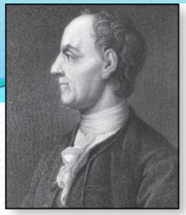
Euler and Hamiltonian Graphs

Section 10.5

Based on **Discrete Mathematics and Its Applications**, 7^{th} ed., by Kenneth H. Rosen, published by McGraw Hill, Boston, MA, 2011.

Section Summary

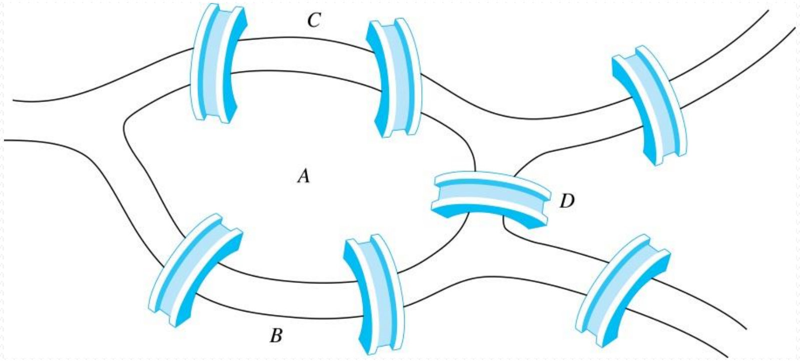
- Euler Paths and Circuits
- Hamilton Paths and Circuits
- Applications of Hamilton Circuits



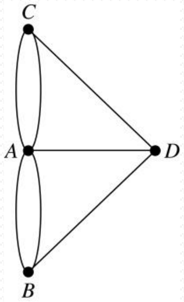
Euler Paths and Circuits

Leonard Euler (1707-1783)

- The town of Königsberg, Prussia (now Kalinigrad, Russia) was divided into four sections by the branches of the Pregel river. In the 18th century seven bridges connected these regions.
- People wondered whether it was possible to follow a path that crosses each bridge exactly once and returns to the starting point.
- The Swiss mathematician Leonard Euler proved that no such path exists. This result is often considered to be the first theorem ever proved in graph theory.



The 7 Bridges of Königsberg



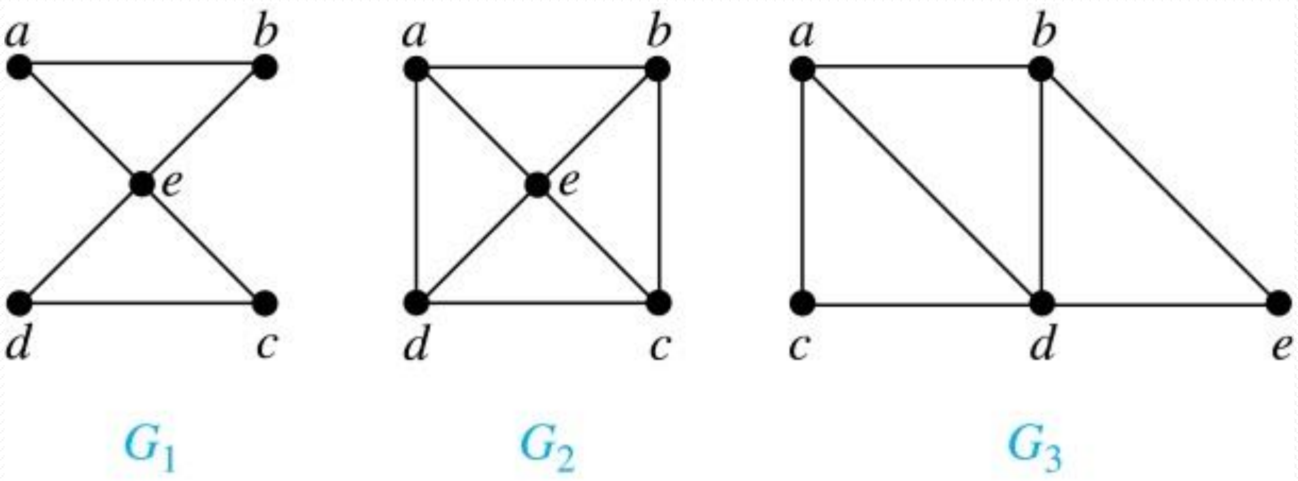
Multigraph Model of the Bridges of Königsberg

Euler Paths and Circuits (*continued*)

Definition: An *Euler circuit* in a graph G is a simple circuit containing every edge of G. An *Euler path* in G is a simple path containing every edge of G.

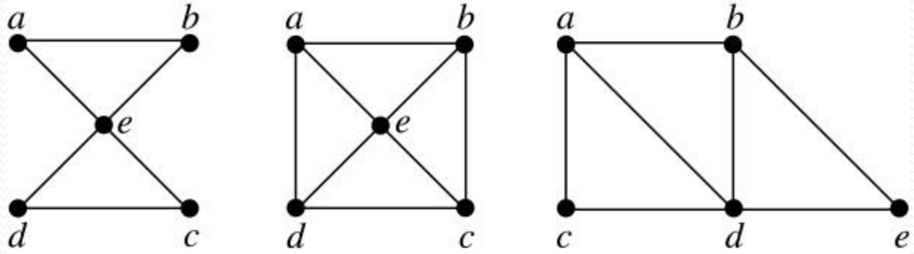
Euler Paths and Circuits (*continued*)

Example-1: Which of the undirected graphs G_1, G_2, and G_3 has a Euler circuit? Of those that do not, which has an Euler path?



