Pair of straight lines (যুগল সরলরেখার সমীকরণ)

① Homogeneous Quadratic/second degree equation.

$$ax^2+2hxy+by^2=0$$

always represents a pair of straight lines real or imaginary through the origin.

② If $ax^2+2hxy+by^2$ represented by the lines are $y-m_1x=0$ and $y-m_2x=0$

then $m_1+m_2 = -\frac{2h}{b}$

$m_1 m_2 = \frac{a}{b}$

③ Angle between the lines represented by the equation $ax^2+2hxy+by^2=0$ is $\theta = \tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$ (Perpendicular)

④ The lines which are represented by $ax^2+2hxy+by^2=0$ will be,

⇒ real and different if $h^2>ab$

⇒ real and coincident or parllel if $h^2=ab$

⇒ Perpendicular, if $a+b=0$

⇒ imaginary, if $h^2<ab$

⑤ The bisectors of the angels between the lines represented by $ax^2+2hxy+by^2=0$ is $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

⑥ General equation of second degree, $ax^2+2hxy+by^2+2gx+2fy+c=0$ may represent a pair of straight lines, if, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

or, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

⑦ The conic is called a parabola and Ellipse or a hyperbola according as the eccentricity $e=1$, $e<1$, or $e>1$ respectively.

⑧ General equation of the second degree, $ax^2+2hxy+by^2+2gx+2fy+c=0$ will represent

⇒ a pair of straight lines if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

or, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

⇒ two parallel lines if, $\Delta = 0$ and $h^2 = ab$

⇒ two perpendicular lines if $\Delta = 0$ and $a+b=0$

⇒ a circle if $\Delta \neq 0$, $a=b$ and $h=0$

⇒ a parabola if, $\Delta \neq 0$ and $h^2 - ab = 0$

⇒ a elipse if $\Delta \neq 0$ and $h^2 - ab < 0$

⇒ a hyperbola if $\Delta \neq 0$ and $h^2 - ab > 0$

⇒ a rectangular hyperbola if $a+b=0$, $h^2 - ab > 0$ and $\Delta \neq 0$.

⇒ দুইটি বিন্দুর মধ্যবর্তী দূরত্ব = $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ /

$\sqrt{(\text{ভুজদ্বয়ের অন্তর})^2 + (\text{কোটিদ্বয়ের অন্তর})^2}$

⇒ পোলার স্থানাঙ্ক (r,θ) = $r = \sqrt{(x)^2+(y)^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

কার্তেসীয় স্থানাঙ্ক, $(x,y) \Rightarrow x = r\cos\theta$

$y = r\sin\theta$

⇒ y অক্ষে অবস্থিত হলে $x=0$; ভুজ = শূন্য

⇒ x অক্ষে অবস্থিত হলে $y=0$; কোটি = শূন্য

⇒ $m_1:m_2$ অনুপাতে অন্তর্বিভক্ত হলে,

$$x = \frac{m_1x_2+m_2x_1}{m_1+m_2}\ ,\quad y = \frac{m_1y_2+m_2y_1}{m_1+m_2}$$

⇒ $m_1:m_2$ অনুপাতে বহিঃবিভক্ত করলে,

$$x = \frac{m_1x_2-m_2x_1}{m_1-m_2}\ ,\quad y = \frac{m_1y_2-m_2y_1}{m_1-m_2}$$

⇒ $\left.\begin{aligned} ax_1+by_1+c=0 \\ ax_2+by_2+c=0 \end{aligned}\right\}$ সরলরেখা দুটির সমদ্বিখণ্ডক

$$\frac{ax_1+by_2+c}{\sqrt{a^2+b^2}} = \pm\frac{ax_2+by_2+c}{\sqrt{a^2+b^2}}$$

⇒ $\tan 45^\circ = \pm\frac{m_1-m_2}{1+m_1m_2}$

⇒ ক্ষেত্রফল = $\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}$

$$= \frac{1}{2} | x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_4 - x_4 y_3 |$$

⇒ সরলরেখা (0,0) মূলবিন্দু সমীকরণ, $y = mx$ | $m = \frac{y_2 - y_1}{x_2 - x_1}$

⇒ দুইটি বিন্দুর স্থানাঙ্ক দেওয়া থাকলে সমীকরণ,

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

Example-1 Prove that the homogeneous second degree equation in xy. i.e $ax^2+2hxy+by^2=0$ always represents a pair of straight lines through the origin.

Solution: Given the equation, $ax^2+2hxy+by^2=0$ —(1)

Now, dividing equation (1) by bx^2, we get,

$$\frac{ax^2}{bx^2}+\frac{2hxy}{bx^2}+\frac{by^2}{bx^2}=0$$

$$\Rightarrow \frac{a}{b}+\frac{2h}{b}\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2=0$$

Putting $y/x=m$ we get,

$$m^2+\frac{2h}{b}m+\frac{a}{b}=0 \quad \text{—(2)}$$

Let m_1 and m_2 be the roots of the quardratic equation (2) so we get, $m_1+m_2=-\frac{2h}{b}$

and $m_1m_2=\frac{a}{b}$

মূলদ্বয়ের যোগফল $=-b/a$

মূলদ্বয়ের গুণফল $=c/a$

Now the equation (2) can be written as,

$$m^2-(m_1+m_2)m+m_1m_2=0$$

$[x^2-(\alpha+\beta)x+\alpha\beta=0]$

$$\Rightarrow m^2-m_1m-m_2m+m_1m_2=0$$

$$\Rightarrow m(m-m_1)-m_2(m-m_1)=0$$

$$\Rightarrow (m-m_1)(m-m_2)=0$$

Either, $m-m_1=0$

$\Rightarrow m=m_1$

$\Rightarrow y/x=m_1$

$\Rightarrow y=m_1x$

$\Rightarrow y-m_1x=0$ —(3)

or, $m-m_2=0$

$\Rightarrow m=m_2$

$\Rightarrow y/x=m_2$

$\Rightarrow y=m_2x$

$\Rightarrow y-m_2x=0.$ —(4)

As the equation ③ and ④ are two straight lines through the ~~again~~ origin. so the equation ① always represents a pair of straight lines through the origin.

