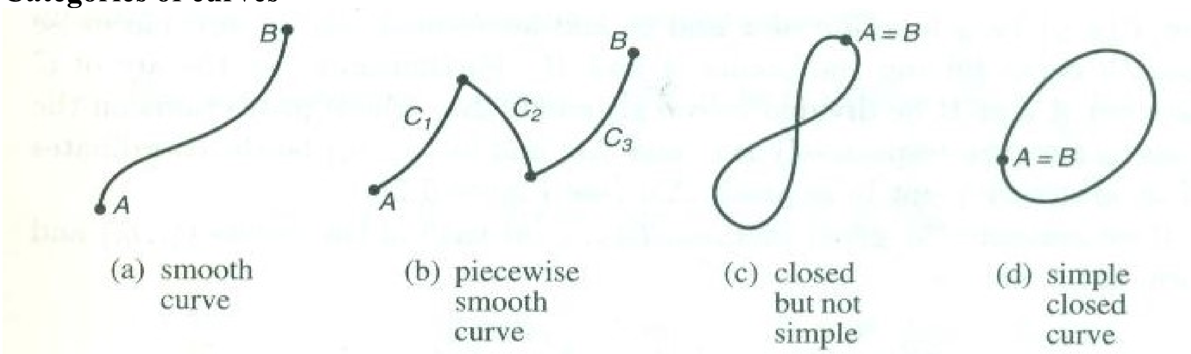


Figure # 79

1. C is smooth curve
2. C is piecewise smooth if it consists of a finite number of smooth curves $\mathbf{C_1, C_2, ..., C_n}$ joined end to end .i.e. $\mathbf{C = C_1 \cup C_2 \cup .... \cup C_n}$
3. C is a closed curve if $\mathbf{A = B}$
4. C is a simple closed curve if $\mathbf{A = B}$ and the curve does not cross itself

First, we approximate the curve **C** by a polygonal path - a path made up of straight-line segments - as shown in figures below.

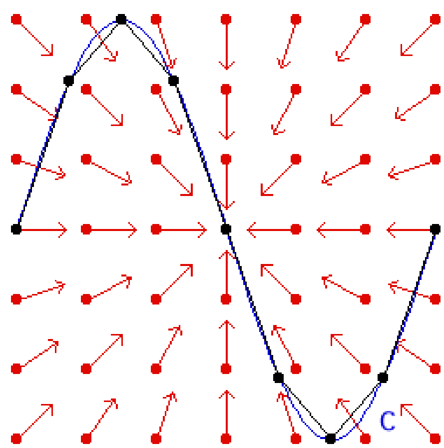


Figure # 80

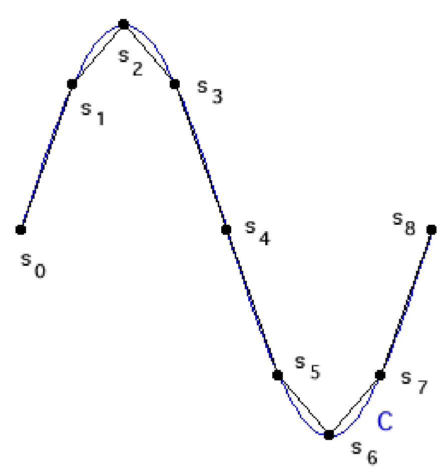


Figure # 81

Path integral may refer to: Line integral, Suppose a force F acting at each point on a smooth curve C.

Line Integral

Any Integral which is evaluated along the curve is called Line Integral, and it is denoted by $\int_C \vec{F}.d\vec{r}$ where $\vec{F}$ is a vector point function, $\vec{r}$ is a position vector and C is the curve

Theorem: the work performed by a vector field on a particle moving along a parametric curve C is obtained by integrating the scalar tangential component of force along C.

$$W = \oint_C \vec{F}.\vec{dr}$$

Q # 74: Establish the path Integral $\int_C \vec{F}.d\vec{S} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}(\vec{S}_{i-1}).(\vec{S}_i - \vec{S}_{i-1})$

Answer:

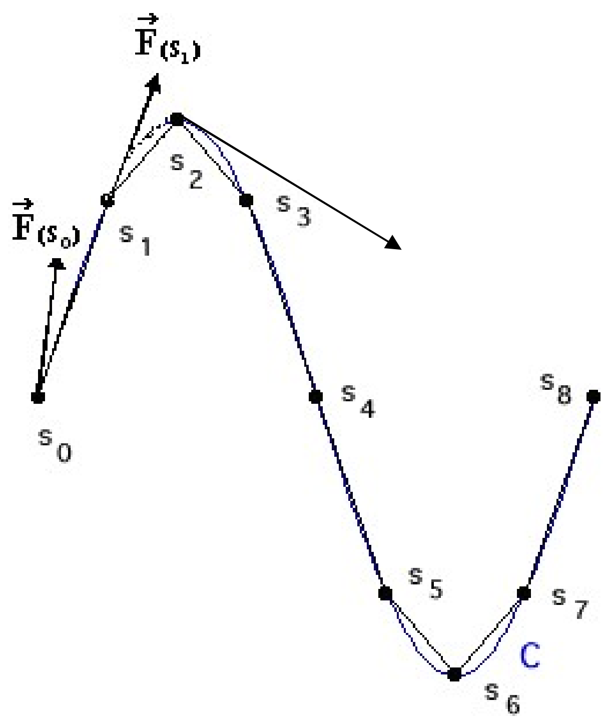


Figure # 82

We choose points $S_0, S_1, S_2, ...S_n$ along the path **C** and then connect these points as shown in the figure above.

and the amount of total work done on the whole path by :

$$W_1 = \vec{F}(S_0).(\vec{S}_1 - \vec{S}_0) = \vec{F}(S_{1-1}).(\vec{S}_1 - \vec{S}_{1-1})$$

$$W_2 = \vec{F}(S_1).(\vec{S}_2 - \vec{S}_1) = \vec{F}(S_{2-1}).(\vec{S}_2 - \vec{S}_{2-1})$$

$$W_3 = \vec{F}(S_2).(\vec{S}_3 - \vec{S}_2) = \vec{F}(S_{3-1}).(\vec{S}_3 - \vec{S}_{3-1})$$

$\mathbf{W_n = \vec{F}_{(S_{n-1})} . (\vec{S}_n - \vec{S}_{n-1})}$

Total Work: $\mathbf{W_1 + W_2 + W_3 + --------- + W_n = \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \vec{F}_{(\vec{S}_{i-1})} . (\vec{S}_i - \vec{S}_{i-1})}$

Then we estimate the Total work done on the **i**-th segment of the path by

$$\mathbf{\sum_{i=1}^{n} W_i = \vec{F}_{(S_{i-1})} . (\vec{S}_i - \vec{S}_{i-1})}$$

By using a large number of small segments we can obtain a very good estimate for the amount of work done. The exact amount of work done is obtained by taking the limit of these estimates. This limit is called the **line integral** of the vector field **F** over the path **C** and the amount of work done on the whole path by:

Total Work done = $\mathbf{\int_C \vec{F} . d\vec{S} = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{F}_{(\vec{S}_{i-1})} . (\vec{S}_i - \vec{S}_{i-1})}$ ***Answer***

[Where** $ds = s_1 - s_0 = s_2 - s_1 = s_3 - s_2 = ... = s_n - s_{n-1}$**]

Mathematical Expression to find out work done along a curve

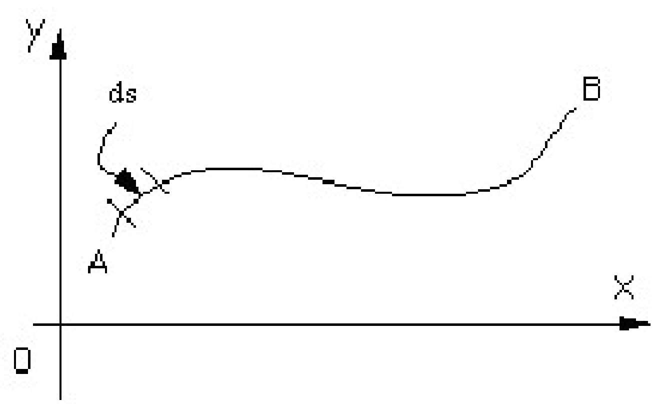


Figure # 83

Then the work done by that force in moving the particle small distance ds is given by: Work Done $= \mathbf{F \times ds}$

Now if we want to know the total work done in moving the particle all the way from A to B, we need to add up all the small contributions, each of the form $\mathbf{F \times ds}$.

However $\mathbf{F}$ may have different strengths at different positions, i.e. $\mathbf{F}$ is a function of position, so what we need to add up are lots of contributions like $\mathbf{F(s) \times ds}$.

It should be familiar to you that when we add up lots of small things like that we do it by **integration**. The integral in this case is then: $\int_{AB} \mathbf{Fds}$

Notice that instead of upper and lower limits we just have AB written at the bottom of the integral sign; to show we're integrating **along the line from A to B**.

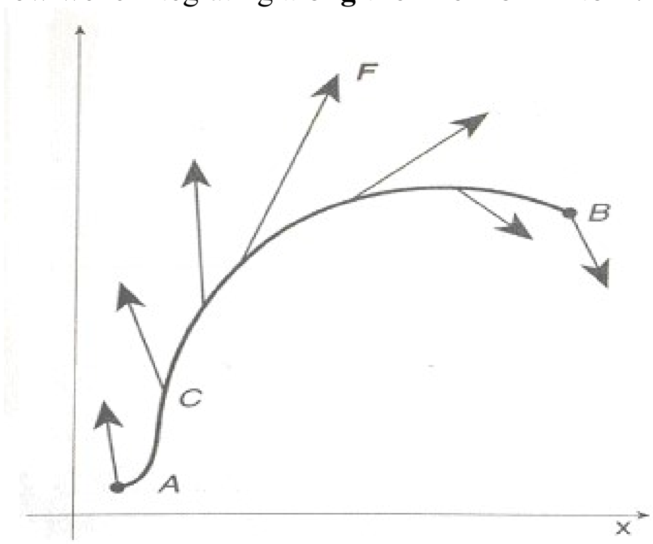


Figure # 84

A force $\vec{\mathbf{F}}$ acting at each point on a smooth curve C.

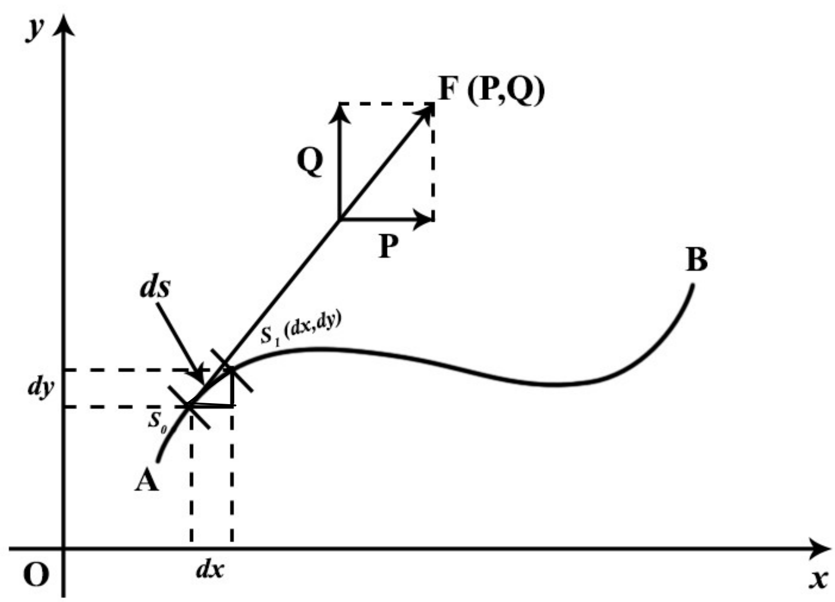


Figure # 85

Here I have divided the force $\vec{\mathbf{F}}$ into a component in the x-direction, called P, and a component in the y-direction, called Q. The work done, $\mathbf{F} \times \mathbf{ds}$, can therefore be written as $\mathbf{Pdx + Qdy}$.

So we can rewrite the integral above: $\int_{AB} \mathbf{Fds} = \int_{AB} \mathbf{Pdx + Qdy}$

Because,

The work done by $\vec{\mathbf{F}}$ in moving the particle from the tail to the head $\mathbf{d}\vec{\mathbf{S}}$ is approximately:

$\vec{\mathbf{F}} . \mathbf{d}\vec{\mathbf{S}}$ ---(i)

If $\vec{\mathbf{F}}(\mathbf{x,y}) = \mathbf{P(x,y)}\hat{\mathbf{i}} + \mathbf{Q(x,y)}\hat{\mathbf{j}}$ (Where P and Q are functions of x and y)

and $\mathbf{d}\vec{\mathbf{S}} = \mathbf{dx}\,\hat{\mathbf{i}} + \mathbf{dy}\,\hat{\mathbf{j}}$

Then, $\vec{\mathbf{F}} . \mathbf{d}\vec{\mathbf{S}} = (\mathbf{P}\,\hat{\mathbf{i}} + \mathbf{Q}\,\hat{\mathbf{j}}).\,(\mathbf{dx}\,\hat{\mathbf{i}} + \mathbf{dy}\,\hat{\mathbf{j}})$

Then, $\vec{\mathbf{F}} . \mathbf{d}\vec{\mathbf{S}} = \mathbf{Pdx + Qdy}$ $[\because \hat{\mathbf{i}}.\hat{\mathbf{i}} = 1; \hat{\mathbf{j}}.\hat{\mathbf{j}} = 1]$----------------(ii)

Since $\mathbf{Pdx + Qdy}$ is a local estimate (from the tail to the head $\mathbf{d}\vec{\mathbf{S}}$) of the work, the total work is represented by a line integral:

Work = $\int_{AB} \vec{F} . d\vec{S} = \int_{AB} (P\,\hat{i} + Q\,\hat{j}).\,(dx\,\hat{i} + dy\,\hat{j}) = \int_{AB} (Pdx + Qdy)$ ----------------(iii)

In three dimension,

Work = $\int_{AB} \vec{\mathbf{F}} . \mathbf{d}\vec{\mathbf{S}} = \int_{AB} (\mathbf{P}\,\hat{\mathbf{i}} + \mathbf{Q}\,\hat{\mathbf{j}} + \mathbf{R}\,\hat{\mathbf{k}}).(\mathbf{dx}\,\hat{\mathbf{i}} + \mathbf{dy}\,\hat{\mathbf{j}} + \mathbf{dz}\,\hat{\mathbf{k}}) = \int_{AB} (\mathbf{Pdx + Qdy + Rdz})$----(iv)

Q # 75: How much work is accomplished by the force $\vec{\mathbf{F}}(\mathbf{x,y}) = \mathbf{xy}\,\hat{\mathbf{i}} + \mathbf{y}\,\hat{\mathbf{j}}$ in pushing a particle from $\mathbf{(0,0)}$ to $\mathbf{(3,9)}$ along the parabola $\mathbf{y = x^2}$?

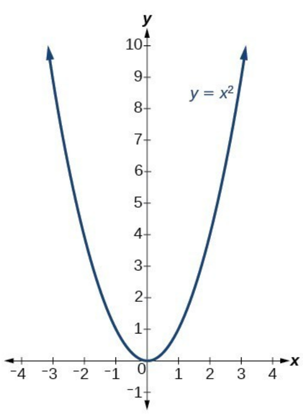


Figure # 86

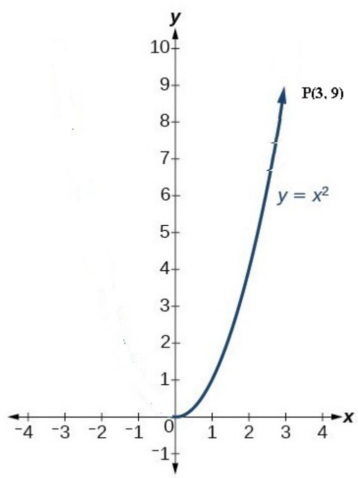


Figure # 87

Answer:

Figure shows the path of the particle, Call this path OP. Then

Here, $\vec{\mathbf{F}}(\mathbf{x},\mathbf{y}) = \mathbf{xy}\,\hat{\mathbf{i}} + \mathbf{y}\,\hat{\mathbf{j}}$,---------------------------------(i)

We know,

$\vec{\mathbf{F}}(\mathbf{x},\mathbf{y}) = \mathbf{P}\,\hat{\mathbf{i}} + \mathbf{Q}\,\hat{\mathbf{j}}$---(ii)

Comparing (i) and (ii),

$\mathbf{p = xy, Q = y}$

We know,

Work = $\int_{OP} \vec{\mathbf{F}}.\mathbf{d}\vec{\mathbf{S}} = \int_{OP} (\mathbf{Pdx + Qdy})$

Work = $\int_{OP} (\mathbf{xydx + ydy})$ $\quad [\because \mathbf{p = xy, Q = y}]$

To evaluate this integral, let us use x as the parameter,

Then, Given $\mathbf{y = x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = 2x$$

$$\therefore dy = 2xdx$$

Work = $\int_o^p (xydx + ydy)$

Work = $\int_0^3 \left[(\mathbf{x.x^2dx + x^2.2xdx}\right]$

Work = $\int_0^3 (\mathbf{x^3dx + 2x^3dx})$

Work = $\int_0^3 \mathbf{3x^3dx} = 3\left[\frac{\mathbf{x^{3+1}}}{\mathbf{3+1}}\right]_0^3 = \frac{3}{4}\left[\mathbf{x^4}\right]_0^3 = \frac{3}{4} \times 3^4 = \frac{243}{4}$ *Answer.*

Q # 76: Evaluate $\int_C \mathbf{xy\,dx}$ from **B(1,0)** to **C(0,1)** along the curve C that is the portion of $\mathbf{x^2 + y^2 = 1}$ in the first quadrant**.**

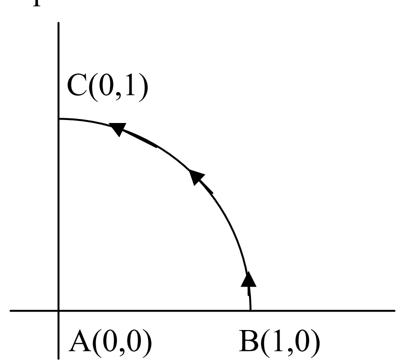


Figure # 88

Anwer:

Given $\mathbf{x^2 + y^2 = 1}$

$\Rightarrow \mathbf{y = \sqrt{1-x^2}}$

The curve BC is the first quadrant of the unit circle as shown in figure. On the curve $\mathbf{y = \sqrt{1-x^2}}$, so that,

$$\int_C \mathbf{xy\,dx} = \int_1^0 \mathbf{x\sqrt{1-x^2}\,dx}$$

Let $\mathbf{1 - x^2 = z}$

$\mathbf{-2xdx = dz}$

$\mathbf{xdx = -\frac{dz}{2}}$

x	1	0
$1-x^2 = z$ $\therefore z = 1-x^2$	$z = 1-x^2$ $z = 1-1$ $z = 0$	$z = 1-x^2$ $z = 1-0$ $z = 1$

$$\int_C \mathbf{xy\,dx} = \int_1^0 \mathbf{x\sqrt{1-x^2}\,dx} =$$

$$-\int_0^1 \sqrt{z}\frac{dz}{2} = -\frac{1}{2}\int_0^1 \sqrt{z}dz = -\frac{1}{2}\left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = -\frac{1}{2}\left[\frac{1^{\frac{3}{2}}}{\frac{3}{2}} - \frac{0^{\frac{3}{2}}}{\frac{3}{2}}\right] = -\frac{1}{2}\left[\frac{1}{\frac{3}{2}} - \frac{0}{\frac{3}{2}}\right]$$

$$= -\frac{1}{2}\times\frac{2}{3}(1-0) = -\frac{1}{3} \textit{ Answer}$$

Q # 77: Find the value of the line integral when $\mathbf{\vec{F}(r) = -y\,\hat{i} - xy\,\hat{j}}$, where $\mathbf{\vec{r}}$ is a function of $\mathbf{t}$ and $\mathbf{C}$ is the circular arc in Figure from A to B.

Or

Find the work done in moving a particle once around a quarter circle C in the xy plane, if the circle has center at the origin and radius 1 and if the force field is given by $\mathbf{\vec{F}(r) = -y\,\hat{i} - xy\,\hat{j}}$

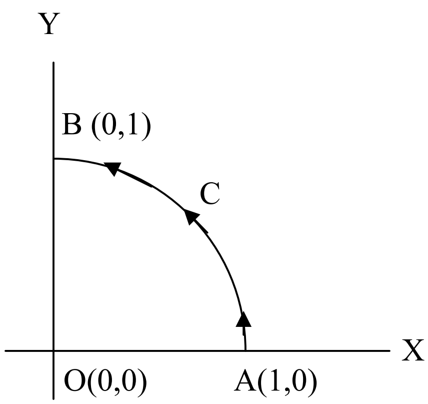


Figure # 89

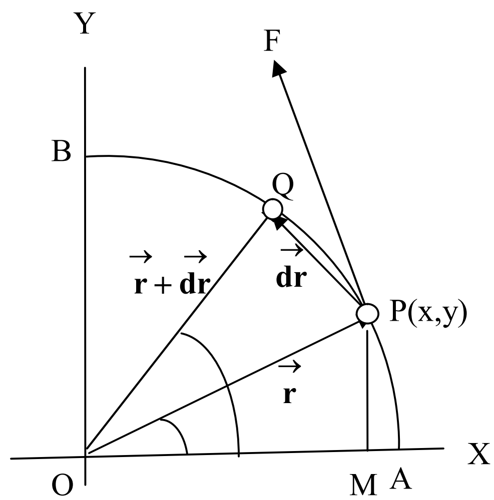


Figure # 90

$\vec{F}(r) = -y\,\hat{i} - xy\,\hat{j}$

Since $\vec{r}$ is a function of t. i.e.