Euler Paths and Circuits (*continued*)

Example-1: Which of the undirected graphs G_1, G_2, and G_3 has a Euler circuit? Of those that do not, which has an Euler path?

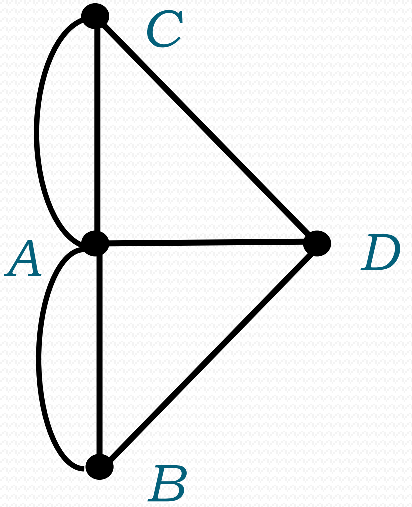


G_1 G_2 G_3

Solution: The graph G_1 has an Euler circuit (e.g., *a*, *e*, *c*, *d*, *e*, *b*, *a*). But, as can easily be verified by inspection, neither G_2 nor G_3 has an Euler circuit. Note that G_3 has an Euler path (e.g., *a*, *c*, *d*, *e*, *b*, *d*, *a*, *b*), but there is no Euler path in G_2, which can be verified by inspection.

Necessary and Sufficient Conditions

- A connected multigraph has an Euler circuit iff *each of its vertices has an even degree*.
- A connected multigraph has an Euler path but not an Euler circuit iff *it has exactly two vertices of odd degree*.



Does this graph have an Euler circuit?

No.

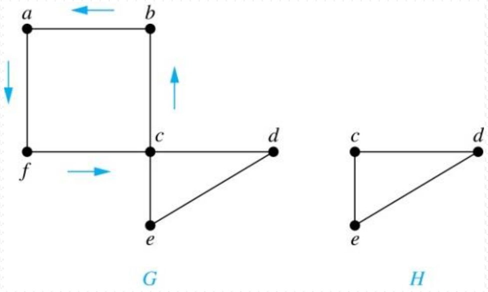
Necessary Conditions for Euler Circuits and Paths

- An Euler circuit begins with a vertex a and continues with an edge incident with a, say $\{a, b\}$. The edge $\{a, b\}$ contributes one to $\deg(a)$.
- Each time the circuit passes through a vertex it contributes two to the vertex's degree.
- Finally, the circuit terminates where it started, contributing one to $\deg(a)$. Therefore $\deg(a)$ must be even.
- We conclude that the degree of every other vertex must also be even.
- By the same reasoning, we see that the initial vertex and the final vertex of an Euler path have odd degree, while every other vertex has even degree. So, a graph with an Euler path has exactly two vertices of odd degree.
- In the next slide we will show that these necessary conditions are also sufficient conditions.

Sufficient Conditions for Euler Circuits and Paths

Suppose that G is a connected multigraph with ≥ 2 vertices, all of even degree. Let $x_0 = a$ be a vertex of even degree. Choose an edge $\{x_0, x_1\}$ incident with a and proceed to build a simple path $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$ by adding edges one by one until another edge can not be added.

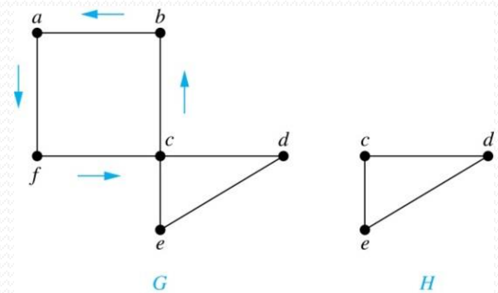
We illustrate this idea in the graph G here. We begin at a and choose the edges $\{a, f\}, \{f, c\}, \{c, b\}$, and $\{b, a\}$ in succession.



- The path begins at a with an edge of the form $\{a, x\}$; we show that it must terminate at a with an edge of the form $\{y, a\}$. Since each vertex has an even degree, there must be an even number of edges incident with this vertex. Hence, every time we enter a vertex other than a, we can leave it. Therefore, the path can only end at a.
- If all of the edges have been used, an Euler circuit has been constructed. Otherwise, consider the subgraph H obtained from G by deleting the edges already used.

In the example H consists of the vertices c, d, e.

Sufficient Conditions for Euler Circuits and Paths (*continued*)



- Because G is connected, H must have at least one vertex in common with the circuit that has been deleted.

> In the example, the vertex is *c*.

