03. Finding Area of a Curve

3.1. Area under a Curve, which are entirely above the x-axis.

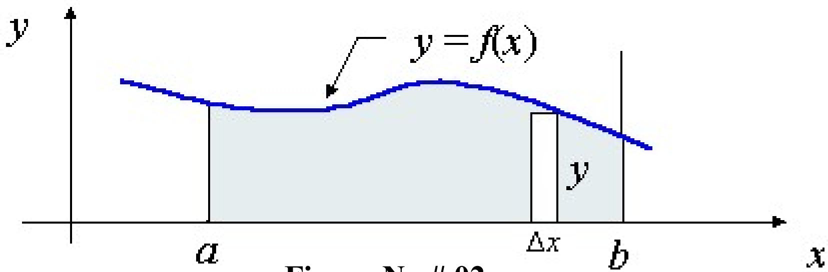


Figure No # 02

In this case, we find the area by simply finding the integral:

$$\textbf{Area} = \int_a^b f(x)dx$$

Example 25: Find the area underneath the curve $y = x^2 + 2$ from *x* = 1 to *x* = 2.
Solution:

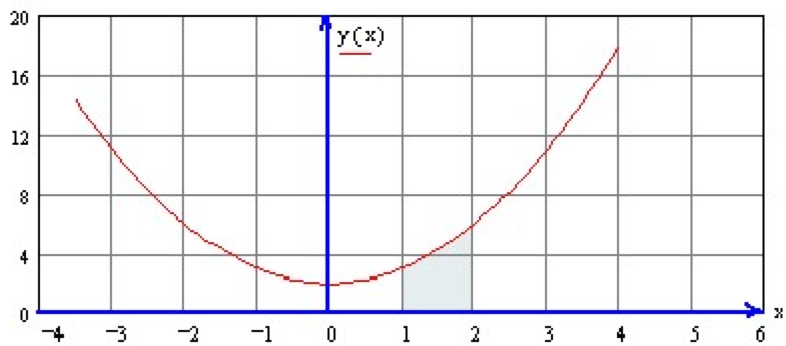


Figure No # 03

$$\text{Area} = \int_a^b f(x)dx$$

$$\text{Area} = \int_1^2 (x^2 + 2)dx = \int_1^2 x^2 dx + \int_1^2 2dx$$

$$\text{Area} = \left[\frac{x^{2+1}}{2+1}\right]_1^2 + [2x]_1^2 \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\ ] \text{ and } [\because \int dx = x]$$

$$\text{Area} = \left[\frac{x^3}{3}\right]_1^2 + [2x]_1^2 = \left[\frac{2^3}{3} - \frac{1^3}{3}\right] + [2 \times 2 - 2 \times 1]$$

$$\text{Area} = \left[\frac{8}{3} - \frac{1}{3}\right] + [4 - 2] = \left[\frac{8-1}{3}\right] + [2] = \left[\frac{7}{3}\right] + [2]$$

Area $= \frac{7}{3} + 2 = \frac{7+6}{3} = \frac{13}{3}$ square unit

Example 26: The region R is bounded by the curve with equation $\mathbf{y = \cos^2 2x}$, the x-axis and the lines $\mathbf{x = 0}$ and $\mathbf{x = \frac{\pi}{6}}$. Find the area of R.

Solution: First draw a sketch, showing R:

Given, $\mathbf{y = f(x) = \cos^2 2x}$

$$\mathbf{Area = \int_a^b f(x)dx}$$

$$\text{Area of R} = \int_0^{\pi/6} \cos^2 2x\,dx$$

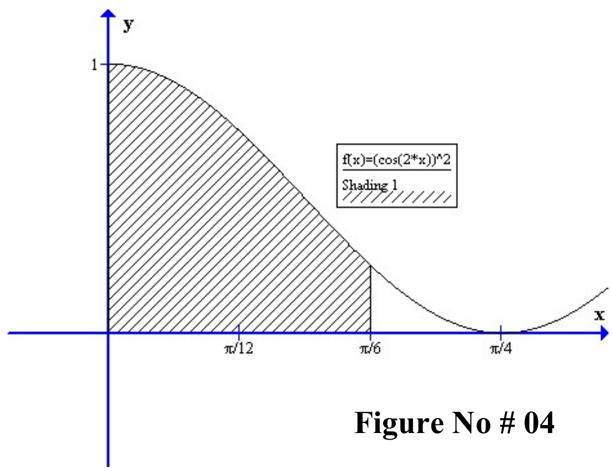


Figure No # 04

Use the identity

$$\cos 2x = 2\cos^2 x - 1.$$

$$\therefore \cos 4x = 2\cos^2 2x - 1$$

$$\therefore \cos 4x + 1 = 2\cos^2 2x$$

$$\therefore \cos^2 2x = \frac{1+\cos 4x}{2}$$

$$\text{Area of R} = \int_0^{\pi/6} \cos^2 2x\,dx = \int_0^{\pi/6} \left(\frac{1+\cos 4x}{2}\right)dx = \int_0^{\pi/6} \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)dx$$

$$\text{Area of R} = \left[\frac{1}{2}x + \frac{1}{2}.\frac{1}{4}\sin 4x\right]_0^{\pi/6}$$

$$\text{Area of R} = \left[\frac{1}{2}.\frac{\pi}{6} + \frac{1}{2}.\frac{1}{4}\sin 4.\frac{\pi}{6} - \frac{1}{2}.0 - \frac{1}{2}.\frac{1}{4}\sin 4.0\right]$$

$$\text{Area of R} = \frac{\pi}{12} + \frac{1}{8}\sin\frac{2\pi}{3} - (0+0)$$

$$\text{Area of R} = \frac{\pi}{12} + \frac{1}{8}.\frac{\sqrt{3}}{2} \quad [\because \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}]$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{16} \quad \textit{Answer}$$

Example 27: Find the area of the region under the curve $\mathbf{y = \sin^2 x}$ and above the interval $[\frac{\pi}{6}, \frac{\pi}{3}]$