Δ **OPM,**

Let, $\angle POM = t$

$\therefore \dfrac{OM}{OP} = \cos t \Rightarrow OM = OP\cos t \Rightarrow x = 1.\cos t$ [Since radius is 1, i.e $|\vec{OP}| = 1$]

$x = \cos t$ ---(i)

Δ **OPM,**

$\dfrac{PM}{OP} = \sin t \Rightarrow PM = OP\sin t \Rightarrow y = 1.\sin t$ [Since radius is 1, i.e $|\vec{OP}| = 1$]

$y = \sin t$ ---(ii)

$$\therefore \vec{OP} = \vec{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = x\,\hat{i} + y\,\hat{j} = \cos t\,\hat{i} + \sin t\,\hat{j}$$

$$\frac{d\vec{r}}{dt} = -\sin t\,\hat{i} + \cos t\,\hat{j}$$

$d\vec{r} = (-\sin t\,\hat{i} + \cos t\,\hat{j})dt$ ------------------------------------(iii)

$$\text{Work} = \int_{AB} \vec{F}\,.\,d\vec{r} = \int_{AB} (-y\,\hat{i} - xy\,\hat{j})\,.\,d\vec{r}$$

$$\text{Work} = \int_{AB} \vec{F}\,.\,d\vec{r} = \int_{AB} (-\sin t\,\hat{i} - \cos t.\sin t\,\hat{j})\,.(-\sin t\,\hat{i} + \cos t\,\hat{j})dt$$

[From i. $x = \cos t$ and from ii. $y = \sin t$]

[$\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0$]

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \int_0^{\pi/2} (\sin t \times \sin t - \cos t \times \sin t \cos t)dt \qquad [\text{Here } 0 \le t \le \pi/2\,]$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \int_0^{\pi/2} (\sin^2 t - \cos^2 t.\sin t)dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \int_0^{\pi/2} \sin^2 t dt - \int_0^{\pi/2} \cos^2 t.\sin t dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{1}{2}\int_0^{\pi/2} 2\sin^2 t dt - \int_0^{\pi/2} \cos^2 t.\sin t dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2t)\,dt - \int_0^{\pi/2} \cos^2 t.\sin t dt \qquad [\,2\sin^2 t = 1 - \cos 2t\,]$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} \cos^2 t.\sin t dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin 2\times\frac{\pi}{2}}{2}\right) - \left(0 - \frac{\sin 2\times 0}{2}\right)\right] - \int_0^{\pi/2} \cos^2 t.\sin t dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{\sin \pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right] - \int_0^{\pi/2} \cos^2 t.\sin t dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{1}{2}\left[\left(\frac{\pi}{2} - \frac{0}{2}\right) - (0 - 0)\right] - \int_0^{\pi/2} \cos^2 t.\sin t dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)\right] - \int_0^{\pi/2} \cos^2 t.\sin t dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{\pi}{4} - \int_0^{\pi/2} \cos^2 t.\sin t dt \text{ ----------------------(iv)}$$

Let, $z = \cos t$

$$\Rightarrow \frac{dz}{dt} = \frac{d}{dt}(\cos t) = -\sin t$$

$$\Rightarrow dz = -\sin t dt$$

t	$\frac{\pi}{2}$	0
$z = \cos t$	$z = \cos\frac{\pi}{2}$ $z = 0$	$z = \cos 0$ $z = 1$

From equation (iv),

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{\pi}{4} - \int_0^{\pi/2} \cos^2 t.\sin t dt$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{\pi}{4} + \int_1^{0} z^2 dz$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{\pi}{4} + \left[\frac{z^3}{3}\right]_1^0$$

$$\text{Work} = \int_0^{\pi/2} \vec{F}\,.\,d\vec{r} = \frac{\pi}{4} + [\frac{0}{3} - \frac{1}{3}] = \frac{\pi}{4} - \frac{1}{3} \approx 0.4521 \text{ Answer}$$

Q # 78: Find the work done by a) $\vec{\mathbf{F}} = \mathbf{x}\,\hat{\mathbf{i}} + \mathbf{y}\,\hat{\mathbf{j}}$ and b) $\vec{\mathbf{F}} = \frac{3}{4}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}$ along the curve C traced by $\vec{\mathbf{r}}(\mathbf{t}) = \cos \mathbf{t}\,\hat{\mathbf{i}} + \sin \mathbf{t}\,\hat{\mathbf{j}}$ from $\mathbf{t} = \mathbf{0}$ to $\mathbf{t} = \pi$

Answer:

a) The vector function $\vec{\mathbf{r}}(\mathbf{t})$ gives the parametric equations $\mathbf{x} = \cos \mathbf{t}$, $\mathbf{y} = \sin \mathbf{t}$, $\mathbf{0} \le \mathbf{t} \le \pi$ which recognize as a half circle. As seen in Figure 91, the force field $\vec{\mathbf{F}}$ is perpendicular to C at every point. Because the tangential components of $\vec{\mathbf{F}}$ are zero, the work done along C is zero.

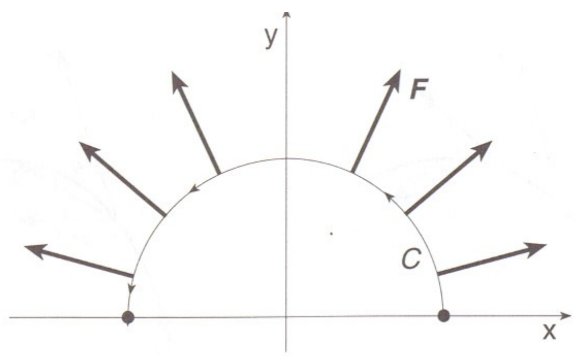


Figure # 91

Half-circle C, with force $\vec{\mathbf{F}}$ perpendicular to C

Given, $\vec{\mathbf{r}}(\mathbf{t}) = \cos \mathbf{t}\,\hat{\mathbf{i}} + \sin \mathbf{t}\,\hat{\mathbf{j}}$

$$\therefore \frac{d}{dt}\{\vec{\mathbf{r}}(\mathbf{t})\} = \frac{d}{dt}\{\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}\}$$

$$\therefore \frac{d}{dt}\{\vec{\mathbf{r}}(\mathbf{t})\} = -\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}}$$

$$\therefore d\,\vec{\mathbf{r}}(\mathbf{t}) = (-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}})dt$$

$$\therefore d\,\vec{\mathbf{r}} = (-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}})dt$$

$$W = \int_C \vec{\mathbf{F}}.d\vec{\mathbf{r}} = \int_C (x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}).d\vec{\mathbf{r}}$$

$$= \int_0^{\pi} (\cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}).(-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}})dt$$

$[\because \hat{\mathbf{i}}.\hat{\mathbf{i}} = 1,\ \hat{\mathbf{j}}.\hat{\mathbf{j}} = 1,\ \hat{\mathbf{k}}.\hat{\mathbf{k}} = 1,\ \hat{\mathbf{i}}.\hat{\mathbf{j}} = 0,\ \hat{\mathbf{i}}.\hat{\mathbf{k}} = 0,\ \hat{\mathbf{j}}.\hat{\mathbf{i}} = 0,\ \hat{\mathbf{j}}.\hat{\mathbf{k}} = 0,\ \hat{\mathbf{k}}.\hat{\mathbf{i}} = 0,\ \hat{\mathbf{k}}.\hat{\mathbf{j}} = 0\ ]$

$$= \int_0^{\pi} (-\cos t \sin t + \sin t \cos t)dt = 0$$

b) In Figure 92 the vectors tangent to the semi-circle are the projections of $\vec{\mathbf{F}}$ on the unit tangent vectors.

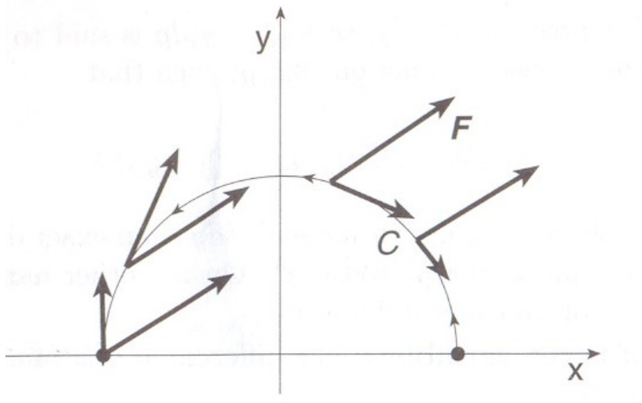


Figure # 92

The work done by $\vec{\mathbf{F}}$ is:

$$\mathbf{W} = \int_C \vec{\mathbf{F}}.\mathbf{d}\vec{\mathbf{r}} = \int_C (\frac{3}{4}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}).\mathbf{d}\vec{\mathbf{r}}$$

$$= \int_0^\pi (\frac{3}{4}\hat{\mathbf{i}} + \frac{1}{2}\hat{\mathbf{j}}).\ (-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}})dt$$

$$= \int_0^\pi (-\frac{3}{4}\sin t + \frac{1}{2}\cos t)dt \qquad [\because \hat{\mathbf{i}}.\hat{\mathbf{i}} = 1;\ \hat{\mathbf{j}}.\hat{\mathbf{j}} = 1]$$

$$= \left[\frac{3}{4}\cos t + \frac{1}{2}\sin t\right]_0^\pi = [\frac{3}{4}\cos\pi + \frac{1}{2}\sin\pi] - [\frac{3}{4}\cos 0 + \frac{1}{2}\sin 0]$$

$$= [\frac{3}{4}(-1) + \frac{1}{2}(0)] - [\frac{3}{4}(1) + \frac{1}{2}(0)] = \left[\frac{-3}{4} + 0 - \frac{3}{4} + 0\right] = -\frac{3}{2} \text{ Answer}$$

Q # 79: Find the work done by the force field $\vec{\mathbf{F}}(\mathbf{x},\mathbf{y}) = \mathbf{x}^3\mathbf{y}\,\hat{\mathbf{i}} + (\mathbf{x}-\mathbf{y})\,\hat{\mathbf{j}}$ on a particle that moves along the parabola $\mathbf{y} = \mathbf{x}^2$ from $(-2,4)$ to $(1,1)$

Answer**:**

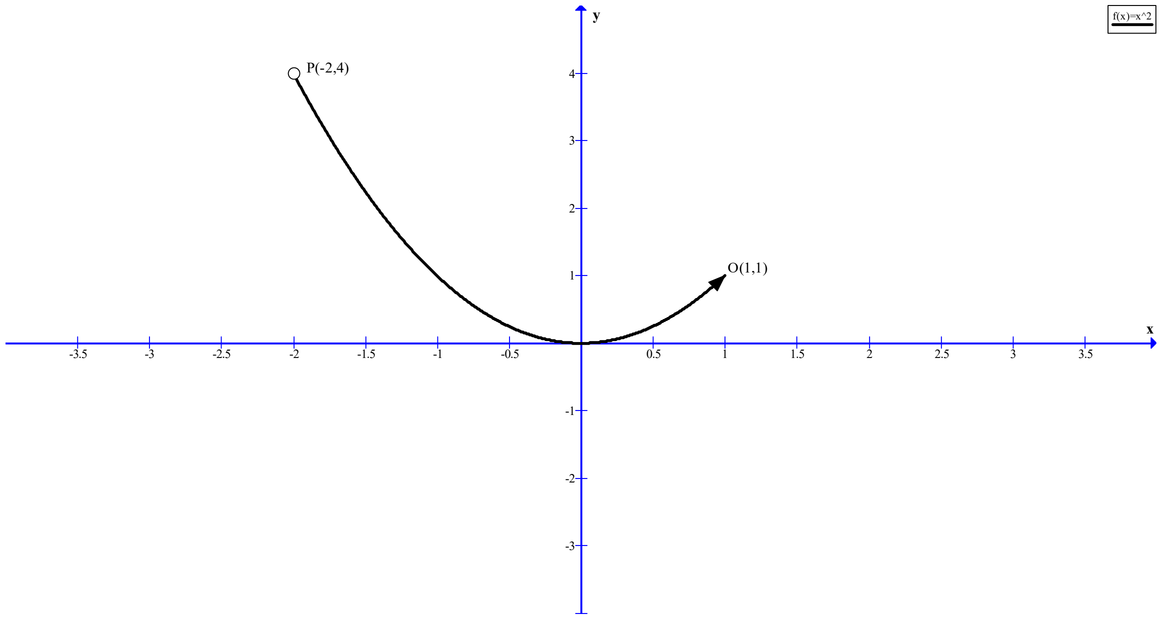


Figure # 93

The work W performed by the field is:

$$\mathbf{W} = \oint_{C} \vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} \qquad \text{-------------------------(i)}$$

We have the position vector

$\therefore \vec{\mathbf{r}} = \mathbf{x}\,\hat{\mathbf{i}} + \mathbf{y}\,\hat{\mathbf{j}}$ **[Page no 48, Figure no 57, Equation no (i)]**

$$\therefore \vec{\mathbf{dr}} = \mathbf{dx}\,\hat{\mathbf{i}} + \mathbf{dy}\,\hat{\mathbf{j}}$$

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = [\mathbf{x^3 y}\,\hat{\mathbf{i}} + (\mathbf{x} - \mathbf{y})\,\hat{\mathbf{j}}] .\ [\mathbf{dx}\,\hat{\mathbf{i}} + \mathbf{dy}\,\hat{\mathbf{j}}]$$

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = [\mathbf{x^3 y}\,\hat{\mathbf{i}} + (\mathbf{x} - \mathbf{y})\,\hat{\mathbf{j}}] .\ [\mathbf{dx}\,\hat{\mathbf{i}} + \mathbf{dy}\,\hat{\mathbf{j}}]$$

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{dr}} = [\mathbf{x^3 y dx} + (\mathbf{x} - \mathbf{y})\mathbf{dy}] \qquad \text{-------------------------(ii)}$$

Let $\mathbf{x = t}$ as the parameter, As $\mathbf{x = t}$,

x	-2	**1**
t	-2	**1**

Then the path C of the particle can be expressed parametrically as

$$\mathbf{x = t,\ y = t^2;\ -2 \le t \le 1}$$

Now,

$$\mathbf{x = t}$$

$$\therefore \frac{\mathbf{dx}}{\mathbf{dt}} = \frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{t})$$

$$\therefore \frac{\mathbf{dx}}{\mathbf{dt}} = \mathbf{1}$$

$\therefore \mathbf{dx = dt}$ ---------------------------(iii)

Again, $\mathbf{y = t^2}$

$\therefore \frac{dy}{dt} = \frac{d}{dt}(t^2)$

$\therefore \frac{dy}{dt} = 2t$

$\therefore dy = 2t\,dt$ ---------------------------(iv)

From (i) & (ii)

$$W = \oint_C \vec{F}.\vec{dr}$$

$$W = \oint_C [x^3 y dx + (x-y)dy]$$

$$W = \int_{-2}^{1} [t^3.t^2 dt + (t-t^2)2t\,dt]$$

$$W = \int_{-2}^{1} [t^5 dt + (2t^2 - 2t^3)\,dt]$$

$$W = \int_{-2}^{1} t^5 dt + \int_{-2}^{1} 2t^2\,dt - \int_{-2}^{1} 2t^3\,dt$$

$$W = \left[\frac{t^6}{6}\right]_{-2}^{1} + \left[\frac{2t^3}{3}\right]_{-2}^{1} - \left[\frac{2t^4}{4}\right]_{-2}^{1}$$

$$W = \left[\frac{1^6}{6} - \frac{(-2)^6}{6}\right] + \left[\frac{2.1^3}{3} - \frac{2(-2)^3}{3}\right] - \left[\frac{2.(1)^4}{4} - \frac{2(-2)^4}{4}\right]$$

$$W = \left[\frac{1}{6} - \frac{64}{6}\right] + \left[\frac{2}{3} + \frac{16}{3}\right] - \left[\frac{2}{4} - \frac{32}{4}\right]$$

$$W = \left[\frac{1-64}{6}\right] + \left[\frac{2+16}{3}\right] - \left[\frac{2-32}{4}\right]$$

$$W = \left[\frac{-63}{6}\right] + \left[\frac{18}{3}\right] + \left[\frac{30}{4}\right]$$

$$W = \frac{-126+72+90}{12}$$

$$W = \frac{-126+162}{12}$$

$W = \frac{36}{12}$ =3; where the units for W depend on the units chosen for force and distance

Or

Figure shows the path of the particle, Call this path PO. Then

Here, $\vec{\mathbf{F}}(\mathbf{x},\mathbf{y}) = \mathbf{x^3 y}\,\hat{\mathbf{i}} + (\mathbf{x}-\mathbf{y})\,\hat{\mathbf{j}}$,--------------------------------(i)

We know,

$\vec{\mathbf{F}}(\mathbf{x},\mathbf{y}) = \mathbf{P}\,\hat{\mathbf{i}} + \mathbf{Q}\,\hat{\mathbf{j}}$ --(ii)

Comparing (i) and (ii),

$\boldsymbol{P = x^3 y, Q = (x - y)}$

We know,

Work = $\int_C \vec{F}.d\vec{s} = \int_C (Pdx + Qdy)$

Work = $\mathbf{W} = \oint\limits_{C} [\mathbf{x^3 ydx + (x-y)dy}]$ $\quad [\,\boldsymbol{P = x^3 y, Q = x - y}\,]$

To evaluate this integral, let us use x as the parameter,

Then, Given $\mathbf{y = x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = 2x$$

$$\therefore dy = 2xdx$$

$$W = \oint\limits_{C} \vec{F}.\vec{ds}$$

$$W = \oint\limits_{C} [x^3 ydx + (x-y)dy]$$

$$W = \int_{-2}^{1} [x^3.x^2 dx + (x - x^2)2x\,dx]$$

$$W = \int_{-2}^{1} [x^5 dx + (2x^2 - 2x^3)\,dx]$$

$$W = \int_{-2}^{1} x^5 dx + \int_{-2}^{1} 2x^2\,dx - \int_{-2}^{1} 2x^3\,dx$$

$$W = \left[\frac{x^6}{6}\right]_{-2}^{1} + \left[\frac{2x^3}{3}\right]_{-2}^{1} - \left[\frac{2x^4}{4}\right]_{-2}^{1}$$

$$W = \left[\frac{1^6}{6} - \frac{(-2)^6}{6}\right] + \left[\frac{2.1^3}{3} - \frac{2(-2)^3}{3}\right] - \left[\frac{2.(1)^4}{4} - \frac{2(-2)^4}{4}\right]$$

$$W = \left[\frac{1}{6} - \frac{64}{6}\right] + \left[\frac{2}{3} + \frac{16}{3}\right] - \left[\frac{2}{4} - \frac{32}{4}\right]$$

$$W = \left[\frac{1-64}{6}\right] + \left[\frac{2+16}{3}\right] - \left[\frac{2-32}{4}\right]$$

$$\mathbf{W} = \left[\frac{-63}{6}\right] + \left[\frac{18}{3}\right] + \left[\frac{30}{4}\right]$$
$$\mathbf{W} = \frac{-126 + 72 + 90}{12}$$
$$\mathbf{W} = \frac{-126 + 162}{12}$$
$\mathbf{W} = \frac{36}{12}$=3; where the units for W depend on the units chosen for force and distance

Q # 80: Find the value of $\int_C \vec{\mathbf{F}}.\vec{\mathbf{dr}}$ where $\vec{\mathbf{F}} = (\mathbf{y} - 2\mathbf{x})\hat{\mathbf{i}} + (3\mathbf{x} + 2\mathbf{y})\hat{\mathbf{j}}$ and C is a circle in the xy- plane with center the origin and radius 2.

