Set Theory

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Set Theory

A set is any well-defined list, collection or class of objects.

The objects of a set are also called its members. The elements of a set can be anything: numbers, people, letters of the alphabet, other sets, and so on. Sets are conventionally denoted with capital letters, for instance A, B and C.

Example: i) The solution of the equation $x^2 + 5x + 6 = 0$
ii) The students Sabrina, Mahe, Zerin, Salma
iii) The students who are absent from the class
iv) The districts in Bangladesh.

Special sets:

There are some sets which hold great mathematical importance :
N = the set of natural numbers 0, 1, 2, ... ;
Z = the set of all integers 0, ±1, ±2, ... ;
Q = the set of all rational numbers
R = the set of real numbers;
C = the set of complex numbers.
P = denoting the set of all primes.
$\mathbb{R}$, denoting the set of all real numbers.

Each of these sets of numbers has an infinite number of elements, and $\mathbb{P} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Notation:
{x $\in$ Z | x > 0} is the set of positive integers, commonly denoted by Z $+$;
{x $\in$ Z | 3 < x < 7} = {4, 5, 6};
{x $\in$ R | x 2 < 1} denotes the open interval $(-1, 1)$ i.e. all real numbers x such that -1 < x < 1;

Finite Set and Infinite Sets:

1. Let D be the set of the days of the week. Then D is finite
2. Let S = {2, 4, 6,8,-------------------}. Then S is infinite

Cardinality: If a set S has n distinct elements for some natural number n, n is the cardinality (size) of S and S is a finite set. The cardinality of S is denoted by |S|.

For example the cardinality of the set {3, 1, 2} is 3.

The number of distinct elements in a finite set is called its cardinal number. For example, the set $\{1, \mathbf{2}, \mathbf{3}\}$ has **three** distinct elements, so its cardinal number is **3**. The set

{1, **2**, **2**, **3**} has **four** elements but only **three** distinct elements (1, **2**, **3**) since **2** is repeated; so its cardinal number is also **3**.

Null / Empty Set:

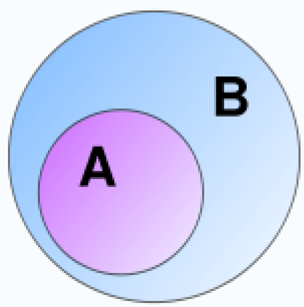
A set, which has no elements, is called an empty set. More formally, an empty set, denoted by ϕ

Example: $\mathbf{let\ \ R = \{x | x^2 = 9,\ x\ is\ even\}}$. Then R is the empty set.

Subsets:
If every member of set A is also a member of set B, then A is said to be a subset of B, written $A \subseteq B$.

Example: $\mathbf{let\ \ H = \{x |\ x\ is\ even\},\ \ K = \{x | x\ is\ a\,positive}$ power of 2}
i.e. H = {2, 4, 6, 8, 10, ---------------} and K = {2, 4, 8, 16, 32, --------------}
Then $\mathbf{K \subset H}$

Proper subset :If A is a subset of, but not equal to, B, then A is called a proper subset of B, written $A \subset B$ (A is a proper subset of B) or $B \supset A$ (B is proper superset of A).



Example: The set of all men is a proper subset of the set of all people.

Problem :The empty set is a subset of every set and every set is a subset of itself:

1. $\phi \subseteq \mathbf{A}$
2. $\mathbf{A} \subseteq \mathbf{A}$

Proof: If A is any set (including the empty set), the empty set is a subset of A: For all x, if x is an element of the empty set, x is also an element of A. (This is vacuously (blankly) true, since there are no elements in the empty set). Therefore, the empty set is a subset of A.

Another Proof: Given any set A, we wish to prove that O is a subset of A. This involves showing that all elements of O are elements of A. But there are no elements of O. For the experienced mathematician, the inference " O has no elements so all elements of O are elements of A" is immediate, but it may be more troublesome for the beginner. Since O has no members at all, how can "they" be members of anything else? It may help to think of it the other way round. In order to prove that O was not a subset of A, we would have to find an element of O which was not also an element of A. Since there are no elements of A, this is impossible and hence O is indeed a subset of A

Equality of Sets:
Two sets A and B are said to be equal if they have the same members; this is written A = B.
Example:

$$\text{Let } E = \{x \mid x^2 - 3x + 2 = 0\}, \ F = \{2,1\}, \text{ and } G = \{1,2,2,1\}$$

Then E = F = G.
Again two sets A and B are equal, i. e. A = B, if and only if $A \subset B$ and $B \subset A$

Universal set:
A set containing all elements of a problem under consideration, denoted by U
Here are 5 examples of such a universal set.

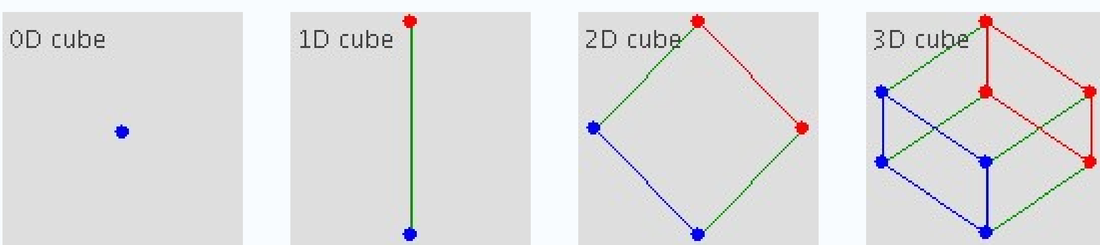
1. The set of all people in the Bush family.
2. The set of all people who have been or are the President of the United States.
3. The set of all living males.
4. The set of all people whose father was a president of the United States.

Power set:

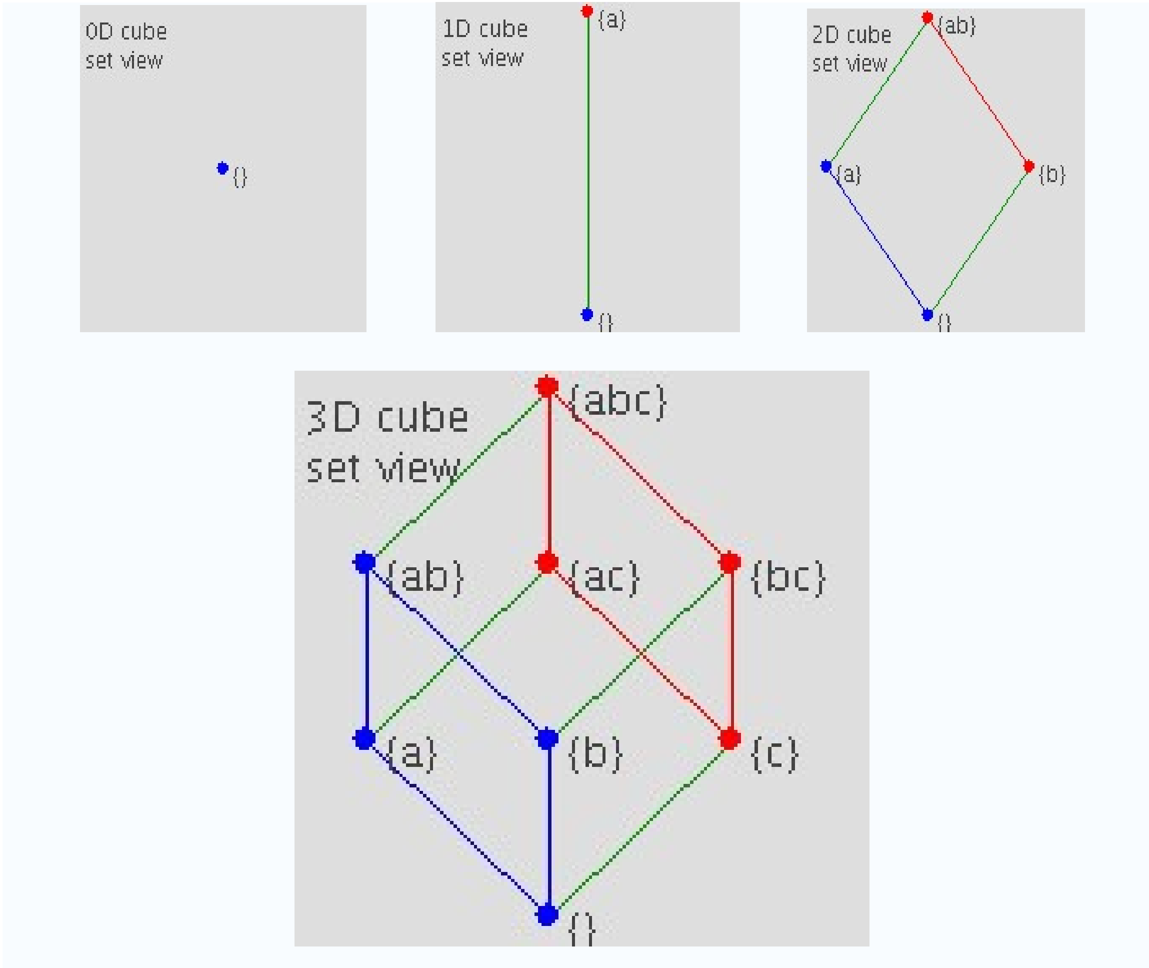
The power set of a set X can be defined as: "the set of all subsets of X". This includes X and the empty set and denoted by 2^A or $\mathcal{P}(A)$.

For example for A = {1, 2}, $\mathcal{P}$(A) = { ϕ , {1}, {2}, {1, 2}}

N-dimensional cubes: rather distorting primitive projection:



N-dimensional cubes as Hasse diagrams of power sets:


0D cube
set view
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1D cube
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2D cube
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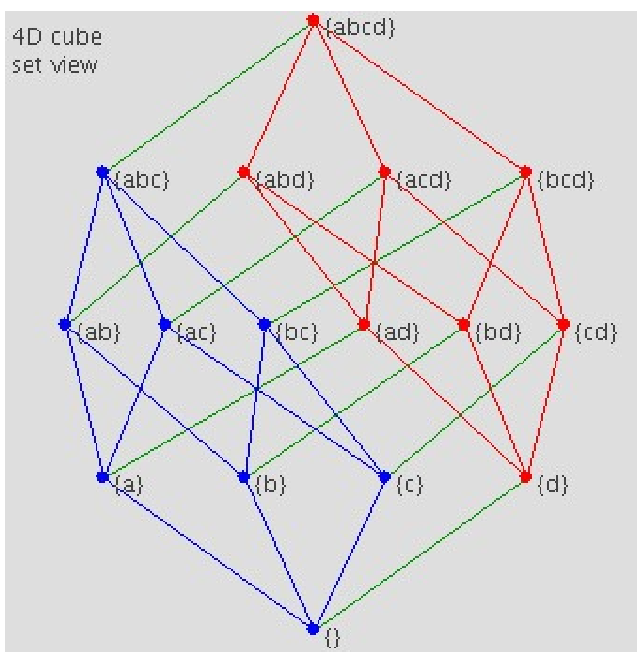
4D cube
set view
{abcd}
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Basic set Functions:

Intersection: $S \cap T = \{x \mid x \in S \text{ and } x \in T\}$

Union: $S \cup T = \{x \mid x \in S \text{ or } x \in T\}$

Difference: $S - T = \{x \mid x \in S \text{ and } x \notin T\}$

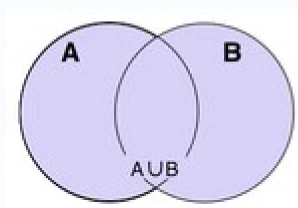
Complement: $-S = \{x \mid x \notin S\} = U - S$

Example

Suppose $S = \{1, 2, 3\}$ and $T = \{1, 3, 5, 7\}$. Then

$S \cap T = \{1, 3\}$; $S \cup T = \{\underline{1}, 2, \underline{3}, \underline{5}, 7\}$; $S - T = \{2\}$; $T - S = \{5, 7\}$

Union: The union of A and B, denoted by A U B, is the set of all things which are members of either A or B.



The union of A and B

Examples:

{1, 2} U {red, white} = {1, 2, red, white}

{1, 2, green} U {red, white, green} = {1, 2, red, white, green}

{1, 2} U {1, 2} = {1, 2}

Some basic properties of unions are:

A U B = B U A

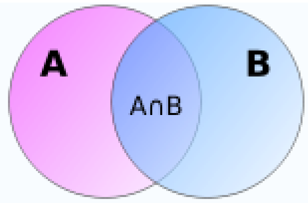
A $\subseteq$ A U B

A U A = A

A U ø = A

$\cup$: But we will instead use "+" for the usual reasons. If A = {a, b, c} and B = {b, c, d} then A + B = {a, b, c, d}

Intersection: The intersection of A and B, denoted by A ∩ B, is the set of all things which are members of both A and B. If A ∩ B = ø, then A and B are said to be disjoint.



The intersection of A and B

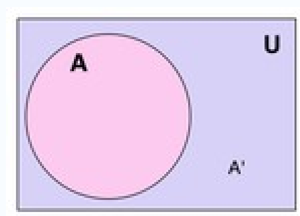
Examples:

$\{1, 2\} \cap \{red, white\} = \varnothing$
$\{1, 2, green\} \cap \{red, white, green\} = \{green\}$
$\{1, 2\} \cap \{1, 2\} = \{1, 2\}$
Some basic properties of intersections:
$A \cap B \;=\; B \cap A$
$A \cap B \;\subseteq\; A$
$A \cap A \;=\; A$
$A \cap \varnothing \;=\; \varnothing$

$\cap$: But we will use the symbol "&" instead (which will help remind us that set intersection is like a logical AND). If A = {a, b, c} and B = {b, c, d} then
A & B = {b, c}

Complement: If U is the set of integers, E is the set of even integers, and O is the set of odd integers, then the complement of E in U is O, or equivalently, $E' = O$.
Some basic properties of complements:

$A \cup A' = U$
$A \cap A' = \varnothing$
$(A')' = A$
$A - A = \varnothing$
$A - B = A \cap B'$



The complement of A in U

Difference: Two sets can also be "subtracted". The relative complement of A in B (also called the set theoretic difference of B and A), denoted by B − A, (or B \ A) is the set of all elements which are members of B, but not members of A.