- Every vertex in H must have even degree because all the vertices in *G* have even degree and for each vertex, pairs of edges incident with this vertex have been deleted. Beginning with the shared vertex construct a path ending in the same vertex (as was done before). Then splice this new circuit into the original circuit.

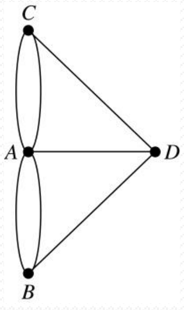
> In the example, we end up with the circuit *a*, *f*, *c*, *d*, *e*, *c*, *b*, *a*.

- Continue this process until all edges have been used. This produces an Euler circuit. Since every edge is included and no edge is included more than once.
- Similar reasoning can be used to show that a graph with exactly two vertices of odd degree must have an Euler path connecting these two vertices of odd degree

Necessary and Sufficient Conditions for Euler Circuits and Paths (*continued*)

Theorem: A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has an even degree and it has an Euler path if and only if it has exactly two vertices of odd degree.

Example: Two of the vertices in the multigraph model of the Königsberg bridge problem have odd degree. Hence, there is no Euler circuit in this multigraph and it is impossible to start at a given point, cross each bridge exactly once, and return to the starting point.



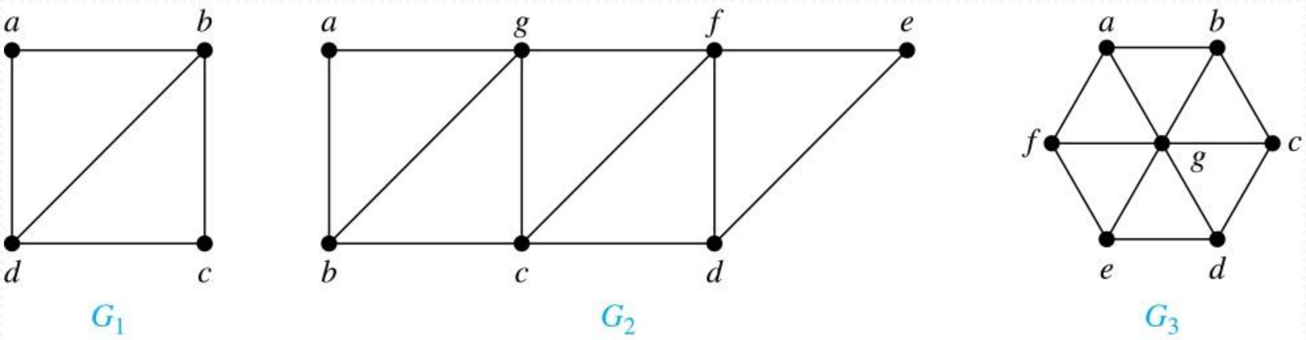
Algorithm for Constructing an Euler Circuits

In our proof we developed this algorithms for constructing a Euler circuit in a graph with no vertices of odd degree.

procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex.
H := G with the edges of this circuit removed
while H has edges
subciruit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge in circuit.
H := H with edges of *subciruit* and all isolated vertices removed
circuit := *circuit* with *subcircuit* inserted at the appropriate vertex.
return *circuit*{*circuit* is an Euler circuit}

Euler Circuits and Paths

Example:



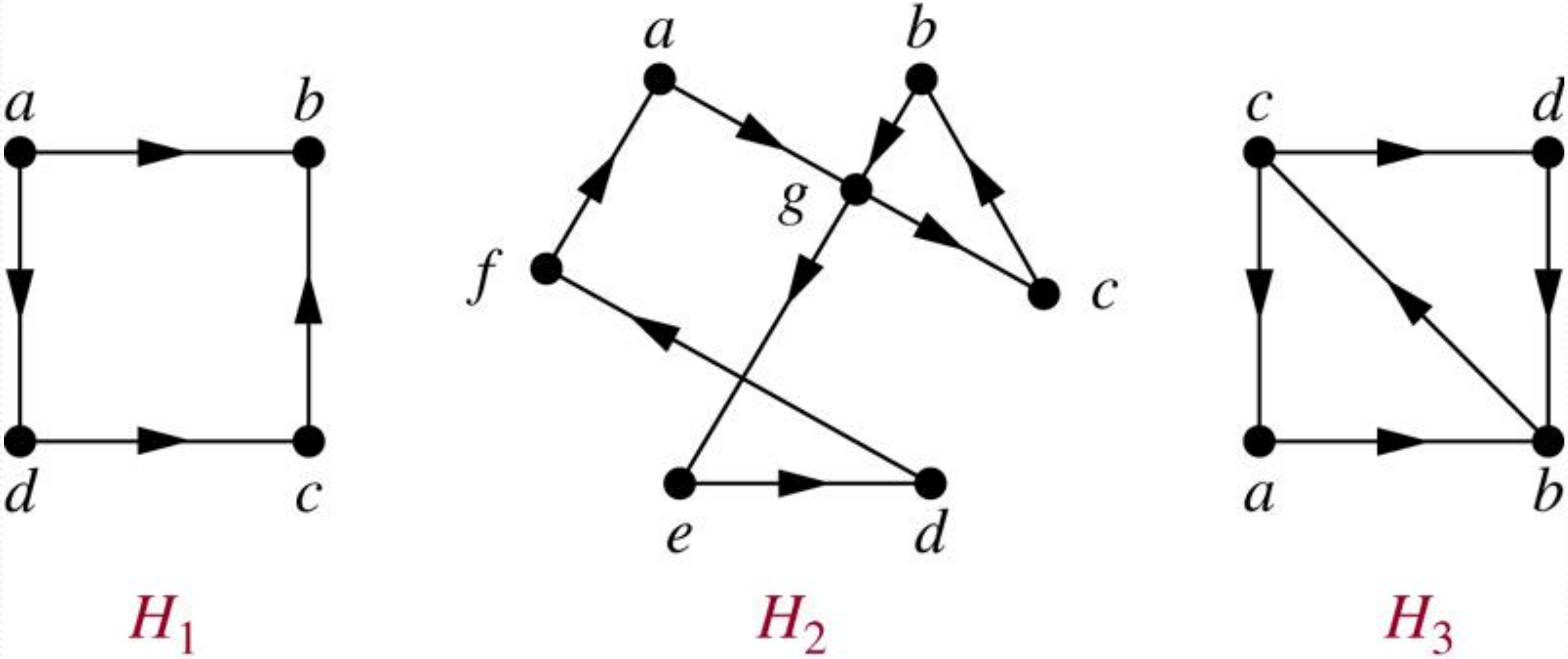
G_1 contains exactly two vertices of odd degree (*b* and *d*). Hence it has an Euler path, e.g., *d*, *a*, *b*, *c*, *d*, *b*.