Example: 2 Find the angle between the straight lines represented by the equation $ax^2+2hxy+by^2=0$

or, Find the angle between the straight lines represented by the homogeneous second degree equation.

Solution: Given the equation,

$$ax^2+2hxy+by^2=0 \quad \text{—— ①}$$

Let the equation ① represented by the equations.

$y-m_1x=0$ and $y-m_2x=0$

$\Rightarrow y=m_1x$ —② and $y=m_2x$ —③

which are straight lines and where,

$$m_1+m_2=-\frac{2h}{b} \quad \text{— ④}$$

$$m_1m_2=\frac{a}{b} \quad \text{— ⑤}$$

Let θ be the angle between the straight lines ② and ③.

So, we get,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2}$$

$$= \frac{\sqrt{(-2h/b)^2 - 4 \cdot a/b}}{1 + a/b} \quad \text{[From ④ and ⑤]}$$

$$= \frac{\sqrt{4h^2/b^2 - 4a/b}}{\frac{b+a}{b}}$$

$$= \frac{\sqrt{4(h^2-ab)/b^2}}{\frac{a+b}{b}}$$

$$= \frac{2\sqrt{h^2-ab}}{b} \times \frac{b}{a+b}$$

$$= \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\therefore \tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$$

$\therefore$ which is the required angle.

Example-3 What is the condition of parallisom and perpendicularity of two lines which the straight lines represented by the homogeneous second degree equation.

Solution: Let us consider the homogeneous quadratic equation is, $ax^2 + 2hxy + by^2 = 0$ —— ①

Let the equation ① represented by the equation.

$y - m_1 x = 0$ and $y - m_2 x = 0$

$\Rightarrow y = m_1 x$ —— ② and $\Rightarrow y = m_2 x$ —— ③

which are # straight lines and where,

$$m_1 + m_2 = -\frac{2h}{b} \quad \text{—— ④}$$

$$m_1 m_2 = \frac{a}{b} \quad \text{—— ⑤}$$

Let θ be the angle between the straight lines ② and ③,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$= \frac{\sqrt{(m_1+m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + a/b} \quad \text{[From ④ and ⑤]}$$

$$= \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + a/b}$$

$$= \frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{a+b}{b}}$$

$$= \frac{\sqrt{\frac{4(h^2-ab)}{b^2}}}{\frac{a+b}{b}}$$

$$= \frac{2\sqrt{h^2-ab}}{b} \times \frac{b}{a+b}$$

$$= \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\therefore \tan\theta = \frac{2\sqrt{h^2-ab}}{a+b} \quad \text{——— (6)}$$

Now the straight lines are perpendicular to each other than $\theta = 90°$, so get from equation (6)

$$\tan 90° = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\Rightarrow 0 = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\Rightarrow 2\sqrt{h^2-ab} = 0$$

$$\Rightarrow \sqrt{h^2-ab} = 0$$

$$\Rightarrow h^2-ab = 0$$

$$\Rightarrow h^2 = ab$$

$\therefore h^2 = ab$; it is the required condition of parallellism.

Again, if the lines will be perpendicular when $\theta = 90°$

$$\tan 90° = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\Rightarrow \infty = \frac{2\sqrt{h^2-ab}}{a+b}$$

$\therefore a+b = 0$ which is the condition of perpendicularity.

Condition of coincidence:

If the straight lines are coincidence, then $\theta = 180^\circ$ so we get from equation ⑥

$$\tan 180^\circ = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\Rightarrow 0 = \frac{2\sqrt{h^2-ab}}{a+b}$$

$$\Rightarrow 2\sqrt{h^2-ab} = 0$$

$$\Rightarrow \sqrt{h^2-ab} = 0$$

$$\Rightarrow h^2 = ab.$$

Example-5 Find the condition that the general equation of the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ may represents a pair of straight lines.

Solution: Given the equation,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{—①}$$

Now transform the origin at the point (α, β), Now putting $x = x+\alpha$ and $y = y+\beta$ in equation ①

$$a(x+\alpha)^2 + 2h(x+\alpha)(y+\beta) + b(y+\beta)^2 + 2g(x+\alpha) + 2f(y+\beta) + c = 0$$

$$\Rightarrow a(x^2+2x\alpha+\alpha^2) + 2h(xy+x\beta+\alpha y+\alpha\beta) + b(y^2+2y\beta+\beta^2)$$
$$+ 2gx + 2g\alpha + 2fy + 2f\beta + c = 0$$

$\Rightarrow ax^2 + 2ax\alpha + a\alpha^2 + 2hxy + 2hx\beta + 2h\alpha y + 2h\alpha\beta + by^2$

$+ 2yb\beta + b\beta^2 + 2gx + 2g\alpha + 2fy + 2f\beta + c = 0$

$\Rightarrow ax^2 + 2hxy + by^2 + 2x(a\alpha + h\beta + g) + 2y(h\alpha + b\beta + f) + (a\alpha^2 + 2h\alpha\beta$

$+ 2g\alpha + 2f\beta + c) = 0$ —— (2)

