Chapter Five

01. Arc Length of a curve

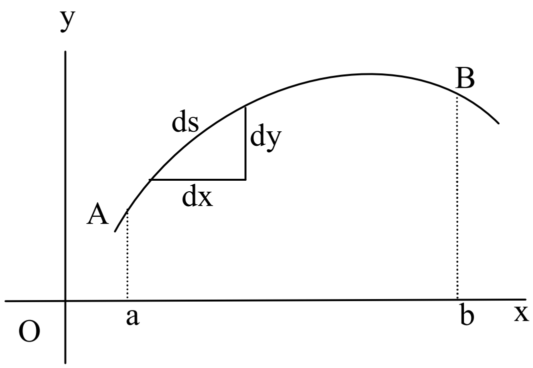


Figure # 30

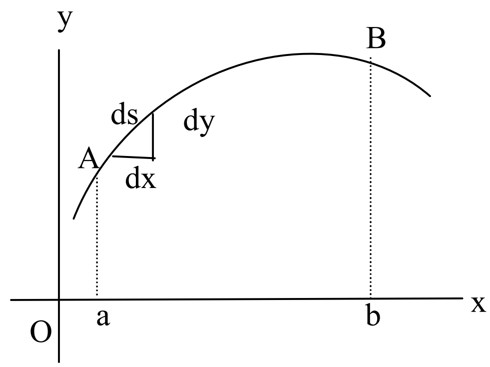


Figure # 31

By Pythagoras theorem

$$ds^2 = dx^2 + dy^2$$

$$\Rightarrow ds = \sqrt{dx^2 + dy^2}$$

$$\Rightarrow \frac{ds}{dx} = \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}} = \sqrt{1 + (\frac{dy}{dx})^2}$$

$$\Rightarrow ds = \sqrt{1 + (\frac{dy}{dx})^2}\, dx \text{ -------------------------------(i)}$$

The length of the curve AB with respect to x axis is:

$$\int ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

$$s = \int ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_a^b \sqrt{1 + f'(x)^2}\, dx \text{ ----(ii)} \qquad [\because \frac{dy}{dx} = f'(x)]$$

Again, with respect to y axis the length of the curve AB is:

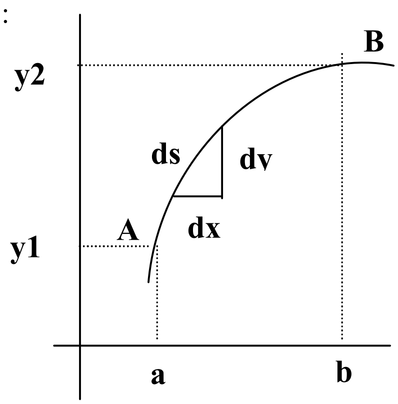


Figure # 32

By Pythagoras theorem

$$ds^2 = dx^2 + dy^2$$

$$\Rightarrow ds = \sqrt{dx^2 + dy^2}$$

$$\Rightarrow \frac{ds}{dy} = \sqrt{\frac{dx^2}{dy^2} + \frac{dy^2}{dy^2}} = \sqrt{(\frac{dx}{dy})^2 + 1}$$

$$\Rightarrow ds = \sqrt{1 + (\frac{dx}{dy})^2}\, dy$$

Prof. Dr. A.N.M. Rezaul Karim/CSE/IIUC

$\therefore$ Arc length of AB is: $\int_{y_1}^{y_2} ds = \int_{y_1}^{y_2} \sqrt{1 + (\frac{dx}{dy})^2}\, dy$

$$s = \int_{y_1}^{y_2} \sqrt{1 + f'(x)^2}\, dy$$

Example 150: Find the circumference of a circle $x^2 + y^2 = R^2$ of radius *R*.

Answer:

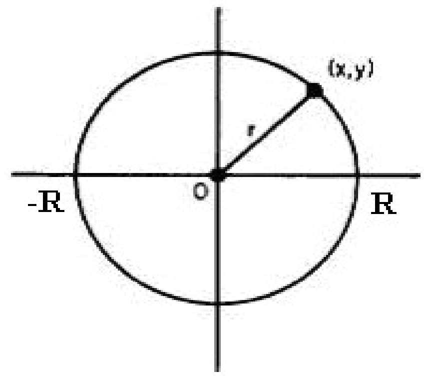


Figure # 33

The equation of the circle is $x^2 + y^2 = R^2$

$\Rightarrow y^2 = R^2 - x^2$

$y = \pm\sqrt{R^2 - x^2}$

Which means that, as a graph, the upper semi-circle is $y = +\sqrt{R^2 - x^2}$

and the lower semi-circle is : $y = -\sqrt{R^2 - x^2}$

We will find the length along this upper semi-circle, which is half the circumference of the circle. We have,

$$s = \int ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx \text{ -----------------------------------(i)}$$

Given, $y = f(x) = \sqrt{R^2 - x^2}$

$$\Rightarrow y = f(x) = (R^2 - x^2)^{1/2}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \frac{1}{2}(R^2 - x^2)^{\frac{1}{2}-1} . \frac{d}{dx}(R^2 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \frac{1}{2}(R^2 - x^2)^{\frac{1}{2}-1} .(0 - 2x)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \frac{1}{2}(R^2 - x^2)^{\frac{1}{2}-1} .(-2x)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (R^2 - x^2)^{-\frac{1}{2}} .(-x)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \frac{-x}{(R^2 - x^2)^{\frac{1}{2}}} \text{ -----------------------------(ii)}$$

$$\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = \frac{(-x)^2}{\{\sqrt{R^2 - x^2}\}^2}$$

$$\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = \frac{x^2}{\{(R^2 - x^2)^{1/2}\}^2}$$

$$\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = \frac{x^2}{(R^2 - x^2)} \text{------------------------(iii)}$$

$$\therefore 1 + (\frac{dy}{dx})^2 = 1 + \{f'(x)\}^2 = 1 + \frac{x^2}{(R^2 - x^2)}$$

$$\therefore 1 + (\frac{dy}{dx})^2 = 1 + \{f'(x)\}^2 = \frac{R^2 - x^2 + x^2}{(R^2 - x^2)}$$

$$\therefore 1 + (\frac{dy}{dx})^2 = 1 + \{f'(x)\}^2 = \frac{R^2}{(R^2 - x^2)} \text{-----------------(iv)}$$

Then we can evaluate the arc length along the upper semi-circle as from (i)

$$s = \int ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

$$\Rightarrow s = \int_{-R}^{R} \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \int_{-R}^{R} \sqrt{\frac{R^2}{R^2 - x^2}}\, dx = \int_{-R}^{R} \frac{R}{\sqrt{R^2 - x^2}}\, dx$$

$$\Rightarrow s = R \int_{-R}^{R} \frac{1}{\sqrt{R^2 - x^2}}\, dx = R \sin^{-1}\left[\frac{x}{R}\right]_{-R}^{R} \qquad [\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}]$$

$$\Rightarrow s = R \sin^{-1}\left[\frac{R}{R}\right] - R \sin^{-1}\left[\frac{-R}{R}\right] = R \sin^{-1} 1 - R \sin^{-1}(-1)$$

$$\Rightarrow s = R \sin^{-1} 1 + R \sin^{-1}(1) \qquad [\because \sin^{-1}(-x) = -\sin^{-1} x]$$

$$\Rightarrow s = R \sin^{-1} \sin\frac{\pi}{2} + R \sin^{-1}(\sin\frac{\pi}{2}) = R\frac{\pi}{2} + R\frac{\pi}{2}$$

$$\Rightarrow s = 2R\frac{\pi}{2} = R\pi$$

Hence the length of half circle is $\mathbf{R\pi}$. So, the length of full circle is $\mathbf{2R\pi}$

This means that the circumference of the circle is $\mathbf{C = 2\pi R}$.

Example 151: we'll compute the length along the parabola $\mathbf{y = x^2}$ between x = 0 and x = 1

We have,

$$s = \int ds = \int_a^b \sqrt{1+(\frac{dy}{dx})^2}\,dx \text{ ----------------------(i)}$$

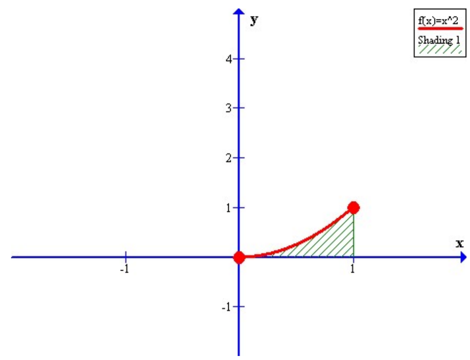


Figure # 34

Given,

$$y = f(x) = x^2$$

$$\Rightarrow \frac{dy}{dx} = f^{'}(x) = 2x$$

$$\therefore 1+(\frac{dy}{dx})^2 = 1+\{f^{'}(x)\}^2 = 1+(2x)^2$$

$$\therefore 1+(\frac{dy}{dx})^2 = 1+\{f^{'}(x)\}^2 = 1+4x^2 \text{ --------(ii)}$$

We have, from (i)

$$s = \int ds = \int_a^b \sqrt{1+(\frac{dy}{dx})^2}\,dx = \int_0^1 \sqrt{1+4x^2}\,dx = \int_0^1 \sqrt{1+(2x)^2}\,dx \text{ --------(iii)}$$

Let,

$$u = 2x$$

$$\Rightarrow du = 2dx$$

$$\Rightarrow dx = \frac{1}{2}du$$

x	0	1
u = 2x	u = 2x u = 2.0 = 0	u = 2x u = 2.1 = 2

From (iii), we get

$$s = \int_0^1 \sqrt{1+4x^2}\,dx = \int_0^1 \sqrt{1+(2x)^2}\,dx = \int_0^2 \sqrt{1+u^2}\,.\frac{1}{2}du$$

$$s = \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du \text{ -------------------------------(iv)}$$

We have, ***[Formula-30]***

$$\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2}+\frac{a^2}{2}\ln(u+\sqrt{a^2+u^2})+C$$

$$\therefore s = \frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du$$

$$\therefore s = \frac{1}{2}\left[\frac{u}{2}\sqrt{1^2+u^2}+\frac{1^2}{2}\ln(u+\sqrt{1^2+u^2})\right]_0^2$$

$$\therefore s = \frac{1}{2}\left[\frac{2}{2}\sqrt{1^2+2^2}+\frac{1^2}{2}\ln(2+\sqrt{1^2+2^2})\right]-\frac{1}{2}\left[\frac{0}{2}\sqrt{1^2+0^2}+\frac{1^2}{2}\ln(0+\sqrt{1^2+0^2})\right]$$

$$\therefore s = \frac{1}{2}\left[\sqrt{5}+\frac{1}{2}\ln(2+\sqrt{5})\right]-\frac{1}{2}\left[0\sqrt{1}+\frac{1}{2}\ln(0+\sqrt{1})\right]$$

$$\therefore s = \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right] - \frac{1}{2}\left[0 + \frac{1}{2}\ln\sqrt{1}\right]$$

$$\therefore s = \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right] - \frac{1}{2}\left[\frac{1}{2}\ln 1\right]$$

$$\therefore s = \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right] - \frac{1}{2}\left[\frac{1}{2}.0\right]$$

$$\therefore s = \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right] - \frac{1}{2}.0$$

$$\therefore s = \frac{1}{2}\left[\sqrt{5} + \frac{1}{2}\ln(2+\sqrt{5})\right] \textit{ Answer}$$

Example 152: Find length of the arc AB of the curve with equation $y = \frac{2}{3}x^{\frac{3}{2}}$

Where the x-coordinates of A and B are 3 and 8 respectively.