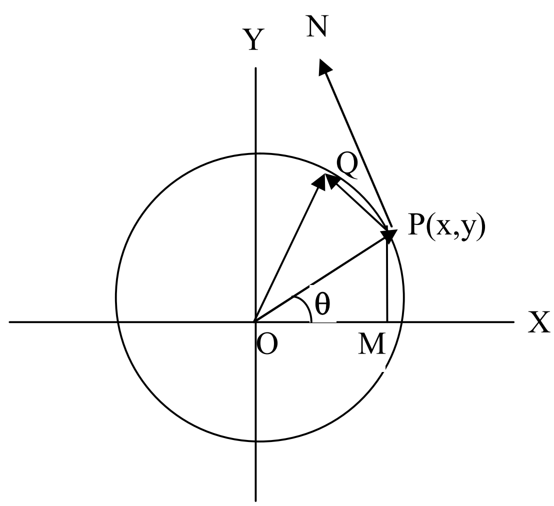


Figure # 94

The position Vector is

$\vec{\mathbf{r}} = \mathbf{x}\hat{\mathbf{i}} + \mathbf{y}\hat{\mathbf{j}}$--------------------------(i) **[Page no 48, Figure no 57, Equation no (i)]**

Let,

$\angle\mathbf{POM} = \theta$, $\mathbf{OP} = 2$

$$\frac{\mathbf{PM}}{\mathbf{OP}} = \sin\theta$$

$$\frac{\mathbf{y}}{2} = \sin\theta \Rightarrow \mathbf{y} = 2\sin\theta$$

Similarly,

$$\frac{\mathbf{OM}}{\mathbf{OP}} = \cos\theta \Rightarrow \frac{\mathbf{x}}{2} = \cos\theta \Rightarrow \mathbf{x} = 2\cos\theta$$

From (i),

$\vec{\mathbf{r}} = \mathbf{x}\hat{\mathbf{i}} + \mathbf{y}\hat{\mathbf{j}}$

$$\vec{r}(\theta) = 2\cos\theta\,\hat{i} + 2\sin\theta\,\hat{j}$$

$$\frac{d\vec{r}}{d\theta} = -2\sin\theta\,\hat{i} + 2\cos\theta\,\hat{j}$$

$$\Rightarrow d\vec{r} = (-2\sin\theta\,\hat{i} + 2\cos\theta\,\hat{j})d\theta$$

Given,

$$\vec{F} = (y - 2x)\,\hat{i} + (3x + 2y)\,\hat{j}$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int \{(y-2x)\,\hat{i} + (3x+2y)\,\hat{j}\}.\,(-2\sin\theta\,\hat{i} + 2\cos\theta\,\hat{j})d\theta$$

$$[\because \hat{i}.\hat{i} = 1,\ \hat{j}.\hat{j} = 1,\ \hat{k}.\hat{k} = 1,\ \hat{i}.\hat{j} = 0\ ,\ \hat{i}.\hat{k} = 0\ ,\ \hat{j}.\hat{i} = 0\ ,\ \hat{j}.\hat{k} = 0\ ,\ \hat{k}.\hat{i} = 0\ ,\ \hat{k}.\hat{j} = 0\ ]]$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int \{(y-2x)(-2\sin\theta)d\theta + (3x+2y)(2\cos\theta)d\theta\}.$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int \{(2\sin\theta - 2\times 2\cos\theta)(-2\sin\theta)d\theta + (3\times 2\cos\theta + 2\times 2\sin\theta)(2\cos\theta)d\theta\}.$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int \{(2\sin\theta - 4\cos\theta)(-2\sin\theta)d\theta + (6\cos\theta + 4\sin\theta)(2\cos\theta)d\theta\}.$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int \{(-4\sin^2\theta + 8\sin\theta\cos\theta)d\theta + (12\cos^2\theta + 8\sin\theta\cos\theta)d\theta\}.$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int (-4\sin^2\theta + 16\sin\theta\cos\theta + 12\cos^2\theta)d\theta.$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int_0^{2\pi} (-4\sin^2\theta + 16\sin\theta\cos\theta + 12\cos^2\theta)d\theta.$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int_0^{2\pi} (-2\times 2\sin^2\theta + 8\times 2\sin\theta\cos\theta + 6\times 2\cos^2\theta)d\theta.$$

$$\therefore \int_C \vec{F}.\vec{dr} = \int_0^{2\pi} \{-2(1-\cos 2\theta) + 8\times\sin 2\theta + 6(1+\cos 2\theta)\}d\theta.$$

$$\therefore \int_C \vec{F}.\vec{dr} = -2\int_0^{2\pi} d\theta + 2\int_0^{2\pi}\cos 2\theta\,d\theta + 8\times\int_0^{2\pi}\sin 2\theta\,d\theta + 6\int_0^{2\pi} d\theta + 6\int_0^{2\pi}\cos 2\theta\,d\theta.$$

$$\therefore \int_C \vec{F}.\vec{dr} = \left[-2\theta + 2\frac{\sin 2\theta}{2} - 8\times\frac{\cos 2\theta}{2} + 6\theta + 6\frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$\therefore \int_C \vec{F}.\vec{dr} = \left[-2(\theta - \frac{\sin 2\theta}{2}) - 8\times\frac{\cos 2\theta}{2} + 6(\theta + \frac{\sin 2\theta}{2})\}\right]_0^{2\pi}$$

$$\therefore \int_C \vec{F}.\vec{dr}$$

$$= -2\left[\left(2\pi - \frac{\sin 2\times 2\pi}{2}\right) - \left(0 - \frac{\sin 2\times 0}{2}\right)\right] - 8\left[\left(\frac{\cos 2\times 2\pi}{2} - \frac{\cos 2\times 0}{2}\right)\right] + 6\left[\left(2\pi + \frac{\sin 2\times 2\pi}{2}\right) - \left(0 + \frac{\sin 2\times 0}{2}\right)\right]$$

$$= -2\left[\left(2\pi - \frac{\sin 4\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right] - 8\left[\left(\frac{\cos 4\pi}{2} - \frac{\cos 0}{2}\right)\right] + 6\left[\left(2\pi + \frac{\sin 4\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$
$$= -2\left[\left(2\pi - \frac{0}{2}\right) - \left(0 - \frac{0}{2}\right)\right] - 8\left[\left(\frac{1}{2} - \frac{1}{2}\right)\right] + 6\left[\left(2\pi + \frac{0}{2}\right) - \left(0 + \frac{0}{2}\right)\right]$$
$$= -2\left[2\pi - \left(-\frac{0}{2}\right)\right] - 8[0] + 6[2\pi - 0]$$
$$= -2[2\pi + 0] - 0 + 6[2\pi]$$
$$= -2[2\pi] + 6[2\pi]$$
$$\therefore \int_C \vec{F}.\vec{dr}\ [= -4\pi + 12\pi = 8\pi \text{ Answer}$$

Q # 81: Find the value of $\int_C \vec{F}.\vec{dr}$ where $\vec{F} = x^2\,\hat{i} + 3xy\,\hat{j}$ if

a) C is the straight line path from $(0,0)$ to $(1,2)$
b) C is the parabolic path $y = 2x^2$ from $(0,0)$ to $(1,2)$
c) C is composed of two straight-line paths the x axis from $(0,0)$ to $(1,0)$ and then a line parallel to the y-axis from $(1,0)$ to $(1,2)$

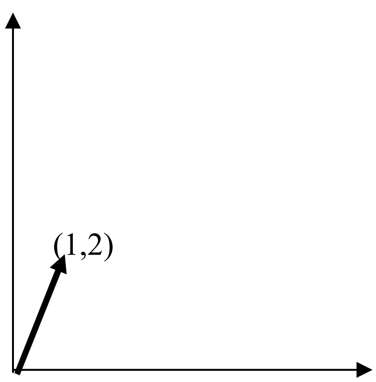


Figure # 95

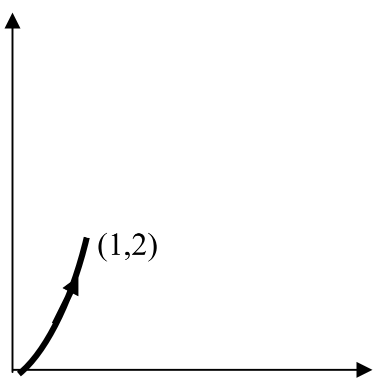


Figure # 96

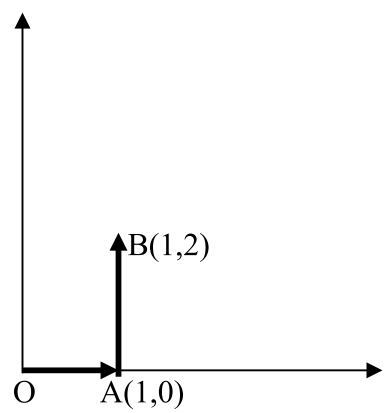


Figure # 97

Let, the position vector is: $\vec{r} = x\,\hat{i} + y\,\hat{j}$ **[Page no 48, Figure no 57, Equation no (i)]**

Since the line element $\vec{dr}$ lies in the xy-plane, we can express it as $\vec{dr} = dx\,\hat{i} + dy\,\hat{j}$, so that

$$\int_C \vec{F}.\vec{dr} = \int_C (x^2\,\hat{i} + 3xy\,\hat{j}).(dx\,\hat{i} + dy\,\hat{j})$$
$$= \int_C (x^2 dx + 3xy dy) \text{ -------------------------------------(i)}$$

a) On this path $y = 2x$, so we can convert (i) to a definite integral with respect to x, this is effectively a simple parameterization, $x = t, y = 2t, 0 \le t \le 1$
Hence,

$$\int_C (x^2dx + 3xydy) = \int_0^1 \left[(x^2dx + 3x(2x)2\,dx\right] \text{ [since } y = 2x\,;\therefore dy = 2xdx\,]$$

$$= \int_0^1 (x^2dx + 12x^2\,dx) = \int_0^1 13x^2dx = 13\left[\frac{x^3}{3}\right]_0^1 = 13\left[\frac{1^3}{3} - \frac{0^3}{3}\right] = \frac{13}{3}$$

Alternatively, of course, we could choose y as the integration variable or parameter and put $x = \frac{y}{2}$ instead of $y = 2x$, in which case (i) becomes,

$$\int_C (x^2dx + 3xydy) = \int_0^2 (x^2dx + 3xy\,dy) = \int_0^2 (\frac{y^2}{4}\frac{dy}{2} + 3\frac{y}{2}ydy)$$

$$\text{[Since } y = 2x\,;\therefore dy = 2dx\ \ \therefore dx = \frac{dy}{2};\ x = \frac{y}{2}]$$

$$= \int_0^2 (\frac{y^2}{8}dy + \frac{3}{2}y^2dy) = \int_0^2 \frac{13y^2}{8}dy = \frac{13}{8}\left[\frac{y^3}{3}\right]_0^2 = \frac{13}{8}\left[\frac{2^3}{3} - \frac{0^3}{3}\right] = \frac{13}{8}\times\frac{8}{3} = \frac{13}{3}$$

b) Here we put $y = 2x^2$ and $\therefore dy = 4xdx$ in (i), giving the definite integral:

$$\int_C (x^2dx + 3xydy) = \int_0^1 (x^2dx + 3x(2x^2)4x\,dx = \int_0^1 (x^2dx + 24x^4\,dx)$$

$$= \left[\frac{x^3}{3} + 24\frac{x^5}{5}\right]_0^1 = \left[\frac{1^3}{3} + 24\frac{1^5}{5} - \frac{0^3}{3} - 24\frac{0^5}{5}\right] = \left[\frac{1}{3} + \frac{24}{5} - \frac{0}{3} - \frac{0}{5}\right] = \frac{77}{5}$$

c) Referring to figure (c), we must integrate on the horizontal and vertical portions separately and add the two contributions. On the horizontal section, y is a constant i.e the equation of horizontal line is $y = 0$ so $dy = 0$ and (i) is simply:

$$\int_C (x^2dx + 3xydy) = \int_0^1 (x^2dx + 3x(0).0 = \int_0^1 x^2dx = \frac{1}{3}$$

On the vertical section, x is constant at 1, i.e the equation of the line parallel to y axis from $(1,0)$ to $(1,2)$ is $x = 1$ so $dx = 0$ and (i) yield:

$$\int_C (x^2dx + 3xydy) = \int_0^2 (1^2.0 + 3.1ydy) = \int_0^2 3ydy = 3\left[\frac{y^2}{2}\right]_0^2 = 3\left[\frac{2^2}{2} - \frac{0^2}{2}\right] = 6$$

Adding the two values, we conclude that the value of the line integral along OA and then AB is $= \frac{1}{3} + 6 = \frac{19}{3}$

Q # 82: Evaluate the following integrals:

1. $\int_C xy^2 dx$
2. $\int_C xy^2 dy$

On the quarter circle C defined by $x = 4\cos t, y = 4\sin t, 0 \le t \le \frac{\pi}{2}$

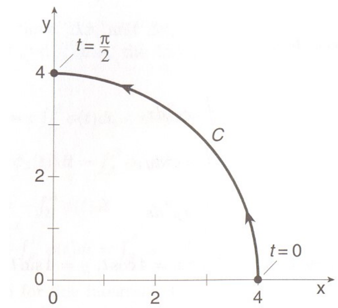


Figure # 98

Given,

$$x = 4\cos t, y = 4\sin t, 0 \le t \le \frac{\pi}{2}$$
$$\therefore dx = -4\sin t dt, dy = 4\cos t dt$$

1. $\int_C xy^2 dx = \int_0^{\pi/2} (4\cos t)(16\sin^2 t)(-4\sin t dt) = -256\int_0^{\pi/2} \sin^3 t \cos t dt$

Let, $z = \sin t$
$dz = \cos t dt$

$t = 0$	$z = \sin t$ $z = \sin 0 = 0$
$t = \pi/2$	$z = \sin t$ $z = \sin \pi/2 = 1$

$$= -256\int_0^1 z^3 dt = -256\left[\frac{z^4}{4}\right]_0^1 = -256[\frac{1^4}{4} - 0] = -64 \text{ Answer}$$

2. $\int_C xy^2 dy = \int_0^{\pi/2} (4\cos t)(16\sin^2 t)(4\cos t dt) dt = 256\int_0^{\pi/2} \sin^2 t \cos^2 t dt$

$$= 256\int_0^{\pi/2} \frac{1}{4}(4\sin^2 t \cos^2 t)dt = 256\int_0^{\pi/2} \frac{1}{4}(2\sin t\cos t)(2\sin t \cos t)dt$$

$$= 256\int_0^{\pi/2} \frac{1}{4}(\sin 2t)(\sin 2t)dt = 256\int_0^{\pi/2} \frac{1}{4}\sin^2 2t dt = 64\int_0^{\pi/2} \sin^2 2t dt$$

$$= 64\int_0^{\pi/2}\frac{1}{2}(2\sin^2 2t)dt = 64\int_0^{\pi/2}\frac{1}{2}(1-\cos 4t)dt = 32\left[t-\frac{\sin 4t}{4}\right]_0^{\pi/2}$$

$$= 32\left[\frac{\pi}{2}-\frac{\sin 4\times\frac{\pi}{2}}{4}-0+0\right] = 32\left[\frac{\pi}{2}-\frac{\sin 2\pi}{4}\right] = 32\left[\frac{\pi}{2}-0\right] = 16\pi \text{ Answer}$$

Q # 83: Evaluate $\oint_C \mathbf{y^2dx - x^2dy}$ on the closed curve C that is shown in the figure

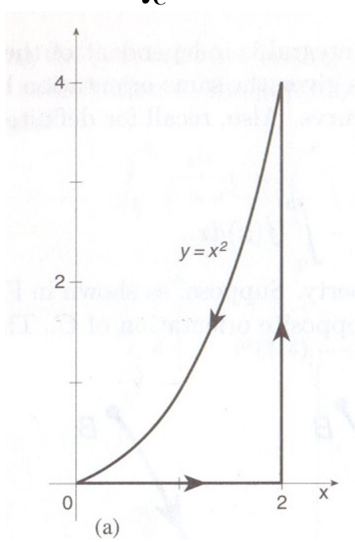


Figure # 99

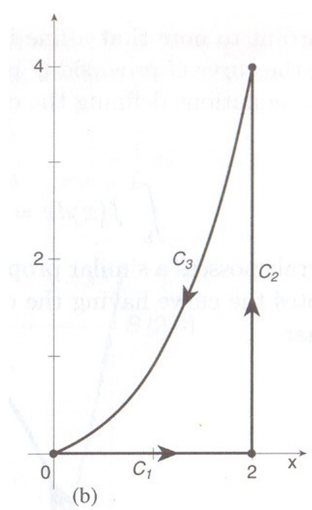


Figure # 100

Answer: Because C is piecewise smooth, we can express the integral as a sum of integrals. Symbolically, we write: $\oint_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$ where $\mathbf{C_1, C_2}$ and $\mathbf{C_3}$ are the curves shown in figure.

i. On $\mathbf{C_1}$, we use $\mathbf{x}$ as a parameter. Because $\mathbf{y = 0, dy = 0;}$ therefore

$$\oint_{C_1}\left[y^2dx - x^2dy\right] = \int_0^2\left[(0)dx - x^2(0)\right] = 0$$

ii. On $\mathbf{C_2}$, we use y as a parameter. From $\mathbf{x = 2, dx = 0;}$ therefore

$$\oint_{C_2}\left[y^2dx - x^2dy\right] = \int_0^4\left[y^2(0) - 4dy\right] = -\int_0^4 4dy = -\left[4y\right]_0^4 = -16$$

iii. Finally, on $\mathbf{C_3}$, we again use x as a parameter. From

$\mathbf{y = x^2; dy = 2xdx;}$ therefore

$$\oint_{C_3}\left[y^2dx - x^2dy\right] = \int_2^0\left[x^4dx - x^2(2xdx)\right] = \int_2^0(x^4dx - 2x^3)dx$$

$$= \left[\frac{1}{5}x^5 - \frac{1}{2}x^4\right]_2^0 = \frac{8}{5}$$

Therefore, $\oint_C y^2dx - x^2dy = 0 - 16 + \frac{8}{5} = -\frac{72}{5}$

Q # 84: Evaluate the line integral, $\int \vec{F}\cdot\vec{dr}$ where the force field is given by $\vec{F}(x,y) = 3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$ along the curve, $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$

Answer:
We have the position vector

$\therefore \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ **[Page no 48, Figure no 57, Equation no (i)]**

$\therefore \vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\vec{F}\cdot\vec{dr} = [3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}].[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}]$

$\vec{F}\cdot\vec{dr} = 3xydx - 5zdy + 10xdz$-------------------------(i)

Given

$x = t^2 + 1$

$\therefore \frac{dx}{dt} = \frac{d}{dt}(t^2 + 1)$

$\therefore \frac{dx}{dt} = 2t$

$\therefore dx = 2tdt$

$y = 2t^2$

$\therefore \frac{dy}{dt} = \frac{d}{dt}(2t^2)$

$\therefore \frac{dy}{dt} = 4t$

$\therefore dy = 4tdt$

$z = t^3$

$\therefore \frac{dz}{dt} = \frac{d}{dt}(t^3)$

$\therefore \frac{dz}{dt} = 3t^2$

$\therefore dz = 3t^2 dt$

From (i),

$\vec{F}.\vec{dr} = 3xydx - 5zdy + 10xdz$

$\vec{F}.\vec{dr} = 3(t^2+1)(2t^2)2tdt - 5t^3 4tdt + 10(t^2+1)3t^2dt$

$\vec{F}.\vec{dr} = 3(t^2+1)(4t^3)dt - 20t^4dt + 10(t^2+1)3t^2dt$

$\vec{F}.\vec{dr} = 3(4t^5+4t^3)dt - 20t^4dt + 10(3t^4+3t^2)dt$

$\vec{F}.\vec{dr} = (12t^5+12t^3)dt - 20t^4dt + (30t^4+30t^2)dt$

$\vec{F}.\vec{dr} = 12t^5dt + 12t^3dt - 20t^4dt + 30t^4dt + 30t^2dt$

$\vec{F}.\vec{dr} = 12t^5dt + 10t^4dt + 12t^3dt + 30t^2dt$

Total work done= $\int_1^2 \vec{F}.\vec{dr} = \int_1^2 (12t^5dt + 10t^4dt + 12t^3dt + 30t^2dt)$

$$= \int_1^2 12t^5dt + \int_1^2 10t^4dt + \int_1^2 12t^3dt + \int_1^2 30t^2dt$$

$$= \left[12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\frac{t^4}{4} + 30\frac{t^3}{3}\right]_1^2$$

$$= [2t^6 + 2t^5 + 3t^4 + 10t^3]_1^2$$

$$= (2\times 2^6 + 2\times 2^5 + 3\times 2^4 + 10\times 2^3) - (2\times 1^6 + 2\times 1^5 + 3\times 1^4 + 10\times 1^3)$$

$$= (2\times 64 + 2\times 32 + 3\times 16 + 10\times 8) - (2+2+3+10)$$

$$= (128+64+48+80) - (2+2+3+10)$$

$$= (320) - (17)$$

$$= 303$$

Q # 85: If $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve curve, $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, then evaluate the line integral, $\int_c \vec{F}\times\vec{dr}$

Answer: We have the position vector

$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ **[Page no 48, Figure no 57, Equation no (i)]**

$$\therefore \vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Given

$$x = t^2$$

$$\therefore \frac{dx}{dt} = \frac{d}{dt}(t^2)$$

$$\therefore \frac{dx}{dt} = 2t$$

$$\therefore dx = 2tdt$$

$$y = 2t$$

$$\therefore \frac{dy}{dt} = \frac{d}{dt}(2t)$$

$$\therefore \frac{dy}{dt} = 2$$

$$\therefore dy = 2dt$$

$$z = t^3$$

$$\therefore \frac{dz}{dt} = \frac{d}{dt}(t^3)$$

$$\therefore \frac{dz}{dt} = 3t^2$$

$$\therefore dz = 3t^2dt$$

$$\therefore \vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$$

$$\Rightarrow \vec{F} = (t^2 \times 2t)\hat{i} - t^3\hat{j} + (t^2)^2\hat{k}$$

$$\Rightarrow \vec{F} = 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k} \text{ --------------------------------(i)}$$

$$\therefore \vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\therefore \vec{dr} = 2tdt\,\hat{i} + 2dt\,\hat{j} + 3t^2dt\,\hat{k} \text{ ------------------------(ii)}$$