Examples:

$\{1, 2\} - \{red, white\} = \{1, 2\}$
$\{1, 2, green\} - \{red, white, green\} = \{1, 2\}$
$\{1, 2\} - \{1, 2\} = \varnothing$

Cartesian product:

$S \times T = \{(x, y) \mid x \in S \text{ and } y \in T\}$

= the set of all ordered pairs whose components come from S and T respectively

Example:

Suppose $S = \{1, 2, 3\}$ and $T = \{1, 3, 5, 7\}$. Then

$S \times T = \{(1, 1), (1, 3), (1, 5), (1, 7), (2, 1), (2, 3), (2, 5), (2, 7), (3, 1), (3, 3), (3, 5), (3, 7)\}$

1. Set union becomes the Boolean sum (OR Gate)
2. Set intersection becomes the Boolean product (AND Gate)
3. Set complement becomes the Boolean complement (NOT Gate)

	Set Theory	**Boolean Algebra**
Identities	A + 0 = A	a + 0 = a
	A & U = A	a * 1 = a
Boundedness	A + U = U	a + 1 = 1
	A & 0 = 0	a * 0 = 0
Commutative	A & B = B & A	a * b = b * a
	A + B = B + A	a + b = b + a
Associative	(A + B) + C = A + (B + C)	(a + b) + c = a + (b + c)
	(A & B) & C = A & (B & C)	(a * b) * c = a * (b * c)
Distributive	A + (B & C) = (A + B) & (A + C)	a + (b * c) = (a + b) * (a + c)
	A & (B + C) = (A & B) + (A & C)	a * (b + c) = (a * b) + (a * c)
Complement Laws	$A + A^c = U$	a + a' = 1
	$A \& A^c = 0$	a * a' = 0
Uniqueness of Complement	$A + B = U, A \& B = 0 \rightarrow B = A^c$	a + x = 1, a * x = 0 -> x = a'
Involution	$(A^c)^c = A$	(a')' = a
	$0^c = U$	0' = 1
	$U^c = 0$	1' = 0
Idempotent	A + A = A	a + a = a
	A & A = A	a * a = a
Absorption	A + (A & B) = A	a + (a * b) = a
	A & (A + B) = A	a * (a + b) = a

DeMorgan's	$(A + B)^c = A^c \& B^c$	(a + b)' = a' * b'
	$(A \& B)^c = A^c + B^c$	(a * b)' = a' + b'

Q-1: State whether each of the following statements is correct or incorrect. Here S is any set that is not empty.
i) $S \in 2^S$ ii) $S \subset 2^S$ iii) $\{S\} \in 2^S$ iv) $\{S\} \subset 2^S$

i) Correct
ii) Correct
iii) Incorrect
iv) Incorrect

Q-2: If $A = \phi$ Find P(A)

Solution: Here A contains no element, Therefore P (A) or $2^A = 2^0 = 1$ element, i.e. P(A) = ϕ

Q-3: Let $E = \{1,0\}$, State whether each of the following statements is correct or incorrect
i) $\{0\} \in E$ ii) $\phi \in E$ iii) $\{0\} \subset E$ iv) $0 \in E$ v) $0 \subset E$

Answer:

i) Incorrect
ii) Incorrect
iii) Correct
iv) Correct
v) Incorrect

Q-4: What is meant by the symbol {{2, 3}}?

Answer: We have a set which contains one element, that element {2, 3} belongs to {{2, 3}}; it is not a subset of {{2, 3}}, Also we can say that {{2, 3}} is a set of sets.

Q-5: Let A = {2, {4, 5}, 4}, which statements are incorrect and why?
i) $\{4,5\} \subset A$ Ii) $\{4,5\} \in A$ iii) $\{\{4,5\}\} \subset A$

Answer:
i) Incorrect
ii) Correct
iii) Correct

Q-6: Let E = {2, {4, 5}, 4}, which statements are incorrect and why?
i) $\mathbf{5 \in E}$ Ii) $\mathbf{\{5\} \in E}$ iii) $\mathbf{\{5\} \subset E}$
i) Incorrect

ii) Incorrect
iii) Incorrect

As for example
$U = \{1,2,3,4,5,6,7,8,9,10\}$
Let,
$A = \{1,2,3,4\}$
$A^c = \{5,6,7,8,9,10\}$
$B = \{3,4,5\}$
$B^c = \{1,2,6,7,8,9,10\}$
$A \cup B = \{1,2,3,4,5\}$
$(A \cup B)^c = \{6,7,8,9,10\}$
$A \cap B = \{3,4\}$

Again,

$A = \underbrace{\{1,2,3,4\}}_{x}$

That is, $x \in A = \underbrace{\{1,2,3,4\}}_{x}$

$B = \underbrace{\{3,4,5\}}_{y}$

That is, $y \in B = \underbrace{\{3,4,5\}}_{y}$

$A \cup B = \underbrace{\{1,2,3,4,5\}}_{z}$

That is, $z \in A \cup B = \underbrace{\{1,2,3,4,5\}}_{z}$

$A \cap B = \underbrace{\{3,4\}}_{s}$

That is, $s \in A \cap B = \underbrace{\{3,4\}}_{s}$

That is, $x \in A = \underbrace{\{1,2,3,4\}}_{x}$

$x \notin A^c = \{5,6,7,8,9,10\}$

That is, $y \in B = \underbrace{\{3,4,5\}}_{y}$

$y \notin B^c = \{1,2,6,7,8,9,10\}$

That is, $z \in A \cup B = \underbrace{\{1,2,3,4,5\}}_{z}$

$z \in A$ or $z \in B$

That is, $s \in A \bigcap B = \underbrace{\{3,4\}}_{s}$

$s \in A$ and $s \in B$

Q-7: Prove De-Morgan's theorem:
1) $(\mathbf{A} \bigcup \mathbf{B})' = \mathbf{A}' \bigcap \mathbf{B}'$
2) $(\mathbf{A} \bigcap \mathbf{B})' = \mathbf{A}' \bigcup \mathbf{B}'$
A= {1,2,3}
B={1,2,3}
B SUBSET OF A
A IS SUBSET OF B
A=B

Proof (1):

Let $\mathbf{x} \in (\mathbf{A} \cup \mathbf{B})'$

$\Rightarrow \mathbf{x} \notin (\mathbf{A} \cup \mathbf{B})$

$\Rightarrow \mathbf{x} \notin \mathbf{A}$ **and** $\mathbf{x} \notin \mathbf{B}$

$\Rightarrow \mathbf{x} \in \mathbf{A}'$ **and** $\mathbf{x} \in \mathbf{B}'$

$\Rightarrow \mathbf{x} \in (\mathbf{A}' \cap \mathbf{B}')$

By definition of subset, we get $(\mathbf{A} \bigcup \mathbf{B})' \subseteq \mathbf{A}' \bigcap \mathbf{B}'$---------------------------(1)

Again Let $\mathbf{y} \in \mathbf{A}' \cap \mathbf{B}'$

Then $\mathbf{y} \in \mathbf{A}'$ **and** $\mathbf{y} \in \mathbf{B}'$

$\Rightarrow \mathbf{y} \notin \mathbf{A}$ **and** $\mathbf{y} \notin \mathbf{B}$

$\Rightarrow \mathbf{y} \notin (\mathbf{A} \cup \mathbf{B})$

$\Rightarrow \mathbf{y} \in (\mathbf{A} \cup \mathbf{B})'$

By definition of subset, we get $\mathbf{A}' \bigcap \mathbf{B}' \subseteq (\mathbf{A} \bigcup \mathbf{B})'$-----------------------------(2)

From (1) and (2), we get,
$(\mathbf{A} \bigcup \mathbf{B})' = \mathbf{A}' \bigcap \mathbf{B}'$ (Proved)

Proof (2):

Let $\mathbf{x} \in (\mathbf{A} \cap \mathbf{B})'$

$\Rightarrow \mathbf{x} \notin (\mathbf{A} \cap \mathbf{B})$

$\Rightarrow \mathbf{x} \notin \mathbf{A}$ **and** $\mathbf{x} \notin \mathbf{B}$

$\Rightarrow \mathbf{x} \in \mathbf{A}'$ **and** $\mathbf{x} \in \mathbf{B}'$

$\Rightarrow \mathbf{x} \in (\mathbf{A}' \cup \mathbf{B}')$

By definition of subset, we get $(A \cap B)' \subseteq A' \cup B'$ --------------------------------(1)

Again Let $y \in A' \cup B'$

Then $y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

By definition of subset, we get $A' \cup B' \subseteq (A \cap B)'$------------------------------(2)

From (1) and (2), we get,

$(A \cap B)' = A' \cup B'$ (Proved)

Q-8: If $A = \{0, \{0,1\}\}$, Find the power set of A.

Answer: Here A contains two elements 0 and the set $\{0,1\}$, Therefore 2^A contain $2^2 = 4$ elements.

i.e. $2^A = \{A, \{0\}, \{0,1\}, \phi\}$

Q-9: If $E = \{0,1\}$ whether the following statements are true or false

i) $\phi \in E$ ii) $\phi \subset E$ iii) $\phi \in 2^E$ iv) $\phi \subset 2^E$ v) $\{\phi\} \subset 2^E$

Answer:

Here, $E = \{0,1\}$

and $2^E = \{E, \{0\}, \{1\}, \phi\}$

So,

i) $\phi \in E \rightarrow$ False ii) $\phi \subset E \rightarrow$ True iii) $\phi \in 2^E \rightarrow$ True iv) $\phi \subset 2^E \rightarrow$ False

v) $\{\phi\} \subset 2^E \rightarrow$ True

Q-10: Let P (A) denote set of $A = \{1,2\}$, find P {P (A)}.

Solution: Here A contains two elements 1 and 2, therefore 2^A contains $2^2 = 4$ elements, i.e. P (A) or 2^A = {A, {1}, {2}, ϕ}

Again, P {P (A)} contains $2^4 = 16$ elements.

P {P (A)} = {P(A), {A,{1}}, {A,{2}}, {A, ϕ}, {{1},{2}}, {{1}, ϕ}, {{2}, ϕ}, {A,{1},{2}}, {A,{1}, ϕ}, {A,{2}, ϕ}, {{1},{2}, ϕ}, {A}, {{1}}, {{2}}, { ϕ}, ϕ}

Q-11: If A = {1, 2, 3}, B = { $x \mid x$ is even integer} and C = { $x \mid x$ is divisible by 3} are three sets, Find the value of $A \cup (B \cap C)$ and $A \cap (B \cup C)$

Answer:

Here, A = {1, 2, 3},

B = {-6, -4, -2, --------------, 2, 4, 6, 8, ---------- ∞ }

and C = {-6, -3, 3, 6, 12, ------------------------------- ∞ }

Now, $\mathbf{B \cap C} = \{-12, -6, 6, 12, - - - - - - - - - - -\}$

$\mathbf{A \cup (B \cap C)}$ = {1, 2, 3, -6, -12, 6, 12, 18, ----------------------}

$\mathbf{A \cup (B \cap C)}$ = { -6, -12,1, 2, 3, 6, 12, 18, ----------------------}

Again, $\mathbf{B \cup C}$ = {-6, -4, -3, -2, ----------, 2, 3, 4, 6, 8, 9, 10, 12, ------------}

$\mathbf{A \cap (B \cup C) = \{2,3\}}$ Answer

Q-12: If A, B, C are the sets of integer and
A = $\{\mathbf{x} | \mathbf{x^2 - 7x + 12 = 0}\}$, $\mathbf{B = \{x | 2 < x < 12\}}$, $\mathbf{C = \{x | x}$ is divisible by 2} then find the values of $\mathbf{i)\ A \cap (B \cup C)}$ $\mathbf{ii)\ (A \cup B) \cap}$ C

Answer:

Here, A = $\mathbf{\{x | x^2 - 7x + 12 = 0\} = \{3,4\}}$

$\mathbf{B = \{x | 2 < x < 12\} = \{3, 4, 5, 6, 7, 8, 9, 10, 11\}}$

C = {-6, -4, -2, --------------, 2, 4, 6, 8, ---------- ∞ }

We have, $\mathbf{(B \cup C)}$ = {-6, -4, -2, ----------, 2,3,4,5,6,7,8,9,10,11,12,14, ------------------}

$\mathbf{i)\ A \cap (B \cup C) = \{3,4\} = A}$

Again $\mathbf{(A \cup B) = (3,4,5,6,7,8,9,10,11\} = B}$

Hence, $\mathbf{(A \cup B) \cap}$ C = {4, 6, 8, 10} Answer

Q-13: Prove that (B-A) is a subset of A'

Answer: Let $\mathbf{x \in B - A}$

$\mathbf{\Rightarrow x \in B \ \text{ and } \ x \notin A}$

$\mathbf{\Rightarrow x \in B \ \text{ and } \ x \in A'}$

$\mathbf{\text{Since } x \in B - A \ \text{ implies } \ x \in A'}$

Therefore B-A is a subset of A'

$\mathbf{\therefore B - A \subset A'}$ (Proved)

Q-14: Prove that $\mathbf{B - A' = B \cap A}$

Answer:

Let $\mathbf{x \in B - A'}$

Then,

$\Rightarrow \mathbf{x} \in \mathbf{B}$ **and** $\mathbf{x} \notin \mathbf{A}'$

$\Rightarrow \mathbf{x} \in \mathbf{B}$ **and** $\mathbf{x} \in \mathbf{A}$

$\Rightarrow \mathbf{x} \in (\mathbf{B} \cap \mathbf{A})$

$\therefore (\mathbf{B} - \mathbf{A}') \subseteq (\mathbf{B} \cap \mathbf{A})$ -----------------(1)

Conversely, Let $\mathbf{y} \in \mathbf{B} \cap \mathbf{A}$

$\Rightarrow y \in B$ and $y \in A$

$\Rightarrow \mathbf{y} \in \mathbf{B}$ **and** $\mathbf{y} \notin \mathbf{A}'$

$\Rightarrow \mathbf{y} \in \mathbf{B} - \mathbf{A}'$

$\therefore (\mathbf{B} \cap \mathbf{A}) \subseteq \mathbf{B} - \mathbf{A}'$ ------------------(2)

From (1) and (2), we get,

$\mathbf{B} - \mathbf{A}' = \mathbf{B} \cap \mathbf{A}$ (Proved)

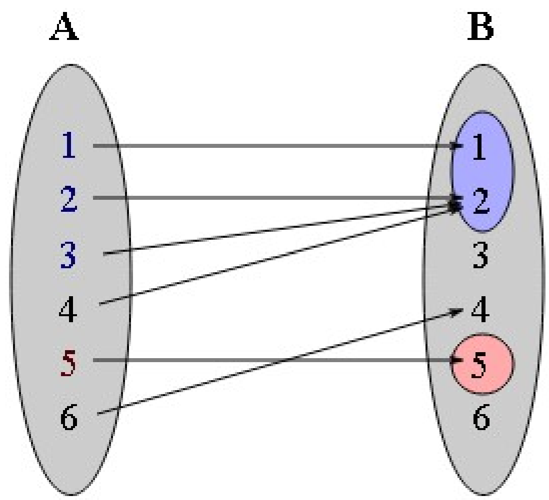
Functions

Definition: Suppose that to each element in a set A there is assigned, by some manner or other, at least a unique element of a set B. We call such assignments a function. Let f denote these assignments, and then we write $\mathbf{f : A \rightarrow B}$ (reads f is a function of A into B). Every element of domain should have corresponding image. Each element of domain cannot have different image. The set A is called the domain of the function f, and B is called the co-domain of f. Further if $a \in A$ then the element in B which is assigned to a is called the image of a and is denoted by f (a) (reads f of a)

One input cannot have two corresponding outputs

1. Let f assign to each real number its square, that is, for every real number x let $\mathbf{f(x) = x^2}$. The domain and co-domain of f are both the real numbers, so we can write: $\mathbf{f : R^{\#} \rightarrow R^{\#}}$
   The image of -3 is 9; hence we can also write $\mathbf{f(-3) = 9}$ **or** $\mathbf{f : -3 \rightarrow 9}$
2. Let f assign to each country in the world its capital city. Here the domain of f is the set of countries in the world; the co-domain of f is the list of capital cities in the world. The image of France is Paris, that is f (France) = Paris.