We have,

$$s = \int ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx \text{ ------------------------------------(i)}$$

Given,

$$y = f(x) = \frac{2}{3}x^{\frac{3}{2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{\frac{3}{2}-1} = x^{\frac{3-2}{2}} = x^{\frac{1}{2}}$$

So: $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(x^{\frac{1}{2}}\right)^2 = 1 + x$

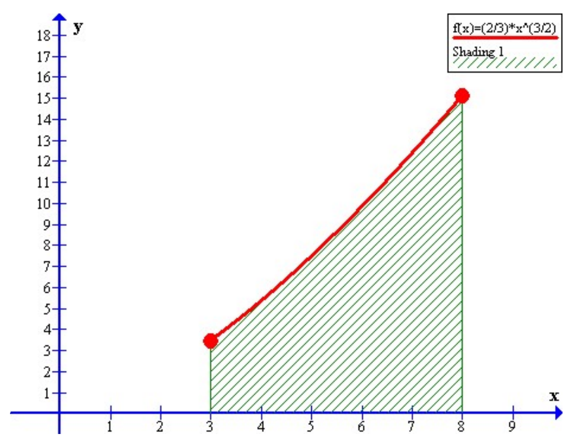


Figure # 35

The length of the curve AB is: $\int ds = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$ **[from (i)]**

Are length AB $= \int_3^8 \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$

$$= \int_3^8 (1+x)^{\frac{1}{2}}\,dx \qquad \left[\because 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(x^{\frac{1}{2}}\right)^2 = 1+x\right]$$

$$= \left[\frac{(1+x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_3^8 = \left[\frac{(1+x)^{\frac{3}{2}}}{\frac{3}{2}}\right]_3^8 = \frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_3^8 = \frac{2}{3}\left[(1+8)^{\frac{3}{2}}\right] - \frac{2}{3}\left[(1+3)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[(9)^{\frac{3}{2}}\right] - \frac{2}{3}\left[(4)^{\frac{3}{2}}\right] = \frac{2}{3}\left[(9)^{\frac{3}{2}}\right] - \frac{2}{3}\left[(4)^{\frac{3}{2}}\right] = \frac{2}{3}\left[(3^2)^{\frac{3}{2}}\right] - \frac{2}{3}\left[(2^2)^{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left[3^3\right] - \frac{2}{3}\left[2^3\right] = \frac{2}{3}(27-8) = \frac{38}{3}$$

02. Areas of Surfaces of Revolution

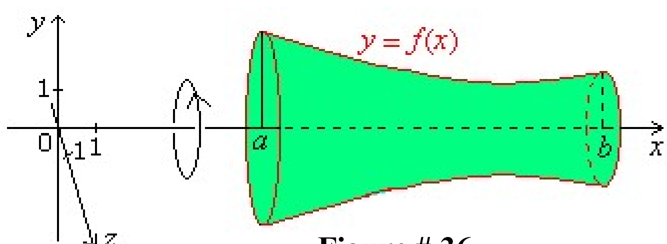


Figure # 36

Figure # 36: Graph of f, when the graph of a function is revolved (rotated) about the x-axis or y-axis, it ***generates a surface, called a surface of revolution.***

a. Revolution About The *x*-axis

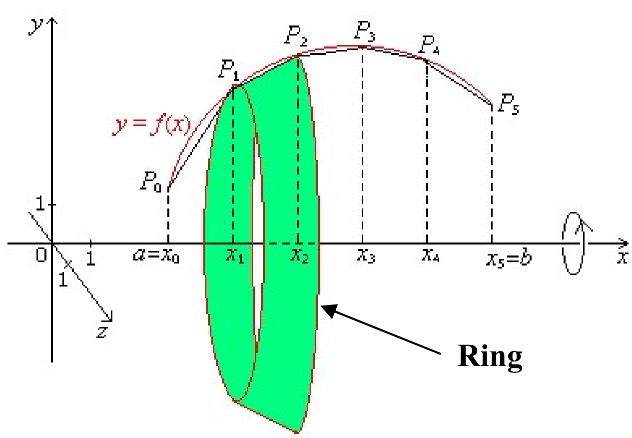


Figure # 37 : Area of element of surface of revolution swept out by a chord.

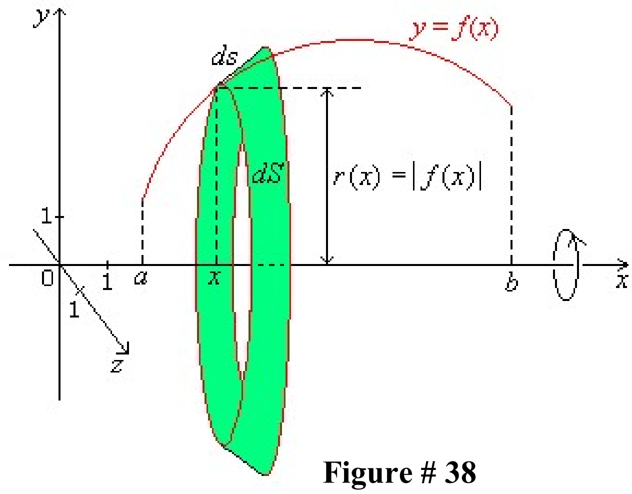


Figure # 38

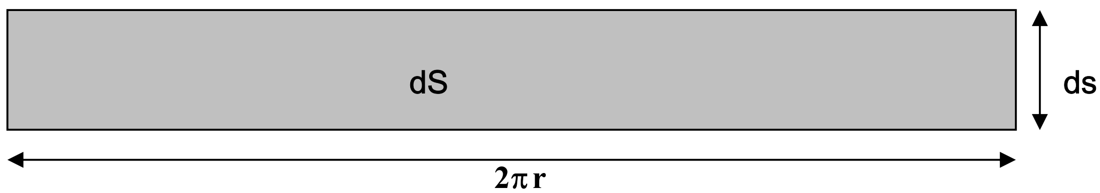


Figure # 39

The area $\mathbf{dS}$ of the rectangle is $\mathbf{2\pi r \times ds}$

The area $\mathbf{dS}$ of the rectangle is

$$= \text{Circumference of revolution at } \mathbf{(x, y).ds}$$

$$= \mathbf{2\pi.}(\text{Radius of revolution at } \mathbf{(x, y)})$$

$$= \mathbf{2\pi r \times ds}$$

$$= \mathbf{2\pi r \times \sqrt{1+(\frac{dy}{dx})^2}dx} \qquad [\because \mathbf{ds = \sqrt{1+(\frac{dy}{dx})^2}dx}]$$

Here, $\mathbf{r = |f(x)| = y}$ since the revolution for $y = f(x)$ about the x-axis.

$\therefore$ The area S of the surface of revolution for $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is: The area $\boldsymbol{S}$ is the integral of the differential area $\boldsymbol{dS}$. We have:

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi r \times \sqrt{1+(\frac{dy}{dx})^2}dx}$$

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1+(\frac{dy}{dx})^2}dx \; [\because r = |f(x)| = y]}$$

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi |f(x)| \times \sqrt{1+(\frac{dy}{dx})^2}dx}$$

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1+(\frac{dy}{dx})^2}dx} \text{ --------------------------------(i)}$$

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi |f(x)| \times \sqrt{1+\{f'(x)\}^2}dx \quad [\because \frac{dy}{dx} = f'(x)]} \text{ --------(ii)}$$

Example 153: Show that the surface area of a sphere having radius a unit is $\mathbf{S = 4\pi a^2}$ square units

Solution

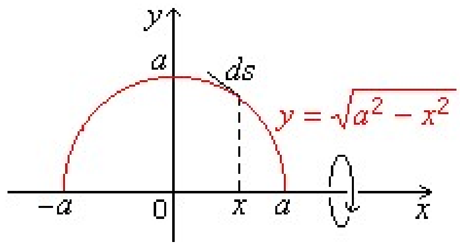


Figure # 40

We have,

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1 + (\frac{dy}{dx})^2}\, dx \text{ ------------------------(i)}$$

Let the equation of the circle is $\mathbf{x^2 + y^2 = a^2}$

$$\Rightarrow y^2 = a^2 - x^2$$

$$y = \pm\sqrt{a^2 - x^2}$$

Which means that, as a graph, the upper semi-circle is $\mathbf{y = +\sqrt{a^2 - x^2}}$. The sphere's surface can be considered as generated by revolving the upper semi-circle $\mathbf{y = +\sqrt{a^2 - x^2}}$ about the x-axis. So, the radius of revolution is $\mathbf{r = y = \sqrt{a^2 - x^2}}$

$$y = f(x) = \sqrt{a^2 - x^2}$$

$$\Rightarrow y = f(x) = (a^2 - x^2)^{1/2}$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \frac{1}{2}(a^2 - x^2)^{\frac{1}{2}-1} . \frac{d}{dx}(a^2 - x^2)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \frac{1}{2}(a^2 - x^2)^{\frac{1}{2}-1} .(0 - 2x)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \frac{1}{2}(a^2 - x^2)^{\frac{1}{2}-1} .(-2x)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = (a^2 - x^2)^{-\frac{1}{2}} .(-x)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = \frac{-x}{(a^2 - x^2)^{\frac{1}{2}}} \text{ ------------------------------(ii)}$$

$$\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = \frac{(-x)^2}{\{\sqrt{a^2 - x^2}\}^2}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \{f'(x)\}^2 = \frac{x^2}{\{(a^2 - x^2)^{1/2}\}^2}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \{f'(x)\}^2 = \frac{x^2}{(a^2 - x^2)} \text{ ------------------------(iii)}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \{f'(x)\}^2 = 1 + \frac{x^2}{(a^2 - x^2)}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \{f'(x)\}^2 = \frac{a^2 - x^2 + x^2}{(a^2 - x^2)}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \{f'(x)\}^2 = \frac{a^2}{(a^2 - x^2)}$$

$$\therefore \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \{f'(x)\}^2} = \sqrt{\frac{a^2}{(a^2 - x^2)}} = \frac{a}{\sqrt{(a^2 - x^2)}} \text{ -----(iv)}$$

Putting the value of $\sqrt{1 + (\frac{dy}{dx})^2}$ in (i), we get,

Let *S* be the desired area:

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\Rightarrow S = \int_{-a}^{a} 2\pi \times \sqrt{a^2 - x^2} \times \frac{a}{\sqrt{(a^2 - x^2)}}\, dx \qquad [\because y = \sqrt{a^2 - x^2}\,]$$

$$\Rightarrow S = \int_{-a}^{a} 2\pi \times a\, dx = 2\pi a \int_{-a}^{a} dx = 2\pi a [x]_{-a}^{+a}$$

$$\Rightarrow S = 2\pi a[a - (-a)] = 2\pi a[a + a] = 2\pi a \times 2a = 4\pi a^2 \; \textit{Answer}$$

Example 154:

Find the area of the surface of revolution obtained by revolving the graph of $\mathbf{y = f(x) = 2x}$ from $\mathbf{x = -3}$ to $\mathbf{x = -1}$ about the *x*-axis.

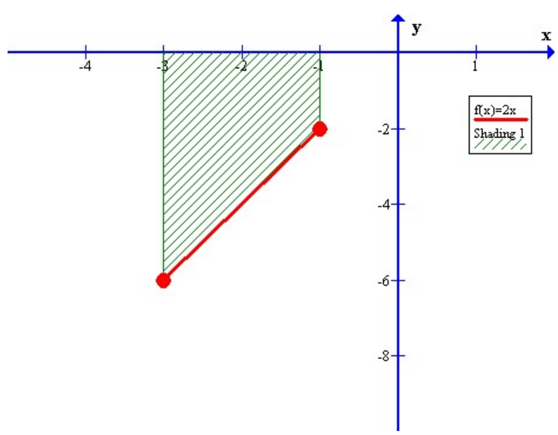


Figure # 41

Let *S* be the desired area:

We have,

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1+(\frac{dy}{dx})^2}\, dx \text{------------------(i)}$$

Given the function: $\mathbf{y = 2x}$

The sphere's surface can be considered as generated by revolving the graph $\mathbf{y = 2x}$ about the x-axis.

So, the radius of revolution is $\mathbf{r = y = 2x}$

Given the function: $\mathbf{y = 2x}$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2$$

$$\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = (2)^2 = 4$$

$$\therefore 1+(\frac{dy}{dx})^2 = 1+\{f'(x)\}^2 = 1+4$$

$$\therefore \sqrt{1+(\frac{dy}{dx})^2} = \sqrt{1+\{f'(x)\}^2} = \sqrt{1+4} = \sqrt{5} \text{------------------------------(ii)}$$

Putting the value of $\sqrt{1+(\frac{dy}{dx})^2}$ in (i), we get,

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1+(\frac{dy}{dx})^2}\, dx$$

$$\Rightarrow S = \int_{-3}^{-1} 2\pi y\sqrt{5}\, dx = \int_{-3}^{-1} 2\pi|r|\sqrt{5}\, dx = \int_{-3}^{-1} 2\pi 2x\sqrt{5}\, dx$$

$$\Rightarrow S = \int_{-3}^{-1} 4\pi x\sqrt{5}\, dx = \int_{-3}^{-1} 4\pi x\sqrt{5}\, dx = 4\pi \times \sqrt{5}\int_{-3}^{-1} x\, dx$$

$$\Rightarrow S = 4\pi \times \sqrt{5}\left[\frac{x^2}{2}\right]_{-3}^{-1} = 4\pi \times \sqrt{5} \times \frac{1}{2}\left[x^2\right]_{-3}^{-1}$$

$$\Rightarrow S = 4\pi \times \sqrt{5} \times \frac{1}{2}\left[(-1)^2 - (-3)^2\right] = 4\pi \times \sqrt{5} \times \frac{1}{2}[1-9]$$

$$\Rightarrow S = -4\pi \times \sqrt{5} \times \frac{1}{2} \times 8 = -4\pi \times \sqrt{5} \times \frac{1}{2} \times 8$$

$$\Rightarrow S = -16\pi \times \sqrt{5} = -16\pi \times \sqrt{5} = -16\sqrt{5}\,\pi \;\; \textit{Answer}$$

Example 155: You are required to find the area of the surface formed by revolving the graph of $\mathbf{y = f(x) = x^3}$ on the interval $\mathbf{[0,1]}$ about the x-axis, seen below.

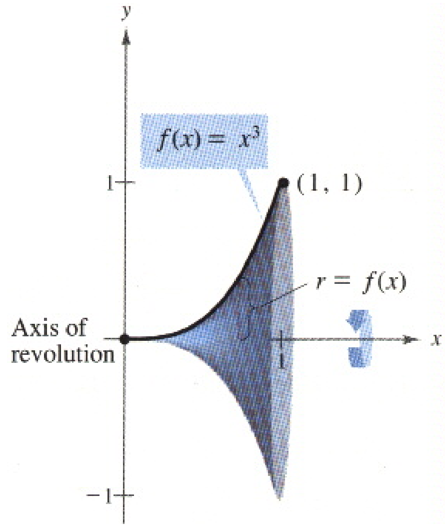


Figure # 42

Let *S* be the desired area:

We have,

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi r \times ds}$$

$$\mathbf{= \int_a^b 2\pi y \times \sqrt{1 + (\frac{dy}{dx})^2}\, dx} \text{------------------------(i)}$$

Given the function: $\mathbf{y = x^3}$

The sphere's surface can be considered as generated by revolving the graph $\mathbf{y = x^3}$ about the x-axis.

