Problem 18: Compact Trigonometric Fourier Series:

A Fourier series is an infinite sum of trigonometric functions that can be used to model realvalued, periodic functions**.**

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega t)+\sum_{n=1}^{\infty}b_n\sin(n\omega t)\text{----------------------------------(i)}$$

The trigonometric Fourier series equation no (i) contains sine and cosine terms of the same frequency. We can combine the two terms in a single term of the same frequency using trigonometric identity:

$$a_n\cos(n\omega t)+b_n\sin(n\omega t)=C_n\cos(n\omega_0 t+\theta_n)\text{------------------------------(ii)}$$

Where, $C_n=\sqrt{a_n^{\,2}+b_n^{\,2}}$ --(iii)

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n+\sum_{n=1}^{\infty}b_n$$

Problem 19: Derive Complex form of Fourier series

We have, the trigonometric form of Fourier series is:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega t)+\sum_{n=1}^{\infty}b_n\sin(n\omega t)\text{----------------(iv)}$$

We have,

$$e^x=1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+----------------$$

Put $x=ix$,

$$e^{ix}=1+\frac{ix}{1!}+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\frac{(ix)^6}{6!}+\frac{(ix)^7}{7!}+--------$$

$$[i^2=-1;\ i^3=i^2.i=-i;\ i^4=i^2.i^2=(-1).(-1)=+1;\ i^5=i^4.i=i]$$

$$e^{ix}=1+\frac{ix}{1!}+\frac{-x^2}{2!}+\frac{-ix^3}{3!}+\frac{x^4}{4!}+\frac{ix^5}{5!}+\frac{-x^6}{6!}+\frac{-ix^7}{7!}+--------$$

$$e^{ix}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+----+\left(\frac{ix^1}{1!}-\frac{ix^3}{3!}+\frac{ix^5}{5!}-\frac{ix^7}{7!}+--------\right)$$

$$e^{ix}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+----+i\left(\frac{x^1}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+--------\right)$$

$$[\because \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+-----;\quad \sin x=\frac{x^1}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}---------]$$

$$\therefore e^{ix}=\cos x+i\sin x\text{ ------------------------------------(v)}$$

Similarly,

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + ----------------$$

Put $x = -ix$,

$$e^{-ix} = 1 + \frac{-ix^1}{1!} + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \frac{(-ix)^5}{5!} + \frac{(-ix)^6}{6!} + \frac{(-ix)^7}{7!} + ----$$

$[(-i)^2 = -1; (-i)^3 = (-i)^2.(-i) = i; (-i)^4 = (-i)^2.(-i)^2 = (-1).(-1) = +1;$

$(-i)^5 = (-i)^4.(-i) = (+1).(-i) = -i]$

$$e^{-ix} = 1 + \frac{-ix^1}{1!} + \frac{-x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{-ix^5}{5!} + \frac{-x^6}{6!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{ix^1}{1!} - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ----------- \frac{ix^1}{1!} + \frac{ix^3}{3!} - \frac{ix^5}{5!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ------------- i(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ----)$$

$$[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + -----; \quad \sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ---------]$$

$\therefore e^{-ix} = \cos x - i\sin x$ --(vi)

Adding (v) and (vi),

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$e^{ix} + e^{-ix} = 2\cos x$

$\therefore \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ --(vii)

Again Subtracting (v) and (vi)

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$e^{ix} - e^{-ix} = 2i\sin x$

$\therefore \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ --(viii)

From equation (vii)

$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

Hence From equation (vii), we can write,

$$\therefore \cos(n\omega t) = \frac{1}{2}(e^{in\omega t} + e^{-in\omega t}) \text{ ------------------------------(ix)}$$

And from equation (vii)

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

Hence from equation (vii), we can write

$$\therefore \sin(n\omega t) = \frac{1}{2i}(e^{in\omega t} - e^{-in\omega t}) \text{ --------------------------(x)}$$

Putting the values of (ix) and (x) in (iv), we get,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \; \{\frac{1}{2}(e^{in\omega t} + e^{-in\omega t})\} + \sum_{n=1}^{\infty} b_n \; \{\frac{1}{2i}(e^{in\omega t} - e^{-in\omega t})\}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \; \{\frac{1}{2}(e^{in\omega t} + e^{-in\omega t})\} + b_n \; \{\frac{1}{2i}(e^{in\omega t} - e^{-in\omega t})\}]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\; \{\frac{1}{2}(a_n e^{in\omega t} + a_n e^{-in\omega t})\} + \{\frac{1}{2i}(b_n e^{in\omega t} - b_n e^{-in\omega t})\}]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\; \frac{1}{2}(a_n e^{in\omega t} + a_n e^{-in\omega t}) + \frac{1}{2i}(b_n e^{in\omega t} - b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\; \frac{1}{2}(a_n e^{in\omega t}) + \frac{1}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{1}{2i}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\; \frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(-1)}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(-1)}{2i}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\; \frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(i^2)}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(i^2)}{2i}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\; \frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(i\;)}{2}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(i\;)}{2}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\; \frac{1}{2}(a_n e^{in\omega t}) + \frac{-i}{2}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{-i}{2}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\; \{\frac{1}{2}(a_n e^{in\omega t}) - \frac{i}{2}(b_n e^{in\omega t})\} + \{\frac{1}{2}(a_n e^{-in\omega t}) + \frac{i}{2}(b_n e^{-in\omega t})\}]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[\frac{1}{2}(a_n - ib_n)e^{in\omega t}) + \frac{1}{2}(a_n + ib_n)e^{-in\omega t})\;] \text{---------------------(xi)}$$

Let, $c_0 = \frac{a_0}{2}$ -------------------------------------(a)

$$c_n = \frac{1}{2}(a_n - ib_n) \quad \text{-------------------------------(b)}$$

$$c^*_n = \frac{1}{2}(a_n + ib_n) \quad \text{------------------------------(c)}$$

Then from (viii), we get the series is:

$$f(t) = c_0 + \sum_{n=1}^{\infty}[c_n e^{in\omega t} + c^*_n\, e^{-in\omega t}]$$

$$f(t) = c_0 + \sum_{n=1}^{\infty}[c_n e^{in\omega t} + c_{-n} e^{-in\omega t}]\ \text{[Say } c^*_n = c_{-n}]$$

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega t} + \sum_{n=1}^{\infty} c_{-n} e^{-in\omega t}$$ -------(d)= ---+------+…….

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega t} + \sum_{n=-1}^{-\infty} c_n e^{in\omega t}$$ ---------(e)=----------+-------+-----

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$ --------------(f)

[We have, $\mathbf{c_n e^{in\omega t}}$ -1 0 1

Put n = 0, then

$$\mathbf{c_n e^{in\omega t}}$$
$$= c_0 e^{i\times 0\times \omega t}$$
$$= c_0 e^{0} = c_0 .1 = c_0\,]$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$ -------------------------------------***(xii) Which is referred to as the complex or exponential form of the Fourier Series expansion of the function f(t)***

Where

$$\because \mathbf{e^{ix} = \cos x + i \sin x}\ \text{[from (v)]}$$

$$\therefore \mathbf{e^{in\omega t} = \cos n\omega t + i \sin n\omega t}$$ -------------------------------(xiii)

$$\mathbf{c_0 = \frac{a_0}{2} = \frac{1}{2}a_0 = \frac{1}{2}\frac{1}{L}\int_{-L}^{L} f(t)dt}$$

$$\mathbf{c_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt}$$

$$\mathbf{c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt}$$ **[Where Period T= 2L]--------------(xiv)**

and

$$\mathbf{c_n = \frac{1}{2}(a_n - ib_n)}$$ ---(xv)

$$\mathbf{c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)}$$ ---------------------------------(xvi)

We have,

$$\mathbf{a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt}$$

$$\mathbf{b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt}$$

$$\therefore c_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt - i\frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\int_{-L}^{L}\{f(t)\cos(n\omega t) - i f(t)\sin(nwt)\}dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\{\cos(n\omega t) - i\sin(nwt)\}dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\omega t}dt \text{ [Since } e^{-ix} = \cos x - i\sin x \text{ from (vi)]}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt \text{ [T = 2L] ----------------------(xvii)}$$

and

$$c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt + i\frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\left(\int_{-L}^{L} f(t)\cos(n\omega t)dt + i.\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\left(\int_{-L}^{L} f(t)\{\cos(n\omega t)dt + i\sin(nwt)dt\right)\}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)e^{in\omega t}dt \text{ [}\because e^{ix} = \cos x + i\sin x \text{ from (2)]}$$

$$\therefore c^*_n = c_{-n} = \frac{1}{2L}\int_{-L}^{L} f(t)e^{in\omega t}dt$$

$$\therefore c^*_n = c_{-n} = \frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega t}dt \text{ [Period T = 2L]-------(xviii)}$$

In summary, the complex form of the Fourier series expansion of a periodic function f (t), of period T, is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

Where,

$$c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c^*_n = c_{-n} = \frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega t}dt$$

Example 37: Find the complex form of the Fourier Series expansion of the periodic function $\mathbf{f(t)}$ and find the trigonometric form of the Fourier Series expansion of the periodic function f(t) is given by:

$$f(t) = \cos\frac{1}{2}t \qquad ;-\pi < t < \pi \;\; [T = 2\pi]$$

Solution:

The complex form of the Fourier series expansion of a periodic function $\mathbf{f(t)}$, of period T, is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \text{--(A)}$$

Where,

$$c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c^*_n = c_{-n} = \frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega t}dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in\frac{2\pi}{T}t}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in\frac{2\pi}{2\pi}t}dt \;\; [T = 2\pi]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\frac{1}{2}t\, e^{-int}dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \{\frac{1}{2}(e^{i(\frac{1}{2}t)} + e^{-i(\frac{1}{2}t)}\}e^{-int}dt \;\; [\therefore \cos x = \frac{1}{2}(e^{ix} + e^{-ix})\;]$$

$$=\frac{1}{4\pi}\int_{-\pi}^{\pi}\{(e^{i(\frac{1}{2}t)}+e^{-i(\frac{1}{2}t)}\}e^{-int}dt$$

$$=\frac{1}{4\pi}\int_{-\pi}^{\pi}\{(e^{i(\frac{1}{2}t)}.e^{-int}+e^{-i(\frac{1}{2}t)}.e^{-int}\}dt$$

$$=\frac{1}{4\pi}\int_{-\pi}^{\pi}\{(e^{i(\frac{1}{2}t-nt)}+e^{-i(\frac{1}{2}t+nt)}\}dt$$

$$=\frac{1}{4\pi}\int_{-\pi}^{\pi}\{(e^{-i(n-\frac{1}{2})t}+e^{-i(n+\frac{1}{2})t}\}dt$$

$$=\frac{1}{4\pi}[\frac{e^{-i(n-\frac{1}{2})t}}{-i(n-\frac{1}{2})}+\frac{e^{-i(n+\frac{1}{2})t}}{-i(n+\frac{1}{2})}]_{-\pi}^{\pi}$$

$$=\frac{1}{4\pi}[\frac{2e^{-i\frac{(2n-1)t}{2}}}{-i(2n-1)}-\frac{2e^{-i\frac{(2n+1)t}{2}}}{i(2n+1)}]_{-\pi}^{\pi}$$

$$=\frac{1}{4\pi}[\frac{-2e^{-int}.e^{\frac{it}{2}}}{i(2n-1)}-\frac{2e^{-int}.e^{-\frac{it}{2}}}{i(2n+1)}]_{-\pi}^{\pi}$$

$$=\frac{-2}{4\pi}[\frac{e^{-int}.e^{\frac{it}{2}}}{i(2n-1)}+\frac{e^{-int}.e^{-\frac{it}{2}}}{i(2n+1)}]_{-\pi}^{\pi}$$

$$=\frac{-2}{4\pi}[\frac{e^{-in\pi}.e^{\frac{i\pi}{2}}}{i(2n-1)}+\frac{e^{-in\pi}.e^{-\frac{i\pi}{2}}}{i(2n+1)}-\frac{e^{-in(-\pi)}.e^{\frac{-i\pi}{2}}}{i(2n-1)}-\frac{e^{-in(-\pi)}.e^{-\frac{i(-\pi)}{2}}}{i(2n+1)}]$$

$$=\frac{-2}{4\pi}[\frac{e^{-in\pi}.e^{\frac{i\pi}{2}}}{i(2n-1)}+\frac{e^{-in\pi}.e^{-\frac{i\pi}{2}}}{i(2n+1)}-\frac{e^{in\pi}.e^{\frac{-i\pi}{2}}}{i(2n-1)}-\frac{e^{in\pi}.e^{\frac{i\pi}{2}}}{i(2n+1)}]\text{--------------------(B)}$$

We have,

$e^{ix}=\cos x+i\sin x$

$\therefore e^{i\frac{\pi}{2}}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=0+i.1=i$ ---------------------(*)

And, $e^{-ix}=\cos x-i\sin x$

$\therefore e^{-i\frac{\pi}{2}}=\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}=0-i.1=-i$ -------------------(*)

$e^{in\pi}=\cos n\pi+i\sin n\pi=(-1)^n+i.0=(-1)^n$ ------------(*)

Since,

if n =1, $\cos n\pi = \cos\pi = -1$
if n =2, $\cos n\pi = \cos 2\pi = 1$
if n =3, $\cos n\pi = \cos 3\pi = -1$

Hence we can write,
$\cos n\pi = (-1)^n$ [n = 1, 2, 3, ...]

Again,
$e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n - i.0 = (-1)^n$ ------------(*)
Putting these values in (B),

$$c_n = \frac{-2}{4\pi}\left[\frac{e^{-in\pi}.e^{\frac{i\pi}{2}}}{i(2n-1)} + \frac{e^{-in\pi}.e^{-\frac{i\pi}{2}}}{i(2n+1)} - \frac{e^{in\pi}.e^{\frac{-i\pi}{2}}}{i(2n-1)} - \frac{e^{in\pi}.e^{\frac{i\pi}{2}}}{i(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{e^{-in\pi}.i}{i(2n-1)} + \frac{e^{-in\pi}.(-i)}{i(2n+1)} - \frac{e^{in\pi}.(-i)}{i(2n-1)} - \frac{e^{in\pi}.(i)}{i(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{e^{-in\pi}.i}{i(2n-1)} - \frac{e^{-in\pi}.(i)}{i(2n+1)} + \frac{e^{in\pi}.(i)}{i(2n-1)} - \frac{e^{in\pi}.(i)}{i(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{(-1)^n.i}{i(2n-1)} - \frac{(-1)^n.(i)}{i(2n+1)} + \frac{(-1)^n.(i)}{i(2n-1)} - \frac{(-1)^n.(i)}{i(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{(-1)^n}{(2n-1)} - \frac{(-1)^n}{(2n+1)} + \frac{(-1)^n}{(2n-1)} - \frac{(-1)^n}{(2n+1)}\right]$$

$$c_n = \frac{-2}{4\pi}\left[\frac{1}{(2n-1)} - \frac{1}{(2n+1)} + \frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right](-1)^n$$

$$c_n = \frac{-2}{4\pi}\left[\frac{2}{(2n-1)} - \frac{2}{(2n+1)}\right](-1)^n$$

$$c_n = \frac{-4}{4\pi}\left[\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right](-1)^n$$

$$c_n = \frac{1}{\pi}\left[\frac{1}{(2n+1)} - \frac{1}{(2n-1)}\right](-1)^n$$

$$c_n = \frac{1}{\pi}\left[\frac{2n-1-(2n+1)}{(2n+1)(2n-1)}\right](-1)^n$$

$$c_n = \frac{1}{\pi}\left[\frac{-2}{(2n+1)(2n-1)}\right](-1)^n$$

$$c_n = \frac{-2}{\pi}\left[\frac{1}{(4n^2-1))}\right](-1)^n$$

$$c_n = \frac{2}{\pi}\left[\frac{1}{(4n^2-1))}\right](-1)^n(-1)$$

$$c_n = \frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1}$$

$\therefore$ From (A), the complex form of the Fourier series expansion of a periodic function f (t), of period T, is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

$$= \sum_{n=-\infty}^{\infty} \{\frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1}\} e^{in\omega t}$$

$$= \sum_{n=-\infty}^{\infty} \{\frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1}\} e^{int} [\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1] \text{ Answer}$$

Problem 20: Conversion of $f(t)$ from complex form to the trigonometric form

We have, the trigonometric form of the Fourier series is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$[\omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1]$$

$$\therefore f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \text{ --------------------------------(B)}$$

We have,

Let, $c_0 = \frac{a_0}{2}$

$\therefore a_0 = 2c_0$

We got,

$$c_n = \frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1}$$

$$\therefore c_0 = \frac{2}{\pi}[\frac{1}{(4\times 0^2-1))}](-1)^{0+1} \text{ [putting } n = 0]$$

$$= \frac{2}{\pi}[\frac{1}{-1}](-1)^1]$$

$$= \frac{2}{\pi}$$

$$\therefore a_0 = 2c_0 = 2\times\frac{2}{\pi} = \frac{4}{\pi}$$

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$\therefore c_n + c_{-n} = \frac{1}{2}a_n + \frac{1}{2}a_n = a_n$$

$$\therefore a_n = c_n + c_{-n}$$

We have,

$$c_n = \frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1}$$

$$\therefore c_{-n} = \frac{2}{\pi}[\frac{1}{(4(-n)^2-1))}](-1)^{-n+1} \text{ [putting } n = -n]$$

$$\therefore a_n = c_n + c_{-n}$$

$$= \frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1} + \frac{2}{\pi}[\frac{1}{(4(-n)^2-1))}](-1)^{-n+1}$$

$$= \frac{2}{\pi}\frac{1}{(4n^2-1))}[(-1)^{n+1} + (-1)^{-n+1}]$$

$$= \frac{2}{\pi}\frac{1}{(4n^2-1))}[(-1)^{n+1} + (-1)^{-n+1}]$$

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$\therefore c_n - c_{-n} = -i\frac{1}{2}b_n - i\frac{1}{2}b_n = -i(\frac{1}{2}b_n + \frac{1}{2}b_n) = -ib_n$$

$$\therefore -ib_n = c_n - c_{-n}$$

$$\therefore b_n = -\frac{1}{i}(c_n - c_{-n}) = \frac{-1}{i}(c_n - c_{-n}) = \frac{i^2}{i}(c_n - c_{-n}) = i(c_n - c_{-n})$$

$$\therefore b_n = i(c_n - c_{-n})$$

$$= i\{\frac{2}{\pi}[\frac{1}{(4n^2-1))}](-1)^{n+1} - \frac{2}{\pi}[\frac{1}{(4(-n)^2-1))}](-1)^{-n+1}\}$$

$$= \frac{2}{\pi}\frac{1}{(4n^2-1))}[(-1)^{n+1} - (-1)^{-n+1}]$$

$$= \frac{2}{\pi}\frac{1}{(4n^2-1))}[(-1)^{n+1} - (-1)^{-n+1}]$$

if n =1, $b_1 = 0$
if n =2, $b_2 = 0$
if n =3 , $b_3 = 0$

Since the given function is even function so the coefficient $\mathbf{b_n}$ will be zero

The trigonometric form of the Fourier series is:

$$\therefore \mathbf{f(t)} = \frac{\mathbf{a_0}}{\mathbf{2}} + \sum_{n=1}^{\infty} \mathbf{a_n \cos nt} + \sum_{n=1}^{\infty} \mathbf{b_n \sin nt} \text{ ---------------------------------(B)}$$

$$= \frac{4}{2\times\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi}\frac{1}{(4n^2-1))} [(-1)^{n+1} + (-1)^{-n+1}] \cos nt + 0$$

$$= \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi}\frac{1}{(4n^2-1))} [(-1)^{n+1} + (-1)^{-n+1}] \cos nt \text{ Answer}$$

Example 38: Obtain the complex form of the Fourier series of the saw tooth function $\mathbf{f(t)}$ defined by $\mathbf{f(t) = \frac{2t}{T}(0 < t < 2T), \quad f(t+2T) = f(t)}$ [Period = 2T]

Answer:

We have, $\mathbf{f(t)} = \sum_{n=-\infty}^{\infty} \mathbf{c_n e^{in\omega t}}$ the complex or exponential form of the Fourier series.

