Relations

Chapter 9

Chapter Summary

- Relations and Their Properties
- *n*-ary Relations and Their Applications (*not currently included in overheads*)
- Representing Relations
- Closures of Relations (*not currently included in overheads*)
- Equivalence Relations
- Partial Orderings

Relations and Their Properties

Section 9.1

Section Summary

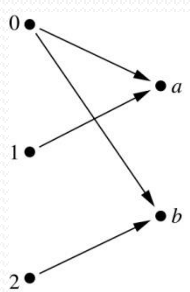
- Relations and Functions
- Properties of Relations
  - Reflexive Relations
  - Symmetric and Antisymmetric Relations
  - Transitive Relations
- Combining Relations

Binary Relations

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B.
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A.

Domain & Range of a relation

- The *domain* of relation R is the set of all first elements of the ordered pairs, which belongs to R. The range of relation R is the set of all second elements of the ordered pairs, which belongs to R.
- Example: Let A = {1,2,3} , B ={x,y,z} and R = {(1,y), (1,z), (3,y)}. Then R is a relation from A to B since R is a subset of $A \times B$.
  - Domain of R is {1,3}
  - Range of R is {y,z}

Inverse Relation

- Let R be any function from A to B. The invert of R, denoted by R^{-1}, is the relation from B to A which consist of those ordered pairs which, when reversed, belong to R; that is

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

- Example: The inverse of the relation $R = \{(1,y), (1,z), (3,y)\}$ from $A = \{1,2,3\}$ to $B =\{x,y,z\}$

follows $R^{-1} = \{(y,1), (z,1), (y,3)\}$

Binary Relation on a Set

Definition: A binary relation R *on a set* A is a subset of $A \times A$ or a relation from A to A.

Example:

- Suppose that $A = \{a,b,c\}$. Then $R = \{(a,a),(a,b), (a,c)\}$ is a relation on A.
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation R $= \{(a,b) \mid a \text{ divides } b\}$ are

  (1,1), (1, 2), (1,3), (1, 4), (2, 2), (2, 4), (3, 3), and (4, 4).

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A?

Solution: Because a relation on A is the same thing as a subset of $A \times A$, we count the subsets of $A \times A$. Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$. Therefore, there are $2^{|A|^2}$ relations on a set A.

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_2 = \{(a,b) \mid a > b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$,
$R_5 = \{(a,b) \mid a = b + 1\}$,
$R_6 = \{(a,b) \mid a + b \leq 3\}$.

Note that these relations are on an infinite set and each of these relations is an infinite set.

Which of these relations contain each of the pairs

$(1,1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution: Checking the conditions that define each relation, we see that the pair $(1,1)$ is in R_1, R_3, R_4, and R_6: $(1,2)$ is in R_1 and R_6: $(2,1)$ is in R_2, R_5, and R_6: $(1, -1)$ is in R_2, R_3, and R_6 : $(2,2)$ is in R_1, R_3, and R_4.

Reflexive Relations

Definition: R is *reflexive* iff $(a,a) \in R$ for every element $a \in$ A. Written symbolically, R is reflexive if and only if

$$\forall x[x \in U \longrightarrow (x,x) \in R]$$

Example: The following relations on the integers are reflexive:

$R_1 = \{(a,b) \mid a \leq b\}$,
$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,
$R_4 = \{(a,b) \mid a = b\}$.

The following relations are not reflexive:

$R_2 = \{(a,b) \mid a > b\}$ (note that $3 \ngtr 3$),
$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $3 \neq 3 + 1$),
$R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that $4 + 4 \nleq 3$).

If $A = \emptyset$ then the empty relation is reflexive vacuously. That is the empty relation on an empty set is reflexive!

Symmetric Relations

Definition: R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$. Written symbolically, R is symmetric if and only if

$$\forall x \forall y\, [(x,y) \in R \longrightarrow (y,x) \in R]$$

Example: The following relations on the integers are symmetric:

$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a,b) \mid a = b\}$,

$R_6 = \{(a,b) \mid a + b \leq 3\}$.

The following are not symmetric:

$R_1 = \{(a,b) \mid a \leq b\}$ (note that $3 \leq 4$, but $4 \nleq 3$),

$R_2 = \{(a,b) \mid a > b\}$ (note that $4 > 3$, but $3 \ngtr 4$),

$R_5 = \{(a,b) \mid a = b + 1\}$ (note that $4 = 3 + 1$, but $3 \neq 4 + 1$).

