# CHAPTER SIX

## RANDOM VARIABLE AND MATHEMATICAL EXPECTATION

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### Random variable with example.

A random variable is a real valued function whose values are determined with the outcomes of a random experiment. It is usually denoted by X,Y,Z etc and the value of the random variable denoted by x,y,z.

Let us consider the experiment of tossing two fair coins. The sample space of the experiment is S= {HH, HT, TH, TT}

Let X denotes the number of heads. So,

| Sample point | Number of Head (X) |
|---|---|
| HH | 2 |
| HT | 1 |
| TH | 1 |
| TT | 0 |

Here, X can take the values 0, 1 and 2. Therefore X is a random variable.

### Types of random variable.

There are two types of random variable:

1. Discrete random variable.
2. Continuous random variable.

**Discrete random variable:** A random variable is called discrete random variable if it can take only isolated values.

Example: Family members, mobile number etc.

**Continuous random variable:** A random variable is called continuous random variable if it can take any values between certain limits.

Example: Age, weight, height etc.

## Probability function and probability density function.

**Probability function:** A function f(x) of a discrete random variable X is called a probability function if it satisfies the following two conditions:

(i) $f(x) \geqslant 0$

(ii) $\sum f(x) = 1$

**Probability density function:** A function f(x) of a discrete random variable X is called a probability function if it satisfies the following two conditions:

(i) $f(x) \geqslant 0$

(ii) $\int_{\infty}^{\infty} f(x)dx = 1$

**Cumulative Distribution Function:** The **cumulative distribution function (cdf)** F(x) of a discrete random variable X with probability distribution f(x) is $F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$

**Cumulative Distribution Function:** The cumulative distribution function $F(x)$ of a continuous random variable $X$ with density function f(x) is $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt \qquad for\ -\infty < x < \infty$

The consequence of above definition

$$P(a < x < b) = \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx = F(b) - F(a) \quad and \quad f(x) = \frac{dF(x)}{dx}$$

**Application problem:** A discrete random variable x has the following probability function:

| | | | | | |
|---|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 |
| f (x) | k | 2k | 3k | 3k | k |

For what value of k the function will be a probability function?

Ans: Since, $\sum f(x) = 1$

Or, (K+2k+3k+3k+k) = 1

Or, 10k =1

Or, k = 0.10

Therefore,

| | | | | | |
|---|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 |
| f (x) | 0.10 | 0.20 | 0.30 | 0.30 | 0.10 |

**Application problem:** A discrete random variable X has the following probability function:

| | | | | | |
|---|---|---|---|---|---|
| Values of X:x | 0 | 1 | 2 | 3 | 4 |
| f (x) | 0.12 | 0.18 | k | 0.30 | 0.16 |

(i) Find the value of k, Compute (ii) P[X>3] ; (iii) P[1< X < 4] ; and (iv) P[X<1].

(i) Since, $\sum f(x) = 1$

Or, (0.76+k) = 1

Or, k =1-0.76

Or, k = 0.24

(ii) P[X>3] = P[X=4] = 0.16

(iii) P[1< X < 4]= P[X=2]+ P[X=3]

= k+0.30

= 0.24+0.30 = 0.54

(iv) P[X<1] = P[X=0]

= 0.12

**Assignment problem:** A discrete random variable X has the following probability function:

| | | | | | |
|---|---|---|---|---|---|
| Values of X:x | 0 | 1 | 2 | 3 | 4 |
| f (x) | 0.10 | 0.30 | 0.20 | 0.25 | 0.15 |

Find the value of (i) P[X=1]; (ii) P[X>3] ; (iii) P[1< X < 4] ; and (iv) P[X<1].

**Assignment problem:** Suppose x is discrete random variable with probability function.

| | | | | | |
|---|---|---|---|---|---|
| Values of X:x | -2 | -1 | 0 | 1 | 2 |
| f (x) | 0.3 | 0.2 | 0.1 | 0.25 | 0.15 |

Find the value of (i) P[ X = 1]; (ii) P[-1< X <2]; (iii) P[ X >0] and P[ X<-1]

**Application problem:** A continuous random variable X has the following probability density function:

$$f(x) = kx^2 \ ; \ 0 \le x \le 1$$

(i) Find the value of K
(ii) probability that X lies between 0.2 and 0.50
(iii) probability that X less than 0.30 and
(iv) probability that X greater than 0.75