So, the radius of revolution is $\mathbf{r = y = x^3}$

Given the function: $\mathbf{y = x^3}$

$$\mathbf{\Rightarrow \frac{dy}{dx} = f'(x) = 3x^2}$$

$$\mathbf{\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = (3x^2)^2 = 9x^4}$$

$$\mathbf{\therefore 1 + (\frac{dy}{dx})^2 = 1 + \{f'(x)\}^2 = 1 + 9x^4}$$

$$\mathbf{\therefore \sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + \{f'(x)\}^2} = \sqrt{1 + 9x^4}} \text{------------------(ii)}$$

Putting the value of $\mathbf{\sqrt{1 + (\frac{dy}{dx})^2}}$ in (i), we get,

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1 + (\frac{dy}{dx})^2}\, dx}$$

$$\mathbf{S = \int_0^1 2\pi |f(x)| \sqrt{1 + 9x^4}\, dx = \int_0^1 2\pi x^3 \sqrt{1 + 9x^4}\, dx}$$

$$\mathbf{S = 2\pi \int_0^1 |x^3| \sqrt{1 + 9x^4}\, dx} \text{--(iii)}$$

Now, $\mathbf{S = 2\pi \int x^3 (\sqrt{1 + 9x^4})\, dx}$ --(iv)

Let, $1 + 9x^4 = z$

$$\Rightarrow \frac{dz}{dx} = 9.4x^{4-1} = 36x^3$$

$$\therefore dz = 36x^3 dx$$

$$x^3 dx = \frac{dz}{36}$$

From (iv), $S = 2\pi\int x^3(\sqrt{1 + 9x^4})\,dx$

$$S = 2\pi \times \frac{1}{36}\int \sqrt{z}dz = 2\pi \times \frac{1}{36}\int z^{1/2}dz = \frac{2\pi}{36} \times \frac{z^{1/2+1}}{1/2+1}$$

$$\Rightarrow S = \frac{2\pi}{36} \times \frac{z^{3/2}}{3/2} = \frac{2\pi}{36} \times \frac{2}{3} \times z^{3/2} = \frac{2\pi}{36} \times \frac{2}{3}(1 + 9x^4)^{3/2} \qquad [\because 1 + 9x^4 = z]$$

$$\Rightarrow S = \frac{\pi}{9} \times \frac{1}{3}(1 + 9x^4)^{3/2} = \frac{\pi}{27}(1 + 9x^4)^{3/2}$$

From (iii), $\therefore S = 2\pi\int_0^1 |x^3|\sqrt{1 + 9x^4}\,dx$

$$\Rightarrow S = \frac{\pi}{27}\left[(1 + 9x^4)^{3/2}\right]_0^1 \qquad [\because 2\pi\int x^3(\sqrt{1 + 9x^4})\,dx = \frac{\pi}{27}(1 + 9x^4)^{3/2}]$$

$$\Rightarrow S = \frac{\pi}{27}\left[(1 + 9.1^4)^{3/2} - (1 + 9.0^4)^{3/2}\right] = \frac{\pi}{27}\left[(1 + 9.1)^{3/2} - (1 + 9.0)^{3/2}\right]$$

$$\Rightarrow S = \frac{\pi}{27}\left[(10)^{3/2} - (1 + 0)^{3/2}\right] = \frac{\pi}{27}\left[(10)^{3/2} - (1)^{3/2}\right]$$

$$\Rightarrow S = \frac{\pi}{27}\left[(10)^{3/2} - (1)^{3/2}\right] \textit{ Answer}$$

Example 156: Calculate the area of the surface generated by revolving the arc of $y = x^3$ from $x = -1$ **to** $x = 2$ about the *x*-axis.

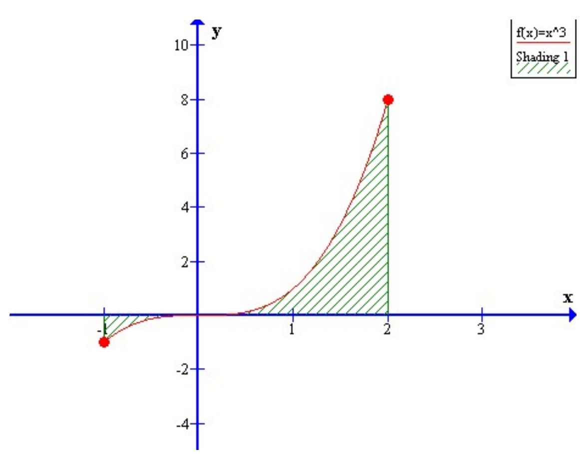


Figure # 43

Solution

Let S be the desired area. Then

We have,

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds$$

$$= \int_a^b 2\pi y \times \sqrt{1 + (\frac{dy}{dx})^2}\, dx \text{ -------------------------(i)}$$

Given the function: $\mathbf{y = x^3}$

The sphere's surface can be considered as generated by revolving the graph $\mathbf{y = x^3}$ about the x-axis.

So, the radius of revolution is $\mathbf{r = y = x^3}$

Given the function: $\mathbf{y = x^3}$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 3x^2$$

$$\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = (3x^2)^2 = 9x^4$$

$$\therefore 1 + (\frac{dy}{dx})^2 = 1 + \{f'(x)\}^2 = 1 + 9x^4$$

$$\therefore \sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + \{f'(x)\}^2} = \sqrt{1 + 9x^4} \text{ ---------(ii)}$$

Putting the value of $\sqrt{1 + (\frac{dy}{dx})^2}$ in (i), we get,

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

$$S = \int_{-1}^{2} 2\pi |f(x)| \sqrt{1 + 9x^4}\, dx = \int_{-1}^{2} 2\pi x^3 \sqrt{1 + 9x^4}\, dx$$

$$S = 2\pi \int_{-1}^{2} \left| x^3 \right| \sqrt{1 + 9x^4}\, dx \text{ ---------------------------------(iii)}$$

Now,

$$S = 2\pi \int x^3 (\sqrt{1 + 9x^4})\, dx \text{ ---(iv)}$$

Let, $1 + 9x^4 = z$

$$\Rightarrow \frac{dz}{dx} = 9.4x^{4-1} = 36x^3$$

$$\therefore dz = 36x^3 dx$$

$$x^3 dx = \frac{dz}{36}$$

From (iv), $S = 2\pi \int x^3 (\sqrt{1 + 9x^4})\, dx$

$$S = 2\pi \times \frac{1}{36} \int \sqrt{z} dz$$

$$S = 2\pi \times \frac{1}{36} \int z^{1/2} dz = \frac{2\pi}{36} \times \frac{z^{1/2 + 1}}{1/2 + 1} = \frac{2\pi}{36} \times \frac{z^{3/2}}{3/2}$$

$$S = \frac{2\pi}{36} \times \frac{2}{3} \times z^{3/2} = \frac{2\pi}{36} \times \frac{2}{3}(1+9x^4)^{3/2} \qquad [\because 1+9x^4 = z]$$

$$S = \frac{\pi}{9} \times \frac{1}{3}(1+9x^4)^{3/2} = \frac{\pi}{27}(1+9x^4)^{3/2}$$

From (iii), $\therefore S = 2\pi \int_{-1}^{2} |x^3| \sqrt{1+9x^4}\, dx$

$$\Rightarrow S = \frac{\pi}{27}\left[(1+9x^4)^{3/2}\right]_{-1}^{2} \qquad [\because 2\pi \int x^3 (\sqrt{1+9x^4})\, dx = \frac{\pi}{27}(1+9x^4)^{3/2}]$$

$$\Rightarrow S = \frac{\pi}{27}\left[(1+9.2^4)^{3/2} - (1+9.(-1)^4)^{3/2}\right] = \frac{\pi}{27}\left[(1+9.16)^{3/2} - (1+9.1)^{3/2}\right]$$

$$\Rightarrow S = \frac{\pi}{27}\left[(145)^{3/2} - (10)^{3/2}\right] \textit{ Answer}$$

Example 157: Find the area of the surface generated by revolving the arc of $y = x^3$ from $(-1,-1)$ to $(0,0)$ about the x-axis.

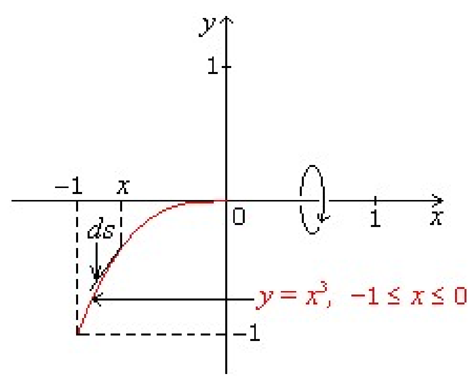


Figure # 44

Solution
Let S be the desired area. Then

We have,

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds$$

$$= \int_a^b 2\pi y \times \sqrt{1+(\frac{dy}{dx})^2}\, dx \text{ --------------------------(i)}$$

Given the function: $y = x^3$

The sphere's surface can be considered as generated by revolving the graph $y = x^3$ about the x-axis.

So, the radius of revolution is $r = y = x^3$

Given the function: $y = x^3$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 3x^2$$

$$\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = (3x^2)^2 = 9x^4$$

$$\therefore 1+(\frac{dy}{dx})^2 = 1+\{f'(x)\}^2 = 1+9x^4$$

$$\therefore \sqrt{1+(\frac{dy}{dx})^2} = \sqrt{1+\{f'(x)\}^2} = \sqrt{1+9x^4} \text{ -----------(ii)}$$

Putting the value of $\sqrt{1+(\frac{dy}{dx})^2}$ in (i), we get,

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1+(\frac{dy}{dx})^2}\, dx$$

$$S = \int_{-1}^{0} 2\pi |f(x)| \sqrt{1+9x^4}\, dx = \int_{-1}^{0} 2\pi x^3 \sqrt{1+9x^4}\, dx$$

$$S = 2\pi \int_{-1}^{0} |x^3| \sqrt{1+9x^4}\, dx \text{ ----------------------------------(iii)}$$

Now,

$$S = 2\pi \int x^3 (\sqrt{1+9x^4})\, dx \text{ ---(iv)}$$

Let, $1+9x^4 = z$

$$\Rightarrow \frac{dz}{dx} = 9.4x^{4-1} = 36x^3$$

$$\therefore dz = 36x^3 dx$$

$$x^3 dx = \frac{dz}{36}$$

From (iv), $S = 2\pi \int x^3 (\sqrt{1+9x^4})\, dx$

$$S = 2\pi \times \frac{1}{36} \int \sqrt{z} dz = 2\pi \times \frac{1}{36} \int z^{1/2} dz = \frac{2\pi}{36} \times \frac{z^{1/2+1}}{1/2+1}$$

$$S = \frac{2\pi}{36} \times \frac{z^{3/2}}{3/2} = \frac{2\pi}{36} \times \frac{2}{3} \times z^{3/2} = \frac{2\pi}{36} \times \frac{2}{3} (1+9x^4)^{3/2} \qquad [\because 1+9x^4 = z]$$

$$S = \frac{\pi}{9} \times \frac{1}{3} (1+9x^4)^{3/2} = \frac{\pi}{27} (1+9x^4)^{3/2}$$

From (iii), $\therefore S = 2\pi \int_{-1}^{0} |x^3| \sqrt{1+9x^4}\, dx$

$$\Rightarrow S = \frac{\pi}{27} \left[(1+9x^4)^{3/2} \right]_{-1}^{0} \qquad [\because 2\pi \int x^3 (\sqrt{1+9x^4})\, dx = \frac{\pi}{27} (1+9x^4)^{3/2}]$$

$$\Rightarrow S = \frac{\pi}{27} \left[(1+9.0^4)^{3/2} - (1+9.(-1)^4)^{3/2} \right]$$

$$\Rightarrow S = \frac{\pi}{27} \left[(1+0)^{3/2} - (1+9.1)^{3/2} \right] = \frac{\pi}{27} \left[(1)^{3/2} - (10)^{3/2} \right] \textit{ Answer}$$

a. Revolution about the *y*-axis

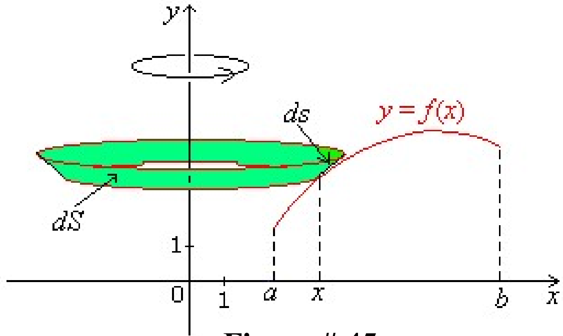


Figure # 45

Differential surface area *dS* are described by differential arc length *ds*.