From (i) and (ii)

$$\vec{F} \times \vec{dr} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2tdt & 2dt & 3t^2dt \end{vmatrix}$$

$$\vec{F}\times\vec{dr} = (-3t^5 - 2t^4)dt\hat{i} - (6t^5 - 2t^5)dt\hat{j} + (4t^3 + 2t^4)dt\hat{k}$$

$$\vec{F}\times\vec{dr} = (-3t^5 - 2t^4)dt\hat{i} - (4t^5)dt\hat{j} + (4t^3 + 2t^4)dt\hat{k}$$

$$\int_c \vec{F}\times\vec{dr} = \int_0^1 (-3t^5 - 2t^4)dt\hat{i} - (4t^5)dt\hat{j} + (4t^3 + 2t^4)dt\hat{k}$$

$$\int_c \vec{F}\times\vec{dr} = [-3\frac{t^6}{6} - 2\frac{t^5}{5}]_0^1\hat{i} - [4\frac{t^6}{6}]_0^1\hat{j} + [4\frac{t^4}{4} + 2\frac{t^5}{5}]_0^1\hat{k}$$

$$\int_c \vec{F}\times\vec{dr} = [-3\frac{1^6}{6} - 2\frac{1^5}{5}]\hat{i} - [4\frac{1^6}{6}]\hat{j} + [4\frac{1^4}{4} + 2\frac{1^5}{5}]\hat{k}$$

$$\int_c \vec{F}\times\vec{dr} = [-\frac{1}{2} - \frac{2}{5}]\hat{i} - [\frac{2}{3}]\hat{j} + [1 + \frac{2}{5}]\hat{k}$$

$$\int_c \vec{F}\times\vec{dr} = -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k} \quad \textit{Answer}$$

Q # 86 :Home Task:

01. Find the work done in moving a particle once around a circle C in the xy plane, If the circle has center at the origin and radius 3 and the force field is given by

$$\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$$

Hints:

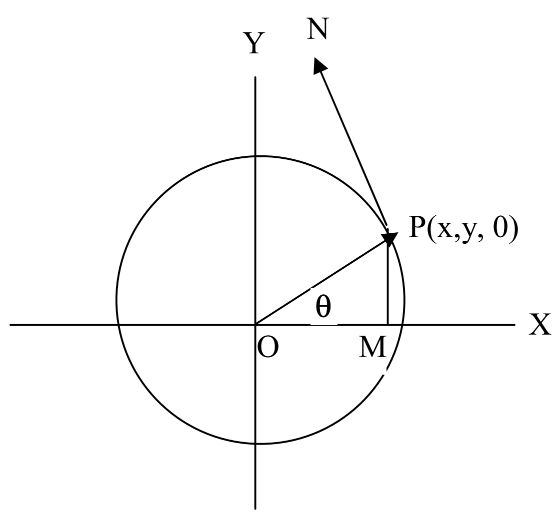


Figure # 101

Answer: Let the equation of the circle is : $x^2 + y^2 = 3^2$

Let,

$\angle \mathbf{POM} = \theta$, $\mathbf{OP = 3}$

$$\frac{\mathbf{PM}}{\mathbf{OP}} = \sin\theta$$

$$\frac{y}{3} = \sin\theta$$

$$\Rightarrow y = 3\sin\theta$$

$$\Rightarrow dy = 3\cos\theta\, d\theta$$

Similarly,

$$\frac{OM}{OP} = \cos\theta$$

$$\frac{x}{3} = \cos\theta$$

$$\Rightarrow x = 3\cos\theta$$

$$\Rightarrow dx = -3\sin\theta\, d\theta$$

Since the circle is in two dimentional, hence, z = 0 $\Rightarrow$ dz = 0

Let, the position Vector is

$$\vec{r}(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$$

[**Page no 48, Figure no 57, Equation no (i)**]

$$\therefore \vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Let, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\vec{F}.\vec{dr} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}).(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\vec{F}.\vec{dr} = F_1dx + F_2dy + F_3dz$$

$$\vec{F}.\vec{dr} = F_1dx + F_2dy + F_3dz$$

Given, $\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$

So, $\vec{F}.\vec{dr} = F_1dx + F_2dy + F_3dz$

$$\vec{F}.\vec{dr} = (2x - y + z)dx + (x + y - z^2)dy + (3x - 2y + 4z)dz$$

$$\vec{F}.\vec{dr} = (2x - y)dx + (x + y)dy + (3x - 2y).0 \qquad [\ z = 0,\ dz = 0]$$

$$\vec{F}.\vec{dr} = (2x - y)dx + (x + y)dy \qquad [z = 0\ ]$$

Hence, total work $= \int_0^{2\pi} \vec{F}.\vec{dr}$

$= \int_0^{2\pi} (2x - y)dx + (x + y)dy$

$= \int_0^{2\pi} \{(2.3\cos\theta - 3\sin\theta)(-3\sin\theta)d\theta + (3\cos\theta + 3\sin\theta)3\cos\theta d\theta\}$

=……………………………………………….

02. Find the work done in moving a particle once around a circle C in the xy plane, if the circle has a centre (0, 0) and radius 1 and if the force field $\vec{F}$ is given by

$\vec{F} = (2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$

Answer: Let the equation of the circle is $x^2 + y^2 = 1$

Let, $x = r\cos\theta$ & $y = r\sin\theta$

Here, radius $r = 1$

$x = \cos\theta$ $y = \sin\theta$

then, $dx = -\sin\theta d\theta$ and $dy = \cos\theta d\theta$

since the circle is in two dimentional, hence, z = 0 $\Rightarrow$ dz = 0

let, the position Vector is

$\vec{r}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}$

[Page no 48, Figure no 57, Equation no (i)]

$\therefore \vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Let, $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\vec{F}.\vec{dr} = (F_1\hat{i} + F_2\hat{j} + F_3\hat{k}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$\vec{F}.\vec{dr} = F_1dx + F_2dy + F_3dz$

$\vec{F}.\vec{dr} = F_1dx + F_2dy + F_3dz$

Given, $\vec{F} = (2x - y + 2z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y - 5z)\hat{k}$

So, $\vec{F}.\vec{dr} = F_1dx + F_2dy + F_3dz$

$\vec{F}.\vec{dr} = (2x - y + 2z)dx + (x + y - z)dy + (3x - 2y - 5z)dz$

$\vec{F}.\vec{dr} = (2x - y + 2z)dx + (x + y - z)dy + 0$ [dz = 0]

$\vec{F}.\vec{dr} = (2x - y)dx + (x + y)dy$ [z = 0]

Hence, total work $= \int_0^{2\pi} \vec{F}.\vec{dr}$

$= \int_0^{2\pi} (2x - y)dx + (x + y)dy$

$= \int_0^{2\pi} \{(2\cos\theta - \sin\theta)(-\sin\theta)d\theta + (\cos\theta + \sin\theta)\cos\theta d\theta\}$

=...................................

Surface Integrals

Surface Integral

The Integral which is evaluated over a surface is called Surface Integral.

If S is any surface and $\hat{\eta}$ is the outward drawn unit normal vector to the surface S then $\int_S \vec{F} \cdot \hat{\eta}\, dS$ is called the Surface Integral.

Normal Vector:

A surface normal, or simply normal, to a flat surface is a vector that is perpendicular to that surface.

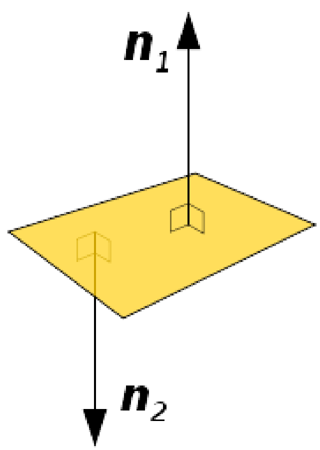


Figure # 102

A normal to a non-flat surface at a point *P* on the surface is a vector perpendicular to the tangent plane to that surface at *P*.

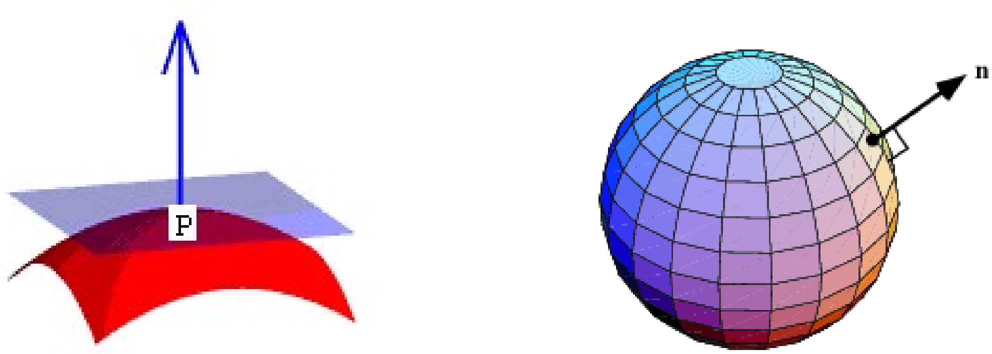


Figure # 103: A normal to a surface at a point is the same as a normal to the tangent plane to that surface at that point P

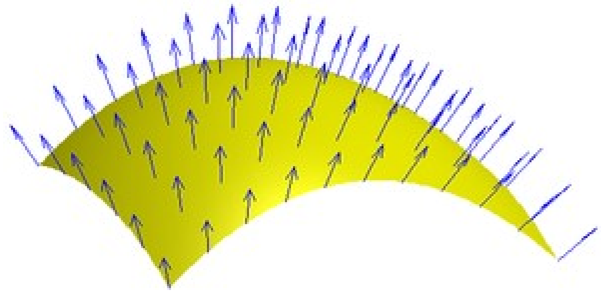


Figure # 104: A vector field of normals to a surface

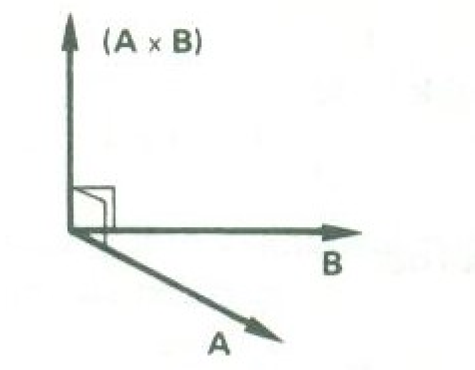


Figure # 105

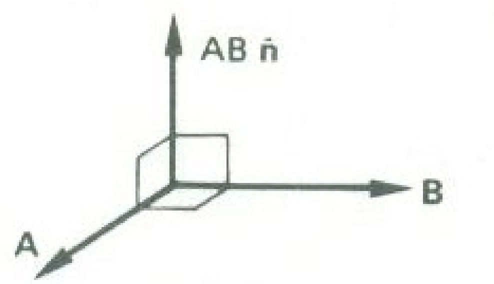


Figure # 106

We know, $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\mathbf{\sin\theta}\,\hat{\eta}$

If $\theta = \frac{\pi}{2}$, then $\vec{\mathbf{A}}\times\vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\mathbf{\sin\theta}\,\hat{\eta} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\mathbf{\sin}\frac{\pi}{\mathbf{2}}\hat{\eta} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\mathbf{.1.}\hat{\eta} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right|\hat{\eta}$, where $\hat{\eta}$ is a unit normal to the plane A and B.

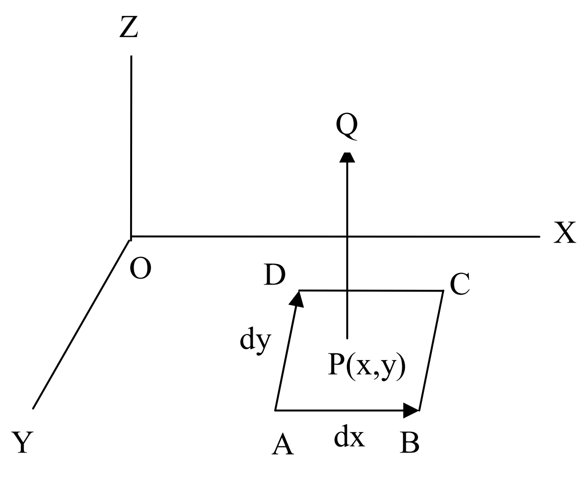


Figure # 107

If $\mathbf{P(x,y)}$ is a point in the $\mathbf{xy}$ plane, the element of area $\mathbf{dx\,dy}$ has a vector area

$$\mathbf{dS} = \left|\vec{\mathbf{dS}}\right|$$

From figure

$$\vec{\mathbf{PQ}} = \left|\vec{\mathbf{PQ}}\right|\hat{\eta} \qquad [\because \textbf{Any Vector} = \textbf{Length of this Vector} \times \textbf{Unit Vector}]$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right| \mathbf{\sin\theta}\,\hat{\eta}$$

$$\vec{\mathbf{PQ}} = \vec{\mathbf{AB}} \times \vec{\mathbf{AD}} = \left|\vec{\mathbf{AB}}\right|\left|\vec{\mathbf{AD}}\right| \mathbf{\sin\theta}\,\hat{\eta} \text{------------------------(i)}$$

Again, $\vec{\mathbf{AB}} = \left|\vec{\mathbf{AB}}\right|\hat{\eta}$ $\qquad [\because \textbf{Any Vector} = \textbf{Length of this Vector} \times \textbf{Unit Vector}]$

$$\vec{\mathbf{AB}} = \mathbf{dx}\,\hat{\eta} \qquad [\because \textbf{Any Vector} = \textbf{Length of this Vector} \times \textbf{Unit Vector}]$$

$$\vec{\mathbf{AB}} = \mathbf{dx}\,\hat{\mathbf{i}} \qquad [\hat{\eta} = \hat{\mathbf{i}}] \text{-------------------(ii)}$$

Again,

$$\vec{\mathbf{AD}} = \left|\vec{\mathbf{AD}}\right|\hat{\eta}$$

$$\vec{\mathbf{AD}} = \mathbf{dy}\,\hat{\eta}$$

$$\vec{\mathbf{AD}} = \mathbf{dy}\,\hat{\mathbf{j}} \qquad [\hat{\eta} = \hat{\mathbf{j}}] \text{---------------------------(iii)}$$

From (i)

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right| \mathbf{\sin\theta}\,\hat{\eta}$$

$$\vec{\mathbf{PQ}} = \vec{\mathbf{AB}} \times \vec{\mathbf{AD}} = \left|\vec{\mathbf{AB}}\right|\left|\vec{\mathbf{AD}}\right| \mathbf{\sin\theta}\,\hat{\eta}$$

$$\Rightarrow \vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \left|\mathbf{dx}\,\hat{\mathbf{i}}\right|\left|\mathbf{dy}\,\hat{\mathbf{j}}\right| \mathbf{\sin\theta}\,\hat{\eta} \qquad \text{[From (ii) \& (iii)]}$$

$$\Rightarrow \vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \mathbf{dxdy\,\sin\theta}\,\hat{\eta} \qquad [\because \left|dx\,\hat{i}\right| = dx \;\&\; \left|dy\,\hat{j}\right| = dy]$$

$$\Rightarrow \vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \mathbf{dxdy\,\sin(\frac{\pi}{2})}\,\hat{\eta}\ [\theta = \frac{\pi}{2}]$$

$$\Rightarrow \vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \mathbf{dxdy.1.}\hat{\eta}$$

$$\Rightarrow \vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \mathbf{dxdy}\,\hat{\eta}$$

$\Rightarrow \vec{PQ} = dx\,\hat{i} \times dy\,\hat{j} = dxdy\,\hat{k}$ ---------------(iv) $[\hat{\eta} = \hat{k}]$

$\Rightarrow \vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \mathbf{dS}\,\hat{\eta}$ ---------------(v) **[Let $\mathbf{dS = dxdy}$]**

$\Rightarrow \vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \mathbf{dS}\,\hat{\mathbf{k}}$ ---------------(vi) [Here $\hat{\eta} = \hat{k}$ since $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$] and $\mathbf{dS = dxdy}$]

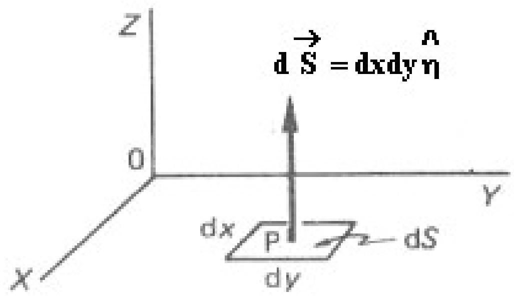


Figure # 108

$\mathbf{d\vec{S}}$ is the vector which is perpendicular to the plane $\mathbf{dxdy}$ and $\mathbf{dS}$ is the length of the perpendicular vector $\mathbf{d\vec{S}}$ as well as $\mathbf{dS}$ is the area of parallelogram $\mathbf{dxdy}$ and $\hat{\eta}$ is the unit normal vector of $\mathbf{d\vec{S}}$ to the plane $\mathbf{dxdy}$

$\Rightarrow \vec{\mathbf{dS}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \mathbf{dS}\,\hat{\eta}$ ----------------------(vii) [Here $\vec{\mathbf{PQ}} = \vec{\mathbf{dS}}$]

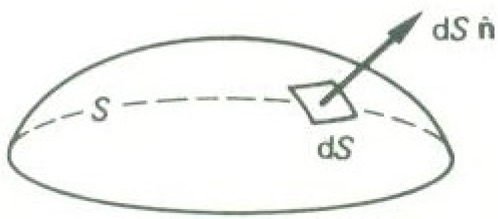


Figure # 109

So we can write, From Figure # 108

01. $\vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dy}\,\hat{\mathbf{j}} = \mathbf{dxdy}\,\hat{\mathbf{k}}$ [Anti-Clockwise; xy-plane; $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$]; $\mathbf{dS = dxdy}$]

02. $\vec{\mathbf{PQ}} = \mathbf{dy}\,\hat{\mathbf{j}} \times \mathbf{dx}\,\hat{\mathbf{i}} = -\mathbf{dxdy}\,\hat{\mathbf{k}}$ [Clockwise; xy-plane; $\hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}$; $\mathbf{dS = dxdy}$]

03. $\vec{\mathbf{PQ}} = \mathbf{dz}\,\hat{\mathbf{k}} \times \mathbf{dx}\,\hat{\mathbf{i}} = \mathbf{dxdz}\,\hat{\mathbf{j}}$ [Anti-Clockwise; xz-plane; $\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$; $\mathbf{dS = dxdz}$]

04. $\vec{\mathbf{PQ}} = \mathbf{dx}\,\hat{\mathbf{i}} \times \mathbf{dz}\,\hat{\mathbf{k}} = -\mathbf{dxdz}\,\hat{\mathbf{j}}$ [Clockwise; xz-plane; $\hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}$; $\mathbf{dS = dxdz}$]

05. $\vec{\mathbf{PQ}} = \mathbf{dy}\,\hat{\mathbf{j}} \times \mathbf{dz}\,\hat{\mathbf{k}} = \mathbf{dydz}\,\hat{\mathbf{i}}$ [Anti-Clockwise; yz-plane; $\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$; $\mathbf{dS = dydz}$]

06. $\vec{\mathbf{PQ}} = \mathbf{dz}\,\hat{\mathbf{k}} \times \mathbf{dy}\,\hat{\mathbf{j}} = -\mathbf{dydz}\,\hat{\mathbf{i}}$ [Clockwise; yz-plane; $\hat{\mathbf{k}} \times \hat{\mathbf{j}} = -\hat{\mathbf{i}}$; $\mathbf{dS = dydz}$]

Example:

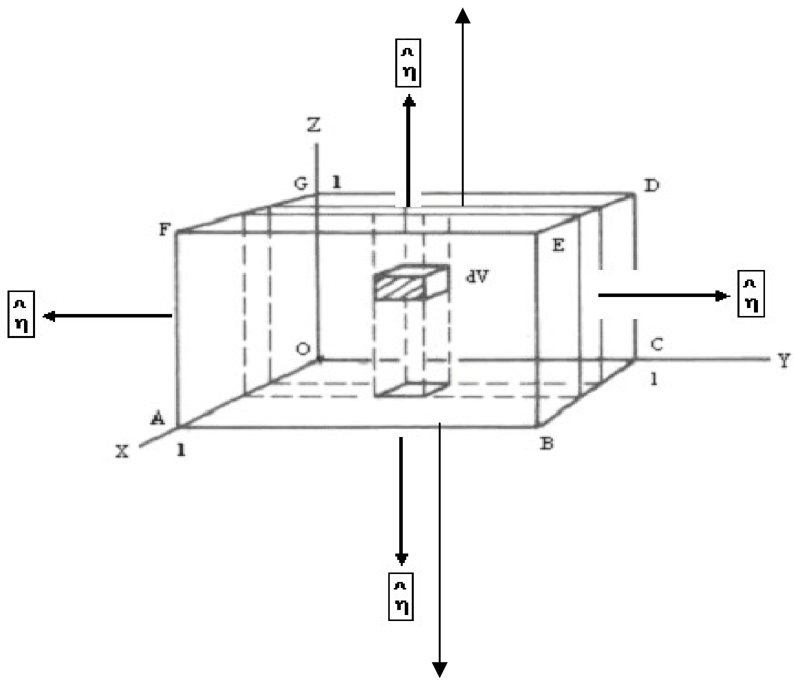


Figure # 110

From Figure # 110

Serial no	Surface	$\hat{\eta} = ?$	**ds**	Plane
1	OABC	$\hat{\eta} = -\hat{\mathbf{k}}$	**dxdy**	**z = 0**
2	DEFG	$\hat{\eta} = \hat{\mathbf{k}}$	**dxdy**	**z = 1**
3	OAFG	$\hat{\eta} = -\hat{\mathbf{j}}$	**dxdz**	**y = 0**
4	BCDE	$\hat{\eta} = \hat{\mathbf{j}}$	**dxdz**	**y = 1**
5	OCDG	$\hat{\eta} = -\hat{\mathbf{i}}$	**dydz**	**x = 0**
6	ABEF	$\hat{\eta} = \hat{\mathbf{i}}$	**dydz**	**x = 1**

Volume Integrals

Volume Integral

If $\vec{F}$ is a vector point function bounded by the region R with volume V, then $\int_v \vec{F}\, dV$ is called as Volume Integral

If V is a closed region bounded by a surface S and $\vec{F}$ is a vector field at each point of V and on its boundary surface S, then $\int_V \vec{F}\, dV$ is the volume integral of $\vec{F}$ throughout the region.