G_2 has exactly two vertices of odd degree (*b* and *d*). Hence it has an Euler path, e.g., *b*, *a*, *g*, *f*, *e*, *d*, *c*, *g*, *b*, *c*, *f*, *d*.

G_3 has six vertices of odd degree. Hence, it does not have an Euler path.

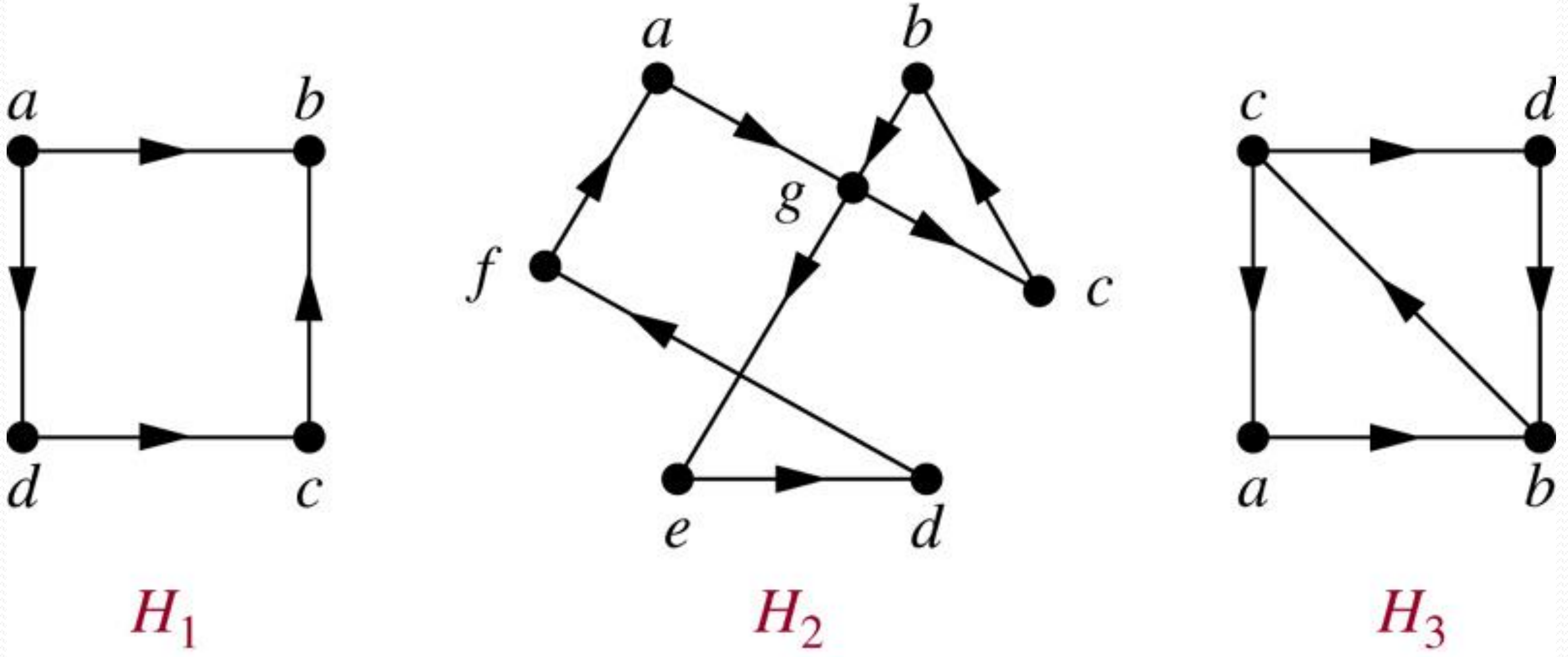
Define Euler Circuits and Paths in Directed Graphs

