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Complex Variable

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Complex Mapping

Complex Number:

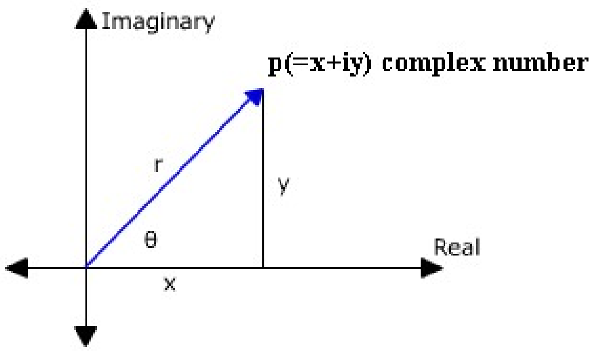

Figure 01

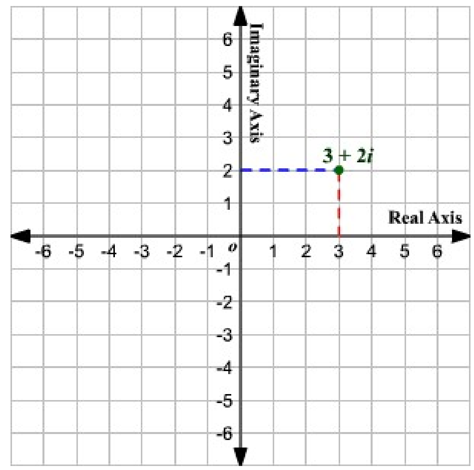

Figure 02

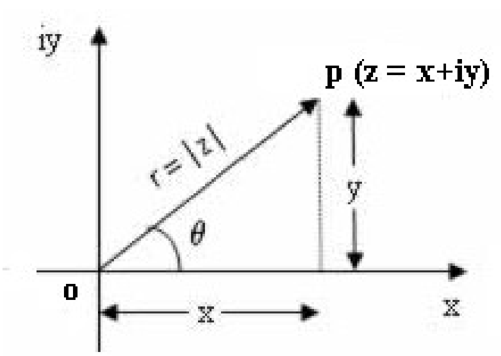

Figure 03

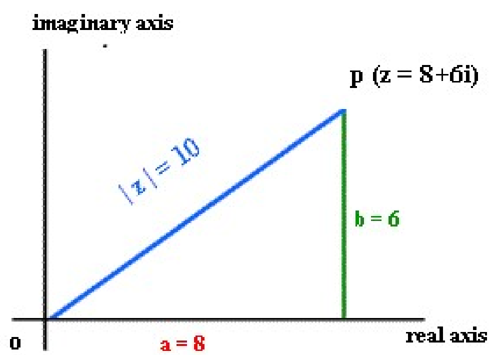

Figure 04

$$OP = |z| = \sqrt{x^2 + y^2}$$

$$OP = |z| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$$

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Transformation

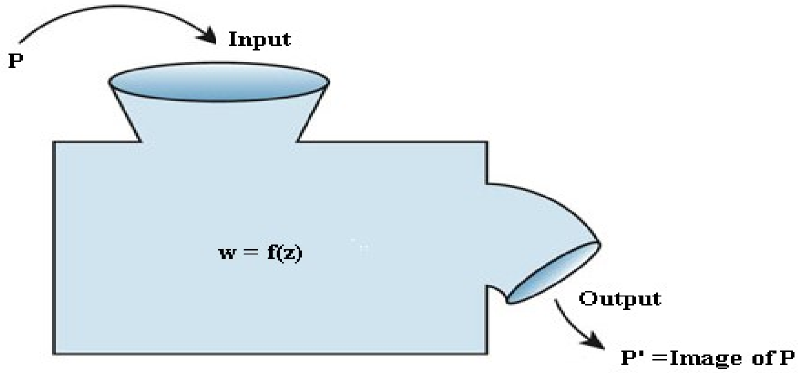

Figure 05

Where the point P in the z-plane and the point P' in the w-plane.

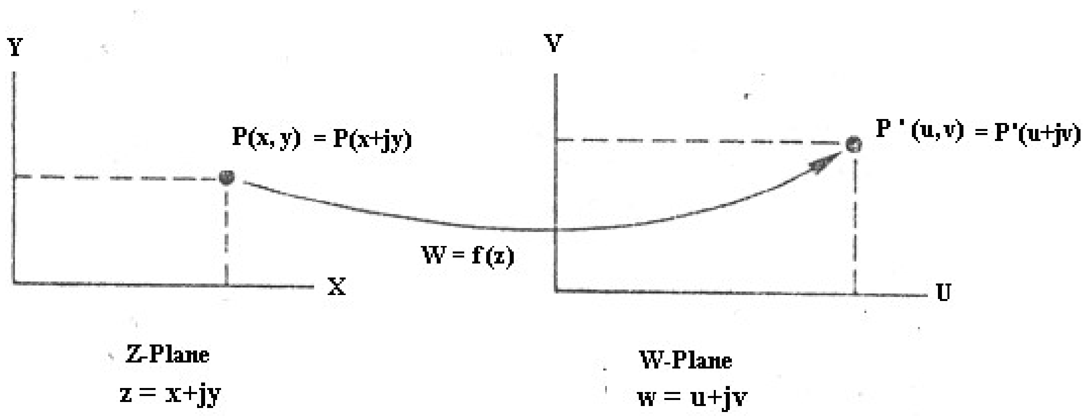

Figure 06

Definition: The transformation of P in the z-plane onto P' in the w-plane is said to be a mapping of P onto $\mathbf{P'}$ under the transformation $\mathbf{w = u + jv = f(z)}$ and $\mathbf{P'}$ is sometimes referred to as the image of P.

Example 01

Determine the image of the point P, $\mathbf{z = 3 + j2,}$ on the w-plane under the transformation $\mathbf{w = 3z + 2 - j}$

Answer:

We have,

$$\mathbf{w = 3z + 2 - j}$$

$$w = f(z) = 3z + 2 - j$$

$$u + jv = f(z) = 3(x + jy) + 2 - j \qquad [\mathbf{z = x + jy}] \quad [w = u + jv]$$

$$u + jv = f(z) = 3x + 3jy + 2 - j$$

$$u + jv = f(z) = 3x + 2 + 3jy - j$$
$$u + jv = f(z) = 3x + 2 + j(3y - 1) \text{ -------------(i)}$$
Equating real and imaginary part, we get
$$\mathbf{u = 3x + 2} \quad \text{-----------------------------(ii)}$$
$$\mathbf{v = 3y - 1} \quad \text{-----------------------------(iii)}$$

Given, the point P, $\mathbf{z = 3 + j2,}$
That is P (3, 2) -----------------------------(iv)
Here, $\mathbf{x = 3, \ y = 2}$
Putting the values of x and y in (ii) & (iii), Then the point P ($\mathbf{z = 3 + j2)}$ transforms onto w-plane is

$\mathbf{u = 3x + 2}$ and $\mathbf{v = 3y - 1}$
$\mathbf{u = 3.3 + 2}$ $\quad$ $\mathbf{v = 3.2 - 1}$
$\mathbf{u = 11}$ $\quad$ $\mathbf{v = 5}$
$\therefore \mathbf{w = u + jv = 11 + j5}$

The image of P is $\mathbf{P'(= u + jv = 11 + j5)}$
That is $\boldsymbol{P'(11,5)}$ -----------------------------(v)

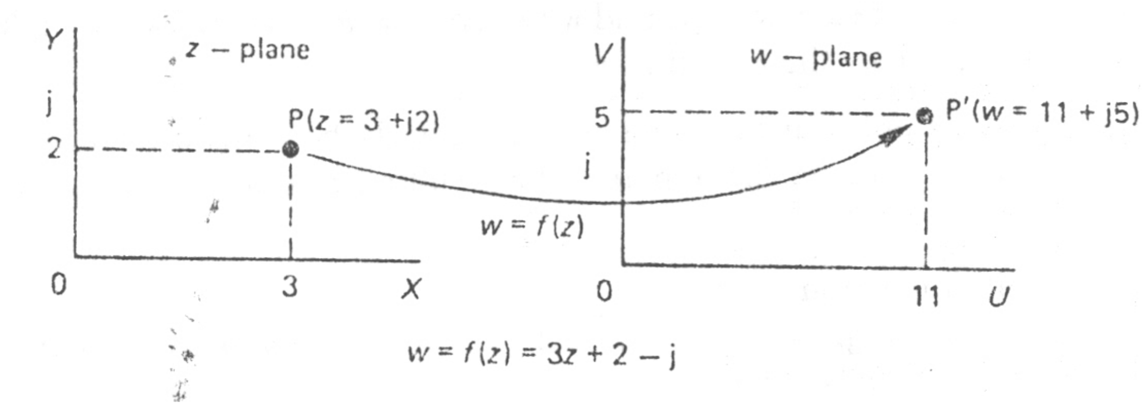

Figure 07

Example 02
Map the points $\mathbf{A(z = -2 + j)}$ and $\mathbf{B(z = 3 + j4)}$ on to the w-plane under the transformation $\mathbf{w = j2z + 3}$ and illustrate the transformation on a diagram.

