Euler's theorem and its applications

Euler's theorem for two variables:

If $u = f(x, y)$ is a homogeneous function of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Example 6.38

Verify Euler's theorem for the function $u = \dfrac{1}{\sqrt{x^2+y^2}}$.

Solution:

$$u(x,y) = \left(x^2+y^2\right)^{-\frac{1}{2}}$$

$$u(tx,ty) = \left(t^2x^2+t^2y^2\right)^{-\frac{1}{2}} = t^{-1}\left(x^2+y^2\right)^{-\frac{1}{2}}$$

$\therefore$ u is a homogeneous function of degree -1

By Euler's theorem , $x.\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = (-1)u = -u$

u is a homogeneous function of degree -1

Example 6.39

Verification:

$$u = \left(x^2+y^2\right)^{-\frac{1}{2}}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(x^2+y^2\right)^{-\frac{3}{2}}\cdot 2x = \frac{-x}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$x\cdot\frac{\partial u}{\partial x} = \frac{-x^2}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$\frac{\partial u}{\partial y} = -\frac{1}{2}\left(x^2+y^2\right)^{-\frac{3}{2}}\cdot 2y = \frac{-y}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$y\cdot\frac{\partial u}{\partial y} = \frac{-y^2}{\left(x^2+y^2\right)^{-\frac{3}{2}}}$$

$$\therefore \quad x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = \frac{-(x^2+y^2)}{(x^2+y^2)^{-\frac{3}{2}}}$$

$$= (-1)\frac{1}{\sqrt{x^2+y^2}} = (-1)u = -u$$

Hence Euler's theorem verified.

. In each of the following cases, determine whether the following function is homogeneous or not. If it is so, find the degree.

(i) $f(x,y) = x^2y+6x^3+7$ (ii) $h(x,y) = \dfrac{6x^2y^3-\pi y^5+9x^4y}{2020x^2+2019y^2}$

(iii) $g(x,y,z) = \dfrac{\sqrt{3x^2+5y^2+z^2}}{4x+7y}$ (iv) $U(x,y,z) = xy+\sin\left(\dfrac{y^2-2z^2}{xy}\right)$.

Solution:

(i) $f(x, y) = x^2y + 6x^3 + 7$

$f(\lambda x, \lambda y) = \lambda^3x^2y + 6\lambda^3x^3 + 7$

There is no common λ in this equation.

$\therefore$ It is not homogeneous

(ii) $$h(x, y) = \frac{6x^2y^3 - \pi y^5 + 9x^4y}{2020x^2 + 2019y^2}$$

$$h(\lambda x, \lambda y) = \frac{6\lambda^2x^2\lambda^3y^3 - \pi\lambda^5y^5 + 9\lambda^4x^4\lambda y}{2020\lambda^2x^2 + 2019\lambda^2y^2}$$

$$= \frac{\lambda^5\left(6x^2y^3 - \pi y^5 + 9x^4y\right)}{\lambda^2\left(2020x^2 + 2019y^2\right)}$$

$$= \lambda^3\, h(x, y)$$

Thus f is homogeneous with degree 3.

(iii) $$g(x, y, z) = \frac{\sqrt{3x^2 + 5y^2 + z^2}}{4x + 7y}$$

$$g(\lambda x, \lambda y, \lambda z) = \frac{\sqrt{3\lambda^2x^2 + 5\lambda^2y^2 + \lambda^2z^2}}{4\lambda x + 7\lambda y}$$

$$= \frac{\lambda\sqrt{3x^2 + 5y^2 + z^2}}{\lambda(4x + 7y)}$$

$$= \lambda^0\, g(x, y, z)$$

Thus g is homogeneous with degree 0.

(iv) $$U(x, y, z) = xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right)$$

$$U(\lambda x, \lambda y, \lambda z) = \lambda x\, \lambda y + \sin\left(\frac{\lambda^2y^2 - 2\lambda^2z^2}{\lambda x\lambda y}\right)$$

$$= \lambda^2 xy + \sin\left(\frac{\lambda^2\left(y^2 - 2z^2\right)}{\lambda^2(xy)}\right)$$

$$= \lambda^2xy + \sin\left(\frac{y^2 - 2z^2}{xy}\right)$$

There is no common λ

$\therefore$ It is not homogeneous.

Prove that $f(x, y) = x^3 - 2x^2y + 3xy^2 + y^3$ is homogeneous; what is the degree? Verify Euler's Theorem for f.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Solution:

$$f(x, y) = x^3 - 2x^2y + 3xy^2 + y^3$$

$$f(\lambda x, \lambda y) = \lambda^3x^3 - 2\lambda^2x^2\lambda y + 3\lambda x\lambda^2y^2 + \lambda^3y^3$$

$$= \lambda^3(x^3 - 2x^2y + 3xy^2 + y^3)$$

f is a homogeneous function of degree 3

By Euler's Theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$$

Verification:

$$f(x, y) = x^3 - 2x^2y + 3xy^2 + y^3$$

$$\frac{\partial f}{\partial x} = 3x^2 - 4xy + 3y^2$$

$$x\frac{\partial f}{\partial x} = 3x^3 - 4x^2y + 3xy^2$$

$$\frac{\partial f}{\partial y} = -2x^2 + 6xy + 3y^2$$

$$y\frac{\partial f}{\partial y} = -2x^2y + 6xy^2 + 3y^2$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3x^3 - 4x^2y + 3xy^2 - 2x^2y + 6xy^2 + 3y^3$$

$$= 3x^3 - 6x^2y + 9xy^2 + 3y^3$$

$$= 3(x^3 - 2x^2y + 3xy^2 + y^3)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f$$

We verified the Euler's Theorem.