Where, $\mathbf{c_n} = \frac{1}{2L}\int_{-L}^{L} f(t) e^{-in\omega t} dt$

$$= \frac{1}{2T}\int_{-T}^{T} \frac{2t}{T} e^{-in\omega t} dt$$

$$= \frac{1}{2T}\int_{-T}^{T} \frac{2t}{T} e^{-in\frac{2\pi}{T}t} dt$$

$$= \frac{1}{2T}\int_{-T}^{T} \frac{2t}{T} e^{-in\frac{2\pi}{2T}t} dt$$

$$= \frac{1}{2T}\int_{0}^{2T} \frac{2t}{T} e^{-in\frac{\pi}{T}t} dt \; [\because \int_{-a}^{a} f(x)dx = \int_{0}^{2a} f(x)dx] \text{ (Proved)}$$

$$= \frac{1}{2T^2}\int_{0}^{2T} 2t e^{-in\frac{\pi}{T}t} dt$$

$$= \frac{1}{T^2}\int_{0}^{2T} t e^{-in\frac{\pi}{T}t} dt$$

Now, $\int t e^{-in\frac{\pi}{T}t} dt = t\int e^{-in\frac{\pi}{T}t} dt - \int\{\frac{d}{dt}(t)\int e^{-in\frac{\pi}{T}t} dt\}$

$$= t\frac{e^{-in\frac{\pi}{T}t}}{-in\frac{\pi}{T}} - \int 1.\frac{e^{-in\frac{\pi}{T}t}}{-in\frac{\pi}{T}} dt$$

$$= t\frac{e^{-in\frac{\pi}{T}t}}{-in\frac{\pi}{T}} + \frac{T}{in\pi}\int e^{-in\frac{\pi}{T}t}dt$$

$$= Tt\frac{e^{-in\frac{\pi}{T}t}}{-in\pi} + \frac{T}{in\pi}.\frac{e^{-in\frac{\pi}{T}t}}{-in\frac{\pi}{T}}$$

$$= Tt\frac{e^{-in\frac{\pi}{T}t}}{-in\pi} - \frac{T^2}{(in\pi)^2}.\frac{e^{-in\frac{\pi}{T}t}}{1}$$

$$c_n = \frac{1}{T^2}\int_0^{2T} te^{-in\frac{\pi}{T}t}dt = \frac{1}{T^2}[Tt\frac{e^{-in\frac{\pi}{T}t}}{-in\pi} - \frac{T^2}{(in\pi)^2}.\frac{e^{-in\frac{\pi}{T}t}}{1}]_0^{2T}$$

$$= \frac{1}{T^2}[T.2T\frac{e^{-in\frac{\pi}{T}2T}}{-in\pi} - \frac{T^2}{(in\pi)^2}.\frac{e^{-in\frac{\pi}{T}2T}}{1} - 0 + \frac{T^2}{(in\pi)^2}.e^{-in\frac{\pi}{T}\times 0}]$$

$$= \frac{1}{T^2}[T.2T\frac{e^{-in2\pi}}{-in\pi} - \frac{T^2}{(in\pi)^2}.\frac{e^{-in2\pi}}{1} + \frac{T^2}{(in\pi)^2}] \; [e^0 = 1]$$

$$= \frac{1}{T^2}[2T^2\frac{e^{-in2\pi}}{-in\pi} + \frac{T^2}{(n\pi)^2}.\frac{e^{-in2\pi}}{1} - \frac{T^2}{(n\pi)^2}] \; [i^2 = -1]$$

We have,

$e^{-ix} = \cos x - i\sin x$

$e^{-i2n\pi} = \cos 2n\pi - i\sin 2n\pi$

$e^{-i2\pi} = \cos 2\pi - i\sin 2\pi = 1 + 0[\text{when } n = 1]$

$e^{-i4\pi} = \cos 4\pi - i\sin 4\pi = 1 + 0[\text{when } n = 2]$

$$c_n = \frac{1}{T^2}[2T^2\frac{e^{-in2\pi}}{-in\pi} + \frac{T^2}{(n\pi)^2}.\frac{e^{-in2\pi}}{1} - \frac{T^2}{(n\pi)^2}] \; [i^2 = -1]$$

$$c_n = \frac{1}{T^2}[2T^2\frac{1}{-in\pi} + \frac{T^2}{(n\pi)^2}.\frac{1}{1} - \frac{T^2}{(n\pi)^2}]$$

$$c_n = \frac{1}{T^2}[\frac{2T^2}{-in\pi} + \frac{T^2}{(n\pi)^2} - \frac{T^2}{(n\pi)^2}]$$

$$c_n = \frac{1}{T^2}[\frac{2T^2}{-in\pi}] = \frac{2}{-in\pi} = \frac{-2}{in\pi} = \frac{i^2 2}{in\pi} = \frac{i.2}{n\pi}$$ ------------------------------(ii)

Again,

$$c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt = \frac{1}{2T}\int_{-T}^{T}\frac{2t}{T}dt = \frac{1}{2T}\int_{0}^{2T}\frac{2t}{T}dt \frac{2}{2T^2}[\frac{t^2}{2}]_0^{2T} = \frac{1}{T^2}[\frac{4T^2}{2}] = 2$$

Hence,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} = c_0 . e^0 + \sum_{n=-\infty}^{-1} c_n e^{in\omega t} + \sum_{n=1}^{\infty} c_n e^{in\omega t}$$

$$= c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{in\omega t}$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i2}{n\pi} e^{in\frac{2\pi}{T}t}$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i.2}{n\pi} e^{in\frac{2\pi}{2T}t}$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{n\pi}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})e^{in\frac{\pi}{T}t} \quad [\because \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i]$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{n\pi}(e^{i\frac{\pi}{2}})e^{in\frac{\pi}{T}t} \quad [\because e^{ix} = \cos x + i\sin x]$$

$$= 2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2}{n\pi}(e^{i\frac{\pi}{2} + i\frac{n\pi t}{T}})$$

$$= 2 + \frac{2}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}(e^{i(\frac{\pi}{2} + \frac{n\pi t}{T})}) \text{--------(iii)-complex form of Fourier series}$$

Representation of Aperiodic Signals by Fourier Integral

Problem 21: Fourier transform

In the last chapter we discussed the spectral representation of periodic signals (Fourier Series). In this chapter we extend this spectral representation of aperiodic signals.

Example : Earth's periodic rotation on its own axis can be made using Fourier Series. So things which don't possess periodicity it can be made periodic and interpreted using Fourier Transform.

Fourier transform is a technique that converts a time domain signal to its equivalent frequency domain signal. For example, a sinusoidal signal in time domain is the corresponding energy (amplitude) at that particular time. So when a fourier transform is applied the spectrum represents the corresponding energy (amplitude) at that particular frequency. **There are many other transforms that give frequency domain representations like laplace transforms.**

Example

An essential mathematical tool we use in signal analysis is called the Fourier transform. Sound waves can be mathematically described in two domains. In the time domain, sound is described as a sequence of pressure changes (oscillations) that occur over time. In other words, the time-domain description of a sound wave specifies how the sound pressure increases and decreases over time. In the frequency domain, the spectrum defines sound in terms of the tonal components that make up the sound. A tonal sound has a time-domain description in which sound pressure changes as a regular (sinusoidal) function of time. If one knows the tonal components of sound as defined in the frequency domain, one can calculate the time-domain description of the sound. Using the same analytic tools, the frequency domain representation of a sound can also be calculated from the time-domain description. Thus, the time and frequency domain descriptions of sound are two different ways of measuring the same thing (i.e., the time and frequency domains are functional equivalents). Thus, one can describe sound as temporal fluctuations in pressure, or one can describe sounds in terms of the frequency components that compose the sound.

Fourier Transform: ***Representation of Aperiodic Signals.*** The extension of a Fourier series for a non-periodic function is known as the Fourier transforms. The basic equations of the Fourier series led to the development of the Fourier transform, which can decompose a nonperiodic function much like the Fourier series decomposes a periodic function.

Fourier transform as a filter. The Fourier transform is useful for extracting a signal from a noisy background. You can also take the inverse Fourier transform to move data from the frequency domain to the time domain. The inverse FFT command is given as ifft. Try taking the Fourier transform of a function, then apply the inverse to see that you get the proper result back.

Example
With these new techniques, Fourier series and Transforms have become an integral part of the toolboxes of mathematicians and scientists. Today, it is used for applications as diverse as file compression (such as the JPEG image format), signal processing in communications and astronomy, acoustics, optics, and cryptography.

The Fourier series representation of periodic signals consists of harmonically related spectral lines spaced at the integer multiples of the fundamental frequency. The Fourier representation of aperiodic signals can be developed by regarding an aperiodic signal as a special case of a periodic signal with an infinite period. If the period of a signal is infinite, then the signal does not repeat itself and is aperiodic

When calculating the Fourier transform, rather than decomposing a signal in terms of sines and cosines, people often use complex exponentials. They can be a little easier to interpret, although they are mathematically equivalent. A complex exponential is defined as $\mathbf{Ae^{i\phi}}$ where $\mathbf{i^2 = -1}$ ($\mathbf{i}$ is the "imaginary" number), $\mathbf{A}$ is the amplitude, and $\boldsymbol{\phi}$ is the phase. A waveform can be decomposed in terms of complex exponentials rather than sines and cosines because of Euler's Theorem, which highlights the surprisingly close relationship between a complex exponential and sines/cosines.

Euler's Theorem

$$\mathbf{e^{i\phi} = \cos\phi + i\sin\phi}$$

The Fourier transform allows you to write any function $\mathbf{f(t)}$ as the integral (sum) across frequencies of complex exponentials of different amplitudes and phases $\mathbf{F(\omega)}$. $\mathbf{f(t)}$ is often called the "time domain" representation while $\mathbf{F(\omega)}$ is called the "frequency domain representation." The key thing to understand about Fourier transforms is that these two representations are different ways of expressing the same information

- An aperiodic/non-periodic signal, like an audio signal. Consider the train whistle
- Fourier transform compute the frequency spectrum
- The Fourier transform of a function produces a spectrum from which the original function can be reconstructed (aka *synthesized*) by an inverse transform. So it is reversible. In order to do that, it preserves not only the magnitude of each frequency component, but also its phase.
- The Fourier transform is an equation to calculate the frequency, amplitude and phase of each sine wave needed to make up any given signal.
- The **Fourier Transform** is a mathematical technique for doing a similar thing - resolving any time-domain function into a frequency spectrum.
- The Fourier Transform (FT) is a mathematical formula using integrals.
- The Discrete Fourier Transform (DFT) is a discrete numerical equivalent using sums instead of integrals.
- The Fast Fourier Transform (FFT) is just a computationally fast way to calculate the DFT

যেহেতু Fourier Transform হচ্ছে Aperiodic/Non-Peridic Signal এর জন্য সেহেতু aperiodic/non periodic signal কে periodic signal করার জন্য ধরে নিব ∞ সময় পরে এ রকম একটি SIGNAL REPEAT পাওয়া যাবে , তার মানে এখানে Period $\mathbf{T} \rightarrow \infty$

Problem 22: Angular Frequency

Angular frequency ω also referred to by the terms angular speed, radial frequency, and radian frequency. If we think of the wave as a rotating wheel, then this means that the wheel makes a full revolution the same number of times per second.

We also know that one full revolution of the wheel is 360 or 2π radians.

Consequently, if we multiply the frequency of the sound wave by 2π, we get the number of radians the wheel turns each second. This value is called the *angular frequency* or the *radian frequency*

$\boldsymbol{\omega = 2\pi f}$

If f = 1, that means one cycle occurs

If f =2, that means two cycle occurs

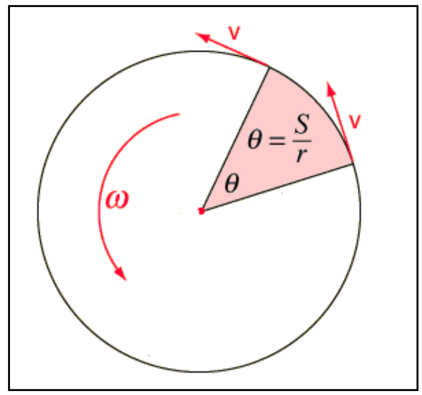

Figure 84

Angular Speed or Angular Frequency =ω; $\omega = \frac{2\pi}{t}$

$$\theta = 2\pi, \therefore \omega = \frac{\theta}{t} \Rightarrow \theta = \omega t$$

Problem 23:

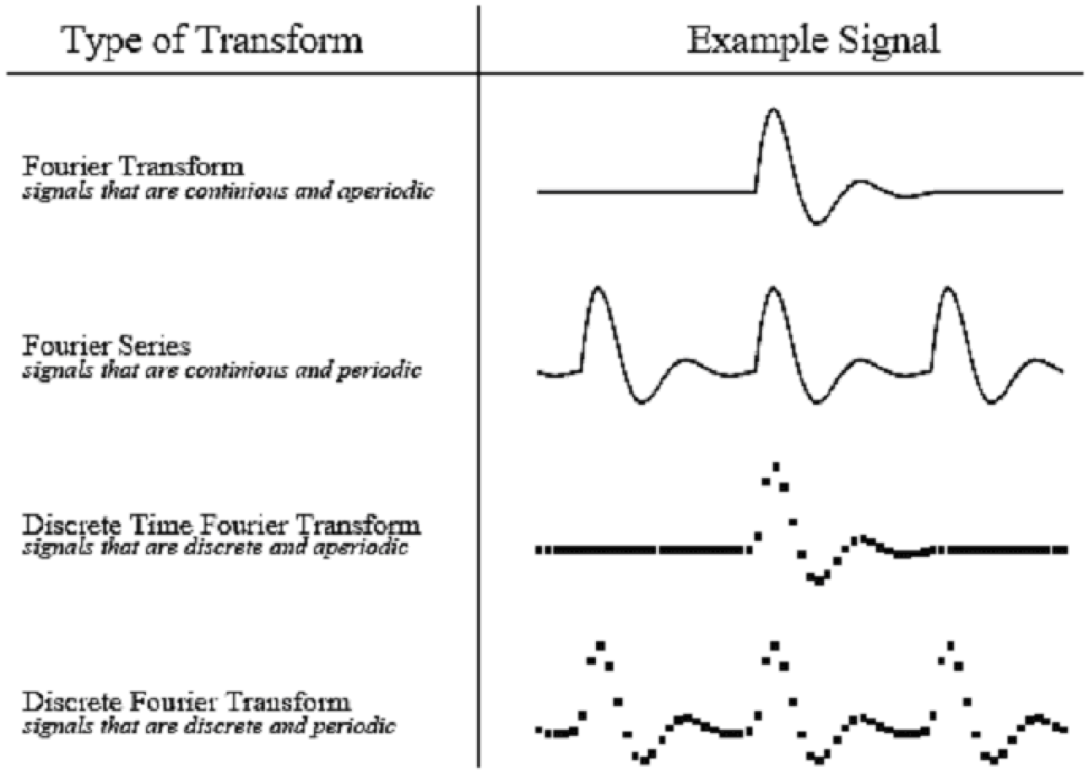

Figure 85

Revision of Previous lecture
Plot the line spectrum (discrete frequency spectra) for the Fourier series:

$$f(t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{1}{2n-1} \cos n\omega t + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \sin n\omega t$$

$$a_n = \frac{1}{2n-1} \qquad b_n = \frac{(-1)^n}{2n}$$

Now, using $R = C_n = \sqrt{a_n^2 + b_n^2}$ for each term, we have:

$a_n = \frac{1}{2n-1}$	$b_n = \frac{(-1)^n}{2n}$	$C_n = \sqrt{a_n^2 + b_n^2}$
$a_1 = 1$	$b_1 = -\frac{1}{2}$	$C_1 = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = 1.118$
$a_2 = \frac{1}{3}$	$b_2 = \frac{1}{4}$	$C_2 = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2} = 0.4167$
$a_3 = \frac{1}{5}$	$b_3 = -\frac{1}{6}$	$C_3 = \sqrt{\left(\frac{1}{5}\right)^2 + \left(-\frac{1}{6}\right)^2} = 0.260$
$a_4 = \frac{1}{7}$	$b_4 = \frac{1}{8}$	$C_4 = \sqrt{\left(\frac{1}{7}\right)^2 + \left(\frac{1}{8}\right)^2} = 0.190$

The resulting line spectrum is:
Amplitude of different harmonics/frequencies

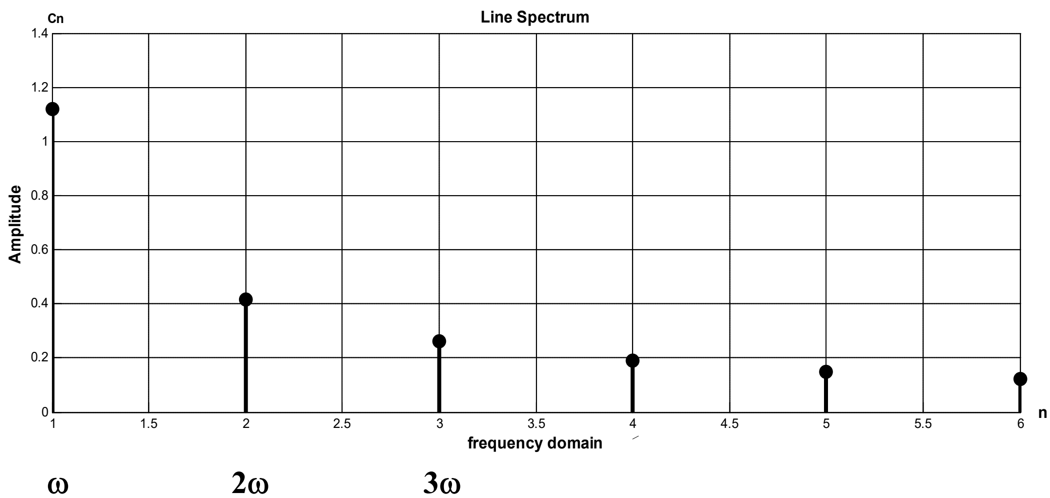

Figure 86

Hence Fundamental Frequency/ Harmonic/First Harmonic $\omega_1 = \omega$ $[\omega = 2\pi f]$

Hence Second Harmonic $\omega_2 = 2\omega$ $[2\omega = 2\pi * 2f]$

Hence Third Harmonic $\omega_3 = 3\omega$ $[3\omega = 2\pi * 3f]$

Hence Fourth Harmonic $\omega_4 = 4\omega$ $[4\omega = 2\pi * 4f]$

Problem 24: **Derive the equation of Fourier Transform from Fourier series**
Answer: we have the exponential form of Fourier series is:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$ --------------------------------------(i) [see equation no xii, **page no 86**]

Where,

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$ --------------------------------(ii) [see equation no xvii, **page no 87**]

As period $\mathbf{T} \to \infty$, $\boldsymbol{\omega}$ tends to zero. [See explanation part why $\mathbf{T} \to \infty$]
Since, We have,

$$\omega = \frac{2\pi}{T}$$

If $\mathbf{T} \to \infty$, then $\omega = \frac{2\pi}{\infty} = 0$

Explanation:
An aperiodic signal may be looked at as a periodic signal with an infinite period.

That means an aperiodic signal (pulses) repeats after an infinite interval (say). That is $\mathbf{T} \to \infty$; then this aperiodic signal will be converted into periodic signal.

$\mathbf{T} \to \infty$

Figure 87

The Fourier transform is an extension of the Fourier series that results when the period of the represented function is stretched and allowed to approach infinity. Fourier Transform was essentially developed for non periodic signals assuming the signals to be of infinite period.