The area $\mathbf{dS}$ of the rectangle is $\mathbf{2\pi r \times ds}$

The area $\mathbf{dS}$ of the rectangle is

$$= \text{Circumference of revolution at } \mathbf{(x, y).ds}$$
$$= \mathbf{2\pi.}(\text{Radius of revolution at } \mathbf{(x, y)})$$
$$= \mathbf{2\pi\, r \times ds}$$
$$= \mathbf{2\pi r \times \sqrt{1+(\frac{dy}{dx})^2}\,dx} \quad [\because \mathbf{ds = \sqrt{1+(\frac{dy}{dx})^2}\,dx}]$$

Here, $\mathbf{r = x}$ since the revolution for $y = f(x)$ about the y-axis.

$\therefore$ The area S of the surface of revolution for $y = f(x)$ from $x = a$ to $x = b$ about the y-axis is: The area ***S*** is the integral of the differential area ***dS***. We have:

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi r \times \sqrt{1+(\frac{dy}{dx})^2}\,dx}$$

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi\, r \times ds = \int_a^b 2\pi\, x \times \sqrt{1+(\frac{dy}{dx})^2}\,dx\ [\because r = x]}$$

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi\, r \times ds = \int_a^b 2\pi\, x \times \sqrt{1+(\frac{dy}{dx})^2}\,dx}$$

$$\mathbf{S = \int_a^b dS = 2\pi\int_a^b |f(x)|\sqrt{1+(f'(x))^2}\,dx} \quad [\because \mathbf{\frac{dy}{dx} = f'(x)}]$$

Example 158: The graph of $\mathbf{y = x^2}$ over $\mathbf{[0,1]}$ is revolved about the y-axis. Find the area of the generated surface.

Solution

We have,

$$\mathbf{S = \int_a^b dS = \int_a^b 2\pi\, r \times ds}$$

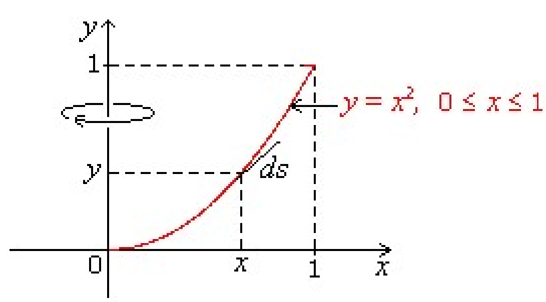


Figure # 46

$$= \int_a^b 2\pi y \times \sqrt{1+(\frac{dy}{dx})^2}\,dx \text{------------------------------(i)}$$

Given the function: $\mathbf{y = x^2}$

The sphere's surface can be considered as generated by revolving the graph $\mathbf{y = x^2}$ about the y-axis.

So, the radius of revolution is $\mathbf{r = x}$

Given the function: $\mathbf{y = x^2}$

$$\Rightarrow \frac{dy}{dx} = f'(x) = 2x$$

$$\therefore (\frac{dy}{dx})^2 = \{f'(x)\}^2 = (2x)^2 = 4x^2$$

$$\therefore 1+(\frac{dy}{dx})^2 = 1+\{f'(x)\}^2 = 1+4x^2$$

$$\therefore \sqrt{1+(\frac{dy}{dx})^2} = \sqrt{1+\{f'(x)\}^2} = \sqrt{1+4x^2} \text{ ---------------(ii)}$$

Putting the value of $\sqrt{1+(\frac{dy}{dx})^2}$ in (1), we get,

$$S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi y \times \sqrt{1+(\frac{dy}{dx})^2}\,dx$$

$$\Rightarrow S = \int_0^1 2\pi |f(x)| \sqrt{1+4x^2}\,dx$$

$$\Rightarrow S = \int_0^1 2\pi x \sqrt{1+4x^2}\,dx \text{ ---------------------------------------(iii)}$$

Let, $\mathbf{1+4x^2 = u}$

$$\Rightarrow u = 1+4x^2$$

$$\Rightarrow \frac{du}{dx} = 8x$$

$$\Rightarrow du = 8xdx$$

$$\Rightarrow 8xdx = du$$

$$\Rightarrow xdx = \frac{du}{8}$$

x	0	1
$u = 1+4x^2$	$u = 1+4x^2$ $u = 1+4.0^2 = 1$	$u = 1+4x^2$ $u = 1+4.1^2 = 5$

From (iii), we get, $\mathbf{S = \int_0^1 2\pi x \sqrt{1+4x^2}\,dx}$

$$\Rightarrow S = \int_1^5 2\pi \sqrt{u^2}\,\frac{du}{8} = \int_1^5 2\pi u \frac{du}{8} = 2\pi \times \frac{1}{8}\int_1^5 udu$$

$$\Rightarrow S = 2\pi \times \frac{1}{8} \times \left[\frac{u^2}{2}\right]_1^5 = 2\pi \times \frac{1}{8} \times \frac{1}{2}\left[u^2\right]_1^5$$

$$\Rightarrow S = 2\pi \times \frac{1}{8} \times \frac{1}{2}\left[5^2 - 1^2\right] = 2\pi \times \frac{1}{8} \times \frac{1}{2}\left[25 - 1\right]$$

$$\Rightarrow S = 2\pi \times \frac{1}{8} \times \frac{1}{2} \times 24 = 2\pi \times \frac{1}{8} \times 12$$

$$\Rightarrow S = \pi \times \frac{1}{4} \times 12 = 3\pi \text{ } \textit{Answer}$$

b. Revolution about A Non-Coordinate Axis

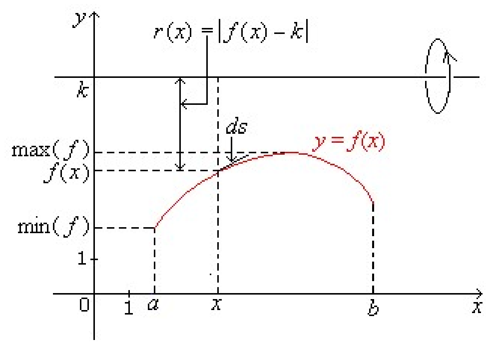


Figure # 47

Area of revolution is integral of differential area.
Radius of revolution at x is $r(x) = |f(x) - k|$. Here $r(x) = k - f(x)$ for all x in $[a, b]$.

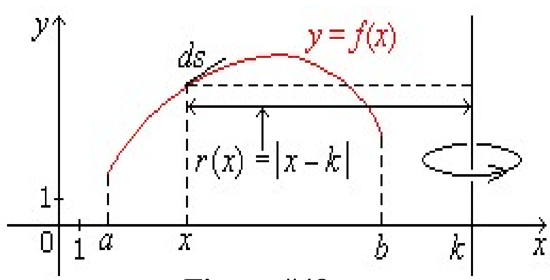


Figure #48

Area of revolution is integral of differential area.
Radius of revolution at x is $r(x) = |x - k|$. Here $r(x) = k - x$ for all x in $[a, b]$.

Suppose the graph of the continuously differentiable function f over $[a,b]$ is revolved about the horizontal line $y = k$, where k is constant. If $k = 0$, then the line is the x-axis. Let's find the area S of the generated surface. Let x be an arbitrary point in $[a,b]$. The differential arc length ds at $(x, f(x))$ generates the differential area dS. The radius of revolution at x is $r(x) = |f(x) - k|$

$$\therefore S=\int_a^b dS=\int_a^b 2\pi r\times ds=\int_a^b 2\pi|f(x)-k|\times\sqrt{1+(\frac{dy}{dx})^2}\,dx \text{ ------------------------(i)}$$

Similarly, suppose the axis of revolution is the vertical line $x = k$. If $k = 0$ then the line is the y-axis. The radius of revolution at x is $r(x) = |x-k|$

Then

$$\therefore S=\int_a^b dS=\int_a^b 2\pi r\times ds=\int_a^b 2\pi|x-k|\times\sqrt{1+(\frac{dy}{dx})^2}\,dx \text{ ----------------------------(ii)}$$

Example 159: Find the area of the surface generated by revolving the line segment $y = x + 2$ from $y = 2$ to $y = 5$ about the line $y = 3$

Answer:

The *x*-coordinate of the point of intersection of $y = x + 2$ and $y = 3$

$y = x + 2$ ----------------------(i)

$y = 3$ ---------------------------(ii)

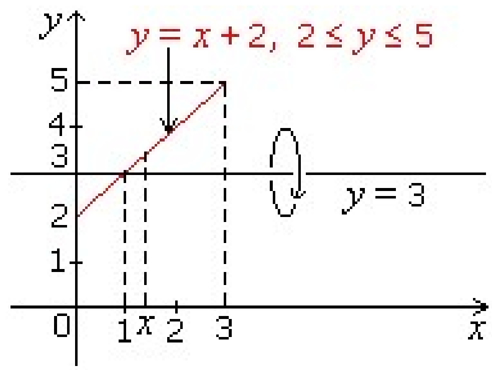


Figure # 49

From (i) and (ii), we get,

$y = y$

$\Rightarrow x + 2 = 3$

$\Rightarrow x = 3 - 2$

$\Rightarrow x = 1$

When *y* increases from 2 to 5 along the line $y = x + 2$

$y = x + 2$ -----------------------(iii)

Putting $y = 2$ in (iii)

$y = x + 2$

$\Rightarrow 2 = x + 2$

$\Rightarrow x = 2 - 2$

$\Rightarrow x = 0$

Again Putting $y = 5$ in (iii)

$y = x + 2$

$\Rightarrow 5 = x + 2$

$\Rightarrow x = 5 - 2$

$\Rightarrow x = 3$

That is, *x* increases from 0 to 3.

Let *x* be any point in [0, 3].

The radius of revolution is $r(x) = |f(x) - k| = |y - k| = |(x + 2) - 3| = |x - 1|$

We have,

$$\therefore S=\int_a^b dS=\int_a^b 2\pi r\times ds=\int_a^b 2\pi|f(x)-k|\times\sqrt{1+(\frac{dy}{dx})^2}\,dx \text{ ----------------------------(iv)}$$

Given , $y = x + 2$

$$\Rightarrow \frac{dy}{dx} = 1 + 0$$

$$\Rightarrow \frac{dy}{dx} = 1$$

$$\Rightarrow (\frac{dy}{dx})^2 = (1)^2 = (1)^2 = 1$$

$$\therefore 1 + (\frac{dy}{dx})^2 = 1 + 1 = 2$$

$$\therefore \sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{2}$$

From (iv),

$$\therefore S = \int_a^b dS = \int_a^b 2\pi r \times ds = \int_a^b 2\pi |f(x) - k| \times \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

$$\therefore S = \int_0^3 2\pi |f(x) - k| \times \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

$$\therefore S = \int_0^3 2\pi |x - 1| \times \sqrt{2}\, dx = 2\pi \times \sqrt{2} \int_0^3 |x - 1|\, dx$$

$$\therefore S = 2\pi \times \sqrt{2} \int_0^3 |x - 1|\, dx \text{ ------------------------------------(v)}$$

For $0 \le x < 1$;

$x - 1 < 0$
$\Rightarrow |x - 1| = -(x - 1) = 1 - x$

For $1 \le x \le 3$;
$x - 1 \ge 0$
$\Rightarrow |x - 1| = +(x - 1) = x - 1$

From (v), $\therefore S = 2\pi \times \sqrt{2} \int_0^3 |x - 1|\, dx$

$$\therefore S = 2\pi \times \sqrt{2} \int_0^1 (1 - x)\, dx + 2\pi \times \sqrt{2} \int_1^3 (x - 1)\, dx$$

$$\therefore S = 2\pi \times \sqrt{2} \left[x - \frac{x^2}{2} \right]_0^1 + 2\pi \times \sqrt{2} \left[\frac{x^2}{2} - x \right]_1^3$$

$$\therefore S = 2\pi \times \sqrt{2} \left[1 - \frac{1^2}{2} - 0 + 0 \right] + 2\pi \times \sqrt{2} \left[\frac{3^2}{2} - 3 - \frac{1}{2} + 1 \right]$$

$$\therefore S = 2\pi \times \sqrt{2} \left[1 - \frac{1}{2} \right] + 2\pi \times \sqrt{2} \left[\frac{9}{2} - 3 - \frac{1}{2} + 1 \right]$$

$$\therefore S = 2\pi \times \sqrt{2}\left[\frac{1}{2}\right] + 2\pi \times \sqrt{2}\left[\frac{9}{2} - 2 - \frac{1}{2}\right]$$

$$\therefore S = 2\pi \times \sqrt{2}\left[\frac{1}{2}\right] + 2\pi \times \sqrt{2}\left[\frac{9-4}{2} - \frac{1}{2}\right]$$

$$\therefore S = 2\pi \times \sqrt{2}\left[\frac{1}{2}\right] + 2\pi \times \sqrt{2}\left[\frac{5}{2} - \frac{1}{2}\right] = 2\pi \times \sqrt{2}\left[\frac{1}{2}\right] + 2\pi \times \sqrt{2}\left[\frac{5-1}{2}\right]$$

$$\therefore S = 2\pi \times \sqrt{2}\left[\frac{1}{2}\right] + 2\pi \times \sqrt{2}\left[\frac{4}{2}\right] = 2\pi \times \sqrt{2}\left[\frac{1}{2}\right] + 2\pi \times \sqrt{2} \times 2$$

$$\therefore S = 2\pi \times \sqrt{2} \times \frac{1}{2} + 2\pi \times \sqrt{2} \times 2 = 2\pi \times \sqrt{2}(\frac{1}{2} + 2)$$

$$\therefore S = 2\pi \times \sqrt{2}(\frac{5}{2}) = \pi \times \sqrt{2} \times 5 = 5\pi\sqrt{2} \ \textit{Answer}$$