The equation (2) may represent a pair of straight line, if it is reduced to a homogeneous equation in x and y. This is possible if the co-efficients of x and y and the constant terms are separately zero,

i.e, $a\alpha + h\beta + g = 0$ —— (3)

$h\alpha + b\beta + f = 0$ —— (4)

$b\beta^2 + a\alpha^2 + 2h\alpha\beta + 2g\alpha + 2f\beta + c = 0$ —— (5)

The relation (5) can be written as

$\alpha(a\alpha + 2h\beta + g) + \beta(h\alpha + b\beta + f) + g\alpha + f\beta + c = 0$

$\Rightarrow \alpha.0 + \beta.0 + g\alpha + f\beta + c = 0$ [From equation (3) and (4)

$\Rightarrow g\alpha + f\beta + c = 0$ —— (6)

Now if we eliminate α and β from eq (3), (4) and (6)

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\Rightarrow \Delta \equiv a(bc - f^2) - h(hc - gf) + g(hf - bg) = 0$

$$\Rightarrow \Delta \equiv abc - af^2 - ch^2 + fgh + fgh - bg^2 = 0$$

$$\Rightarrow \Delta \equiv abc + 2fgh - af^2 - ch^2 = 0$$

which is the required equation / condition.

① Prove that homogeneous quadratic equations represents two straight lines.

Let us consider homogeneous quadratic equation as,

$$ax^2 + 2hxy + by^2 = 0 \quad \text{--- ①}$$

If $b \neq 0$ dividing both sides of the equations ① by x^2b, we have,

$$\frac{ax^2}{x^2b} + \frac{2hxy}{x^2b} + \frac{by^2}{x^2b} = 0$$

$$\Rightarrow \frac{a}{b} + 2\cdot\frac{h}{b}\cdot\frac{y}{x} + \left(\frac{y}{x}\right)^2 = 0$$

$$\Rightarrow \left(\frac{y}{x}\right)^2 + 2\cdot\frac{h}{b}\cdot\frac{y}{x} + \frac{a}{b} = 0 \quad \text{--- ②}$$

Let, m_1 and m_2 be the roots of quadratic equation ② in $\left(\frac{y}{x}\right)$ so, that,

$$m_1 + m_2 = -\frac{2h}{b}$$

$$m_1m_2 = \frac{a}{b}$$

The equation ① must be equivalent to,

$$\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right) = 0$$

$$\therefore \frac{y}{x} - m_1 = 0 \qquad \frac{y}{x} - m_2 = 0$$

$$\Rightarrow y - m_1x = 0 \qquad \Rightarrow y - m_2x = 0$$

which passes through the origin.

Thus, the homogenerus quadratic equation $ax^2 + 2hxy + by^2 = 0$ always represent a pair of straight line, real or imaginary, passes through the origin.

(Proved)

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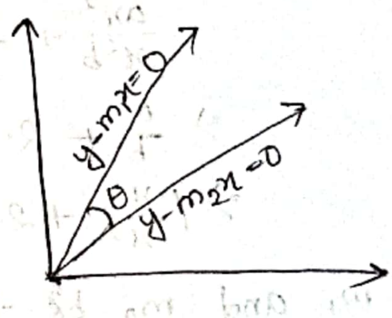
(2) Find an angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$

Given the equation $ax^2 + 2hxy + by^2 = 0$ —— (1)

The axes assumed to be rectangular, Let the lines represented by (1) be

$y - m_1x = 0$ and $y - m_2x = 0$, so that equation (1) can be written as,



$(y - m_1x)(y - m_2x) = 0$ —— (2)

Comparing equations (1) and (2) we get

$m_1 + m_2 = -\frac{2h}{b}$

$m_1 m_2 = \frac{a}{b}$

If θ be the angle between the straight lines $y - m_1x = 0$ and $y - m_2x = 0$ then we know

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$= \frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

$$= \frac{\sqrt{(-\frac{2h}{b})^2 - 4(\frac{a}{b})}}{1 + \frac{a}{b}}$$

$$= \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{\frac{a+b}{b}}$$

$$\text{or, } \tan\theta = \frac{\frac{2}{b}\sqrt{h^2 - ab}}{\frac{a+b}{b}} = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2\sqrt{h^2 - ab}}{a+b}\right)$$

Q. Find the condition that the general equation of
(5) the second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represent a pair of straight line.

Solution: Given the equation,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{--- (1)}$$

If we transfer the origin to a point (α, β), the point of intersection of straight lines and keep the direction of the axes unaltered, equation (1) reduces to,

$$a\cancel{(x}(x+\alpha)^2+2h(x+\alpha)(y+\beta)+b(y+\beta)^2+2g(x+\alpha)+2f(y+\beta)+c=0$$

$$\text{or, } ax^2+2a\alpha x+a\alpha^2+2hxy+2h\alpha y+2h\beta x+2h\alpha\beta+by^2+2by\beta+b\beta^2+2gx+2g\alpha+2fy+2f\beta+c=0$$

$$\text{or, } ax^2+2hxy+by^2+2(a\alpha+h\beta+g)x+2(h\alpha+b\beta+f)y+a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \quad \text{--- (2)}$$

equation (2) may represent a pair of straight lines, if it is reduced to a homogeneous equation in x and y. This is possible if the co-efficients of x, y and constant terms are separately zero,

i.e,
$$a\alpha+h\beta+g=0 \quad \text{--- (3)}$$
$$h\alpha+b\beta+f=0 \quad \text{--- (4)}$$

and
$$a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c=0 \quad \text{--- (5)}$$

The relation ⑤ may be written as

$$\alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+g\alpha+f\beta+c=0 \quad \text{---} ⑥$$

By equations ③ and ④ the relation ⑥ becomes,

$$g\alpha+f\beta+c=0 \quad \text{---} ⑦$$

Now if we eliminate α and β from equations ③, ④ and ⑦ then the required condition is obtained as,

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or, $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

which is the required condition that the general equation of second degree may represents a pair of straight lines,

By solving equation ③ and ④ we get,

$$\frac{\alpha}{hf-bg} = \frac{\beta}{gh-af} = \frac{1}{ab-h^2}$$

$$\therefore \alpha = \frac{hf-bg}{ab-h^2} \text{ and } \beta = \frac{gh-af}{ab-h^2}$$

$\therefore$ where (α,β) is the point of intersection of the straight lines represented by ①.