Euler Circuit in Directed Graphs



NO (a, g, c, b, g, e, d, f, a) NO

Euler Path in Directed Graphs

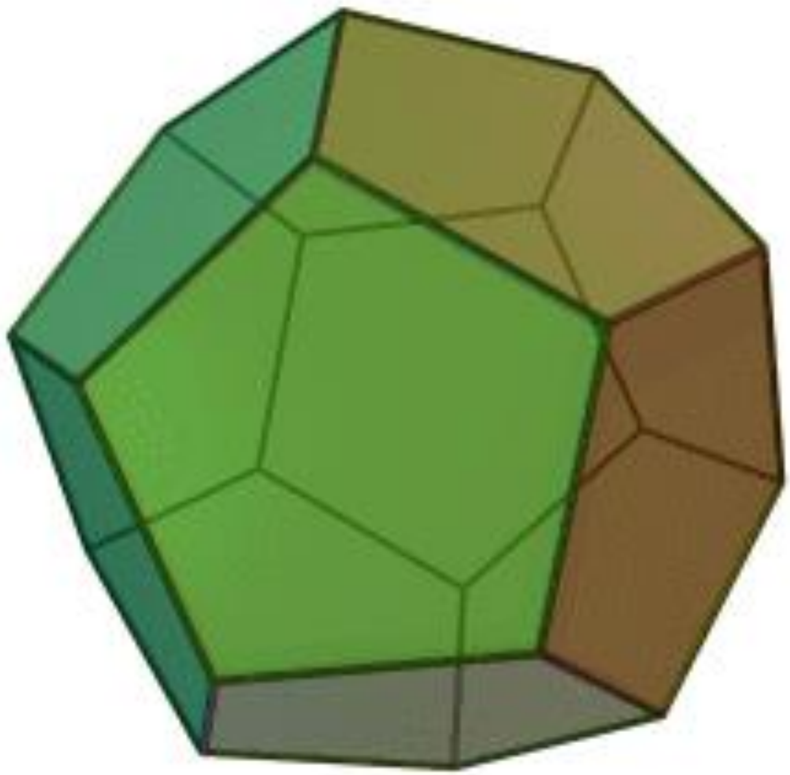


NO (a, g, c, b, g, e, d, f, a) (c, a, b, c, d, b)

Applications of Euler Paths and Circuits

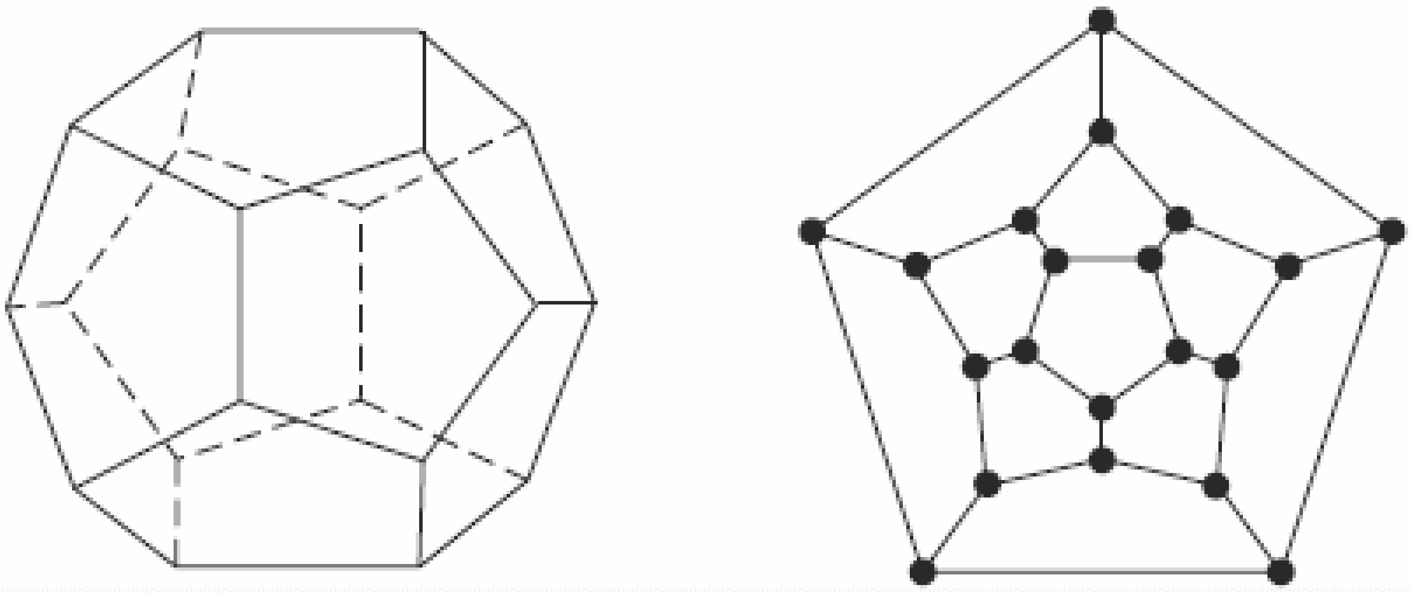
- Euler paths and circuits can be used to solve many practical problems such as finding a path or circuit that traverses each
  - street in a neighborhood,
  - road in a transportation network,
  - connection in a utility grid,
  - link in a communications network.
- Other applications are found in the
  - layout of circuits,
  - network multicasting,
  - molecular biology, where Euler paths are used in the sequencing of DNA.