03. Volume of a Solid of Revolution/Volume by Disk

a) Volume of a Solid of Revolution:

The volume (**V**) of a solid generated by revolving the region bounded by $\mathbf{y = f(x)}$ and the *x*-axis on the interval [a, *b*] about the *x*-axis is

Horizontal axis of revolution

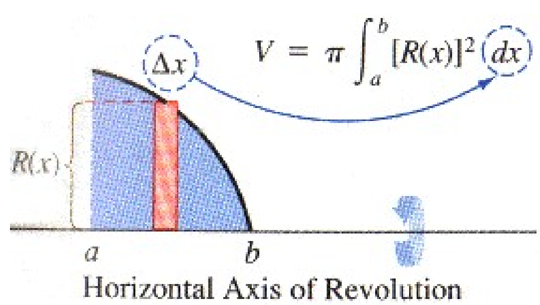


Figure # 50

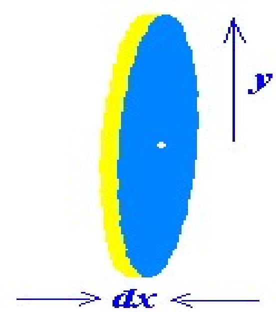
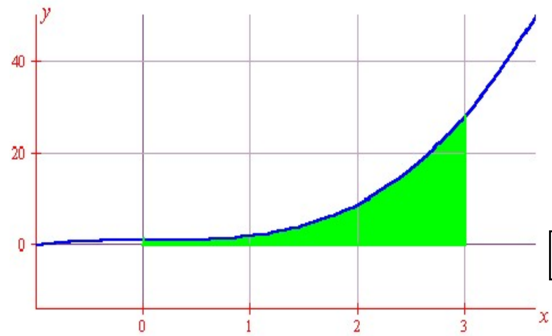


Figure # 51

Volume $= \int_a^b \pi(\text{radius})^2 dx = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi(f(x))^2 dx$ -------------------(i)

[With respect to x-axis r = y]

Example 160: Let's try another example of an area bounded by a curve $\mathbf{y = x^3 + 1}$, the x-axis and the limits of $x = 0$ and $x = 3$:
Answer



After Rotating

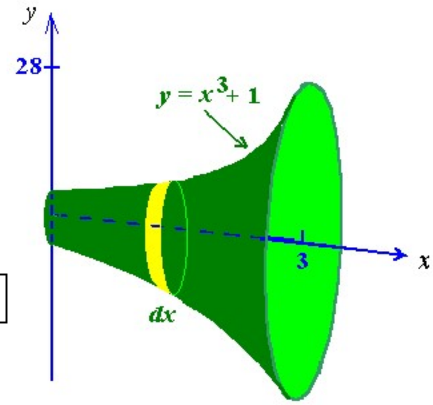


Figure # 52

Figure # 53

When the shaded area (Figure # 52) is rotated 360° about the x-axis, we again observe that a volume (Figure # 53) is generated:
Applying the above formula, we get

$$\text{Volume} = \int_a^b \pi \mathbf{r}^2 \mathbf{dx} = \int_a^b \pi \mathbf{y}^2 \mathbf{dx} = \int_a^b \pi(\mathbf{f(x)})^2 \mathbf{dx}$$

Given, $\mathbf{y = x^3 + 1}$

$$V = \pi \int_a^b y^2 dx$$

$$= \pi \int_0^3 (x^3 + 1)^2 dx$$

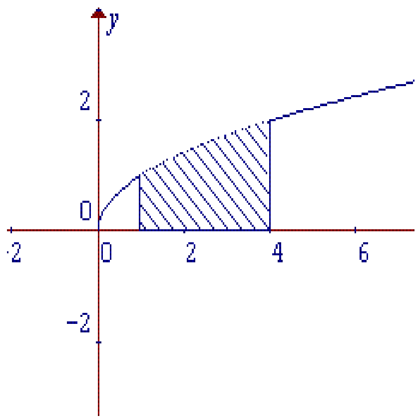
$$= \pi \int_0^3 (x^6 + 2x^3 + 1) dx$$

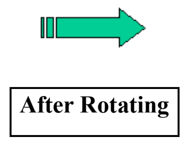
$$= \pi \left[\frac{x^7}{7} + \frac{x^4}{2} + x \right]_0^3$$

$$= \pi([355.93] - [0])$$

$$= 1118.2 \text{ units}^3$$

Example 161: The graph below shows the curve $\mathbf{y = \sqrt{x}; y \geq 0}$ and is shaded in the region $\mathbf{1 \leq x \leq 4}$ .What volume does the shaded area sweep out, when the curve is rotated completely about the x-axis?





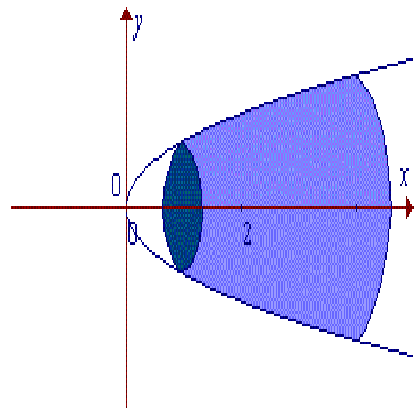


Figure # 54

Figure # 55

The volume that we are looking for is shown in the diagram (Figure # 55)

Applying the above formula, we get,

$$\mathbf{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given, $\mathbf{y = \sqrt{x}}$

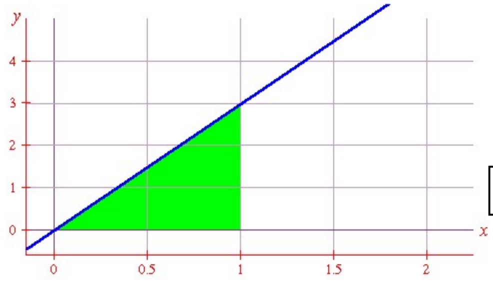
$$\mathbf{Volume} = \int_1^4 \pi(\sqrt{x})^2 dx$$

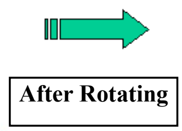
$$\mathbf{Volume} = \int_1^4 \pi x dx = \pi \int_1^4 x dx = \pi \left[\frac{x^{1+1}}{1+1}\right]_1^4 = \pi \left[\frac{x^2}{2}\right]_1^4$$

$$\mathbf{Volume} = \pi \left[\frac{4^2}{2} - \frac{1^2}{2}\right] = \pi \left[\frac{16}{2} - \frac{1}{2}\right] = \pi \left[\frac{16-1}{2}\right]$$

$$\mathbf{Volume} = \pi \left[\frac{15}{2}\right] = \frac{15\pi}{2} \quad \textit{Answer}$$

Example 162: Consider the area bounded by the straight line $\mathbf{y = 3x}$, the *x*-axis, and $x = 1$:





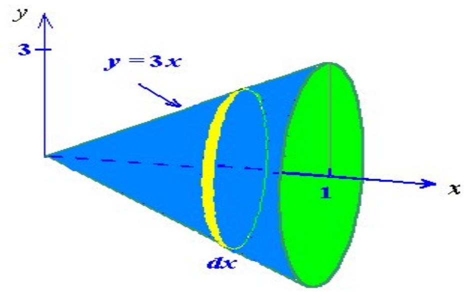


Figure # 56

Figure # 57

The resulting solid is a cone:

To find this volume, we could take slices (the *yellow* disk shown above), each *dx* wide and radius *y*:

Applying the above formula, we get,

$$\mathbf{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given,

$\mathbf{y = 3x}$

$$\mathbf{Volume} = \int_0^1 \pi (3x)^2 dx$$

$$\mathbf{Volume} = \int_0^1 \pi 9x^2 dx$$

$$\mathbf{Volume} = 9\pi \int_0^1 x^2 dx$$

$$\mathbf{Volume} = 9\pi \left[\frac{x^3}{3}\right]_0^1 = 9\pi \left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$\mathbf{Volume} = 9\pi \left[\frac{1}{3}\right] = 3\pi$$

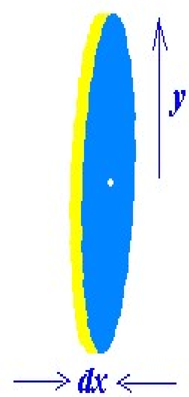


Figure # 58

CHECK: Does the method work? We can find the volume of the cone using

$$V = \frac{\pi r^2 h}{3} = \frac{\pi (3)^2 (1)}{3} = \frac{9\pi}{3} = 3\pi \ \text{unit}^3$$

Example 163: Consider the solid of revolution formed by the graph of $\mathbf{y = x^2}$ from x = 0 to x = 2:

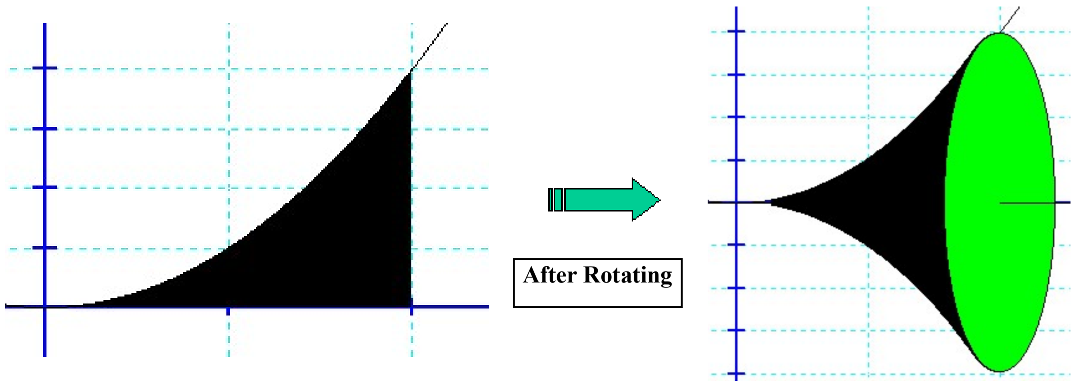


Figure # 59

Figure # 60

Applying the above formula, we get,

$$\textbf{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given, $\mathbf{y = x^2}$

$$\textbf{Volume} = \int_0^2 \pi (x^2)^2 dx$$

$$\textbf{Volume} = \int_0^2 \pi x^4 dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{x^5}{5}\right]_0^2$$

$$\textbf{Volume} = \frac{\pi}{5}[2^5 - 0] = \frac{\pi \times 2^5}{5} \quad \textit{Answer}$$

Example 164: what area is swept out when the curve of $\mathbf{y = \sin x}$ in the range $0 \le x \le \pi$ is rotated completely about the x-axis?

Applying the above formula, we get,

$$\textbf{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given, $\mathbf{y = \sin x}$

$$\textbf{Volume} = \int_0^\pi \pi (\sin x)^2 dx$$

$$\textbf{Volume} = \int_0^\pi \pi \sin^2 x dx \text{ ----------------(i)}$$

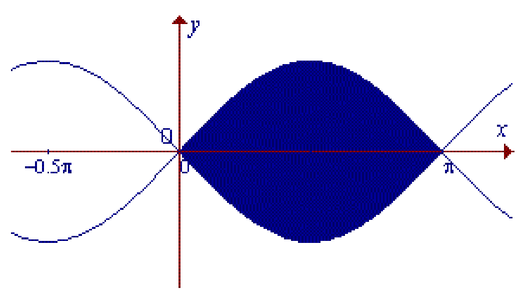


Figure # 61

We have,

$$\cos 2x = 1 - 2\sin^2 x$$

$$\Rightarrow 1 - 2\sin^2 x = \cos 2x$$

$$\Rightarrow -2\sin^2 x = \cos 2x - 1$$

$$\Rightarrow 2\sin^2 x = -\cos 2x + 1$$

$$\Rightarrow 2\sin^2 x = 1 - \cos 2x$$

$$\Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$$

From (i),

$$\text{Volume} = \int_0^{\pi} \pi \sin^2 x dx$$

$$\text{Volume} = \int_0^{\pi} \pi\left[\frac{1-\cos 2x}{2}\right]dx$$

$$\text{Volume} = \int_0^{\pi} \pi\left[\frac{1}{2}-\frac{\cos 2x}{2}\right]dx = \int_0^{\pi} \pi \times \frac{1}{2}dx - \int_0^{\pi} \frac{\pi}{2}\cos 2x\, dx$$

$$\text{Volume} = \frac{\pi}{2}\int_0^{\pi} dx - \frac{\pi}{2}\int_0^{\pi} \cos 2x\, dx = \frac{\pi}{2}[x]_0^{\pi} - \frac{\pi}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\text{Volume} = \frac{\pi}{2}[\pi - 0] - \frac{\pi}{2}\left[\frac{\sin 2\pi}{2} - \frac{\sin 2.0}{2}\right] = \frac{\pi}{2}[\pi] - \frac{\pi}{2}\left[\frac{\sin 2\pi}{2} - 0\right]$$

$$\text{Volume} = \frac{\pi}{2}[\pi] - \frac{\pi}{2} \times \frac{1}{2}\sin 2\pi = \frac{\pi}{2}[\pi] - \frac{\pi}{4}\sin 2\pi$$

$$\text{Volume} = \frac{\pi}{2}[\pi] - \frac{\pi}{4}.0 \qquad [\because \sin 2\pi = 0]$$

$$\text{Volume} = \frac{\pi}{2}[\pi] = \frac{\pi^2}{2} \quad \textit{Answer}$$

Example 165: Find the volume generated by the areas bounded by the given curves $y = x, y = 0$ and $x = 2$ if they are revolved about the *x*-axis:

Applying the above formula, we get,

$$\text{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given, $y = x$

$$\text{Volume} = \int_0^2 \pi (x)^2 dx$$

$$\text{Volume} = \int_0^2 \pi x^2 dx \quad \text{Volume} = \pi\int_0^2 x^2 dx = \pi\left[\frac{x^3}{3}\right]_0^2$$

$$\text{Volume} = \pi\left[\frac{2^3}{3} - \frac{0^3}{3}\right] = \pi\left[\frac{8}{3} - \frac{0}{3}\right] = \pi\left[\frac{8}{3} - 0\right] = \frac{8\pi}{3} \quad \textit{Answer}$$

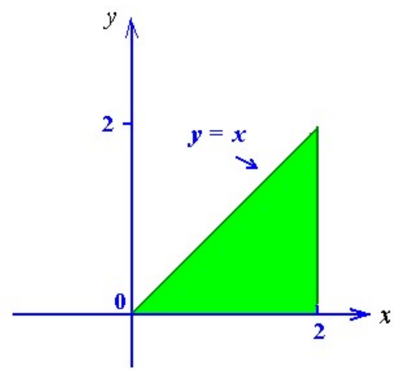


Figure # 62

Example 166: Find the volume generated by the areas bounded by the given curves $y = 2x - x^2$ and $y = 0$ if they are revolved about the *x*-axis:

Solution: From the diagram, we can see that the limits of the bounded area are $x = 0$ and $x = 2$

Applying the above formula, we get,

$$\textbf{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given,

$$y = 2x - x^2$$

$$\textbf{Volume} = \int_0^2 \pi (2x - x^2)^2 dx$$

$$\textbf{Volume} = \int_0^2 \pi (4x^2 - 4x^3 + x^4) dx$$

$$\textbf{Volume} = \int_0^2 \pi 4x^2 dx - \pi \int_0^2 4x^3 dx + \pi \int_0^2 x^4 dx$$

$$\textbf{Volume} = 4\pi \int_0^2 x^2 dx - 4\pi \int_0^2 x^3 dx + \pi \int_0^2 x^4 dx$$

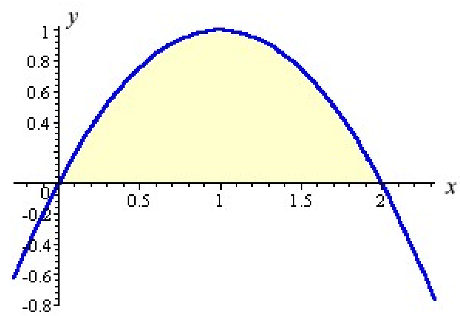


Figure # 63

$$\textbf{Volume} = 4\pi \left[\frac{x^{2+1}}{2+1}\right]_0^2 - 4\pi \left[\frac{x^{3+1}}{3+1}\right]_0^2 + \pi \left[\frac{x^{4+1}}{4+1}\right]_0^2$$

$$\textbf{Volume} = 4\pi \left[\frac{x^3}{3}\right]_0^2 - 4\pi \left[\frac{x^4}{4}\right]_0^2 + \pi \left[\frac{x^5}{5}\right]_0^2$$

$$\textbf{Volume} = 4\pi \left[\frac{2^3}{3} - \frac{0^3}{3}\right] - 4\pi \left[\frac{2^4}{4} - \frac{0^4}{4}\right] + \pi \left[\frac{2^5}{5} - \frac{0^5}{5}\right]$$

$$\textbf{Volume} = 4\pi \left[\frac{8}{3} - 0\right] - 4\pi \left[\frac{16}{4} - 0\right] + \pi \left[\frac{32}{5} - 0\right]$$

$$\textbf{Volume} = 4\pi \left[\frac{8}{3}\right] - 4\pi \left[\frac{16}{4}\right] + \pi \left[\frac{32}{5}\right] = \pi \left[\frac{32}{3} - \frac{64}{4} + \frac{32}{5}\right]$$

$$\textbf{Volume} = \pi \left[\frac{640 - 960 + 384}{60}\right] = \pi \left[\frac{64}{60}\right] = \pi \left[\frac{16}{15}\right] \textit{ Answer}$$

Example 167: you need to find the volume of the solid formed by revolving the region bounded by the graph of $y = f(x) = \sqrt{\sin x}$ and the *x*-axis $(0 \le x \le \pi)$ about the *x*-axis.

Solution: Applying the above formula, we get,

$$\textbf{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given, $y = f(x) = \sqrt{\sin x}$

From the representative rectangle above, you can see that the radius of this solid is $y = f(x) = \sqrt{\sin x}$, In turn, the volume of the solid of revolution is

$$v = \int_0^{\pi} \pi(\text{radius})^2 dx$$

$$= \int_0^{\pi} \pi[R(x)]^2 dx = \pi\int_0^{\pi} (\sqrt{\sin x})^2 dx$$

$$= \pi\int_0^{\pi} \sin x \, dx = \pi[-\cos x]_0^{\pi}$$

$$= \pi[-\cos\pi + \cos 0]$$

$$= \pi[1+1] = 2\pi$$

$[\because \cos\pi = -1, \cos 0 = 1]$ ***Answer***

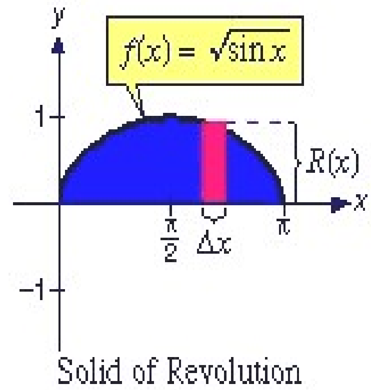


Figure # 64

Example 168:

Solution: Applying the above formula, we get,

$$\text{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given,

$$y = f(x) = \sqrt{x}$$

For $f(x) = \sqrt{x}$, $1 \le x \le 4$,

$$\text{Volume} = \int_a^b \pi(\text{radius})^2 dx$$

$$= \int_1^4 \pi(\sqrt{x})^2 dx$$

$$= \pi\int_1^4 x \, dx$$

$$= \pi \frac{x^2}{2}\Big]_1^4 = \frac{15}{2}\pi$$

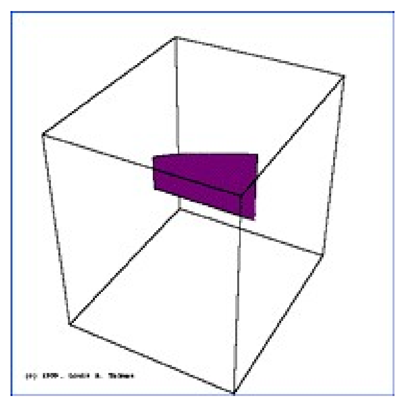


Figure # 65

Example 169: Find the volume V of the solid that is obtain when the region under the curve $y = \sin^2 x$ over the interval $[0, \pi]$ is revolved about the x-axis

Solution: Using the method of disks,
Applying the above formula, we get,

$$\text{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given,

$$y = \sin^2 x$$

$$\text{volume} = \pi\int_0^{\pi}[f(x)]^2dx$$

$$= \pi\int_0^{\pi}(\sin^2 x)^2dx$$

$$= \pi/4\int_0^{\pi}(1-\cos 2x)^2dx$$

$$[\because 2\sin^2 x = 1-\cos 2x;\ \sin^2 x = \frac{1-\text{co}}{2}$$

$$= \pi/4\int_0^{\pi}(1-2\cos 2x+\cos^2 2x)dx$$

$$[\because \cos 2x = 2\cos^2 x - 1;\ \cos^2 x = \frac{\cos 2x+1}{2};\ \cos^2 2x = \frac{\cos 4x+1}{2}]$$

$$= \pi/4\int_0^{\pi}(1-2\cos 2x+\frac{\cos 4x+1}{2})dx = \pi/4\int_0^{\pi}(1-2\cos 2x+\frac{1}{2}\cos 4x+\frac{1}{2})dx$$

$$= \pi/4\left[x-\frac{2\sin 2x}{2}+\frac{1}{2}*\frac{1}{4}\sin 4x+\frac{1}{2}x\right]_0^{\pi} = \pi/4\left[\frac{3}{2}x-\sin 2x+\frac{1}{8}\sin 4x\right]_0^{\pi}$$

$$= \frac{\pi}{4}[(\frac{3}{2}\pi-\sin 2\pi+\tfrac{1}{8}\sin 4\pi)-\frac{3}{2}.0-0+0)] = \frac{3}{8}\pi^2\quad [\because \sin 2\pi = \sin 4\pi = 0]$$

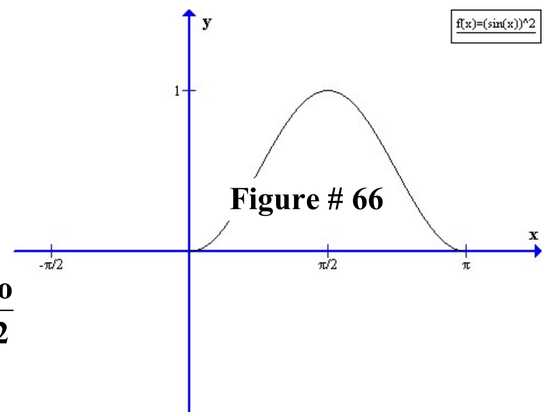


Figure # 66

Example 170: There area enclosed by the curve $y = 1+\cos x$; the x and y-axes and the line $x = \frac{\pi}{2}$ is rotated through 4 right angles about the x-axis. Fund the volume generated.

Solution:

Applying the above formula, we get,

$$\text{Volume} = \int_a^b \pi r^2dx = \int_a^b \pi y^2dx = \int_a^b \pi(f(x))^2dx$$

Given, $y = 1+\cos x$

Volume generated is

$$V = \pi\int_0^{\pi/2}(1+\cos x)^2dx$$

$$= \pi\int_0^{\pi/2}(1+2\cos x+\cos^2 x)dx$$

But, $[\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2}+\frac{\cos 2x}{2}]$

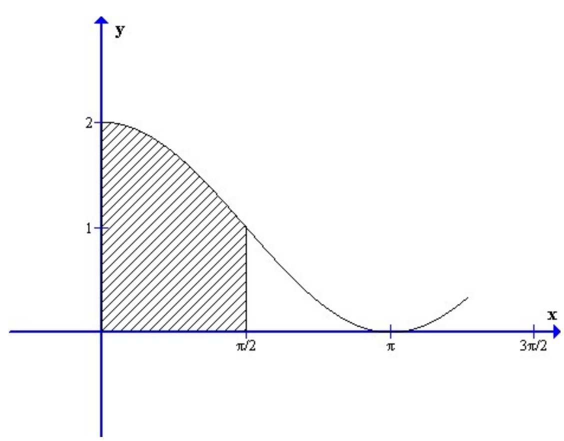


Figure # 67

$$V=\pi\int_0^{\pi/2}\left(1+2\cos x+\frac{1}{2}+\frac{1}{2}\cos 2x\right)dx$$

$$\therefore V=\pi\int_0^{\pi/2}\left(\frac{3}{2}+2\cos x+\frac{1}{2}\cos 2x\right)dx$$

$$=\pi\left(\frac{3}{2}x+2\sin x+\frac{1}{2}\frac{\sin 2x}{2}\right)_0^{\pi/2}$$

$$=\pi\left[\frac{3}{2}\left(\frac{\pi}{2}\right)+2\sin\left(\frac{\pi}{2}\right)+\frac{\sin 2\left(\frac{\pi}{2}\right)}{4}\right]-\pi\left[\frac{3}{2}(0)+2\sin 0+\frac{\sin 0}{4}\right]$$

$$=\pi\left[\frac{3\pi^2}{4}+2.1+\frac{\sin\pi}{4}\right]-\pi[0+0+0]=\pi\left[\frac{3\pi^2}{4}+2.1+0\right]-\pi[0]$$

$$=\left(\frac{3\pi^2}{4}+2\pi\right)\text{units}^3\ [\because \sin\pi=0]\ \textit{Answer}$$

Example 171: If the function $y=\frac{1}{x}$ is revolved around the *x*-axis for $x \geq 1$; find the figure has a finite volume.