⑥ Find the angle between the lines given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

or, Find the angle between the line g represented by general equation of second degree.

Solution: Given the equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{——— (1)}$$

Let the straight lines represented by the equation (1) are,

$$l_1 x^2 + m_1 y + n_1 = 0 \quad \text{——— (2)}$$

$$l_2 x^2 + m_2 y + n_2 = 0 \quad \text{——— (3)}$$

So, we have,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (l_1 x + m_1 y + n_1)(l_2 x + m_2 y + n_2)$$

$$= l_1 l_2 x^2 + l_1 m_2 xy + l_2 x n_2 + l_2 m_1 xy + m_1 m_2 y^2 + m_1 n_2 y + l_2 n_1 x + m_2 n_1 y + n_1 n_2$$

$$= l_1 l_2 x^2 + xy(l_1 m_2 + l_2 m_1) + m_1 m_2 y^2 + x(l_1 n_2 + l_2 n_1) + y(m_1 n_2 + m_2 n_1) + n_1 n_2$$

Now, comparing the co-efficients of both sides we get,

$l_1 l_2 = a$, $l_1 m_2 + l_2 m_1 = 2h$, $m_1 m_2 = b$, $l_1 n_2 + l_2 n_1 = 2g$.

$m_1 n_2 + m_2 n_1 = 2f$, $c = n_1 n_2$

Again from equation ② we get,

$$l_1 x + m_1 y + n_1 = 0$$

$$\Rightarrow m_1 y = -l_1 x - n_1$$

$$\Rightarrow y = -\frac{l_1}{m_1} - \frac{n_1}{m_1}$$

So, slope of the line ② is

$$m_1' = \tan\theta = -\frac{l_1}{m_1}$$

Again from ③ we get, $l_2 x + m_2 y + n_2 = 0$

$$\Rightarrow m_2 y = -l_2 x - n_2$$

$$\Rightarrow y = -\frac{l_2}{m_2}x - \frac{n_2}{m_2}$$

∴ Slope of the line ③,

$$m_2' = \tan\theta_2 = -\frac{l_2}{m_2}$$

Let θ be the angle between the straight lines ② and ③,

So, we have, $\tan\theta = \dfrac{m_1' - m_2'}{1 + m_1' m_2'}$

$$\Rightarrow \tan\theta = \frac{-\frac{l_1}{m_1} + \frac{l_2}{m_2}}{1 + \frac{l_1}{m_1} \times \frac{l_2}{m_2}}$$

$$= \frac{\frac{-l_1 m_1 + l_2 m_1}{m_1 m_2}}{1 + \frac{l_1 l_2}{m_1 m_2}}$$

$$= \frac{-l_1m_2 + l_2m_1}{m_1m_2} \times \frac{m_1m_2}{m_1m_2 + l_1l_2}$$

$$= \frac{-l_1m_2 + l_2m_1}{m_1m_2 + l_1l_2}$$

$$= \frac{l_2m_1 - l_1m_2}{m_1m_2 + l_1l_2}$$

$$= \frac{\sqrt{(l_2m_1 + l_1m_2)^2 - 4l_1l_2m_1m_2}}{m_1m_2 + l_1l_2}$$

$$= \frac{\sqrt{(2h)^2 - 4ab}}{a+b}$$

$$\Rightarrow \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$\therefore \theta = \tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right)$ which is the required angle.

Condition for parallisom:

Two lines will be parallel if $\theta = 0°$

i.e. $\tan\theta = \tan 0°$

$$\Rightarrow \frac{2\sqrt{h^2-ab}}{a+b} = 0$$

$$\Rightarrow h^2 = ab$$

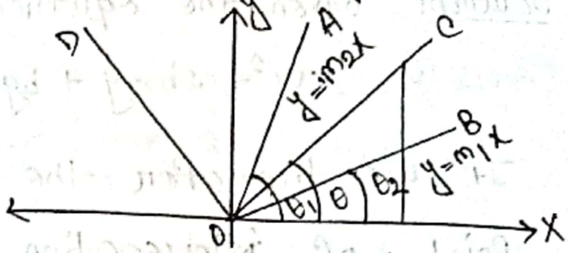
④ 田) Find the bisectors of the angles between the lines represented by $ax^2+2hxy+by^2=0$