Hamilton Paths and Circuits



Dodecahedron is a polyhedron with twelve flat faces

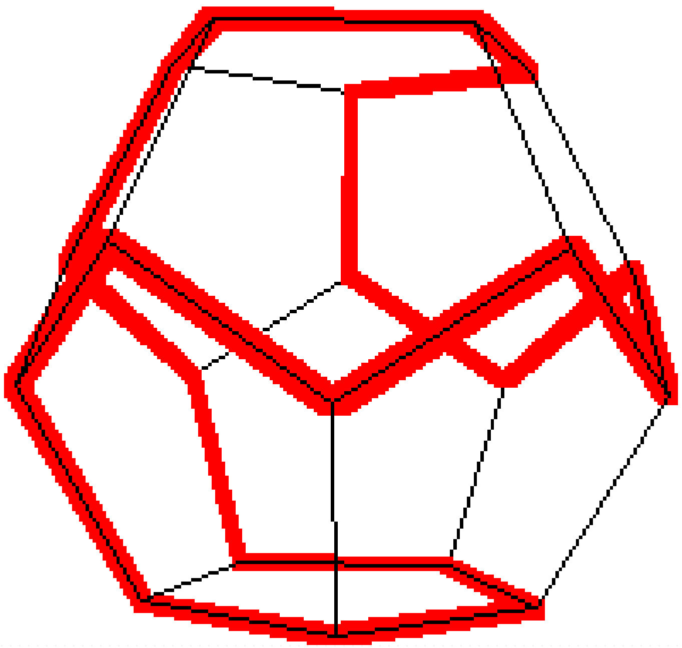
Hamilton Circuits

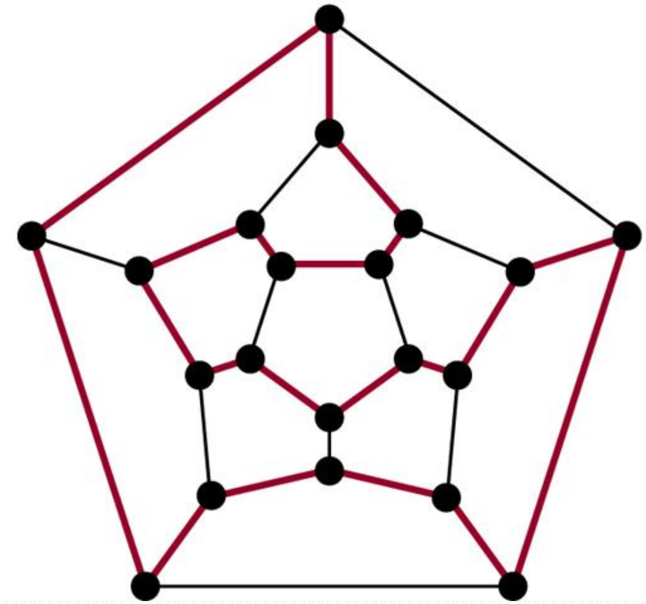


Dodecahedron puzzle and it equivalent graph

Is there a circuit in this graph that passes through each vertex exactly once?

Hamilton Circuits



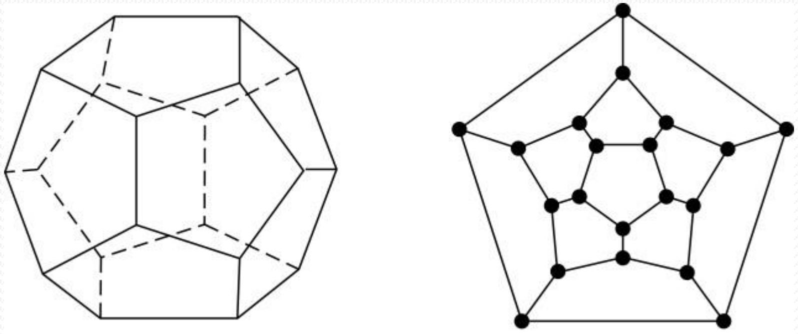


Yes; this is a circuit that passes through each vertex exactly once.