Ans:(i) Since, $\int_{\infty}^{\infty} f(x)dx = 1$

**Or,** $\int_{0} kx^2 dx = 1$

Or, $k[\ \frac{x^3}{3}\ ]_0^1 = 1$

Or, $\frac{k}{3} = 1$

Or, $k = 3$

(ii) probability that X lies between 0.2 and 0.50

P[0.20< X < 0.50] = $\int_{0.20}^{0.50} kx^2 dx$

$= k[\ \frac{x^3}{3}\ ]_{0.20}^{0.50}$

$= 3[\frac{1}{3}(0.50^3 - 0.20^3)]$

= 0.125-0.008 =0.117

(iii) probability that X less than 0.30

P[X<0.3] = $\int_{0}^{0.3} kx^2 dx$

$= k[\ \frac{x^3}{3}\ ]_0^{0.30} = 3[\frac{1}{3}(0.30^3 - 0^3)] = 0.027$

(iv) probability that X greater than 0.75 is P[X > 0.75] = $\int_{0.75} kx^2 dx$

$$= k[\frac{x^3}{3}]_{00.75}^{1} = 3[\frac{1}{3}(0.30^3 - 0^3)] = 0.578$$

**Assignment problem:** The following is the probability density function of a random variable x:

$$f(x) = \frac{3}{4}(2x - x^2) ; 0 < x < 2$$

Find (i) the value of 'K' ; (ii) P(x>1) and (iii) P( 1.5 < x < 2.25)

**Assignment problem:** A continuous random variable X has the following probability density function:

$$f(x) = K(x-1); \qquad 2 \le x \le 6$$

Compute (i) the value of 'K' ; (ii) P(X >3) and (iii) P(3 < X < 4)

**Assignment problem:** A continuous random variable X has the following probability density function:

$$f(x) = K(x+1); \qquad 2 \le x \le 5$$

Compute (i) the value of 'K' ; (ii) P(X >3); (iii) P(X = 4) and (iv) P(3 < X < 4)

**Assignment problem:** Let X be a continues random variable with probability density function

f(x) = kx ; $0 \le x \leqslant 4$ Find (i) the value of k; (ii) P($x \ge 1$); (iii) P($x \le 2$)

# MATHEMATICAL EXPECTATION

If X is a discrete or continuous random variable with probability function or probability density function f(x).then the mathematical expectation of X is usually denoted by E[X] or μ and defined by

$$\mu = E[X] \;=\; \sum xf(x) \text{ ; If X is a discrete random variable.}$$

$$= \int_{\infty}^{\infty} xf(x)dx \text{ ; If X is a continuous random variable.}$$

**Properties of mathematical expectation of a random variable:**

(i) If b is a constant then E[b] = b
(ii) If X is a random variable with expectation E[X], then E[aX+b] = a E[X] + b, Where a and b constant.
(iii) If X is a random variable with expectation E[X], then E[X-E(X)] = 0
(iv) If X and Y are random variables then E[X+Y] = E[X] + E[Y]
(v) If X and Y are random variables then E[X-Y] = E[X] - E[Y]

**Properties of mathematical expectation of a random variable:**

(i) If b is a constant then V[b] = 0
(ii) If X is a random variable, then V [aX+b] = $a^2$ V[X] , Where a and b constant.
(iii) If X is a random variable with expectation E[X], then
$V(X) = E[X-E(X)]^2 = E(X^2) - [E(X)]^2$
(iv) If X and Y are random variables then V[X+Y] = V[X] + V[Y]
(v) If X and Y are random variables then V[X-Y] = V[X] + V[Y]

**Application problem:** Find mean variance and standard deviation of the following probability function:

Or, Find (i) E[X] (ii) V(X) and (iii) standard deviation

| Values of X:x | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| f(x) | 0.125 | 0.375 | 0.375 | 0.125 |

Solution: (i) **Mean:**

We know, $E[X] = \sum x f(x)$

$= (0)(0.125)+(1)(0.375)+(2)(0.375)+(3)(0.125)$

$= 1.5$

**(ii) Variance:**

We know, $V(X) = E(X^2) - [E(X)]^2$

Here, $E(X^2) = \sum x^2 f(x)$

$= (0)^2(0.125) + (1)^2(0.375) + (2)^2(0.375) + (3)^2(0.125)$

$= 3$

$V(X) = E(X^2) - [E(X)]^2$

$= 3-(1.5)^2 = 0.75$

**(iii) Standard deviation:** $\sqrt{V(X)} = \sqrt{0.75} = 0.87$

**Assignment problem:** Suppose x is discrete random variable with probability function.

| | | | | | |
|---|---|---|---|---|---|
| Values of X:x | -2 | -1 | 0 | 1 | 2 |
| f (x) | 0.3 | 0.2 | 0.1 | 0.25 | 0.15 |

Compute (i) E[x] ; (ii) E[3x+3] ; (iii) V[x] and (iv) V[2x-3]

**Assignment problem:** Let X be a continues random variable with probability density function

$f(x) = kx$ ; $0 \le x \leqslant 4$ Find (i) the value of k; (ii) $P(x \ge 1)$; (iii) $P(x \le 2)$

(iv) Mean, variance and standard deviation of x