Answer

Applying the above formula, we get,

$$\textbf{Volume}=\int_a^b \pi r^2 dx=\int_a^b \pi y^2 dx=\int_a^b \pi(f(x))^2 dx$$

Given, $y=\frac{1}{x}$

Volume generated is $V=\pi\int_1^\infty\left(\frac{1}{x}\right)^2 dx$

Volume generated is $V=\pi\int_1^\infty\frac{1}{x^2}dx=\pi\int_1^\infty x^{-2}dx=\pi\times\left[\frac{x^{-2+1}}{-2+1}\right]_1^\infty$

Volume generated is $V=\pi\times\left[\frac{x^{-1}}{-1}\right]_1^\infty=-\pi\times\left[\frac{1}{x}\right]_1^\infty=-\pi\times\left[\frac{1}{\infty}-\frac{1}{1}\right]_1^\infty$

Volume generated is $V=-\pi\times[0-1]=-\pi\times[-1]=\pi$ ***Answer***

Example 172: The region enclosed by the curve $\mathbf{y = x^2 + 2}$ and the line $\mathbf{y = 2 + 3x}$ is rotated through 360^0 about the x-axis Calculate the volume of the solid thus formed.

Solution:

In the sketch the line intersects the curve at the points A and B

$$\mathbf{y = x^2 + 2 ----------(i)}$$
$$\mathbf{y = 2 + 3x ----------(ii)}$$

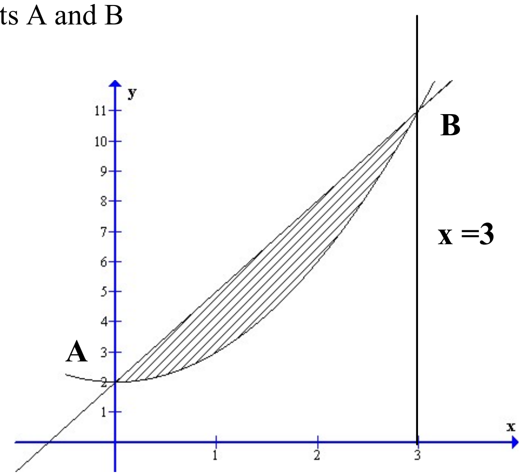


Figure # 68

From (i) and (ii)

$$\mathbf{2 + 3x = x^2 + 2}$$

$$\Rightarrow \mathbf{x^2 - 3x = 0}$$

$$\Rightarrow \mathbf{x(x-3) = 0}$$

$$\Rightarrow \mathbf{x = 0 \quad or \quad x = 3}$$

Putting the value in (i) and (ii),

$$\Rightarrow \mathbf{y = 2 \text{ or } y = 2 + 3.3 = 11}$$

$\therefore$ The coordinates of $\mathbf{A(x,y) = (0,2)}$ and the coordinates of $\boldsymbol{B(x,y) = (3,11)}$

Applying the above formula, we get,

$$\textbf{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

$\therefore$ Volume of the solid generated is

$$V = \int_0^3 \pi\left[(2+3x) - (x^2+2)\right]^2 dx$$

$$V = \int_0^3 \pi\left[2 + 3x - x^2 - 2\right]^2 dx$$

$$V = \int_0^3 \pi(3x - x^2)^2 dx$$

$$V = \int_0^3 \pi[(3x)^2 - 2.3x.x^2 + (x^2)^2]\, dx$$

$$V = \int_0^3 \pi[9x^2 - 6x^3 + x^4]\, dx$$

$$= \pi\left[\frac{9x^3}{3} - \frac{6x^4}{4} + \frac{x^5}{5}\right]_0^3 \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1}]$$

$$= \pi\left[[\frac{9.3^3}{3} - \frac{6.3^4}{4} + \frac{3^5}{5}] - [\frac{9.0^3}{3} - \frac{6.0^4}{4} + \frac{0^5}{5}] \right]$$

$$= \pi\left[\frac{9.27}{3} - \frac{6.81}{4} + \frac{243}{5} - 0 \right]$$

$$V = \pi\left[\frac{9.27}{3} - \frac{6.81}{4} + \frac{243}{5} \right] \textit{ Answer}$$

Example 173: Find the volume of the solid generated by revolving the region bounded by $\mathbf{y = x^2}$ and the x-axis on [- 2, 3] about the x-axis.

Solution: Because the x-axis is a boundary of the region, you can use the disk method (see Figure # 69)

Applying the above formula, we get,

$$\textbf{Volume} = \int_a^b \pi r^2 dx = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Given, $\mathbf{y = x^2}$

$$\textbf{Volume} = \int_{-2}^{3} \pi (x^2)^2 dx = \int_{-2}^{3} \pi x^4 dx$$

$$\textbf{Volume} = \pi \int_{-2}^{3} x^4 dx$$

$$\textbf{Volume} = \pi \left[\frac{x^5}{5} \right]_{-2}^{3}$$

$$\textbf{Volume} = \pi \left[\frac{3^5}{5} - \frac{(-2)^5}{5} \right]_{-2}^{3} = \pi \left[\frac{243}{5} - \frac{-32}{5} \right]$$

$$\textbf{Volume} = \pi \left[\frac{243}{5} + \frac{32}{5} \right] = \pi \left[\frac{243+32}{5} \right]$$

$$\textbf{Volume} = \pi \left[\frac{275}{5} \right] = 55\pi \textit{ Answer}$$

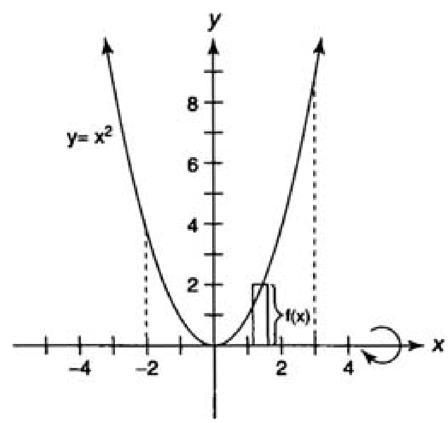


Figure # 69

Vertical Axis of Revolution

If the region bounded by $\mathbf{x = f(y) = R(y)}$ and the *y*-axis on $\mathbf{[c, d]}$ is revolved about the *y*-axis, then its volume (V) is

$$\textbf{Volume} = \mathbf{V} = \pi\int_c^d [f(y)]^2 dy = \pi\int_c^d r^2 dy = \pi\int_c^d x^2 dy$$

[With respect to y-axis r = x]

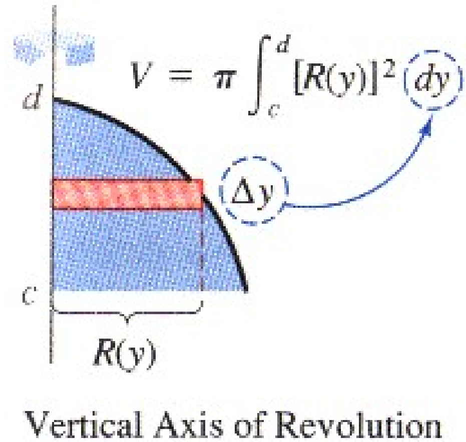


Figure # 70

Example 174: Find the volume of the solid of revolution generated by rotating the curve $\mathbf{y = x^3}$ between $\mathbf{y = 0}$ and $\mathbf{y = 4}$ about the *y*-axis.

Applying the above formula, we get,

Solution: $\textbf{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi (f(y))^2 dy$ ----------------------(i)

Given, $\mathbf{y = x^3}$

$$\Rightarrow y^{1/3} = (x^3)^{1/3}$$
$$\Rightarrow y^{1/3} = x$$
$$\Rightarrow x = y^{1/3}$$
$$\Rightarrow x = y^{1/3} = f(y)$$

Here is the region we need to rotate:

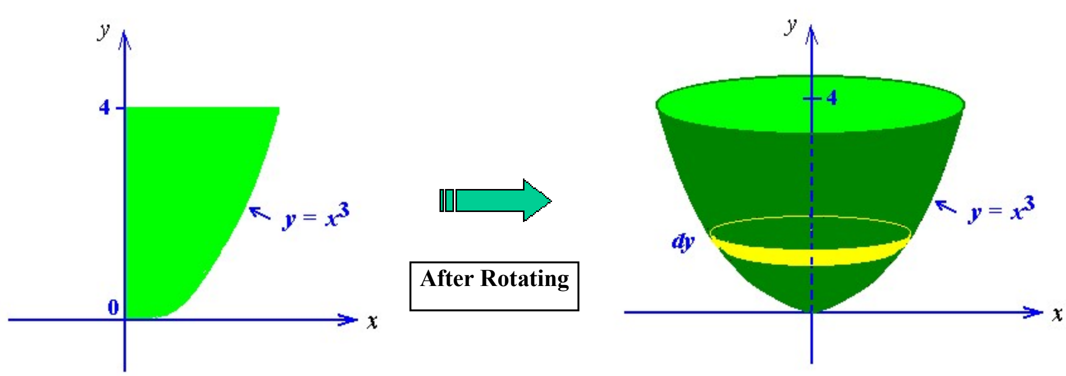


Figure # 71 **Figure # 72**

And here is the volume generated:

From (i), We get ,

$$\textbf{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi(f(y)^2 dy$$

$$\textbf{Volume} = \int_0^4 \pi x^2 dy = \int_0^4 \pi(y^{1/3})^2 dy = \int_0^4 \pi(y^{2/3})dy = \pi\int_0^4 y^{2/3} dy$$

$$\textbf{Volume} = \pi\left[\frac{y^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right]_0^4 = \pi\left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^4 = \pi\times\frac{3}{5}\left[y^{\frac{5}{3}}\right]_0^4 = \pi\times\frac{3}{5}\left[4^{\frac{5}{3}} - 0^{\frac{5}{3}}\right]$$

$$\textbf{Volume} = \pi\times\frac{3}{5}\left[4^{\frac{5}{3}}\right] = \pi\times\frac{3}{5}\times 4^{\frac{5}{3}} = \frac{3}{5}\pi\times 4^{\frac{5}{3}} \;\; \textit{Answer}$$

Example 175: Find the volume generated by the areas bounded by the given curves if they are revolved about the *y*-axis:

a) $y^2 = x$ **,** $y = 4$ and $x = 0$ [revolved about the *y*-axis]

Answer

Hence, the volume generated is:
Applying the above formula, we get,

$$\textbf{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi(f(y)^2 dy \text{ ---------------------------------(i)}$$

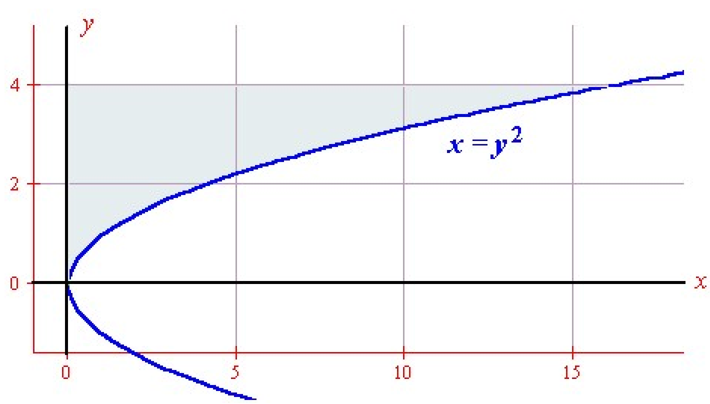


Figure # 73

Given, $y^2 = x$

$$\Rightarrow x = y^2 = f(y)$$

From (i), we get

$$\textbf{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi (f(y))^2 dy$$

$$\textbf{Volume} = \int_0^4 \pi (f(y))^2 dy = \int_0^4 \pi x^2 dy = \int_0^4 \pi (y^2)^2 dy = \int_0^4 \pi y^4 dy = \pi \left[\frac{y^{4+1}}{4+1} \right]_0^4$$

$$\textbf{Volume} = \int_0^4 \pi (f(y))^2 dy = \pi \left[\frac{y^5}{5} \right]_0^4 = \pi \left[\frac{4^5}{5} - \frac{0^5}{5} \right] = \pi \left[\frac{4^5}{5} \right] \textit{ Answer}$$

(b) $\mathbf{x^2 + 4y^2 = 4}$ (Quadrant I) [revolved around *y*-axis]

Answer

We recognize that this is an ellipse. The question tells us to draw the first quadrant area only. From the diagram, we can see that the limits of the bounded area are $y = 0$ and $y = 1$.