Antisymmetric Relations

Definition:A relation R on a set A such that for all $a,b \in A$ if $(a,b) \in R$ and $(b,a) \in R$, then $a = b$ is called *antisymmetric*.
Written symbolically, R is antisymmetric if and only if

$\forall x \forall y\, [(x,y) \in R \wedge (y,x) \in R \longrightarrow x = y]$

- **Example**: The following relations on the integers are antisymmetric:

  $R_1 = \{(a,b) \mid a \leq b\}$,
  $R_2 = \{(a,b) \mid a > b\}$,
  $R_4 = \{(a,b) \mid a = b\}$,
  $R_5 = \{(a,b) \mid a = b + 1\}$.

  For any integer, if a $a \leq b$ and $a \leq b$, then $a = b$.

  The following relations are not antisymmetric:
  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
  (note that both $(1,-1)$ and $(-1,1)$ belong to R_3),
  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

Symmetric &Antisymmetric Relations

- The term symmetric and antisymmetric is not opposite.
- For example, $R = \{(1,3), (3,1), (2,3)\}$ is neither symmetric not antisymmetric.
- On the other hand, $R_1 = \{(1,1), (2,2)\}$ is both symmetric and antisymmetric

Transitive Relations

Definition: A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$. Written symbolically, R is transitive if and only if

$$\forall x \forall y \forall z[(x,y) \in R \land (y,z) \in R \longrightarrow (x,z) \in R]$$

- **Example**: The following relations on the integers are transitive:

  $R_1 = \{(a,b) \mid a \leq b\}$,

  $R_2 = \{(a,b) \mid a > b\}$,

  $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$,

  $R_4 = \{(a,b) \mid a = b\}$.

  For every integer, $a \leq b$ and $b \leq c$, then $b \leq c$.

  The following are not transitive:

  $R_5 = \{(a,b) \mid a = b + 1\}$ (note that both (3,2) and (4,3) belong to R_5, but not (3,3)),

  $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both (2,1) and (1,2) belong to R_6, but not (2,2)).

Combining Relations

- Given two relations R_1 and R_2, we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- **Example**: Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1),(2,2),(3,3)\}$ and $R_2 = \{(1,1),(1,2),(1,3),(1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2),(3,3)\}$$