Solution: Given the equation is,

$ax^2+2hxy+by^2=0$ which represent the lines $y-m_1x=0$ and $y-m_2x=0$, so that,

$m_1+m_2 = -\frac{2h}{b}$

$m_1m_2 = \frac{a}{b}$



The equation of the required bisectors are, obtained by

$$\frac{y-m_1x}{\sqrt{(1)^2+(-m)^2}} = \pm \frac{y-m_2x}{\sqrt{(1)^2+(-m_2)^2}}$$

or, $\frac{(y-m_1x)^2}{1+m_1^2} = \pm \frac{(y-m_2x)^2}{1+m_2^2}$ [By squaring]

or, $(y^2-2m_1xy+m_1x^2)(1+m_2^2) = (y^2-2m_2xy+m_2^2x^2)(1+m_1^2)$

or, $y^2-2m_1xy+m_1^2x^2+m_2^2y^2-2m_1m_2^2xy+m_1^2m_2^2x^2 = y^2-2m_2xy+m_2^2x^2+m_1^2y^2-2m_1^2m_2xy+m_1^2m_2^2x^2$

or, $x^2(m_1^2-m_2^2)-y(m_1^2-m_2^2) = 2xy(m_1-m_2)-2xy\, m_1m_2(m_1-m_2)$

or, $(m_1^2-m_2^2)(x^2-y^2) = 2xy(m_1-m_2)(1-m_1m_2)$

or, $(m_1+m_2)(x^2-y^2) = 2xy(1-m_1m_2)$

or, $(-\frac{2h}{b})(x^2-y^2) = 2xy(1-\frac{a}{b})$

or, $(-\frac{2h}{b})(x^2-y^2) = 2xy(\frac{b-a}{b})$

$\therefore \frac{x^2-y^2}{a-b} = \frac{xy}{h}$

which is the required equation of the bisectors.

⑨ Prove that the straight line $y^3-x^3+3xy(y-x)=0$ represents three straight lines equally inclained to other.

Solution: Given the equation,

$$y^3-x^3+3xy(y-x)=0 \quad \text{—} ①$$

It is the homogeneous equation of third degree in $x.y$, so, it represents three straight lines through the origin.

Changing the given equation in polar coordinates,

Let $x = r\cos\theta$

$y = r\sin\theta$

$\therefore y = x\tan\theta$

Now putting the value of $y = x\tan\theta$ in eq. ① we get

$$y^3-x^3+3xy(y-x)=0$$

$\Rightarrow (x\tan\theta)^3 - x^3 + 3x(x\tan\theta)(x\tan\theta - x) = 0$

$\Rightarrow x^3\tan^3\theta - x^3 + 3x^3\tan^2\theta - 3x^3\tan\theta = 0$

$\Rightarrow x^3(\tan^3\theta - 1 + 3\tan^2\theta - 3\tan\theta) = 0$

$\Rightarrow \tan^3\theta - 1 + 3\tan^2\theta - 3\tan\theta = 0$

$\Rightarrow 3\tan\theta - \tan^3\theta = -(1-3\tan^2\theta)$

$\Rightarrow \dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = -1$

$\Rightarrow \tan 3\theta = -1$

$\Rightarrow \tan 3\theta = \tan \frac{3\pi}{4}$

$\Rightarrow \tan 3\theta = \tan\left(2n\pi + \frac{3\pi}{4}\right)$ [where, $n = 0, 1$]

$\therefore 3\theta = 2n\pi + 3\pi/4$

Putting $n = 0, 1, 2,$ we get the inclination of the lines with as follows:-

$3\theta_1 = 2.0.\pi + \frac{3\pi}{4}$; $3\theta_2 = 2.1.\pi + 3\pi/4$; $3\theta_3 = 2.2.\pi + 3\pi/4$

$\Rightarrow 3\theta_1 = 3\pi/4$

$\Rightarrow \theta_1 = \pi/4$

$\Rightarrow 3\theta_2 = 2\pi + 3\pi/4$

$\Rightarrow 3\theta_2 = \frac{8\pi + 3\pi}{4}$

$\Rightarrow 3\theta_2 = \frac{11\pi}{4}$

$\Rightarrow \theta_2 = \frac{11\pi}{12}$

$\Rightarrow 3\theta_3 = 4\pi + \frac{3\pi}{4}$

$\Rightarrow 3\theta_3 = \frac{16\pi + 3\pi}{4}$

$\Rightarrow 3\theta_3 = \frac{19\pi}{4}$

$\Rightarrow \theta_3 = \frac{19\pi}{12}$

Therefore the angle between first and second line is $\theta_2 - \theta_1 = \frac{11\pi}{12} - \frac{\pi}{4}$

$$= \frac{11\pi - 3\pi}{12}$$

$$= \frac{8\pi}{12}$$

$$= \frac{2\pi}{3}$$

The angle between 2nd and 3rd line is

$$\theta_3 - \theta_2 = \frac{19\pi}{12} - \frac{11\pi}{12} = \frac{19\pi - 11\pi}{12} = \frac{8\pi}{12} = \frac{2\pi}{3}$$

The angle between 3rd and 1st line is

$$\theta_3 - \theta_1 = 2\pi - POR$$

$$\Rightarrow POR = 2\pi - \theta_3 + \theta_2$$

$$= 2\pi - \frac{19\pi}{12} + \pi/4$$

$$= \frac{24\pi - 19\pi + 3\pi}{12}$$

$$= \frac{8\pi}{12}$$

$$= \frac{2\pi}{3}$$

Hence, the given lines are inclined to each other at an angle $2\pi/3$.

(10) Show that the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel lines if $a:h = h:b = g:f$. Also show that the distance between them is $\frac{2\sqrt{g^2-ac}}{\sqrt{a(a+b)}}$ or, $2\sqrt{\{(g^2-ac)/a(a+b)\}}$.