Solution: We first find an antiderivative of $\mathbf{\sin^2 x}$

$[\because 1-\cos 2x = 2\sin^2 x; \therefore \sin^2 x = \frac{1-\cos 2x}{2}]$

Now, $\int \sin^2 x dx$

$$= \int \frac{1-\cos 2x}{2} dx$$

$$= \int \frac{dx}{2} - \int \frac{\cos 2x}{2} dx$$

$$= \frac{x}{2} - \frac{1}{2}\frac{\sin 2x}{2} + c.$$

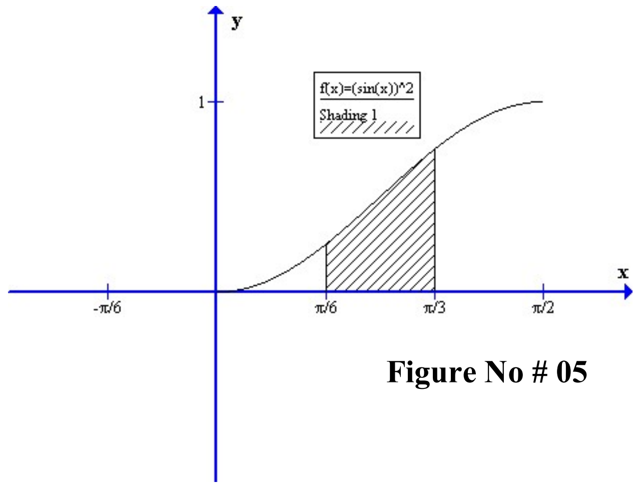


Figure No # 05

Given, $y = f(x) = \sin^2 x$

$$\text{Area} = \int_a^b f(x)dx$$

Thus, the area $= \int_{\pi/6}^{\pi/3} \sin^2 x dx$

$$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \qquad [\because \int \sin^2 x dx = \frac{x}{2} - \frac{1}{2}\frac{\sin 2x}{2}]$$

$$= \left[\frac{\frac{\pi}{3}}{2} - \frac{\sin 2.\frac{\pi}{3}}{4} - \left[\frac{\frac{\pi}{6}}{2} - \frac{\sin 2.\frac{\pi}{6}}{4}\right]\right] = \left[\frac{\frac{\pi}{3}}{2} - \frac{\sin 2.\frac{\pi}{3}}{4} - \frac{\frac{\pi}{6}}{2} + \frac{\sin 2.\frac{\pi}{6}}{4}\right]$$

$$= \frac{\pi}{6} - \frac{\sin\frac{2\pi}{3}}{4} - \frac{\pi}{12} + \frac{\sin\frac{\pi}{3}}{4} = \frac{\pi}{6} - \frac{\pi}{12} + \frac{\sin\frac{\pi}{3}}{4} - \frac{\sin\frac{2\pi}{3}}{4}$$

$$= \frac{2\pi - \pi}{12} + \frac{\frac{\sqrt{3}}{2}}{4} - \frac{\frac{\sqrt{3}}{2}}{4} = \frac{\pi}{12} + \frac{\sqrt{3}/2}{4} - \frac{\sqrt{3}/2}{4} = \frac{\pi}{12} = \frac{\pi}{12}\textit{Answer}$$

3.2. Curves which are entirely below the x-axis

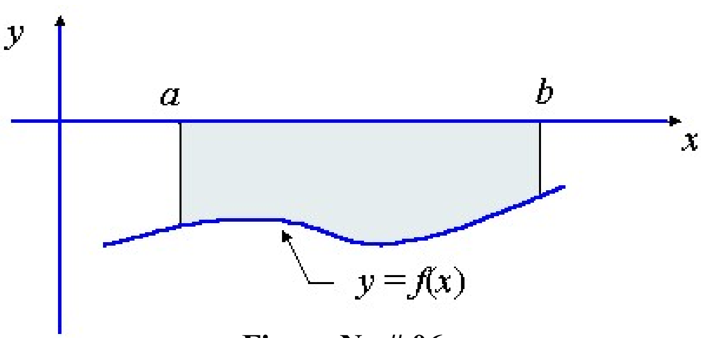


Figure No # 06

In this case, the integral gives a negative number. We need to take the absolute value of this to find our area:

$$\mathbf{Area} = \left|\int_a^b f(x)dx\right|$$

Example 28: Find the area bounded by $\mathbf{y = x^2 - 4}$, the *x*-axis and the lines $\mathbf{x = -1}$ and $\mathbf{x = 2}$

Solution:

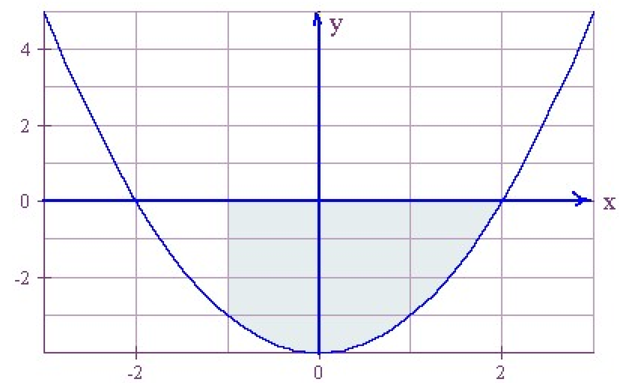


Figure No # 07

$$\mathbf{Area} = \left|\int_a^b f(x)dx\right|$$

$$\Rightarrow \mathbf{Area} = \left|\int_{-1}^{2}(x^2-4)dx\right| = \left|\int_{-1}^{2} x^2dx - \int_{-1}^{2}4dx\right|$$

$$\Rightarrow \mathbf{Area} = \left|\left[\frac{x^{2+1}}{2+1}\right]_{-1}^{2} - [4x]_{-1}^{2}\right| \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\ ] \text{ and } [\because \int dx = x]$$

$$\Rightarrow \mathbf{Area} = \left|\left[\frac{x^3}{3}\right]_{-1}^{2} - [4x]_{-1}^{2}\right| = \left|\left[\frac{2^3}{3} - \frac{(-1)^3}{3}\right] - [4\times 2 - 4(-1)]\right|$$

$$\Rightarrow \mathbf{Area} = \left|\left[\frac{8}{3} - \frac{-1}{3}\right] - [8+4]\right| = \left|\left[\frac{8}{3} + \frac{1}{3}\right] - [8+4]\right| = \left|\left[\frac{8+1}{3}\right] - [12]\right|$$

$$\Rightarrow \mathbf{Area} = \left|\left[\frac{9}{3}\right] - [12]\right| = |[3] - [12]| = |-9| \text{ square unit}$$

3.3. Part of the curve is below the x-axis and part of the curve is above the x-axis.

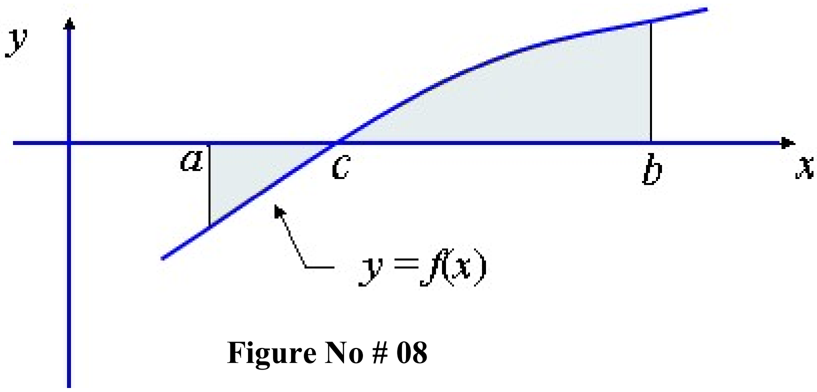


Figure No # 08

In this case, we have to sum the individual parts, taking the absolute value for the section where the curve is below the x-axis (from $x = a$ to $x = c$).

$$\text{Area } = \left|\int_a^c \mathbf{f(x)dx}\right| + \int_c^b \mathbf{f(x)dx}$$