Let A and B be two sets. A function f from A to B is a relation between A and B such that for each a ∈ A there is one and only one associated b ∈ B. The set A is called the domain of the function; B is called its range.



Domain = {1, 2 , 3, 4, 5, 6}, Co-domain = {1, 2 , 3, 4, 5, 6}, Range = {1 , 2, 4, 5}

x is the image of b and c that is, f (b) = x and f(c) = x, y is the image of a , that is f(a) = y.

Often a function is denoted as y = f(x) or simply f(x), indicating the relation {(x, f(x))}.

Image and Pre-image

- Let A and B be two sets and f a function from A to B. Then the image of f is defined as

  image(f) = {b $\in$ B : there is an a $\in$ A with f(a) = b}.

- Let A and B be two sets and f a function from A to B. If C is a subset of the range B then the pre-image, or inverse image, of C under the function f is the set defined as
  - $f^{-1}(C) = \{x \in A : f(x) \in C\}$

Equal Functions: If f and g are functions defined on the same domain D and if f (a) = g (a) for every $\mathbf{a} \in \mathbf{D}$**, then** the functions f and g are equal and we write f = g.

Example:

1. Let $\mathbf{f(x) = x^2}$ where x is a real number. Let $\mathbf{g(x) = x^2}$ where x is a complex number. Then the function f is not equal to g since they have different domains.
2. Let $\mathbf{f : R^\# \to R^\#}$ and $\mathbf{g : R^\# \to R^\#}$. Suppose f is defined by $\mathbf{f(x) = x^2}$ and g by $g(y) = y^2$. Then f and g are equal functions, that is, f = g.

Range of a function:

Let f be a mapping of A into B, that is, let $\mathbf{f : A \to B}$. Each element in B need not appear as the image of an element in A. We define the range of f to consist precisely of those elements in B which appear as the image of at least one element in A. we denote the range of $\mathbf{f : A \to B}$ by f (A), Notice that f(A) is a subset of B.

Example

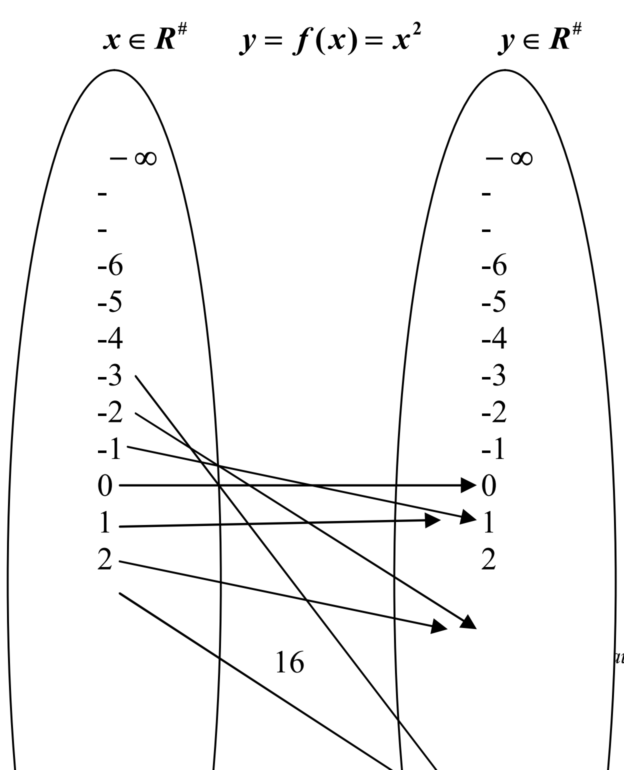
Let the function $\mathbf{f : R^{\#} \to R^{\#}}$ be defined by the equation $\mathbf{f(x) = x^2}$. Then the range of f consists of the positive real numbers and zero.

One-One Functions:

- Let f map A into B. Then f is <u>called a one-one function</u> if different elements in B are assigned to different elements in A, that is, if no two different elements in A have the same image. More briefly, $f : A \to B$ is one-one if f (a) = $f(a^{'})$ implies a = $\mathbf{a}^{'}$ or, equivalently a $\neq a^{'}$ implies a $f(a) \neq f(a^{'})$
- A function f from A to B is called one to one (or one- one) if whenever f(a) = f(b) then a = b. Such functions are also called injections.

Example:

1. Let the function $\mathbf{f : R^{\#} \to R^{\#}}$ be defined by the formula y $= \mathbf{f(x) = x^2}$. Then f is not a one-one function since f (2) = f (-2) =4, that is, since the image of two different real numbers, 2 and -2, is the same number 4.



3	3
4	4
5	5
6	6
7	7
8	8
9	9
10	10
11	11
12	12
13	
-	-
-	-
∞	∞

2. Let the function $\mathbf{f : R^{\#} \to R^{\#}}$ be defined by the formula $\mathbf{f(x) = x^3}$. Then f is a one-one function since the cubes of two different real numbers are themselves different.
3. The function f which assigns to each country in the world its capital city is one-one since different countries have different capitals, that is, no city is the capital of two different countries.

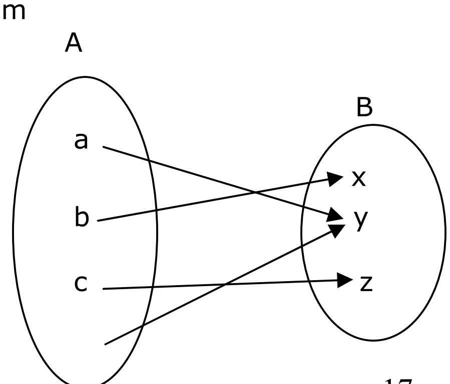
Onto Functions:

- Let f be a function of A into B. Then the range f (A) of the function f is a subset of B, that is, $\mathbf{f(A) \subset B}$. **If** $\mathbf{f(A) = B}$, that is, if every member of B appears as the image of at least one element of A, then we say " f is a function of A onto B", or " f maps A onto B", or " f is an onto function".

- A function f from A to B is called onto if for all b in B there is an a in A such that f(a) = b. Such functions are also called surjection.

Example:

1. Let the function $\mathbf{f : R^{\#} \to R^{\#}}$ be defined by the formula $\mathbf{f(x) = x^2}$. Then f is not an onto function since the negative numbers do not appear in the range of f, that is, no negative number is the square of a real number.
2. Let A = {a, b, c, d} and B = {x, y, z}. Let $f : A \to B$ be defined by the diagram



d

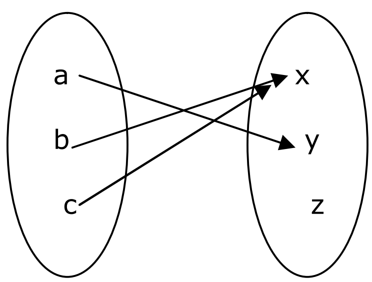
Note that f (A) = {x , y , z } = B
that is, the range of f is equal to the co-domain B. Thus f maps A onto B, i.e f is an onto mapping.

Constant Functions:
A function f of A into B that is, let $\mathbf{f : A \rightarrow B}$ is called a constant function if the same element $b \in B$ is assigned to every element in A. i.e. if the range of f consists of only one element.

Example:
Let the function $\mathbf{f : R^{\#} \rightarrow R^{\#}}$ be defined by the formula $\mathbf{f(x) = 5}$. Then f is a constant function since 5 is assigned to every element.

Inverse of a function:
Let f be a function of A into B and let $b \in B$. Then the inverse of b, denoted by $\mathbf{f^{-1}(b)}$ consists of those elements in A which are mapped onto b, that is, those elements in A which have b as their image. More briefly, if $\mathbf{f : A \rightarrow B}$ then
$\mathbf{f^{-1}(b) = \{x | x \in A,\ f(x) = b\}}$

Example:
1.

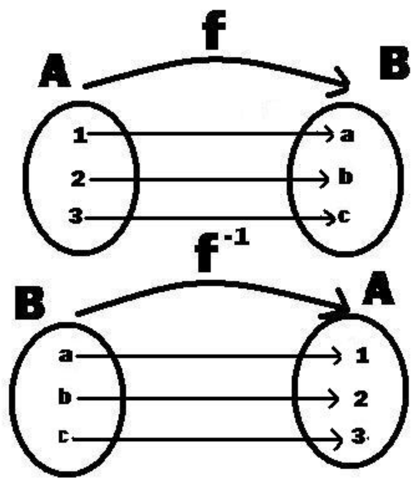


Then $f^{-1}(x)$ = {b, c}, since both b and c have x as their image point. Also $f^{-1}(y)$ = {a}, as only a is mapped into y. The inverse of z, $f^{-1}(z)$, is the null set ϕ, since no element of A is mapped into z.

1. Let the function $\mathbf{f : R^{\#} \rightarrow R^{\#}}$ be defined by the formula $\mathbf{f(x) = x^2}$. Then $\mathbf{f^{-1}(4)}$ = {2, -2}, since 4 is the image of both 2 and -2 and there is no other real number whose square is four. Notice that $\mathbf{f^{-1}(-3)}$ $= \phi$, since there is no element in $R^{\#}$ whose square is -3.

Inverse Function:

If $\mathbf{f : A \rightarrow B}$ is a one-one function and an onto function, then $\mathbf{f^{-1} : B \rightarrow A}$



Note that f is one-one and onto. Therefore f^{-1}, the inverse function, exists

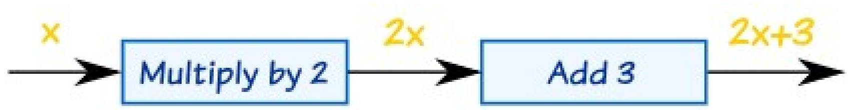
Definition: The inverse of a function is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original function. If f is a given function, then f^{-1} denotes the inverse of f. (If the original function is a one-to-one function, the inverse will also be a function.)

Inverse Functions

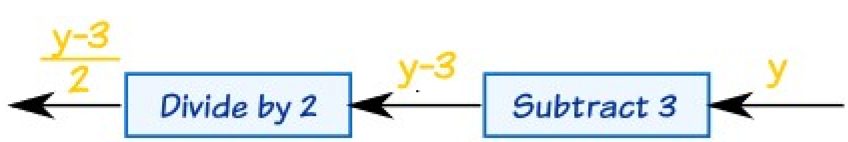
An inverse function goes the other way!

Let us start with an example:

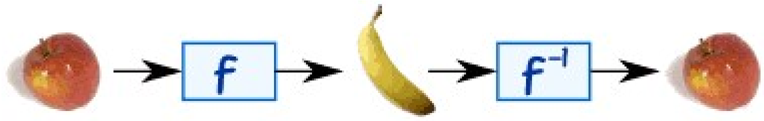
Here we have the function **f(x) = 2x+3**, written as a flow diagram:



The **Inverse Function** just goes the other way:



So the inverse of: 2x+3 is: (y-3)/2



When the function f turns the apple into a banana,
Then the **inverse** function f^{-1} turns the banana back to the apple

Examples:

a. Given function f, find the inverse. Is the inverse also a function?

$f(x) = \{(3,4),(1,-2),(5,-1),(0,2)\}$

Answer:
Function f is a one-to-one function since the x and y values are used only once.
The inverse is

$f^{-1}(x) = \{(4,3),(-2,1),(-1,5),(2,0)\}$

Since function f is a one-to-one function, the inverse is also a function.

b. Determine the inverse of this function. Is the inverse also a function?

x	1	-2	-1	0	2	3	4	-3
f (x)	2	0	3	-1	1	-2	5	1

Answser: Swap the x and y variables to create the inverse. Since function f was not a one-to-one function (the y value of 1 was used twice), the inverse will NOT be a function (because the x value of 1 now gets mapped to two separate y values which is not possible for functions).

x	2	0	3	-1	1	-2	5	1
$f^{-1}(x)$	1	-2	-1	0	2	3	4	-3

Find the inverse of the function $f(x) = x - 4$

Answer:

$y = x - 4$

$x = y - 4$

$x + 4 = y$

$f^{-1}(x) = x + 4$

Remember:
Set = y.
Swap the variables.
Solve for y.

"Is it a function?" - Quick answer without the graph

Think of all the graphing that you've done so far. The simplest method is to solve for "y =", make a T-chart, pick some values for x, solve for the corresponding values of y, plot your points, and connect the dots, yadda, yadda, yadda. Not only is this useful for graphing, but this methodology gives yet another way of identifying functions: If you can solve for "y =", then it's a function. In other words, if you can enter it into your graphing calculator, then it's a function. (The calculator can only handle functions.) For example, 2y + 3x = 6 is a function, because you can solve for y:

$$2y + 3x = 6$$
$$2y = -3x + 6$$
$$y = (-3/2)x + 3$$

On the other hand, $y^2 + 3x = 6$ is not a function, because you can not solve for a unique y:

$$y^2 + 3x = 6$$
$$y^2 = -3x + 6$$
$$y = \pm\sqrt{-3x + 6}$$

I mean, yes, this is solved for "y =", but it's not unique. Do you take the positive square root, or the negative? Besides, where's the "±" key on your graphing calculator? So, in this case, the relation is not a function. (You can also check this by using our first definition from above. Think of "x = –1". Then we get $y^2 - 3 = 6$, so $y^2 = 9$, and then y can be either –3 or +3. That is, if we did an arrow chart, there would be two arrows coming from x = –1.)

Definition: One-one, Onto, Bijection

- A function f from A to B is called one to one (or one- one) if whenever f(a) = f(b) then a = b. Such functions are also called injections.
- A function f from A to B is called onto if for all b in B there is an a in A such that f(a) = b. Such functions are also called surjections.
- A function f from A to B is called a bijection if it is one to one and onto, i.e. bijections are functions that are injective and surjective.

Examples:

- If the graph of a function is known, how can you decide whether a function is one-to-one (injective) or onto (surjective) ?
- Which of the following functions are one-one, onto, or bijections ? The domain for all functions is R.
    1. f(x) = 2x + 5
    2. g(x) = arctan(x)
    3. g(x) = sin(x)
    4. $h(x) = 2x^3 + 5x^2 - 7x + 6$

Bijection, injection and surjection

In mathematics, injections, surjections and bijections are classes of functions distinguished by the manner in which arguments (input expressions from the domain) and images (output expressions from the codomain) are related or mapped to each other.

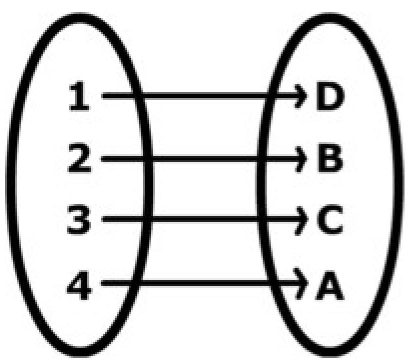
A function $f: A \rightarrow B$ is injective (one-to-one) if $f(x) = f(y) \rightarrow x = y$ or, equivalently, if $x \neq y \rightarrow f(x) \neq f(y)$ One could also say that elements of the codomain (sometimes called range by mistake) are mapped to by at most one element (argument) of the domain; not every element of the codomain, however, need have an argument mapped to it. An injective function is an injection.