Solution:
We have,
$$w = f(z) = j2z + 3$$
$$u + jv = f(z) = j2z + 3 \qquad [w = u + jv]$$
$$u + jv = \mathbf{j2(x + jy) + 3} \qquad [z = x + jy]$$
$$u + jv = \mathbf{j2x + 2j^2y + 3}$$
$$u + jv = \mathbf{j2x - 2y + 3[\because j^2 = -1]}$$
$$u + jv = \mathbf{(3 - 2y) + j2x} \text{ ------------------(i)}$$
Equating the coefficient of real and imaginary part, we get

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$\therefore$ ***u = 3 - 2y - - - - - - - - - - - - - - - - (ii)***

$\therefore$ ***v = 2x - - - - - - - - - - - - - - - - - - (iii)***

Given, $\mathbf{A(z = -2 + j.1)}$

That is, A (-2, 1) -----------------------(iv)

Here,

x = -2, y = 1

Putting the value of x and y in (ii) and (iii),

u = 3 - 2y	**v = 2x**
u = 3 - 2.1	**v = 2.(-2)**
u = 1	**v = -4**

$\therefore \mathbf{w = u + jv = 1 - j4}$

The image of A is $\mathbf{A'(w = 1 - j.4)}$

That is $\mathbf{A'(1,-4)}$ -----------------------(v)

Again,

$\mathbf{B(z = 3 + j4)}$

That is, B (3, 4) -----------------------(vi)

Here, ***x = 3, y = 4***

Putting the value of x and y in (ii) and (iii),

u = 3 - 2y	**v = 2x**
u = 3 - 2.4	**v = 2.3**
u = -5	**v = 6**

$\therefore \mathbf{w = u + jv = -5 + j.6}$

The image of B is $B'(w = u + jv = -5 + j.6)$

That is $B'(-5,6)$ - - - - - - - - - - - - - - - - ***(vii)***

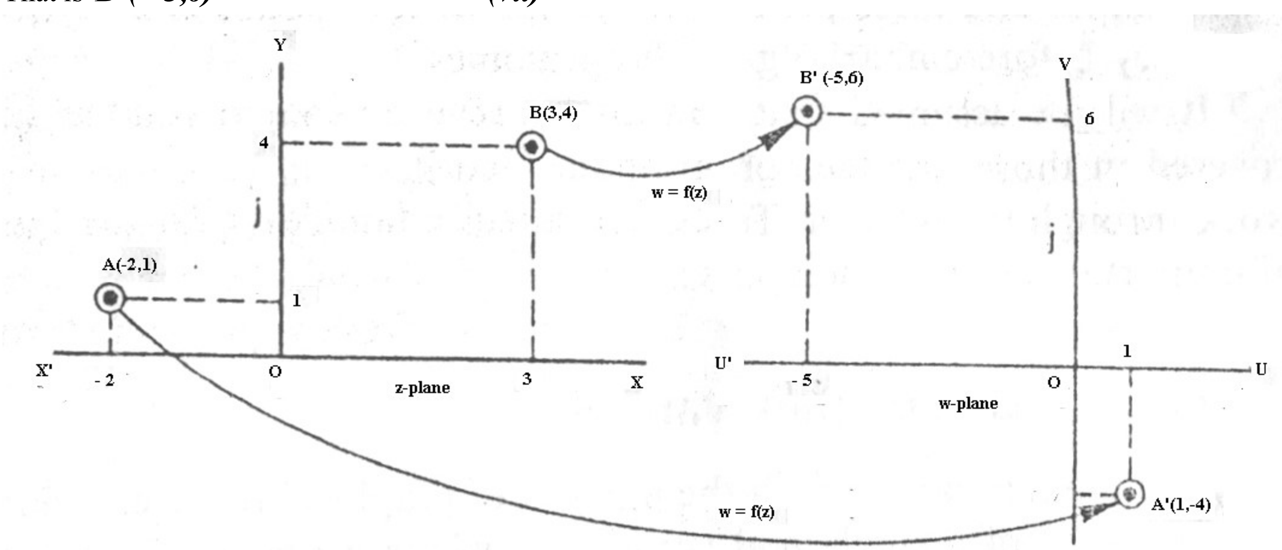

Figure 08

Example 03

Map the straight line joining $\mathbf{A(-2+j)}$ and $\mathbf{B(3+j6)}$ in the z-plane on to the w-plane when $\mathbf{w = u + jv = f(z) = 3 + j2z}$

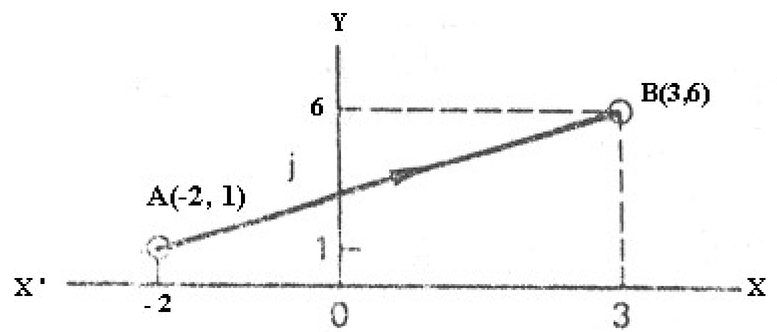

Figure 09

Answer:

We have,

$$w = f(z) = 3 + j2z$$

$$u + jv = f(z) = 3 + j2(x + jy) \qquad [w = u + jv;\ z = x + jy]$$

$$u + jv = f(z) = 3 + j2x + j^2 2y$$

$$u + jv = f(z) = 3 + j2x - 2y [\because j^2 = -1]$$

$$u + jv = f(z) = 3 + j2x - 2y$$

$$u + jv = f(z) = 3 - 2y + j2x$$

$$u + jv = f(z) = (3 - 2y) + j2x \text{ ---------------------(i)}$$

Equating real and imaginary part, we get

$\mathbf{u = 3 - 2y}$ ----------------------------------(ii)

$\mathbf{v = 2x}$ ----------------------------------(iii)

Given, the point A, $\mathbf{z = -2 + j.1}$

That is A (-2, 1) ----------------------------------(iv)

Here, $\mathbf{x = -2,\ y = 1}$

Putting the value of x and y in (ii) and (iii),

Then the point A, $\mathbf{(z = -2 + j)}$ transforms onto w-plane is

$\mathbf{u = 3 - 2y}$ and $\mathbf{v = 2x}$

$\mathbf{u = 3 - 2.1}$ $\quad$ $\mathbf{v = 2.(-2)}$

$\mathbf{u = 1}$ $\quad$ $\mathbf{v = -4}$

$\therefore$ The image of A is $\mathbf{A'(w = u + jv = 1 - 4j)}$

That is $A'(1,-4)$ ----------------------------------(v)

Again,

Given, the point B, $\mathbf{z = 3 + j6}$

That is B (3, 6) ----------------------------------(vi)

Here, $\mathbf{x = 3,\ y = 6}$

Putting the value of x and y in (ii) and (iii), Then the point B, $\mathbf{z = 3 + j6}$ transforms onto w-plane is

$$\mathbf{u = 3 - 2y} \qquad \text{and} \qquad \mathbf{v = 2x}$$
$$\mathbf{u = 3 - 2.6} \qquad \mathbf{v = 2.3}$$
$$\mathbf{u = -9} \qquad \mathbf{v = 6}$$

$\therefore$ **The image of B is** $\mathbf{B'(w = u + jv = -9 + 6j)}$

That is $\boldsymbol{B'(-9,6)}$ **---------------------------------(vii)**

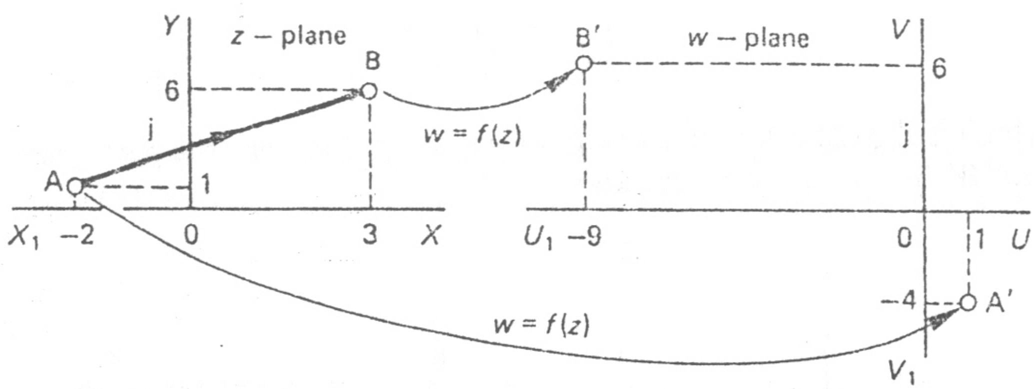

Figure 10

Again,

From Figure 09:

Given, the point A, $\mathbf{z = -2 + j.1}$

That is A (-2, 1)

Here, $\boldsymbol{x_1 = -2,\ y_1 = 1}$

Given, the point B, $\mathbf{z = 3 + j6}$

That is, B (3, 6),

Here, $\boldsymbol{x_2 = 3,\ y_2 = 6}$

The equation of the straight line AB is:

$$\frac{\mathbf{y - y_1}}{\mathbf{y_1 - y_2}} = \frac{\mathbf{x - x_1}}{\mathbf{x_1 - x_2}}$$
$$\Rightarrow \frac{\mathbf{y - 1}}{\mathbf{1 - 6}} = \frac{\mathbf{x - (-2)}}{\mathbf{-2 - 3}}$$
$$\Rightarrow \frac{\mathbf{y - 1}}{\mathbf{-5}} = \frac{\mathbf{x + 2}}{\mathbf{-5}}$$
$$\Rightarrow \mathbf{y - 1 = x + 2}$$
$$\Rightarrow \mathbf{y = x + 3} \text{---------------------------------(viii)}$$

We have, from (ii) & (iii)

$$\mathbf{u = 3 - 2y} \qquad \mathbf{v = 2x}$$
$$\Rightarrow \mathbf{3 - 2y = u} \qquad \Rightarrow \mathbf{2x = v}$$
$$\Rightarrow \mathbf{-2y = u - 3} \qquad \Rightarrow \mathbf{x = \frac{v}{2}}$$

$$\Rightarrow \mathbf{2y = -u + 3}$$
$$\Rightarrow \mathbf{2y = 3 - u}$$
$$\Rightarrow \mathbf{y = \frac{3-u}{2}}$$

Putting the value of **x** and **y** in (viii),

$$\Rightarrow \mathbf{y = x + 3}$$
$$\Rightarrow \mathbf{\frac{3-u}{2} = \frac{v}{2} + 3}$$
$$\Rightarrow \mathbf{\frac{3-u}{2} = \frac{v+6}{2}}$$
$$\Rightarrow \mathbf{3 - u = v + 6}$$
$$\Rightarrow \mathbf{v + 6 = 3 - u}$$
$$\Rightarrow \mathbf{v = -6 + 3 - u}$$
$$\Rightarrow \mathbf{v = -3 - u}$$
$$\Rightarrow \mathbf{v = -u - 3} \text{ -------------------------------------(ix)}$$

The equation (ix) is the equation of the straight line $\mathbf{A'B'}$

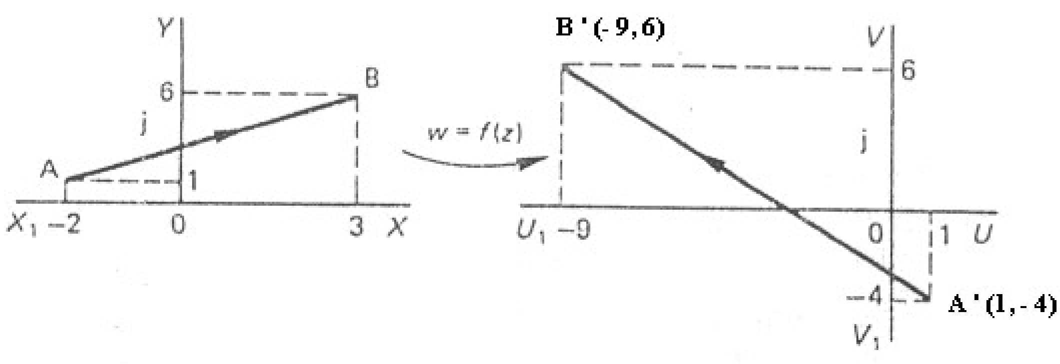

Figure 11

Justification:

We have

The image of A is $\mathbf{A'(w = u + jv = 1 - 4j)}$

That is $\boldsymbol{A'(1,-4)}$

Here, $\boldsymbol{u_1 = 1, v_1 = -4}$

and

The image of B is $\mathbf{B'(w = u + jv = -9 + 6j)}$

That is $\boldsymbol{B'(-9,6)}$

Here, $\boldsymbol{u_2 = -9, v_2 = 6}$

The equation of the straight line $\mathbf{A'B'}$ is:

$$\mathbf{\frac{v - v_1}{v_1 - v_2} = \frac{u - u_1}{u_1 - u_2}} \qquad \text{[As we know: } \frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}\text{]}$$

$$\Rightarrow \frac{v-(-4)}{-4-6} = \frac{u-1}{1-(-9)}$$
$$\Rightarrow \frac{v+4}{-10} = \frac{u-1}{1+9}$$
$$\Rightarrow \frac{v+4}{-10} = \frac{u-1}{10}$$
$$\Rightarrow \frac{v+4}{-1} = \frac{u-1}{1}$$
$$\Rightarrow v+4 = -1(u-1)$$
$$\Rightarrow v+4 = -u+1$$
$$\Rightarrow v = -u+1-4$$
$$\Rightarrow v = -u-3 \text{-------------------------------------}(x)$$

Since the equation (ix) and (x) is same. ***Hence proved***

Example 04: Graph of Parabola

$$y^2 = 4ax$$

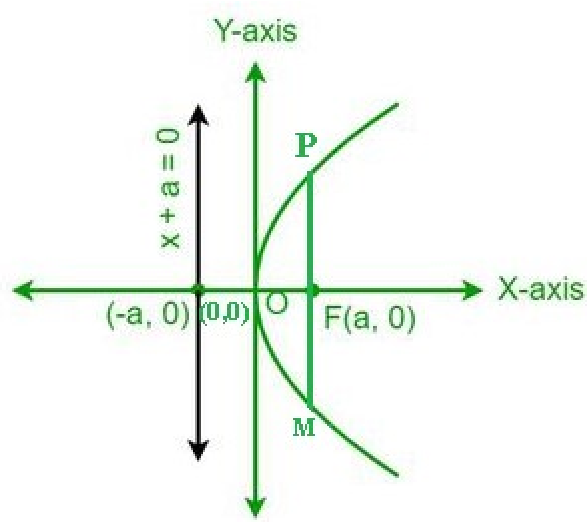

Figure 12

Focus F (a, 0)
PM= Latus Rectum = 4a

Example 05: Graph of Parabola

$$y^2 = -4ax$$

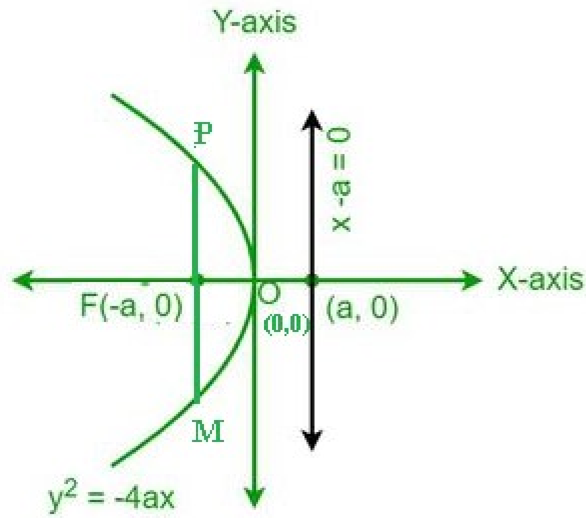

Figure 13

Focus F (- a, 0)
PM= Latus Rectum = 4a

Example 06: Graph of Parabola

$$x^2 = 4ay$$

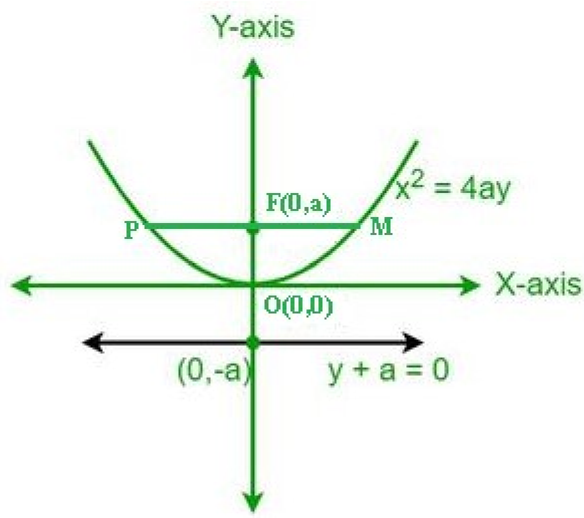

Figure 14

Focus F (0, a)
PM= Latus Rectum = 4a

Example 07: Graph of Parabola

$$x^2 = -4ay$$

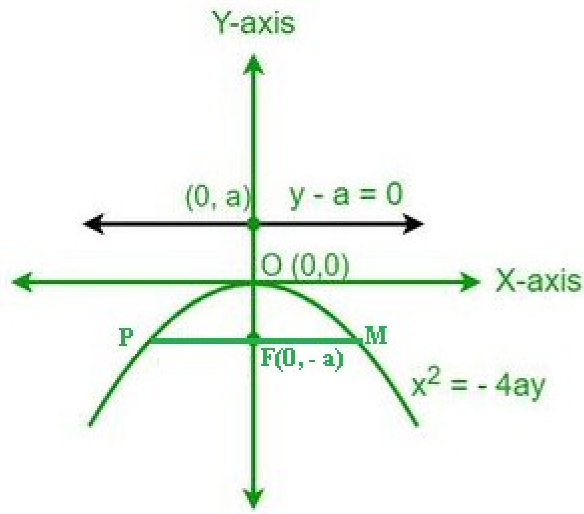

Figure 15

Focus F (0, -a)
PM= Latus Rectum = 4a

Example 08

If $\mathbf{w} = \mathbf{z}^2$, find the path traced out by $\mathbf{w}$ as $\mathbf{z}$ move along the straight line joining $\mathbf{A(2+0.j)}$ and $\mathbf{B(0+2j)}$

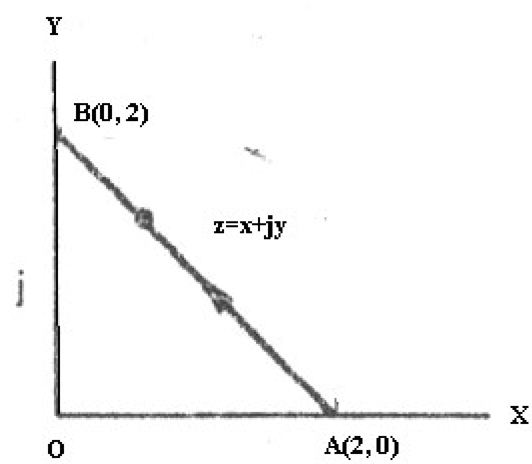

Figure 16

Solution:

We have, $w = f(z) = z^2$

$$u + jv = f(z) = (x + jy)^2 \qquad [w = u + jv;\ z = x + jy]$$

$$u + jv = x^2 + j2xy + j^2y^2$$

$$u + jv = x^2 + j2xy - y^2 \qquad [\because j^2 = -1]$$

$$u + jv = x^2 - y^2 + j2xy \text{ ----------------(i)}$$

Equating the coefficient of real and imaginary part, we get

$\therefore u = x^2 - y^2$ ----------------- *(ii)*

$\therefore v = 2xy$ ----------------- *(iii)*

Given, $\mathbf{A(2 + j.0)}$

That is, $A(2,0)$ ----------------------------(iv)

Here x = 2, y = 0

Putting the values of x and y in (ii) and (iii),

$$\begin{array}{ll} \mathbf{u = x^2 - y^2} & \mathbf{v = 2xy} \\ \mathbf{u = 2^2 - 0^2} & \mathbf{v = 2.2.0} \\ \mathbf{u = 4 - 0} & \mathbf{v = 0} \\ \mathbf{u = 4} & \end{array}$$

$\therefore \mathbf{w = u + jv = 4 + j.0}$

The image of A is $\mathbf{A'(w = 4 + j.0)}$

That is $A'(4,0)$ ----------------------------(v)

Again,

$\mathbf{B(z = 0 + j2)}$

That is, $B(0,2)$ ----------------------------(vi)

Putting the values of x and y in (ii) and (iii),

$$\begin{array}{ll} \mathbf{u = x^2 - y^2} & \mathbf{v = 2xy} \\ \mathbf{u = 0^2 - 2^2} & \mathbf{v = 2.0.2} \\ \mathbf{u = 0 - 4} & \mathbf{v = 0} \\ \mathbf{u = -4} & \end{array}$$

$\therefore \mathbf{w = u + jv = -4 + j.0}$

The image of B is $B'(w = u + jv = -4 + j.0)$

That is $B'(-4,0)$ ----------------------------(vii)

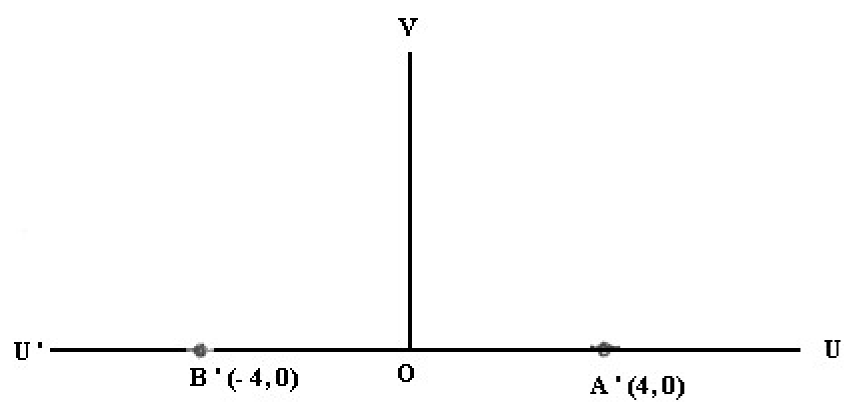

Figure 17

Given, From Figure 16:
We have: A (2, 0) and B (0, 2)

$$A : x_1 = 2, y_1 = 0$$

$$B : x_2 = 0, y_2 = 2$$

The equation of the line AB is,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\Rightarrow \frac{y - 0}{0 - 2} = \frac{x - 2}{2 - 0}$$

$$\Rightarrow \frac{y}{-2} = \frac{x - 2}{2}$$

$$\Rightarrow 2y = -2(x - 2)$$

$$\Rightarrow y = \frac{-2(x - 2)}{2}$$

$$\therefore y = -(x - 2)$$

$$\therefore y = -x + 2$$

$$\therefore y = 2 - x \qquad \text{-----------------------------(viii)}$$

Putting the value of y in (ii),

$$\therefore u = x^2 - y^2$$

$$\Rightarrow u = x^2 - (2 - x)^2$$

$$\Rightarrow u = x^2 - (4 - 4x + x^2)$$

$$\Rightarrow u = x^2 - 4 + 4x - x^2$$

$$\Rightarrow u = 4x - 4$$

$$\Rightarrow u + 4 = 4x$$

$$\Rightarrow x = \frac{u + 4}{4} \qquad \text{------------------------------(ix)}$$