Example 29: What is the area bounded by the curve $\mathbf{y = x^3}$, $\mathbf{x = -2}$ and $\mathbf{x = 1}$

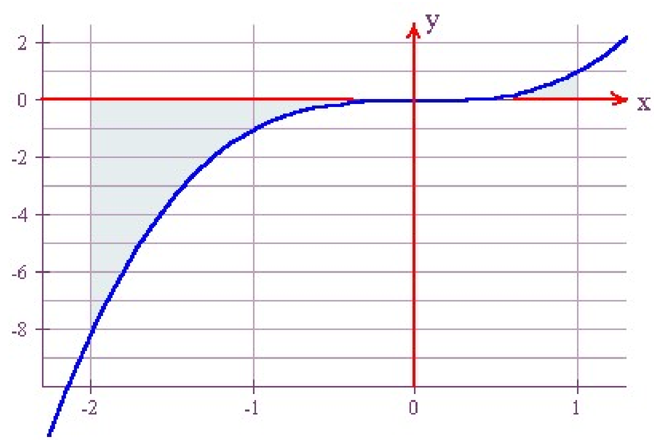


Figure No # 09

Solution: We can see from the graph that the portion between $\mathbf{x = -2}$ and $\mathbf{x = 0}$ is below the *x*-axis, so we need to take the absolute value for that portion.

$$\text{Area } = \left|\int_a^c \mathbf{f(x)dx}\right| + \int_c^b \mathbf{f(x)dx}$$

$$\Rightarrow \text{Area } = \left|\int_{-2}^{0} \mathbf{f(x)dx}\right| + \int_0^1 \mathbf{f(x)dx} = \left|\int_{-2}^{0} \mathbf{x^3dx}\right| + \int_0^1 \mathbf{x^3dx} = \left|\left[\frac{\mathbf{x^{3+1}}}{\mathbf{3+1}}\right]_{-2}^{0}\right| + \left[\frac{\mathbf{x^{3+1}}}{\mathbf{3+1}}\right]_0^1$$

$$\Rightarrow \text{Area} = \left| \left[\frac{x^4}{4} \right]_{-2}^{0} \right| + \left[\frac{x^4}{4} \right]_{0}^{1} = \left| \left[\frac{0^4}{4} - \frac{(-2)^4}{4} \right] \right| + \left[\frac{1^4}{4} - \frac{0^4}{4} \right]$$

$$\Rightarrow \text{Area} = \left| \left[0 - \frac{16}{4} \right] \right| + \left[\frac{1}{4} - 0 \right] = \left| \left[-\frac{16}{4} \right] \right| + \left[\frac{1}{4} \right] = \frac{16}{4} + \frac{1}{4} = \frac{16+1}{4} = \frac{17}{4} \text{ Square unit}$$

We have,

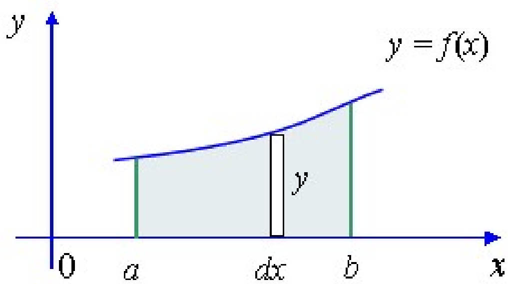


Figure No # 10

We are (effectively) finding the area by horizontally adding the areas of the rectangles, width dx and heights y (which we find by substituting values of x into $f(x)$).
So

$$Area = \int_a^b f(x)dx$$

(With absolute value signs where necessary).

3.4. Certain curves are much easier to sum vertically

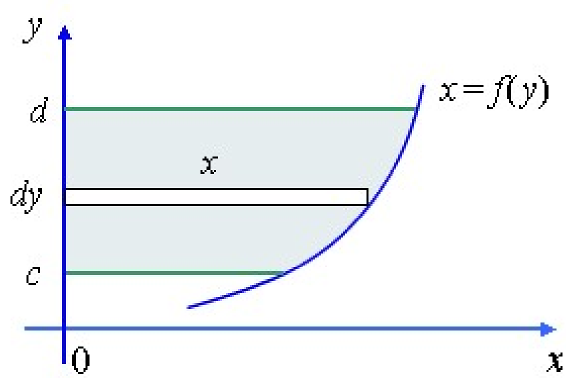


Figure No # 11

In this case, we find the area is the sum of the rectangles, heights $\mathbf{x = f(y)}$ and width $\mathbf{dy}$
If we are given $\mathbf{x = f(y)}$, then we need to re-express this as $\mathbf{x = f(y)}$ and we need to sum from bottom to top.

So, in case 4 we have: $\textbf{Area} = \int_c^d f(y)dy$

Example 30: Find the area of the region bounded by the curve, $y = \sqrt{x-1}$, the *y*-axis and the lines $y = 1$ and $y = 5$

Solution:

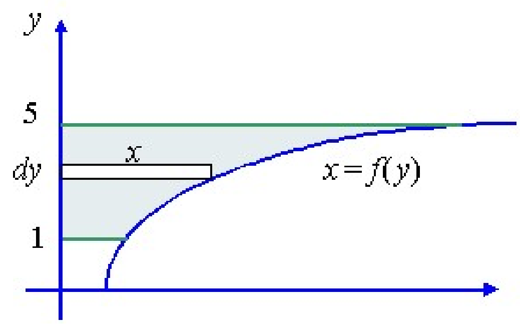


Figure No # 12

In this case, we express *x* as a function of *y:*

Given, $y = \sqrt{x-1}$

$\Rightarrow y^2 = x - 1$

$\Rightarrow x = y^2 + 1$

$\therefore x = f(y) = y^2 + 1$

So the area is given by

$$\text{Area} = \int_c^d f(y)dy$$

$$\Rightarrow \text{Area} = \int_1^5 f(y)dy = \int_1^5 (y^2+1)dy = \int_1^5 y^2 dy + \int_1^5 1\,dy$$

$$\Rightarrow \text{Area} = \left[\frac{y^{2+1}}{2+1}\right]_1^5 + [y]_1^5 \;[\because \int dy = y]$$

$$\Rightarrow \text{Area} = \left[\frac{y^3}{3}\right]_1^5 + [y]_1^5 = \left[\frac{5^3}{3} - \frac{1^3}{3}\right] + [5-1] = \left[\frac{125}{3} - \frac{1}{3}\right] + [5-1]$$

$$\Rightarrow \text{Area} = \left[\frac{125-1}{3}\right] + [4] = \left[\frac{124}{3}\right] + [4] = \frac{124}{3} + 4 = \frac{124+12}{3}$$

$$\Rightarrow \text{Area} = \frac{136}{3} \text{ Square unit}$$

Example 31: The region R shown is bounded by the curve with equation $y = \ln x$, the y-axis and the lines $y = 2$ and $y = 5$. Find the area of R