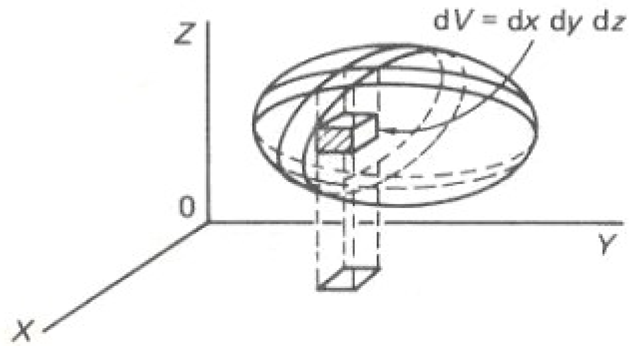


Figure # 111

$$dV = dxdydz$$

Then, $$\int_V \vec{F}\, dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \vec{F}\, dzdydx$$

Double Integral

Q # 87:

Example: $$\int_0^1\int_0^2 x^2 dxdy$$

$$= \int_0^1 \left[\int_0^2 x^2 dx\right] dy$$

$$= \int_0^1 \left[\frac{x^3}{3}\right]_0^2 dy$$

$$= \int_0^1 \left[\frac{2^3}{3} - \frac{0^3}{3}\right] dy$$

$$= \int_0^1 \left[\frac{8}{3} - 0\right] dy$$

$$= \int_0^1 \left[\frac{8}{3}\right] dy$$

$$= \frac{8}{3}\int_0^1 dy$$

$$= \frac{8}{3}[y]_0^1$$

$$= \frac{8}{3}[1-0]$$

$$= \frac{8}{3}[1]$$

$$= \frac{8}{3}$$

Or

Example: $\int_0^2\int_0^1 x^2 dydx$

$$= \int_0^2 \left[\int_0^1 x^2 dy\right] dx$$

$$= \int_0^2 \left[x^2 \int_0^1 dy\right] dx$$

$$= \int_0^2 x^2 [y]_0^1 dx$$

$$= \int_0^2 x^2 [1-0] dx$$

$$= \int_0^2 x^2 \, dx$$

$$= \left[\frac{x^3}{3}\right]_0^2$$

$$= \left[\frac{2^3}{3} - \frac{0^3}{3}\right]$$

$$= \left[\frac{8}{3} - 0\right]$$

$$= \frac{8}{3}$$

Q # 88: If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, Evaluate $\int_V \vec{F}\,dV$ where v is the bounded by the surfaces.

$$x = 0, x = 2, y = 0, y = 4, z = x^2, z = 2.$$

Answer: $\int_V \vec{F}\,dV = \iiint_V (2z\hat{i} - x\hat{j} + y\hat{k})dzdydx$

$$= \int_0^2 \int_0^4 \int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})dzdydx$$

$$= \int_0^2 \int_0^4 \left[(2\,z^2\!/_2\,\hat{i} - xz\hat{j} + yz\hat{k})\right]_{x^2}^2 dydx$$

$$= \int_0^2 \int_0^4 \left[z^2\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^2 dydx$$

$$= \int_0^2 \int_0^4 \left[4\hat{i} - 2x\hat{j} + 2y\hat{k} - x^4\hat{i} + x^3\hat{j} - x^2y\hat{k}\right]dydx$$

$$= \int_0^2 \left[4y\hat{i} - 2xy\hat{j} + y^2\hat{k} - x^4y\hat{i} + x^3y\hat{j} - x^2\frac{y^2}{2}\hat{k}\right]_0^4 dx$$

$$= \int_0^2 \left[16\hat{i} - 8x\hat{j} + 16\hat{k} - 4x^4\hat{i} + 4x^3\hat{j} - 8x^2\hat{k}\right]dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k} - \frac{4}{5}x^5\hat{i} + x^4\hat{j} - \frac{8}{3}x^3\hat{k}\right]_0^2$$

$$= 32\hat{i} - 16\hat{j} + 32\hat{k} - \frac{128}{5}\hat{i} + 16\hat{j} - \frac{8}{3}\times 8\hat{k}$$

$$= \frac{160-128}{5}\hat{i} + \frac{96-64}{3}\hat{k}$$

$$= \frac{32}{5}\hat{i} + \frac{32}{3}\hat{k} \text{ Answer}$$

Q # 89: Evaluate $\int_V \vec{F}\,dV$ where V is the region bounded by the planes:

$x = 0, x = 2, y = 0, y = 3, z = 0, z = 4$ and $\vec{F} = xy\hat{i} + z\hat{j} - x^2\hat{k}$.

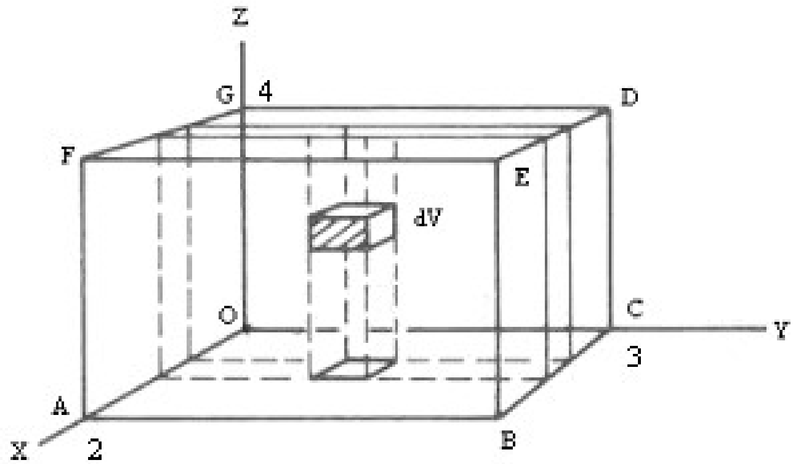


Figure # 112

Answer: $\int_V \vec{\mathbf{F}}\,\mathbf{dV} = \iiint_V (\mathbf{xy\hat{i} + z\hat{j} - x^2\hat{k})dxdydz}$

$$= \int_0^4\int_0^3\left[\int_0^2 (xy\hat{i} + z\hat{j} - x^2\hat{k})dx\right]dydz$$

$$= \int_0^4\int_0^3\left[(y\frac{x^2}{2}\hat{i} + xz\hat{j} - \frac{x^3}{3}\hat{k})\right]_0^2 dydz \qquad [\int x^n dx = \frac{x^{n+1}}{n+1}; \int dx = x]$$

$$= \int_0^4\int_0^3\left[(y\frac{2^2}{2}\hat{i} + 2.z\hat{j} - \frac{2^3}{3}\hat{k}) - (y\frac{0^2}{2}\hat{i} + 0.z\hat{j} - \frac{0^3}{3}\hat{k})\right]dydz$$

$$= \int_0^4\int_0^3\left[({}^{4y}\!/_2\,\hat{i} + 2z\hat{j} - \frac{8}{3}\hat{k})\right]dydz$$

$$= \int_0^4\left[\int_0^3 (2y\hat{i} + 2z\hat{j} - \frac{8}{3}\hat{k})dy\right]dz$$

$$= \int_0^4\left[2\frac{y^2}{2}\hat{i} + 2yz\hat{j} - \frac{8}{3}y\hat{k}\right]_0^3 dz \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1}; \int dy = y]$$

$$= \int_0^4\left[y^2\hat{i} + 2yz\hat{j} - \frac{8}{3}y\hat{k}\right]_0^3 dz$$

$$= \int_0^4\left[(3^2\hat{i} + 2\times 3.z.\hat{j} - \frac{8}{3}\times 3\hat{k}) - (0^2\hat{i} + 2\times 0.z\hat{j} - \frac{8}{3}\times 0.\hat{k})\right]dz$$

$$= \int_0^4 \left[9\hat{i} + 6z\hat{j} - 8\hat{k}\right]dz$$

$$= \int_0^4 9\hat{i}dz + \int_0^4 6z\hat{j}dz + \int_0^4 -8\hat{k}dz$$

$$= \left[9z\hat{i} + 6\frac{z^2}{2}\hat{j} - 8z\hat{k}\right]_0^4 \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1}; \int dz = z]$$

$$= \left[9z\hat{i} + 3z^2\hat{j} - 8z\hat{k}\right]_0^4$$

$$= \left[(9\times 4\hat{i} + 3\times 4^2\hat{j} - 8\times 4\hat{k}) - (9\times 0\hat{i} + 3\times 0^2\hat{j} - 8\times 0\hat{k})\right]$$

$$= \left[36\hat{i} + 3\times 16\hat{j} - 32\hat{k}\right]$$

$$= 4\left[9\hat{i} + 12\hat{j} - 8\hat{k}\right]$$

Q # 90: Show that $\iint \vec{F}\cdot\hat{\eta}\,ds = \frac{3}{2}$; Where $\vec{F} = 4xz\,\hat{i} - y^2\,\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

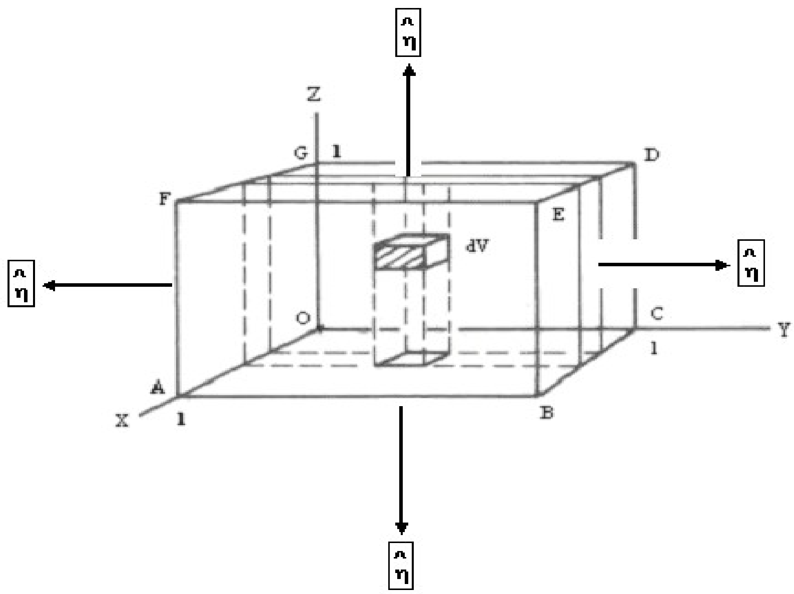


Figure # 113

Serial no	Surface	$\hat{\eta} = ?$	ds	Plane
1	OABC	$\hat{\eta} = -\hat{k}$	dxdy	$z = 0$
2	DEFG	$\hat{\eta} = \hat{k}$	dxdy	$z = 1$
3	OAFG	$\hat{\eta} = -\hat{j}$	dxdz	$y = 0$
4	BCDE	$\hat{\eta} = \hat{j}$	dxdz	$y = 1$
5	OCDG	$\hat{\eta} = -\hat{i}$	dydz	$x = 0$
6	ABEF	$\hat{\eta} = \hat{i}$	dydz	$x = 1$

Now,

$$\iint_S \vec{F}.\hat{\eta}\,ds = \iint_{OABC} \vec{F}.\hat{\eta}\,ds + \iint_{DEFG} \vec{F}.\hat{\eta}\,ds + \iint_{OAFG} \vec{F}.\hat{\eta}\,ds + \iint_{BCDE} \vec{F}.\hat{\eta}\,ds + \iint_{OCDG} \vec{F}.\hat{\eta}\,ds + \iint_{ABEF} \vec{F}.\hat{\eta}\,ds \text{ -(1)}$$

1. $$\iint_{OABC} \vec{F}.\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).(-\hat{k})dxdy$$

$[\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0]$

$$= \int_0^1\int_0^1 -yzdxdy$$

$$= -\int_0^1\int_0^1 yzdxdy = -\int_0^1 [yzx]_0^1 dy \qquad [\because \int dx = x]$$

$$= -\int_0^1 [yz \times 1 - yz \times 0]dy = -\int_0^1 [yz(1-0)]dy$$

$$= -\left[z\frac{y^2}{2}\right]_0^1 = -\left[z\frac{1^2}{2} - z\frac{0^2}{2}\right] = -\frac{z}{2}(1-0) = -\frac{z}{2} = 0 [\because z = 0]$$

2.

$$\iint_{DEFG} \vec{F}.\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).(\hat{k})dxdy = \int_0^1\int_0^1 yzdxdy$$

$[\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0]$

$$= \int_0^1\int_0^1 yzdxdy = \int_0^1 [yzx]_0^1 dy = \int_0^1 [yz \times 1 - yz \times 0]dy \qquad [\because \int dx = x]$$

$$= \int_0^1 [yz]dy = \left[z\frac{y^2}{2}\right]_0^1 = \left[z\frac{1^2}{2} - z\frac{0^2}{2}\right] = \frac{z}{2}(1-0) = \frac{z}{2} = \frac{1}{2} [\because z = 1]$$

3.

$$\iint_{OAFG} \vec{F}.\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).(-\hat{j})dxdz = \int_0^1\int_0^1 y^2dxdz$$

$[\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0]$

$$= \int_0^1\int_0^1 y^2dxdz = \int_0^1 [xy^2]_0^1 dz \qquad [\because \int dx = x]$$

$$= \int_0^1 [1 \times y^2 - 0 \times y^2]dz$$

$$= \int_0^1 [y^2 - 0]dz = [y^2z]_0^1 = [y^2 \times 1 - y^2 \times 0] \qquad [\because \int dz = z]$$

$$= y^2(1-0) = y^2 = 0 \qquad [\because y = 0]$$

4.

$$\iint_{BCDE} \vec{F}.\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).(\hat{j})dxdz = \int_0^1\int_0^1 -y^2 dxdz$$

$[\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0]$

$$= -\int_0^1\int_0^1 y^2 dxdz = -\int_0^1 \left[xy^2\right]_0^1 dz = -\int_0^1 \left[1\times y^2 - 0\times y^2\right] dz \qquad [\because \int dx = x]$$

$$= -\int_0^1 \left[y^2 - 0\right]dz = -\left[y^2 z\right]_0^1 \qquad [\because \int dz = z]$$

$$= -\left[y^2 \times 1 - y^2 \times 0\right]$$

$$= -y^2(1-0) = -y^2 = -1 \qquad [\because y = 1]$$

5. $$\iint_{OCDG} \vec{F}.\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).(-\hat{i})dydz$$

$[\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0]$

$$= \int_0^1\int_0^1 -4xzdydz$$

$$= -\int_0^1\int_0^1 4xzdydz = -\int_0^1 \left[4xyz\right]_0^1 dz = -\int_0^1 \left[4x\times 1\times z - 4x\times 0\times z\right]dz \qquad [\because \int dy = y]$$

$$= -\int_0^1 \left[4xz(1-0)\right]dz = -\left[4x\frac{z^2}{2}\right]_0^1 = -\left[4x\frac{1^2}{2} - 4x\frac{0^2}{2}\right]$$

$$= -4x(\frac{1}{2} - 0) = -2x = 0 \qquad [\because x = 0]$$

6. $$\iint_{ABEF} \vec{F}.\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).\hat{\eta}\,ds = \iint_S (4xz\hat{i} - y^2\hat{j} + yz\hat{k}).(\hat{i})dydz$$

$[\because \hat{i}.\hat{i} = 1, \hat{j}.\hat{j} = 1, \hat{k}.\hat{k} = 1, \hat{i}.\hat{j} = 0, \hat{i}.\hat{k} = 0, \hat{j}.\hat{i} = 0, \hat{j}.\hat{k} = 0, \hat{k}.\hat{i} = 0, \hat{k}.\hat{j} = 0]$

$$= \int_0^1\int_0^1 4xzdydz$$

$$= \int_0^1\int_0^1 4xzdydz = \int_0^1 \left[4xyz\right]_0^1 dz = \int_0^1 \left[4x\times 1\times z - 4x\times 0\times z\right]dz \qquad [\because \int dy = y]$$

$$= \int_0^1 \left[4xz(1-0)\right]dz = \left[4x\frac{z^2}{2}\right]_0^1 = \left[4x\frac{1^2}{2} - 4x\frac{0^2}{2}\right]$$

$$= 4x(\frac{1}{2} - 0) = 2x = 2 \qquad [\because x = 1]$$

Putting the values in (1),

Now,

$$\iint\limits_{S} \vec{F}.\hat{\eta}ds = \iint\limits_{OABC} \vec{F}.\hat{\eta}ds + \iint\limits_{DEFG} \vec{F}.\hat{\eta}ds + \iint\limits_{OAFG} \vec{F}.\hat{\eta}ds + \iint\limits_{BCDE} \vec{F}.\hat{\eta}ds + \iint\limits_{OCDG} \vec{F}.\hat{\eta}ds + \iint\limits_{ABEF} \vec{F}.\hat{\eta}ds$$

$$\iint\limits_{S} \vec{F}.\hat{\eta}ds = 0 + \frac{1}{2} + 0 - 1 + 0 + 2 = \frac{1}{2} + 1 = \frac{3}{2} \text{ (Proved)}$$

Area Enclosed by a closed Curve:

One of the earliest applications of integration is finding the area of a plane figure bounded by the x-axis, the curve $y = f(x)$ and ordinates at $x = x_1$ and $x = x_2$.

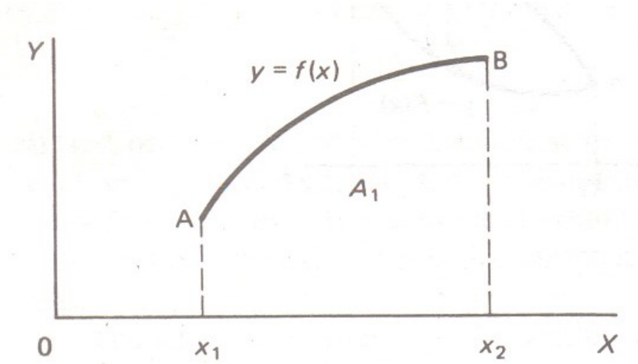


Figure # 114

$$A_1 = \int_{x_1}^{x_2} ydx = \int_{x_1}^{x_2} f(x)dx \text{ ------------------------------------(i)}$$

If points A and B are joined by another curve $y = F(x)$

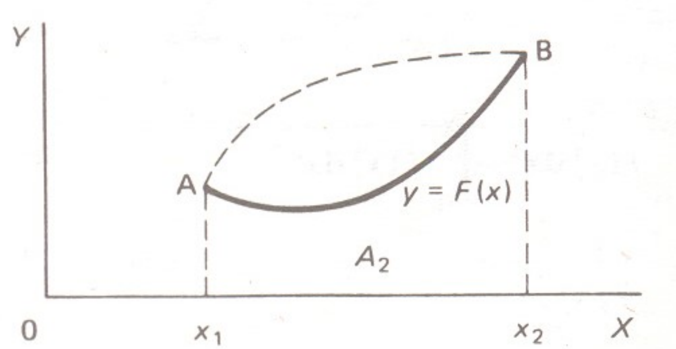


Figure # 115

$$A_2 = \int_{x_1}^{x_2} ydx = \int_{x_1}^{x_2} F(x)dx \text{ ---------------------------------(ii)}$$

Combining the two figures, we have

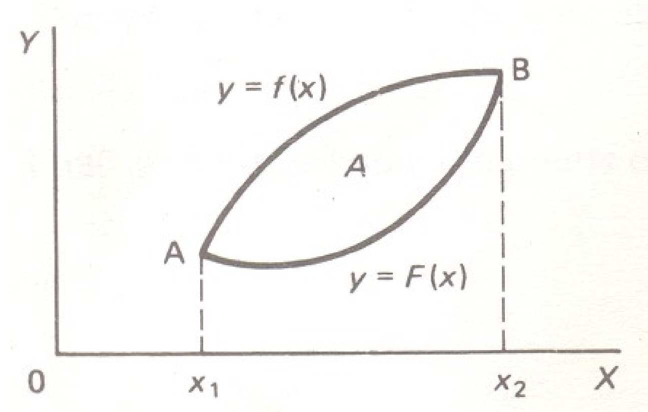


Figure # 116

$$\mathbf{A = A_1 - A_2}$$

$$\therefore \mathbf{A} = \int_{x_1}^{x_2} \mathbf{f(x)dx} - \int_{x_1}^{x_2} \mathbf{F(x)dx} \text{ ----------------------------------(iii)}$$

It is convenient on occasions to arrange the limits so that the integration follows the path round the enclosed area in a regular order.