$$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$$

Composition

Definition: Suppose

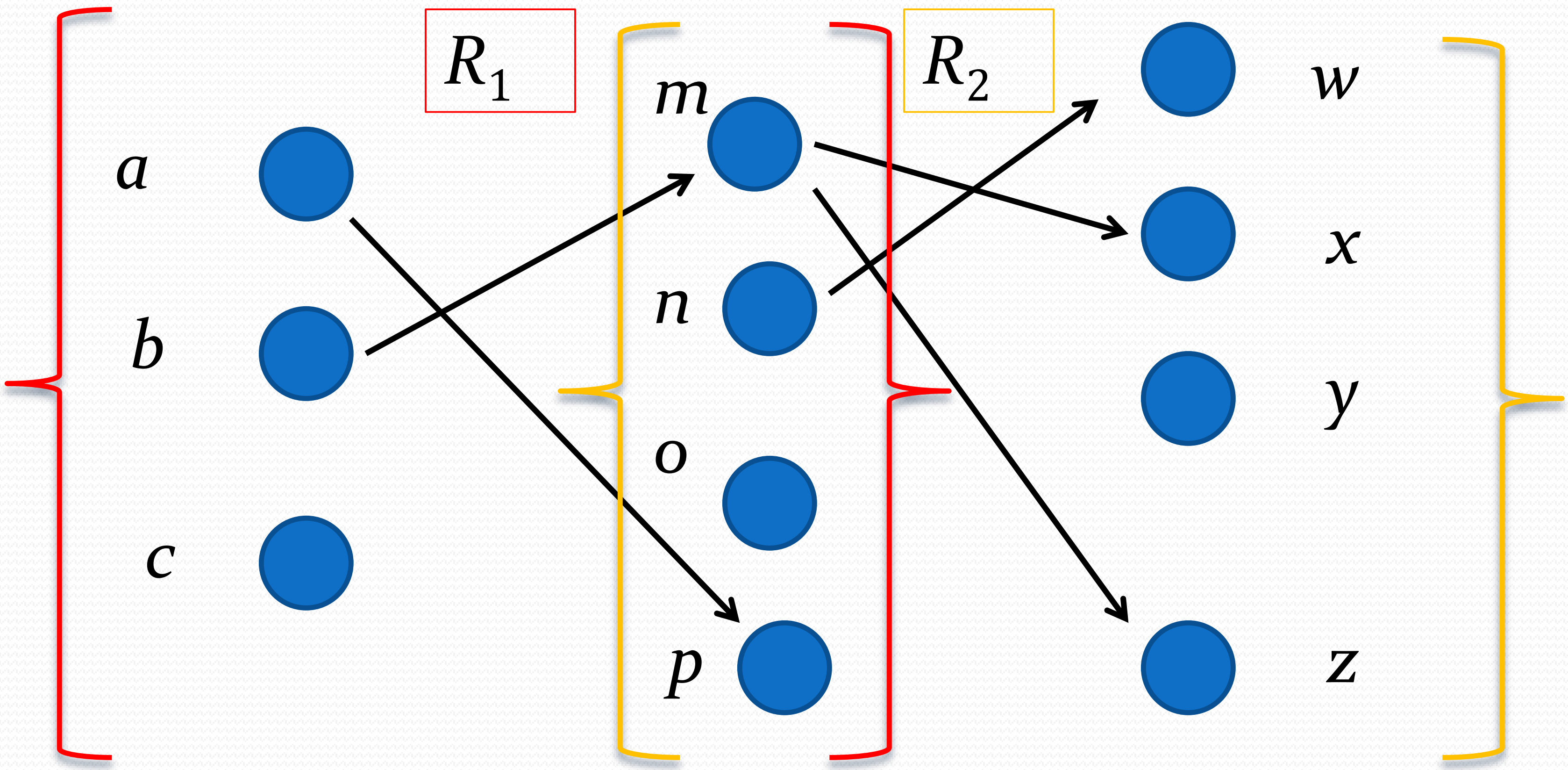
- R_1 is a relation from a set A to a set B.
- R_2 is a relation from B to a set C.

Then the *composition* (or *composite*) of R_2 with R_1, is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2, then (x,z) is a member of $R_2 \circ R_1$.

- Example: Let A ={1, 2, 3, 4}, B= {a, b, c, d} and C = {x, y, z} and let $R_{1\,=}$ ={(1,a),(2,d),(3,a) ,(3,b)} and $R_{2\,=}$ ={(b,x),(b,z),(c,y) ,(d,z)}. Find $R_2 \circ R_1$.
- Solution: $R_2 \circ R_1$= {(2,z),(3,x),(3,z)}

Representing the Composition of a Relation



$$R_1 \circ R_2 = \{(b,D),(b,B)\}$$

Powers of a Relation

Definition: Let R be a binary relation on A. Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

(*see the slides for Section* 9.3 *for further insights*)

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem 1: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1,2,3 \ldots$.

(*see the text for a proof via mathematical induction*)

Powers of a Relation (*Continue*)

- Example: Let R = {(1,1), (2,1), (3,2), (4,3)}. Find R^n, n= 2,3,…
- Solution:
- $R^2 = R \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$
- $R^3 = R^2 \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$
- $R^4 = R^3 \circ R = \{(1,1), (2,1), (3,1), (4,2)\}$

n-ARY Relation

- Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subsets of $A_1 \times A_2 \times \ldots \times A_n$. These sets $A_1, A_2, \ldots, A_n$ are called the domain of the relation and n is called its degree.
- Example: Let R be the relation consisting of the triples (a, b, c) where a, b and c are integers with $a<b<c$. Then $(1,2,3) \in R$ but $(2,4,3) \notin R$. The degree of relation is 3. Its domains are equal to the set of integers.

Representing Relations

Section 9.3

Section Summary

- Representing Relations using Matrices
- Representing Relations using Digraphs

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- Suppose R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
  - The elements of the two sets can be listed in any particular arbitrary order. When $A = B$, we use the same ordering.
- The relation R is represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j.