Putting the value of y in (iii),

$$v = 2xy$$

$$v = 2x(2 - x)$$

$$\therefore v = 4x - 2x^2 \qquad \text{--------------------------------------(x)}$$

Now putting the value of x in equation (x)

$$\therefore v = 4x - 2x^2$$

$$v = 4\left(\frac{u + 4}{4}\right) - 2\left(\frac{u + 4}{4}\right)^2$$

$$\Rightarrow v = u + 4 - 2\left(\frac{u^2 + 8u + 16}{16}\right)$$

$$\Rightarrow v = u + 4 - \frac{1}{8}\left(u^2 + 8u + 16\right)$$

$$\Rightarrow v = u + 4 - \frac{1}{8}u^2 - u - 2$$

$$\Rightarrow v = 2 - \frac{1}{8}u^2$$

$$\Rightarrow v = \frac{16 - u^2}{8}$$

$$\Rightarrow v = -\frac{1}{8}(u^2 - 16)$$

$$\Rightarrow 8v = -(u^2 - 16)$$

$$\Rightarrow 8v = -u^2 + 16$$

$$\Rightarrow 8v = -u^2 + 16$$

$$\Rightarrow -u^2 = 8v - 16$$

$$\Rightarrow u^2 = -8(v - 2)$$

$$\Rightarrow u^2 = -4.2(v - 2) \quad \text{-----------------------------(xi)}$$

The equation (xi) represents an equation of a parabola.

Let,

$$U = u \quad and \quad V = v - 2 \quad \text{-----------------------------(xii)}$$

From (xii),
When,

$$U = 0 \text{ then } u = 0$$

and

$$V = 0 \text{ then}$$

$$\Rightarrow 0 = v - 2$$

$$\Rightarrow v = 2$$

$$\therefore \text{Vertex} = (u, v) = (0,2)$$

And latus rectum $A'B' =$

$$= 4a$$

$$= 4 \cdot 2$$

$$= 8$$

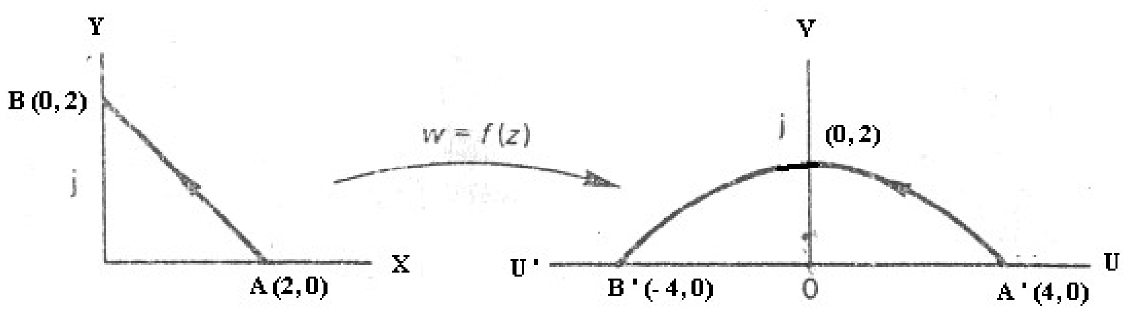

Figure 18

Example 09

The straight line AB in the z-plane as shown is mapped onto the w-plane by $\mathbf{w = z^2}$

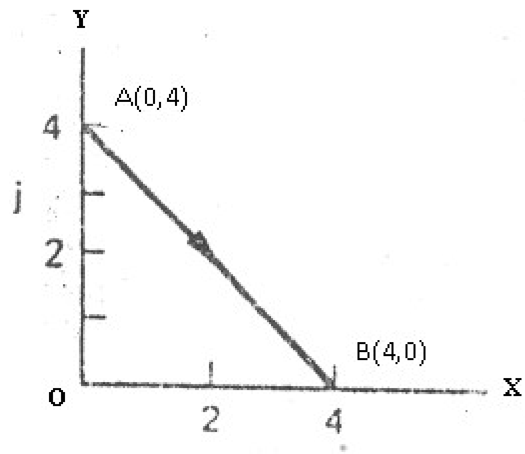

Figure 19

Solution:

We have,

$$w = f(z) = z^2$$

$$w = f(z) = (x + jy)^2 \qquad [z = x + jy]$$

$$u + jv = f(z) = x^2 + j2xy + j^2y^2 \qquad [w = u + jv]$$

$$u + jv = x^2 + j2xy - y^2 \; [\because j^2 = -1]$$

$$u + jv = x^2 - y^2 + j2xy \text{ ----------------------------(i)}$$

Equating the coefficient of real and imaginary part, we get

$$\therefore u = x^2 - y^2 \text{ ---------------- (ii)}$$

$$\therefore v = 2xy \text{ ------------------ (iii)}$$

Given, $\mathbf{A(0 + j.4)}$

That is, $\boldsymbol{A}\mathbf{(0,4)}$ ---------------------------------(iv)

Here $\mathbf{x = 0, y = 4}$

Putting the value of x and y in (ii) and (iii),

$$u = x^2 - y^2 \qquad v = 2xy$$

$$u = 0^2 - 4^2 \qquad v = 2.0.4$$

$$u = 0 - 16 \qquad v = 0$$

$$u = -16$$

$$\therefore w = u + jv = -16 + j.0$$

The image of A is $\mathbf{A'(w = -16 + j.0)}$

That is $\boldsymbol{A'(-16,0)}$ ---------------------------------(v)

Again, $\mathbf{B(z = 4 + j.0)}$

That is, $\boldsymbol{B}\mathbf{(4,0)}$ ---------------------------------(vi)

Here, $\mathbf{x = 4, y = 0}$

Putting the value of x and y in (ii) and (iii),

$$\mathbf{u = x^2 - y^2} \qquad \mathbf{v = 2xy}$$
$$\mathbf{u = 4^2 - 0^2} \qquad \mathbf{v = 2.4.0}$$
$$\mathbf{u = 16} \qquad \mathbf{v = 0}$$
$$\mathbf{u = 16}$$
$$\therefore \mathbf{w = u + jv = 16 + j.0}$$

The image of B is $B'(\mathbf{w = u + jv = 16 + j.0})$

That is $B'(16,0)$ --------------------------------(vii)

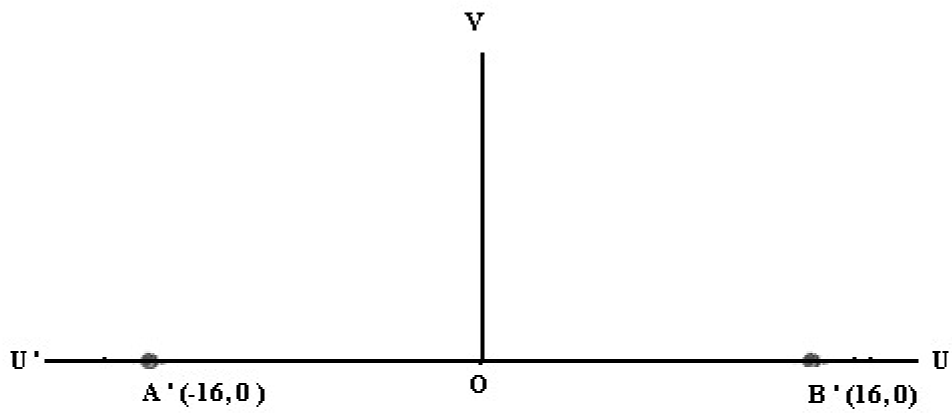

Figure 20

Given, From Figure 19:

For A(0,4) and B(4,0)

$$A : x_1 = 0, y_1 = 4$$
$$\mathrm{B} : x_2 = 4, y_2 = 0$$

The equation of the line AB is,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$
$$\Rightarrow \frac{y - 4}{4 - 0} = \frac{x - 0}{0 - 4}$$
$$\Rightarrow \frac{y - 4}{4} = \frac{x}{-4}$$
$$\Rightarrow -4(y - 4) = 4x$$
$$\Rightarrow -(y - 4) = x$$
$$\Rightarrow (y - 4) = -x$$
$$\Rightarrow y = 4 - x \quad \text{--------------------------------(viii)}$$

Putting the value of y in (ii),

$$\therefore \mathbf{u = x^2 - y^2}$$

$$\Rightarrow u = x^2 - (4-x)^2$$
$$\Rightarrow u = x^2 - (16 - 8x + x^2)$$
$$\Rightarrow u = x^2 - 16 + 8x - x^2$$
$$\Rightarrow u = 8x - 16$$
$$\Rightarrow u + 16 = 8x$$
$$\Rightarrow x = \frac{u+16}{8} \text{-------------------} (ix)$$

Putting the value of y in (iii),

$$v = 2xy$$
$$v = 2x(4-x)$$
$$\therefore v = 8x - 2x^2 \text{-------------------} (x)$$

Putting the value of x in (x), we get

$$\Rightarrow v = 8\left(\frac{u+16}{8}\right) - 2\left(\frac{u+16}{8}\right)^2$$
$$\Rightarrow v = (u+16) - \frac{2}{64}(u+16)^2$$
$$\Rightarrow v = (u+16) - \frac{1}{32}(u+16)^2$$
$$\Rightarrow v = u + 16 - \frac{1}{32}(u^2 + 32u + 256)$$
$$\Rightarrow v = u + 16 - \frac{1}{32}u^2 - u - 8$$
$$\Rightarrow v = 8 - \frac{1}{32}u^2$$
$$\Rightarrow v = -\frac{1}{32}u^2 + 8$$
$$\Rightarrow v = -\left(\frac{u^2 - 256}{32}\right)$$
$$\Rightarrow 32v = -(u^2 - 256)$$
$$\Rightarrow -u^2 + 256 = 32v$$
$$\Rightarrow -u^2 = 32v - 256$$
$$\Rightarrow u^2 = -32v + 256$$
$$\Rightarrow u^2 = -32(v-8)$$
$$\Rightarrow u^2 = -4 \cdot 8(v-8) \text{-------------------} (xi)$$