When $\mathbf{T} \to \infty$ Then $\omega \to 0$,

We have,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\frac{1}{f}} = 2\pi f$$

$$\omega = \omega 1 = 2\pi f$$
$$2\omega = \omega 2 = 2\pi * 2f$$
$$3\omega = \omega 3 = 2\pi * 3f$$
$$4\omega = \omega 4 = 2\pi * 4f$$
$$5\omega = \omega 5 = 2\pi * 5f$$
$$6\omega = \omega 6 = 2\pi * 6f$$

When $\mathbf{T} \to \infty$ Then $\omega \to 0$ means,

Say

$$T \to 10000000000000000000000000000000000000000000000000000000000000010000$$

We have,

$$\omega = \frac{2\pi}{T}$$

$$\omega = \frac{2\pi}{10000000000000000000000000000000000000000000000000000000000000010000}$$

1st harmonic say

$$\omega = \omega 1 \to 0.000000000000000000000000000000000000000000000001100000000000$$
$$2\omega = \omega 2 \to 0.00000000000000000000000000000000000000000000002200000000000$$
$$3\omega = \omega 3 \to 0.00000000000000000000000000000000000000000000003300000000000$$
$$4\omega = \omega 4 \to 0.000000000000000000000000000000000000000000000044000000000000$$

So প্রত্যেকটি harmonic এর কম্পাংক খুবই কম, এবং

$\omega 2 - \omega 1, \omega 3 - \omega 2, \omega 4 - \omega 3,$ ....একটি harmonic থেকে আর একটি harmonic এর মধ্যে ব্যবধান খুবই কম, একদম লাগালাগি, কাছাকাছি, পাশাপাশি, ঘেষাঘেষি, ঠাসাঠাসি। ফলে প্রত্যেকটি harmonic এর against এ যে amplitude আঁকা হবে তা continuous spectrum হবে|

Fourier Transform এ $T \to \infty$, then $\omega = \frac{2\pi}{\infty} \to 0$ ফলে $\omega, 2\omega, 3\omega$ এদের মধ্যে পার্থক্য tends to zero ফলে Fourier Transform এ line spectrum draw করলে line spectrum গুলি একদম কাছাকাছি একটার সাথে আর একটা লেগে থাকে ফলে line spectrum গুলি আর আলাদা করা যায়না যেমন এখানে আলাদা করা যায় অর্থাৎ line spectrum গুলি continuous হয়।

----

and n becomes meaningless as T approaches infinity. and the lines will become so dense that that the discrete frequency spectrum eventually (finally/ ultimately) approaches a smooth curve, resulting in a continuous spectrum

So, for a continuous spectrum, $n\omega \to \omega$ (n becomes meaningless as T approaches infinity) and Spacing between adjacent components $\omega \to \Delta\omega$

Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\Delta\omega}$

From (1),

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

$$f(t) = \sum_{\omega=-\infty}^{\infty} c_\omega e^{i\omega t} \text{-------------------------------------(iii)} \qquad [n\omega \to \omega]$$

Where,

$$c_n = \frac{1}{T} \int_{-L}^{L} f(t) e^{-in\omega t} dt \text{ [From (ii)]}$$

$$\therefore c_\omega = \frac{1}{\frac{2\pi}{\Delta\omega}} \int_{-L}^{L} f(t)e^{-i\omega t}dt$$

$$\Rightarrow c_\omega = \frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(t)e^{-i\omega t}dt \text{ ---------------------------(iv)}$$

Hence from (iii), we get,

$$f(t) = \sum_{\omega=-\infty}^{\infty} c_\omega e^{i\omega t}$$

$$= \sum_{\omega=-\infty}^{\infty} c_\omega e^{i\omega t}$$

$$= \sum_{\omega=-\infty}^{\infty} [\frac{\Delta\omega}{2\pi} \int_{-L}^{L} f(t)e^{-i\omega t}dt\,]\,e^{i\omega t} \text{ [Putting the value of } c_\omega]$$

$$= \frac{1}{2\pi}[\sum_{\omega=-\infty}^{\infty} \int_{-L}^{L} f(t)e^{-i\omega t}dt\,]\,e^{i\omega t}\,\Delta\omega \text{ ----------(v)}$$

As $T \to \infty$, $\Delta\omega \to d\omega$ and $\sum \to \int$ [That is the discrete summation sign Σ should be replaced by the continuous summation integral $\int$]

Then equation (v) becomes,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt\,]\,e^{i\omega t}\,d\omega \text{ ----------------------(vi)}$$

Here, we have, $T = 2L$

$$\therefore L = \frac{T}{2}, \text{ If } T \to \infty, \text{then } L = \frac{\infty}{2} = \infty$$

Equation (vi) is form of the Fourier Integral.

The Fourier Integral (6) can be broken into a pair of relations:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [g(\omega)]e^{i\omega t} d\omega \qquad \text{-------------------------(vii)}$$

Where $g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$ ------------------------(viii) ***is called the Fourier transform of f (t)***

The graph of $(\omega, |g(\omega)|)$ is called the amplitude spectrum of $f(t)$, ω is called the frequency of the spectrum

The equation (vii) and (viii) are called a Fourier transform pair; $g(\omega)$ ***is called the Fourier transform of f (t) and conversely, f (t) is called the inverse Fourier transform of*** $g(\omega)$**.**

The Fourier transform is an equation to calculate the frequency, amplitude and phase of each sine wave needed to make up any given signal f (t)
That is:

$\mathbf{f(t) = F^{-1}[g(\omega)]}$ ---(ix)

$\mathbf{g(\omega) = F[f(t)]}$ --(x) ***Answer***

The plot of $|\mathbf{g(\omega)}|$ versus ω shows the relative frequency distribution of f (t).

Summary:

01. The Fourier transform transforms a function of t (in the time domain) into a function of ω (in the frequency domain), and the inverse Fourier transformation does the reverse, $\mathbf{g(\omega)}$ is also called the spectrum function of f (t)
02. We call $\mathbf{g(\omega)}$ the direct Fourier Transform of f (t) and f (t) the inverse Fourier Transform of $\mathbf{g(\omega)}$. The Transform $\mathbf{g(\omega)}$ is the frequency –domain specification of f (t)
03. Fourier Transform: $\mathbf{g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt}$
04. Inverse Fourier Transform: $\mathbf{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\ g(\omega)\ ]\, e^{i\omega t}\, d\omega}$

Remember:

01. The Fourier Spectrum of a signal indicates the relative amplitudes and phases of the sinusoids that are required to synthesize that signal

02. ***A periodic signal Fourier Spectrum has finite amplitudes and exists at discrete frequencies (*** $\omega 1, \omega 2, \omega 3, \ldots\ldots\ldots\ldots$ ***). Such a spectrum is easy to visualize, but the spectrum of a non periodic signal is not easy to visualize because it has a continuous spectrum that exists at every frequency (*** $\omega 1, \omega 2, \omega 3, \ldots\ldots\ldots\ldots$ ***)***

03. ***Spectral representation***—the frequency representation of periodic and aperiodic signals indicates how their power or energy is allocated to different frequencies. Such a distribution over frequency is called the *spectrum of the signal*. For a periodic signal the spectrum is discrete, as its power is concentrated at frequencies multiples of a so-called *fundamental frequency*, directly related to the period of the signal. On the other hand, the spectrum of an aperiodic signal is a continuous function of frequency. The concept of spectrum is similar to the one used in optics for light, or in material science for metals, each indicating the distribution of power or energy over frequency.

04. ***Spectrum Analysis*** **or** ***Fourier analysis*** is the process of analyzing some time-domain waveform to find its spectrum. We also say that the time domain waveform is converted into a frequency spectrum by means of the *Fourier transform*. This process is reversible: using the *inverse Fourier transform* a spectrum may be converted back into a time-domain waveform**.**

Example 39: Consider the rectangular pulse figure 84 described as

$$\mathbf{f(t) = 1; \qquad -T \le t \le T}$$
$$\mathbf{= 0; \qquad |t| > T}$$

Find the Fourier Transform of $\mathbf{f(t)}$

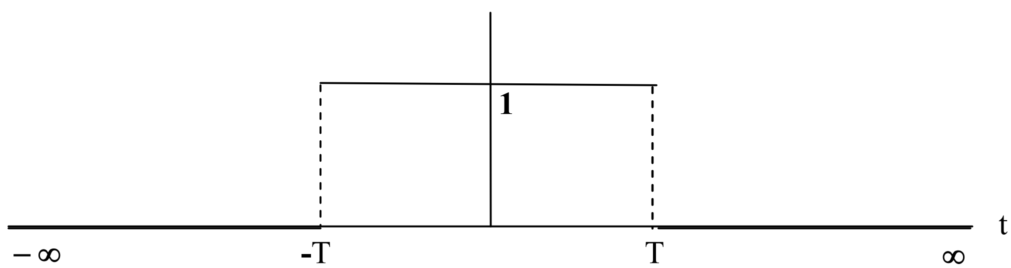

Figure 88: Rectangular Pulse

Answer:
We have,

$$\mathbf{f(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[\,g(\omega)\,]e^{i\omega t}\,d\omega$$

Where $g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$g(\omega) = \int_{-\infty}^{-T} f(t)e^{-i\omega t}dt + \int_{-T}^{T} f(t)e^{-i\omega t}dt + \int_{T}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{-T} 0.e^{-i\omega t}dt + \int_{-T}^{T} 1.e^{-i\omega t}dt + \int_{T}^{\infty} 0.e^{-i\omega t}dt$$

$$g(\omega) = \int_{-T}^{T} 1.e^{-i\omega t}dt = [\frac{e^{-i\omega t}}{-i\omega}]_{-T}^{T} \qquad [\because \int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-1}{i\omega}(e^{-i\omega T} - e^{i\omega T}) = \frac{1}{i\omega}(e^{i\omega T} - e^{-i\omega T})$$

$$g(\omega) = \frac{2.1}{\omega}\{\frac{1}{2i}(e^{i\omega T} - e^{-i\omega T})\} = \frac{2}{\omega}\sin(\omega T) \text{------------------------(i)}$$

$$[\because \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}), \text{ page no 84, equation no viii}]$$

OR

[Since we have, $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$]

$$g(\omega) = \frac{2.1}{\omega}\{\frac{1}{2i}(e^{i\omega T} - e^{-i\omega T})\}$$

$$= \frac{2}{\omega}\sin(\omega T)$$

$$= \frac{2T}{\omega T}\sin(\omega T)$$

$$g(\omega) = 2T\frac{\sin(\omega T)}{\omega T}$$

Thus we write for all ω , $g(\omega) = 2T\frac{\sin(\omega T)}{\omega T}$ ------------------------------------(ii)

Both (i) & (ii) are correct

For $\omega = 0,$ the integral simplifies to 2T. It is straightforward to show using L' Hopital's rule that, from (i)

$$\lim_{\omega\to 0}\frac{2}{\omega}\sin(\omega T)$$

$$= \lim_{\omega\to 0}\frac{2\sin(\omega T)}{\omega}[\frac{0}{0};\text{ Indeterminate Form}]$$

$$= \lim_{\omega\to 0}\frac{2\cos(\omega T).T}{1}\text{ [Differentiate numerator and denominator with respect to } \omega\text{]}$$

$$= \frac{2\cos 0.T}{1}$$

$$= 2.1.T = 2T$$

$$\therefore \lim_{\omega\to 0}\frac{2}{\omega}\sin(\omega T) = 2T$$

Again, From (ii),

$$g(\omega) = 2T\frac{\sin(\omega T)}{\omega T}$$

$$\lim_{\omega\to 0} g(\omega) = \lim_{\omega\to 0} 2T\frac{\sin(\omega T)}{\omega T}$$

$$\lim_{\omega\to 0} g(\omega) = 2T\lim_{\omega\to 0}\frac{\sin(\omega T)}{\omega T}$$

$$\lim_{\omega\to 0} g(\omega) = 2T.1 \qquad \left[\because \lim_{\theta\longrightarrow 0}\frac{\sin\theta}{\theta} = 1\right]$$

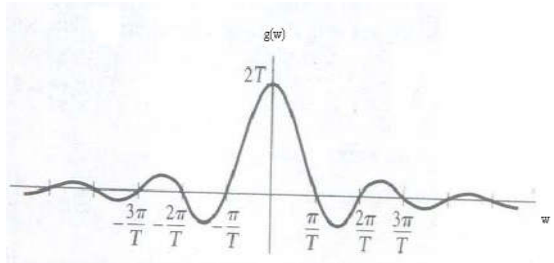

Figure 89: Fourier Transform of f (t)

Example 40: Find the Fourier Transform of $f(t) = e^{-at}u(t)$

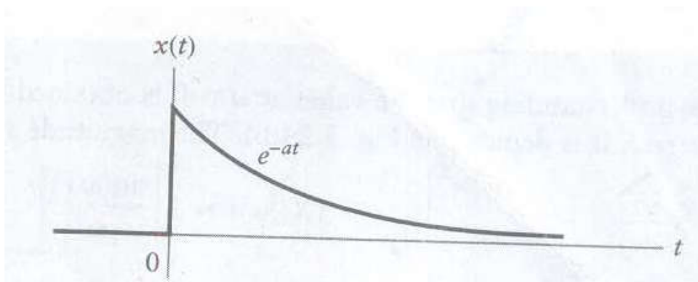

Figure 90: Exponential Signal

Answer: The Fourier Transform does not converge for $\mathbf{a \leq 0}$, since f (t) is not absolutely integrable, as shown by:

i. **For a>0, we have,**

$$g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{0} e^{-at}u(t)e^{-i\omega t}dt + \int_{0}^{\infty} e^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{0} e^{-at}.0.e^{-i\omega t}dt + \int_{0}^{\infty} e^{-at}.1.e^{-i\omega t}dt$$ [By definition from unit step function]

$$g(\omega) = 0 + \int_{0}^{\infty} e^{-at}.1.e^{-i\omega t}dt$$

$$g(\omega) = \int_{0}^{\infty} e^{-at}e^{-i\omega t}dt$$

$$g(\omega) = \int_0^\infty e^{-at} e^{-i\omega t} dt = \int_0^\infty e^{-(a+i\omega)t} dt \text{ ----------------------------------(i)}$$

$$= \frac{-1}{a+i\omega}\left[e^{-(a+i\omega)t}\right]_0^\infty$$

$$g(\omega) = \frac{-1}{a+i\omega}[e^{-(a+i\omega)\infty} - e^{-(a+i\omega)0}]$$

$$g(\omega) = \frac{-1}{a+i\omega}[e^{-\infty} - e^{-0}]$$

$$= \frac{-1}{a+i\omega}[\frac{1}{e^\infty} - \frac{1}{e^0}] = \frac{-1}{a+i\omega}\left[\frac{1}{\infty} - \frac{1}{1}\right] = \frac{-1}{a+i\omega}[0-1] = \frac{1}{a+i\omega} \quad [\because e^\infty = \infty]$$

$$|g(\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

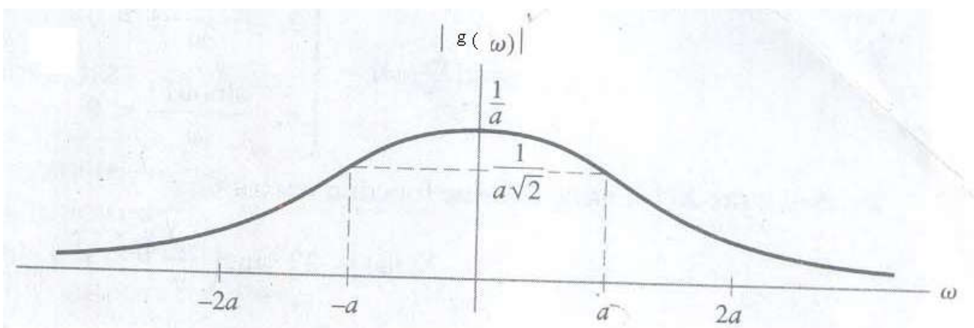

Figure 91: Fourier Transform of f (t) or Magnitude Spectrum

ii. for a < 0

$$g(\omega) = \int_0^\infty e^{-(a+i\omega)t} dt$$

$$g(\omega) = \int_0^\infty e^{-at-i\omega t} dt$$

$$g(\omega) = \int_0^\infty e^{at-i\omega t} dt \qquad [\text{for } a < 0]$$

$$g(\omega) = \int_0^\infty e^{(a-i\omega)t} dt$$

$$g(\omega) = \frac{1}{a-i\omega}\left[e^{(a-i\omega)t}\right]_0^\infty$$

$$g(\omega) = \frac{1}{a-i\omega}\left[e^{(a-i\omega)\infty} - e^{(a-i\omega)0}\right]_0^\infty$$

$$g(\omega) = \frac{1}{a-i\omega}\left[e^\infty - e^0\right]$$

$$g(\omega) = \frac{1}{a - i\omega}[\infty - 1] = \infty$$

Example 41: Find Fourier Transform of

$$f(t) = 1 \qquad ; 0 \le t < 1$$
$$= -1 \qquad ; -1 \le t < 0$$
$$= 0 \qquad ; |t| > 1$$

We have $g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$$g(\omega) = \int_{-\infty}^{-1} f(t) e^{-i\omega t} dt + \int_{-1}^{0} f(t) e^{-i\omega t} dt + \int_{0}^{1} f(t) e^{-i\omega t} dt + \int_{1}^{\infty} f(t) e^{-i\omega t} dt$$

$$g(\omega) = \int_{-\infty}^{-1} 0.e^{-i\omega t} dt + \int_{-1}^{0} (-1) e^{-i\omega t} dt + \int_{0}^{1} 1.e^{-i\omega t} dt + \int_{1}^{\infty} 0.e^{-i\omega t} dt$$

$$g(\omega) = -\int_{-1}^{0} e^{-i\omega t} dt + \int_{0}^{1} e^{-i\omega t} dt$$

$$g(\omega) = -\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{0} + \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{0}^{1} \qquad [\because \int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$g(\omega) = -\left[\frac{e^{-i\omega.0}}{-i\omega} - \frac{e^{-i\omega(-1)}}{-i\omega}\right] + \left[\frac{e^{-i\omega.1}}{-i\omega} - \frac{e^{-i\omega.0}}{-i\omega}\right]$$

$$g(\omega) = -\left[\frac{e^{-0}}{-i\omega} - \frac{e^{i\omega}}{-i\omega}\right] + \left[\frac{e^{-i\omega}}{-i\omega} - \frac{e^{-0}}{-i\omega}\right]$$

$$g(\omega) = -\left[\frac{\frac{1}{e^{0}}}{-i\omega} - \frac{e^{i\omega}}{-i\omega}\right] + \left[\frac{e^{-i\omega}}{-i\omega} - \frac{\frac{1}{e^{0}}}{-i\omega}\right]$$

$$g(\omega) = -\left[\frac{\frac{1}{1}}{-i\omega} - \frac{e^{i\omega}}{-i\omega}\right] + \left[\frac{e^{-i\omega}}{-i\omega} - \frac{\frac{1}{1}}{-i\omega}\right]$$

$$g(\omega) = -\left[\frac{1}{-i\omega} - \frac{e^{i\omega}}{-i\omega}\right] + \left[\frac{e^{-i\omega}}{-i\omega} - \frac{1}{-i\omega}\right]$$

$$g(\omega) = \left[\frac{1}{i\omega} - \frac{e^{i\omega}}{i\omega}\right] + \left[-\frac{e^{-i\omega}}{i\omega} + \frac{1}{i\omega}\right]$$

$$g(\omega) = \frac{1}{i\omega} + \frac{1}{i\omega} - \frac{e^{i\omega}}{i\omega} - \frac{e^{-i\omega}}{i\omega}$$

$$g(\omega)=\frac{2}{i\omega}-\frac{1}{i\omega}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{1}{i\omega}\frac{2}{2}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{2}{i\omega}\frac{1}{2}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{2}{i\omega}\cos\omega \qquad [\because \cos x=\frac{1}{2}(e^{ix}+e^{-ix})]$$

$$g(\omega)=\frac{2}{i\omega}(1-\cos\omega)$$ *Answer*

Example 42: Find Fourier Transform of $f(t)=e^{-|t|}$

Or

Find Fourier Transform of

$$f(t)=e^{-t} \qquad ;t>0$$
$$=e^{t} \qquad ;t<0$$

Answer:

Given

$$f(t)=e^{-t} \qquad ;t>0$$
$$=e^{t} \qquad ;t<0 \qquad \text{--------------------------(i)}$$