Solution: $\text{Area} = \int_c^d f(y)dy$

$\Rightarrow \text{Area} = \int_2^5 f(y)dy$

Given, $y = \ln x$

$\Rightarrow e^y = e^{\ln x}$

$\Rightarrow e^y = x \ln e$

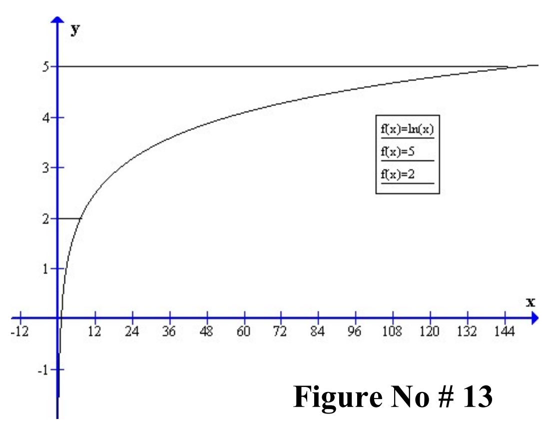


Figure No # 13

$$\Rightarrow e^{y} = x.1$$

$$\Rightarrow e^{y} = x$$

$$\Rightarrow x = f(y) = e^{y} \text{ ----------------(1)}$$

$$\textbf{Area} = \int_{2}^{5} f(y)dy$$

$$\Rightarrow \text{Area} = \int_{2}^{5} x dy = \int_{2}^{5} e^{y} dy \ \text{[From (1)]}$$

$$\Rightarrow \textbf{Area of R} = \left[e^{y}\right]_{2}^{5} = e^{5} - e^{2} \approx 141 \text{units}^{2}$$

3.5. Area between 2 Curves

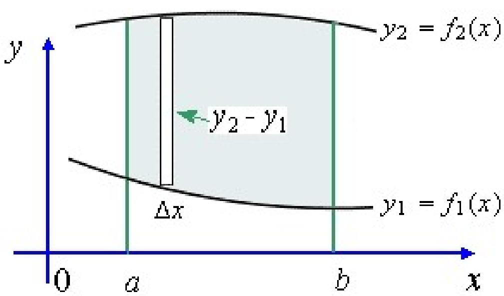


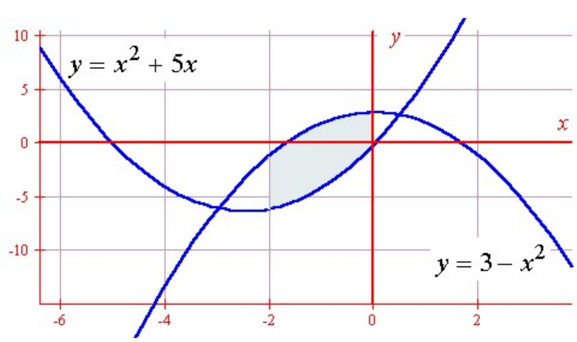
Figure No # 14

We are trying to find the area between 2 curves, $y_1 = f_1(x)$ and $y_2 = f_2(x)$, the lines $x = a$ and $x = b$. We see that if we subtract the area under lower curve $y_1 = f_1(x)$
From the area under the upper curve $y_2 = f_2(x)$
Then we will find the required area. This can be achieved in one step:

$$\textbf{Area} = \textbf{upper curve} - \textbf{lower curve} = \int_{a}^{b} (y_2 - y_1)dx$$

Example 32: Find the area between the curves $y = x^2 + 5x$ and $y = 3 - x^2$ between $x = -2$ and $x = 0$

Solution: Sketching first:



Here, Upper curve, $y_2 = f_2(x) = 3 - x^2$
and Lower Curve, $y_1 = f_1(x) = x^2 + 5x$

Figure No # 15

$$\mathbf{Area = upper\ curve - lower\ curve} = \int_a^b (y_2 - y_1)\,dx$$

So we need to find:

$$\mathbf{Area} = \int_a^b (y_2 - y_1)\,dx$$

$$\Rightarrow \mathbf{Area} = \int_{-2}^{0} \{(3 - x^2) - (x^2 + 5x)\}\,dx = \int_{-2}^{0} \{3 - x^2 - x^2 - 5x\}\,dx$$

$$\Rightarrow Area = \int_{-2}^{0} \{3 - 2x^2 - 5x\}\,dx = \int_{-2}^{0} 3dx - 2x^2dx - 5xdx$$

$$\Rightarrow \mathbf{Area} = \int_{-2}^{0} 3dx - \int_{-2}^{0} 2x^2dx - \int_{-2}^{0} 5xdx = 3\int_{-2}^{0} dx - 2\int_{-2}^{0} x^2dx - 5\int_{-2}^{0} xdx$$

$$\Rightarrow \mathbf{Area} = 3[x]_{-2}^{0} - 2\left[\frac{x^{2+1}}{2+1}\right]_{-2}^{0} - 5\left[\frac{x^{1+1}}{1+1}\right]_{-2}^{0} \quad [\because \int dx = x]$$

$$\Rightarrow \mathbf{Area} = 3[0 - (-2)] - 2\left[\frac{0^{2+1}}{2+1} - \frac{(-2)^{2+1}}{2+1}\right] - 5\left[\frac{0^{1+1}}{1+1} - \frac{(-2)^{1+1}}{1+1}\right]$$

$$\Rightarrow \mathbf{Area} = 3[0 + 2] - 2\left[\frac{0}{3} - \frac{(-2)^3}{3}\right] - 5\left[\frac{0}{2} - \frac{(-2)^2}{2}\right]$$

$$\Rightarrow \mathbf{Area} = 3[2] - 2\left[0 - \frac{-8}{3}\right] - 5\left[0 - \frac{4}{2}\right] = 3[2] - 2\left[0 + \frac{8}{3}\right] - 5[0 - 2]$$

$$\Rightarrow \mathbf{Area} = 3[2] - 2\left[\frac{8}{3}\right] - 5[-2] = 3 \times 2 - 2 \times \frac{8}{3} + 5 \times 2$$

$$\Rightarrow \mathbf{Area} = 6 - \frac{16}{3} + 10 = \frac{18 - 16 + 30}{3} = \frac{2 + 30}{3} = \frac{32}{3} \text{ Square unit}$$

Example 33: Find the area bounded by $\mathbf{y = x^3}$, $\mathbf{y = 0}$ and $\mathbf{y = 3}$

Solution:

We need to use: $\mathbf{Area} = \int_c^d f(y)dy$

In this case, $\mathbf{c = 0}$ and $\mathbf{d = 3}$

We need to express *x* in terms of *y*:

Given, $\quad \mathbf{y = x^3}$

$$\Rightarrow y^{\frac{1}{3}} = (x^3)^{\frac{1}{3}}$$

$$\Rightarrow y^{\frac{1}{3}} = x$$

$$\Rightarrow x = f(y) = y^{\frac{1}{3}}$$

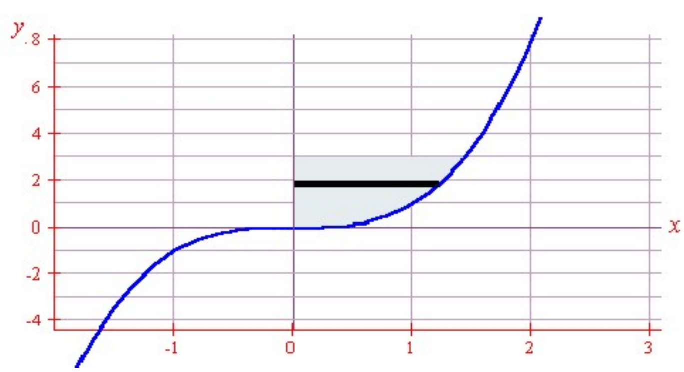
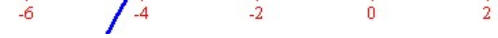


Figure No # 16



So, $\mathbf{Area} = \int_c^d \mathbf{f(y)dy}$

$$\Rightarrow \mathbf{Area} = \int_0^3 \mathbf{y^{\frac{1}{3}}dy}$$

$$\Rightarrow \mathbf{Area} = \left[\frac{y^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right]_0^3 = \left[\frac{y^{\frac{1+3}{3}}}{\frac{1+3}{3}}\right]_0^3 = \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_0^3 = \frac{3}{4}\left[y^{\frac{4}{3}}\right]_0^3$$

$$\Rightarrow \mathbf{Area} = \frac{3}{4}\left[3^{\frac{4}{3}} - 0^{\frac{4}{3}}\right] = \frac{3}{4}\left[3^{\frac{4}{3}} - 0^{\frac{4}{3}}\right] \text{ Square unit}$$

Example 34: Find the area bounded by the curves $\mathbf{y = x^2 + 5x}$ and $\mathbf{y = 3 - x^2}$

Solution: Sketch first:

We need to use: $\mathbf{Area} = \int_a^b \mathbf{(y_2 - y_1)dx}$

Here, Upper curve, $\mathbf{y_2 = f_2(x) = 3 - x^2}$

and Lower Curve, $\mathbf{y_1 = f_1(x) = x^2 + 5x}$

We note that $\mathbf{y = 3 - x^2}$ is above $\mathbf{y = x^2 + 5x}$ so we take

$$\mathbf{y_2 = f_2(x) = 3 - x^2} \text{ -------------------------------(i)}$$

$$\mathbf{y_1 = f_1(x) = x^2 + 5x} \text{ ------------------------------(ii)}$$

Points of intersection occur where:

$$3 - x^2 = x^2 + 5x$$

$$\Rightarrow 3 - x^2 - x^2 - 5x = 0$$

$$\Rightarrow 3 - 2x^2 - 5x = 0$$

$$\Rightarrow -2x^2 - 5x + 3 = 0$$

$$\Rightarrow 2x^2 + 5x - 3 = 0$$

$$\Rightarrow 2x^2 + 6x - x - 3 = 0$$

$$\Rightarrow 2x(x+3) - 1(x+3) = 0$$

$$\Rightarrow (x+3)(2x-1) = 0$$

$$\Rightarrow (x+3) = 0 \text{ and } (2x-1) = 0$$

$$\Rightarrow x = -3 \text{ and } 2x = 1$$

$$\Rightarrow x = -3 \text{ and } x = \frac{1}{2} = 0.5$$

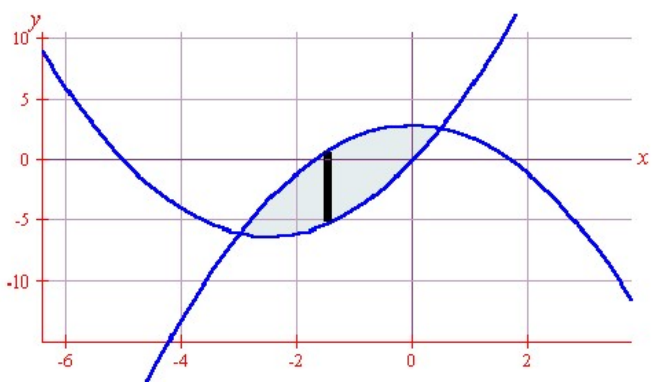


Figure No # 17

Putting the value of $\mathbf{x = -3}$ in (i), we get

$$\mathbf{y_1 = 3 - x^2}$$

$$\Rightarrow y_1 = 3-(-3)^2$$
$$\Rightarrow y_1 = 3-9$$
$$\Rightarrow y_1 = -6$$

Putting the value of $\mathbf{x=-3}$ in (ii), we get

$$y_2 = x^2+5x$$
$$\Rightarrow y_2 = (-3)^2+5.(-3)$$
$$\Rightarrow y_2 = (-3)^2-15$$
$$\Rightarrow y_2 = 9-15$$
$$\Rightarrow y_2 = -6$$

Hence the point of intersection $\mathbf{(-3,-6)}$

Again, putting the value of $\mathbf{x=\frac{1}{2}}$ in (i), we get

$$y_1 = 3-x^2$$
$$\Rightarrow y_1 = 3-(\frac{1}{2})^2$$
$$\Rightarrow y_1 = 3-\frac{1}{4}$$
$$\Rightarrow y_1 = \frac{12-1}{4} = \frac{11}{4}$$

Again, putting the value of $\mathbf{x=\frac{1}{2}}$ in (ii), we get

$$y_2 = x^2+5x$$
$$\Rightarrow y_2 = (\frac{1}{2})^2+5.\frac{1}{2}$$
$$\Rightarrow y_2 = \frac{1}{4}+\frac{5}{2}$$
$$\Rightarrow y_2 = \frac{1+10}{4}$$
$$\Rightarrow y_2 = \frac{11}{4}$$

Hence the point of intersection $\mathbf{(\frac{1}{2},\frac{11}{4})}$

Hence the point of intersection $\mathbf{(-3,-6)}$ and $\mathbf{(\frac{1}{2},\frac{11}{4})}$ between the curves $\mathbf{y=x^2+5x}$ and $\mathbf{y=3-x^2}$.