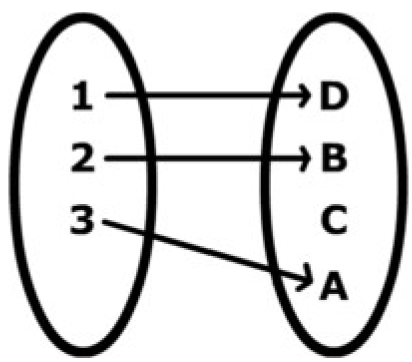
A function is surjective (onto) if every element of the codomain is mapped to by some element (argument) of the domain; some images may be mapped to by more than one argument. (Equivalently, a function where the range is equal to the codomain.) A surjective function is a surjection.

A function is bijective (one-to-one and onto) if and only if (iff) it is both injective and surjective. (Equivalently, every element of the codomain is mapped to by exactly one element of the domain.) A bijective function is a bijection (one-to-one correspondence).

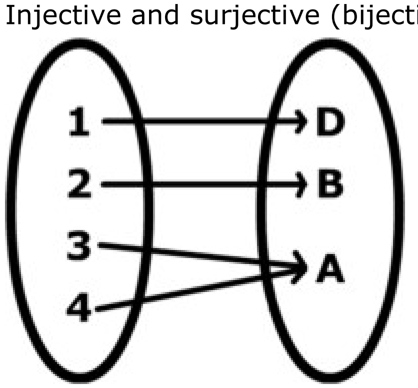
(Note: a one-to-one function is injective, but may fail to be surjective, while a one-to-one correspondence is both injective and surjective.)

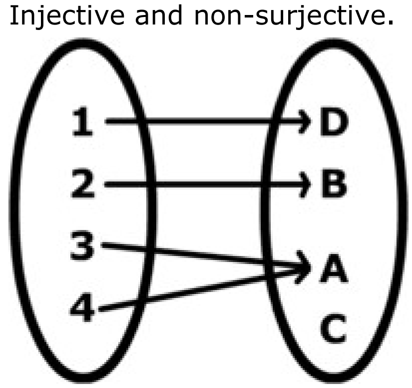
An injective function need not be surjective (not all elements of the codomain may be associated with arguments), and a surjective function need not be injective (some images may be associated with more than one argument). The four possible combinations of injective and surjective features are illustrated in the following diagrams.





Injective and surjective (bijective). Injective and non-surjective.





Non-injective and surjective. Non-injective and non-surjective.

Bijection: A function f from A to B is called a bijection if it is one to one and onto, bijections are functions that are injective and surjective.

Definition: Function, Domain, and Range

- Let A and B be two sets. A function f from A to B is a relation between A and B such that for each $a \in A$ there is one and only one associated $b \in B$. The set A is called the domain of the function; B is called its range.
- Often a function is denoted as y = f(x) or simply f(x), indicating the relation { (x, f(x)) }.

Examples:

Let A = {1, 2, 3, 4}, B = {14, 7, 234}, C = {a, b, c}, and R = real numbers. Define the following relations:

1. r is the relation between A and B that associates the pairs 1 ~ 234, 2 ~ 7, 3 ~ 14, 4 ~ 234, 2 ~ 234
2. f is the relation between A and C that relates the pairs {(1,c), (2,b), (3,a), (4,b)}
3. g is the relation between A and C consisting of the associations {(1,a), (2,a), (3,a)}
4. h is the relation between R and itself consisting of pairs {(x,sin(x))}

Which of those relations are functions?

The outcomes of a function (i.e. the elements from the range associated to elements in the domain) do not only depend on the rule of the function (such as x is associated with sin(x)) but also on the domain of the function. Therefore, we need to specify those outcomes that are possible for a given rule and a given domain:

Definition: Image and Pre-image

- Let A and B be two sets and f a function from A to B. Then the image of f is defined as
  - imag(f) = $\{b \in B$: there is an $a \in A$ with $f(a) = b\}$.
- Let A and B be two sets and f a function from A to B. If C is a subset of the range B then the pre-image, or inverse image, of C under the function f is the set defined as
  - $f^{-1}(C) = \{x \in A : f(x) \in C\}$

As an example, consider the following functions:

Example:

- Let f(x) = 0 if x is rational and f(x) = 1 if x is irrational. This function is called Dirichlet's Function. The range for f is R.
  - Find the image of the domain of the Dirichlet Function when:

1. The domain of f is Q
2. The domain of f is R
3. The domain of f is [0, 1] (the closed interval between 0 and 1)

- What is the pre-image of R? What is the pre-image of [-1/2, 1/2]?

- Let f (x) = x^2, with domain and range being R. Then use the graph of the function to determine:
  1. What is the image of [0, 2] and the pre-image of [1, 4] ?
  2. Find the image and the pre-image of [-2, 2].

Functions can be classified into three groups: those for which every element in the image has one pre-image, those for which the range is the same as the image, and those, which have both of these properties. Accordingly, we make the following definitions:

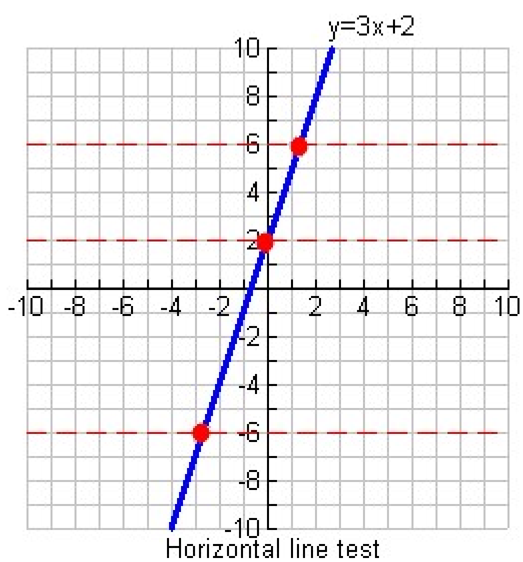
The inverse of a function may not always be a function!
The original function must be a one-to-one function to guarantee that its inverse will also be a function.

Definition: A function is a one-to-one function if and only if each second element corresponds to one and only one first element. (each x and y value is used only once)

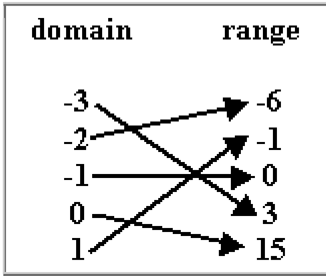
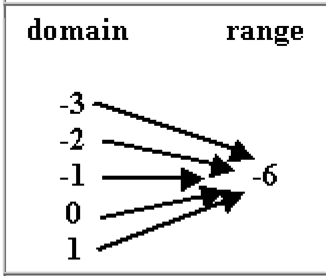
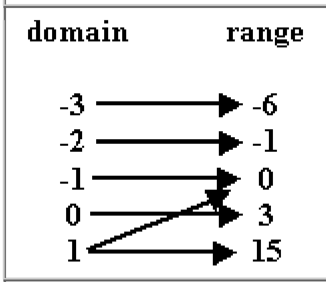
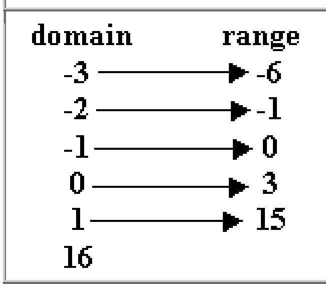
Use the horizontal line test to determine if a function is a one-to-one function. If ANY horizontal line intersects your original function in ONLY ONE location, your function will be a one-to-one function and its inverse will also be a function.

The function y = 3x + 2, shown at the right, IS a one-to-one function and its inverse will also be a function.

(Remember that the vertical line test is used to show that a relation is a function.)

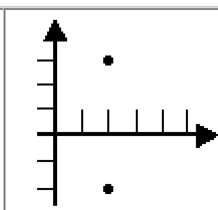


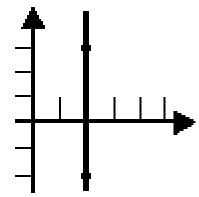
Definition: The inverse of a function is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original function. Notation: If f is a given function, then f^{-1} denotes the inverse of f. (If the original function is a one-to-one function, the inverse will also be a function.)

<table>
<tr><td>
</td><td>This is a function. You can tell by tracing from each x to each y. There is only one y for each x; there is only one arrow coming from each x.</td></tr>
<tr><td>
</td><td>Ha! Bet I fooled some of you on this one! This is a function! There is only one arrow coming from each x; there is only one y for each x. It just so happens that it's always the same y, but it is only one y. So this is a function; it's just an extremely boring function!</td></tr>
<tr><td>
</td><td>This one is not a function. If you'll notice, there are two arrows coming from the number 1; the number 1 is associated with two different range elements. So this is a relation, but it is not a function.</td></tr>
<tr><td>
</td><td>Okay, this one's a trick question. Each element of the domain that has a pair in the range is nicely well-behaved. But what about that 16? It is in the domain, but it has no range element that corresponds to it! This won't work! So then this is not a function. Heck, it ain't even a relation!</td></tr>
</table>

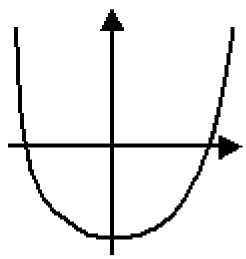
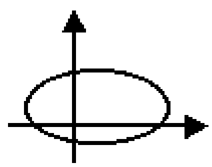
The "Vertical Line Test"

Looking at this function stuff graphically, what if we had the relation {(2, 3), (2, –2)}? We already know that this is not a function, since x = 2 goes to both y = 3 and y = –2.

<table>
<tr><td>If we graph this relation, it looks like:</td><td></td></tr>
</table>

Notice that you can draw a vertical line through the two points, like this:	

This characteristic of non-functions was noticed by I-don't-know-who, and was codified in The Vertical Line Test: Given the graph of a relation, if you can draw a vertical line that crosses the graph in more than one place, then the relation is not a function. Here are a couple examples:

	This one is a function. There is no vertical line that will cross this graph twice.
	This one is not a function. Any number of vertical lines will intersect this oval twice. For instance, the y-axis intersects twice.

- State the domain and range of the following relation. Is the relation a function?
  {(2, –3), (4, 6), (3, –1), (6, 6), (2, 3)}

This list of points, being a relationship between certain x's and certain y's, is a relation. The domain is all the x-values, and the range is all the y-values. You list the values without duplication:

domain: {2, 3, 4, 6}

range: {–3, –1, 3, 6}

(It is customary to list these values in numerical order, but it is not required. Sets are called "unordered lists", so you can list the numbers in any order you feel like. Just don't duplicate.)

While this is a relation (because x's and y's are being related to each other), you have two points with the same x-value: (2, –3) and (2, 3). Since x = 2 gives you two possible destinations, then this relation is not a function.

Note that all I had to do to check whether the relation was a function was to look for duplicate x-values. If you find a duplicate x-value, then the different y-values mean that you do not have a function.

- State the domain and range of the following relation. Is the relation a function?
  {(–3, 5), (–2, 5), (–1, 5), (0, 5), (1, 5), (2, 5)}

  List the x-values for the domain and the y-values for the range:

  domain: {–3, –2, –1, 0, 1, 2}

  range: {5}

This is another example of a "boring" function (like we saw in the first chart higher up in this page), because all the x-values go to the exact same y-value. But each x-value is different, so, while boring, this relation is indeed a function. (In point of fact, these points lie on the horizontal line y = 5.)

There is one other case for finding the domain and range of functions. They will give you a function and ask you to find the domain (and maybe the range, too). I have only ever seen (or can even think of) two things at this stage in your mathematical career that you'll have to check in order to determine the domain of the function they'll give you, and those two things are denominators and square roots.

- Determine the domain and range of the given function:

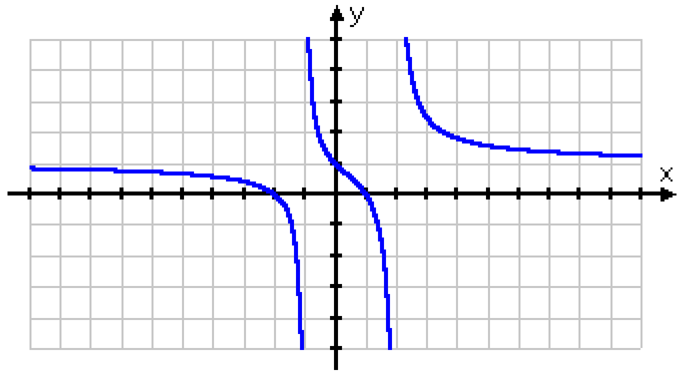
$$\frac{x^2+x-2}{x^2-x-2}$$

  The domain is all the values that x is allowed to take on. The only problem you have with this function is that you have to be careful not to divide by zero. So the only values that x can not take on are those which would cause division by zero. So set the denominator equal to zero and solve; the domain will be everything else.

$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2 \quad \text{or} \quad x = -1$$

Then the domain is "all x not equal to –1 or 2".

The range is a bit trickier, which is why they may not ask for it. In general, though, they'll want you to graph the function and find the range from the picture. In this case:



As you can see from the picture, the graph covers all y-values (that is, the graph will go as low as you like, and will also go as high as you like). Since the graph will eventually cover all possible values of y, then the range is "all real numbers".

- Determine the domain and range of the given function:

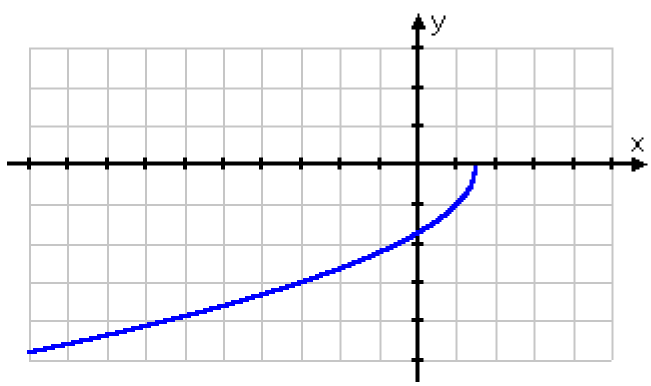
$$-\sqrt{-2x+3}$$

The domain is all values that x can take on. The only problem you have with this function is that you cannot have a negative inside the square root. So set the insides greater-than-or-equal-to zero, and solve. This will be the domain:

$$-2x + 3 \geq 0$$
$$-2x \geq -3$$
$$2x \leq 3$$
$$x \leq 3/2 = 1.5$$

Then the domain is "all x $\leq$ 3/2".