We have,

$$g(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} f(t)e^{-i\omega t}dt+\int_{0}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} e^{t}\,e^{-i\omega t}dt+\int_{0}^{\infty} e^{-t}\,e^{-i\omega t}dt \qquad \text{[Given equation no (i)]}$$

$$g(\omega)=\int_{-\infty}^{0} e^{t}\,e^{-i\omega t}dt+\int_{0}^{\infty} e^{-t}\,e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} e^{t-i\omega t}dt+\int_{0}^{\infty} e^{-t-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} e^{(1-i\omega)t}dt+\int_{0}^{\infty} e^{-(1+i\omega)t}dt$$

$$g(\omega)=\left[\frac{e^{(1-i\omega)t}}{(1-i\omega)}\right]_{-\infty}^{0}+\left[\frac{e^{-(1+i\omega)t}}{-(1+i\omega)}\right]_{0}^{\infty}$$

$$g(\omega)=\left[\frac{e^{(1-i\omega).0}}{(1-i\omega)}-\frac{e^{(1-i\omega)(-\infty)}}{(1-i\omega)}\right]+\left[\frac{e^{-(1+i\omega)\infty}}{-(1+i\omega)}-\frac{e^{-(1+i\omega).0}}{-(1+i\omega)}\right]$$

$$g(\omega)=\left[\frac{e^{0}}{(1-i\omega)}-\frac{e^{-\infty}}{(1-i\omega)}\right]+\left[\frac{e^{-\infty}}{-(1+i\omega)}-\frac{e^{-0}}{-(1+i\omega)}\right]$$

$$g(\omega)=\left[\frac{1}{(1-i\omega)}-\frac{1}{e^{\infty}(1-i\omega)}\right]+\left[\frac{e^{-\infty}}{-(1+i\omega)}-\frac{e^{-0}}{-(1+i\omega)}\right]$$

$$g(\omega)=\frac{1}{(1-i\omega)}\left[1-\frac{1}{e^{\infty}}\right]+\frac{-1}{1+i\omega}\left[\frac{1}{e^{\infty}}-\frac{1}{e^{0}}\right]$$

$$g(\omega)=\frac{1}{(1-i\omega)}\left[1-\frac{1}{\infty}\right]+\frac{-1}{1+i\omega}\left[\frac{1}{\infty}-\frac{1}{1}\right]$$

$$g(\omega)=\frac{1}{(1-i\omega)}[1-0]+\frac{-1}{1+i\omega}\left[\frac{1}{\infty}-\frac{1}{1}\right]$$

$$g(\omega)=\frac{1}{(1-i\omega)}+\frac{-1}{1+i\omega}[0-1]$$

$$g(\omega)=\frac{1}{(1-i\omega)}+\frac{1}{1+i\omega}$$

$$g(\omega)=\frac{1+i\omega+1-i\omega}{(1-i\omega)(1+i\omega)}$$

$$g(\omega)=\frac{2}{(1-i\omega)(1+i\omega)}$$

$$g(\omega)=\frac{2}{(1-i^{2}\omega^{2})}$$

$$g(\omega)=\frac{2}{1+\omega^{2}} \qquad [\,i^{2}=-1\,]$$

Answer

Example 43: Find Fourier Transform for the given functions

$\mathbf{f(t)=e^{-2t}}$; $\mathbf{t\ge 0}$

$\mathbf{f(t)=0}$; $\mathbf{t<0}$

Answer:

Given

$\mathbf{f(t)=e^{-2t}}$; $\mathbf{t\ge 0}$

$\mathbf{f(t)=0}$; $\mathbf{t<0}$ --------------------------(i)

We have,

$$g(\omega)=\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt$$

$$g(\omega)=\int_{-\infty}^{0} f(t)\,e^{-i\omega t}\,dt+\int_{0}^{\infty} f(t)\,e^{-i\omega t}\,dt$$

$$g(\omega)=\int_{-\infty}^{0} 0.e^{-i\omega t}dt+\int_{0}^{\infty} e^{-2t}\, e^{-i\omega t}dt$$ [Given equation no (i)]

$$g(\omega)=\int_{0}^{\infty} e^{-2t}\, e^{-i\omega t}dt$$

$$g(\omega)=\int_{0}^{\infty} e^{-2t-i\omega t}dt$$

$$g(\omega)=\int_{0}^{\infty} e^{-(2+i\omega)t}dt$$

$$g(\omega)=\left[\frac{e^{-(2+i\omega)t}}{-(2+i\omega)}\right]_{0}^{\infty}$$

$$g(\omega)=\frac{-1}{2+i\omega}\left[e^{-\infty}-e^{-0}\right]$$

$$g(\omega)=\frac{-1}{2+i\omega}\left[\frac{1}{e^{\infty}}-\frac{1}{e^{0}}\right]$$

$$g(\omega)=\frac{-1}{2+i\omega}\left[\frac{1}{\infty}-\frac{1}{1}\right]$$

$$g(\omega)=\frac{-1}{2+i\omega}\left[0-1\right]$$

$$g(\omega)=\frac{1}{2+i\omega}$$

$$|\,g(\omega)\,|=\frac{1}{\sqrt{2^2+\omega^2}}$$

$$|\,g(\omega)\,|=\frac{1}{\sqrt{4+\omega^2}}$$ *Answer*

Example 44: Find Fourier Transform of

$$f(t)=1+\frac{t}{a} \qquad ;-a<t<0$$

$$=1-\frac{t}{a} \qquad ;\ 0<t<a$$

$$=0 \qquad ;\ otherwise$$

Answer:

We have,

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[g(\omega)]e^{i\omega t}d\omega$$

Where $g(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$g(\omega)=\int_{-\infty}^{-a} f(t)e^{-i\omega t}dt+\int_{-a}^{0} f(t)e^{-i\omega t}dt+\int_{0}^{a} f(t)e^{-i\omega t}dt+\int_{a}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{-a} 0.e^{-i\omega t}dt+\int_{-a}^{0}(1+\frac{t}{a}).e^{-i\omega t}dt+\int_{0}^{a}(1-\frac{t}{a}).e^{-i\omega t}dt+\int_{a}^{\infty} 0.e^{-i\omega t}dt$$

$$g(\omega)=\int_{-a}^{0}(1+\frac{t}{a}).e^{-i\omega t}dt+\int_{0}^{a}(1-\frac{t}{a}).e^{-i\omega t}dt$$

$$g(\omega)=\int_{-a}^{0}1.e^{-i\omega t}dt+\int_{-a}^{0}\frac{t}{a}.e^{-i\omega t}dt+\int_{0}^{a}1.e^{-i\omega t}dt+\int_{0}^{a}(-\frac{t}{a}).e^{-i\omega t}dt$$

$$g(\omega)=\int_{-a}^{0} e^{-i\omega t}dt+\frac{1}{a}\int_{-a}^{0}t.e^{-i\omega t}dt+\int_{0}^{a} e^{-i\omega t}dt-\frac{1}{a}\int_{0}^{a}t.e^{-i\omega t}dt$$

$$g(\omega)=\int_{-a}^{0} e^{-i\omega t}dt+\int_{0}^{a} e^{-i\omega t}dt+\frac{1}{a}\int_{-a}^{0}t.e^{-i\omega t}dt-\frac{1}{a}\int_{0}^{a}t.e^{-i\omega t}dt\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(i)$$

Now,

$$\int t.e^{-i\omega t}dt$$

$$=t\int e^{-i\omega t}dt-\int\left\{\frac{d}{dt}(t)\int e^{-i\omega t}dt\right\}dt \qquad [\int \mathbf{uvdx}=\mathbf{u}\int \mathbf{vdx}-\int\{\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{u})\int \mathbf{vdx}\}\mathbf{dx}]$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]-\int\left\{1.\left[\frac{e^{-i\omega t}}{-i\omega}\right]\right\}dt$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]+\int\frac{e^{-i\omega t}}{i\omega}dt$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]+\frac{1}{i\omega}\int e^{-i\omega t}\,dt$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]+\frac{1}{i\omega}\left[\frac{e^{-i\omega t}}{-i\omega}\right]$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]-\frac{1}{i^2\omega^2}e^{-i\omega t}$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]-\frac{1}{-\omega^2}e^{-i\omega t} \qquad [i^2=-1]$$

$$=t\left[\frac{e^{-i\omega t}}{-i\omega}\right]+\frac{1}{\omega^2}e^{-i\omega t}$$

$$=\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t} \qquad \text{----------------------------------(ii)}$$

Putting the result of $\int t.e^{-i\omega t}dt$ in (i), we get

$$g(\omega)=\int_{-a}^{0} e^{-i\omega t}dt+\int_{0}^{a} e^{-i\omega t}dt+\frac{1}{a}\int_{-a}^{0} t.e^{-i\omega t}dt-\frac{1}{a}\int_{0}^{a} t.e^{-i\omega t}dt$$

$$g(\omega)=\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-a}^{0}+\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{0}^{a}+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\left[\frac{e^{-i\omega.0}}{-i\omega}-\frac{e^{-i\omega(-a)}}{-i\omega}\right]+\left[\frac{e^{-i\omega.a}}{-i\omega}-\frac{e^{-i\omega.0}}{-i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\left[\frac{e^{-0}}{-i\omega}-\frac{e^{+ai\omega}}{-i\omega}\right]+\left[\frac{e^{-i\omega.a}}{-i\omega}-\frac{e^{-0}}{-i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\left[\frac{1}{-i\omega}-\frac{e^{+ai\omega}}{-i\omega}\right]+\left[\frac{e^{-i\omega.a}}{-i\omega}-\frac{1}{-i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$[e^{-0}=\frac{1}{e^0}=\frac{1}{1}=1]$$

$$g(\omega)=\left[\frac{1}{-i\omega}+\frac{e^{+ai\omega}}{i\omega}\right]+\left[\frac{e^{-i\omega.a}}{-i\omega}+\frac{1}{i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\left[-\frac{1}{i\omega}+\frac{e^{+ai\omega}}{i\omega}\right]+\left[-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{i\omega}\right]+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=-\frac{1}{i\omega}+\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{i\omega}+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{-a}^{0}-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{-i\omega}0.e^{-i\omega.0}+\frac{1}{\omega^2}e^{-i\omega.0}-\frac{1}{-i\omega}(-a)e^{-i\omega(-a)}-\frac{1}{\omega^2}e^{-i\omega(-a)}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[0+\frac{1}{\omega^2}e^{-0}-\frac{1}{-i\omega}(-a)e^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}.1+\frac{1}{-i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_{0}^{a}$$

$$[e^{-0}=\frac{1}{e^0}=\frac{1}{1}=1]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}te^{-i\omega t}+\frac{1}{\omega^2}e^{-i\omega t}\right]_0^a$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}ae^{-i\omega a}+\frac{1}{\omega^2}e^{-i\omega a}-\frac{1}{-i\omega}.0.e^{-i\omega.0}-\frac{1}{\omega^2}e^{-i\omega.0}\right]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}ae^{-i\omega a}+\frac{1}{\omega^2}e^{-i\omega a}-\frac{1}{\omega^2}e^{-0}\right]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}ae^{-i\omega a}+\frac{1}{\omega^2}e^{-i\omega a}-\frac{1}{\omega^2}.1\right]$$

$$[e^{-0}=\frac{1}{e^0}=\frac{1}{1}=1]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a}\left[\frac{1}{\omega^2}-\frac{1}{i\omega}ae^{i\omega a}-\frac{1}{\omega^2}e^{i\omega a}\right]-\frac{1}{a}\left[\frac{1}{-i\omega}ae^{-i\omega a}+\frac{1}{\omega^2}e^{-i\omega a}-\frac{1}{\omega^2}\right]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\left[\frac{1}{a\omega^2}-\frac{ae^{i\omega a}}{ai\omega}-\frac{e^{i\omega a}}{a\omega^2}\right]-\left[\frac{-ae^{-i\omega a}}{ai\omega}+\frac{e^{-i\omega a}}{a\omega^2}-\frac{1}{a\omega^2}\right]$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega.a}}{i\omega}+\frac{1}{a\omega^2}-\frac{ae^{i\omega a}}{ai\omega}-\frac{e^{i\omega a}}{a\omega^2}+\frac{ae^{-i\omega a}}{ai\omega}-\frac{e^{-i\omega a}}{a\omega^2}+\frac{1}{a\omega^2}$$

$$g(\omega)=\frac{1}{i\omega}(e^{+ai\omega}-e^{-i\omega a})+\frac{1}{a\omega^2}+\frac{1}{a\omega^2}+\frac{ae^{-i\omega a}}{ai\omega}-\frac{ae^{i\omega a}}{ai\omega}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{1}{i\omega}(e^{+ai\omega}-e^{-i\omega a})+\frac{2}{a\omega^2}+\frac{ae^{-i\omega a}}{ai\omega}-\frac{ae^{i\omega a}}{ai\omega}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{1}{i\omega}(e^{+ai\omega}-e^{-i\omega a})+\frac{2}{a\omega^2}+\frac{e^{-i\omega a}}{i\omega}-\frac{e^{i\omega a}}{i\omega}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{e^{+ai\omega}}{i\omega}-\frac{e^{-i\omega a}}{i\omega}+\frac{2}{a\omega^2}+\frac{e^{-i\omega a}}{i\omega}-\frac{e^{i\omega a}}{i\omega}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{2}{a\omega^2}-\frac{e^{i\omega a}}{a\omega^2}-\frac{e^{-i\omega a}}{a\omega^2}$$

$$g(\omega)=\frac{1}{a\omega^2}(2-e^{i\omega a}-e^{-i\omega a})$$

Answer

Example 45: Find Fourier Transform of $\mathbf{f(t)=te^{-at}u(t)}$ for $\mathbf{a>0}$

Answer:

We have $g(\omega)=\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt$

$$g(\omega)=\int_{-\infty}^{\infty}te^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} te^{-at}u(t)e^{-i\omega t}dt+\int_{0}^{\infty} te^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} te^{-at}.0.e^{-i\omega t}dt+\int_{0}^{\infty} te^{-at}.1.e^{-i\omega t}dt$$ *[As per the definition of unit function]*

$$g(\omega)=0+\int_{0}^{\infty} te^{-at}.1.e^{-i\omega t}dt$$

$$g(\omega)=\int_{0}^{\infty} te^{-(a+i\omega)t}dt \quad \text{`} \qquad ....(i)$$

Let $I_n=\int_{0}^{\infty} te^{-(a+i\omega)t}dt$ ....(ii)

Now, $\int te^{-(a+i\omega)t}dt$

$$=t\int e^{-(a+i\omega)t}dt-\int\{\frac{d}{dt}(t)\int e^{-(a+i\omega)t}dt\}dt$$

$$=t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}-\int\left\{1.\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right\}dt$$

$$=t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}+\frac{1}{a+i\omega}\int e^{-(a+i\omega)t}dt$$

$$=t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}+\frac{1}{a+i\omega}.\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}$$

$$=t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}-\frac{1}{(a+i\omega)^2}.e^{-(a+i\omega)t} \quad ....(iii)$$

Putting the value of $\int t^n e^{-(a+i\omega)t}dt$ in (ii),

$$I_n=\int_{0}^{\infty} t^n e^{-(a+i\omega)t}dt$$

$$=\left[t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_0^{\infty}-\left[\frac{1}{(a+i\omega)^2}.e^{-(a+i\omega)t}\right]_0^{\infty}$$

$$=\left[t\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_0^{\infty}-\frac{1}{(a+i\omega)^2}\left[e^{-(a+i\omega)t}\right]_0^{\infty}$$

$$=\left[\infty\frac{e^{-(a+i\omega).\infty}}{-(a+i\omega)}-0.\frac{e^{-(a+i\omega).0}}{-(a+i\omega)}\right]-\frac{1}{(a+i\omega)^2}\left[e^{-(a+i\omega)t}\right]_0^{\infty}$$

$$=\left[\infty.\frac{e^{-\infty}}{-(a+i\omega)}-0.\frac{e^{-0}}{-(a+i\omega)}\right]-\frac{1}{(a+i\omega)^2}\left[e^{-(a+i\omega).\infty}-e^{-(a+i\omega).0}\right]$$

$$=\left[\infty.\frac{1}{e^{\infty}}.\frac{1}{-(a+i\omega)}-0.\frac{1}{e^{0}}.\frac{1}{-(a+i\omega)}\right]-\frac{1}{(a+i\omega)^2}\left[e^{-\infty}-e^{-0}\right]$$

$$=\left[\infty.\frac{1}{\infty}.\frac{1}{-(a+i\omega)}-0.\frac{1}{1}.\frac{1}{-(a+i\omega)}\right]-\frac{1}{(a+i\omega)^2}\left[\frac{1}{e^{\infty}}-\frac{1}{e^{0}}\right]$$

$$=\left[\infty.0.\frac{1}{-(a+i\omega)}-0\right]-\frac{1}{(a+i\omega)^2}\left[\frac{1}{\infty}-\frac{1}{e^{0}}\right] \qquad [e^{\infty}=\infty]$$

$$=\left[\infty.0.\frac{1}{-(a+i\omega)}-0\right]-\frac{1}{(a+i\omega)^2}\left[0-\frac{1}{1}\right]$$

$$=\left[\infty.0.\frac{1}{-(a+i\omega)}-0\right]-\frac{1}{(a+i\omega)^2}\left[0-1\right]$$

$$=0+\frac{1}{(a+i\omega)^2}$$

$$=\frac{1}{(a+i\omega)^2}$$

$$\therefore g(\omega)=I_n=\int_0^{\infty} te^{-(a+i\omega)t}dt=\frac{1}{(a+i\omega)^2}$$ *Answer*

Example 46: Find Fourier Transform of $f(t)=t^n e^{-at}u(t)$ for $a>0$

Answer:

We have $g(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$g(\omega)=\int_{-\infty}^{\infty} t^n e^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} t^n e^{-at}u(t)e^{-i\omega t}dt+\int_{0}^{\infty} t^n e^{-at}u(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{0} t^n e^{-at}.0.e^{-i\omega t}dt+\int_{0}^{\infty} t^n e^{-at}.1.e^{-i\omega t}dt$$ *[As per the definition of unit function]*

$$g(\omega)=0+\int_{0}^{\infty} t^n e^{-at}.1.e^{-i\omega t}dt$$

$$g(\omega)=\int_{0}^{\infty} t^n e^{-(a+i\omega)t}dt \qquad \text{.......................(i)}$$

Let $I_n=\int_{0}^{\infty} t^n e^{-(a+i\omega)t}dt$ ....(ii)

$$\therefore I_{n-1}=\int_{0}^{\infty} t^{n-1} e^{-(a+i\omega)t}dt \qquad \text{.......................(iii)}$$

Now, $\int t^n e^{-(a+i\omega)t}dt$

$$= t^n \int e^{-(a+i\omega)t}dt - \int\{\frac{d}{dt}(t^n)\int e^{-(a+i\omega)t}dt\}dt$$

$$= t^n \frac{e^{-(a+i\omega)t}}{-(a+i\omega)} - \int\left\{nt^{n-1}.\frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right\}dt$$

$$= t^n \frac{e^{-(a+i\omega)t}}{-(a+i\omega)} + \frac{n}{a+i\omega}\int t^{n-1}e^{-(a+i\omega)t}dt$$

$$= t^n \frac{e^{-(a+i\omega)t}}{-(a+i\omega)} + \frac{n}{a+i\omega}I_{n-1} \text{ ....(iv)} \qquad \text{[from (iii)]}$$

Putting the value of $\int t^n e^{-(a+i\omega)t}dt$ in (ii),

$$I_n = \int_0^\infty t^n e^{-(a+i\omega)t}dt$$

$$= \left[t^n \frac{e^{-(a+i\omega)t}}{-(a+i\omega)}\right]_0^\infty + \frac{n}{a+i\omega}I_{n-1}$$

$$= \left[\infty^n \frac{e^{-(a+i\omega).\infty}}{-(a+i\omega)} - 0^n \frac{e^{-(a+i\omega).0}}{-(a+i\omega)}\right] + \frac{n}{a+i\omega}I_{n-1}$$

$$= \left[\infty.\frac{e^{-\infty}}{-(a+i\omega)} - 0.\frac{e^{-0}}{-(a+i\omega)}\right] + \frac{n}{a+i\omega}I_{n-1}$$

$$= \left[\infty.\frac{1}{e^\infty}.\frac{1}{-(a+i\omega)} - 0.\frac{1}{e^0}.\frac{1}{-(a+i\omega)}\right] + \frac{n}{a+i\omega}I_{n-1}$$

$$= \left[\infty.\frac{1}{\infty}.\frac{1}{-(a+i\omega)} - 0.\frac{1}{1}.\frac{1}{-(a+i\omega)}\right] + \frac{n}{a+i\omega}I_{n-1}$$

$$= \left[\infty.0.\frac{1}{-(a+i\omega)} - 0\right] + \frac{n}{a+i\omega}I_{n-1}$$

$$= \left[\infty.0.\frac{1}{-(a+i\omega)} - 0\right] + \frac{n}{a+i\omega}I_{n-1}$$

$$= \left[\infty.0.\frac{1}{-(a+i\omega)} - 0\right] + \frac{n}{a+i\omega}I_{n-1}$$

$$= [0] + \frac{n}{a+i\omega}I_{n-1}$$

$$I_n = \frac{n}{a+i\omega}I_{n-1} \qquad \text{.................................(v)}$$