So the area is given by: $\mathbf{A=\int_a^b (y_2-y_1)\,dx}$

$$A = \int_a^b (y_2 - y_1)dx$$
$$= \int_{-3}^{0.5} ([3 - x^2] - [x^2 + 5x])dx$$
$$= \int_{-3}^{0.5} (3 - 5x - 2x^2)dx$$
$$= \left[3x - \frac{5x^2}{2} - \frac{2x^3}{3}\right]_{-3}^{0.5}$$
$$= 14.29 \text{ sq units}$$

Example 35: Find the area bounded by the curves $\mathbf{y = x^2}$, $\mathbf{y = 2 - x}$ and $\mathbf{y = 1}$

Solution: Sketch first:

We need to solve $y = x^2$ for x:

Given, $\mathbf{y = x^2}$

$$\Rightarrow \mathbf{y^{\frac{1}{2}} = (x^2)^{\frac{1}{2}}}$$
$$\Rightarrow \mathbf{y^{\frac{1}{2}} = x}$$
$$\Rightarrow \mathbf{\pm\sqrt{y} = x}$$
$$\Rightarrow \mathbf{x = \pm\sqrt{y}} \text{ ------------------(i)}$$

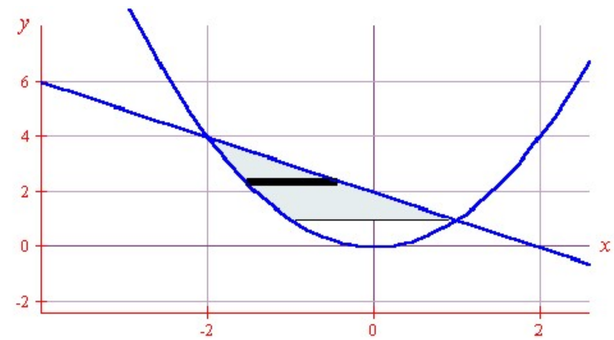


Figure No # 18

Given, $\mathbf{y = 2 - x}$

$$\Rightarrow \mathbf{2 - x = y}$$
$$\Rightarrow \mathbf{-x = y - 2}$$
$$\Rightarrow \mathbf{x = -y + 2}$$
$$\Rightarrow \mathbf{x = 2 - y} \text{ ------------------(ii)}$$

We need the left hand portion, so $\mathbf{x = -\sqrt{y}}$.

Notice that $\Rightarrow \mathbf{x = 2 - y}$ is to the *right* of $\mathbf{x = -\sqrt{y}}$, so we choose

$$\mathbf{x_2 = 2 - y} \text{ --------------------(iii)}$$
$$\mathbf{x_1 = -\sqrt{y}} \text{ ---------------------(iv)}$$

The intersection of the graphs occurs at

$$\mathbf{2 - y = -\sqrt{y}}$$
$$\Rightarrow \mathbf{2 - y = -\sqrt{y}}$$
$$\Rightarrow \mathbf{(2 - y)^2 = (-\sqrt{y})^2}$$
$$\Rightarrow \mathbf{(2 - y)^2 = y}$$
$$\Rightarrow \mathbf{4 - 4y + y^2 = y}$$

$$\Rightarrow 4-4y+y^2-y=0$$
$$\Rightarrow y^2-5y+4=0$$
$$\Rightarrow y^2-4y-y+4=0$$
$$\Rightarrow y(y-4)-1(y-4)=0$$
$$\Rightarrow (y-4)(y-1)=0$$
$$\Rightarrow y=4 \text{ and } y=1$$

Putting the value of $y=4$ in (iii), we get

$$x_2=2-y$$
$$\Rightarrow x_2=2-4=-2$$

And putting the value of $y=4$ in (iv), we get

$$x_1=-\sqrt{y}$$
$$\Rightarrow x_1=-\sqrt{4}=-2$$

Hence the point of intersection $(-2,4)$

Again, putting the value of $y=1$ in (iii), we get

$$x_2=2-y$$
$$\Rightarrow x_2=2-1=1$$

And putting the value of $y=1$ in (iv), we get

$$x_1=-\sqrt{y}$$
$$\Rightarrow x_1=-\sqrt{1}=-1$$

Hence the point of intersection $(-1,1)$

Hence the point of intersection $(-2,4)$ and $(-1,1)$ between the curves $y=x^2$ and $y=2-x$.

So we have $c=1$ and $d=4$

$$A=\int_a^b (x_2-x_1)\,dy$$

$$A=\int_c^d (x_2-x_1)dy$$
$$=\int_1^4 ([2-y]-[-\sqrt{y}])dy$$
$$=\int_1^4 (2-y+\sqrt{y})dy$$
$$=\left[2y-\frac{y^2}{2}+\frac{2}{3}y^{3/2}\right]_1^4$$
$$=\left(\frac{16}{3}\right)-\left(\frac{13}{6}\right)$$
$$=\frac{19}{6} \text{ sq units}$$

Example 36: Fig shows the curve $\mathbf{y = x^2 - 5x + 4}$

$\mathbf{A_1}$ is the area bounded by the co-ordinate axes and the curve, $\mathbf{A_2}$ is the area bounded by the curve and the x-axis and $\mathbf{A_3}$ is the area bounded by the curve, the x-axis and the line $\mathbf{x = 5}$

Find the area (i) $\mathbf{A_1}$ (ii) $\mathbf{A_2}$ (iii) $\mathbf{A_3}$

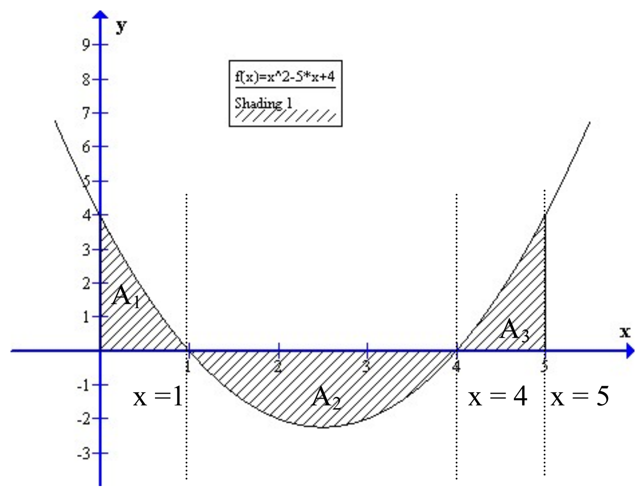


Figure No # 19

Solution: We have,

$$\mathbf{A = \int_a^b (y_2 - y_1)\,dx}$$

Where, $\mathbf{y_1 = f_1(x)}$ and $\mathbf{y_2 = f_2(x)}$

(i) In the first area: $\mathbf{A_1}$

Here, Upper curve, $\mathbf{y_2 = f_2(x) = x^2 - 5x + 4}$

and Lower Curve, $\mathbf{y_1 = f_1(x) = 0}$ [Since the equation of the x-axis is $\mathbf{y = 0}$]

For the area $\mathbf{A_1}$:

Here, $\mathbf{y_1 = f_1(x) = 0}$ [Since the equation of the x-axis is $\mathbf{y = 0}$] and $\mathbf{y_2 = f_2(x) = x^2 - 5x + 4}$

So, Area $\mathbf{A_1 = \int_0^1 (y_2 - y_1)dx = \int_0^1 (x^2 - 5x + 4 - 0)dx = \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_0^1}$

$$\mathbf{= \left[\frac{1^3}{3} - \frac{5\times 1^2}{2} + 4\times 1 - \left[\frac{0^3}{3} - \frac{5\times 0^2}{2} + 4\times 0\right]\right]}$$

$$\mathbf{= \left[\frac{1}{3} - \frac{5}{2} + 4 - \left[\frac{0}{3} - \frac{0}{2} + 0\right]\right] = \left[\frac{1}{3} - \frac{5}{2} + 4 - 0\right]}$$

$$\mathbf{= \left[\frac{1}{3} - \frac{5}{2} + 4\right] = \frac{2 - 15 + 24}{6} = \frac{11}{6}}$$