Solution: 1st part:

Given the equation is,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{—(1)}$$

We see that the given equation is general equation of second degree. So, the equation (1) represents two straight lines if $abc+2fgh-af^2-bg^2-ch^2=0$ —(2)

And the straight lines will parallel if,

$$h^2=ab \quad \text{—(3)}$$

Now, putting $h^2=ab$ in eq.(2) we get,

$$abc + 2fg\sqrt{ab} - af^2 - bg^2 - cab = 0$$

$$\Rightarrow af^2 - 2fg\sqrt{ab} + bg^2 = 0$$

$$\Rightarrow (\sqrt{a}f)^2 - 2\sqrt{a}f\sqrt{b}g + (\sqrt{b}g)^2 = 0$$

$$\Rightarrow (\sqrt{a}f - \sqrt{b}g)^2 = 0$$

$$\Rightarrow \sqrt{a}f = \sqrt{b}g$$

$$\Rightarrow \frac{\sqrt{a}}{\sqrt{b}} = \frac{g}{f}$$

$$\Rightarrow \frac{\sqrt{ab}}{\sqrt{b^2}} = \frac{g}{f}$$

$$\Rightarrow \frac{h}{b} = \frac{g}{f}$$

$$\Rightarrow h:b = g:f \quad \text{——} (4)$$

Again from (3) we get,

$$h^2 = ab$$

$$\Rightarrow \frac{h}{b} = \frac{a}{h}$$

$$\Rightarrow h:b = a:h \quad \text{——} (5)$$

Now, from (4) and (5) we get, $a:h = h:b = g:f$

[Showed]

2nd part:

Let the two parallel lines which are represented by the equation (1) are,

$$lx + my + n_1 = 0 \quad \text{——} (6)$$

$$lx + my + n_2 = 0 \quad \text{——} (7)$$

So, we have,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n_1)(lx + my + n_2)$$

$$= l^2x^2 + lmxy + ln_2x + mlxy + m^2y^2 + mn_2y + n_1lx + n_1my + n_1n_2$$

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Now comparing the co-officients from both sides, we get.

$l^2 = a$; $h = lm$; $m^2 = b$; $2g = ln_1 + ln_2$; $2f = mn_1 + mn_2$,

$c = n_1 n_2$

Now the distance between ⑥ and ⑦ is,

$$\frac{n_1 - n_2}{\sqrt{l^2+m^2}}$$

$$= \frac{\sqrt{(n_1+n_2)^2 - 4n_1n_2}}{\sqrt{l^2+m^2}}$$

$$= \frac{\sqrt{(2g/l)^2 - 4c}}{\sqrt{a+b}}$$

$$= \frac{\sqrt{\frac{4g^2 - 4l^2c}{l^2}}}{\sqrt{a+b}}$$

$$= \frac{2\sqrt{g^2 - ac}}{\sqrt{a(a+b)}}$$

[showed]

(11) Prove that the straight lines represented by the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ will be equidistant from the origin if $f^4-g^4=c(bf^2-ag^2)$

Solution: Given the equation,

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Let the straight lines which represented by (1) are

$$lx+my+n=0 \quad \text{---(2)}$$

and $l_1x+m_1y+n_1=0$ ---(3)

So, we get,

$$ax^2+2hxy+by^2+2gx+2fy+c=(lx+my+n)(l_1x+m_1y+n_1)$$

$$=ll_1x^2+lm_1xy+n_1lx+l_1mxy+mm_1y^2+mn_1y+nl_1x+m_1ny+nn_1$$

$$=ll_2x^2+xy(lm_1+l_1m)+mm_1y^2+x(n_1l+nl_1)+y(mn_1+m_1n)+nm_1$$

Now, comparing the co-efficient from both sides we get,

$a=ll_1$, $2h=lm_1+l_1m$, $b=mm_1$, $2g=n_1l+nl_1$, $2f=mn_1+m_1n$, $c=nn_1$

Again, since the straight lines (2) and (3) are equidistance from the origin.

$$\text{so, } \frac{n}{\sqrt{l^2+m^2}} = \frac{n_1}{\sqrt{l_1^2+m_1^2}}$$

$$\Rightarrow \frac{n^2}{l^2+m^2} = \frac{n_1^2}{l_1^2+m_1^2}$$

$$\Rightarrow n^2(l_1^2+m_1^2) = n_1^2(l^2+m^2)$$

$$\Rightarrow n^2l_1^2+n^2m_1^2 = n_1^2l^2+n_1^2m^2$$

$$\Rightarrow (n^2l_1^2-n_1^2l^2) = (n_1^2m^2-n^2m_1^2)$$

$$\Rightarrow (nl_1+n_1l)(nl_1-n_1l) = (n_1m+nm_1)(n_1m-nm_1)$$

$$\Rightarrow 2g(nl_1-n_1l) = 2f(mn_1-m_1n)$$

$$\Rightarrow g^2(nl_1-n_1l)^2 = f(mn_1-m_1n)^2$$

$$\Rightarrow g^2\{(nl_1+n_1l)^2-4ll_1nn_1\} = f^2\{(mn_1+m_1n)^2-4mm_1nn_1\}$$

$$\Rightarrow g^2\{(2g)^2-4.a.c\} = f^2\{(2f)^2-4.b.c\}$$

$$\Rightarrow g^2(4g^2-4ac) = f^2(4f^2-4bc)$$

$$\Rightarrow g^4-g^2ac = f^4-f^2bc$$

$$\Rightarrow f^4-g^4 = c(bf^2-ag^2)$$

Here, the straight lines represented by the eq.① will be equidistance from the origin when,

$$f^4-g^4 = c(bf^2-ag^2)$$

[Proved]

(12) The axes being rectangular find the equation to the pair of straight lines meeting at the origin which are perpendicular to the pair given by the equation $ax^2+2hxy+by^2=0$