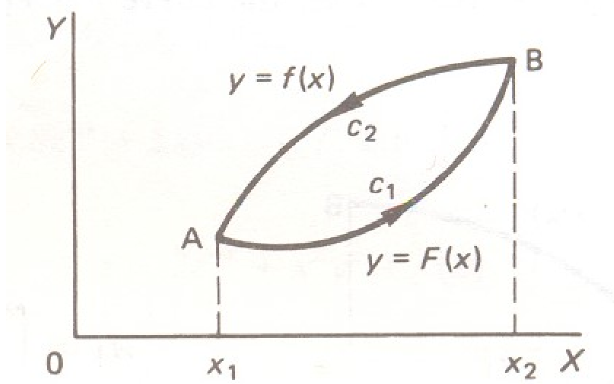


Figure # 117

For example

$\int_{x_1}^{x_2} \mathbf{F(x)dx}$ gives $\mathbf{A_2}$ as before , but integrating from B to A along $\mathbf{c_2}$ with $\mathbf{y = f(x)}$, i.e.

$\int_{x_1}^{x_2} \mathbf{f(x)dx}$, is the integral for $\mathbf{A_1}$ with the sign changed, i.e.

We can write, For Upper Curve $\boldsymbol{y = f(x)}$;

$$\int_{x_2}^{x_1} \mathbf{f(x)dx} = -\int_{x_1}^{x_2} \mathbf{f(x)dx} \qquad \text{--(iv)}$$

4-2 = 2
-(2-4)= -(-2)=2

$$\Rightarrow -\int_{x_2}^{x_1} \mathbf{f(x)dx} = -\left[-\int_{x_1}^{x_2} \mathbf{f(x)dx}\right]$$ [Multiplying by negative sign on both sides]

$$\Rightarrow -\int_{x_2}^{x_1} \mathbf{f(x)dx} = \int_{x_1}^{x_2} \mathbf{f(x)dx}$$

$$\therefore \int_{x_1}^{x_2} \mathbf{f(x)dx} = -\int_{x_2}^{x_1} \mathbf{f(x)dx}$$ [Side change]------------------------(v)

From (iii),

$$\therefore \text{The result } \mathbf{A = A_1 - A_2} = \int_{x_1}^{x_2} \mathbf{f(x)dx} - \int_{x_1}^{x_2} \mathbf{F(x)dx}$$ [From (iii)]

$$= -\int_{x_2}^{x_1} \mathbf{f(x)dx} - \int_{x_1}^{x_2} \mathbf{F(x)dx}$$ [From (v)]

$$= -\{\int_{x_2}^{x_1} \mathbf{f(x)dx} + \int_{x_1}^{x_2} \mathbf{F(x)dx}\}$$

$$= -\{\int_{x_1}^{x_2} \mathbf{F(x)dx} + \int_{x_2}^{x_1} \mathbf{f(x)dx}\}$$ --------------------------(vi)

If we proceed round the boundary in an anticlockwise manner, the enclosed area is kept on the left-hand side and the resulting area is considered positive. If we proceed round the boundary in a clockwise manner, the enclosed area remains on the right-hand side and the resulting area is negative.

The final result above can be written in the form

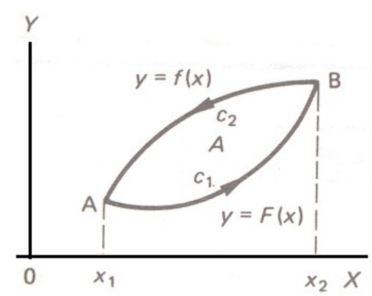


Figure # 118

$\mathbf{A} = -\oint \mathbf{ydx}$; Where the symbol $\oint$ indicates that the integral is to be evaluated round the closed boundary in the positive (i.e. anticlockwise) direction.

$$\therefore \mathbf{A} = -\oint \mathbf{ydx} = -\{\int_{x_1}^{x_2} \mathbf{F(x)dx} + \int_{x_2}^{x_1} \mathbf{f(x)dx}\}$$ ----------------------------(vii)

Q # 91: Determine the area enclosed by the graphs of $\mathbf{y = x^3}$ and $\mathbf{y = 4x}$ for $\mathbf{x \geq 0}$

Answer: First we need to know the points of intersection.

Given, $\mathbf{y = x^3}$ and $\mathbf{y = 4x}$

Then, we can write, $\mathbf{x^3 = 4x}$

$$\Rightarrow \mathbf{x^3 - 4x = 0}$$
$$\Rightarrow \mathbf{x(x^2 - 4) = 0}$$
$$\Rightarrow \mathbf{x(x+2)(x-2) = 0}$$

$\Rightarrow \mathbf{x = 0}$ and $\mathbf{x = -2}$, $\mathbf{x = 2}$

But $\mathbf{x \geq 0}$

$\therefore \mathbf{x = 0}$ and $\mathbf{x = 2}$

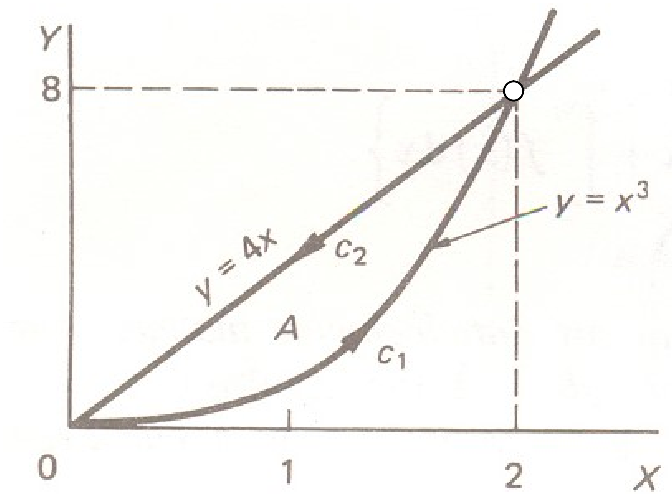


Figure # 119

We integrate in an anticlockwise manner

$\mathbf{c_1 : y = x^3,}$ Limits $\mathbf{x = 0}$ to $\mathbf{x = 2}$

$\mathbf{c_2 : y = 4x,}$ Limits $\mathbf{x = 2}$ to $\mathbf{x = 0}$

We have,

$$\therefore \mathbf{A = -\oint y dx = -\{\int_{x_1}^{x_2} F(x)dx + \int_{x_2}^{x_1} f(x)dx\}}$$

$$\mathbf{= -\{\int_0^2 F(x)dx + \int_2^0 f(x)dx\}}$$

$$\mathbf{= -\{\int_0^2 x^3 dx + \int_2^0 4x dx\}}$$

$$\mathbf{= -\{\left[\frac{x^4}{4}\right]_0^2 + 4\left[\frac{x^2}{2}\right]_2^0\}}$$

$$= -\left\{\left[\frac{2^4}{4} - 0\right] + 4\left[\frac{0^2}{2} - \frac{2^2}{2}\right]\right\}$$

$$= -\left\{\left[\frac{16}{4}\right] + 4\left[-\frac{4}{2}\right]\right\}$$

$= \mathbf{4}$ Answer

Q # 92: Find the area of the triangle with vertices **(0,0)** ,**(5,3)** and **(2,6)**

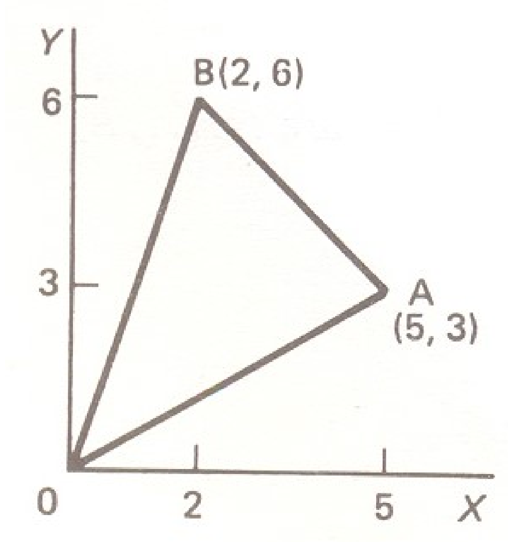


Figure # 120

Answer: We have,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

∴ The equation of **OA** or **AO** is

$$\frac{y-0}{0-3} = \frac{x-0}{0-5}$$

$$\Rightarrow \frac{y}{-3} = \frac{x}{-5}$$

$$\Rightarrow \frac{y}{3} = \frac{x}{5}$$

$$\Rightarrow y = \frac{3x}{5}$$

The equation of **BA** or **AB** is

$$\frac{y-6}{6-3} = \frac{x-2}{2-5}$$

$$\Rightarrow \frac{y-6}{3} = \frac{x-2}{-3}$$

$$\Rightarrow \frac{y-6}{1} = \frac{x-2}{-1}$$

$$\Rightarrow \frac{y-6}{1} = -x + 2$$

$$\Rightarrow y = 8 - x$$

The equation of **OB** or **BO** is
$$\frac{y-0}{0-6} = \frac{x-0}{0-2}$$
$$\Rightarrow \frac{y-0}{-6} = \frac{x-0}{-2}$$
$$\Rightarrow \frac{y}{6} = \frac{x}{2}$$
$$\Rightarrow \frac{y}{3} = \frac{x}{1}$$
$$\Rightarrow y = 3x$$

Then,

$$\therefore A = -\oint ydx = -[\int_{OA} f(x)dx + \int_{AB} f(x)dx + \int_{BO} f(x)dx]$$

$$\therefore A = -\oint ydx = -\{\int_0^5 \frac{3}{5}xdx + \int_5^2 (8-x)dx + \int_2^0 3xdx$$

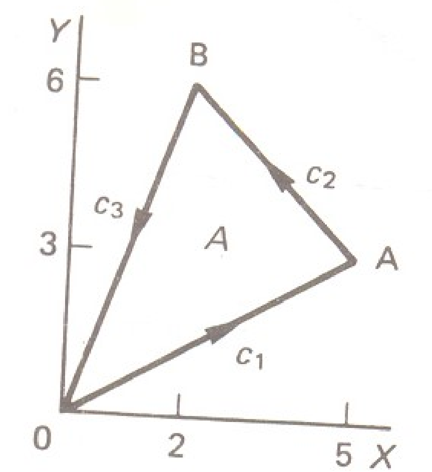


Figure # 121

The limits chosen must progress the integration round the boundary of the figure in an anticlockwise manner.

We get, $\therefore A = -\oint ydx = -\{\int_0^5 \frac{3}{5}xdx + \int_5^2 (8-x)dx + \int_2^0 3xdx\}$

$$= -\{\frac{3}{5}\left[\frac{x^2}{2}\right]_0^5 + \left[8x - \frac{x^2}{2}\right]_5^2 + 3\left[\frac{x^2}{2}\right]_2^0\}$$

$$= -\{\frac{3}{5}\left[\frac{5^2}{2} - 0\right] + \left[(8\times 2 - \frac{2^2}{2}) - (8\times 5 - \frac{5^2}{2})\right] + 3\left[\frac{0^2}{2} - \frac{2^2}{2}\right]\}$$

$$= -\{\frac{3}{5}\left[\frac{25}{2}\right] + \left[(16 - \frac{4}{2}) - (40 - \frac{25}{2})\right] + 3\left[0 - \frac{4}{2}\right]\}$$

$$= -\{\frac{3}{5}\left[\frac{25}{2}\right] + \left[(16 - 2) - (40 - \frac{25}{2})\right] + 3\left[-\frac{4}{2}\right]\}$$

$$= -\{\frac{3}{5}\left[\frac{25}{2}\right] + \left[14 - (40 - \frac{25}{2})\right] - 3\left[\frac{4}{2}\right]\}$$

$$= -\{3\left[\frac{5}{2}\right] + \left[14 - (40 - \frac{25}{2})\right] - 6\}$$

$$= -\{\left[\frac{15}{2}\right] + \left[14 - (40 - \frac{25}{2})\right] - 6\}$$

$$= -\{\left[\frac{15 + 28 - 80 + 25 - 12}{2}\right]$$

$$= -\{\left[\frac{-24}{2}\right]$$

$= \mathbf{12}$ Square units

Another method of a line Integral

It is often more convenient to integrate with respect to x or y than to take arc length as the variable.

We have,

Work done = $\int_{AB} \vec{F}\,.\,d\vec{S} = \int_{AB} (\mathbf{Pdx + Qdy})$ -------- -----------(iii)

In three dimension,

Work done = $\int_{AB} \vec{F}\,.\,d\vec{S} = \int_{AB} (\mathbf{Pdx + Qdy + Rdz}))$ ------------(iv)

Q # 93: Evaluate $\int_{c} \mathbf{(x + 3y)dx}$ from $\mathbf{A(0,1)}$ to $\mathbf{B(2,5)}$ along the curve $\mathbf{y = 1 + x^2}$

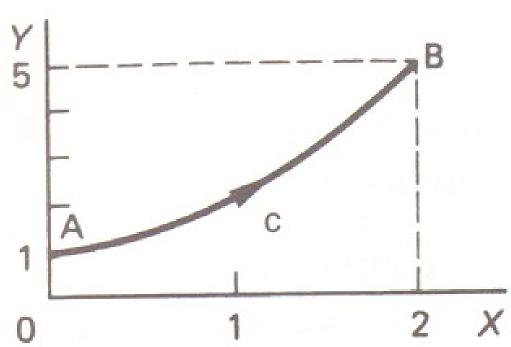


Figure # 122

Answer:

Work done = $\int_{AB} \vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{S}} = \int_{AB} (\mathbf{Pdx} + \mathbf{Qdy})$

Given, $\int_{c} \mathbf{(x + 3y)dx}$

The line integral is of the form: $\int_c \mathbf{(Pdx + Qdy)}$ where, in this case, $\mathbf{P = x + 3y}$ and $\mathbf{Q = 0}$ and c is the curve $\mathbf{y = 1 + x^2}$

It can be converted at once into an ordinary integral by substituting for y and applying the appropriate limits of x.

$$I = \int_c (Pdx + Qdy) = \int_c (x + 3y)dx = \int_0^2 \{x + 3(1 + x^2)\}dx = \int_0^2 (x + 3 + 3x^2)dx$$

$$= \left[\frac{\mathbf{x^2}}{\mathbf{2}} + \mathbf{3x} + \mathbf{3}\frac{\mathbf{x^3}}{\mathbf{3}}\right]_0^2 = \left[(\frac{\mathbf{2^2}}{\mathbf{2}} + \mathbf{3 \times 2} + \mathbf{3 \times}\frac{\mathbf{2^3}}{\mathbf{3}}) - \mathbf{0}\right] = \mathbf{16}$$

Line Integrals round a closed curve

We have already introduced the symbol $\oint$ to indicate that an integral is to be evaluated round a closed curve in the positive (anticlockwise) direction.

Positive (anticlockwise) direction line integral denoted by $\oint$

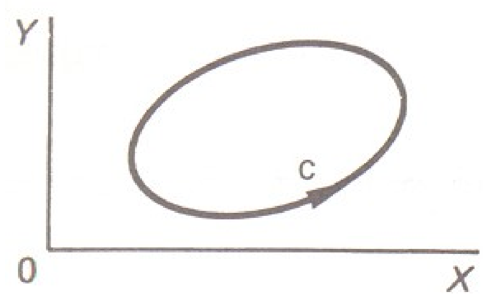


Figure # 123

Negative (clockwise) direction line integral denoted by $-\oint$

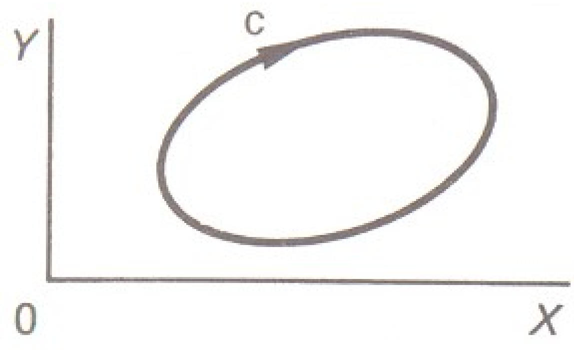
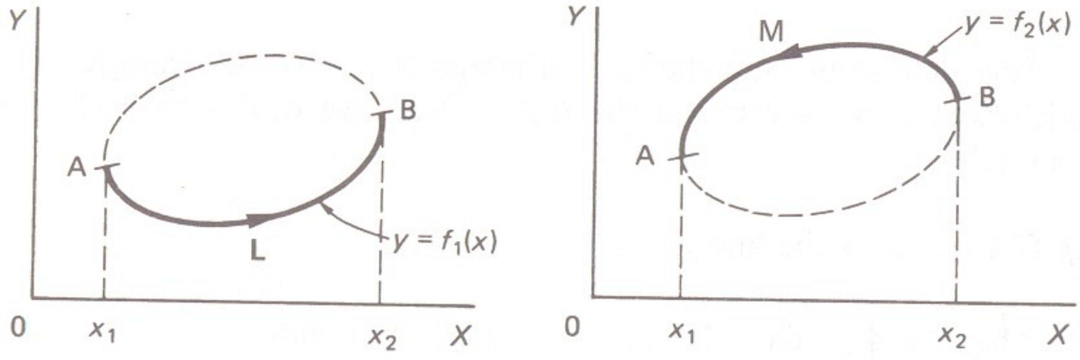


Figure # 124

With a closed curve, the path c cannot be single-valued. Therefore, we divide the path into two or more parts and treat each separately as a single-valued curve.



(i) Use $y = f_1(x)$ for ALB (ii) Use $y = f_2(x)$ for BMA.

Figure # 125

Unless specially required otherwise, we always proceed round the closed curve in an anticlockwise direction

Q # 94: Evaluate the line integral $\mathbf{I = \oint_C (x^2 dx - 2xydy)}$ where c comprises the three sides of the triangle joining **O(0,0), A(1,0)** and **B(0,1)**

Answer:

Work done = $\int_{AB} \vec{F} \cdot d\vec{S} = \int_{AB} (Pdx + Qdy)$

First draw the diagram and mark in c_1, c_2 and c_3 the proposed directions of integration. Do just that.

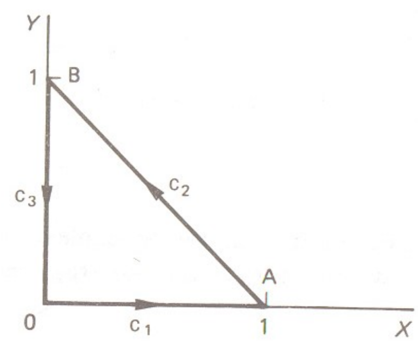


Figure # 126

The three sections of the path of integration must be arranged in an anticlockwise manner round the figure. Now we deal with each part separately.

a) The equation of OA is:

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2} \Rightarrow \frac{y - 0}{0 - 0} = \frac{x - 0}{0 - 1} \Rightarrow \frac{y}{0} = \frac{x}{-1} \Rightarrow -y = x.0 \Rightarrow y = 0$$

OA : $\mathbf{c_1}$ is the line $\mathbf{y = 0}\ \therefore \mathbf{dy = 0}$, Then

$\mathbf{I_1 = \oint (x^2dx - 2xydy)}$ for this part becomes

$$\mathbf{I_1 = \int_0^1 (x^2dx - 2x.0.0) = \int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}}$$

b) **AB :** $\mathbf{c_2}$ is the line: $\mathbf{\frac{y-0}{0-1} = \frac{x-1}{1-0} \Rightarrow \frac{y}{-1} = \frac{x-1}{1} \Rightarrow y = -x+1}$

$$\mathbf{\therefore \frac{dy}{dx} = -1 + 0 \Rightarrow dy = -dx}$$

Then

$\mathbf{I_2 = \oint (x^2dx - 2xydy)}$ for this part becomes

$$\mathbf{I_2 = \int_1^0 \{x^2dx - 2x(-x+1)(-dx)\} = \int_1^0 (x^2 - 2x^2 + 2x)(dx) = \int_1^0 (2x - x^2)dx = \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^0 = -\frac{2}{3}}$$

c) **BO :** $\mathbf{c_3}$ is the line $\mathbf{x = 0}\ \therefore \mathbf{dx = 0}$, Then

$I_3 = \oint (x^2dx - 2xydy)$ for this part becomes

$$I_3 = \oint (0^2.0 - 2.0.ydy) = 0$$

Finally $\mathbf{I = I_1 + I_2 + I_3 = \frac{1}{3} - \frac{2}{3} + 0 = -\frac{1}{3}}$

Q # 95: Evaluate the area of a circle $\mathbf{x^2 + y^2 = 4}$.