The equation (xi) represents an equation of a parabola.
Let,

$$U = u \;\; and \;\; V = v - 8 \text{-------------------} (xii)$$

From (xii),
When,

$$U = 0 \text{ then } u = 0$$

and

$$\mathbf{V = 0 \;\; then}$$
$$\Rightarrow \mathbf{0 = v - 8}$$
$$\Rightarrow \mathbf{v = 8}$$
$$\therefore \mathbf{Vertex = (u, v) = (0,8)}$$

And latus rectum $\mathbf{A'B' =}$

$$\mathbf{= 4a}$$
$$\mathbf{= 4 \cdot 8}$$
$$\mathbf{= 32}$$

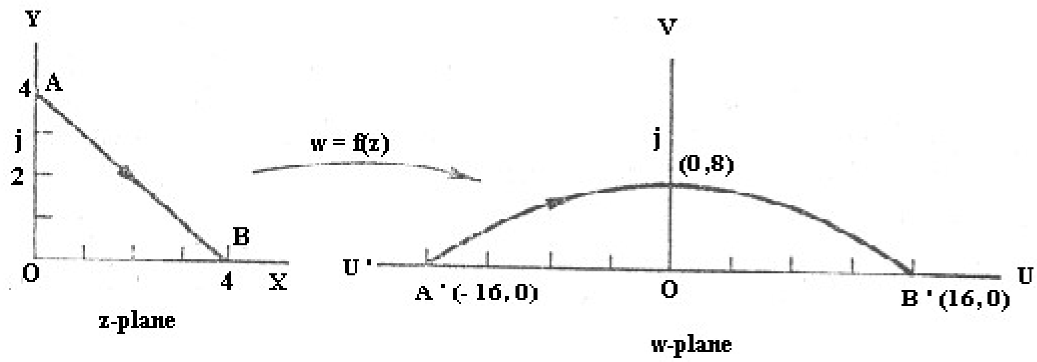

Figure 21

Example 10

A triangle consisting of AB, BC and CA in the z-plane is mapped to the w-plane by the transformation $\mathbf{w = z^2}$

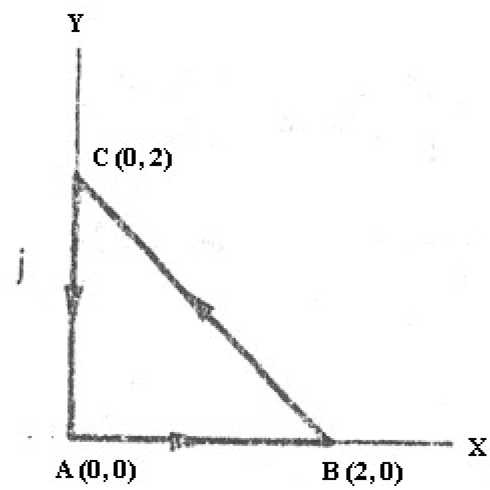

Figure 22

Solution:

We have,

$$w = f(z) = z^2$$
$$w = f(z) = (x + jy)^2 \qquad [z = x + jy]$$
$$u + jv = f(z) = x^2 + j2xy + j^2y^2 \qquad [w = u + jv]$$

$$u + jv = f(z) = x^2 + j2xy - y^2 \; [\because j^2 = -1]$$

$$u + jv = x^2 - y^2 + j2xy \qquad \text{----------------------(i)}$$

Equating the coefficient of real and imaginary part, we get

$\therefore$ $u = x^2 - y^2$ ----------------------(ii)

$\therefore$ $v = 2xy$ ----------------------(iii)

Given, $A(0,0)$ ----------------------(iv)

That is, $x = 0, y = 0$

Putting the value of x and y in (ii) and (iii)

$$u = x^2 - y^2 \qquad v = 2xy$$
$$u = 0^2 - 0^2 \qquad v = 2.0.0$$
$$u = 0 - 0 \qquad v = 0$$
$$u = 0$$
$$\therefore w = u + jv = 0 + j.0$$

The image of A is $A'(w = 0 + j.0)$

That is $A'(0,0)$ ----------------------(v)

Again, $B(2,0)$ ----------------------(vi)

That is, $x = 2, y = 0$

Putting the value of x and y in (ii) and (iii),

$$u = x^2 - y^2 \qquad v = 2xy$$
$$u = 2^2 - 0^2 \qquad v = 2.2.0$$
$$u = 4 \qquad v = 0$$
$$u = 4$$
$$\therefore w = u + jv = 4 + j.0$$

The image of B is $B'(w = u + jv = 4 + j.0)$

That is $B'(4,0)$ ----------------------(vii)

Again, $C(0,2)$ ----------------------(viii)

That is, $x = 0, y = 2$

Putting the value of x and y in (ii) and (iii)

$$u = x^2 - y^2 \qquad v = 2xy$$
$$u = 0^2 - 2^2 \qquad v = 2.0.2$$
$$u = -4 \qquad v = 0$$
$$u = -4$$
$$\therefore w = u + jv = -4 + j.0$$

The image of C is

$C'(w = u + jv = -4 + j.0)$

That is

$C'(-4,0)$ ----------------------(ix)

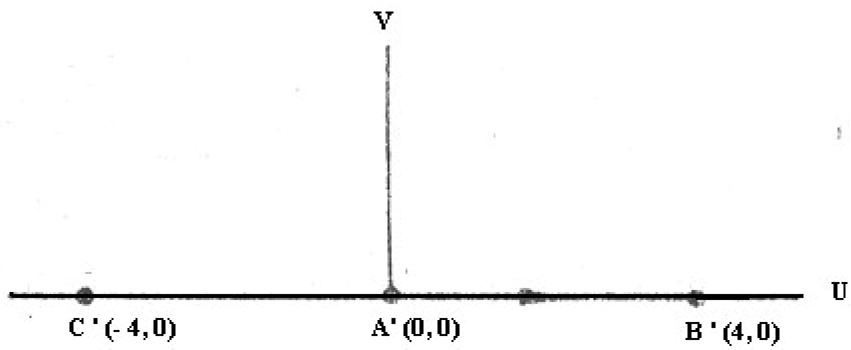

Figure 23

From Figure 22:
Given: $A(0,0)$
That is,
$x_1 = 0, y_1 = 0$
$B(2,0)$
That is, $x_2 = 2, y_2 = 0$
The equation of the line AB is,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\Rightarrow \frac{y - 0}{0 - 0} = \frac{x - 0}{0 - 2}$$

$$\Rightarrow \frac{y}{0} = \frac{x}{-2}$$

$$\Rightarrow -2y = 0$$

$$\Rightarrow y = 0 \qquad \text{----------------------(x)}$$

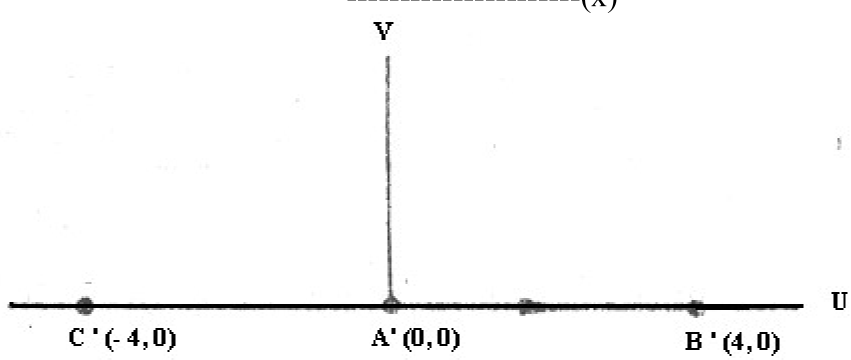

Figure 24

Now, Putting the value of y from (x) in (ii) & (iii),

$$u = x^2 - y^2 \qquad\qquad v = 2xy$$

$$\Rightarrow u = x^2 - 0 \qquad\qquad \therefore v = 0$$

$$\Rightarrow u = x^2$$

Given:
From Figure 22
$B(2,0)$
Then, $x_1 = 2, y_1 = 0$
Again
$C(0,2)$
Then, $x_2 = 0, y_2 = 2$
The equation of the line BC is,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\Rightarrow \frac{y - 0}{0 - 2} = \frac{x - 2}{2 - 0}$$

$$\Rightarrow \frac{y}{-2} = \frac{x - 2}{2}$$

$$\Rightarrow 2y = -2(x - 2)$$

$$\Rightarrow y = \frac{-2(x - 2)}{2}$$

$$\therefore y = -(x - 2)$$

$$\therefore y = 2 - x \quad \text{----------------------(xi)}$$

Putting the value of y in (ii),

$$\therefore u = x^2 - y^2$$

$$\Rightarrow u = x^2 - (2 - x)^2$$

$$\Rightarrow u = x^2 - (4 - 4x + x^2)$$

$$\Rightarrow u = x^2 - 4 + 4x - x^2$$

$$\Rightarrow u = 4x - 4$$

$$\Rightarrow u + 4 = 4x$$

$$\Rightarrow x = \frac{u + 4}{4} \qquad ------------(xii)$$

Putting the value of y in (iii),

$$v = 2xy$$

$$v = 2x(2 - x)$$

$$\therefore v = 4x - 2x^2 \qquad ------------(xiii)$$

Now putting the value of x in equation (xiii)