The range requires a graph. Be sure to be careful when graphing radicals:



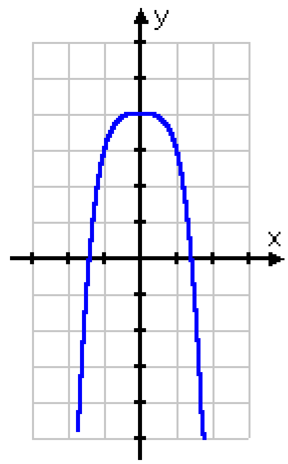
The graph starts at y = 0 and goes down from there. While the graph goes down very slowly, you know that, eventually, you can go as low as you like (by picking an x that is sufficiently big). Also, from your experience with graphing, you know that the graph will never start coming back up. Then the range is "y < 0".

- Determine the domain and range of the given function:

$$y = -x^4 + 4$$

This is just a garden-variety polynomial. There are no denominators (so no division-by-zero problems) and no radicals (so no square-root-of-a-negative problems). There are no problems with a polynomial. There are no values that you can't plug in for x. When you have a polynomial, the answer is always that the domain is "all x".

The range will vary from polynomial to polynomial, and they probably won't even ask, but, if they do, look at the picture:



Q-1: Which if these statements is different from the others, and why?

(1) f is a function of A into B.   (3) f: $x \to f(x)$

(2) f: $A \to B$ (4) $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B}$

(5) f is a mapping of A into B.

Solution:

(3) is different from the others. We are not told what is the domain is and the co-domain in (3), whereas in all the others we are told that A is the domain and B is the co-domain.

Q-2: Let $\mathbf{f(x) = x^2}$ define a function on the closed interval $\mathbf{-2 \le x \le 8.}$ Find

(1) f (4), (2) f (-3) (3) f $(t-3)$.

Solution:

(1) $f(4) = 4^2 = 16.$

(2) $f(-3)$ has no meaning, i.e. is undefined, since -3 is not in the domain of the function.

(3) $f(t-3) = (t-3)^2 = t^2 - 6t + 9.$ But this formula is true only $t-3$ is in the domain, i.e. when $-2 \le t-3 \le 8.$ In other words, t must satisfy $1 \le t \le 11.$

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Q-3: Let the function $f:\Re^{\#} \to \Re^{\#}$ be defined by

$$f(x) = \begin{cases} \mathbf{3x-1} & \mathbf{if} \quad \mathbf{x>3} \\ \mathbf{x^2-2} & \mathbf{if} \quad \mathbf{-2 \le x \le 3} \\ \mathbf{2x+3} & \mathbf{if} \quad \mathbf{x<-2} \end{cases}$$

Find (a) $\mathbf{f(2)}$, (b) $f(4)$, **(c)** $\mathbf{f(-1)}$, (d) $\mathbf{f(-3)}$.

Solution:

(a) Since 2 belongs to the closed interval $[-2,3]$, we use the formula $f(x)=x^2-2.$ hence $\mathbf{f(2)=2^2-2=4-2=2.}$

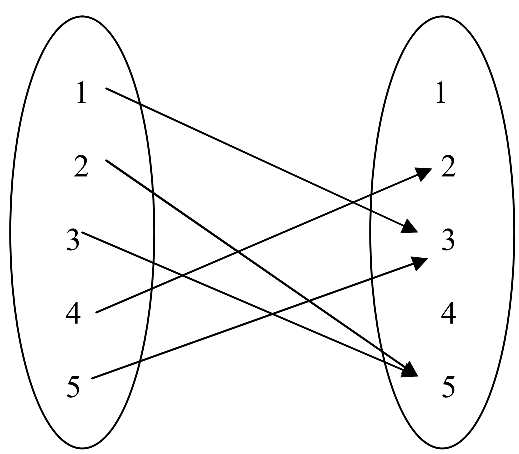
(b) Since 4 belongs to $(3,\infty)$, we use the formula $\mathbf{f(x)=3x-1.}$ Thus $\mathbf{f(4)=3(4)-1=12-1=11.}$

(c) Since -1 is in interval $[-2,3]$, we use the formula $\mathbf{f(x)=x^2-2.}$ Computing, $\mathbf{f(-1)=(-1)^2-2=1-2=-1.}$

(d) Since -3 is less than -2, i.e. -3 belongs to $(-\infty,-2)$, we use the formula $\mathbf{f(x)=2x+3.}$ Thus $\mathbf{f(-3)=2(-3)+3=-6+3=-3.}$

Notice that there is only one function f defined even though there are three formulas, which are used to define f.

Q-4 : Is the function $f:A \to A$ onto?



Solution:

The numbers 1 and 4 in the co-domain are not the images of any elements in the domain: hence f is not an onto function. In other words, $\mathbf{f(A)=\{2,3,5\}}$ is a proper subset of A?

Q-5: Let $\mathbf{A=[-1,1]}$.. Let functions f, g and h of A into A be defined by:

(1) $\mathbf{f(x)=x^2}$, (2) $\mathbf{g(x)=x^3}$, (3) $\mathbf{h(x)=\sin x}$

Which function, if any, is onto?

Solution:
No negative numbers appear in the range of f: hence f is not an onto function

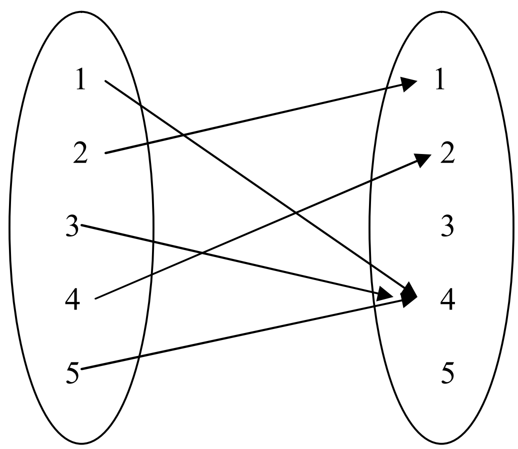
(2) The function g is onto, that is $\mathbf{g(A) = A.}$

(3) The function h is not onto. In fact, there is no number x in A such that sin $x = 1$.

Q-6: Can a constant function be an onto function?

Solution:
If the co-domain of a function f consists of a single element, then f is always a constant function and is onto.

Q-7: Let $\mathbf{A = \{1,2,3,5\}}$. let the function $\mathbf{f : A \rightarrow A}$ be defined by the diagram



Find (1) $\mathbf{f^{-1}(2)}$, (2) $\mathbf{f^{-1}(3)}$, (3) $\mathbf{f^{-1}(4)}$, (4) $\mathbf{f^{-1}\{1,2\}}$, (5) $\mathbf{f^{-1}\{2,3,4\}}$.

Solution:

(1) $\mathbf{f^{-1}(2)}$ Consists of those elements whose image is 2. Only 4 has the image 2; hence $\mathbf{f^{-1}(2) = \{4\}}$.

(2) $\mathbf{f^{-1}(3)} = \phi$ Since 3 is not the image of any element in the domain.

(3) $\mathbf{f^{-1}(4) = \{1,3,5\}}$ Since $f(1) = 4$, $\mathbf{f(3) = 4}$, $\mathbf{f(5) = 4}$ and since 4 is not the image of any other element.

(4) $\mathbf{f^{-1}\{1,2\}}$ Consists of those elements whose image is either 1 or 2; hence $\mathbf{f^{-1}\{1,2\} = \{2,4\}}$.

(5) $\mathbf{f^{-1}(2,3,4) = \{4,1,3,5\}}$ Since each of these numbers, and no others, has 2, 3 or 4 as an image point.

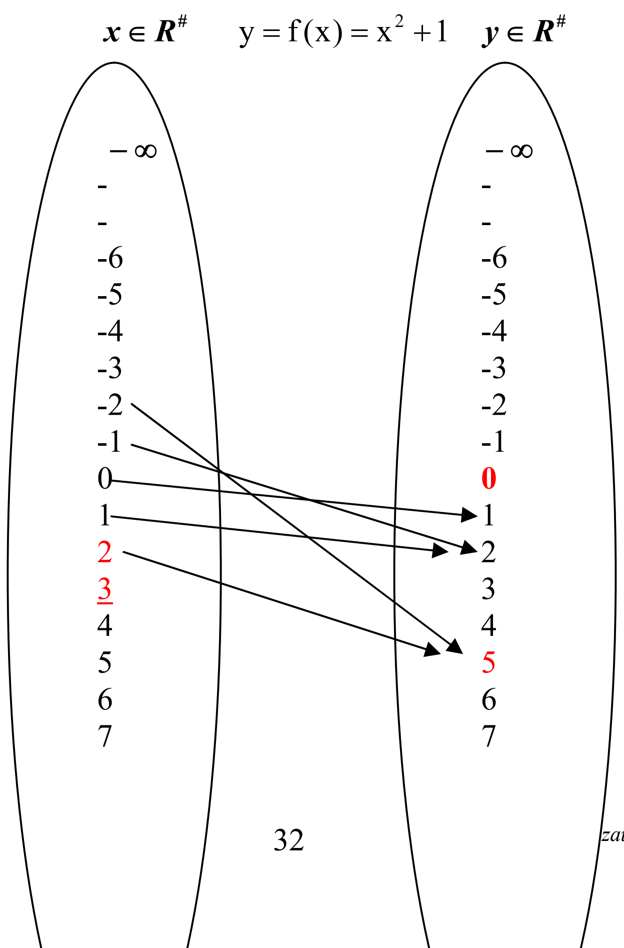
Q-8: Let the function $\mathbf{f}:\Re^{\#}\rightarrow\Re^{\#}$ be defined by $\mathbf{f}(\mathbf{x})=\mathbf{x}^2$. Find:
(1) $\mathbf{f}^{-1}(\mathbf{25})$, (2) $\mathbf{f}^{-1}(-\mathbf{9})$, (3) $\mathbf{f}^{-1}([-\mathbf{1},\mathbf{1}])$, (4) $\mathbf{f}^{-1}((-\infty,\mathbf{0}))$. (5) $\mathbf{f}^{-1}([\mathbf{4},\mathbf{25}])$.
Solution:

(1) $f^{-1}(25)=\{5,-5\}$ Since $f(5)=25$ and $f(-5)=25$ and since the square of no other number is 25.
(2) $\mathbf{f}^{-1}(-\mathbf{9})=\phi$ Since there is no real number whose square is -9, i.e. the equation $x^2=-9$ has no real root.
(3) $\mathbf{f}^{-1}([-\mathbf{1},\mathbf{1}])=[-\mathbf{1},\mathbf{1}]$ Since $|\mathbf{x}|\leq\mathbf{1}$ implies $|\mathbf{x}^2|\leq\mathbf{1}$, i.e. if x belongs to $[-\mathbf{1},\mathbf{1}]$ then $f(x)=x^2$ also belongs to $[-1,1]$.
(4) $\mathbf{f}^{-1}((-\infty,\mathbf{0}])=\{\mathbf{0}\}$ Since $\mathbf{0}^2=\mathbf{0}\varepsilon(-\infty,\mathbf{0}]$ and since no other number squared belongs to $(-\infty,\mathbf{0}]$.
(5) $\mathbf{f}^{-1}([\mathbf{4},\mathbf{25}])$ Consists of those numbers whose squares belong to $[\mathbf{4},\mathbf{25}]$, i.e. those numbers $\mathbf{x}$ such that $\mathbf{4}\leq\mathbf{x}^2\leq\mathbf{25}$. Hence

$$\mathbf{f}^{-1}([\mathbf{4},\mathbf{25}]) = \{\mathbf{x}\mid \mathbf{2}\leq\mathbf{x}\leq\mathbf{5}\ \text{or}\ -\mathbf{5}\leq\mathbf{x}\leq-\mathbf{2}\}$$

Q-9: Let the function $\mathbf{f}:\Re^{\#}\rightarrow\Re^{\#}$ be defined by $\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{x}^2+\mathbf{1}$ Find:
(1) $\mathbf{f}^{-1}(\mathbf{5})$, (2) $\mathbf{f}^{-1}(\mathbf{0})$, (3) $\mathbf{f}^{-1}(\mathbf{10})$, (4) $\mathbf{f}^{-1}(-\mathbf{5})$. (5) $\boldsymbol{f}^{-1}([\mathbf{10},\mathbf{26}])$. (6) $\mathbf{f}^{-1}([\mathbf{0},\mathbf{5}])$.
(7) $\mathbf{f}^{-1}([-\mathbf{5},\mathbf{1}])$. (8) $\mathbf{f}^{-1}([-\mathbf{5},\mathbf{5}])$.

Solution:



8	8
9	9
10	10
11	11
12	12
13	
-	-
-	-
∞	∞

1) $f^{-1}(\underset{y}{\underset{\downarrow}{5}}) = \{x \in R^{\#};\ y = 5\}$

$= \{x \in R^{\#};\ x^2 + 1 = 5\}$

$= \{x \in R^{\#};\ x^2 = 4\}$

$= \{x \in R^{\#};\ x = \pm 2\}$

$= \{-2, 2\}$ Answer

2) $f^{-1}(\underset{y}{\underset{\downarrow}{0}}) = \{x \in R^{\#};\ y = 0\}$

$= \{x \in R^{\#};\ x^2 + 1 = 0\}$

$= \{x \in R^{\#};\ x^2 = -1\}$

$= \{x \in R^{\#};\ x = \pm\sqrt{-1}\}$

$= \{x \in R^{\#};\ x = \pm\sqrt{i^2}\}$

$= \{x \in R^{\#};\ x = \pm i\}$

$= \phi$ (Since $\pm\sqrt{-1}$ are not real numbers.) Answer

3) $f^{-1}(10) = \{x \in R^{\#};\ x^2 + 1 = 10\}$

$= \{x \in R^{\#};\ x^2 = 9\}$

$= \{x \in R^{\#};\ x = \pm 3\}$

$= \{-3, 3\}$ Answer

4) $f^{-1}(-5) = \{x \in R^{\#};\ x^2 + 1 = -5\}$

$= \{x \in R^{\#};\ x^2 = -6\}$

$= \{x \in R^{\#};\ x = \pm\sqrt{-6}\}$

$= \phi$ (Since $\pm\sqrt{-6}$ are not real numbers.) Answer

5) 4) $f^{-1}(\underset{y}{\underset{\downarrow}{[10,26]}}) = \{x \in R^{\#};\ y \in [10,26]\}$

$= \{x \in R^{\#};\ 10 \le y \le 26\}$

$= \{x \in R^{\#};\ 10 \le x^2 + 1 \le 26\}$

$= \{x \in R^{\#};\ 9 \le x^2 \le 25\}$

$$= \{x \in R^{\#};\ \ \pm 3 \le x \le \pm 5\}$$
$$= \{x |\ \ 3 \le x \le 5,\ \ -5 \le x \le -3\}$$

6) $f^{-1}([0,5]) = \{x \in R^{\#};\ \ f(x) \in [0,5]\}$
$$= \{x \in R^{\#};\ \ 0 \le x^2 + 1 \le 5\}$$
$$= \{x \in R^{\#};\ \ -1 \le x^2 \le 4\}$$
$$= \{x \in R^{\#};\ \ \pm\sqrt{-1} \le x \le \pm 2\}$$
$$= \{x \in R^{\#};\ \ x \le \pm 2\}$$

Since $\pm\sqrt{-1}$ are not real numbers.