Solution: Given the equation,

$$ax^2+2hxy+by^2=0 \text{ ——— } (1)$$

Let the straight lines which are represented by (1),

$$y-m_1x=0 \text{ —— } (2)$$

$$y-m_2x=0 \text{ —— } (3)$$

where, $m_1+m_2 = -\frac{2h}{b}$ —— (4)

and $m_1m_2 = \frac{a}{b}$ —— (5)

Again the straight lines which are perpendicular to (2) and (3) and passing through the origin are,

$$m_1y+x=0 \text{ —— } (6)$$

$$m_2y+x=0 \text{ —— } (7)$$

So, the equation which represented by the straight lines (6) and (7) is,

$$(m_1y+x)(m_2y+x)=0$$

$$\Rightarrow m_1m_2y^2+m_1xy+m_2xy+x^2=0$$

$$\Rightarrow \left(\frac{a}{b}\right)+\left(-\frac{2h}{b}\right)xy+x^2=0$$

$$\Rightarrow bx^2-2hxy+ay^2=0 \quad \text{(Ans)}$$

(17) If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of straight lines equidistant from the origin. Show that $h(g^2-f^2)=fg(a-b)$.

Solution: Given the equation,

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{--- (1)}$$

Let the straight lines which represented by (1) are,

$$lx+my+n=0 \quad \text{--- (2)}$$
$$l_1x+m_1y+n_1=0 \quad \text{--- (3)}$$

So, we get,

$$ax^2+2hxy+by^2+2gx+2fy+c=(lx+my+n)(l_1x+m_1y+n_1)$$
$$=ll_1x^2+lm_1xy+ln_1x+l_1mxy+mm_1y^2$$
$$+mn_1y+nl_1x+m_1ny+nn_1$$
$$=ll_1x^2+(lm_1+l_1m)xy+mm_1y^2+(ln_1+nl_1)x$$
$$+(mn_1\ m_1n)y+nn_2$$

Now comparing the co-efficient from both sides we get,

$ll_1=a$, $lm_1+l_1m=2h$, $mm_1=b$, $l_1n+n_1l=2g$,

$mn_1+m_2n=2f$, $nn_1=c$.

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Again, since the straight lines ① and ③ and equidistance from the origin so,

$$\frac{n}{\sqrt{l^2+m^2}} = \frac{n_1}{\sqrt{l_1^2+m_1^2}}$$

$$\Rightarrow \frac{n^2}{l^2+m^2} = \frac{n_1^2}{l_1^2+m_1^2}$$

$$\Rightarrow n^2(l_1^2+m_1^2) = n_1^2(l^2+m^2)$$

$$\Rightarrow n^2l_1^2 + n^2m_1^2 - n_1^2l^2 + n_1^2m^2 = 0$$

$$\Rightarrow (nl_1)^2 - (n_1l)^2 = (n_1m)^2 - (nm_1)^2$$

$$\Rightarrow (nl_1+n_1l)(nl_1-n_1l) = (n_1m+nm_1)(n_1m-nm_1)$$

$$\Rightarrow 2g(l_1n-ln_1) = 2f(mn_1-m_1n)$$

$$\Rightarrow g^2(l_1n-ln_1)^2 = f^2(mn_1-m_1n)^2$$

$$\Rightarrow g^2\{(l_1n+ln_1)^2 - 4l_1nln_1\} = f^2\{(mn_1+m_1n)^2 - 4mm_1nn_1\}$$

$$\Rightarrow g^2(4g^2-4ac) = f^2(4f^2-4bc)$$

$$\Rightarrow 4g^4 - 4g^2ac = 4f^4 - 4f^2bc$$

$$\rightarrow f^4 - g^4 = c(bf^2-ag^2) \quad \text{—— ④}$$

$\rightarrow$ ~~4g~~ Again we have,

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow c(ab-h^2) = af^2 + bg^2 - 2fgh$$

$$\Rightarrow c = \frac{af^2+bg^2-2fgh}{ab-h^2}$$

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Now putting the value of c in eq.(4) we get

$$-f^4 g^4 = \frac{af^2+bg^2-2fgh}{ab-h^2} \times bf^2-ag^2$$

$$\Rightarrow f^4ab - f^4h^2 - g^4ab + g^4h^2 = -f^4ab - a^2f^2g^2 + b^2f^2g^2 - g^4ab - 2bf^3gh + 2afg^3h$$

$$\Rightarrow g^4h^2 - 2afg^3h + a^2f^2g^2 = f^4h^2 - 2bf^3gh + b^2f^2g^2$$

$$\cancel{\Rightarrow g^4h^2 - 2afg^3h + a^2f^2g^2 = f^4h^2 - 2bf^3gh + b^2f^2g^2}$$

$$\Rightarrow (g^2h - afg)^2 = (f^2h - bfg)^2$$

$$\Rightarrow g^2h - afg = -f^2h - bfg$$

$$\Rightarrow h(g^2 - f^2) = fg(a-b)$$

[showed]

(22) If one of the lines $ax^2+2hxy+by^2=0$ be perpendicular to one of the lines $a_1x^2+2h_1xy+b_1y^2$. Prove that $(aa_1-bb_1)^2+4(a_1h+bh_1)(ah_1+b_1h)=0$

Solution: Given the equation are

$$ax^2+2hxy+by^2=0 \quad \text{——(1)}$$

$$a_1x^2+2h_1xy+b_1y^2=0 \quad \text{——(2)}$$

Let $y-mx=0$ be a straight line of the equation (1).

Then the perpendicular line of it be $my+x=0$.

which is one of the lines of the equation (2).