Hamilton Paths and Circuits

William Rowan Hamilton (1805- 1865)

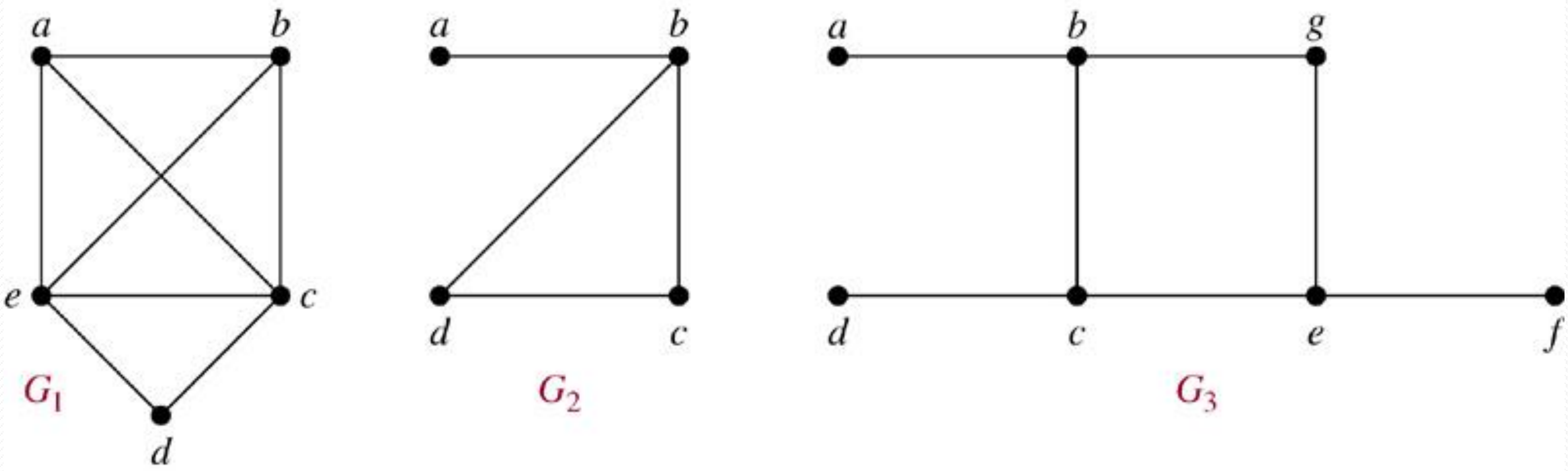
- Euler paths and circuits contained every edge only once. Now we look at paths and circuits that contain every vertex exactly once.
- William Hamilton invented the *Icosian puzzle* in 1857. It consisted of a wooden dodecahedron (with 12 regular pentagons as faces), illustrated in (a), with a peg at each vertex, labeled with the names of different cities. String was used to used to plot a circuit visiting 20 cities exactly once
- The graph form of the puzzle is given in (b).



(a) (b)

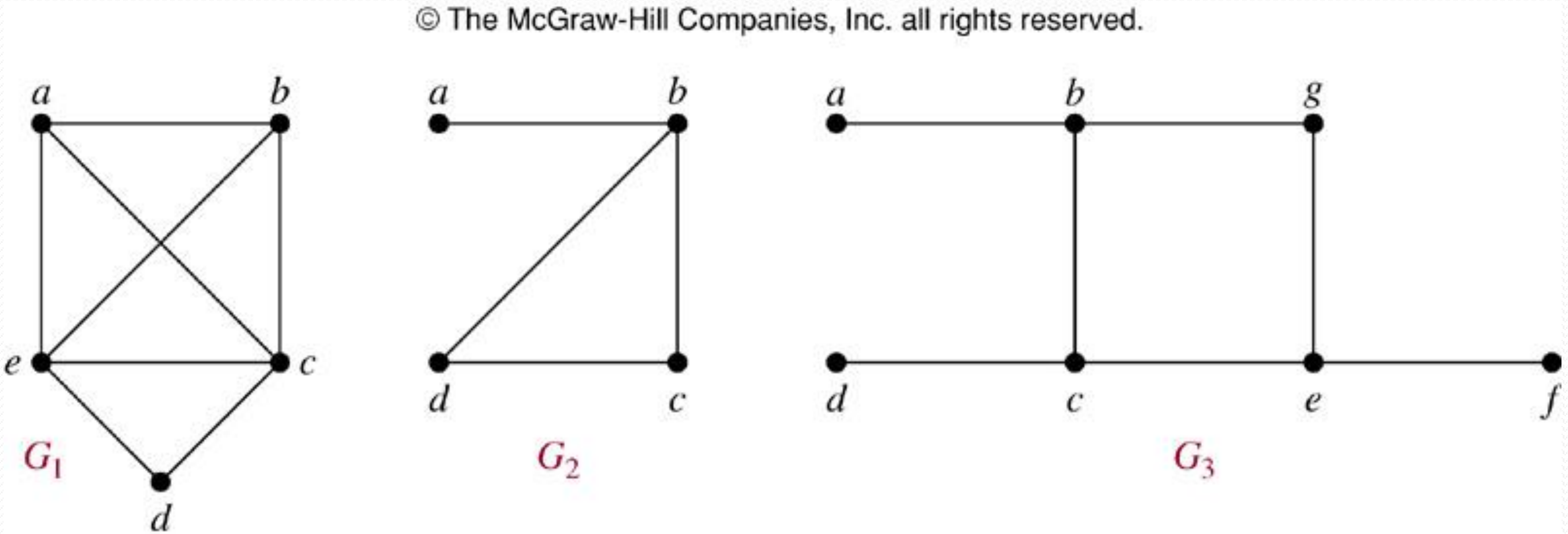
Definition: A simple path in a graph G that passes through every vertex exactly once is called a *Hamilton path*, and a simple circuit in a graph G that passes through every vertex exactly once is called a *Hamilton circuit*.