Examples of Representing Relations Using Matrices

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1),(3,2)\}$, the matrix is

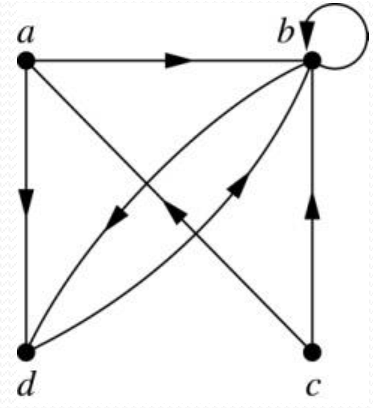
$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Representing Relations Using Digraphs

Definition: A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b), and the vertex b is called the *terminal vertex* of this edge.

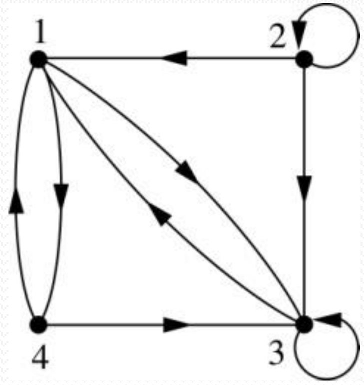
- An edge of the form (a,a) is called a *loop*.

Example 7: A drawing of the directed graph with vertices a, b, c, and d, and edges (a, b), (a, d), (b, b), (b, d), $(c,$ a$)$, (c, b), and (d, b) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?



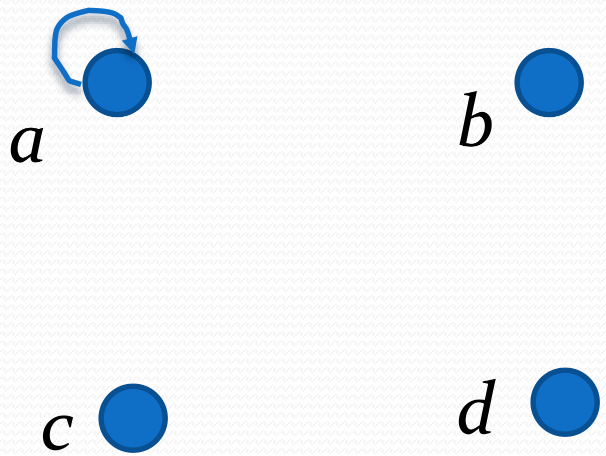
Solution: The ordered pairs in the relation are

(1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), and (4, 3)

Determining which Properties a Relation has from its Digraph

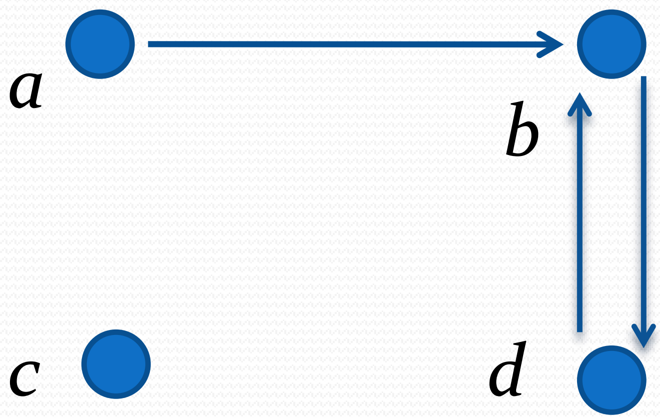
- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x).
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z).

Determining which Properties a Relation has from its Digraph – Example 1



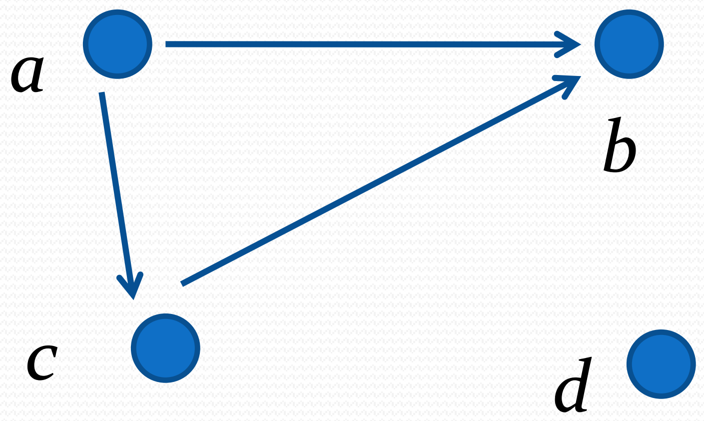
- *Reflexive?* No, not every vertex has a loop
- *Symmetric?* Yes (trivially), there is no edge from one vertex to another
- *Antisymmetric?* Yes (trivially), there is no edge from one vertex to another
- *Transitive?* Yes, (trivially) since there is no edge from one vertex to another