$$v = 4\left(\frac{u + 4}{4}\right) - 2\left(\frac{u + 4}{4}\right)^2$$

$$\Rightarrow v = u + 4 - 2\left(\frac{u^2 + 8u + 16}{16}\right)$$

$$\Rightarrow v = u + 4 - \frac{1}{8}(u^2 + 8u + 16)$$

$$\Rightarrow v = u + 4 - \frac{1}{8}u^2 - u - 2$$

$$\Rightarrow v = 2 - \frac{1}{8}u^2$$

$$\Rightarrow v = \frac{16 - u^2}{8}$$

$$\Rightarrow v = -\frac{1}{8}(u^2 - 16)$$

$$\Rightarrow 8v = -(u^2 - 16)$$

$$\Rightarrow 8v = -u^2 + 16$$

$$\Rightarrow 8v = -u^2 + 16$$

$$\Rightarrow -u^2 = 8v - 16$$

$$\Rightarrow u^2 = -8(v - 2)$$

$$\Rightarrow u^2 = -4.2(v - 2) \qquad ----------- (xiv)$$

The equation (xiv) represents an equation of a parabola.
Let,

$$U = u \;\; and \;\; V = v - 2 \qquad ----------- (xv)$$

From (xv),
When,

$$U = 0 \text{ then } u = 0$$

and

$$V = 0 \text{ then}$$

$$\Rightarrow 0 = v - 2$$

$$\Rightarrow v = 2$$

$$\therefore \text{Vertex} = (u, v) = (0,2)$$

And latus rectum $B'C' =$

$$= 4a$$

$$= 4 \cdot 2 = 8$$

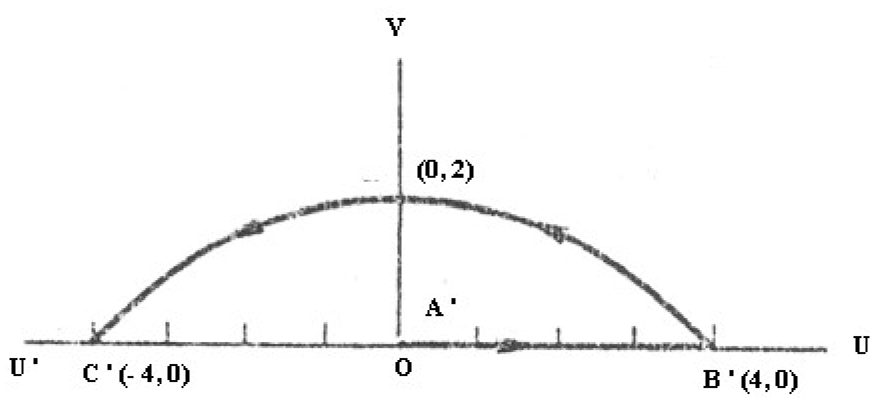

Figure 25

So, finally we get

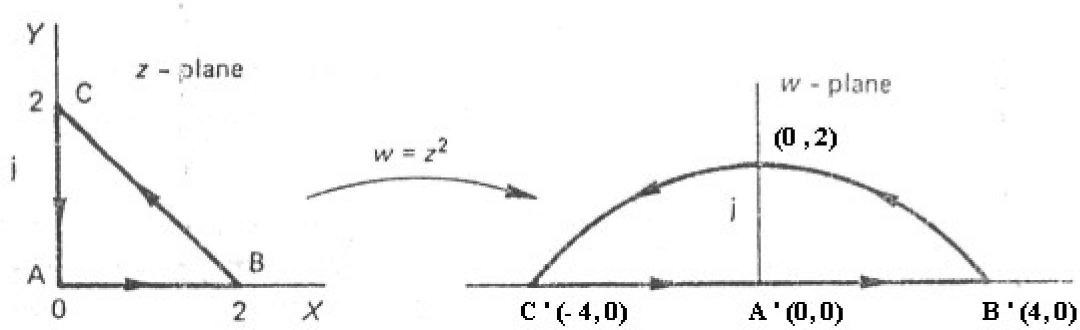

Figure 26

Example 11

A straight line joining $\mathbf{A(-j)}$ and $\mathbf{B(2+j)}$ in the z-plane is mapped onto the w-plane by the transformation equation $\mathbf{w = \frac{1}{z}}$

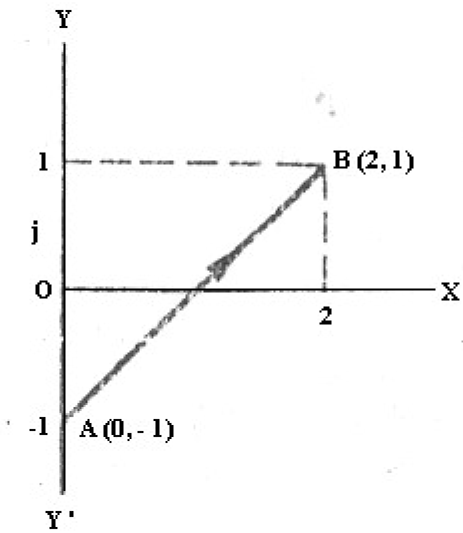

Figure 27

Solution:

Given,

$$w = \frac{1}{z}$$

$$w = \frac{1}{x+jy} \qquad [z = x + jy]$$

$$w = \frac{x-jy}{(x+jy)(x-jy)} \qquad \text{[Multiplying by } x - jy \text{]}$$

$$w = \frac{x-jy}{x^2 - jxy + jxy - j^2y^2}$$

$$w = \frac{x-jy}{x^2+y^2} \qquad [\because j^2 = -1]$$

$$u + jv = \frac{x - jy}{x^2 + y^2} \qquad [w = u + jv]$$

$$u + jv = \frac{x}{x^2 + y^2} - j\frac{y}{x^2 + y^2} \qquad \text{------------------------(i)}$$

Equating the coefficient of real and imaginary part, we get,

$$u = \frac{x}{x^2 + y^2} \qquad -----------(ii)$$

$$v = \frac{-y}{x^2 + y^2} \qquad ----------(iii)$$

Given, $\mathbf{A(0 - j.1)}$

That is, $A(0,-1)$ ------------------------(iv)

Here,

$\mathbf{x = 0, y = -1}$

Putting the value of x and y in (ii) and (iii)

$$u = \frac{x}{x^2 + y^2} \qquad v = \frac{-y}{x^2 + y^2}$$

$$u = \frac{0}{0 + (-1)^2} \qquad v = \frac{-(-1)}{0^2 + (-1)^2}$$

$$u = \frac{0}{0 + 1} \qquad v = \frac{1}{1}$$

$$u = \frac{0}{1} \qquad v = 1$$

$$u = 0 \qquad v = 1$$

$$\therefore w = u + jv = 0 + j.1$$

The image of A is $\mathbf{A'(w = 0 + j.1)}$

That is $A'(0,1)$ ------------------------(v)

Again,

$\mathbf{B(z = 2 + j.1)}$

That is, $B(2,1)$ ------------------------(vi)

Here, $\mathbf{x = 2, y = 1}$

Putting the value of x and y in (ii) and (iii),

$$u = \frac{x}{x^2 + y^2} \qquad v = \frac{-y}{x^2 + y^2}$$

$$u = \frac{2}{2^2 + 1^2} \qquad v = \frac{-1}{2^2 + 1^2}$$

$$u = \frac{2}{5} \qquad v = \frac{-1}{5}$$

$$\therefore \mathbf{w = u + jv = \frac{2}{5} - j\frac{1}{5}}$$

The image of B is $B'(\ w = \frac{2}{5} - j\frac{1}{5})$

That is $B'(\frac{2}{5}, -\frac{1}{5})$ ------------------------(vii)

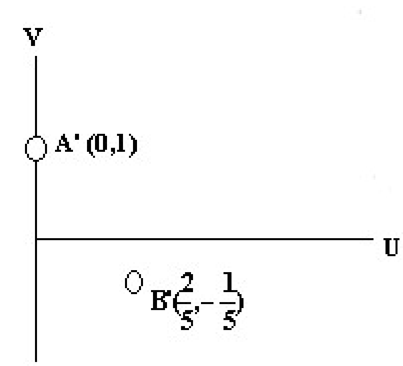

Figure 28

From Figure 27:
Given A(0,-1) and B(2,1)
The equation of the line AB is,

$$\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$$

$$\Rightarrow \frac{y - (-1)}{-1 - 1} = \frac{x - 0}{0 - 2}$$

$$\Rightarrow \frac{y + 1}{-1 - 1} = \frac{x - 0}{0 - 2}$$

$$\Rightarrow \frac{y + 1}{-2} = \frac{x}{-2}$$

$$\Rightarrow y + 1 = x$$

$$\therefore y = x - 1 \qquad ----------(viii)$$

Again, Given

$$w = \frac{1}{z}$$

$$\therefore z = \frac{1}{w}$$

$$z = \frac{1}{u + jv} \qquad [w = u + jv]$$

$$z = \frac{u - jv}{(u + jv)(u - jv)}$$

$$z = \frac{u - jv}{u^2 - (jv)^2}$$

$$z = \frac{u - jv}{u^2 + v^2} \qquad [\because j^2 = -1]$$

$$x + jy = \frac{u - jv}{u^2 + v^2} \qquad [z = x + jy]$$

$$i.e.\ x + jy = \frac{u}{u^2 + v^2} - j\frac{v}{u^2 + v^2} - - - - - - - - -(ix)$$

Equating the coefficient of real and imaginary part, we get,

$$x = \frac{u}{u^2 + v^2}\ ; \qquad y = \frac{-v}{u^2 + v^2} - - - - - - - - - - (x)$$

Putting the value of x and y in (viii),

$$\mathbf{y = x - 1}$$

$$\Rightarrow \frac{-v}{u^2 + v^2} = \frac{u}{u^2 + v^2} - 1$$

$$\Rightarrow \frac{-v}{u^2 + v^2} = \frac{u - u^2 - v^2}{u^2 + v^2}$$

$$\Rightarrow -v = u - u^2 - v^2$$

$$\Rightarrow u - u^2 - v^2 + v = 0$$

$$\Rightarrow -u + u^2 + v^2 - v = 0$$

$$\Rightarrow u^2 - u + v^2 - v = 0$$

$$\Rightarrow (u^2 - u) + (v^2 - v) = 0$$

$$\Rightarrow u^2 - 2 \cdot u \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + v^2 - 2 \cdot v \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0$$

$$\Rightarrow u^2 - 2 \cdot u \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 + v^2 - 2 \cdot v \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{4} = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 - \frac{2}{4} = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 - \frac{1}{2} = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v - \frac{1}{2}\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 - - - - - - - - - - -(xi)$$

The equation (xi) represents an equation of a circle whose centre $C\left(\frac{1}{2},\frac{1}{2}\right)$ and