Put $n = n-1$ in **(v)**

$$\therefore I_{n-1} = \frac{n-1}{a+i\omega}I_{n-2} \text{ --(vi)}$$

Again, Put $\boldsymbol{n = n - 2}$ in $\mathbf{(v)}$

$$\therefore \mathbf{I_{n-2} = \frac{n-2}{a+i\omega} I_{n-3}} \text{ -- } \mathbf{(vii)}$$

Again Put, $\boldsymbol{n = n - 3}$ in $\mathbf{(v)}$

$$\therefore \mathbf{I_{n-3} = \frac{n-3}{a+i\omega} I_{n-4}} \text{ --- } \mathbf{(viii)}$$

Put $\boldsymbol{n = 2}$ in $\mathbf{(v)}$

$$\mathbf{I_2 = \frac{2}{a+i\omega} I_{2-1}}$$

$$\therefore \mathbf{I_2 = \frac{2}{a+i\omega} I_1} \text{ --- } (ix)$$

Put $\boldsymbol{n = 1}$ in $\mathbf{(v)}$

$$\therefore \mathbf{I_1 = \frac{1}{a+i\omega} I_{1-1}}$$

$$\mathbf{= \frac{1}{a+i\omega} I_0} \text{ -- } (x)$$

Putting in values of $\boldsymbol{I_{n-1}, I_{n-2}, \ldots\ldots\ldots, I_2, I_1}$ in (v)

$$\therefore \mathbf{I_n = \frac{n}{a+i\omega} I_{n-1}}$$

$$\mathbf{= \frac{n}{a+i\omega} \cdot \frac{n-1}{a+i\omega} I_{n-2}} \qquad \mathbf{[form(vi)]}$$

$$\mathbf{= \frac{n}{a+i\omega} \cdot \frac{n-1}{a+i\omega} \cdot \frac{n-2}{a+i\omega} I_{n-3}} \qquad \mathbf{[form(vii)]}$$

$$\mathbf{= \frac{n}{a+i\omega} \cdot \frac{n-1}{a+i\omega} \cdot \frac{n-2}{a+i\omega} \cdot \frac{n-3}{a+i\omega} I_{n-4}} \qquad \mathbf{[form(viii)]}$$

$\mathbf{[form(ix)\ and\ (x)]}$

$$\mathbf{= \frac{n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots\ldots\ldots\ldots 2.1}{(a+i\omega)^n} I_0}$$

$$\therefore \mathbf{I_n = \frac{n!}{(a+i\omega)^n} I_0} \text{ ------------------------------------- } \mathbf{(xi)}$$

We have, $\mathbf{I_n = \int_0^\infty t^n e^{-(a+i\omega)t} dt}$

Put $\boldsymbol{n = 0}$

$$I_0 = \int_0^\infty t^0 e^{-(a+i\omega)t} dt$$

$$I_0 = \int_0^\infty 1.e^{-(a+i\omega)t} dt$$

$$I_0 = \int_0^\infty e^{-(a+i\omega)t} dt$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[e^{-(a+i\omega)t}\right]_0^\infty$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[e^{-(a+i\omega).\infty} - e^{-(a+i\omega).0}\right]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[e^{-\infty} - e^{-0}\right]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[\frac{1}{\infty} - \frac{1}{e^0}\right] \qquad [e^\infty = \infty]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[0 - \frac{1}{1}\right]$$

$$I_0 = \frac{1}{-(a+i\omega)}\left[0 - 1\right]$$

$$I_0 = \frac{1}{(a+i\omega)}$$

From (xi),

$$\therefore I_n = \frac{n!}{(a+i\omega)^n} I_0$$

$$\therefore I_n = \frac{n!}{(a+i\omega)^n} . \frac{1}{(a+i\omega)}$$

$$\therefore I_n = \frac{n!}{(a+i\omega)^{n+1}}$$

$$\therefore g(\omega) = I_n = \int_0^\infty t^n e^{-(a+i\omega)t} dt = \frac{n!}{(a+i\omega)^{n+1}}$$ *Answer*

Example 47: Find Fourier Transform of

$f(t) = 1 - t^2$ for $|t| < 1$

$= 0$ for $|t| > 1$

We have $g(\omega)=\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$

$$g(\omega)=\int_{-\infty}^{-1} f(t)e^{-i\omega t}dt+\int_{-1}^{1} f(t)e^{-i\omega t}dt++\int_{1}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega)=\int_{-\infty}^{-1} 0.e^{-i\omega t}dt+\int_{-1}^{1}(1-t^2)e^{-i\omega t}dt+\int_{1}^{\infty} 0.e^{-i\omega t}dt$$

$$g(\omega)=0+\int_{-1}^{1}(1-t^2)e^{-i\omega t}dt+0$$

$$g(\omega)=\int_{-1}^{1} e^{-i\omega t}dt-\int_{-1}^{1} t^2.e^{-i\omega t}dt$$

$$g(\omega)=\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{1}-\int_{-1}^{1} t^2.e^{-i\omega t}dt$$

$$g(\omega)=\left[\frac{e^{-i\omega}}{-i\omega}-\frac{e^{-i\omega(-1)}}{-i\omega}\right]-\int_{-1}^{1} t^2.e^{-i\omega t}dt$$

$$g(\omega)=\left[\frac{e^{-i\omega}}{-i\omega}-\frac{e^{i\omega}}{-i\omega}\right]-\int_{-1}^{1} t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{1}{-i\omega}\left[e^{-i\omega}-e^{i\omega}\right]-\int_{-1}^{1} t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{1}{i\omega}\left[e^{i\omega}-e^{-i\omega}\right]-\int_{-1}^{1} t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{1}{\omega}\frac{2}{2i}\left[e^{i\omega}-e^{-i\omega}\right]-\int_{-1}^{1} t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{2}{\omega}\frac{1}{2i}\left[e^{i\omega}-e^{-i\omega}\right]-\int_{-1}^{1} t^2.e^{-i\omega t}dt$$

$$g(\omega)=\frac{2}{\omega}\sin\omega-\int_{-1}^{1} t^2.e^{-i\omega t}dt \text{ ....(i)} \qquad [\because \sin x=\frac{1}{2i}(e^{ix}-e^{-ix})]$$

Now, $\int t^2.e^{-i\omega t}dt$

$$=t^2\int e^{-i\omega t}dt-\int\{\frac{d}{dt}(t^2)\int e^{-i\omega t}dt\}dt$$

$$=t^2\frac{e^{-i\omega t}}{-i\omega}-\int 2t\frac{e^{-i\omega t}}{-i\omega}dt$$

$$=t^2\frac{e^{-i\omega t}}{-i\omega}+2\int t\frac{e^{-i\omega t}}{i\omega}dt$$

$$= t^2 \frac{e^{-i\omega t}}{-i\omega} + 2\frac{1}{i\omega}\int te^{-i\omega t}dt$$

$$= t^2 \frac{e^{-i\omega t}}{-i\omega} + \frac{2}{i\omega}\int te^{-i\omega t}dt \qquad \text{.............................(ii)}$$

Now,

$$\int t.e^{-i\omega t}dt$$

$$= t\int e^{-i\omega t}dt - \int\left\{\frac{d}{dt}(t)\int e^{-i\omega t}dt\right\}dt \qquad [\int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] - \int\left\{1.\left[\frac{e^{-i\omega t}}{-i\omega}\right]\right\}dt$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] + \int\frac{e^{-i\omega t}}{i\omega}dt$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] + \frac{1}{i\omega}\int e^{-i\omega t}\,dt$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] + \frac{1}{i\omega}\left[\frac{e^{-i\omega t}}{-i\omega}\right]$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] - \frac{1}{i^2\omega^2}e^{-i\omega t}$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] - \frac{1}{-\omega^2}e^{-i\omega t} \qquad [i^2 = -1]$$

$$= t\left[\frac{e^{-i\omega t}}{-i\omega}\right] + \frac{1}{\omega^2}e^{-i\omega t}$$

$$= \frac{1}{-i\omega}te^{-i\omega t} + \frac{1}{\omega^2}e^{-i\omega t} \qquad \text{-----------------------------------(iii)}$$

Putting the value of $\int t.e^{-i\omega t}dt$

From (ii)

$$\int t^2.e^{-i\omega t}dt = t^2\frac{e^{-i\omega t}}{-i\omega} + \frac{2}{i\omega}\int te^{-i\omega t}dt$$

$$\int t^2.e^{-i\omega t}dt = t^2\frac{e^{-i\omega t}}{-i\omega} + \frac{2}{i\omega}\left[\frac{1}{-i\omega}te^{-i\omega t} + \frac{1}{\omega^2}e^{-i\omega t}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \left[t^2\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{1} + \frac{2}{i\omega}\left[\frac{1}{-i\omega}te^{-i\omega t} + \frac{1}{\omega^2}e^{-i\omega t}\right]_{-1}^{1}$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \left[1^2\frac{e^{-i\omega.1}}{-i\omega} - 1^2\frac{e^{-i\omega.(-1)}}{-i\omega}\right] +$$
$$\frac{2}{i\omega}\left[\frac{1}{-i\omega}1.e^{-i\omega.1} + \frac{1}{\omega^2}e^{-i\omega.1} - \frac{1}{-i\omega}(-1)e^{-i\omega(-1)} - \frac{1}{\omega^2}e^{-i\omega(-1)}\right]$$

$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \left[-\frac{e^{-i\omega}}{i\omega} + \frac{e^{i\omega}}{i\omega}\right] + \frac{2}{i\omega}\left[\frac{1}{-i\omega}e^{-i\omega} + \frac{1}{\omega^2}e^{-i\omega} - \frac{1}{i\omega}e^{i\omega} - \frac{1}{\omega^2}e^{i\omega}\right]$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right] + \frac{2}{i\omega}\left[-\frac{1}{i\omega}e^{-i\omega} + \frac{1}{\omega^2}e^{-i\omega} - \frac{1}{i\omega}e^{i\omega} - \frac{1}{\omega^2}e^{i\omega}\right]$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right] + \frac{2}{i\omega}\left[-\frac{1}{i\omega}e^{i\omega} - \frac{1}{i\omega}e^{-i\omega} - \frac{1}{\omega^2}e^{i\omega} + \frac{1}{\omega^2}e^{-i\omega}\right]$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right] + \frac{2}{i\omega}.\frac{1}{i\omega}\left[-e^{i\omega} - e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{1}{\omega^2}\left[e^{i\omega} - e^{-i\omega}\right]$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{i\omega}\left[e^{i\omega} - e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{1}{i\omega}\left[e^{i\omega} + e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{1}{\omega^2}\left[e^{i\omega} - e^{-i\omega}\right]$$
$$[\therefore \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})\,;\cos x = \frac{1}{2}(e^{ix} + e^{-ix})\,]$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{1}{\omega}\frac{2}{2i}\left[e^{i\omega} - e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{1}{i\omega}\frac{2}{2}\left[e^{i\omega} + e^{-i\omega}\right] - \frac{2}{\omega}.\frac{1}{\omega^2}\frac{2}{2i}\left[e^{i\omega} - e^{-i\omega}\right]$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\frac{1}{2i}\left[e^{i\omega} - e^{-i\omega}\right] - \frac{2}{i\omega}.\frac{2}{i\omega}\frac{1}{2}\left[e^{i\omega} + e^{-i\omega}\right] - \frac{2}{\omega}.\frac{2}{\omega^2}\frac{1}{2i}\left[e^{i\omega} - e^{-i\omega}\right]$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\sin\omega - \frac{2}{i\omega}.\frac{2}{i\omega}\cos\omega - \frac{2}{\omega}.\frac{2}{\omega^2}\sin\omega$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\sin\omega - \frac{4}{i^2\omega^2}\cos\omega - \frac{4}{\omega^3}\sin\omega$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\sin\omega - \frac{4}{-\omega^2}\cos\omega - \frac{4}{\omega^3}\sin\omega$$
$$\therefore \int_{-1}^{1} t^2.e^{-i\omega t}dt = \frac{2}{\omega}\sin\omega + \frac{4}{\omega^2}\cos\omega - \frac{4}{\omega^3}\sin\omega$$
Answer

Example 48: Find Fourier Transform of the unit impulse $\delta(t)$

Answer:

The Impulse function $\delta(t)$ is defined as

$\delta(t) = 1 \qquad t = 0$

$= 0$ **Otherwise**

Here, $f(t) = \delta(t)$

We have,

$$g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$g(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t} dt$$

$$g(\omega) = \left[e^{-i\omega t}\right]_{t = location\ of\ \delta(t)\ function}$$

$$g(\omega) = \left[e^{-i\omega t}\right]_{t=0}$$

$$g(\omega) = \left[e^{-0}\right]$$

$$g(\omega) = \frac{1}{e^{0}}$$

$$g(\omega) = \frac{1}{1}$$

$$g(\omega) = 1$$

Summary:

01. Spectral analysis of periodic functions is achieved through the *Fourier series.* The three forms are:

i. cosine-sine or trigonometric form: $\mathbf{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t + \sum_{n=1}^{\infty} b_n \sin(n\omega t)}$

ii. amplitude-phase form $\mathbf{f(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \phi_n)}$

iii. Complex exponential form: $\mathbf{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$

iv. These sine terms are combined to create the time-varying waveform in the display. And if the opposite operation were to be performed on the result (something called a Fourier transform), these individual harmonic elements would reappear. It is important to reemphasize that waveform generation, and the Fourier transform, are reciprocal operations. You can use frequency components to generate a waveform in the time domain, then transform the result back to the frequency domain and recover what you started with. This reciprocal relationship is to Fourier analysis what the Fundamental Theorem of Calculus (the idea that integration and derivation are reciprocal operations) is to Calculus

02. Why would you want to take a Fourier transform?

i. Fourier analysis can be very useful for two main reasons**.** Many calculations are simpler in the frequency domain than the time domain. For example: filtering (convolving) becomes trivial in the frequency domain. We'll talk about this next chapter

ii. Many neural processes can be described more effectively in the frequency domain. For example: The cochlea transforms a time domain signal (the sound's waveform) into a frequency domain signal. The strength of the response in the auditory nerve fiber tuned to a particular frequency reflects the amplitude of the sound's waveform at that frequency. In other words, the auditory system takes a Fourier transform of the incoming signal, decomposing the sound into amplitudes as a function of frequency.

iii. Many brain regions have oscillations of a particular frequency that can be easily characterized with Fourier analysis

03. Distingush between Fourier Series and Fourier Transform

i. The Fourier series is used only for periodic functions. The Fourier fransform is used for many classes of non-periodic functions.

ii. Fourier series is just the representation of your periodic signal in sine and cosine waves. It just decomposes your signal into these sine-cosine waves. Fourier transform comes into the picture. It transforms aperiodic signal into the continuous frequency domain.

iii. In short, fourier series is for periodic signals and fourier transform is for aperiodic signals. For evaluation of aperiodic signal we solve in a similar manner as that of periodic signal with the assumption that the period is infinite.

What is Inverse Fourier transform?

We also say that the time domain waveform is converted into a frequency spectrum by means of the *Fourier transform*. This process is reversible: using the *inverse Fourier transform* a spectrum may be converted back into a time-domain waveform.

04. What are the applications of Fourier analysis and Fourier Transform?

Application of Fourier analysis—the frequency representation of signals and systems is extremely important in signal processing and in communications. It explains filtering, modulation of messages in a communication system, the meaning of bandwidth, and how to design filters. Likewise, the frequency representation turns out to be essential in the sampling of analog signals—the bridge between analog and digital signal processing.

The applications of the Fourier transform include filtering. telecommunication, music processing, pitch modification, signal coding and signal synthesis feature extraction for pattern identification as in speech recognition, image processing, spectral analysis in astrophysics, radar signal processing. The Fourier transform is useful for extracting a signal from a noisy background.

Fourier methods have revolutionized fields of science and engineering, from radio astronomy to medical imaging, from seismology to spectroscopy. The wide application of Fourier methods is credited principally to the existence of the fast Fourier transform (FFT). The most direct application of the FFT are to the convolution or de convolution of data, correlation and autocorrelation, optimal filtering, power spectrum estimation, and the computational of Fourier integrals.

Example 49: Draw the graph of $\mathbf{y = f(t) = e^{5t}}$

Answer:

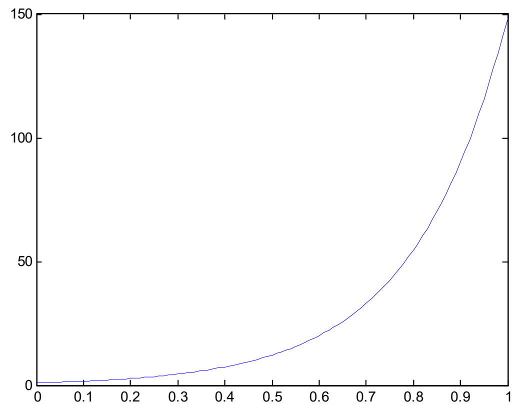

Figure 92

Example 50: Draw the graph of $\mathbf{y = f(t) = e^{-5t}}$

Answer:

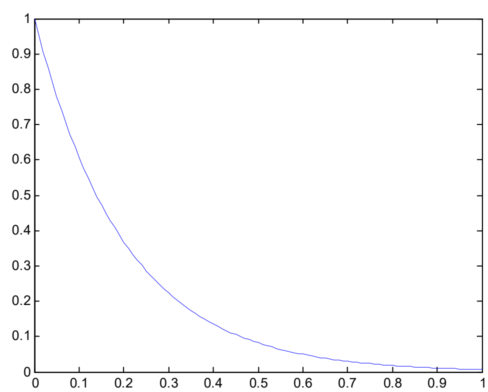

Figure 93

Example 51: Draw the graph of $\mathbf{y = f(t) = e^{-7t}}$

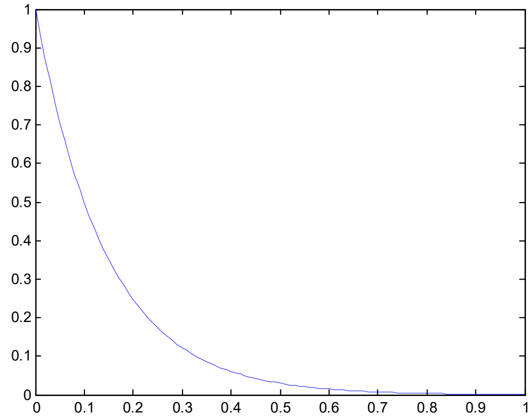

Figure 94

Example 52: Draw the graph of $\mathbf{y = f(t) = e^{5t} * e^{-7t}}$

Answer:

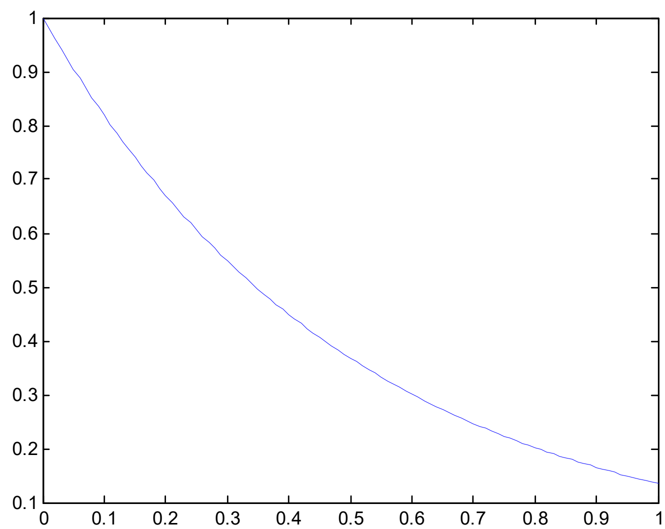

Figure 95

Example 53: Integrate the function of $\mathbf{y = f(t) = e^{5t}}$

Answer:

$$\int_0^\infty e^{5t}dt$$

$$= \left[\frac{e^{5t}}{5}\right]_0^\infty = \frac{1}{5}\left[e^\infty - e^0\right] = \frac{1}{5}[\infty - 1] = \infty$$

Example 54: Integrate the function of $\mathbf{y = f(t) = e^{-5t}}$

Answer:

$$\int_0^\infty e^{-5t}dt$$

$$=\left[\frac{e^{-5t}}{-5}\right]_0^\infty = -\frac{1}{5}\left[e^{-\infty}-e^{-0}\right] = -\frac{1}{5}\left[\frac{1}{e^\infty}-\frac{1}{e^0}\right] = -\frac{1}{5}\left[\frac{1}{\infty}-\frac{1}{1}\right] = -\frac{1}{5}\left[0-1\right] = \frac{1}{5}$$