Determining which Properties a Relation has from its Digraph – Example 2



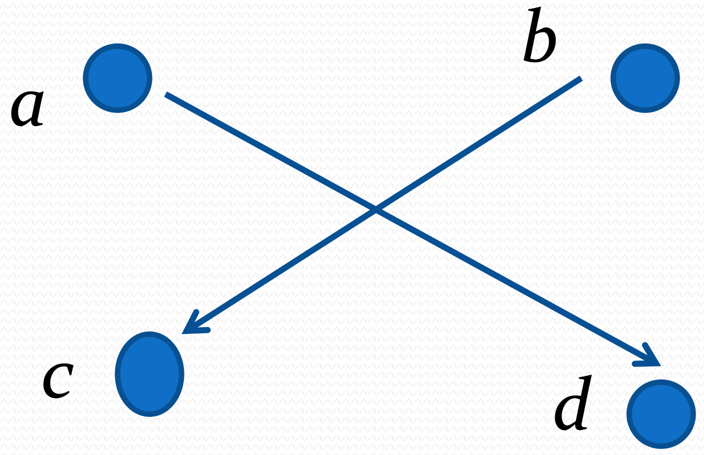
- *Reflexive?* No, there are no loops
- *Symmetric?* No, there is an edge from *a* to *b*, but not from *b* to *a*
- *Antisymmetric?* No, there is an edge from *d* to *b* and *b* to *d*
- *Transitive?* No, there are edges from *a* to *c* and from *c* to *b*, but there is no edge from *a* to *d*

Determining which Properties a Relation has from its Digraph – Example 3



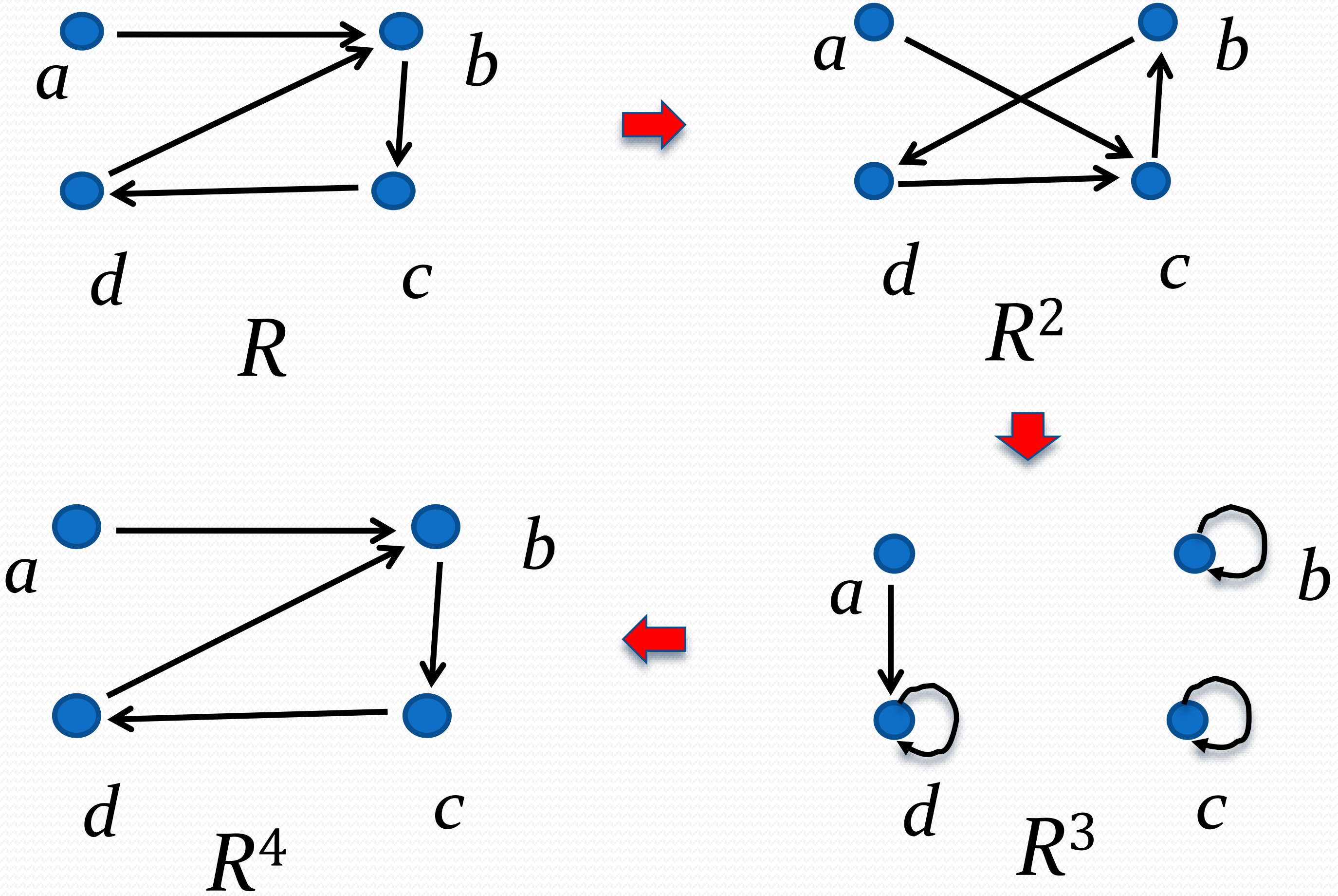
Reflexive? No, there are no loops
Symmetric? No, for example, there is no edge from *c* to *a*
Antisymmetric? Yes, whenever there is an edge from one vertex to another, there is not one going back
Transitive? No, there is no edge from *a* to *b*