So, these two equation satisfy the equation ① and ② respectively,

$$ax^2 + 2hmx^2 + bm^2x^2 = 0$$

$$\Rightarrow bm^2 + 2hm + a = 0 \quad \text{——③}$$

and $a_1x^2 + 2h_1x(-x/m) + b_1(-x/m)^2 = 0$

$$\Rightarrow a_1x^2 - \frac{2h_1x^2}{m} + \frac{b_1x^2}{m^2} = 0$$

$$\Rightarrow a_1x^2m^2 - 2h_1x^2m + b_1x^2 = 0$$

$$\Rightarrow a_1m^2 - 2h_1m + b_1 = 0 \quad \text{——④}$$

Now by cross multiplication ③ and ④ we get,

$$\frac{m^2}{2hb_1 - 2h_1a} = \frac{m}{aa_1 - bb_1} = \frac{1}{2h_1b - 2ha_1}$$

$$\Rightarrow m = \frac{2(hb_1 + h_1a)}{aa_1 - bb_1}, \quad m = \frac{aa_1 - bb_1}{-2(h_1b + ha_1)}$$

$$\therefore \frac{2(hb_1 + h_1a)}{aa_1 - bb_1} = \frac{aa_1 - bb_1}{-2(h_1b + ha_1)}$$

$$\Rightarrow (aa_1 - bb_1)^2 = -4(a_1h + bh_1)(ah_1 + b_1h)$$

$$\therefore (aa_1 - bb_1)^2 + 4(a_1h + bh_1)(ah_1 + b_1h) = 0$$

[Proved]

(26) If one of the lines $ax^2+2hxy+by^2=0$ be coincident with one of the lines $a_1x^2+2h_1xy+b_1y^2=0$ Then Prove that $(ab_1-a_1b)^2 = 4(a_1h-ah_1)(bh_1-b_1h)$

Solution: Given the equation,

$$ax^2+2hxy+by^2=0 \quad \text{——} (1)$$

$$a_1x^2+2h_1xy+b_1y^2=0 \quad \text{——} (2)$$

Let $y-mx=0$ be the line given by the eq. (1) and (2)

Now putting $y=mx$ in the eq. (1) and (2) we get,

$$ax^2+2hmx^2+bm^2x^2=0$$

$$\therefore bm^2+2hm+a=0 \quad \text{——} (3)$$

Again, $a_1x^2+2h_1mx^2+b_1m^2x^2=0$

$$\Rightarrow b_1m^2+2h_1m+a_1=0 \quad \text{——} (4)$$

So, solving (3) and (4) we get,

$$\frac{m^2}{2(a_1h-ah_1)} = \frac{m}{ab_1-a_1b} = \frac{1}{2(bh_1-b_1h)}$$

$$\therefore m = \frac{2(a_1h-ah_1)}{ab_1-a_1b} \text{ and } m = \frac{ab_1-a_1b}{2(bh_1-b_1h)}$$

So, $$\frac{2(a_1h-ah_1)}{ab_1-a_1b} = \frac{ab_1-a_1b}{2(bh_1-b_1h)}$$

$$\therefore (ab_1-a_1b)^2 = 4(a_1h-ah_1)(bh_1-b_1h)$$

[Proved]

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If the straight lines represented by the equation $x^2(\tan^2\varphi + \cos^2\varphi) - 2xy\tan\varphi + y^2\sin^2\varphi = 0$ makes angles α and β with the axis of x then show that $\tan\alpha - \tan\beta = 2$.

Solution: Given the equation,

$$x^2(\tan^2\varphi + \cos^2\varphi) - 2xy\tan\varphi + y^2\sin^2\varphi = 0 \quad \text{--- (1)}$$

Equation (1) represents a pair of straight lines, and Suppose these are,

$$y - m_1x = 0 \quad \text{--- (2)}$$

and $y - m_2x = 0$ --- (3)

According to question, equations (2) and (3) makes an angles α and β with the axis of x, so that we can write, $m_1 = \tan\alpha$ and $m_2 = \tan\beta$.

Since equations (2) and (3) represents the equation (1) we have,

$$m_1 + m_2 = \frac{-2(-\tan\varphi)}{\sin^2\varphi} = \frac{2\tan\varphi}{\sin^2\varphi}$$

and $m_1m_2 = \dfrac{\tan^2\varphi + \cos^2\varphi}{\sin^2\varphi}$

Now, $\tan\alpha - \tan\beta = m_1 - m_2$

$$= \sqrt{(m_1+m_2)^2 - 4m_1m_2}$$

$$= \sqrt{\left(\frac{2\tan\varphi}{\sin^2\varphi}\right)^2 - 4\left(\frac{\tan^2\varphi + \cos^2\varphi}{\sin^2\varphi}\right)}$$

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$$= 2\sqrt{\frac{\tan^2\varphi - (\tan^2\varphi + \cos^2\varphi)\sin^2\varphi}{\sin^4\varphi}}$$

or, $\tan\alpha - \tan\beta = 2 \cdot \dfrac{\sqrt{\tan^2\varphi - \tan^2\varphi \cdot \sin^2\varphi - \sin^2\varphi\cos^2\varphi}}{\sin^2\varphi}$

$$= 2 \cdot \frac{\sqrt{\tan^2\varphi(1 - \sin^2\varphi) - \sin^2\varphi\cos^2\varphi}}{\sin^2\varphi}$$

$$= 2\frac{\sqrt{\sin^2\varphi - \sin^2\varphi\cos^2\varphi}}{\sin^2\varphi}$$

$$= 2 \cdot \frac{\sqrt{\sin^2\varphi \cdot (1 - \cos^2\varphi)}}{\sin^2\varphi}$$

$$= 2 \cdot \frac{\sqrt{\sin^4\varphi}}{\sin^2\varphi}$$

$$= 2$$