$= \{x |\ \ -2 \le x \le 2\}$ Answer

7) $f^{-1}([-5,1]) = \{x \in R^{\#};\ \ f(x) \in [-5,1]\}$
$$= \{x \in R^{\#};\ \ -5 \le x^2 + 1 \le 1\}$$
$$= \{x \in R^{\#};\ \ -6 \le x^2 \le 0\}$$
$$= \{x \in R^{\#};\ \ \pm\sqrt{-6} \le x \le \pm 0\}$$

$= \{0\}$ Answer

Since $\pm\sqrt{-6}$ are not real numbers.

7) $f^{-1}([-5,5]) = \{x \in R^{\#};\ \ f(x) \in [-5,5]\}$
$$= \{x \in R^{\#};\ \ -5 \le x^2 + 1 \le 5\}$$
$$= \{x \in R^{\#};\ \ -6 \le x^2 \le 4\}$$
$$= \{x \in R^{\#};\ \ \pm\sqrt{-6} \le x \le \pm 2\}$$
$$= \{x \in R^{\#};\ \ x \le \pm 2\}$$

Since $\pm\sqrt{-6}$ are not real numbers.

$= \{x |\ \ -2 \le x \le 2\}$ Answer

Q-10: Let the function $f : \Re^{\#} \to \Re^{\#}$ be defined by y= $f(x) = \sin x$ Find:
(1) $f^{-1}(0)$, (2) $f^{-1}(1)$, (3) $f^{-1}(2)$. 4) $f^{-1}([-1,1])$.

Solution:

1) $f^{-1}(0) = \{x \in R^{\#};\ \ f(x) = 0\}$

$$= \{x \in R^{\#};\ \ \sin x = 0\}$$
$$= \{x \in R^{\#};\ \ \sin x = \sin 0, \sin\pi, \sin 2\pi, -\sin 4\pi, -\sin 2\pi, ------\}$$
$$= \{x \in R^{\#};\ \ x = 0, \pi, 2\pi, -4\pi, -2\pi, ------\}$$
$$= \{-4\pi, -2\pi, -\pi,\ 0,\ \pi\ \ 2\pi, 4\pi, ------\}$$

2) $f^{-1}(1) = \{x \in R^{\#};\ f(x) = 1\}$

$= \{x \in R^{\#};\ \sin x = 1\}$

$= \{x \in R^{\#};\ \sin x = \sin\frac{\pi}{2}, \sin\frac{5\pi}{2}, ------\}$

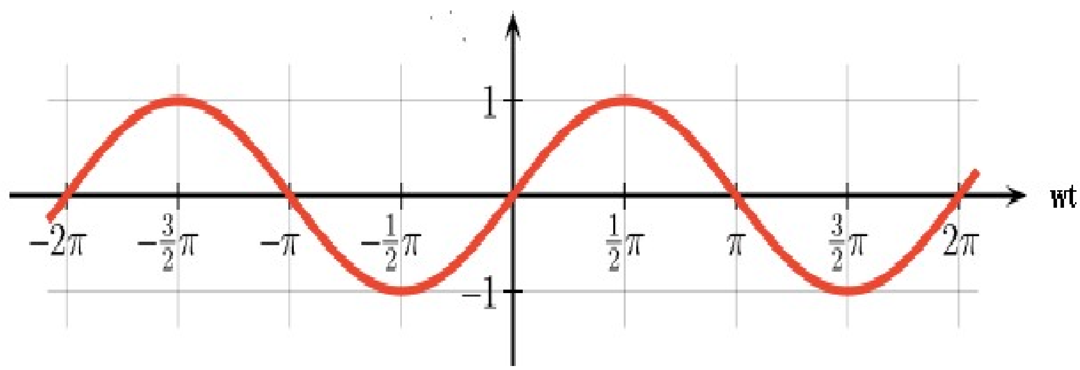
$= \{x \in R^{\#};\ x = \frac{\pi}{2}, \frac{5\pi}{2}, ------\}$

$= \{x |\ x = \frac{\pi}{2} + 2\pi n,\ \text{Where}\ n \in z\}$ Answer

3) $f^{-1}(2) = \{x \in R^{\#};\ f(x) = 2\}$

$= \{x \in R^{\#};\ \sin x = 2\}$

$= \phi$



4) $f^{-1}([-1,1])$ = The set of all real numbers.

Q-11: Let the function $f : \Re^{\#} \to \Re^{\#}$ be defined by $f(x) = x^2 + x - 2$ Find:
(1) $f^{-1}(10)$, (2) $f^{-1}(4)$, (3) $f^{-1}(-5)$.

Solution:

1) $f^{-1}(10) = \{x \in R^{\#};\ x^2 + x - 2 = 10\}$

$= \{x \in R^{\#};\ x^2 + x - 12 = 0\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm \sqrt{1+48}}{2}\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm \sqrt{49}}{2}\}$

$= \{x \in R^{\#};\ x = \frac{-1 \pm 7}{2}\}$

$= \{x \in R^{\#};\ x = -4, 3\}$

$= \{-4, 3\}$ Answer

2) $f^{-1}(4) = \{x \in R^{\#};\ \ x^2 + x - 2 = 4\}$

$= \{x \in R^{\#};\ \ x^2 + x - 6 = 0\}$

$= \{x \in R^{\#};\ \ x = \frac{-1 \pm \sqrt{1+24}}{2}\}$

$= \{x \in R^{\#};\ \ x = \frac{-1 \pm \sqrt{25}}{2}\}$

$= \{x \in R^{\#};\ \ x = \frac{-1 \pm 5}{2}\}$

$= \{x \in R^{\#};\ \ x = -3, 2\}$

$= \{-3, 2\}$ Answer

3) $f^{-1}(-5) = \{x \in R^{\#};\ \ x^2 + x - 2 = -5\}$

$= \{x \in R^{\#};\ \ x^2 + x + 3 = 0\}$

$= \{x \in R^{\#};\ \ x = \frac{-1 \pm \sqrt{1-12}}{2}\}$

$= \{x \in R^{\#};\ \ x = \frac{-1 \pm \sqrt{-11}}{2}\}$

$= \phi$

Since $\frac{-1 \pm \sqrt{-11}}{2}$ are not real numbers.

Q-12: Let $A = R^{\#} - \{-\frac{1}{2}\}$ and $B = R^{\#} - \{\frac{1}{2}\}$, Let $f : A \to B$ be defined by $f(x) = \frac{(x-3)}{(2x+1)}$, Then f is one-one and onto. Find a formula that defines f^{-1}

Solution:

1) Domain = Set of all real numbers except $x = -\frac{1}{2}$, i.e. $R^{\#} - \{-\frac{1}{2}\}$

2) Let $y = f(x) = \frac{(x-3)}{(2x+1)}$

$\Rightarrow 2xy + y = x - 3$

$\Rightarrow x = \frac{-3-y}{2y-1}$

This is undefined for $y = \frac{1}{2}$

Hence Range = $R^{\#} - \{\frac{1}{2}\}$

3) $\mathbf{f(x)=\dfrac{(x-3)}{(2x+1)}}$

When x is any real number except $\mathbf{x=-\frac{1}{2}}$ then we have $\mathbf{y=f(x)=\frac{1}{2}}$

Hence Co-domain = $\mathbf{R^{\#}-\{\frac{1}{2}\}}$

Here Range = Co-domain
So, we can say that f is onto.

Again f is one-one since for two different elements in the domain have no the same image.

4) Let y be the image of x under the function of f, Then $\mathbf{f(x)=\dfrac{(x-3)}{(2x+1)}}$

$$\mathbf{y=f(x)=\frac{(x-3)}{(2x+1)}}$$
$$\mathbf{\Rightarrow 2xy+y=x-3}$$
$$\mathbf{\Rightarrow x=\frac{-3-y}{2y-1}}$$

Again x will be the image of y under the function f-1, Then

$\mathbf{\Rightarrow x=f^{-1}(y)=\dfrac{-3-y}{2y-1}=\dfrac{3+y}{1-2y}}$

Hence $\mathbf{f^{-1}(x)=\dfrac{3+x}{1-2x}}$ Answer.

More Questions:

Ex 3A1: Let A = {1, 3, 5, 9, 11}, B = {n $\in$ N| n < 10}, C = {n $\in$ N| n is prime}, D = A $\cap$ C and E = A – C. List the elements of:
(a) B; (b) D; (c) E; (d) B – A; (e) A $\cup$ (B $\cap$ C); (f) D $\times$ E; (g) $\wp$(D).
Ex 3A2: Simplify each of the following (each answer is either S or $\varnothing$):
(a) S $\cap$ S; (b) S – S; (c) S $\cap$ $\varnothing$; (d) S $\times$ $\varnothing$; (e) S $\cup$ $\varnothing$
Ex 3A3: Which of the following statements are true for all sets R, S and T?
(a) S $\cap$ T = T $\cap$ S; (b) S – T = T – S; (c) S $\times$ T = T $\times$ S;
(d) R $\cap$ (S $\cap$ T) = (R $\cap$ S) $\cap$ T; (e) – (S $\cap$ T) = –S $\cap$ –T; (f) –(S $\cup$ T) = –S $\cap$ –T;
(g) R $\cap$ (S – T) = (R $\cap$ S) – (R $\cap$ T); (h) R $\cup$ (S – T) = (R $\cup$ S) – (R $\cup$ T)
Ex 3A4: Let A = {3, 4, 5}, B = {n$\in$N | $\exists$q$\in$N[(n = 2q – 1) $\wedge$ (q < 5)]}.
(a)Find the elements of: $\wp$(A $\cap$ B) $\times$ (A – B).
(b) How many elements are there in $\wp$(A $\times$ $\wp$(A)) $\cup$ A?
(c) Find the elements of [$\wp$(A $\cap$ A) $\times$ $\varnothing$] $\cup$ [$\wp$($\varnothing$) $\times$ (A $\cup$ $\varnothing$)]
Ex 3A5: If A = {1, 2, 3} and B = {2, 4} write down:
(a) A $\cap$ B; (b) A $\cup$ B; (c) A $\times$ B; (d) $\wp$(A).
Ex 3A6: If A = {1, 2, 3}, B = {0, 1} and C = {0, 2, 4} write down:

(a) $A \cap (B \cup C)$; (b) $B \times C$; (c) $\wp(C) - \wp(A \cup B)$.
Ex 3A7: Let A, B be the following binary languages (sets of binary strings):
$A = \{1, 10, 101\}$, $B = \{1, 11, 011\}$. Find each of the following sets:
(a) $A \cap B$; (b) $A \cup B$; (c) $A - B$; (d) $A \times B$; (e) AB; (f) $\wp(A)$.

Solutions:

Ex 3A1: (a) 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; (b) 3, 5, 11; (c) 1, 9; (d) 0, 2, 4, 6, 7, 8;
(e) 1, 2, 3, 5, 7, 9, 11; (f) (3,1), (3,9), (5,1), (5,9), (11,1), (11,9);
(g) $\emptyset$, {3}, {5}, {11}, {3,5}, {3,11}, {5,11}, {3,5,11}.
Ex 3A2: (a)(e) S; (b)(c)(d) $\emptyset$.
Ex 3A3: (a), (d), (f), (g).
Ex 3A4: (a) $A = \{3, 4, 5\}$; $B = \{1, 3, 5, 7\}$; $A \cap B = \{3, 5\}$; $A - B = \{4\}$;
$\wp(A \cap B) = \{\emptyset, \{3\}, \{5\}, \{3, 5\}\}$;
$\wp(A \cap B) \times (A - B) = \{(\emptyset, 4), (\{3\}, 4), \{\{5\}, 4), (\{3, 5\}, 4)\}$
(b) $\#A = 3$; $\#\wp(A) = 2^3$; $\#(A \times \wp(A)) = 3 \times 2^3 = 24$; $\#\wp(A \times \wp(A)) = 2^{24}$. Since A and $\wp(A \times \wp(A))$ are disjoint, $\#(\wp(A \times \wp(A)) \cup A) = 2^{24} + 3$.
(c) $\wp(A \cap A) \times \emptyset = \emptyset$ so the set can be simplified to $\wp(\emptyset) \times (A \cup \emptyset)$. As $\wp(\emptyset) = \{\emptyset\}$, the set can be simplified to $\{\emptyset\} \times A = \{(\emptyset, 3), (\emptyset, 4), (\emptyset, 5), (\emptyset, 6)\}$.

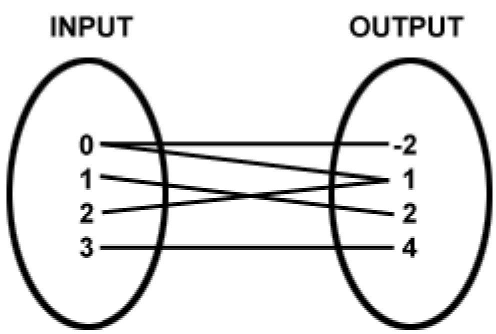
Ex 3A5: (a) $A \cap B = \{2\}$; (b) $A \cup B = \{1, 2, 3, 4\}$;
(c) $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$;
(d) $\wp(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
Ex 3A6: (a) $A \cap (B \cup C) = \{1, 2, 3\} \cap \{0, 1, 2, 4\} = \{1, 2\}$;
(b) $B \times C = \{(0, 0), (0, 2), (0, 4), (1, 0), (1, 2), (1, 4)\}$;
(c) $\wp(C) - \wp(A \cup B) = \{\{4\}, \{0, 4\}, \{2, 4\}, \{0, 2, 4\}\}$.
Ex 3A7: (i) {1}; (ii) {1, 10, 11, 011, 101}; (iii) {10, 101};
(iv) {(1, 1), (1, 11), (1, 011), (10, 1), (10, 11), (10, 011), (101, 1), (101, 11), (101, 011);
(v) {11, 111, 1011, 101, 10011, 10111, 101011};
(vi) {$\emptyset$, {1}, {01}, {101}, {1, 10}, {1, 101}, {10, 101}, {1, 10, 101}}.

Relations

Relations and Functions

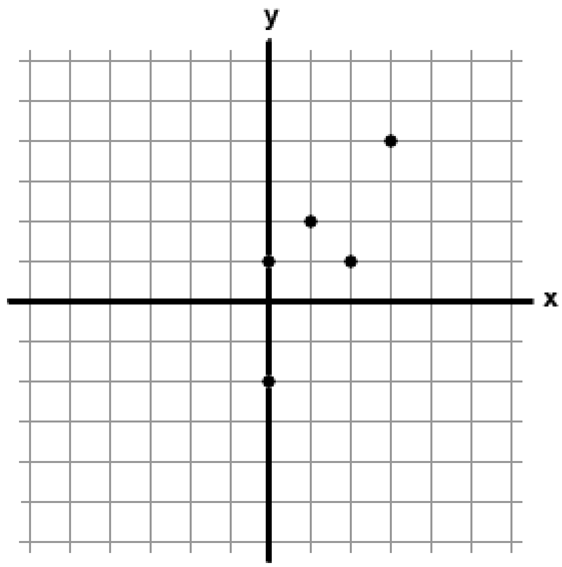
Relations

A relation is a set of inputs and outputs, often written as ordered pairs (input, output). We can also represent a relation as a mapping diagram or a graph. For example, the relation can be represented as:



Mapping Diagram of

Lines connect the inputs with their outputs. The relation can also be represented as:



Graph of Relation

In mathematics, what distinguishes a function from a relation?