Applying the above formula, we get,

$$\textbf{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi (f(y))^2 dy \text{ ----------------------------------(i)}$$

Given,

$$x^2 + 4y^2 = 4$$
$$x^2 = 4 - 4y^2$$

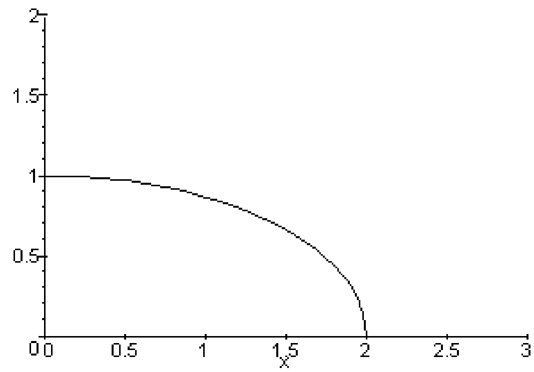


Figure # 74

From (i), we get

$$\textbf{Volume} = \int_c^d \pi (f(y))^2 dy = \int_c^d \pi x^2 dy$$

$$= \int_0^1 \pi (4 - 4y^2) dy$$

$$= \pi \int_0^1 (4 - 4y^2) dy$$

$$= 4\pi \int_0^1 dy - 4\pi \int_0^1 y^2 dy = 4\pi [y]_0^1 - 4\pi \left[\frac{y^{2+1}}{2+1} \right]_0^1 = 4\pi [1 - 0] - 4\pi \left[\frac{y^3}{3} \right]_0^1$$

$$= 4\pi - 4\pi \left[\frac{1^3}{3} - \frac{0^3}{3} \right] = 4\pi - 4\pi \left[\frac{1^3}{3} \right] = 4\pi - 4\pi \times \frac{1}{3} = 4\pi (1 - \frac{1}{3})$$

$$= 4\pi (\frac{3-1}{3}) = 4\pi (\frac{2}{3}) = \frac{8\pi}{3} \textit{ Answer}$$

Example 176: The region R shown is bounded by the curve with equation $\mathbf{y = \ln x}$, the y-axis and the lines $\mathbf{y = 2}$ and $\mathbf{y = 5}$

Find a) the area of R (b) the volume generated when R is rotated through 2π about the y-axis, giving your answers to 3 significant figures.

a) Given, $y = \ln x$

$\Rightarrow e^y = e^{\ln x}$

$\Rightarrow e^y = e^{\ln x^1}$

$\Rightarrow e^y = x \ln e$

$\Rightarrow e^y = x.1$

$\Rightarrow e^y = x$

The area of R: $\int_2^5 x dy = \int_2^5 e^y dy$

Area of R = $\left[e^y\right]_2^5 = e^5 - e^2 \approx 141 \text{units}^2$

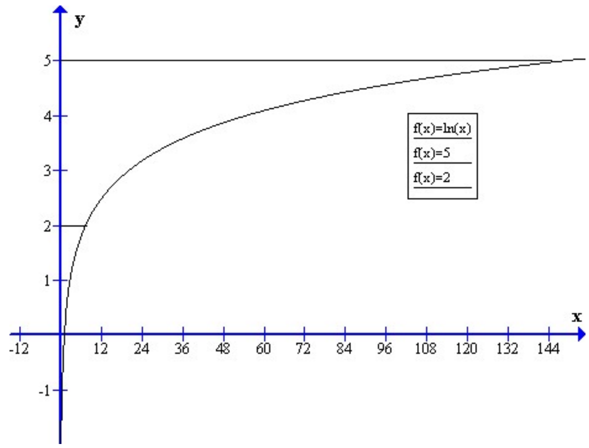


Figure # 75

b) The volume generated when R is rotated through 2π about the y-axis is
Applying the above formula, we get,

$$\textbf{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi (f(y))^2 dy \text{ -----------------------------(i)}$$

$$\textbf{Volume} = \int_2^5 \pi x^2 dy = \int_2^5 \pi (e^y)^2 dy = \int_2^5 \pi e^{2y} dy = \pi \left[\frac{1}{2} e^{2y}\right]_2^5$$

$$\textbf{Volume} = \pi\left[\frac{1}{2}e^{2\times 5} - \frac{1}{2}e^{2\times 2}\right] = \pi\left[\frac{1}{2}e^{10} - \frac{1}{2}e^4\right] = \frac{\pi}{2}\left[e^{10} - e^4\right] \textit{ Answer}$$

Example 177: The region bounded by the curve $y = x^2$ and the lines $y = 1$ and $y = 2$ is rotated through 4 right angles about the y-axis Find the volume of the solid generated

Solution: Applying the above formula, we get,

$$\textbf{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi (f(y))^2 dy \text{ ----------------------------------(i)}$$

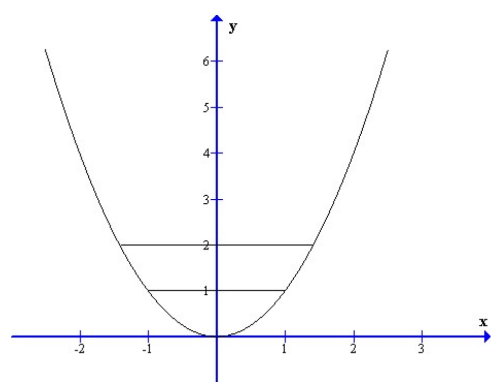
Given, $y = x^2$

$\Rightarrow x^2 = y$

Putting the value of x^2 in (i)

The volume generated is

$$\textbf{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi (f(y))^2 dy$$



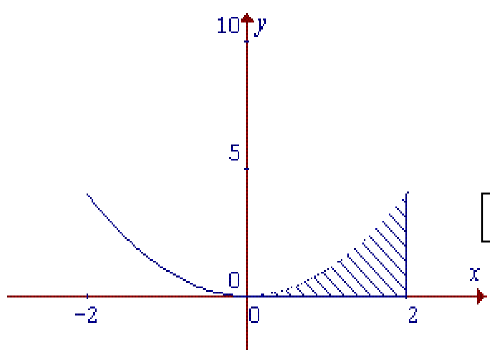
$$\text{Volume} = \int_1^2 \pi(f(y))^2 dy = \int_1^2 \pi x^2 dy$$

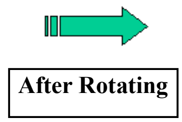
Figure # 76

$$\text{Volume} = \int_1^2 \pi(f(y))^2 dy = \int_1^2 \pi y dy$$

$$\text{Volume} = \pi\left(\frac{y^2}{2}\right)_1^2 = \pi\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{3\pi}{2} \text{ Units}^3 \quad \textbf{\textit{Answer}}$$

Example 178: You may be asked to rotate a curve $\mathbf{y = x^2;\ 0 \le x \le 2}$ about the y-axis, for example if you rotate the shaded area below, about the y-axis.





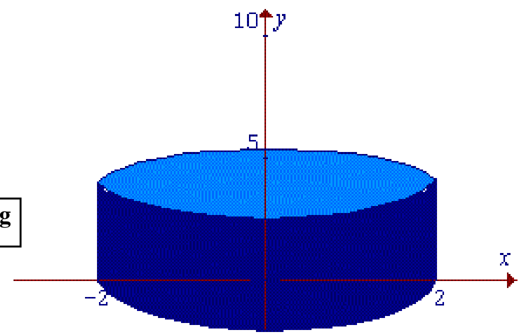


Figure # 77 **Figure # 78**

All that you need to, to find a volume rotated about the y-axis, is slightly alter the formula to

Applying the above formula, we get,

$$\text{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi(f(y))^2 dy \text{ ---------------------------------(i)}$$

Given, $\mathbf{y = x^2}$

$\Rightarrow \mathbf{x^2 = y}$

The curve in question is: $\mathbf{y = x^2}$ in the range: $\mathbf{0 \le x \le 2}$

Before we can integrate, we must find y_1 and y_2.

From (1),

$$\text{Volume} = \int_c^d \pi r^2 dy = \int_c^d \pi x^2 dy = \int_c^d \pi(f(y))^2 dy$$

x	0	2
$y = x^2$	$y = x^2$ $y = 0^2 = 0$	$y = x^2$ $y = 2^2 = 4$

$$\text{Volume} = \int_c^d \pi(f(y))^2 dy = \int_0^4 \pi y dy$$

$$\text{Volume} = \pi\int_0^4 y dy = \pi\left[\frac{y^2}{2}\right]_0^4 = \pi\left[\frac{4^2}{2} - \frac{0^2}{2}\right] = \pi\left[\frac{16}{2} - \frac{0}{2}\right]$$

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$$\text{Volume} = \pi\left[\frac{16}{2}\right] = 8\pi \ \textit{Answer}$$

Example 179: Find Area of an ellipse
Solution:

We need integral calculus to find the area of the ellipse. If we think of the area in the first quadrant with x and y both positive, the area is given by $\int_0^a y\,dx$

Let the equation of the ellipse in parametric form be:

$$x = a\cos\theta$$
$$y = b\sin\theta$$

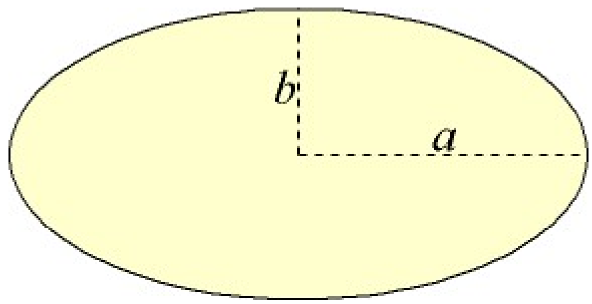

Figure # 79

Given,

$$x = a\cos\theta$$
$$\Rightarrow dx = -a\sin\theta$$

$$\text{Area} = \int_0^a y\,dx$$

$$= \int_{\pi/2}^{0} (b\sin\theta) * (-a\sin\theta)d\theta$$

$$= -\int_{\pi/2}^{0} (b\sin\theta) * (a\sin\theta)d\theta$$

$$= \int_0^{\pi/2} ab\sin^2\theta\ d\theta$$

$$= \int_0^{\pi/2} \frac{ab}{2} 2\sin^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{ab}{2}(1 - \cos 2\theta)d\theta$$

$$= \frac{ab}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$x = a\cos\theta$	0	a
$x = a\cos\theta$ $\Rightarrow \cos\theta = \frac{x}{a}$ $\Rightarrow \theta = \cos^{-1}(\frac{x}{a})$	$\theta = \cos^{-1}(\frac{x}{a})$ $\theta = \cos^{-1}(\frac{0}{a})$ $\theta = \cos^{-1} 0$ $\theta = \cos^{-1}\cos\frac{\pi}{2}$ $\theta = \frac{\pi}{2}$	$\theta = \cos^{-1}(\frac{x}{a})$ $\theta = \cos^{-1}(\frac{a}{a})$ $\theta = \cos^{-1}(1)$ $\theta = \cos^{-1}(\cos 0)$ $\theta = 0$

$$= \frac{ab}{2}\left[\frac{\pi}{2} - \frac{\sin 2\times\frac{\pi}{2}}{2} - 0 + \frac{\sin 2\times 0}{2}\right] = \frac{ab}{2}\left[\frac{\pi}{2} - \frac{\sin\pi}{2} - 0 + \frac{\sin 0}{2}\right]$$

$$= \frac{ab}{2}\left[\frac{\pi}{2} - 0 + 0\right] = \frac{ab}{2}(\frac{\pi}{2} - 0) = \frac{\pi ab}{4}$$

$\therefore$ The total area of the ellipse will be 4 times this area, so: Area of ellipse = $4 * \frac{\pi ab}{4} = \pi ab$

Or Find the area of Ellipse

The equation of the ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$

$$\Rightarrow y^2 = b^2 \times (\frac{a^2 - x^2}{a^2})$$

$$\Rightarrow y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\Rightarrow y = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

Solution: Because the ellipse is symmetric about both axes, its area A is four times the area in the first quadrant (Figure).If we solve the equation of the ellipse for y in terms of x, we obtain

Where the positive square root gives the equation of the upper half

Thus, the area A is given by

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx \text{ ------------------------------(i)}$$

Let, $x = a\sin\theta$

x	0	a
$x = a\sin\theta$ $\Rightarrow \sin\theta = \frac{x}{a}$ $\Rightarrow \theta = \sin^{-1}(\frac{x}{a})$	$\theta = \sin^{-1}(\frac{x}{a})$ $\theta = \sin^{-1}(\frac{0}{a})$ $\theta = \sin^{-1} 0$ $\theta = \sin^{-1}\sin 0$ $\theta = 0$	$\theta = \sin^{-1}(\frac{x}{a})$ $\theta = \sin^{-1}(\frac{a}{a})$ $\theta = \sin^{-1}(1)$ $\theta = \sin^{-1}(\sin\frac{\pi}{2})$ $\theta = \frac{\pi}{2}$

Given,

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

Thus, we obtain, From (i),

$$A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$\Rightarrow A = \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2-(a\sin\theta)^2}.(a\cos\theta)d\theta$$

$$\Rightarrow A = \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}.(a\cos\theta)d\theta$$

$$\Rightarrow A = \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2(1-\sin^2\theta)}.(a\cos\theta)d\theta$$

$$\Rightarrow A = \frac{4b}{a}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta}.(a\cos\theta)d\theta = \frac{4b}{a}\int_0^{\pi/2}a\cos\theta.\,a\cos\theta\,d\theta$$

$$\Rightarrow A = \frac{4b}{a}\times a^2\int_0^{\pi/2}\cos\theta.\cos\theta\,d\theta = 4ab\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$\Rightarrow A = 4ab\times\frac{1}{2}\int_0^{\pi/2}2\cos^2\theta\,d\theta = 4ab\times\frac{1}{2}\int_0^{\pi/2}(1+\cos 2\theta)d\theta$$

$$\Rightarrow A = 2ab\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2ab\left[\frac{\pi}{2}+\frac{1}{2}\sin 2\times\frac{\pi}{2}-0-\frac{1}{2}\sin 2\times 0\right]$$

$$\Rightarrow A = 2ab\left[\frac{\pi}{2}+\frac{1}{2}\sin\pi-0-\frac{1}{2}\sin 0\right] = 2ab\left[\frac{\pi}{2}+0+0\right]$$

$$\Rightarrow A = 2ab\left[\frac{\pi}{2}\right] = 2ab\times\frac{\pi}{2} = \pi ab \;\; \textit{Answer}$$