Determining which Properties a Relation has from its Digraph – Example 4



- *Reflexive?* No, there are no loops
- *Symmetric?* No, for example, there is no edge from d to a
- *Antisymmetric?* Yes, whenever there is an edge from one vertex to another, there is not one going back
- *Transitive?* Yes (trivially), there are no two edges where the first edge ends at the vertex where the second edge begins

Example of the Powers of a Relation



The pair (x,y) is in R^n if there is a path of length n from x to y in R
(following the direction of the arrows).

Equivalence Relations

Section 9.5

Section Summary

- Equivalence Relations
- Equivalence Classes
- Equivalence Classes and Partitions

Equivalence Relations

Definition 1: A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a, and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Strings

Example: Suppose that R is the relation on the set of strings of English letters such that aRb if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution: Show that all of the properties of an equivalence relation hold.

- *Reflexivity*: Because $l(a) = l(a)$, it follows that aRa for all strings a.
- *Symmetry*: Suppose that aRb. Since $l(a) = l(b)$, $l(b) = l(a)$ also holds and bRa.
- *Transitivity*: Suppose that aRb and bRc. Since $l(a) = l(b)$,and $l(b) = l(c)$, $l(a) = l(a)$ also holds and aRc.

Congruence Modulo *m*

Example: Let m be an integer with $m > 1$. Show that the relation

$R = \{(a,b) \mid a \equiv b \pmod{m}\}$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity*: $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry*: Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m, and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)\, m$, so $b \equiv a \pmod{m}$.
- *Transitivity*: Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:

  $$a - c = (a - b) + (b - c) = km + lm = (k + l)\, m.$$

  Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the "divides" relation on the set of positive integers is not an equivalence relation.

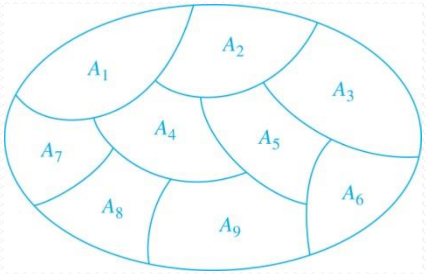
Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, "divides" is not an equivalence relation.

- *Reflexivity*: $a \mid a$ for all a.
- *Not Symmetric*: For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S.$



A Partition of a Set

Partition of a Set (continued)

- Example: Consider the following collections of subsets S = {1, 2, 3, 4, 5, 6, 7, 8, 9}
  i. [{1, 3, 5}, {2, 6}, {4, 8, 9}]
  ii. [{1, 3, 5}, {2, 4, 6, 8}, {5, 7, 9}]
  iii. [{1, 3, 5}, {2, 4, 6, 8}, {7, 9}]
- Solution:
  - (i) is not a partition of S since 7 in S does not belongs to any of the subsets.
  - (ii) is not a partition of S since {1, 3, 5} and {5, 7, 9} are not disjoint.
  - (iii) is a partition of S.