Finding Hamilton Circuits



Which of these three figures has a Hamilton circuit?
Or, if no Hamilton circuit, a Hamilton path?

Finding Hamilton Circuits



- G_1 has a Hamilton circuit: a, b, c, d, e, a
- G_2 does not have a Hamilton circuit, but does have a Hamilton path: a, b, c, d
- G_3 has neither.

Euler versus Hamilton

Property	Euler	Hamilton
Repeated visits to a given node allowed?	Yes	No
Repeated traversals of a given edge allowed?	No	No
Omitted nodes allowed?	No	No
Omitted edges allowed?	No	Yes

Necessary Conditions for Hamilton Circuits

Gabriel Andrew Dirac
(1925-1984)

- Unlike for an Euler circuit, no simple necessary and sufficient conditions are known for the existence of a Hamiton circuit.
- However, there are some useful necessary conditions. We describe two of these now.

Dirac's Theorem: If G is a simple graph with $n \geq 3$ vertices such that the degree of every vertex in G is $\geq n/2$, then G has a Hamilton circuit.

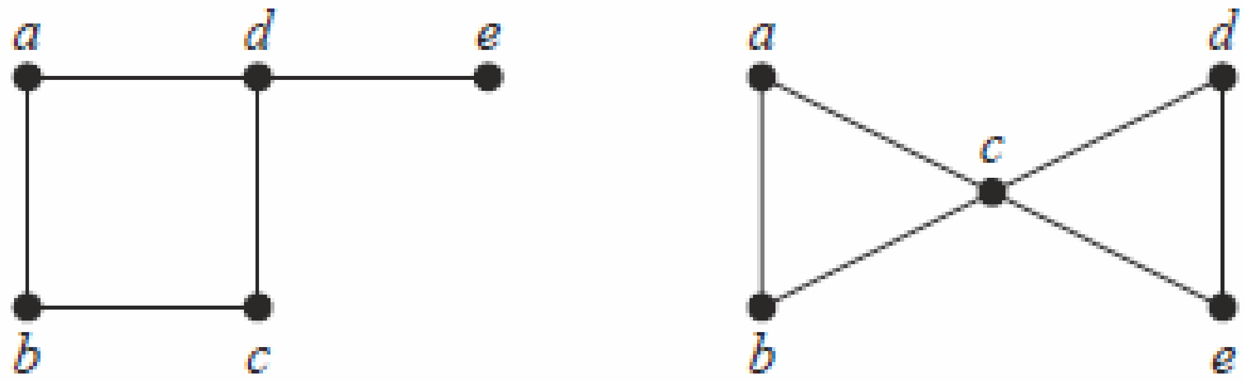
Ore's Theorem: If G is a simple graph with $n \geq 3$ vertices such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices, then G has a Hamilton circuit.

Øysten Ore
(1899-1968)

Properties to look for ...

- No vertex of degree 1
- No cut edges
- If a node has degree 2, then both edges incident to it must be in any Hamilton circuit.
- No smaller circuits contained in any Hamilton circuit (the start/endpoint of any smaller circuit would have to be visited twice).

Show that neither graph displayed below has a Hamilton circuit.



G H

There is no Hamilton circuit in G because G has a vertex of degree one: e.

Now consider *H. Because the degrees of the vertices a, b, d, and e are all two, every edge* incident with these vertices must be part of any Hamilton circuit. No Hamilton circuit can exist in *H,* for any Hamilton circuit would have to contain four edges incident with *c,* which is impossible.

Time Complexity

The best algorithms known for finding a Hamilton circuit in a graph or determining that no such circuit exists have exponential worst-case time complexity (in the number of vertices of the graph).
Finding an algorithm that solves this problem with polynomial worst-case time complexity would be a major accomplishment because it has been shown that **this problem is NP-complete**. Consequently, the existence of such an algorithm would imply that many other seemingly intractable problems could be solved using algorithms with polynomial worst-case time complexity.

Applications of Hamilton Paths and Circuits

- Applications that ask for a path or a circuit that visits each intersection of a city, each place pipelines intersect in a utility grid, or each node in a communications network exactly once, can be solved by finding a Hamilton path in the appropriate graph.
- The famous *traveling salesperson problem* (*TSP*) asks for the shortest route a traveling salesperson should take to visit a set of cities. This problem reduces to finding a Hamilton circuit such that the total sum of the weights of its edges is as small as possible.
- A family of binary codes, known as *Gray codes*, which minimize the effect of transmission errors, correspond to Hamilton circuits in the n-cube Q_n. (*See the text for details*.)