3D Rectangular Co-ordinates (আয়তাকার স্থানাঙ্ক)

1. If the three numbers x, y, z are called the co-ordinates of any point p then the point represented by $p(x, y, z)$.

2. Distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $PQ = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}$

3. Distance between origin $O(0,0,0)$ and $P(x_1, y_1, z_1)$ is

   $OP = \sqrt{x_1^2+y_1^2+z_1^2}$

4. Co-ordinates of the point which divides the straight line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

   Internal section ratio $= \left(\dfrac{m_1x_2+m_2x_1}{m_1+m_2}, \dfrac{m_1y_2+m_2y_1}{m_1+m_2}, \dfrac{m_1z_2+m_2z_1}{m_1+m_2}\right)$

   External section ratio $= \left(\dfrac{m_1x_2-m_2x_1}{m_1-m_2}, \dfrac{m_1y_2-m_2y_1}{m_1-m_2}, \dfrac{m_1z_2-m_2z_1}{m_1-m_2}\right)$

5. Centre of gravity of a triangle,

   $$= \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

6. Direction cosine are denoted by l, m, n

   where $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ also $l^2+m^2+n^2=1$

7. Direction ratio are denoted by a, b, c,

8. Reletion between direction cosine and direction ratio.

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

9. The direction cosine of the line joining the two Pointers (x_1, y_1, z_1) and (x_2, y_2, z_2) are projectional to $x_2-x_1,\ y_2-y_1,\ z_2-z_1$

10. Angle between two lines.

According to direction cosine, $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$

According to direction ratio, $\cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$

11. Condition for perpendicularity of two lines.

According to direction cosine, $l_1l_2+m_1m_2+n_1n_2=0$

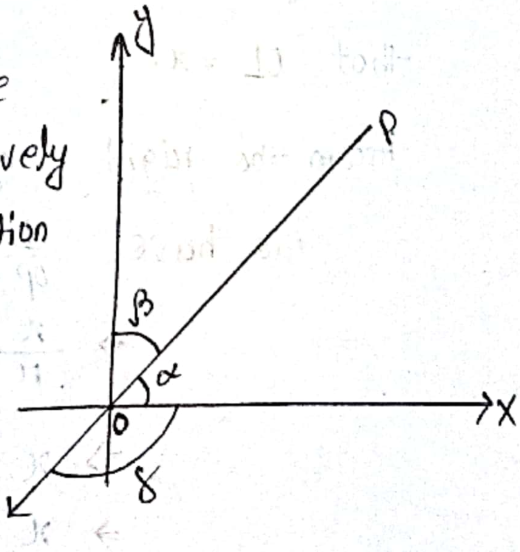
According to direction ratio, $a_1a_2+b_1b_2+c_1c_2=0$

12. Condition for parallelism of two lines.

According to direction cosine, $l_1=l_2$; $m_1=m_2$; $n_1=n$

According to direction ratio, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

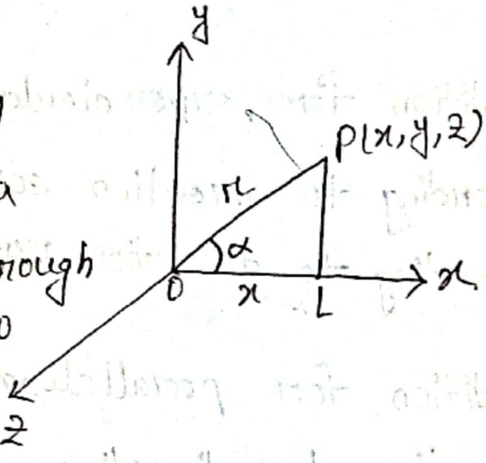
<u>Direction cosine:</u> If a given line OP makes angles α, β, γ with the positive direction of axes of x, y, z respectively then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the line OP. They are generally denoted by the letters l, m, n.



Direction ratios: If any three numbers a, b, c which are proportional to the direction cosines l, m, n respectively of a given line are called the direction ratios of the given lines.

Ⓠ If a line makes angle α, β, γ with the axes Show that $l^2+m^2+n^2=1$ or $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ or $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.

Solution:



If $O(0,0,0)$ be the origin and (x,y,z) the co-ordinates of a point P then OP be drawn through the origin to the givien line. so that l, m, n are the direction cosine of the line OP and r be the length of OP

Through P draw PL perpendicular to x axes so that $OL = x$.