(ii) In the 2nd area: $\mathbf{A_2}$

Upper Curve, $\mathbf{y_2 = f_2(x) = 0}$ [Since the equation of the x-axis is $\mathbf{y = 0}$]

and Lower curve, $\mathbf{y_1 = f_1(x) = x^2 - 5x + 4}$

The area $\mathbf{A_2}$ is bounded above $\mathbf{y = 0}$, below by the curve $\mathbf{y = x^2 - 5x + 4}$ and the line $\mathbf{x = 1}$ and $\mathbf{x = 4}$

Here, $\mathbf{y_2 = f_2(x) = 0}$ [Since the equation of the x-axis is $y = 0$] and $\mathbf{y_1 = f_1(x) = x^2 - 5x + 4}$

So, Area: $\mathbf{A_2 = \int_1^4 (y_2 - y_1)\,dx = \int_1^4 \{0 - (x^2 - 5x + 4)\}dx} = \left[-\frac{\mathbf{x^3}}{\mathbf{3}} + \frac{\mathbf{5x^2}}{\mathbf{2}} - \mathbf{4x}\right]_1^4$

$$= \left[-\frac{4^3}{3} + \frac{5\times 4^2}{2} - 4\times 4 - \left[-\frac{1^3}{3} + \frac{5\times 1^2}{2} - 4\times 1\right]\right]$$

$$= \left[-\frac{64}{3} + \frac{80}{2} - 16\right] + \left[\frac{1}{3} - \frac{5}{2} + 4\right]$$

$$= \frac{-128 + 240 - 96}{6} + \frac{2 - 15 + 24}{6} = \frac{16 + 11}{6} = \frac{27}{6} \text{ Answer}$$

(iii) In the third area: $\mathbf{A_3}$

Here, Upper curve, $\mathbf{y_2 = f_2(x) = x^2 - 5x + 4}$

and Lower Curve, $\mathbf{y_1 = f_1(x) = 0}$ [Since the equation of the x-axis is $\mathbf{y = 0}$]

The area $\mathbf{A_3 = \int_4^5 (y_2 - y_1)\,dx = \int_4^5 \{(x^2 - 5x + 4) - 0\}dx}$

$$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right]_4^5$$

$$= \left[\frac{5^3}{3} - \frac{5.5^2}{2} + 4.5 - \left[\frac{4^3}{3} - \frac{5.4^2}{2} + 4.4\right]\right]$$

$$= \left[\frac{125}{3} - \frac{125}{2} + 20 - \frac{64}{3} + \frac{80}{2} - 16\right] = \left[\frac{125}{3} - \frac{125}{2} + 4 - \frac{64}{3} + \frac{80}{2}\right]$$

$$= \left[\frac{125}{3} - \frac{64}{3} + \frac{80}{2} - \frac{125}{2} + 4\right] = \left[\frac{125 - 64}{3} + \frac{80 - 125}{2} + 4\right]$$

$$= \left[\frac{61}{3} - \frac{45}{2} + 4\right] = \left[\frac{122 - 135 + 24}{6}\right] = \left[\frac{146 - 135}{6}\right] = \frac{11}{6} \textit{ Answer}$$

Example 37: Calculate the area enclosed by the curves $\mathbf{y = 4 - x^2}$ and $\mathbf{y = x^2 - 2x}$.

Solution:

Here, Upper curve, $\mathbf{y_2 = f_2(x) = 4 - x^2}$

and Lower Curve, $\mathbf{y_1 = f_1(x) = x^2 - 2x}$

Here, $y_2 = f_2(x) = 4 - x^2$

x	0	1	2	-1	-2	3
$y_2 = f_2(x) = 4 - x^2$	4	3	0	3	0	-5

Here, $y_1 = f_1(x) = x^2 - 2x$

x	0	1	2	-1	-2	3
$y_1 = f_1(x) = x^2 - 2x$	0	-1	0	3	8	3

The x co-ordinate of the point A and B (the points at which the curves intersects) in the figure # 20 are found as follows:

$$4 - x^2 = x^2 - 2x$$
$$\Rightarrow 4 - x^2 = x^2 - 2x$$
$$\Rightarrow 4 - x^2 - x^2 + 2x = 0$$
$$\Rightarrow 4 - 2x^2 + 2x = 0$$
$$\Rightarrow -2x^2 + 2x + 4 = 0$$
$$\Rightarrow 2x^2 - 2x - 4 = 0$$
$$\Rightarrow 2x^2 - 4x + 2x - 4 = 0$$
$$\Rightarrow 2x(x-2) + 2(x-2) = 0$$
$$\Rightarrow (x-2)(2x+2) = 0$$
$$\Rightarrow x - 2 = 0 \quad \text{and} \quad 2x + 2 = 0$$
$$\Rightarrow x = 2 \quad \text{and} \quad 2x = -2$$
$$\Rightarrow x = 2 \quad \text{and} \quad x = -1$$

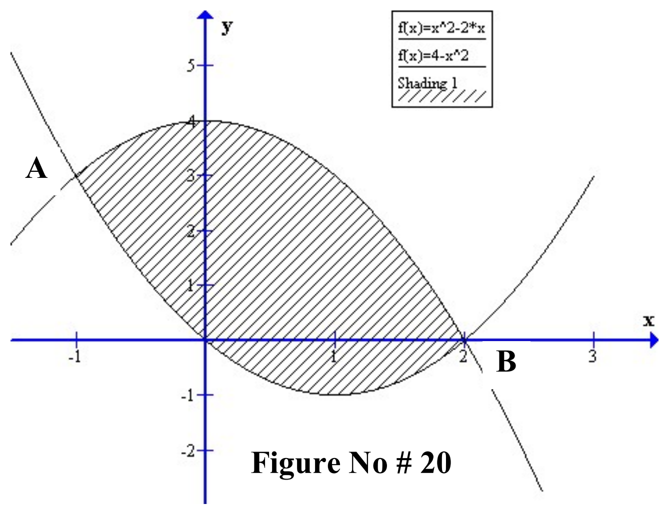


Figure No # 20

The x co-ordinate of A is -1 and of B is 2 .

$$\text{Area} = \int_{-1}^{2} (y_2 - y_1)\,dx = \int_{-1}^{2} \{(4 - x^2) - (x^2 - 2x)\}dx$$

$$= \int_{-1}^{2} (4 - x^2 - x^2 + 2x)dx$$

$$= \int_{-1}^{2} (4 - 2x^2 + 2x)dx = \left[4x - 2\frac{x^3}{3} + 2\frac{x^2}{2}\right]_{-1}^{2}$$

$$= \left[4.2 - 2\frac{2^3}{3} + 2\frac{2^2}{2} - \left[4.(-1) - 2\frac{(-1)^3}{3} + 2\frac{(-1)^2}{2}\right]\right]$$

$$= \left[4.2 - 2\frac{2^3}{3} + 2\frac{2^2}{2} - 4.(-1) + 2\frac{(-1)^3}{3} - 2\frac{(-1)^2}{2}\right]$$

$$=\left[8-2\frac{8}{3}+2\frac{4}{2}+4-\frac{2}{3}-2\frac{1}{2}\right]=\left[8-\frac{16}{3}+4+4-\frac{2}{3}-1\right]$$
$$=\left[15-\frac{16}{3}-\frac{2}{3}\right]=\left[\frac{45-16-2}{3}\right]=\left[\frac{45-18}{3}\right]$$
$$=\left[\frac{27}{3}\right]=9\ \textit{Answer}$$

3.6. Finding Area using derivative method

Example 38: Find the area of the region bounded by the curve $\mathbf{y=2x-x^2}$ and the x-axis. Figure shows a sketch of the curve which cuts the axis of x at (0.0) and (2.0).