Function: A function is a relation in which each input has only one output
Relation: A relation is a set of inputs and outputs, often written as ordered pairs (input, output)

Example of Relations include:

3. { (0,1) , (55,22), (3,-50) }
4. { (0, 1) , (5, 2), (-3, 9) }

Practice Problem one:
Which relations below arefunctions?
Relation #1 { (-1,2), (-4,51), (1,2), (8,-51) }
Relation #2 { ~~(13,14)~~, ~~(13,5)~~, (16,7), (18,13) }
Relation #3 { (3,90), (4,54), (6,71), (8,90) }

Relation #1 and Relation #3 are both functions.

Practice Problem Two:

Which relations below are functions?

Relation #1 { (3,4), (4,5), (6,7), (8,9) }
Relation #2 { ~~(3,4)~~, (4,5), (6,7), ~~(3,9)~~ }

Relation #4 { $\boxed{\cancel{(8, 11)}}$, (34,5), (6,17), $\boxed{\cancel{(8,19)}}$ }

Relation #1 and Relation #3 are functions because each x value, each element in the domain, has one and only only one y value, or one and only number in the range.

Definition:
A relation R consists of the following:

(1) a set A
(2) a set B
(3) an open sentence P (x , y) in which P (a, b) is either true or false for any ordered pair (a, b) belonging to $\mathbf{A} \times \mathbf{B}$

[P means ordered pair]
We then call R a relation from A to B and denote it by $\mathbf{R} = (\mathbf{A}, \mathbf{B}, \mathbf{P}(\mathbf{x}, \mathbf{y}))$

Example 01: Let $\mathbf{R} = (\mathbf{N}, \mathbf{N}, \mathbf{P}(\mathbf{x}, \mathbf{y}))$ where N is the natural numbers and $\mathbf{P}(\mathbf{x}, \mathbf{y})$ reads "x divides y". then R is a relation. Furthermore, 3 R 12, 2 $\not\mathrm{R}$ 7, 5 R 15, 6 $\not\mathrm{R}$ 13

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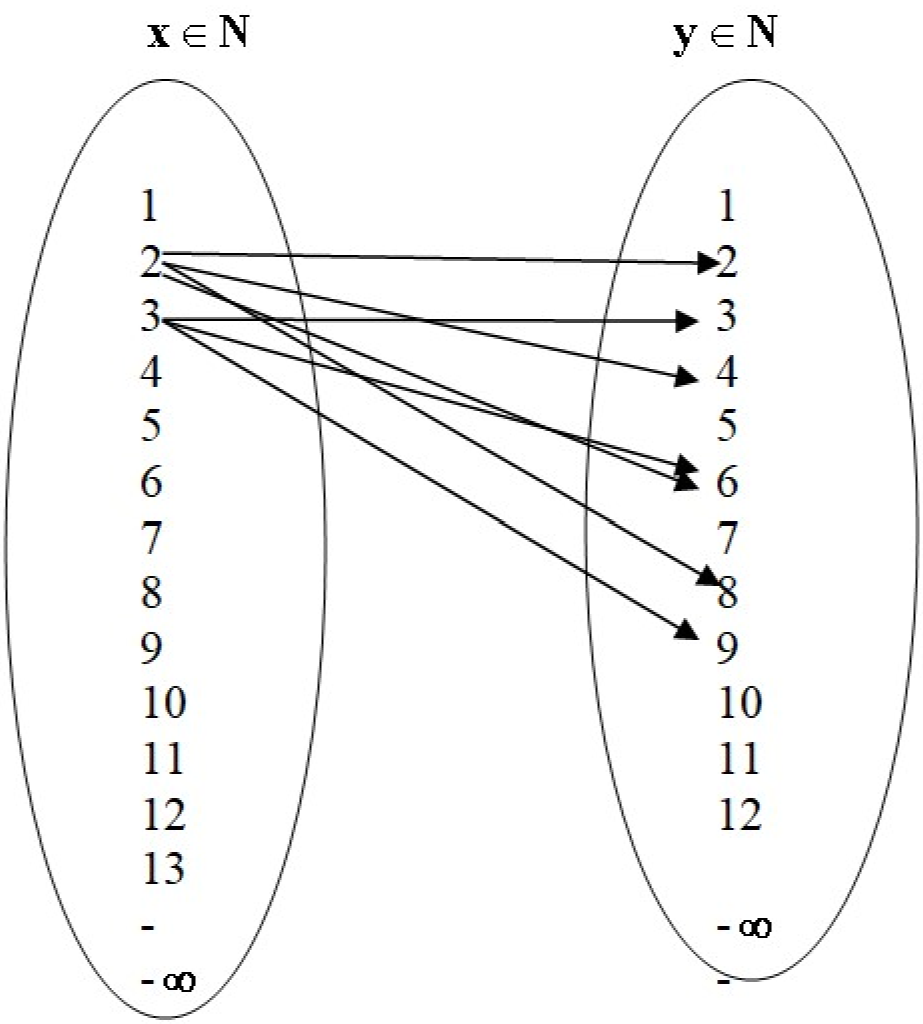


Example 02: Let $\mathbf{R} = (\mathbf{A}, \mathbf{B}, \mathbf{P}(\mathbf{x}, \mathbf{y}))$ where A is the set of man, B is the set of woman and $\mathbf{P}(\mathbf{x}, \mathbf{y})$ reads "x is the husband of y". then R is a relation.

Example 03: Let $\mathbf{R} = (\mathbf{A}, \mathbf{B}, \mathbf{P}(\mathbf{x}, \mathbf{y}))$ where A is the set of man, B is the set of woman and $\mathbf{P}(\mathbf{x}, \mathbf{y})$ reads "x divides y". then R is not a relation since P(a, b) has no meaning if a is a man and b is a woman

Example 04: Let $\mathbf{R} = (\mathbf{N}, \mathbf{N}, \mathbf{P}(\mathbf{x}, \mathbf{y}))$ where N is the natural numbers and $\mathbf{P}(\mathbf{x}, \mathbf{y})$ reads "**x** is less than y". then R is a relation.

Inverse Relations: Every relation R from A to B has an inverse relation R^{-1} from B to A which is defined by $\mathbf{R}^{-1} = \{(\mathbf{b}, \mathbf{a}) | (\mathbf{a}, \mathbf{b}) \in \mathbf{R}\}$.

In other words, the inverse relation R^{-1} consists of those ordered pairs which when reversed.

Example 1: Let $\mathbf{A} = \{1,2,3\}$ and $\mathbf{B} = \{\mathbf{a}, \mathbf{b}\}$. Then

$\mathbf{R} = \{(\mathbf{1},\mathbf{a}), \quad (\mathbf{1},\mathbf{b}) \quad (\mathbf{3},\mathbf{a})\}$ is a relation from A to B.
The inverse relation of R is $\mathbf{R}^{-1} = \{(\mathbf{a,1}), \quad (\mathbf{b,1}), \quad (\mathbf{a,3})\}$

Example 2: Let $\mathbf{W} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}.$ Then
$\mathbf{R} = \{(\mathbf{a},\mathbf{b}), \quad (\mathbf{a},\mathbf{c}), \quad (\mathbf{c},\mathbf{c}) \quad (\mathbf{c},\mathbf{b})\}$ is a relation in W. The inverse relation of R is
$\mathbf{R}^{-1} = \{(\mathbf{b},\mathbf{a}), \quad (\mathbf{c},\mathbf{a}), \quad (\mathbf{c},\mathbf{c}) \quad (\mathbf{b},\mathbf{c})\}$

1. **Reflexive:** Each element is related to itself.

2. **Symmetric:** If any one element is related to any other element, then the second element is related to the first.

3. **Transitive:** If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

Reflexive Relations:

Let $\mathbf{R} = (\mathbf{A},\mathbf{A},\mathbf{P}(\mathbf{x},\mathbf{y}))$ be a relation in a set A, i.e. Let R be a subset of $\mathbf{A} \times \mathbf{A}$. Then R is called a reflexive relation if, for every $\mathbf{a} \in \mathbf{A}, \quad (\mathbf{a},\mathbf{a}) \in \mathbf{R}$

In other words, R is reflexive if every element in A is related to itself.

Example 1: Let $\mathbf{V} = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}$ and $\mathbf{R} = \{(\mathbf{1,1}), \quad (\mathbf{2,4}), \quad (\mathbf{3,3}), \quad (\mathbf{4,1}), \quad (\mathbf{4,4})\}$
Then R is not a reflexive relation since (2, 2) does not belong to R. Notice that all ordered pairs (a, a) must belong to R in order for R to be reflexive.

Example 2: Let R be the relation in the real numbers defined by the open sentence "x is less than y", i.e. "x<y". Then R is not reflexive since $a < a$ for any real number a.

Example 3: Let W = {1, 2, 3, 4} and R = {(1, 1), (1, 3), (2, 2), (3, 1), (4, 4)}. Is R reflexive?
Solution: R is not reflexive since $\mathbf{3} \in \mathbf{W}$ and $(\mathbf{3,3}) \notin \mathbf{R}.$

Example 4: Let. E = {1, 2, 3}. Consider the following relations in E:
R_1 = {(1, 2), (3, 2), (2, 2), (2, 3)}
R_2 = {(1, 2), (2, 3), (1, 3)
R_3 = {(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)}
R_4 = {(1, 2)}
R_5 = E × E
State whether or not each of these relations is reflexive.
Solution: If a relation in E is reflexive, then (1, 1), (2, 2) and (3, 3) must belong to the relation. Therefore only R_3 and R_5 are reflexive.

Symmetric Relations:

Let R be a subset of $\mathbf{A} \times \mathbf{A}$, i.e. let R be a relation in A. Then R is called a symmetric relation if $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}$ implies $(\mathbf{b}, \mathbf{a}) \in \mathbf{R}$

That is, if a is related to b then b is also related to a.

Example 1: When is a relation R in a set A not symmetric?

Solution: R is not symmetric if there are elements $\mathbf{a} \in \mathbf{A}, \quad \mathbf{b} \in \mathbf{A}$ such that $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}, \quad (\mathbf{b}, \mathbf{a}) \notin \mathbf{R}$

Note $a \neq b$, otherwise $(a, b) \in R$ implies $(\mathbf{b}, \mathbf{a}) \in \mathbf{R}.$

Example 2: Let V = {1, 2, 3, 4} and R = {(1,2), (3,4), (2,1), (3,3)}. Is R symmetric?

Solution: R is not symmetric, since $\mathbf{3} \in \mathbf{V}, \quad \mathbf{4} \in \mathbf{V}, \quad (\mathbf{3,4}) \in \mathbf{R}$ and $(\mathbf{4,3}) \notin \mathbf{R}.$

Example 3: Let E = {1, 2, 3}. Consider the following relations in E:

R_1 = {(1,1), (2,1), (2,2), (3,2), (2,3)} R_4 = {(1,1), (3,2), (2,3)}

R_2 = {(1, 1)} R_5 = E × E

R_3 = {(1, 2)}

State whether or not each of these relations is symmetric.

Solution:

(1) R_1 is not symmetric since (2, 1) $\in R_1$ but (1, 2) $\notin R_1$.

(4) R_4 is symmetric.

(2) R_2 is symmetric.

(5) R_5 is symmetric.

(3) R_3 is not symmetric since (1, 2) $\in \mathbf{R}_3$ but (2, 1) $\notin \mathbf{R}_3$.

Anti-Symmetric Relations:

A relation R in a set A, i.e. a subset of $\mathbf{A} \times \mathbf{A}$, is called an anti-symmetric relation

i. if $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}$ and $(\mathbf{b}, \mathbf{a}) \in \mathbf{R}$ implies a = b.

ii. In other words, if $\mathbf{a} \neq \mathbf{b}$ then possibly a is related to b or possibly b is related to a, but never both.

Example 1: Let N be the natural numbers and let R be the relation in N defined by "x divides. y". Then R is anti-symmetric since a divides b and b divides a implies a = b

Example 2: Let $\mathbf{W} = \{\mathbf{1,2,3,4}\}$, let $\mathbf{R} = \{(\mathbf{1,3}), \quad (\mathbf{4,2}), \quad (\mathbf{4,4}), \quad (\mathbf{2,4})\}$

Then R is not an anti-symmetric relation in W since $(\mathbf{4,2}) \in \mathbf{R}$ and $(\mathbf{2,4}) \in \mathbf{R}$

Example 3: Where is a relation R in a set A not anti-symmetric?

Solution:

R is not anti-symmetric if there exists elements $\mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{A}, \mathbf{a} \neq \mathbf{b}$ such that $(\mathbf{a}, \mathbf{b}) \in \mathbf{R}$ and $(\mathbf{b}, \mathbf{a}) \in \mathbf{R}.$

Example 4: Let $\mathbf{W} = \{1,2,3,4\}$ and $\mathbf{R} = \{(1,2),\ \ (3,4),\ \ (2,2),\ \ (3,3),\ \ (2,1)\}$. Is R anti-symmetric?
Solution: R is not anti-symmetric, science $\mathbf{1 \in W, 2 \in W, 1 \neq 2, (1,2) \in R}$ and $\mathbf{(2,1) \in R}$.

Example 5: Let E = {1, 2, 3}. Consider the following relations in E:

R_1 = {(1,1), (2,1), (2,2), (3,2), (2,3)} R_4 = {(1,1), (2,3), (3,2)}
R_2 = {(1, 1)} R_5 = E × E
R_3 = {(1, 2)}

State whether or not each of these relations is anti-symmetric.
Solution:
(1) R_1 is not anti-symmetric since (3, 2) $\in \mathbf{R_1}$ and (2, 3) $\in \mathbf{R_1}$.
(2) R_2 is anti-symmetric.
(3) R_3 is anti-symmetric.
(4) R_4 is not symmetric since (2, 3) $\in \mathbf{R_4}$ and (3, 2) $\in \mathbf{R_4}$.
(5) R_5 is not anti-symmetric for the same reasons as for R_4.

- reflexive e shob element er repetation thakte hobe
- symmetric e (a,b) thale (b,a) tahaka lagbe
- anti te repetetion, (a,b) allowed but (a,b) er shonge (b,a) allowed na
- transitive (a,b),(b,c) thakle jodi (a,c) pawa jai

TRANSITIVE RELATIONS

A relation R in a set A is called a transitive relation if $\mathbf{(a,b) \in R}$ and $\mathbf{(b,c) \in R}$ implies $\mathbf{(a,c) \in R}$.

In other words, if a is related to b and b is related to c, then a is related to c.