From the right angle triangle OPL,

we have, $\frac{OL}{OP} = \cos \angle LOP$

$\Rightarrow \frac{x}{r} = \cos\alpha$

$\Rightarrow x = r\cos\alpha$

$\Rightarrow x = lr$ —①

x অক্ষের $\cos\alpha = l$

y অক্ষের $\cos\beta = m$

z অক্ষের $\cos\gamma = n$

Similarly, $y = mr$ —②

$z = nr$ —③

Now squaring and adding eq. ①, ② and ③ we get

$$x^2+y^2+z^2 = l^2r^2 + m^2r^2 + n^2r^2$$

$$\Rightarrow x^2+y^2+z^2 = r^2(l^2+m^2+n^2)$$

$$\Rightarrow r^2 = r^2(l^2+m^2+n^2)$$

$$\therefore l^2+m^2+n^2 = 1$$

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$$r = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow r^2 = x^2+y^2+z^2$$

or, $(\cos\alpha)^2 + (\cos\beta)^2 + (\cos\gamma)^2 = 1$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ $[l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma]$

or,

$$(1-\sin^2\alpha) + (1-\sin^2\beta) + (1-\sin^2\gamma) = 1$$

$$\Rightarrow 1-\sin^2\alpha + 1-\sin^2\beta + 1-\sin^2\gamma = 1$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 1+1+1-1$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$

$$\therefore l^2+m^2+n^2 = 1$$

or, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

[showed]

② If a line makes an angle (θ, θ, θ) with the axes, then show that, $\sin\theta = \pm\sqrt{2/3}$

Solution: If a line makes an angle α, β, γ with then we know, that,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Here, according to the Question, $\alpha = \beta = \gamma = \theta$

$$\cos^2\theta + \cos^2\theta + \cos^2\theta = 1$$

$$\Rightarrow 3\cos^2\theta = 1$$

$$\Rightarrow \cos^2\theta = \frac{1}{3}$$

$$\Rightarrow 1 - \sin^2\theta = \frac{1}{3}$$

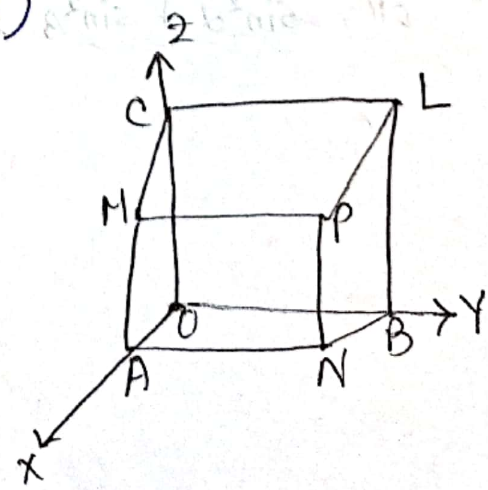
$$\Rightarrow \sin^2\theta = 1 - \frac{1}{3}$$

$$\Rightarrow \sin^2\theta = \frac{2}{3}$$

$$\Rightarrow \sin\theta = \pm\sqrt{2/3}$$

[showed]

③ If the edges of a rectangular parallelopiped are a, b, c show that angles between the four diagonals are given by $\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$



Solution: Given that the length of the edges of the Parallelopiped are a, b, c so that the vertix are,

are $O(0,0,0)$, $P(a,b,c)$, $L(0,b,c)$, $M(a,0,c)$, $N(a,b,0)$, $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$.

The diagonals are OP, AL, BM and CN having direction ratio are

$$(a-0),(b-0),(c-0) = a,b,c$$

$[a,b,c]$; $[-a,b,c]$; $[a,-b,c]$; $[a,b,-c]$ [() use করা যাবে না]

Let $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$ ------ are the angle between the diagonals.

$$\cos\theta_1 = \frac{-a^2+b^2+c^2}{\sqrt{\Sigma a^2 \cdot \Sigma a^2}} = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

$$\cos\theta_2 = \frac{a^2-b^2+c^2}{a^2+b^2+c^2}$$

$$\cos\theta_3 = \frac{a^2+b^2-c^2}{a^2+b^2+c^2}$$

$$\cos\theta_4 = \frac{-a^2-b^2+c^2}{a^2+b^2+c^2}$$

$$\cos\theta_5 = \frac{-a^2+b^2-c^2}{a^2+b^2+c^2}$$

$$\cos\theta_6 = \frac{a^2-b^2-c^2}{a^2+b^2+c^2}$$

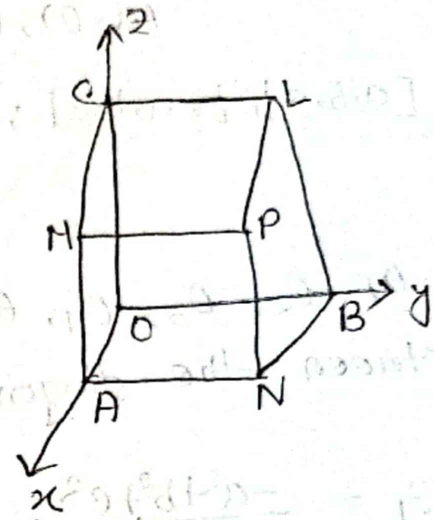
$$\therefore \cos\theta = \frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