3. Prove that $g(x, y) = x \log(y/x)$ is homogeneous; what is the degree? Verify Euler's Theorem for g.

Solution:

$$g(x, y) = x\log\left(\frac{y}{x}\right)$$

$$g(tx, ty) = tx\log\left(\frac{ty}{tx}\right).$$

g is a homogeneous function of degree 1.

$\therefore$ By Euler's Theorem,

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = g$$

Verification:

$$g(x, y) = x\log\left(\frac{y}{x}\right)$$

$$= x(\log y - \log x) = x\log y - x\log x$$

$$\frac{\partial g}{\partial x} = \log y - \log x - x \times \frac{1}{x}$$

$$= \log y - \log x - 1$$

$$x\frac{\partial g}{\partial x} = x\log y - x\log x - x$$

$$\frac{\partial g}{\partial y} = x \times \frac{1}{y}$$

$$y\frac{\partial g}{\partial y} = x$$

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = x\log y - x\log x - x + x$$

$$= x\log\left(\frac{y}{x}\right)$$

$$= g$$

$$x\frac{\partial g}{\partial y} + y\frac{\partial g}{\partial y} = g$$

Hence verified.

4. If $u(x, y) = \dfrac{x^2 + y^2}{\sqrt{x + y}}$, Prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u.$$

Solution:

$$u(x, y) = \frac{x^2 + y^2}{\sqrt{x+y}}$$

$$u(\lambda x, \lambda y) = \frac{\lambda^2 x^2 + \lambda^2 y^2}{\sqrt{\lambda x + \lambda y}}$$

$$= \frac{\lambda^2 \left(x^2 + y^2\right)}{\lambda^{1/2} \sqrt{x + y}}$$

$$= \frac{\lambda^{3/2} \left(x^2 + y^2\right)}{\sqrt{x + y}}$$

u is a homogeneous function of degree $\frac{3}{2}$

By Euler's Theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{3}{2} u$$

5. If v (x, y) = $\log\left(\dfrac{x^2+y^2}{x+y}\right)$, Prove that

$x\dfrac{\partial v}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$

Solution:

$$v(x, y) = \log\left(\frac{x^2+y^2}{x+y}\right)$$

Change into exponential function

$$\text{Let } e^v = \frac{x^2+y^2}{x+y} = f(x, y)$$

$$f(x, y) = \frac{\lambda^2x^2+\lambda^2y^2}{\lambda x+\lambda y}$$

$$= \frac{\lambda^2\left(x^2+y^2\right)}{\lambda\left(x+y\right)}$$

f is a homogeneous function of degree 1.

By Euler's Theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1 \times f = f$$

$$x\frac{\partial}{\partial x}e^v + y\frac{\partial}{\partial y}e^v = e^v \text{ exists.}$$

$$x\,e^v\frac{\partial f}{\partial x} + y\,e^v\frac{\partial f}{\partial y} = e^v$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{e^v}{e^v} = 1$$

Hence Proved

6. If w (x, y, z) = log $\left(\dfrac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right)$

find x $\dfrac{\partial w}{\partial x}$ + y $\dfrac{\partial w}{\partial y}$ + z $\dfrac{\partial w}{\partial z}$.

Solution:

$$w(x, y, z) = \log\left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right)$$

Convert into exponential function

$$e^w = \left(\frac{5x^3y^4 + 7y^2xz^4 - 75y^3z^4}{x^2 + y^2}\right) = f(x, y)$$

$$f(\lambda x, \lambda y) = \frac{5\lambda^3x^3\lambda^4y^4 + 7\lambda^2y^2\lambda x\lambda^4z^4 - 75\lambda^3y^3\lambda^4z^4}{\lambda^2x^2 + \lambda^2y^2}$$

$$= \frac{\lambda^7\left(5x^3y^4 + 7y^2xz^4 - 75y^3z^4\right)}{\lambda^2\left(x^2 + y^2\right)}$$

f is a homogeneous function of degree 5

By Eulers' Theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 5f$$

$$x\frac{\partial}{\partial x}e^w + y\frac{\partial}{\partial y}e^w + z\frac{\partial}{\partial z}e^w = 5e^w$$

$$e^w x\frac{\partial w}{\partial x} + e^w y\frac{\partial w}{\partial y} + e^w z\frac{\partial w}{\partial z} = 5e^w$$

Divided by e^w

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = \frac{5e^w}{e^w}$$

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} + z\frac{\partial w}{\partial z} = 5$$