Example 1: When is a relation in a set A not transitive?
Solution: R in not transitive if there exist elements a, b and c belonging to A, not necessarily distinct, such that $\mathbf{(a,b) \in R,\ \ (b,c) \in R}$ but $\mathbf{(a,c) \notin R}$

Example 2: Let W = {1, 2, 3, 4} and R = {(1,2), (4,3), (2,2), (2,1), (3,1)}. Is R transitive?
Solution: R is not transitive since $\mathbf{(4,3) \in R,\ \ (3,1) \in R}$ but $\mathbf{(4,1) \notin R.}$

Example 3: Let W = {1, 2, 3, 4} and R = {(2,2), (2,3), (1,4), (3,2)}. Is R transitive?
Solution: R is not transitive since $\mathbf{(3,2) \in R,\ \ (2,3) \in R}$ but $\mathbf{(3,3) \notin R.}$

Example 4: Let E = $\mathbf{\{1,2,3\}}$.Consider the following relations in E:
R_1 = {(1, 2), (2, 2)}
R_4 = {(1, 1)}
R_2 = {(1,2), (2,3), (1,3), (2,1), (1,1)}
R_5 = E × E
R_3 = {(1, 2)}
State whether or not each of these relations is transitive.
Solution:

Each of the relations is transitive except R_2.
R_2 is not transitive, since

$$(2,1) \in R_2, (1,2) \in R_2 \quad \text{But } (2,2) \notin R_2$$

Example: Let. E = {a, b, c}. Consider the following relations in E:
R = {(a, a), (b, a), (b, b), (c, b), (b,c)}

State whether or not each of these relations is reflexive, Symmetric, Anti-Symmetric, and Transitive.

Answer:
1. R is not reflexive since $(c,c) \notin R$
2. R is not symmetric since $(b,a) \in R$ but $(a,b) \notin R$
3. R is not anti-symmetric since $(c,b) \in R$, $(b,c) \in R$ but $b \neq c$
4. R is not transitive since $(c,b) \in R$, $(b,c) \in R$ but $(c,c) \notin R$

Example: Let. E = {1, 2, 3}. Consider the following relations in E:
R = {(1, 1), (1, 2)}

State whether or not each of these relations is reflexive, Symmetric, Anti-Symmetric, and Transitive.

Answer:
5. R is not reflexive since $(2,2) \notin R$, $(3,3) \notin R$
6. R is not symmetric since $(1,2) \in R$ but $(2,1) \notin R$
7. R is anti-symmetric since $1 \neq 2$, $(1,2) \in R$,
8. R is transitive since $(1,1) \in R$, $(1,2) \in R$ implies $(1,2) \in R$

Example: Let. E = {1, 2, 3, 4}. Consider the following relations in E:
R = {(1, 1), (2, 3), (4, 1)}

State whether or not each of these relations is reflexive, Symmetric, Anti-Symmetric, and Transitive.

Answer:
9. R is not reflexive since $(2,2) \notin R$, $(3,3) \notin R$, $(4,4) \notin R$
10. R is not symmetric since $(2,3) \in R$ but $(3,2) \notin R$, $(4,1) \in R$ but $(1,4) \notin R$
11. R is anti-symmetric since $2 \neq 3$ $(2,3) \in R$, $4 \neq 1$ $(4,1) \in R$
12. R is transitive since $(1,1) \in R$, $(4,1) \in R$, $(2,3) \in R$

Equivalence Relations:

A relation R in a set A is an equivalence relation if

i) R is reflexive, that is, for every $a \in R$, $(a,a) \in R$,

ii) R is symmetric, that is, $(a,b) \in R$ implies $(b,a) \in R$,

iii) R is transitive, that is, $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$

স্মরনীয়:

iv) ধরি, R = {(1, 2)} এক্ষেত্রে R is transitive কারণ $(a,b) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$ অর্থাৎ $(a,b) \in R$ and $(b,c) \in R$ থাকলেই দেখব $(a,c) \in R$ আছে কিনা, কিন্তু $(b,c) \in R$ ই যদি না থাকে তাহলে $(a,c) \in R$ আছে কিনা দেখার প্রশ্নই আসেনা। অর্থাৎ R = {(a, b)} is a transitive relation.

v) Anti-Symmetric এ একটি pair নিয়ে চিন্ড়া করব যদি R ={(a,a)}একটি pair হয় তখন Anti-Symmetric হবে, যদি R={(a, b)} হয় তাহলেও Anti-Symmetric হবে কিন্তু Relation এ যদি R={$(a,b),(b,a)$} থাকে যেহেতু $a \neq b$ এ ধরণের R is not anti-symmetric. উলে খ্য anti-symmetric relation এর দুটি সজ্ঞা রয়েছে যে কোন satisfy একটি করলেই anti-symmetric relation বলব।

Determine whether the relation is reflexive, symmetric, antisymmetric, transitive, and/or a partial order.

$(x,y)\in R$ if $x\geq y$ when defined on the set of positive integers.

I'll check two of the properties to give you the idea. I'm assuming that R is a relation on the set of real numbers.

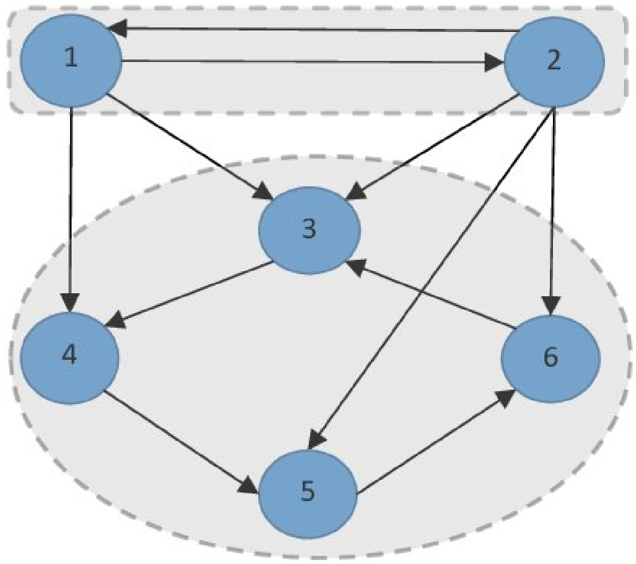
- **Reflexivity:** R is reflexive if $\langle x,x\rangle\in R$ for every real number x. By the definition of R, $\langle x,x\rangle\in R$ if and only if $x\geq x$; is this true for every real number x? Definitely, so R is reflexive.
- **Symmetry:** R is symmetric if it has the following property: for any real numbers x and y, if $\langle x,y\rangle\in R$, then $\langle y,x\rangle\in R$. For this specific relation that property says: for any real numbers x and y, if $x\geq y$, then $y\geq x$. Is that true? Of course not: take $x=2$ and $y=1$, and we certainly have $x\geq y$, since $2\geq 1$, but it's clearly not true that $y\geq x$, because $1\ngeq 2$. Thus, R is **not** symmetric.

I'll leave transitivity to you, just reminding you of the definitions.

- **Anti symmetry:** For any real numbers x and y, if $\langle x,y\rangle\in R$ **and** $\langle y,x\rangle\in R$, then $x=y$. Is this true for this relation? Just translate $\langle x,y\rangle\in R$ and $\langle y,x\rangle\in R$ into more familiar terms, and it should be very clear.
- **Transitivity:** For any real numbers x,y, and z, if $\langle x,y\rangle\in R$ and $\langle y,z\rangle\in R$, then $\langle x,z\rangle\in R$. Again, if you translate the hypothesis into more familiar terms, you should have no trouble deciding whether the statement is true of this relation or

not.

Determine whether the relation is reflexive, symmetric, antisymmetric, transitive



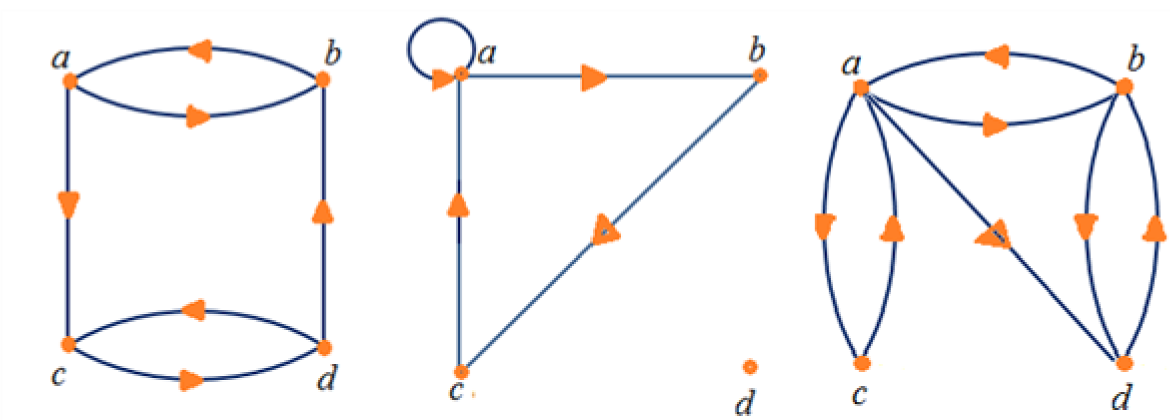
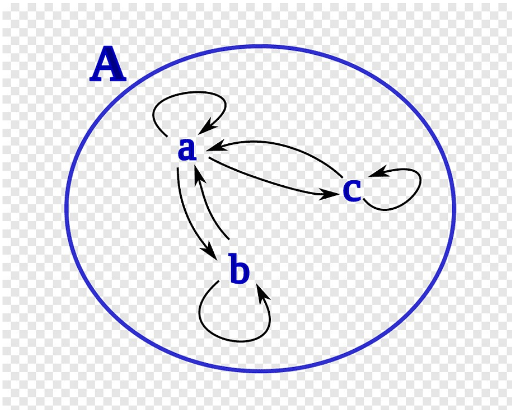


Fig 1 Fig 2 Fig 3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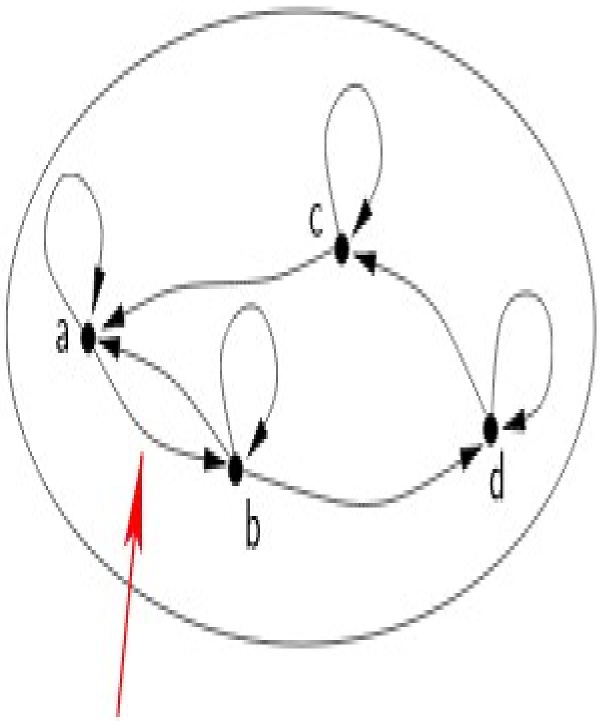
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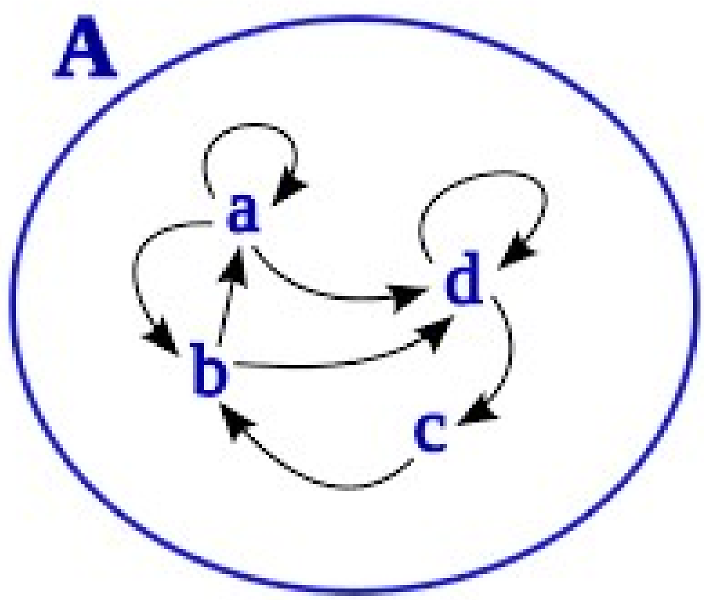
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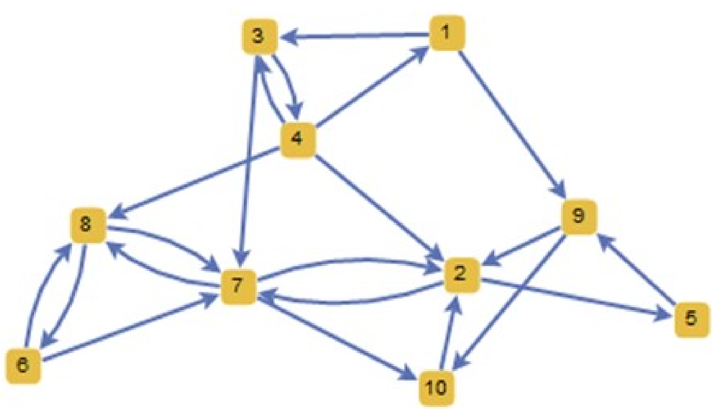
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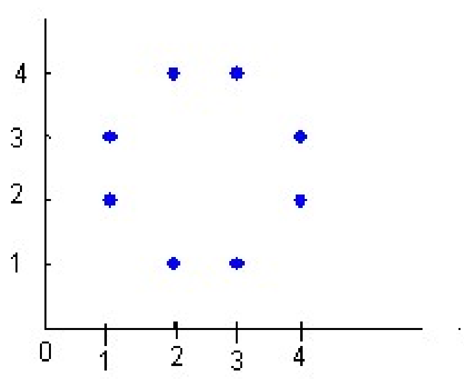
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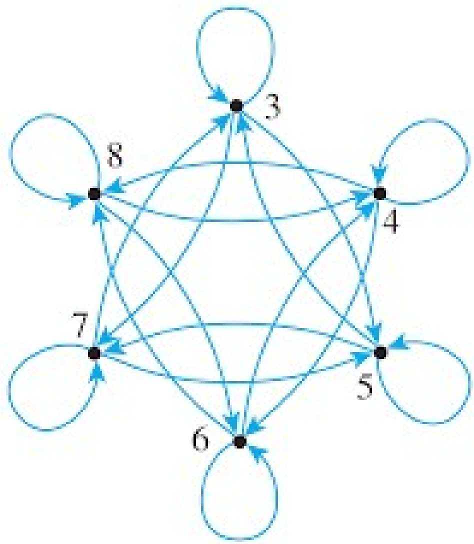
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