[showed]

(4) Find the angle between the two diagonals of a cube.

or, Prove that the angle between two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.



__Solution:__ Let a be the length of the cube. So that the co-ordinates of the vertix are $O(0,0,0)$; $P(a,a,a)$; $L(0,a,a)$; $M(a,0,a)$; $N(a,a,0)$; $A(a,0,0)$; $B(0,a,0)$; $C(0,0,a)$

The diagonals are OP, AL, BM and CN.

Now the direction of OP and AL are

$$[a,a,a];\ [-a,a,a]$$

i.e, $1,1,1$ and $-1,1,1$ [a দ্বারা ভাগ করে]

Let θ be the angle between the diagonals OP and AL

$$\therefore \cos\theta = \frac{1(-1)+1.1+1.1}{\sqrt{(1^2+1^2+1^2).(1^2+1^2+1^2)}}$$

$$= \frac{1}{3}$$

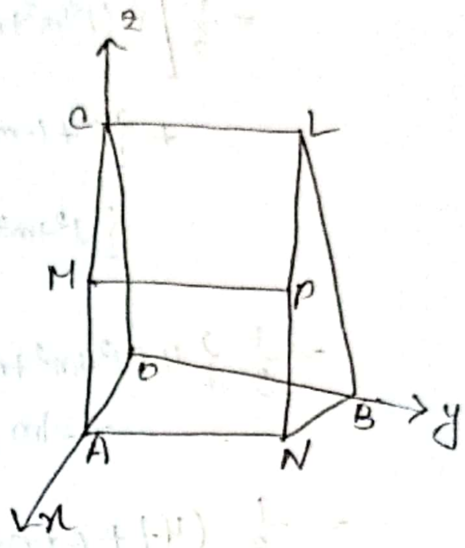
$$\therefore \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

[Proved]

⑤ A line makes angles α, β, γ with the four diagonals of a cube. prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 4/3$.

Solution:

Let a be the length of the cube.
So that it's co-ordinates of the vertix are,



$O(0,0,0)$; $P(a,a,a)$; $L(0,a,a)$;
$M(a,0,a)$; $N(a,a,0)$; $A(a,0,0)$;
$B(0,a,0)$; $C(0,0,a)$

Here the four diagonals of the cube are OP, AL, BM and CN whose directions are

$[a,a,a]$; $[-a,a,a]$; $[a,-a,a]$; $[a,a,-a]$;

i.e, Directions of the four diagonals are $[1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]$;

$[-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}]$; $[1/\sqrt{3}, 1/\sqrt{3}, -\frac{1}{\sqrt{3}}]$

Let the directions of the given line are l, m, n

$$\cos\alpha = \frac{1}{\sqrt{3}}(l+m+n)$$

$$\cos\beta = \frac{1}{\sqrt{3}}(-l+m+n)$$

$$\cos\gamma = \frac{1}{\sqrt{3}}(l-m+n)$$

$$\cos\delta = \frac{1}{\sqrt{3}}(l+m-n)$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{1}{3}\left\{(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2\right\}$$

$$= \frac{1}{3}\Big[\left\{(l^2+m^2+n^2+2lm+2mn+2nl)\right\} + \left\{(-l)^2+m^2+n^2-2lm+2mn-2nl\right\}$$
$$+\left\{l^2+(-m)^2+n^2 - 2lm - 2mn + 2nl\right\} +$$
$$\left\{l^2+m^2+(-n)^2+2lm-2mn-2nl\right\}\Big]$$

$$= \frac{1}{3}\left\{4(l^2+m^2+n^2) + 2mn(1+1-1+1) + 2nl(1-1+1-1) + 2lm(1-1-1+1)\right\}$$

$$= \frac{1}{3}(4\cdot 1 + 0 + 0 + 0)$$

$$= \frac{4}{3}$$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\delta + \cos^2\delta = 4/3$

[Proved]