Solution: We wish to find the area A when $\mathbf{x=2}$.Given that $\mathbf{\frac{dA}{dx}=2x-x^2}$ ***(Since integration and differentiation vice versa)*** and that A = 0 when x = 0.

$$\frac{dA}{dx}=2x-x^2$$
$$\Rightarrow dA=(2x-x^2)dx$$
$$\Rightarrow \int dA=\int(2x-x^2)dx$$
$$\Rightarrow A=2\frac{x^2}{2}-\frac{x^3}{3}+c$$
$$\Rightarrow A=x^2-\frac{x^3}{3}+c$$

When x = 0, A = 0. $\therefore$ c = 0

$$\Rightarrow A=x^2-\frac{x^3}{3}+c$$
$$\Rightarrow A=x^2-\frac{x^3}{3}+0$$
$$\Rightarrow A=x^2-\frac{x^3}{3}$$

When $\mathbf{x=2}$

$$\Rightarrow A=x^2-\frac{x^3}{3}$$
$$\Rightarrow A=2^2-\frac{2^3}{3}=4-\frac{8}{3}=\frac{12-8}{3}=\frac{4}{3}$$

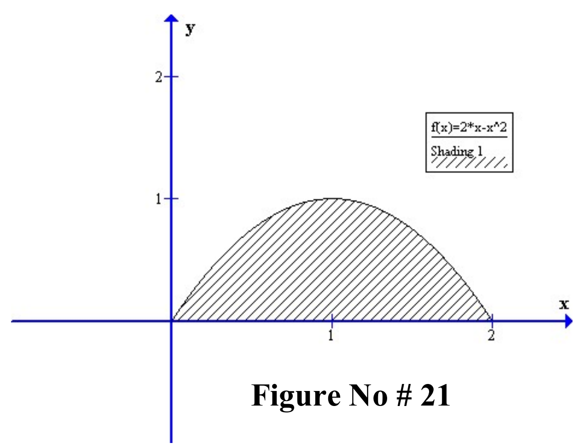


Figure No # 21

Example 39: Find the area of the region bounded by the curve $\mathbf{y = x^2}$ the line $\mathbf{y = x}$ and the x-axis

Solution: Figure shows a sketch of the curve and line. They meet where $\mathbf{y = x^2}$ i.e. at the points (0, 0) and (1, 1)

To find the area of the region under the curve between x = 0 and x = 1 we find the value of A when x = 1 given that $\mathbf{\frac{dA}{dx} = x^2}$ and that A = 0 when x = 0.

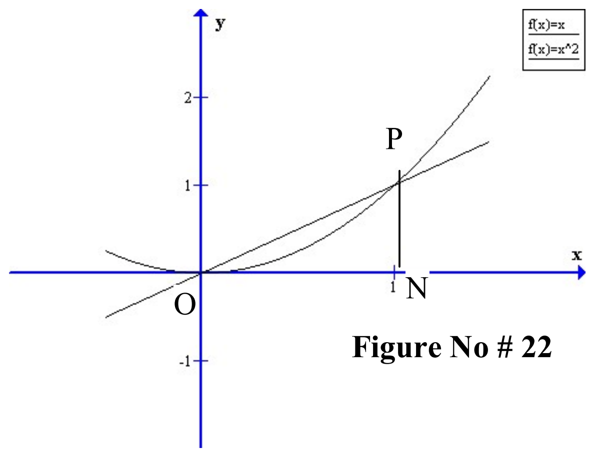


Figure No # 22

$$\mathbf{\frac{dA}{dx} = x^2}$$

$$\mathbf{\Rightarrow dA = x^2 dx}$$

$$\mathbf{\Rightarrow \int dA = \int x^2 dx}$$

$$\mathbf{\Rightarrow A = \frac{x^3}{3} + c}$$

Since A = 0, when x = 0, c = 0 $\therefore \mathbf{A = \frac{1}{3}} x^3$

When x = 1, A = $\mathbf{\frac{1}{3}}$

The area of the triangle OPN = $\mathbf{\frac{1}{2} \times 1 \times 1 = \frac{1}{2}}$

The area of the region bounded by the curve $\mathbf{y = x^2}$, the line y = x and the x-axis is

$\mathbf{(\frac{1}{2} - \frac{1}{3}) = \frac{1}{6}}$ Square units

Example 40: Find the area of the region bounded by the curve $\mathbf{y = x^2 + 1}$, the ordinates x = 1, x = 2 and the x-axis.

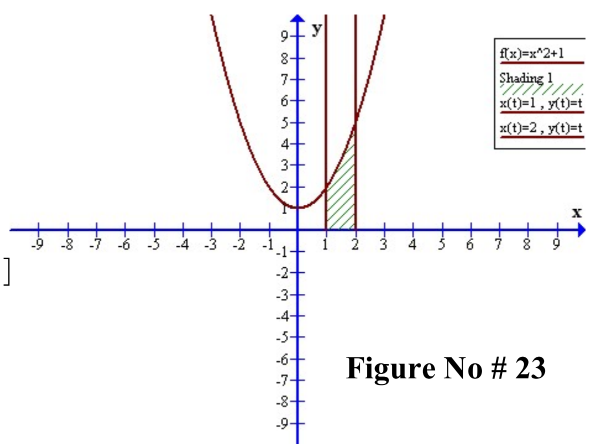


Figure No # 23

$$\mathbf{\frac{dA}{dx} = x^2 + 1}$$

$$\mathbf{\Rightarrow dA = (x^2 + 1)dx}$$

$$\mathbf{\Rightarrow \int dA = \int (x^2 + 1)dx}$$

$$\mathbf{\Rightarrow A = \frac{x^3}{3} + x + c} \left[\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$\mathbf{\therefore A = \int_1^2 \left(x^2 + 1\right)dx = \left[\frac{x^3}{3} + x\right]_1^2}$$

$$= \mathbf{\left(\frac{8}{3} + 2\right) - \left(\frac{1}{3} + 1\right) = 3\frac{1}{3}}$$