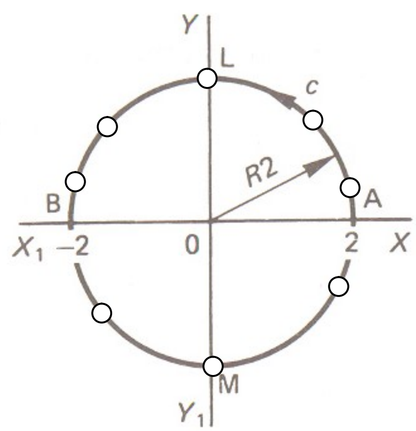


Figure # 127

Answer: Given $\mathbf{x^2 + y^2 = 4}$

$$\mathbf{\Rightarrow y^2 = 4 - x^2}$$

$$\mathbf{\Rightarrow y = \pm\sqrt{4 - x^2}}$$

The Equation of the upper curve ALB: $\mathbf{y = +\sqrt{4-x^2}}$ between $\mathbf{x = 2}$ and $\mathbf{x = -2}$

And The equation of the lower curve BMA : $\mathbf{y = -\sqrt{4 - x^2}}$ between $\mathbf{x = -2}$ and $\mathbf{x = 2}$

We have, Area: $A = -\oint y dx = -\{\int_{x_1}^{x_2} \mathbf{F(x)dx} + \int_{x_2}^{x_1} \mathbf{f(x)dx}\}$

$$\therefore \mathbf{A} = -[\int_{-2}^{2} \{-(\sqrt{4 - x^2})dx + \int_{2}^{-2} \sqrt{4 - x^2}dx]$$

$$\therefore A = -[\int_{2}^{-2} \sqrt{4 - x^2}\,dx + \int_{-2}^{2} \{-(\sqrt{4 - x^2})dx]$$

$$\therefore A = -[\int_{2}^{-2} \sqrt{4 - x^2}\,dx - \int_{-2}^{2} \sqrt{4 - x^2}\,dx]$$

$$\therefore A = -[\int_{2}^{-2} \sqrt{4 - x^2}\,dx + \int_{2}^{-2} \sqrt{4 - x^2}\,dx]$$

$$\therefore A = -[2\int_{2}^{-2} \sqrt{4 - x^2}\,dx]$$

$$\therefore A = (-2)(-1)\int_{-2}^{2} \sqrt{4 - x^2}\,dx$$

$$\therefore A = 2\int_{-2}^{2} \sqrt{4 - x^2}\,dx \text{ ------------------------------(i)}$$

$$\therefore A = 2 \times 2\int_{0}^{2} \sqrt{4 - x^2}\,dx \qquad [\because \int_{-a}^{+a} f(x)dx = 2\int_{0}^{a} f(x)dx]$$

$$\therefore A = 4\int_{0}^{2} \sqrt{4 - x^2}\,dx \text{ ------------------------------(ii)}$$

Let, $\mathbf{x = 2\sin\theta}$

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}(2\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 2\cos\theta$$

$$\Rightarrow dx = 2\cos\theta d\theta$$

Now, $\sqrt{4 - x^2} = \sqrt{4 - (2\sin\theta)^2} = \sqrt{4 - 4\sin^2\theta} = \sqrt{4(1 - \sin^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta$

Given, $\mathbf{x = 2\sin\theta}$

$$\Rightarrow \theta = \sin^{-1}(x/2)$$

x	0	2
$\theta = \sin^{-1}(x/2)$	$\theta = \sin^{-1}(x/2)$ $\Rightarrow \theta = \sin^{-1}(0/2)$ $\Rightarrow \theta = \sin^{-1}(0)$	$\theta = \sin^{-1}(x/2)$ $\Rightarrow \theta = \sin^{-1}(2/2)$ $\Rightarrow \theta = \sin^{-1}(1)$

	$\Rightarrow \theta = \sin^{-1}\sin 0$ $\Rightarrow \theta = 0$	$\Rightarrow \theta = \sin^{-1}(\sin\frac{\pi}{2})$ $\Rightarrow \theta = \frac{\pi}{2}$

From (ii),

$$\therefore A = 4\int_0^2 \sqrt{4-x^2}\,dx$$

$$\therefore A = 4\int_0^{\pi/2} \sqrt{4-(2\sin\theta)^2}\ \ 2\cos\theta\, d\theta$$

$$\therefore A = 4\int_0^{\pi/2} \sqrt{4-4\sin^2\theta}\ \ 2\cos\theta\, d\theta$$

$$\therefore A = 4\int_0^{\pi/2} \sqrt{4(1-\sin^2\theta)}\ \ 2\cos\theta\, d\theta$$

$$\therefore A = 4\int_0^{\pi/2} \sqrt{4\cos^2\theta}\ \ 2\cos\theta\, d\theta$$

$$\therefore A = 4\int_0^{\pi/2} 2\cos\theta\times 2\cos\theta\, d\theta$$

$$\therefore A = 4\int_0^{\pi/2} 4\cos^2\theta\, d\theta$$

$$\therefore A = 4\times 2\int_0^{\pi/2} 2\cos^2\theta\, d\theta$$

$$\therefore A = 8\int_0^{\pi/2} 2\cos^2\theta\, d\theta$$

$$\therefore A = 8\int_0^{\pi/2} (1+\cos 2\theta)d\theta \qquad [[2\cos^2\theta = 1+\cos 2\theta]]$$

$$\therefore A = 8\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$\therefore A = 8\left[(\frac{\pi}{2}-0)+(\frac{\sin 2\times\frac{\pi}{2}}{2}-0)\right]$$

$$\therefore A = 8\left[\frac{\pi}{2} + \frac{\sin\pi}{2}\right]$$

$$\therefore A = 8\left[\frac{\pi}{2} + \frac{0}{2}\right]$$

$$\therefore A = 8\left[\frac{\pi}{2}\right]$$

$\therefore A = 4\pi$ Answer

Or from (i),

$$\therefore A = 2\int_{-2}^{2}\sqrt{4 - x^2}\,dx \text{ ------------------------------(i)}$$

Let, $x = 2\sin\theta$

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}(2\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 2\cos\theta$$

$\Rightarrow dx = 2\cos\theta d\theta$

Now, $\sqrt{4 - x^2} = \sqrt{4 - (2\sin\theta)^2} = \sqrt{4 - 4\sin^2\theta} = \sqrt{4(1 - \sin^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta$

Given, $x = 2\sin\theta$

$$\frac{x}{2} = \sin\theta$$

$$\sin\theta = \frac{x}{2}$$

$\Rightarrow \theta = \sin^{-1}(x/2)$

x	-2	2
$\theta = \sin^{-1}(x/2)$	$\theta = \sin^{-1}(x/2)$ $\Rightarrow \theta = \sin^{-1}(-2/2)$ $\Rightarrow \theta = \sin^{-1}(-1)$ $\Rightarrow \theta = -\sin^{-1}(1)$ $\Rightarrow \theta = -\sin^{-1}(\sin\frac{\pi}{2})$ $\Rightarrow \theta = -\frac{\pi}{2}$	$\theta = \sin^{-1}(x/2)$ $\Rightarrow \theta = \sin^{-1}(2/2)$ $\Rightarrow \theta = \sin^{-1}(1)$ $\Rightarrow \theta = \sin^{-1}(\sin\frac{\pi}{2})$ $\Rightarrow \theta = \frac{\pi}{2}$

From (i),

$$\therefore A = 2\int_{-2}^{2} \sqrt{4 - x^2}\, dx$$

$$\therefore A = 2\int_{-\pi/2}^{\pi/2} \sqrt{4 - (2\sin\theta)^2}\; 2\cos\theta\, d\theta$$

$$\therefore A = 2\int_{-\pi/2}^{\pi/2} \sqrt{4 - 4\sin^2\theta}\; 2\cos\theta\, d\theta$$

$$\therefore A = 2\int_{-\pi/2}^{\pi/2} \sqrt{4(1 - \sin^2\theta)}\; 2\cos\theta\, d\theta$$

$$\therefore A = 2\int_{-\pi/2}^{\pi/2} \sqrt{4\cos^2\theta}\; 2\cos\theta\, d\theta$$

$$\therefore A = 2\int_{-\pi/2}^{\pi/2} 2\cos\theta \times 2\cos\theta d\theta$$

$$\therefore A = 2\int_{-\pi/2}^{\pi/2} 4\cos^2\theta d\theta$$

$$\therefore A = 2 \times 2\int_{-\pi/2}^{\pi/2} 2\cos^2\theta d\theta$$

$$\therefore A = 4\int_{-\pi/2}^{\pi/2} 2\cos^2\theta d\theta$$

$$\therefore A = 4\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta) d\theta \qquad [2\cos^2\theta = 1 + \cos 2\theta]$$

$$\therefore A = 4\int_{-\pi/2}^{\pi/2} 1 d\theta + 4\int_{-\pi/2}^{\pi/2} \cos 2\theta d\theta$$

$$\therefore A = 4[\theta]_{-\pi/2}^{\pi/2} + 4\left[\frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$\therefore A = 4\left[(\frac{\pi}{2} - (-\pi/2)) \right] + 4\left[(\frac{\sin 2 \times \frac{\pi}{2}}{2} - \frac{\sin 2 \times (-\frac{\pi}{2})}{2}) \right]$$

$$\therefore A = 4\left[\left(\frac{\pi}{2}+\frac{\pi}{2}\right)\right] + 4\left[\left(\frac{\sin\pi}{2}-\frac{\sin(-\pi)}{2}\right)\right]$$

$$\therefore A = 4\left[\left(\frac{2\pi}{2}\right)\right] + 4\left[\left(\frac{\sin\pi}{2}+\frac{\sin\pi}{2}\right)\right] \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$\therefore A = 4[\pi] + 4\left[\left(\frac{0}{2}+\frac{0}{2}\right)\right] \qquad [\because \sin\pi = 0]$$

$$\therefore A = 4[\pi] + 4.0$$

$\therefore A = 4\pi$ Answer

Justification

We know Area of a circle is πr^2

Here radius = r =2

$\therefore$ Area of a circle is $\pi r^2 = \pi \times 2^2 = \pi \times 4 = 4\pi$

Q # 96: If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$. Evaluate $\iint_s \vec{F}.\hat{\eta}\,ds$

Answer: Given, $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and the scalar function $y^2 = 8x$ .i.e.
$y^2 - 8x = 0$ implies $8x - y^2 = 0$

Let the scalar function $\phi(x, y) = 8x - y^2$ of the given parabolic surface.

We have, $\vec{\nabla}\phi$ **is** normal (perpendicular) vector to the surface.

Given, $\phi(x, y) = 8x - y^2$

$$\therefore \vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi = \left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right) = \hat{i}\frac{\delta}{\delta x}\phi + \hat{j}\frac{\delta}{\delta y}\phi + \hat{k}\frac{\delta}{\delta z}\phi$$

$$= \hat{i}\frac{\delta}{\delta x}(8x - y^2) + \hat{j}\frac{\delta}{\delta y}(8x - y^2) + \hat{k}\frac{\delta}{\delta z}(8x - y^2)$$

$$= 8\hat{i} - 2y\hat{j}$$

Let, $\hat{\eta}$ is the unit vector of $\vec{\nabla}\phi$

We can write,

$$\hat{\eta} = \frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}$$

$$\hat{\eta} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{8^2 + (-2y)^2}} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{64 + 4y^2}} = \frac{2(4\hat{i} - y\hat{j})}{\sqrt{4(16 + y^2)}}$$

$$= \frac{2(4\hat{i} - y\hat{j})}{2\sqrt{(16 + y^2)}} = \frac{(4\hat{i} - y\hat{j})}{\sqrt{(16 + y^2)}}$$

Given, $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$

$$\therefore \vec{F}.\hat{\eta} = (2y\hat{i} - z\hat{j} + x^2\hat{k}).\frac{(4\hat{i} - y\hat{j})}{\sqrt{(16 + y^2)}}$$

$$\therefore \vec{F}.\hat{\eta} = (2y\hat{i} - z\hat{j} + x^2\hat{k}).\frac{(4\hat{i} - y\hat{j} + o.\hat{k})}{\sqrt{(16 + y^2)}}$$

$$\therefore \vec{F}.\hat{\eta} = \frac{(8y + yz)}{\sqrt{(16 + y^2)}}$$

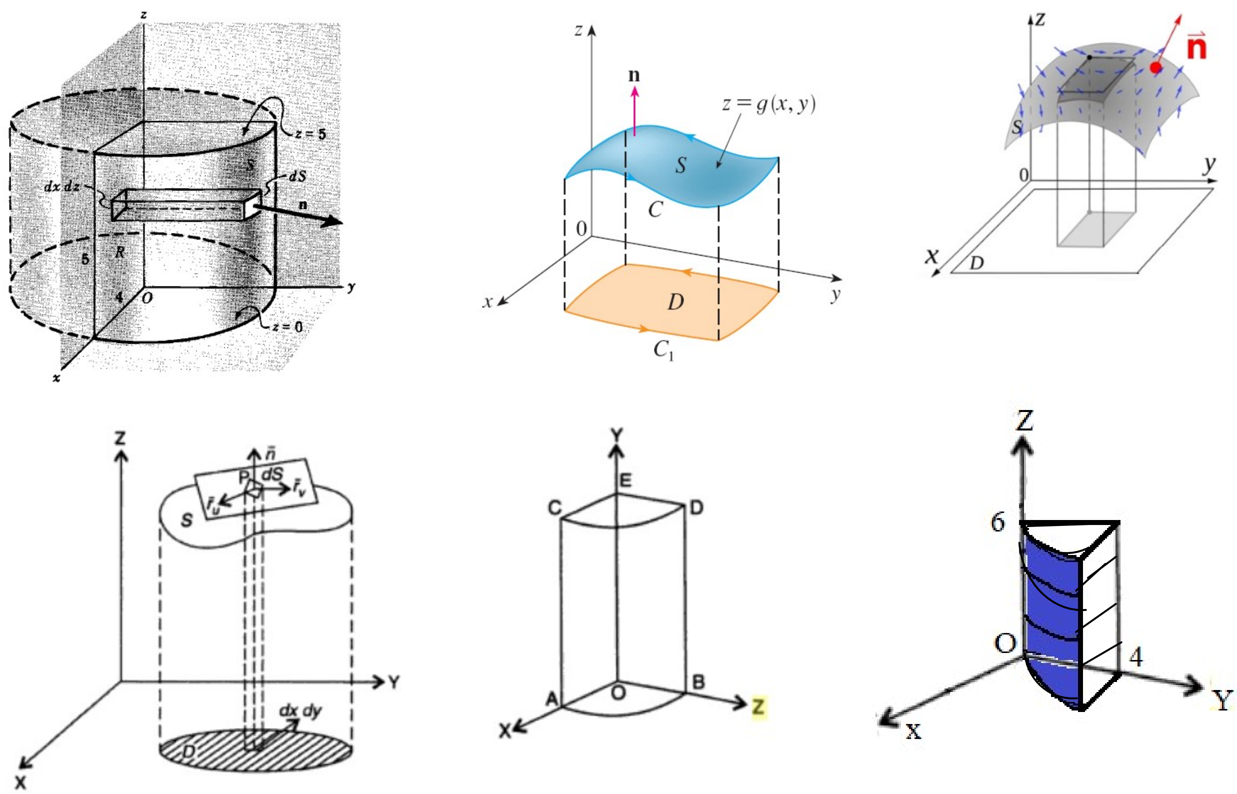


Figure # 128

A surface integral is evaluated by reducing it to a double integral by projecting the given surface S onto one of the coordinate planes. Let D be the projection of S onto the xy-plane . Then $dS = \dfrac{dxdy}{\left|\hat{\eta}.\hat{k}\right|}$; where $\hat{\eta}$ is the unit outward drawn normal to S. If D be the

projection of S onto the yz-plane . Then $dS = \dfrac{dydz}{\left|\hat{\eta}.\hat{i}\right|}$;

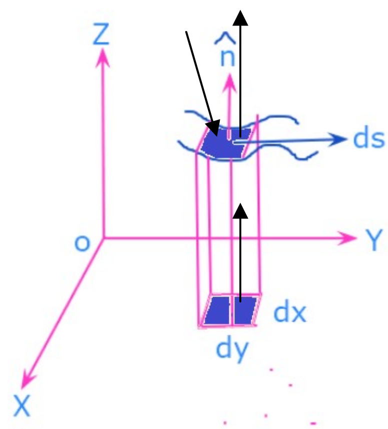


Figure # 129

$\therefore$ Projection of $\vec{\mathbf{A}}$ on $\vec{\mathbf{B}}$ **is** $= \dfrac{\vec{A}.\vec{B}}{\left|\vec{B}\right|}$ ----------------(i)

Here, $\vec{A} = ds\,\hat{\eta}$; $\vec{B} = dxdy\,\hat{k}$; dxdy is the projection of ds

From (i), $\dfrac{\vec{A}.\vec{B}}{\left|\vec{B}\right|} = dxdy$

$\Rightarrow \dfrac{ds\,\hat{\eta}.dxdy\,\hat{k}}{dxdy} = dxdy$

$\Rightarrow \dfrac{ds\left|\hat{\eta}.\hat{k}\right|}{1} = dxdy$

$\Rightarrow ds = \dfrac{dxdy}{\left|\hat{\eta}.\hat{k}\right|}$ --------------------(ii)

Similarly, If D be the projection of S onto the yz-plane . Then $dS = \dfrac{dydz}{\left|\hat{\eta}.\hat{i}\right|}$;

We have, $\hat{\eta} = \dfrac{2(4\hat{i} - y\hat{j})}{2\sqrt{(16+y^2)}} = \dfrac{(4\hat{i} - y\hat{j})}{\sqrt{(16+y^2)}}$

$$\therefore \hat{\eta}.\hat{i} = \frac{(4\hat{i} - y\hat{j})}{\sqrt{(16+y^2)}}.\hat{i} = \frac{4}{\sqrt{(16+y^2)}}$$

$$\therefore \iint_s \vec{F}.\hat{\eta}\,ds = \iint_s \vec{F}.\hat{\eta}\frac{dydz}{\left|\hat{\eta}.\hat{i}\right|}$$

$$\therefore \iint_s \vec{F}.\hat{\eta}\,ds = \iint_s \frac{(8y+yz)}{\sqrt{(16+y^2)}}\frac{dydz}{\left|\hat{\eta}.\hat{i}\right|}$$

$$\therefore \iint_s \vec{F}.\hat{\eta}\,ds = \iint_s \frac{(8y+yz)}{\sqrt{(16+y^2)}}\frac{dydz}{\frac{4}{\sqrt{(16+y^2)}}}$$

$$\therefore \iint_s \vec{F}.\hat{\eta}\,ds = \iint_s \frac{(8y+yz)}{\sqrt{(16+y^2)}}\frac{\sqrt{(16+y^2)}dydz}{4}$$

$$\therefore \iint_s \vec{F}.\hat{\eta}\,ds = \frac{1}{4}\iint_s (8y+yz)dydz$$

$$\therefore \iint_s \vec{F}.\hat{\eta}\,ds = \frac{1}{4}\int_0^6\int_0^4 (8y+yz)dydz$$

$$\therefore \iint_s \vec{F}.\hat{\eta}\,ds = \frac{1}{4}\int_0^6\int_0^4 (8+z)ydydz = 132$$

Q # 97. Evaluate $\iint\limits_S \vec{A}.\hat{\eta}\,ds$ over the entire surface S of the region bounded by the cylinder $x^2+z^2=9, x=0, y=0, z=0$ and $y=8$ if $\vec{A}=6z\hat{i}+(2x+y)\hat{j}-x\hat{k}$

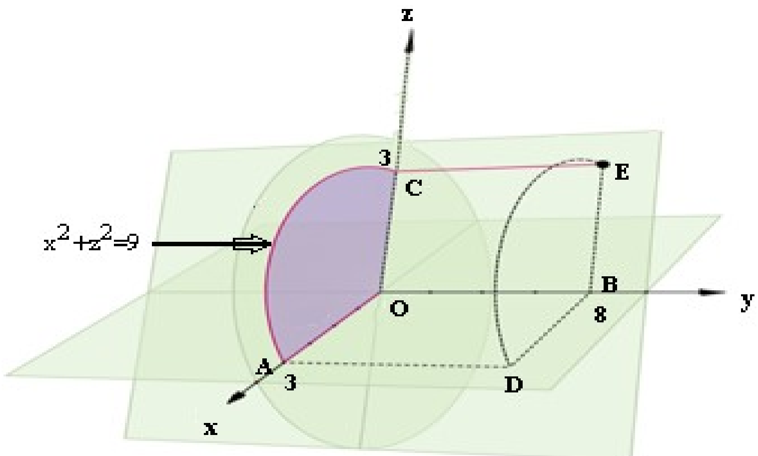


Figure # 130

Given,

S_1: Surface-1 (ACED): $x^2+z^2=9$

Let the scalar function $\phi(x,z)=x^2+z^2-9$ of the given surface.

We have, $\vec{\nabla}\phi$ is normal (perpendicular) vector to the surface.