$\textbf{Radius} = \frac{1}{\sqrt{2}}$

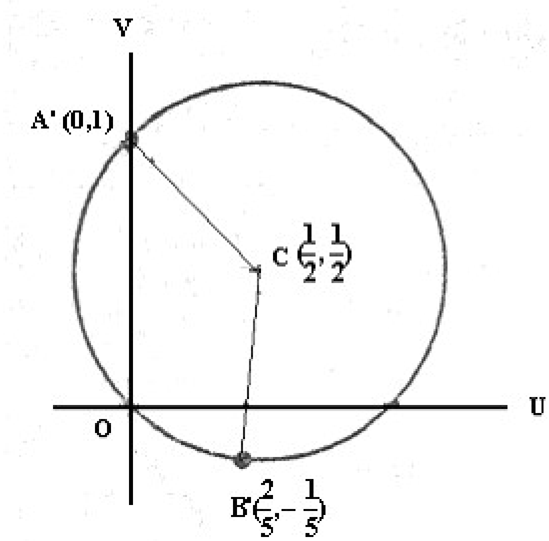

Figure 29

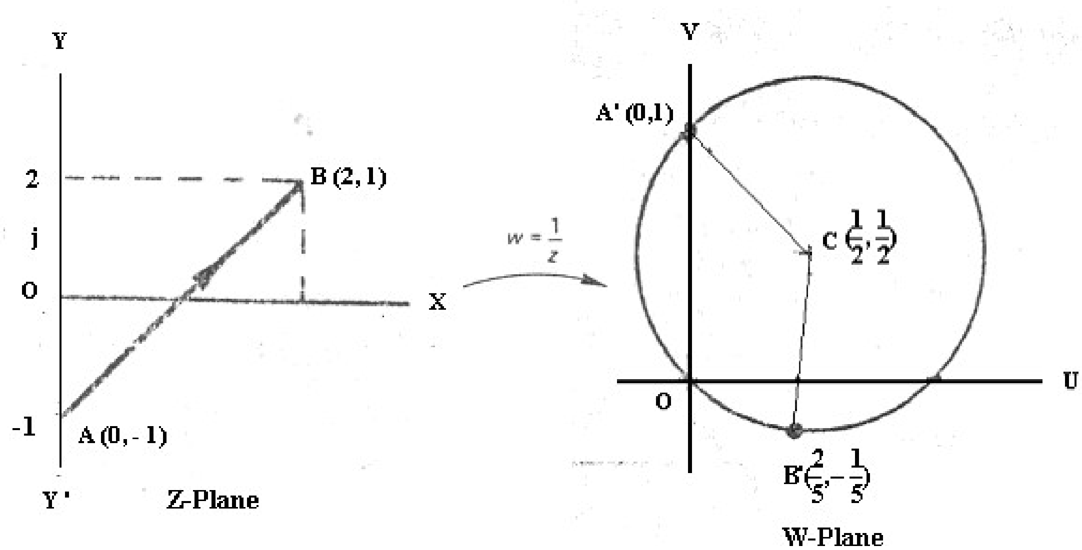

Figure 30

Justification of radius of the circle:

We have $\mathbf{A'(w = 0 + j.1)}$ that is the coordinate of $\mathbf{A'(0,1)}$ and the center $\mathbf{C}\left(\frac{1}{2},\frac{1}{2}\right)$

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$$\therefore A'C = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$$

$$\therefore \mathbf{A'C} = \sqrt{(0 - \frac{1}{2})^2 + (1 - \frac{1}{2})^2}$$

$$\therefore \mathbf{A'C} = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$$

$$\therefore \mathbf{A'C} = \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$\therefore \mathbf{A'C} = \sqrt{\frac{2}{4}}$$

$$\therefore \mathbf{A'C} = \sqrt{\frac{1}{2}}$$

$$\therefore A'C = \frac{1}{\sqrt{2}}$$ **(Proved)**

$$\therefore \text{Radius} = \frac{1}{\sqrt{2}}$$

We have $B'(\frac{2}{5}, -\frac{1}{5})$ and $C(\frac{1}{2}, \frac{1}{2})$

$$\therefore B'C = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$$

$$B'C = \sqrt{(\frac{2}{5} - \frac{1}{2})^2 + (-\frac{1}{5} - \frac{1}{2})^2}$$

$$B'C = \sqrt{(\frac{4-5}{10})^2 + (\frac{-2-5}{10})^2}$$

$$B'C = \sqrt{(\frac{-1}{10})^2 + (\frac{-7}{10})^2}$$

$$B'C = \sqrt{\frac{1}{100} + \frac{49}{100}}$$

$$B'C = \sqrt{\frac{50}{100}}$$

$$B'C = \sqrt{\frac{1}{2}}$$

$$\therefore B'C = \frac{1}{\sqrt{2}}$$ (Proved)

$$\therefore \text{Radius} = \frac{1}{\sqrt{2}}$$

Example 12
A circle in the z-plane has its centre at $\mathbf{z = 3}$ and a radius of 2 units. Determine its image in the w-plane when transformation by $\mathbf{w = \frac{1}{z}}$

Where c is the circle $|z-3| = 2$

We have,

$$z = x + jy$$
$$z - 3 = x + jy - 3$$
$$z - 3 = x - 3 + jy$$
$$\therefore |z-3| = \sqrt{(x-3)^2 + y^2}$$

Given,

$$|z-3| = 2$$
$$\therefore |z-3| = \sqrt{(x-3)^2 + y^2} = 2$$
$$\therefore \sqrt{(x-3)^2 + y^2} = 2$$
$$\therefore (x-3)^2 + y^2 = 2^2$$
$$\therefore (x-3)^2 + (y-0)^2 = 2^2$$

This is the equation of the circle whose centre (3,0) and radius 2

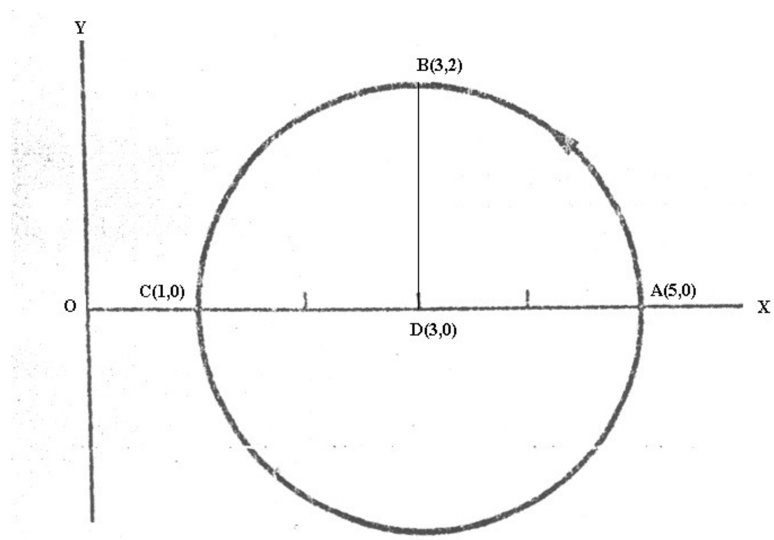

Figure 31

Solution:
Given,

$$\mathbf{z = 3}$$
$$\Rightarrow \mathbf{x + jy = 3}$$
$$\Rightarrow \boldsymbol{x + jy = 3 + 0.j} \text{ ------------------------(i)}$$

Equating the coefficient of real and imaginary part, we get,

$$x = 3, \qquad y = 0$$

Hence, we can write,

$$(x, y) = (3,0)$$

and given, radius =2

So, Equation of the circle

$$(x-3)^2 + (y-0)^2 = 2^2 \qquad \text{------------------------(ii)}$$

$$[\because (x-a)^2 + (y-b)^2 = r^2]$$

That is centre of the circle is (3,0) and radius 2

$$(x-3)^2 + y^2 = 4$$

$$\Rightarrow x^2 - 6x + 9 + y^2 = 4$$

$$\Rightarrow x^2 + y^2 - 6x + 5 = 0 \qquad ------------(iii)$$

Again, Given,

$$w = \frac{1}{z}$$

$$w = \frac{1}{x + jy} \qquad [z = x + jy]$$

$$w = \frac{x - jy}{(x + jy)(x - jy)} \qquad \text{[Multiplying by } x - jy\text{]}$$

$$w = \frac{x - jy}{x^2 - jxy + jxy - j^2 y^2}$$

$$w = \frac{x - jy}{x^2 + y^2} \qquad [\because j^2 = -1]$$

$$u + jv = \frac{x - jy}{x^2 + y^2} \qquad [w = u + jv]$$

$$u + jv = \frac{x}{x^2 + y^2} - j\frac{y}{x^2 + y^2} \qquad \text{------------------------(iv)}$$

Equating the coefficient of real and imaginary part, we get,

$$u = \frac{x}{x^2 + y^2} \qquad -----------(v)$$

$$v = \frac{-y}{x^2 + y^2} \qquad ----------(vi)$$

Again, Given

$$w = \frac{1}{z}$$

$$\therefore z = \frac{1}{w}$$

$$z = \frac{1}{u + jv} \qquad [w = u + jv]$$

$$z=\frac{u-jv}{(u+jv)(u-jv)}$$

$$z=\frac{u-jv}{u^2-(jv)^2}$$

$$z=\frac{u-jv}{u^2+v^2} \qquad [\because j^2=-1]$$

$$x+jy=\frac{u-jv}{u^2+v^2} \qquad [z=x+jy]$$

$$\text{i.e. } x+jy=\frac{u}{u^2+v^2}-j\frac{v}{u^2+v^2}---------(vii)$$

Equating the coefficient of real and imaginary part, we get,

$$x=\frac{u}{u^2+v^2} \;;\quad y=\frac{-v}{u^2+v^2} ----------(viii)$$

Substituting the values of x and y in (iii),

$$x^2+y^2-6x+5=0$$

$$\Rightarrow\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2-6\left(\frac{u}{u^2+v^2}\right)+5=0$$

$$\Rightarrow\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}-\frac{6u}{u^2+v^2}+5=0$$

$$\Rightarrow\frac{u^2+v^2}{(u^2+v^2)^2}-\frac{6u}{u^2+v^2}+5=0$$

$$\Rightarrow\frac{1}{u^2+v^2}-\frac{6u}{u^2+v^2}+5=0$$

$$\Rightarrow\frac{1-6u+5(u^2+v^2)}{u^2+v^2}=0$$

$$\Rightarrow 5(u^2+v^2)-6u+1=0$$

$$\Rightarrow u^2+v^2-\frac{6}{5}u+\frac{1}{5}=0 \quad [\text{dividing by } 5]$$

$$\Rightarrow u^2+v^2-2\cdot\frac{3}{5}\cdot u+2\cdot 0\cdot v+\frac{1}{5}=0$$

$$\Rightarrow u^2+v^2+2\cdot(-\frac{3}{5})\cdot u+2\cdot 0\cdot v+\frac{1}{5}=0------------(ix)$$

We know the general equation of the circle is

$$x^2+y^2+2gx+2fy+c=0$$

Whose centre is $(-g,-f)$ and radius is $\sqrt{g^2+f^2-c}$

Hence from (ix)

Here $\mathbf{g} = -\dfrac{3}{5}$, $\mathbf{f} = \mathbf{0}$ and $\mathbf{c} = \dfrac{1}{5}$