$\mathbf{e^{5t}} \rightarrow$ This is unstable system. Unstable system কে absolute integrate করা যায়না, মানে মান পাওয়া যায়না। সেক্ষেত্রে unstable system কে এ জোর করে stable করে নিতে হয় মানে $\mathbf{e^{-at}}$ দ্বারা গুন করলে signal টা decrease হবে, Unstable system এ signal কে transform করার জন্য stable signal দ্বারা multiple করে নিতে হয়। যেমন: $\mathbf{e^{5t} * e^{-7t}}$

Problem 25: Derive Laplace transform from Fourier transform

Answer:

We have

$$g(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \text{ ---------------------------------------(i)}$$

Let $g(i\omega) = \int_0^\infty f(t)e^{-i\omega t}dt$ ------------------------------------(ii)

$\Rightarrow g(\sigma + i\omega) = \int_{-\infty}^{\infty} \{f(t)e^{-\sigma t}.\}e^{-i\omega t}dt$ [Multiplying **by** $e^{-\sigma t}$ to make stable,

Let $y = f(t) = e^{5t}$]

$$\Rightarrow g(\sigma + i\omega) = \int_{-\infty}^{\infty} f(t)e^{-\sigma t - i\omega t}dt$$

$$\Rightarrow g(\sigma + i\omega) = \int_{-\infty}^{\infty} f(t)e^{-(\sigma t + i\omega t)}dt$$

$$\Rightarrow g(\sigma + i\omega) = \int_{-\infty}^{\infty} f(t)e^{-(\sigma + i\omega)t}dt$$

$\Rightarrow g(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$ **[let, $\sigma + i\omega = s$]** ----------------------------------(iii)

Note that in Fourier Transform, $\sigma = 0$, $\sigma \rightarrow$ Initial Condition

Hence $\mathcal{L}(f(t)) = g(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt$ is called Laplace Transform of $f(t)$ ---------------(iv)

Laplace transform also represents a signal in frequency domain with the frequency variable being s=*σ* +j*ω* , *σ* and *ω* being real numbers. Fourier transform is a special case of Laplace transform when *σ=0.*

z-transform does the same task for discrete-time signals as Laplace transform does for continuous-time signals.//

Example 55: Find Laplace Transform of $f(t) = 1$

Answer:

We have,

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

Given,

$$f(t) = 1$$

$$\therefore \; L(1) = \int_0^\infty 1.e^{-st}dt$$

$$= \int_0^\infty e^{-st}dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{1}{s}\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right]$$

$$= -\frac{1}{s}\left[\frac{1}{\infty} - \frac{1}{1}\right] \qquad [e^\infty = \infty]$$

$$= -\frac{1}{s}[0-1] \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \frac{1}{s}$$

$$\therefore L(f(t)) = L(1) = \frac{1}{s}$$

Example 56: Find Laplace Transform of $f(t) = a$

Answer:

We have,

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

Given,

$$f(t) = a$$

$$\therefore \; L(a) = \int_0^\infty a.e^{-st}dt$$

$$= a\int_0^{\infty} e^{-st} dt$$

$$= a\left[\frac{e^{-st}}{-s}\right]_0^{\infty} \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= -\frac{a}{s}\left[e^{-\infty} - e^{-0}\right]$$

$$= -\frac{a}{s}\left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right]$$

$$= -\frac{a}{s}\left[\frac{1}{\infty} - \frac{1}{1}\right] \qquad [e^{\infty} = \infty]$$

$$= -\frac{a}{s}\left[0 - 1\right] \qquad \left[\frac{1}{\infty} = 0\right]$$

$$= \frac{a}{s}$$

$\therefore L(f(t)) = L(a) = \frac{a}{s}$ *Answer*

Example 57: Find Laplace Transform of $f(t) = t$

Answer

We have,

$$L(f(t)) = \int_0^{\infty} f(t)e^{-st} dt$$

Given,

$$f(t) = t$$

$\therefore\ L(t) = \int_0^{\infty} t.e^{-st} dt$ ---(i)

Now, $\int t e^{-st} dt$

$$= t\int e^{-st}dt - \int\left\{\frac{d}{dt}(t)\int e^{-st}dt\right\}dt \quad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}dt \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-t}{s}e^{-st} + \frac{1}{s}\int e^{-st}dt$$

$$= \frac{-t}{s}e^{-st} + \frac{1}{s}.\frac{e^{-st}}{-s}$$

$$= \frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

$\therefore$ From (i)

$$L(t)=\int_0^\infty t\,e^{-st}\,dt$$

$$=\left[\frac{-t}{s}e^{-st}\right]_0^\infty+\left[-\frac{1}{s^2}e^{-st}\right]_0^\infty$$

$$=0+\left[-\frac{1}{s^2}e^{-\infty}+\frac{1}{s^2}e^0\right]$$

$$=0+\left[-\frac{1}{s^2}\frac{1}{e^\infty}+\frac{1}{s^2}.1\right]$$

$$=0+\left[-\frac{1}{s^2}\frac{1}{\infty}+\frac{1}{s^2}.1\right] \qquad [e^\infty=\infty]$$

$$=0+\left[0+\frac{1}{s^2}\right] \qquad \left[\frac{1}{\infty}=0\right]$$

$$=\left[\frac{1}{s^2}\right]$$

$$=\frac{1}{s^2}$$

$$[\because e^{-\infty}=\frac{1}{e^\infty}=\frac{1}{\infty}=0]$$

$$\therefore L(t)==\frac{1}{s^2}$$

$$\therefore \mathrm{L(f(t))=L(t)}=\frac{1}{\mathrm{s}^2}$$ *Answer*

Details Example 57: Find Laplace Transform of $f(t)=t$

Answer

We have,

$$L(f(t))=\int_0^\infty f(t)e^{-st}\,dt$$

Given,

$$f(t)=t$$

$$\therefore\ L(t)=\int_0^\infty t.e^{-st}\,dt \text{ --(i)}$$

Now, $\int t\,e^{-st}\,dt$

$$=t\int e^{-st}dt-\int\left\{\frac{d}{dt}(t)\int e^{-st}dt\right\}dt\ [\because \int uvdx=u\int vdx-\int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t\frac{e^{-st}}{-s} - \int 1.\frac{e^{-st}}{-s}dt \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-t}{s}e^{-st} + \frac{1}{s}\int e^{-st}dt$$

$$= \frac{-t}{s}e^{-st} + \frac{1}{s}.\frac{e^{-st}}{-s}$$

$$= \frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

$\therefore$ From (i)

$$L(t) = \int_0^\infty t\,e^{-st}dt \quad = \left[\frac{-t}{s}e^{-st}\right]_0^\infty + \left[-\frac{1}{s^2}e^{-st}\right]_0^\infty \text{..............................(ii)}$$

Now,

$$\left[\frac{-t}{s}e^{-st}\right]_0^\infty$$

$$= \lim_{t\to\infty}\left[\frac{-t}{s}e^{-st}\right] - \lim_{t\to 0}\left[\frac{-t}{s}e^{-st}\right]$$

$$= -\frac{1}{s}\lim_{t\to\infty}[t.e^{-st}] + \lim_{t\to 0}\frac{1}{s}[t.e^{-st}]$$

$$= -\frac{1}{s}\lim_{t\to\infty}\left[\frac{t}{e^{st}}\right] + \lim_{t\to 0}\frac{1}{s}\left[\frac{t}{e^{st}}\right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{e^{s\infty}}\right] + \frac{1}{s}\left[\frac{0}{e^{s.0}}\right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{e^{\infty}}\right] + \frac{1}{s}\left[\frac{0}{e^{0}}\right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right] + \frac{1}{s}\left[\frac{0}{1}\right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right] + 0 \quad [Indeterminate\ form]$$

By L'Hospital's rule, Differentiate numerator and Denominator separately,

$$-\frac{1}{s}\lim_{t\to\infty}\left[\frac{t}{e^{st}}\right]$$

$$= \frac{-1}{s}\lim_{t\to\infty}\left[\frac{1}{se^{st}}\right]$$

$$= \frac{-1}{s}\left[\frac{1}{se^{s.\infty}}\right] \qquad \textit{[Differentiate numerator and Denominator separately]}$$

$$= \frac{-1}{s}\left[\frac{1}{se^{\infty}}\right]$$

$$= \frac{-1}{s}\left[\frac{1}{s.e^{\infty}}\right]$$

$$= \frac{-1}{s}\left[\frac{1}{s.\infty}\right]$$

$$= \frac{-1}{s}\left[\frac{1}{\infty}\right]$$

$$= \frac{-1}{s}[0]$$

$= 0$

$\therefore$ From (ii)

$$L(t)=\int_0^\infty t\,e^{-st}\,dt$$

$$=\left[\frac{-t}{s}e^{-st}\right]_0^\infty+\left[-\frac{1}{s^2}e^{-st}\right]_0^\infty$$

$$=\left[\frac{-t}{s}e^{-st}\right]_0^\infty+\left[-\frac{1}{s^2}e^{-st}\right]_0^\infty$$

$$=0+\left[-\frac{1}{s^2}e^{-\infty}+\frac{1}{s^2}e^{0}\right]$$

$$=0+\left[-\frac{1}{s^2}\frac{1}{e^\infty}+\frac{1}{s^2}.1\right]$$

$$=0+\left[-\frac{1}{s^2}\frac{1}{\infty}+\frac{1}{s^2}.1\right] \qquad [e^\infty=\infty]$$

$$=0+\left[0+\frac{1}{s^2}\right] \qquad \left[\frac{1}{\infty}=0\right]$$

$$=\left[\frac{1}{s^2}\right]$$

$$=\frac{1}{s^2}$$

$$[\because e^{-\infty}=\frac{1}{e^\infty}=\frac{1}{\infty}=0]$$

$$\therefore L(t)==\frac{1}{s^2}$$

$$\therefore L(f(t))=L(t)=\frac{1}{s^2}\ \textit{Answer}$$

Example 58: Find Laplace Transform of $f(t)=e^{at}$

Answer:

We have,

$$L(f(t))=\int_0^\infty f(t)\,e^{-st}\,dt$$

Given,

$$f(t)=e^{at}$$

$$\therefore\ L(e^{at})=\int_0^\infty e^{at}.e^{-st}\,dt$$

$$=\int_0^\infty e^{at-st}\,dt$$

$$= \int_0^\infty e^{-st+at} dt$$

$$= \int_0^\infty e^{-(st-at)} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{1}{-(s-a)} \left[e^{-\infty} - e^{-0} \right]$$

$$= -\frac{1}{(s-a)} \left[\frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= -\frac{1}{(s-a)} \left[\frac{1}{\infty} - \frac{1}{1} \right] \qquad [e^\infty = \infty]$$

$$= -\frac{1}{(s-a)} [0-1] \qquad \left[\frac{1}{\infty} = 0 \right]$$

$$= \frac{1}{s-a} \text{ Answer}$$

$$\therefore L(f(t)) = L(e^{at}) = \frac{1}{s-a} \text{ Answer}$$

Example 59: Find Laplace Transform of $f(t) = t^n$

We have,

$$L(f(t)) = \int_0^\infty f(t) e^{-st} dt$$

Given,

$$f(t) = t^n$$

$$\therefore \; L(t^n) = \int_0^\infty t^n e^{-st} dt$$

Let, $I_n = L(t^n) = \int_0^\infty t^n e^{-st} dt$ --------------------------------(i)

Now, $\int t^n e^{-st} dt$

$$= t^n \int e^{-st} dt - \int \left\{ \frac{d}{dt}(t^n) \int e^{-st} dt \right\} dt \quad [\because \int uv dx = u \int v dx - \int \{ \frac{d}{dx}(u) \int v dx \} dx]$$

$$= t^n \frac{e^{-st}}{-s} - \int n t^{n-1} \frac{e^{-st}}{-s} dt \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-t^n}{s}e^{-st} + \frac{n}{s}\int t^{n-1}e^{-st}dt \text{ ----------------------------}(ii)$$

Since $I_n = L(t^n) = \int_0^\infty t^n e^{-st}dt$

$$\therefore I_{n-1} = \int_0^\infty t^{n-1}e^{-st}dt \text{ --}(iii)$$

$\therefore$ From (i)

$$I_n = L(t^n) = \int_0^\infty t^n e^{-st}dt$$

$$= \left[\frac{-t^n}{s}e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \text{ [From(ii): } \int t^n e^{-st}dt = \frac{-t^n}{s}e^{-st} + \frac{n}{s}\int t^{n-1}e^{-st}dt \text{]}$$

$$= 0 + \frac{n}{s}I_{n-1} \qquad \text{[From (iii): } I_{n-1} = \int_0^\infty t^{n-1}e^{-st}dt \text{]}$$

$$[\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0 \text{ and } 0^n = 0]$$

$$\therefore I_n = \frac{n}{s}I_{n-1} \text{ --}(iv)$$

Put $n = n-1$ in (iv)

$$\therefore I_{n-1} = \frac{n-1}{s}I_{n-2} \text{ ---}(v)$$

Again, Put $n = n-2$ in (iv)

$$\therefore I_{n-2} = \frac{n-2}{s}I_{n-3} \text{ --}(vi)$$

Again Put, $n = n-3$ in (iv)

$$\therefore I_{n-3} = \frac{n-3}{s}I_{n-4} \text{ --}(vii)$$

Put $n = 2$ in (iv)

$$I_2 = \frac{2}{s}I_{2-1}$$

$$\therefore I_2 = \frac{2}{s}I_1 \text{ ---}(viii)$$

Put $n = 1$ in (iv)

$$\therefore I_1 = \frac{1}{s} I_{1-1}$$

$$= \frac{1}{s} I_0 \text{ --- } (ix)$$

Putting in values of $I_{n-1}, I_{n-2}, ..., I_2, I_1$ in (iv)

$$\therefore I_n = \frac{n}{s} I_{n-1}$$

$$= \frac{n}{s} . \frac{n-1}{s} I_{n-2} \qquad [form(v)]$$

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} I_{n-3} \qquad [form(vi)]$$

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \frac{n-3}{s} I_{n-4} \quad [form(vii)]$$

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \frac{n-3}{s} . \frac{n-4}{s} .... \frac{2}{s} . \frac{1}{s} I_0 \quad [form(viii) \text{ and } (ix)]$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)......................2.1}{s^n} I_0$$

$$\therefore I_n = \frac{n!}{s^n} I_0 \text{ --- } (x)$$

We have, $I_n = \int_0^\infty t^n e^{-st} dt$

Put $n = 0$

$$I_0 = \int_0^\infty t^0 e^{-st} dt$$

$$I_0 = \int_0^\infty 1.e^{-st} dt$$

$$I_0 = \int_0^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= -\frac{1}{s} \left[e^{-\infty} - e^{-0} \right]$$

$$= -\frac{1}{s} \left[\frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= -\frac{1}{s}\left[\frac{1}{\infty} - \frac{1}{1}\right]$$

$$= -\frac{1}{s}[0-1]$$

$$I_0 = \frac{1}{s} \text{---} (xi)$$

From (x)

$$I_n = \frac{n!}{s^n} I_0$$

$$\mathbf{I_n} = \frac{\mathbf{n!}}{\mathbf{s^n}} \cdot \frac{\mathbf{1}}{\mathbf{s}} \qquad [\text{From}\,(xi)\,]$$

$$I_n = \frac{n!}{s^{n+1}} \text{--}(\text{xii})$$

$$\therefore L(f(t)) = L(t^n) = \frac{n!}{s^{n+1}} \ \textit{Answer}$$

As for example,

if $n = 2$

$$\therefore L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}}$$

That is, $I_2 = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$ [From (xii)]

if $\mathbf{n = 4}$

$$\therefore L(f(t)) = L(t^4) = \frac{4!}{s^{4+1}}$$

That is,, $\mathbf{I_4} = \frac{\mathbf{4!}}{\mathbf{s^{4+1}}} = \frac{\mathbf{4!}}{\mathbf{s^5}}$ [From (xii)]

if $n = 7$

$$\therefore L(f(t)) = L(t^7) = \frac{7!}{s^{7+1}}$$

That is,, $\mathbf{I_7} = \frac{\mathbf{7!}}{\mathbf{s^{7+1}}} = \frac{\mathbf{7!}}{\mathbf{s^8}}$ *Answer*

Details Example 59: Find Laplace Transform of $f(t) = t^n$

We have,

$$L(f(t)) = \int_0^\infty f(t) e^{-st} dt$$

Given,

$$f(t) = t^n$$

$$\therefore\ L(t^n) = \int_0^\infty t^n e^{-st} dt$$

Let, $I_n = L(t^n) = \int_0^\infty t^n e^{-st} dt$ --------------------------------(i)

Now, $\int t^n e^{-st} dt$

$$= t^n \int e^{-st} dt - \int \left\{ \frac{d}{dt}(t^n) \int e^{-st} dt \right\} dt \quad [\because \int uvdx = u\int vdx - \int \{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= t^n \frac{e^{-st}}{-s} - \int nt^{n-1} \frac{e^{-st}}{-s} dt \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{-t^n}{s} e^{-st} + \frac{n}{s} \int t^{n-1} e^{-st} dt \text{ ---------------------------} (ii)$$

Since $I_n = L(t^n) = \int_0^\infty t^n e^{-st} dt$

$$\therefore I_{n-1} = \int_0^\infty t^{n-1} e^{-st} dt \text{ ---} (iii)$$

$\therefore$ From (i)

$$I_n = L(t^n) = \int_0^\infty t^n e^{-st} dt$$

$$= \left[\frac{-t^n}{s} e^{-st} \right]_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt \text{.........................(iv)}$$

[since From (ii): $\int t^n e^{-st} dt = \frac{-t^n}{s} e^{-st} + \frac{n}{s} \int t^{n-1} e^{-st} dt$]

Here,

$$\left[-\frac{t^n}{s} e^{-st} \right]_0^\infty$$

$$= -\frac{1}{s} [t^n . e^{-st}]_0^\infty$$

$$= -\frac{1}{s} \lim_{t\to\infty} [t^n . e^{-st}] - (\frac{-1}{s}) \lim_{t\to 0} [t^n . e^{-st}]$$

$$= -\frac{1}{s} \lim_{t\to\infty} [t^n . e^{-st}] + \frac{1}{s} \lim_{t\to 0} [t^n . e^{-st}]$$

$$= -\frac{1}{s} \lim_{t\to\infty} \left[\frac{t^n}{e^{st}} \right] + \frac{1}{s} \lim_{t\to 0} \left[\frac{t^n}{e^{st}} \right]$$

$$= -\frac{1}{s} \left[\frac{\infty^n}{e^{s.\infty}} \right] + \frac{1}{s} \left[\frac{0^n}{e^{s.0}} \right]$$

$$= -\frac{1}{s} \left[\frac{\infty}{e^\infty} \right] + \frac{1}{s} \left[\frac{0}{e^0} \right]$$

$$= -\frac{1}{s} \left[\frac{\infty}{\infty} \right] + \frac{1}{s} \left[\frac{0}{1} \right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right] + \frac{1}{s} \times 0$$

$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right]$$

It is an indeterminate form.