Given, $\phi(x,z)=x^2+z^2-9$

$$\therefore \vec{\nabla}\phi=\left(\frac{\delta}{\delta x}\hat{i}+\frac{\delta}{\delta y}\hat{j}+\frac{\delta}{\delta z}\hat{k}\right)\phi=\left(\frac{\delta\phi}{\delta x}\hat{i}+\frac{\delta\phi}{\delta y}\hat{j}+\frac{\delta\phi}{\delta z}\hat{k}\right)=\hat{i}\frac{\delta}{\delta x}\phi+\hat{j}\frac{\delta}{\delta y}\phi+\hat{k}\frac{\delta}{\delta z}\phi$$

$$=\hat{i}\frac{\delta}{\delta x}(x^2+z^2-9)+\hat{j}\frac{\delta}{\delta y}(x^2+z^2-9)+\hat{k}\frac{\delta}{\delta z}(x^2+z^2-9)$$

$$=2x\hat{i}+0\hat{j}+2z\hat{k}$$

$$=2x\hat{i}+2z\hat{k}$$

Let, $\hat{\eta}$ is the unit vector of $\vec{\nabla}\phi$

We can write,

$$\hat{\eta}=\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}$$

$$\hat{\eta}=\frac{2x\hat{i}+2z\hat{k}}{\sqrt{(2x)^2+(2z)^2}}=\frac{2x\hat{i}+2z\hat{k}}{\sqrt{4x^2+4z^2}}=\frac{2(x\hat{i}+z\hat{k})}{\sqrt{4(x^2+z^2)}}$$

$$\hat{\eta}=\frac{2(x\hat{i}+z\hat{k})}{2\sqrt{(x^2+z^2)}}$$

$$\hat{\eta} = \frac{(x\hat{i} + z\hat{k})}{\sqrt{(x^2 + z^2)}}$$

$$\hat{\eta} = \frac{(x\hat{i} + z\hat{k})}{\sqrt{9}} \qquad [\because x^2 + z^2 = 9]$$

$$\hat{\eta} = \frac{x\hat{i} + z\hat{k}}{3}$$

Given,

$$\vec{A} = 6z\hat{i} + (2x + y)\hat{j} - x\hat{k}$$

$$\therefore \vec{A}.\hat{\eta} = [6z\hat{i} + (2x + y)\hat{j} - x\hat{k}).\frac{(x\hat{i} + z\hat{k})}{3}$$

$$\therefore \vec{A}.\hat{\eta} = \frac{6}{3}xz - \frac{1}{3}xz$$

$$\therefore \vec{A}.\hat{\eta} = \frac{6xz - xz}{3}$$

$$\therefore \vec{A}.\hat{\eta} = \frac{5xz}{3}$$

Now, $\hat{\eta} = \frac{x\hat{i} + z\hat{k}}{3}$

So, $\hat{\eta}.\hat{k} = \frac{(x\hat{i} + z\hat{k})}{3}.\hat{k}$

$$\hat{\eta}.\hat{k} = \frac{z}{3}$$

Now,

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \iint_{s_1} \vec{A}.\hat{\eta} \frac{dxdy}{|\hat{\eta}.\hat{k}|}$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \iint_{s_1} \frac{5xz}{3} \frac{dxdy}{\left|\frac{z}{3}\right|}$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \iint_{s_1} \frac{5xz}{3} \frac{3dxdy}{z}$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \iint_{s_1} 5xdxdy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_0^8 \int_0^3 5xdxdy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_0^8 [\frac{5x^2}{2}]_0^3 dy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_0^8 [\frac{5.3^2}{2} - \frac{5.0^2}{2}]dy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_0^8 [\frac{45}{2}]dy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \frac{45}{2}[y]_0^8$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \frac{45}{2}[8-0]$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = 180$$

S_2: Surface-2 (OADB): $z = 0$

Given,

$$\vec{A} = 6z\,\hat{i} + (2x+y)\,\hat{j} - x\,\hat{k};\ \hat{\eta} = -\hat{k}$$

$$\vec{A} = 6.0\,\hat{i} + (2x+y)\,\hat{j} - x\,\hat{k} \qquad [z=0]$$

$$\vec{A} = (2x+y)\,\hat{j} - x\,\hat{k}$$

$$\therefore \vec{A}.\hat{\eta} = [(2x+y)\,\hat{j} - x\,\hat{k}].(-\hat{k})$$

$$\therefore \vec{A}.\hat{\eta} = x$$

Now,

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \iint_{s_2} \vec{A}.\hat{\eta}\, dxdy$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \iint_R x\, dxdy$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \int_0^8\int_0^3 x\, dxdy$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \int_0^8\int_0^3 x\, dxdy$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \int_0^8 [\frac{x^2}{2}]_0^3 dy$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \int_0^8 [\frac{3^2}{2} - \frac{5.0^2}{2}]dy$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \int_0^8 [\frac{9}{2}]dy$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \frac{9}{2}[y]_0^8$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = \frac{9}{2}[8-0]$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\, ds_2 = 36$$

S_3: Surface-3 (OCEB): $x = 0$

Given,

$$\vec{A} = 6z\hat{i} + (2x+y)\hat{j} - x\hat{k};\ \hat{\eta} = -\hat{i}$$

$$\vec{A} = 6z\hat{i} + (2.0+y)\hat{j} - 0.\hat{k} \qquad\qquad [x = 0]$$

$$\vec{A} = 6z\hat{i} + y\hat{j}$$

$$\therefore \vec{A}.\hat{\eta} = [(6z\hat{i} + y\hat{j}).(-\hat{i})$$

$$\therefore \vec{A}.\hat{\eta} = -6z$$

Now,

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = \iint_{s_3} \vec{A}.\hat{\eta}\ dzdy$$

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = \int_0^8\int_0^3 -6zdzdy$$

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = \int_0^8 [-6\frac{z^2}{2}]_0^3 dy$$

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = \int_0^8 [-3z^2]_0^3 dy$$

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = \int_0^8 [-3.3^2]dy$$

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = -27\int_0^8 dy$$

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = -27[y]_0^8$$

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = -27[8-0]$$

$$\therefore \iint_{s_3} \vec{A}.\hat{\eta}\, ds_3 = -216$$

S_4: Surface-4 (OAC): $y = 0$

Given,

$$\vec{A} = 6z\hat{i} + (2x+y)\hat{j} - x\hat{k};\ \hat{\eta} = -\hat{j}$$
$$\vec{A} = 6z\hat{i} + (2x+0)\hat{j} - x\hat{k};\ \hat{\eta} = -\hat{j}\ \ [y=0]$$
$$\vec{A} = 6z\hat{i} + 2x\hat{j} - x\hat{k}$$

$$\therefore \vec{A}.\hat{\eta} = [(6z\hat{i} + 2x\hat{j} - x\hat{k}).(-\hat{j})$$
$$\therefore \vec{A}.\hat{\eta} = -2x$$

Now,
$$x^2 + z^2 = 9$$
$$x^2 = 9 - z^2$$
$$x = \sqrt{9-z^2}$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = \iint_{s4} \vec{A}.\hat{\eta}\ dxdz$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = \int_0^3 \int_0^{\sqrt{9-z^2}} -2x dxdz$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = \int_0^3 -[2\frac{x^2}{2}]_0^{\sqrt{9-z^2}}\ dz$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = -\int_0^3 [(\sqrt{9-z^2})^2]dz$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = -\int_0^3 (9-z^2)dz$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = -[9z - \frac{z^3}{3}]_0^3$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = -[9\times 3 - \frac{3^3}{3}]$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = -[27 - \frac{27}{3}]$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = -[27-9]$$
$$\therefore \iint_{s4} \vec{A}.\hat{\eta}\, ds_4 = -18$$

S_5: Surface-5 (DEB): $y = 8$
Given,
Given,
$$\vec{A} = 6z\hat{i} + (2x+y)\hat{j} - x\hat{k};\ \hat{\eta} = \hat{j}$$
$$\vec{A} = 6z\hat{i} + (2x+8)\hat{j} - x\hat{k}$$

$$\therefore \vec{A}.\hat{\eta} = [(6z\hat{i} + (2x+8)\hat{j} - x\hat{k})].(\hat{j})$$

$$\therefore \vec{A}.\hat{\eta} = 2x + 8$$

Now,

$$x^2 + z^2 = 9$$

$$x^2 = 9 - z^2$$

$$x = \sqrt{9 - z^2}$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = \iint_{s5} \vec{A}.\hat{\eta}\; dxdz$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = \int_0^3 \int_0^{\sqrt{9-z^2}} (2x+8) dxdz$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = \int_0^3 \int_0^{\sqrt{9-z^2}} 2x dxdz + \int_0^3 \int_0^{\sqrt{9-z^2}} 8\, dxdz$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + \int_0^3 \int_0^{\sqrt{9-z^2}} 8\, dxdz$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 8\int_0^3 [x]_0^{\sqrt{9-z^2}}\, dz$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 8\int_0^3 [\sqrt{9 - z^2}\,] dz$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 8\int_0^3 [\sqrt{3^2 - z^2}\,] dz$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 8\frac{1}{2}[z\sqrt{3^2 - z^2} + 3^2 \sin^{-1}\frac{z}{3}]_0^3$$

$$[\int \sqrt{p^2 - x^2} dx = \frac{1}{2}(x\sqrt{p^2 - x^2} + p^2 \sin^{-1}\frac{x}{p})]$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 4[3\sqrt{3^2 - 3^2} + 3^2 \sin^{-1}\frac{3}{3}]$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 4[3.0 + 3^2 \sin^{-1} 1]$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 4[0 + 3^2 \sin^{-1} \sin\frac{\pi}{2}]$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 4[3^2\frac{\pi}{2}]$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 4[9\frac{\pi}{2}]$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 2[9\pi]$$

$$\therefore \iint_{s5} \vec{A}.\hat{\eta}\, ds_5 = 18 + 18\pi$$

$$\iint_S \vec{A}.\hat{\eta}\,ds = \iint_{s1} \vec{A}.\hat{\eta}\,ds_1 + \iint_{s2} \vec{A}.\hat{\eta}\,ds_2 + \iint_{s3} \vec{A}.\hat{\eta}\,ds_3 + \iint_{s4} \vec{A}.\hat{\eta}\,ds_4 + \iint_{s5} \vec{A}.\hat{\eta}\,ds_5$$

$$\iint_S \vec{A}.\hat{\eta}\,ds = 180 + 36 - 216 - 18 + 18 + 18\pi$$

$$\iint_S \vec{A}.\hat{\eta}\,ds = 18\pi$$

Q # 98. Evaluate $\iint_S \vec{A}.\hat{\eta}\,ds$ over the entire surface S of the region above the xy plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$ if $\vec{A} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$

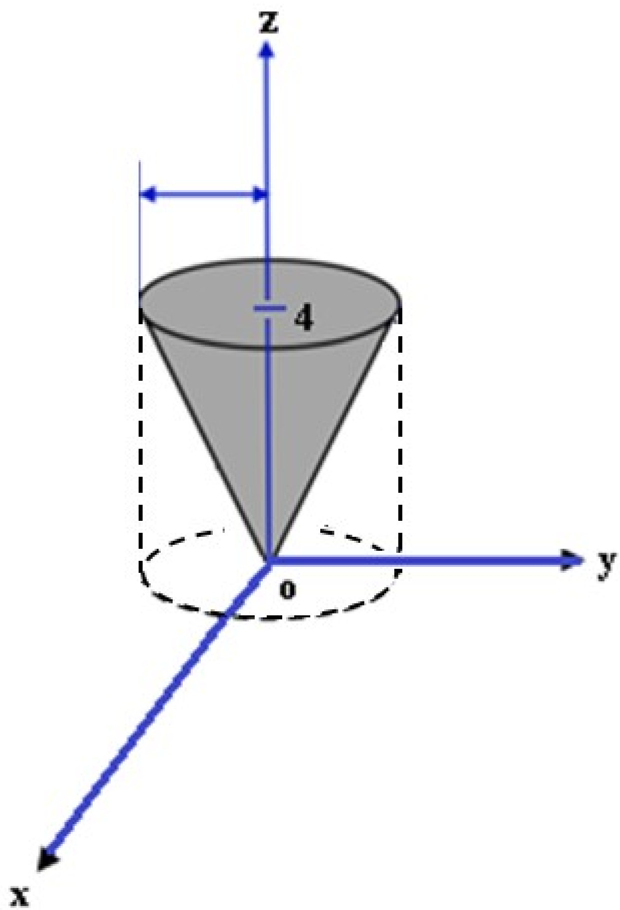


Figure # 131

Given,

$S_1 : z^2 = x^2 + y^2$

$S_1 : x^2 + y^2 = z^2$

$S_1 : x^2 + y^2 - z^2 = 0$

Let the scalar function $\phi(x, y) = x^2 + y^2 - z^2$ of the given surface.

We have, $\vec{\nabla}\phi$ **is** normal (perpendicular) vector to the surface.

Given, $\phi(x,y) = x^2 + y^2 - z^2$

$$\therefore \vec{\nabla}\phi = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right)\phi = \left(\frac{\delta\phi}{\delta x}\hat{i} + \frac{\delta\phi}{\delta y}\hat{j} + \frac{\delta\phi}{\delta z}\hat{k}\right) = \hat{i}\frac{\delta}{\delta x}\phi + \hat{j}\frac{\delta}{\delta y}\phi + \hat{k}\frac{\delta}{\delta z}\phi$$

$$= \hat{i}\frac{\delta}{\delta x}(x^2 + y^2 - z^2) + \hat{j}\frac{\delta}{\delta y}(x^2 + y^2 - z^2) + \hat{k}\frac{\delta}{\delta z}(x^2 + y^2 - z^2)$$

$$= 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

Let, $\hat{\eta}$ is the unit vector of $\vec{\nabla}\phi$

We can write,

$$\hat{\eta} = \frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}$$

$$\hat{\eta} = \frac{2x\hat{i} + 2y\hat{j} - 2z\hat{k}}{\sqrt{(2x)^2 + (2y)^2(-2z)^2}} = \frac{2x\hat{i} + 2y\hat{j} - 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{2(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{4(x^2 + y^2 + z^2)}}$$

$$\hat{\eta} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{(x^2 + y^2 + z^2)}}$$

$$\hat{\eta} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{(z^2 + z^2)}} \qquad [\because x^2 + y^2 = z^2]$$

$$\hat{\eta} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{(2z^2)}}$$

$$\hat{\eta} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{2}\,z}$$

Given,

$$\vec{A} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$$

$$\therefore \vec{A}.\hat{\eta} = (4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}).\frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{2}\,z}$$

$$\therefore \vec{A}.\hat{\eta} = (4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}).\left[\frac{x\hat{i}}{\sqrt{2}\,z} + \frac{y\hat{j}}{\sqrt{2}\,z} + \frac{-z\hat{k}}{\sqrt{2}\,z}\right]$$

$$\therefore \vec{A}.\hat{\eta} = \frac{4xzx}{\sqrt{2}\,z} + \frac{xyz^2y}{\sqrt{2}\,z} - \frac{3zz}{\sqrt{2}\,z}$$

$$\therefore \vec{A}.\hat{\eta} = \frac{4x^2z}{\sqrt{2}\,z} + \frac{xy^2z^2}{\sqrt{2}\,z} - \frac{3z^2}{\sqrt{2}\,z}$$

$$\therefore \vec{A}.\hat{\eta} = \frac{4x^2}{\sqrt{2}} + \frac{xy^2z}{\sqrt{2}} - \frac{3z}{\sqrt{2}}$$

Now, $\hat{\eta} = \dfrac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{2}\,z}$

So, $\hat{\eta}.\hat{k} = \dfrac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{2}\,z}.\hat{k}$

$$\hat{\eta}.\hat{k} = \frac{-z}{\sqrt{2}\,z}$$

$$\hat{\eta}.\hat{k} = \frac{-1}{\sqrt{2}}$$

$$\left|\hat{\eta}.\hat{k}\right| = \frac{1}{\sqrt{2}}$$

Now,

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = \iint_{s_1} \vec{A}.\hat{\eta}\frac{dxdy}{\left|\hat{\eta}.\hat{k}\right|}$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = \iint_{s_1} \left(\frac{4x^2}{\sqrt{2}} + \frac{xy^2z}{\sqrt{2}} - \frac{3z}{\sqrt{2}}\right)\frac{dxdy}{\frac{1}{\sqrt{2}}}$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = \iint_{s_1} \frac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z)\frac{dxdy}{\frac{1}{\sqrt{2}}}$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = \iint_{s_1} (4x^2 + xy^2z - 3z)\,dxdy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = \iint_{s_1} [4x^2 + (xy^2 - 3)z]\,dxdy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = \iint_{s_1} [4x^2 + (xy^2 - 3)\sqrt{x^2 + y^2}\,]\,dxdy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = \iint_{s_1} [4x^2 + (xy^2 - 3)\sqrt{x^2 + y^2}\,]\,dxdy$$

Let $\mathbf{x = r\cos\theta}$ $\mathbf{y = r\sin\theta}$

Using Jacobian **Determinant**

dxdy = rdrd θ

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \iint_{s_1} [4x^2 + (xy^2 - 3)\sqrt{x^2 + y^2}]\,dxdy$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi}\int_{r=0}^{4} [4r^2\cos^2\theta + (r\cos\theta.r^2\sin^2\theta - 3)\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}]\,r\,dr\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi}\int_{r=0}^{4} [4r^2\cos^2\theta + (r\cos\theta.r^2\sin^2\theta - 3)\sqrt{r^2(\cos^2\theta + \sin^2\theta)}]\,r\,dr\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi}\int_{r=0}^{4} [4r^2\cos^2\theta + (r\cos\theta.r^2\sin^2\theta - 3)\sqrt{r^2.1}]\,r\,dr\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi}\int_{r=0}^{4} [4r^2\cos^2\theta + (r\cos\theta.r^2\sin^2\theta - 3)r]\,r\,dr\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi}\int_{r=0}^{4} [4r^2\cos^2\theta + (r^2\cos\theta.r^2\sin^2\theta - 3r)]\,r\,dr\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi}\int_{r=0}^{4} [4r^3\cos^2\theta + (r^5\cos\theta.\sin^2\theta - 3r^2)]\,dr\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi}\int_{r=0}^{4} [4r^3\cos^2\theta + r^5\cos\theta.\sin^2\theta - 3r^2]\,dr\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi} [4\frac{r^4}{4}\cos^2\theta + \frac{r^6}{6}\cos\theta.\sin^2\theta - 3\frac{r^3}{3}]_0^4\, d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi} [r^4\cos^2\theta + \frac{r^6}{6}\cos\theta.\sin^2\theta - r^3]_0^4\, d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi} [4^4\cos^2\theta + \frac{4^6}{6}\cos\theta.\sin^2\theta - 4^3]\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi} [256\cos^2\theta + \frac{4096}{6}\cos\theta.\sin^2\theta - 64]\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi} [128\times 2\cos^2\theta + \frac{2048}{3}\cos\theta.\sin^2\theta - 64]\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi} [128\times(1+\cos 2\theta) + \frac{2048}{3}\cos\theta.\sin^2\theta - 64]\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi} [128 + 128\cos 2\theta + \frac{2048}{3}\cos\theta.\sin^2\theta - 64]\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\, ds_1 = \int_{\theta=0}^{2\pi} [64 + 128\cos 2\theta + \frac{2048}{3}\cos\theta.\sin^2\theta]\,d\theta$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = [64\theta + 128\frac{\sin 2\theta}{2} + \frac{2048}{3}\frac{\sin^3\theta}{3}]_0^{2\pi}$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = [64\times 2\pi + 128\frac{\sin 2\pi}{2} + \frac{2048}{3}\frac{\sin^3\pi}{3}]$$

$$\therefore \iint_{s_1} \vec{A}.\hat{\eta}\,ds_1 = 128\pi \; [\because \sin\pi = \sin 2\pi = 0]$$

Given,

$S_2 : z = 4$

Given,

$$\vec{A} = 4xz\,\hat{i} + xyz^2\,\hat{j} + 3z\,\hat{k}$$

$$\vec{A} = 4x\times 4\,\hat{i} + xy4^2\,\hat{j} + 3\times 4\,\hat{k}$$

$$\vec{A} = 16x\,\hat{i} + 16xy\,\hat{j} + 12\,\hat{k}$$

Now, $\hat{\eta} = \hat{k}$

$$\therefore \vec{A}.\hat{\eta} = (16x\,\hat{i} + 16xy\,\hat{j} + 12\,\hat{k}).\hat{k}$$

$$\therefore \vec{A}.\hat{\eta} = (12\,\hat{k}).\hat{k}$$

$$\therefore \vec{A}.\hat{\eta} = 12$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\,ds_2 = \iint_{s_2} 12\,ds_2$$

$$\therefore \iint_{s_2} \vec{A}.\hat{\eta}\,ds_2 = 12\pi.4^2 = 192\pi$$

$$\therefore \iint_{s} \vec{A}.\hat{\eta}\,ds = \iint_{s_1} \vec{A}.\hat{\eta}\,ds + \iint_{s_2} \vec{A}.\hat{\eta}\,ds = 128\pi + 192\pi = 320\pi$$

https://books.google.com.bd/books?id=yKVi7lMDMpcC&pg=SA16-PA15&lpg=SA16-PA15&dq=the+surface+of+the+parabolic+cylinder++y%5E2%3D8x+in+the+first+octant+bounded+by+the+planes++y+%3D+4+and++z+%3D6&source=bl&ots=owY_JVyr8d&sig=ACfU3U28VFP16pL1iZihCdzDH0iRRq-ZNg&hl=en&sa=X&ved=2ahUKEwjYtpn41bv0AhWZ6nMBHYyKD08Q6AF6BAggEAM#v=onepage&q=the%20surface%20of%20the%20parabolic%20cylinder%20%20y%5E2%3D8x%20in%20the%20first%20octant%20bounded%20by%20the%20planes%20%20y%20%3D%204%20and%20%20z%20%3D6&f=false