Classes of Sets

- Let S be a set. Then class of sets is the collection of some subsets of S. If we consider some of the sets in a given class of sets, then it is called subclass of sets.
- Example: Suppose S = {1,2,3,4}. Let A be the class of subsets of S which contain exactly three elements of S.
- Then A = [{1,2,3}, {1,2,4}, {1,3,4}, {2,3,4}]. Let B be a class of subsets of S which contain 2 and two other elements of S. Then B = [{1,2,3}, {1,2,4}, {2,3,4}]
- Here, B is the subclass of A, since every elements of B is also an elements of A.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$.
When only one relation is under consideration, we can write $[a]$, without the subscript R, for this equivalence class.

Note that $[a]_R = \{s \mid (a,s) \in R\}$.

- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m*. The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\ldots, a-2m, a-m, a+2m, a+2m, \ldots\}$. For example,

$[0]_4 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$ $[1]_4 = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$

$[2]_4 = \{\ldots, -6, -2, 2, 6, 10, \ldots\}$ $[3]_4 = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$

Equivalences Classes (continued)

- The collection of all equivalence classes of elements of S under an equivalence relation R is denoted by $S/R = \{[a] | a \in S\}$. This is called the quotient set of S by R.
- The functional property of a quotient set is contained in the following theorem

Equivalences Classes (continued)

- Theorem: Let R be an equivalence relation on a set. Then the quotient set S/R is a partition of S. Specifically:
  i. For each a in S, we have $a \in [a]$.
  ii. $[a] = [b]$, if and only if $(a,b) \in R$.
  iii. If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint.

Equivalences Classes (continued)

- Example: Let the relation R on S = {1,2,3} is R={(1,1), (1,2), (2,1), (3,3), (2,2)}. Then
- [1] = {1,2}
- [2] = {1,2}
- [3] = {3}
- Here [1] = [2]
- So S/R = {[1],[3]} or {[2],[3]} is a partition of S.

Equivalences Classes (continued)

- Let R be the following equivalence relation on the set A = {1,2,3,4,5,6}. R={(1,1), (1,5), (2,2), (2,3), (2,6), (3,2), (3,3), (3,6), (4,4), (51,), (5,5), (62,), (6,3), (6,6)}. Find the partition of A induced by R i.e. find the equivalence class of R.
- Solution:

[1] = {1,5}	2={2,3,6}	3={2,3,6}
4={4}	5={1,5}	6={2,3,6}

- Here [1]=[5] and [2]=[3]=[6]
- So S/R = {[1],[2],[4]} is a partition induced by R.
- S/R = {{1,5},{2,3,6},{4}} is a partition of A induced by R.

Partial Orderings

Section 9.6

Section Summary

- Partial Orderings and Partially-ordered Sets
- Lexicographic Orderings
- Hasse Diagrams
- Lattices (*not currently in overheads*)
- Topological Sorting (*not currently in overheads*)

Partial Orderings

Definition 1: A relation *R* on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set S together with a partial ordering *R* is called a *partially ordered set*, or *poset*, and is denoted by (*S*, *R*). Members of *S* are called *elements* of the poset.

Partial Orderings (*continued*)

Example 1: Show that the "greater than or equal" relation ($\geq$) is a partial ordering on the set of integers.

- *Reflexivity*: $a \geq a$ for every integer a.
- *Antisymmetry*: If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity*: If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers. (*See Appendix* 1).

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation (|) is a partial ordering on the set of integers.

- *Reflexivity*: $a \mid a$ for all integers a. (*see Example* 9 *in Section* 9.1)
- *Antisymmetry*: If a and b are positive integers with $a \mid b$ and $b \mid a$, then $a = b$. (*see Example* 12 *in Section* 9.1)
- *Transitivity*: Suppose that a divides b and b divides c. Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c. Therefore, the relation is transitive.

- $(\mathbf{Z}^+, |)$ is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation ($\subseteq$) is a partial ordering on the power set of a set S.

- *Reflexivity*: $A \subseteq A$ whenever A is a subset of S.
- *Antisymmetry*: If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity*: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.