Now, Applying L'Hospital's rule in $-\frac{1}{s}\lim_{t\to\infty}\left[\frac{t^n}{e^{st}}\right]$

Differentiate numerator and denominator separately we get,

$$= -\frac{1}{s}\lim_{t\to\infty}\left[\frac{n.t^{n-1}}{s\ e^{st}}\right]$$

$$= -\frac{1}{s}\left[\frac{n.\infty^{n-1}}{s.e^{s.\infty}}\right]$$

$$= -\frac{1}{s}\left[\frac{n\infty}{s.e^{\infty}}\right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{s.\infty}\right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right]$$

It is an indeterminate form

Again, Differentiate numerator and denominator separately we get,

$$-\frac{1}{s}\lim_{t\to\infty}\left[\frac{n.t^{n-1}}{s.e^{st}}\right]$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)t^{n-2}}{s.se^{st}}$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)t^{n-2}}{s^2e^{st}}$$

$$= -\frac{1}{s}\left[\frac{n(n-1).\infty^{n-2}}{s^2e^{s.\infty}}\right]$$

$$= -\frac{1}{s}\left[\frac{n(n-1).\infty}{s^2.e^{\infty}}\right]$$

$$= -\frac{1}{s}\left[\frac{\infty}{\infty}\right] \qquad \text{[Indeterminate Form]}$$

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n- times derivative in,

$$-\frac{1}{s}\lim_{t\to\infty}\left[\frac{t^n}{e^{st}}\right]$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)(n-2)(n-3)\dots\dots\dots\dots\{n-(n-1)\}t^{n-n}}{s^ne^{st}}$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)(n-2)(n-3)\dots\dots\dots\dots\dots..1.t^0}{s^ne^{st}}$$

$$= -\frac{1}{s}\lim_{t\to\infty}\frac{n(n-1)(n-2)(n-3)\dots\dots\dots\dots\dots\dots.1}{s^ne^{st}}$$

$= -\frac{1}{s}\lim_{t\to\infty}\frac{n!}{s^n e^{st}}$

$= -\frac{1}{s}\left[\frac{n!}{s^n e^{s.\infty}}\right]$

$= -\frac{1}{s}\left[\frac{n!}{s^n e^{\infty}}\right]$

$= -\frac{1}{s}\left[\frac{n!}{s^n.\infty}\right]$

$= -\frac{1}{s}\left[\frac{n!}{\infty}\right]$

$= -\frac{1}{s}\times 0$

=0

$\therefore$ From (iv)

$$I_n = L(t^n) = \int_0^\infty t^n e^{-st}dt$$

$$= \left[\frac{-t^n}{s}e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$= 0 + \frac{n}{s}I_{n-1} \qquad \text{[From (iii): } I_{n-1} = \int_0^\infty t^{n-1}e^{-st}dt\text{]}$$

$$[\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0 \text{ and } 0^n = 0]$$

$$\therefore I_n = \frac{n}{s}I_{n-1} \text{ ---(v)}$$

Put $n = n-1$ in (v)

$$\therefore I_{n-1} = \frac{n-1}{s}I_{n-2} \text{ --(vi)}$$

Again, Put $n = n-2$ in (v)

$$\therefore I_{n-2} = \frac{n-2}{s}I_{n-3} \text{ --(vii)}$$

Again Put, $n = n-3$ in (v)

$$\therefore I_{n-3} = \frac{n-3}{s}I_{n-4} \text{ --(viii)}$$

Put $n = 2$ in (v)

$$I_2 = \frac{2}{s}I_{2-1}$$

$$\therefore I_2 = \frac{2}{s}I_1 \text{---(ix)}$$

Put $n = 1$ in (v)

$$\therefore I_1 = \frac{1}{s} I_{1-1}$$

$$= \frac{1}{s} I_0 \text{ --- (x)}$$

Putting in values of $I_{n-1}, I_{n-2}, \ldots\ldots\ldots, I_2, I_1$ in (v)

$$\therefore I_n = \frac{n}{s} I_{n-1}$$

$$= \frac{n}{s} . \frac{n-1}{s} I_{n-2} \qquad [\text{form(vi)}]$$

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} I_{n-3} \qquad [\text{form(vii)}]$$

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \frac{n-3}{s} I_{n-4} \quad [\text{form(viii)}]$$

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \frac{n-3}{s} . \frac{n-4}{s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{2}{s} . \frac{1}{s} I_0 \quad [\text{form(ix) and (x)}]$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots\ldots\ldots\ldots 2.1}{s^n} I_0$$

$$\therefore I_n = \frac{n!}{s^n} I_0 \text{ --- (xi)}$$

We have, $I_n = \int_0^\infty t^n e^{-st} dt$

Put $n = 0$

$$I_0 = \int_0^\infty t^0 e^{-st} dt$$

$$I_0 = \int_0^\infty 1 . e^{-st} dt$$

$$I_0 = \int_0^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s} \right]_0^\infty$$

$$= -\frac{1}{s} \left[e^{-\infty} - e^{-0} \right]$$

$$= -\frac{1}{s} \left[\frac{1}{e^\infty} - \frac{1}{e^0} \right]$$

$$= -\frac{1}{s}\left[\frac{1}{\infty} - \frac{1}{1}\right]$$

$$= -\frac{1}{s}[0-1]$$

$$I_0 = \frac{1}{s} \text{ -- } \mathbf{(xii)}$$

From **(xi)**

$$I_n = \frac{n!}{s^n} I_0$$

$$I_n = \frac{n!}{s^n} \cdot \frac{1}{s} \qquad \text{[From } \mathbf{(xii)} \text{]}$$

$$I_n = \frac{n!}{s^{n+1}} \text{ --(xiii)}$$

$$\therefore L(f(t)) = L(t^n) = \frac{n!}{s^{n+1}} \textit{ Answer}$$

As for example,

if $n = 2$

$$\therefore L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}}$$

That is, $I_2 = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$ [From (xii)]

if $n = 4$

$$\therefore L(f(t)) = L(t^4) = \frac{4!}{s^{4+1}}$$

That is,, $I_4 = \frac{4!}{s^{4+1}} = \frac{4!}{s^5}$ [From (xii)]

if $n = 7$

$$\therefore L(f(t)) = L(t^7) = \frac{7!}{s^{7+1}}$$

That is,, $I_7 = \frac{7!}{s^{7+1}} = \frac{7!}{s^8}$ *Answer*

Example 60: Find Laplace Transform of $f(t) = \sin at$

Answer:

We have,

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt \text{ -------------------------------------(i)}$$

Given,

$$f(t) = \sin at$$

$$L(\sin at) = \int_0^\infty \sin at \; e^{-st} dt \text{ --------------------------------(ii)}$$

We have, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore e^{iat} = \cos at + i\sin at$$

Here, $\cos at$ is the real $(\Re)$ part of e^{iat} and $\sin at$ is the imaginary (I) part of e^{iat}.

That is, $\cos at = \Re(e^{iat})$ and $\sin at = I(e^{iat})$

[$\Re(e^{iat})$ Means real part of e^{iat} and $I(e^{iat})$ means imaginary part of e^{iat}.]

We have, $L(f(t)) = \int_0^\infty f(t) e^{-st} dt$

$$\Rightarrow L(f(t)) = L(\sin at) = L\{I(e^{iat})\}$$

From (ii),

$$L(\sin at) = \int_0^\infty \sin at \; e^{-st} dt$$

$$= \int_0^\infty I\, e^{iat} . e^{-st} dt \qquad [\because \sin at = I(e^{iat})]$$

$$= I\int_0^\infty e^{iat} . e^{-st} dt \qquad \text{----------------------------(iii)}$$

$$= I\int_0^\infty e^{(ia-s)t} dt$$

$$= I\int_0^\infty e^{-(s-ia)t} dt$$

$$= I\left\{\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty\right\} \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= I\left\{\frac{e^{-(s-ia)\infty}}{-(s-ia)} - \frac{e^{-(s-ia)\cdot 0}}{-(s-ia)}\right\}$$

$$= I\left\{0 - \frac{e^{-0}}{-(s-ia)}\right\} \qquad [\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0]$$

$$= I\left[\frac{1}{s-ia}\right] \qquad [\because e^{-0} = \frac{1}{e^0} = \frac{1}{1} = 1]$$

$$= I\left\{\frac{s+ia}{(s+ia)(s-ia)}\right\} \text{[Multiplying by } s+ia \text{ on numerator and denominator]}$$

$$= I\left\{\frac{s+ia}{s^2 - i^2 a^2}\right\}$$

$$= I\left\{\frac{s+ia}{s^2+a^2}\right\} \; [i^2=-1]$$

$$= I\left\{\frac{s}{s^2+a^2}+i\frac{a}{s^2+a^2}\right\} \text{-------------------------(iv)}$$

Taking that imaginary part from (iv)

$$\therefore L(f(t)) = L\ (\sin at) = \frac{a}{s^2+a^2} \quad \textit{Answer}$$

Example 61: Find Laplace Transform of $f(t)=\cos at$

Answer:

We have,

$$\mathbf{L(f(t))} = \int_0^\infty \mathbf{f(t)e^{-st}dt} \quad \text{----------------------------(i)}$$

Given,

$$f(t)=\cos at$$

$$L(\cos at) = \int_0^\infty \cos at\ e^{-st}dt \quad \text{--------------------(ii)}$$

We have, $e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore e^{iat} = \cos at + i\sin at$$

Here, $\cos at$ is the real part of e^{iat} and $\sin at$ is the imaginary part of e^{iat}.

That is, $\cos at = \Re(e^{iat})$ and $\sin at = I(e^{iat})$

[$\Re(e^{iat})$ Means real part of e^{iat} and $I(e^{iat})$ means imaginary part of e^{iat}.]

We have, $L(f(t)) = \int_0^\infty f(t)e^{-st}dt$

$$\Rightarrow L(f(t)) = L(\cos at) = L\left\{\Re(e^{iat})\right\}$$

From (ii),

$$L(\cos at) = \int_0^\infty \cos at\ e^{-st}dt$$

$$= \int_0^\infty \Re e^{iat}.e^{-st}dt \qquad [\because \cos at = \Re(e^{iat})]$$

$$= \Re\int_0^\infty e^{iat}.e^{-st}dt \quad \text{--------------------------(iii)}$$

$$= \Re\int_0^\infty e^{(ia-s)t}dt$$

$$= \Re\int_0^\infty e^{-(s-ia)t}dt$$

$$= \Re\left\{\left[\frac{e^{-(s-ia)t}}{-(s-ia)}\right]_0^\infty\right\} \qquad [\int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$= \Re\left\{\frac{e^{-(s-ia)\infty}}{-(s-ia)} - \frac{e^{-(s-ia)\cdot 0}}{-(s-ia)}\right\}$$

$$= \Re\left\{0 - \frac{e^{-0}}{-(s-ia)}\right\} \quad [\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0]$$

$$= \Re\left[\frac{1}{s-ia}\right] \qquad [\because e^{-0} = \frac{1}{e^0} = \frac{1}{1} = 1]$$

$$= \Re\left\{\frac{s+ia}{(s+ia)(s-ia)}\right\}$$ [Multiplying by $s+ia$ on numerator and denominator]

$$= \Re\left\{\frac{s+ia}{s^2 - i^2a^2}\right\}$$

$$= \Re\left\{\frac{s+ia}{s^2+a^2}\right\} \; [i^2 = -1]$$

$$= \Re\left\{\frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}\right\}$$ --------------------------(iv)

Taking that real part from (iv)

$\therefore L(f(t)) = L\ (\cos at) = \dfrac{s}{s^2+a^2}$ *Answer*

As for example
We get,

$L\ (\sin at) = \dfrac{a}{s^2+a^2}$

$\therefore L\ (\sin 5t) = \dfrac{5}{s^2+5^2}$ *Answer*

and

$L\ (\cos at) = \dfrac{s}{s^2+a^2}$

$L\ (\cos 3t) = \dfrac{s}{s^2+3^2}$ *Answer*

Example 62: Find Laplace Transform of $f(t) = \sinh at$

Given $f(t) = \sinh at$

We have, $f(t) = \sinh at = \dfrac{1}{2}(e^{at} - e^{-at})$

$$\therefore\ \mathcal{L}(\sinh at) = \int_0^{\infty} \sinh at . e^{-st} dt$$

$$= \int_o^{\infty} \frac{1}{2}(e^{at} - e^{-at}) e^{-st} dt$$

$$= \frac{1}{2}\int_o^{\infty} (e^{at} - e^{-at}) e^{-st} dt$$

$$= \frac{1}{2}\int_o^{\infty} (e^{at-st} - e^{-(at+st)}) dt$$

$$= \frac{1}{2}\int_o^{\infty} \left\{e^{-(s-a)t} - e^{-(s+a)t}\right\} dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_o^{\infty} \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s-a)} - \frac{e^{-o}}{-(s-a)} - \frac{e^{-\infty}}{-(s+a)} + \frac{e^{-o}}{-(s+a)}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{e^{\infty}}.\frac{1}{(s-a)} + \frac{1}{e^0}.\frac{1}{(s-a)} + \frac{1}{e^{\infty}}.\frac{1}{(s+a)} - \frac{1}{e^0}.\frac{1}{(s+a)}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{\infty}.\frac{1}{(s-a)} + \frac{1}{1}.\frac{1}{(s-a)} + \frac{1}{\infty}.\frac{1}{(s+a)} - \frac{1}{1}.\frac{1}{(s+a)}\right]$$

$$= \frac{1}{2}\left[-0.\frac{1}{(s-a)} + \frac{1}{(s-a)} + 0.\frac{1}{(s+a)} - \frac{1}{(s+a)}\right] [\because e^{\infty} = \infty, \frac{1}{\infty} = 0, e^0 = 1]$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)} - \frac{1}{(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{(s^2-a^2)}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$$\therefore\ \mathcal{L}(\sinh at) = \left[\frac{a}{s^2-a^2}\right]$$

$$\therefore L(f(t)) = \mathcal{L}(\sinh at) = \frac{a}{s^2-a^2}$$ *Answer*

Example 63: Find Laplace Transform of $f(t) = \cosh at$

Given $f(t) = \cosh at$

We have, $f(t) = \cosh at = \frac{1}{2}(e^{at} + e^{-at})$

$$\therefore \mathcal{L}(\cosh at) = \int_0^\infty \cosh at . e^{-st} dt$$

$$= \int_o^\infty \frac{1}{2}(e^{at} + e^{-at}) e^{-st} dt$$

$$= \frac{1}{2}\int_o^\infty (e^{at} + e^{-at}) e^{-st} dt$$

$$= \frac{1}{2}\int_o^\infty (e^{at-st} + e^{-(at+st)}) dt$$

$$= \frac{1}{2}\int_o^\infty \left\{ e^{-(s-a)t} + e^{-(s+a)t} \right\} dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)} \right]_o^\infty \qquad [\int e^{-mx} dx = \frac{e^{-mx}}{-m}]$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s-a)} - \frac{e^{-o}}{-(s-a)} + \frac{e^{-\infty}}{-(s+a)} - \frac{e^{-o}}{-(s+a)} \right]$$

$$= \frac{1}{2}\left[-\frac{1}{e^\infty} . \frac{1}{(s-a)} + \frac{1}{e^0} . \frac{1}{(s-a)} - \frac{1}{e^\infty} . \frac{1}{(s+a)} + \frac{1}{e^0} . \frac{1}{(s+a)} \right]$$

$$= \frac{1}{2}\left[-\frac{1}{\infty} . \frac{1}{(s-a)} + \frac{1}{1} . \frac{1}{(s-a)} - \frac{1}{\infty} . \frac{1}{(s+a)} + \frac{1}{1} . \frac{1}{(s+a)} \right]$$

$$= \frac{1}{2}\left[-0. \frac{1}{(s-a)} + \frac{1}{(s-a)} - 0. \frac{1}{(s+a)} + \frac{1}{(s+a)} \right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-a)} + \frac{1}{(s+a)} \right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{(s^2-a^2)} \right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2} \right]$$

$$\therefore \mathcal{L}(\cosh at) = \left[\frac{s}{s^2-a^2} \right]$$

$$\therefore L(f(t)) = \mathcal{L}(\cosh at) = \frac{s}{s^2-a^2}$$ *Answer*

Problem 26:

The First Shift Theorem

We have seen that a Laplace transform of $\mathbf{f(t)}$ is a function of $\mathbf{s}$ only, i.e.

$\mathbf{L\{f(t)\} = f(s)}$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $\mathbf{L\{e^{-at}f(t)\} = f(s+a)}$ ---------------------------(ii)

Example 64: Find $L\{e^{-at}t\} = ?$

Answer:

Here, $f(t) = t$

The first shift theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $L\{e^{-at}f(t)\} = f(s+a)$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(t) = \dfrac{1}{s^2}$ **[Here $f(s) = \dfrac{1}{s^2}$]**

If $f(s) = \dfrac{1}{s^2}$

$\therefore f(s+a) = \dfrac{1}{(s+a)^2}$

Hence, according to equation no (ii), we can write

$L\{e^{-at}t\} = f(s+a) = \dfrac{1}{(s+a)^2}$ *Answer*

Example 65: Find $L\{e^{-3t}t\} = ?$

Answer:

Here, $f(t) = t$

The first shift theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $L\{e^{-3t}f(t)\} = f(s+3)$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(t) = \dfrac{1}{s^2}$ **[Here $f(s) = \dfrac{1}{s^2}$]**

If $f(s) = \dfrac{1}{s^2}$

$\therefore f(s+3) = \dfrac{1}{(s+3)^2}$

Hence, according to equation no (ii), we can write

$L\{e^{-3t}t\} = f(s+3) = \dfrac{1}{(s+3)^2}$ *Answer*

Example 66: Find $L\{e^{-at}e^{4t}\} = ?$

Here, $f(t) = e^{4t}$

The first shift theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $\mathbf{L\{e^{-at}f(t)\} = f(s+a)}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(e^{4t}) = \dfrac{1}{(s-4)}}$ **[Here** $\mathbf{f(s) = \dfrac{1}{(s-4)}}$**]**

If $\mathbf{f(s) = \dfrac{1}{(s-4)}}$

$\mathbf{\therefore f(s+a) = \dfrac{1}{(s+a-4)}}$

Hence, according to equation no (ii), we can write

$\mathbf{L\{e^{-at}e^{4t}\} = f(s+a) = \dfrac{1}{(s+a-4)}}$ *Answer*

Example 67: Find $\mathbf{L\{e^{-3t}e^{4t}\} = ?}$

Here, $\mathbf{f(t) = e^{4t}}$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $\mathbf{L\{e^{-3t}f(t)\} = f(s+3)}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(e^{4t}) = \dfrac{1}{(s-4)}}$ **[Here** $\mathbf{f(s) = \dfrac{1}{(s-4)}}$**]**

If $\mathbf{f(s) = \dfrac{1}{(s-4)}}$

$\mathbf{\therefore f(s+3) = \dfrac{1}{(s+3-4)}}$

Hence, according to equation no (ii), we can write

$\mathbf{L\{e^{-3t}e^{4t}\} = f(s+3) = \dfrac{1}{(s+3-4)} = \dfrac{1}{(s-1)}}$ *Answer*

Example 68: Find $\mathbf{L\{e^{-3t} * e^{5t}\} = ?}$

Here, $\mathbf{f(t) = e^{5t}}$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $\mathbf{L\{e^{-3t}f(t)\} = f(s+3)}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(e^{5t}) = \dfrac{1}{(s-5)}}$ **[Here** $\mathbf{f(s) = \dfrac{1}{(s-5)}}$**]**

If $\mathbf{f(s) = \frac{1}{(s-5)}}$

$\therefore \mathbf{f(s+3) = \frac{1}{(s+3-5)}}$

Hence, according to equation no (ii), we can write

$\mathbf{L\{e^{-3t} * e^{5t}\} = f(s+3) = \frac{1}{(s+3-5)} = \frac{1}{(s-2)}}$ *Answer*

Example 69: Find $\mathbf{L\{e^{-4t}t^2\} = ?}$

Answer:

We have,

If $\mathbf{f(t) = t^n}$

Then $\mathbf{L(f(t)) = L(t^n) = \frac{n!}{s^{n+1}}}$

For $\mathbf{n = 2}$;

If $\mathbf{f(t) = t^2}$

Then $\mathbf{L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}}$

We are to find $\mathbf{L\{e^{-4t}t^2\} = ?}$

Here, $\mathbf{f(t) = t^2}$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $\mathbf{L\{e^{-4t}t^2\} = f(s+4)}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}}$ **[Here $\mathbf{f(s) = \frac{2!}{s^3}}$]**

If $\mathbf{f(s) = \frac{2!}{s^3}}$**]**

$\therefore \mathbf{f(s+4) = \frac{2!}{(s+4)^3}}$

Hence, according to equation no (ii), we can write

$\mathbf{L\{e^{-4t} * t^2\} = f(s+4) = \frac{2!}{(s+4)^3}}$ *Answer*

Example 70: Find $\mathbf{L\{e^{-5t} * t^2\} = ?}$

Answer:

Here, $\mathbf{f(t) = t^2}$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then $\mathbf{L\{e^{-5t}t^2\} = f(s+5)}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}}$ **[Here** $\mathbf{f(s) = \frac{2!}{s^3}}$**]**

If $\mathbf{f(s) = \frac{2!}{s^3}}$**]**

$$\therefore \mathbf{f(s+5) = \frac{2!}{(s+5)^3}}$$

Hence, according to equation no (ii), we can write

$$\mathbf{L\{e^{-5t} * t^2\} = f(s+5) = \frac{2!}{(s+5)^3}} \quad \textit{Answer}$$

Problem 27:

Multiplication Theorem: Laplace Transform of $\boldsymbol{t.f(t)}$ (Multiplication by t)

Statement: This theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then, $\mathbf{L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}}$ ---------------------------(ii)

Also, $\boldsymbol{L(t^n.f(t)) = (-1)^n \frac{d^n}{ds^n}\{f(s)\}}$---------------------------(iii)

Proof:

We have, $\boldsymbol{L(f(t)) = f(s) = \int_0^\infty f(t)e^{-st}dt}$ ----------------------(iv)

Differentiating (iv) with respect to s, we get

$$\boldsymbol{L(f(t)) = f(s) = \int_0^\infty f(t)e^{-st}dt}$$

$$\boldsymbol{\frac{d}{ds}\{f(s)\} = \frac{d}{ds}\left[\int_0^\infty f(t)e^{-st}dt\right]}$$

$$\boldsymbol{\frac{d}{ds}\{f(s)\} = \int_0^\infty \frac{d}{ds}\left[f(t)e^{-st}dt\right]}$$

$$\boldsymbol{\frac{d}{ds}\{f(s)\} = \int_0^\infty \frac{d}{ds}\left[e^{-st}\right]f(t)dt}$$

$$\boldsymbol{\frac{d}{ds}\{f(s)\} = \int_0^\infty \left[e^{-st}\right].\frac{d}{ds}\left[-st\right]f(t)dt \quad [\because \frac{d}{dx}(e^{mx}) = e^{mx}.\frac{d}{dx}(mx) = e^{mx}.(m) = me^{mx}]}$$

$$\boldsymbol{\frac{d}{ds}\{f(s)\} = \int_0^\infty \left[e^{-st}\right].(-t)\,f(t)dt}$$

$$\boldsymbol{\frac{d}{ds}\{f(s)\} = -\int_0^\infty t\,f(t)e^{-st}dt}$$

$$\frac{d}{ds}\{f(s)\} = -L(t.f(t)) \qquad [\because L(f(t)) = \int_0^\infty f(t)e^{-st}dt\,]$$

$$L(t.f(t)) = -\frac{d}{ds}\{f(s)\}$$ (Proved equation no ii)

Similarly,

$$L(t^2.f(t)) = (-1)^2\frac{d^2}{ds^2}\{f(s)\}$$

$$L(t^3.f(t)) = (-1)^3\frac{d^3}{ds^3}\{f(s)\}$$

$$L(t^4.f(t)) = (-1)^4\frac{d^4}{ds^4}\{f(s)\}$$

$$L(t^n.f(t)) = (-1)^n\frac{d^n}{ds^n}\{f(s)\}$$ (Proved equation no iii)

Example 71: Find $\mathbf{L\{t.t\} = ?}$

Answer:

Here, $\mathbf{f(t) = t}$

This theorem states that

If $\mathbf{L\{f(t)\} = f(s)}$ ---------------------------(i)

Then, $\mathbf{L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}}$ ---------------------------(ii)

We have, according to equation no (i), $\mathbf{L(f(t)) = L(t) = \frac{1}{s^2}}$ **[Here** $\mathbf{f(s) = \frac{1}{s^2}}$**]**

Hence, according to equation no (ii), we can write

$$\mathbf{L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}}$$

$$\mathbf{L\{t.t\} = -\frac{d}{ds}\{f(s)\}} \qquad \mathbf{[f(t) = t]}$$

$$\mathbf{L\{t.t\} = -\frac{d}{ds}\{f(s)\}}$$

$$\mathbf{L\{t.t\} = -\frac{d}{ds}(\frac{1}{s^2})} \qquad \mathbf{[f(s) = \frac{1}{s^2}]}$$

$$\mathbf{L\{t.t\} = -\frac{d}{ds}(s^{-2})}$$

$$\mathbf{L\{t.t\} = -(-2)s^{-2-1}}$$

$$\mathbf{L\{t.t\} = -(-2)s^{-3}}$$

$$\mathbf{L\{t.t\} = 2s^{-3}}$$

$$L\{t.t\} = \frac{2}{s^3}$$

That is, $L\{t^2\} = \frac{2}{s^3}$ *(Answer)*

Example 72: Find $L\{t.e^{4t}\} = ?$

Here, $f(t) = e^{4t}$

This theorem states that

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then, $L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(e^{4t}) = \frac{1}{(s-4)}$ [Here $f(s) = \frac{1}{(s-4)}$]

Hence, according to equation no (ii), we can write

$$L\{t.f(t)\} = -\frac{d}{ds}\{f(s)\}$$

$$L\{t.e^{4t}\} = -\frac{d}{ds}\{f(s)\} \qquad [f(t) = e^{4t}]$$

$$L\{t.e^{4t}\} = -\frac{d}{ds}\{\frac{1}{(s-4)}\} \qquad [f(s) = \frac{1}{(s-4)}]$$

$$L\{t.e^{4t}\} = -\frac{d}{ds}\{(s-4)^{-1}\}$$

$$L\{t.e^{4t}\} = -(-1)(s-4)^{-1-1}\frac{d}{ds}(s-4)$$

$$L\{t.e^{4t}\} = (s-4)^{-2}(1-0)$$

$$L\{t.e^{4t}\} = (s-4)^{-2}.1$$

$$L\{t.e^{4t}\} = \frac{1}{(s-4)^2} \qquad \textit{Answer}$$

Example 73: Find Laplace Transform of $t^2 \sin 2t$

Here, $f(t) = \sin 2t$

This theorem states that

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then, $L\{t^2.f(t)\} = (-1)^2 \frac{d^2}{ds^2}\{f(s)\}$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(\sin 2t) = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$

[Here $f(s) = \frac{2}{s^2+4}$]

Hence, according to equation no (ii), we can write

$$L\{t^2.f(t)\}=(-1)^2\frac{d^2}{ds^2}\{f(s)\}$$

$$L\{t^2.\sin 2t\}=(-1)^2\frac{d^2}{ds^2}(f(s)) \qquad [f(t)=\sin 2t]$$

$$L\{t^2.\sin 2t\}=(-1)^2\frac{d^2}{ds^2}\{\frac{2}{s^2+4}\} \qquad [f(s)=\frac{2}{s^2+4}]$$

$$L\{t^2.\sin 2t\}=+\frac{d}{ds}\left[\frac{d}{ds}\{\frac{2}{s^2+4}\}\right]$$

$$L\{t^2.\sin 2t\}=\frac{d}{ds}\left[\frac{d}{ds}\{2(s^2+4)^{-1}\}\right]$$

$$L\{t^2.\sin 2t\}=2\frac{d}{ds}\left[(-1)(s^2+4)^{-1-1}.\frac{d}{ds}(s^2+4)\right]$$

$$L\{t^2.\sin 2t\}=2\frac{d}{ds}\left[(-1)(s^2+4)^{-2}.(2s)\right]$$

$$L\{t^2.\sin 2t\}=-4\frac{d}{ds}\left[(s^2+4)^{-2}.s\right]$$

$$L\{t^2.\sin 2t\}=-4\left[s\frac{d}{ds}\left[(s^2+4)^{-2}\right]+\left[(s^2+4)^{-2}\right]\frac{d}{ds}(s)\right]$$

$$L\{t^2.\sin 2t\}=-4\left[s\left[(-2)(s^2+4)^{-2-1}.\frac{d}{ds}(s^2+4)\right]+\left[(s^2+4)^{-2}\right].1\right]$$

$$L\{t^2.\sin 2t\}=-4\left[s\left[(-2)(s^2+4)^{-3}.(2s)\right]+\left[(s^2+4)^{-2}\right]\right]$$

$$L\{t^2.\sin 2t\}=-4\left[s\left[\frac{-4s}{(s^2+4)^3}\right]+\left[\frac{1}{(s^2+4)^2}\right]\right]$$

$$L\{t^2.\sin 2t\}=\left[\left[\frac{16s^2}{(s^2+4)^3}\right]-\left[\frac{4}{(s^2+4)^2}\right]\right]$$

$$L\{t^2.\sin 2t\}=\frac{16s^2}{(s^2+4)^3}-\frac{4}{(s^2+4)^2}$$

$$L\{t^2.\sin 2t\}=\frac{16s^2-4(s^2+4)}{(s^2+4)^3}$$

$$L\{t^2.\sin 2t\}=\frac{16s^2-4s^2-16}{(s^2+4)^3}$$

$$L\{t^2.\sin 2t\}=\frac{12s^2-16}{(s^2+4)^3} \text{ Answer}$$

Example 74: Find the Laplace Transform $L\{t^2\cos 3t\}$

Answer:

Here, $f(t)=\cos 3t$

This theorem states that,

If $L\{f(t)\} = f(s)$ --------------- (i)

Then $L\{t^2 f(t)\} = (-1)^2 \dfrac{d^2}{ds^2}\{f(s)\}$ --------------- (ii)

We have, according to equation no (i), $L\{f(t)\} = L(\cos 3t) = \dfrac{s}{s^2+3^2} = \dfrac{s}{s^2+9}$

[Here $f(s) = \dfrac{s}{s^2+a^2}$]

Hence, according to equation no (ii), we can write,

$$L\{t^2.f(t)\} = (-1)^2 \frac{d^2}{ds^2}\{f(s)\}$$

$$L\{t^2.\cos 3t\} = (-1)^2 \frac{d^2}{ds^2}\{f(s)\} \qquad [\, f(t) = \cos 3t \,]$$

$$L\{t^2.\cos 3t\} = (-1)^2 \frac{d^2}{ds^2}\{\frac{s}{s^2+9}\}$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[\frac{d}{ds}\{\frac{s}{s^2+9}\}\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[\frac{d}{ds}\{s(s^2+9)^{-1}\}\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[s\frac{d}{ds}\{(s^2+9)^{-1}\} + (s^2+9)^{-1}\frac{d}{ds}s\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[s\{(-1)(s^2+9)^{-1-1}.\frac{d}{ds}(s^2+9)\} + (s^2+9)^{-1}.1\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[-s(s^2+9)^{-2}.(2s) + (s^2+9)^{-1}\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[-2s^2(s^2+9)^{-2} + (s^2+9)^{-1}\right]$$

$$L\{t^2.\cos 3t\} = \frac{d}{ds}\left[-2s^2(s^2+9)^{-2}\right] + \frac{d}{ds}\left[(s^2+9)^{-1}\right]$$

$$L\{t^2.\cos 3t\} = -2\frac{d}{ds}\left[s^2(s^2+9)^{-2}\right] + \frac{d}{ds}\left[(s^2+9)^{-1}\right]$$

$$L\{t^2.\cos 3t\} = -2\left[s^2\frac{d}{ds}\left[(s^2+9)^{-2}\right] + (s^2+9)^{-2}\frac{d}{ds}s^2\right] + \frac{d}{ds}\left[(s^2+9)^{-1}\right]$$

$$L\{t^2.\cos 3t\} = -2\left[s^2.(-2).(s^2+9)^{-2-1}\frac{d}{ds}(s^2+9) + (s^2+9)^{-2}.(2s)\right] + (-1)(s^2+9)^{-1-1}\frac{d}{ds}\left[(s^2+9)\right]$$

$$L\{t^2.\cos 3t\} = -2\left[s^2.(-2).(s^2+9)^{-3}.(2s) + (s^2+9)^{-2}.(2s)\right] + (-1)(s^2+9)^{-2}.(2s)$$

$$L\{t^2.\cos 3t\} = -2\left[-4s^3(s^2+9)^{-3} + 2s(s^2+9)^{-2}\right] - 2s(s^2+9)^{-2}$$

$$L\{t^2.\cos 3t\} = 8s^3(s^2+9)^{-3} - 4s(s^2+9)^{-2} - 2s(s^2+9)^{-2}$$

$$L\{t^2.\cos 3t\} = 8s^3(s^2+9)^{-3} - 6s(s^2+9)^{-2}$$

$$L\{t^2.\cos 3t\} = \frac{8s^3}{(s^2+9)^3} - \frac{6s}{(s^2+9)^2}$$

$$L\{t^2.\cos 3t\} = \frac{8s^3 - 6s(s^2+9)}{(s^2+9)^3}$$

$$L\{t^2.\cos 3t\} = \frac{8s^3 - 6s^3 - 54s}{(s^2+9)^3}$$

$$L\{t^2.\cos 3t\} = \frac{2s^3 - 54s}{(s^2+9)^3} \qquad \textbf{Answer}$$

Problem 28: Division Theorem: Laplace transform of $\frac{1}{t}f(t)$ (Division by t)

That is $L(\frac{1}{t}f(t)) = ?$

Division theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ----------------------------(i)

Then $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty f(s)\,ds$ ----------------------------(ii)

Proof: We have $L(f(t)) = \int_0^\infty f(t)e^{-st}dt = f(s)$ ----------------------------(iii)

Integrating (iii) with respect to s

$$\int_s^\infty f(s)\,ds = \int_s^\infty \left[\int_0^\infty f(t)e^{-st}dt\right]ds$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[\int_s^\infty f(t)e^{-st}ds\right]dt \qquad \text{------------------------------(iv)}$$

Now

$$\int f(s)\,ds = \int \left[\int f(t)e^{-st}ds\right]dt$$

$$\int f(s)\,ds = \int \left[f(t)\int e^{-st}ds - \int\left\{\frac{d}{ds}(f(t))\int e^{-st}ds\right\}ds\right]dt$$

$$[\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$\int f(s)\,ds = \int\left[f(t)\left\{\frac{e^{-st}}{-t}\right\} - \int 0.\frac{e^{-st}}{-t}ds\right]dt$$

$$\int f(s)\,ds = \int\left[f(t)\left\{\frac{e^{-st}}{-t}\right\} - 0\right]dt$$

$$\int f(s)\,ds = \int\left[f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt \qquad \text{------------------------------(v)}$$

Putting the values of (v) in (iv), we can write

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[\int_s^\infty f(t)e^{-st}ds\right]dt$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]_s^\infty dt \qquad \text{[From v]}$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[f(t)\left\{\frac{e^{-\infty t}}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[f(t)\left\{\frac{e^{-\infty}}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[f(t)\left\{\frac{1}{e^\infty}.\frac{1}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[f(t)\left\{\frac{1}{\infty}.\frac{1}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[f(t)\left\{0.\frac{1}{-t}\right\} - f(t)\left\{\frac{e^{-st}}{-t}\right\}\right]dt$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[f(t)\{0\} + f(t)\left\{\frac{e^{-st}}{t}\right\}\right]dt$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[f(t)\left\{\frac{e^{-st}}{t}\right\}\right]dt$$

$$\int_s^\infty f(s)\,ds = \int_0^\infty \left[\frac{1}{t}f(t)e^{-st}\right]dt$$

$$\int_s^\infty f(s)\,ds = L\left[\frac{1}{t}f(t)\right] \qquad [\because L(f(t)) = \int_0^\infty f(t)e^{-st}dt = f(s)]$$

(Proved)

That is

$$L\left[\frac{1}{t}f(t)\right] = \int_s^\infty f(s)\,ds$$

Example 75: Find Laplace Transform of $\{\frac{\sin 2t}{t}\}$

Answer:

Here, $f(t) = \sin 2t$

Division theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $L\left[\frac{1}{t}f(t)\right]=\int_s^\infty f(s)\,ds$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t))=L(\sin 2t)=\frac{2}{s^2+4}$ **[Here $f(s)=\frac{2}{s^2+4}$]**

Hence, according to equation no (ii), we can write

$$L\left[\frac{1}{t}f(t)\right]=\int_s^\infty f(s)\,ds$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=\int_s^\infty f(s)\,ds$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=\int_s^\infty \frac{2}{s^2+4}\,ds$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=\int_s^\infty \frac{2}{s^2+2^2}\,ds$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\int_s^\infty \frac{1}{s^2+2^2}\,ds$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\tan^{-1}\frac{\infty}{2}-\tan^{-1}\frac{s}{2}\right]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\tan^{-1}\infty-\tan^{-1}\frac{s}{2}\right]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\tan^{-1}\tan\frac{\pi}{2}-\tan^{-1}\frac{s}{2}\right] \qquad [\because \tan\frac{\pi}{2}=\infty]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=2\times\frac{1}{2}\left[\frac{\pi}{2}-\tan^{-1}\frac{s}{2}\right]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=\left[\frac{\pi}{2}-\tan^{-1}\frac{s}{2}\right]$$

$$L\left[\frac{1}{t}f(t)\right]=L\left[\frac{1}{t}\sin 2t\right]=\left[\frac{\pi}{2}-\tan^{-1}\frac{s}{2}\right] \quad \textit{Answer}$$

Example 76: Prove that $L(f''(t))=s^2L\{f(t)\}-s\,f(0)-f'(0)$

We have

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt \qquad \text{--(i)}$$

$$\therefore L(f'(t)) = \int_0^\infty f'(t)e^{-st}dt \qquad [f(t) = f'(t)] \text{ ------------------------(ii)}$$

Now, $\int f'(t)e^{-st}dt$

$$\int f'(t)e^{-st}dt = e^{-st}\int f'(t)dt - \int\{\frac{d}{dt}(e^{-st})\int f'(t)dt\}dt$$

$$[\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$\int f'(t)e^{-st}dt = e^{-st}\int \frac{d}{dt}(f(t))dt - \int\{\frac{d}{dt}(e^{-st})\int \frac{d}{dt}(f(t))dt\}dt \qquad [\because \frac{d}{dt}(f(t)) = f'(t)]$$

$$\int f'(t)e^{-st}dt = e^{-st}f(t) - \int\{\frac{d}{dt}(e^{-st})f(t)\}dt$$

$$\int f'(t)e^{-st}dt = e^{-st}f(t) - \int -s(e^{-st})f(t)dt$$

$$[\because \frac{d}{dx}(e^{mx}) = e^{mx}.\frac{d}{dx}(mx) = e^{mx}.(m) = me^{mx}]$$

$$\int f'(t)e^{-st}dt = e^{-st}f(t) + \int s(e^{-st})f(t)dt$$

$$\int f'(t)e^{-st}dt = e^{-st}f(t) + s\int e^{-st}f(t)dt$$

From (ii),

$$\therefore L(f'(t)) = \int_0^\infty f'(t)e^{-st}dt = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[e^{-s\times\infty}f(\infty) - e^{-s\times 0}f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[e^{-\infty}f(\infty) - e^{0}f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{e^\infty}f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{e^\infty}f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{\infty}f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[0.f(\infty) - f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = -f(0) + s\int_0^\infty e^{-st}f(t)dt$$

$\therefore L(f^{'}(t)) = -f(0) + sL\{f(t)\}$ $\quad [\because L(f(t)) = \int_{0}^{\infty} f(t)e^{-st}dt]$

$\therefore L(f^{'}(t)) = -f(0) + sL\{f(t)\}$

$\therefore L(f^{'}(t)) = sL\{f(t)\} - f(0)$ ---------------------------------------(iii)

Now replacing $f(t)$ by $f^{'}(t)$ and $f^{'}(t)$ by $f^{''}(t)$ in (iii), we get

$\therefore L(f^{'}(t)) = sL\{f(t)\} - f(0)$

$\therefore L(f^{''}(t)) = sL\{f^{'}(t)\} - f^{'}(0)$ --(iv)

Putting the value of $L(f^{'}(t))$ from (iii) in (iv), we get

$\therefore L(f^{''}(t)) = sL\{f^{'}(t)\} - f^{'}(0)$

$\therefore L(f^{''}(t)) = s[sL\{f(t)\} - f(0)] - f^{'}(0)$ $\quad [L(f^{'}(t)) = sL\{f(t)\} - f(0)]$

$\therefore L(f^{''}(t)) = s^2 L\{f(t)\} - s\,f(0) - f^{'}(0)$ --(v)

$\therefore L(f^{''}(t)) = s^2 L\{f(t)\} - s\,f(0) - f^{'}(0)$ (Proved)

Similarly

$\therefore L(f^{'''}(t)) = s^3 L\{f(t)\} - s^2\, f(0) - s\,f^{'}(0) - f^{''}(0)$

$\therefore L(f^{iv}(t)) = s^4 L\{f(t)\} - s^3\, f(0) - s^2\, f^{'}(0) - s\,f^{''}(0) - f^{'''}(0)$

Summary

Fourier Series: Analyze Periodic Signal
Fourier Transform: Analyze Aperiodic Signal
Laplaces Transform: Analyze Unstable Signal