The centre of the new circle of (ix) is $\mathbf{(-g,-f) = (-(-\frac{3}{5}),-0) = (\frac{3}{5},0)}$

That is, centre of new circle in the w-plane is, $D(\frac{3}{5},0)$ [From figure 32]

Radius is $\sqrt{g^2+f^2-c} = \sqrt{(-\frac{3}{5})^2+0^2-\frac{1}{5}}$

$$= \sqrt{(-\frac{3}{5})^2+0^2-\frac{1}{5}} = \sqrt{\frac{9}{25}+0-\frac{1}{5}}$$

$$= \sqrt{\frac{9}{25}-\frac{1}{5}}$$

$$= \sqrt{\frac{9-5}{25}}$$

$$= \sqrt{\frac{4}{25}}$$

$$= \frac{2}{5}$$

That is, radius of new circle in the w-plane is, $\dfrac{2}{5}$

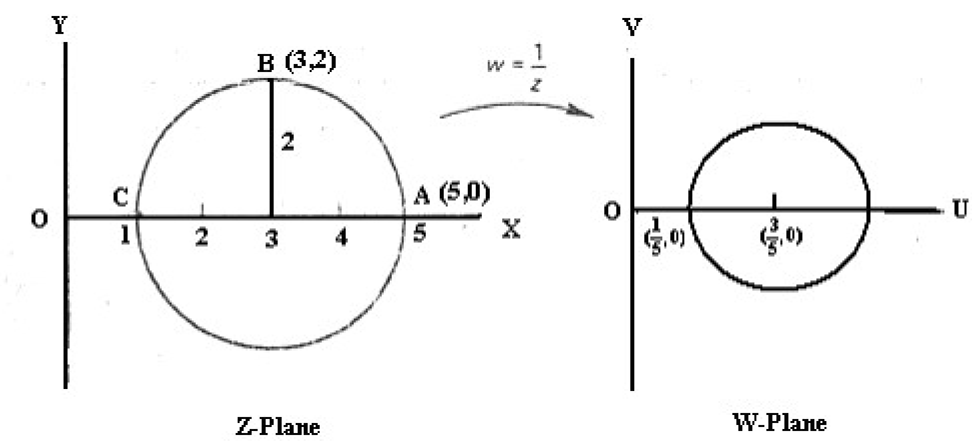

Figure 32

Taking three sample points A, B, C as shown, that is:
A(5,0), **B(3,2)**, **C(1,0)**

Putting the values of **A(5,0)**, **B(3,2)** **C(1,0)** in (v) and (vi)

We have,

$$\mathbf{u} = \frac{\mathbf{x}}{\mathbf{x^2 + y^2}} \qquad \mathbf{v} = \frac{\mathbf{-y}}{\mathbf{x^2 + y^2}}$$

For $\mathbf{A(5,0)}$; $\mathbf{u} = \frac{\mathbf{x}}{\mathbf{x^2+y^2}} = \frac{\mathbf{5}}{\mathbf{5^2+0^2}} = \frac{\mathbf{5}}{\mathbf{25+0}} = \frac{\mathbf{5}}{\mathbf{25}} = \frac{\mathbf{1}}{\mathbf{5}}$

For $\mathbf{A(5,0)}$; $\mathbf{v} = \frac{\mathbf{-y}}{\mathbf{x^2+y^2}} = \frac{\mathbf{-0}}{\mathbf{5^2+0^2}} = \frac{\mathbf{0}}{\mathbf{25+0}} = \mathbf{0}$

$\therefore$ For $\mathbf{A(5,0)}$; $\mathbf{w = u + jv} = \frac{\mathbf{1}}{\mathbf{5}} + \mathbf{j.0}$

The image of A is $\mathbf{A'(w = u + jv} = \frac{\mathbf{1}}{\mathbf{5}} + \mathbf{j.0)} = \frac{\mathbf{1}}{\mathbf{5}} + \mathbf{j.0}$

That is $A'(\frac{1}{5},0)$ ---------------------(x)

For $\mathbf{B(3,2)}$; $\mathbf{u} = \frac{\mathbf{x}}{\mathbf{x^2+y^2}} = \frac{\mathbf{3}}{\mathbf{3^2+2^2}} = \frac{\mathbf{3}}{\mathbf{9+4}} = \frac{\mathbf{3}}{\mathbf{13}}$

For $\mathbf{B(3,2)}$; $\mathbf{v} = \frac{\mathbf{-y}}{\mathbf{x^2+y^2}} = \frac{\mathbf{-2}}{\mathbf{3^2+2^2}} = \frac{\mathbf{-2}}{\mathbf{9+4}} = \frac{\mathbf{-2}}{\mathbf{13}}$

$\therefore$ For $\mathbf{B(3,2)}$; $\mathbf{w = u + jv} = \frac{\mathbf{3}}{\mathbf{13}} + \mathbf{j.(}\frac{\mathbf{-2}}{\mathbf{13}}\mathbf{)}$

The image of B is $\mathbf{B'(w = u + jv} = \frac{\mathbf{3}}{\mathbf{13}} + \mathbf{j.(-}\frac{\mathbf{2}}{\mathbf{13}}\mathbf{))} = \frac{\mathbf{3}}{\mathbf{13}} - \mathbf{j.}\frac{\mathbf{2}}{\mathbf{13}}$

That is $B'(\frac{3}{13},-.\frac{2}{13})$ -------------------------(xi)

For $\mathbf{C(1,0)}$; $\mathbf{u} = \frac{\mathbf{x}}{\mathbf{x^2+y^2}} = \frac{\mathbf{1}}{\mathbf{1^2+0^2}} = \frac{\mathbf{1}}{\mathbf{1}} = \mathbf{1}$

For $\mathbf{C(1,0)}$; $\mathbf{v} = \frac{\mathbf{-y}}{\mathbf{x^2+y^2}} = \frac{\mathbf{-0}}{\mathbf{1^2+0^2}} = \frac{\mathbf{-0}}{\mathbf{1}} = \mathbf{0}$

$\therefore$ For $\mathbf{C(1,0)}$; $\mathbf{w = u + jv = 1 + j.(0)}$

The image of C is $\mathbf{C'(w = u + jv = 1 + j.0 = 1 + j.0}$
That is $C'(1,0)$ --------------------------(xii)

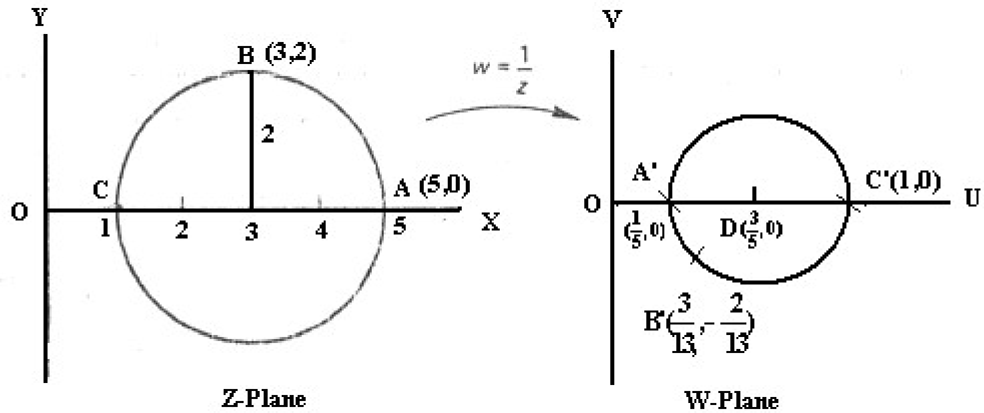

Figure 33

<u>Justification:</u>

We have $A'(\frac{1}{5},0), B'(\frac{3}{13},-\frac{2}{13}), C'(1,0), D(\frac{3}{5},0)$

$\text{Radius} = \frac{2}{5}$

We know the length of a line between two points (x_1, y_1) and (x_2, y_2) is:

$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Here, in the w-plane

$$\therefore A'D = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$$

$$\therefore A'D = \sqrt{(\frac{1}{5} - \frac{3}{5})^2 + (0-0)^2}$$

$$\therefore A'D = \sqrt{(-\frac{2}{5})^2}$$

$$\therefore A'D = \sqrt{\frac{4}{25}}$$

$$\therefore A'D = \frac{2}{5} \text{ (Proved)}$$

Again,

$$\therefore B'D = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$$

$$B'D = \sqrt{(\frac{3}{13} - \frac{3}{5})^2 + (\frac{-2}{13} - 0)^2}$$

$$B'D = \sqrt{(\frac{15-39}{65})^2 + (\frac{4}{169})}$$

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$$B'D = \sqrt{(\frac{-24}{65})^2 + \frac{4}{169}}$$

$$B'D = \sqrt{\frac{576}{4225} + \frac{4}{169}}$$

$$B'D = \sqrt{\frac{114244}{714025}}$$

$$B'D = \frac{2}{5} \quad \text{(Proved)}$$

$$\therefore C'D = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$$

$$C'D = \sqrt{(1 - \frac{3}{5})^2 + (0-0)^2}$$

$$C'D = \sqrt{(\frac{5-3}{5})^2}$$

$$C'D = \sqrt{(\frac{2}{5})^2}$$

$$C'D = \frac{2}{5} \quad \text{(Proved)}$$

Example 13

A circle $|z| = 1$ in the Z-plane is mapped onto the W-plane by $\mathbf{w} = \frac{1}{\mathbf{z-2}}$

Solution: from figure 34

$OP = |z| = \sqrt{x^2 + y^2}$

Given,

$|z| = 1$

$\sqrt{x^2 + y^2} = 1$

$\therefore x^2 + y^2 = 1$

$(x-0)^2 + (y-0)^2 = 1^2$ ----------------(i)

[We have, $(x-a)^2 + (y-b)^2 = r^2$]

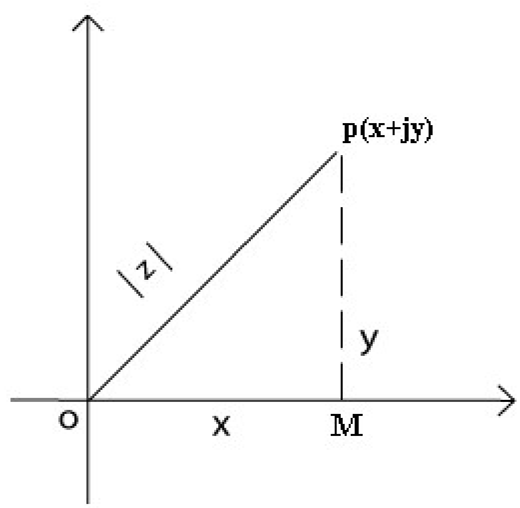
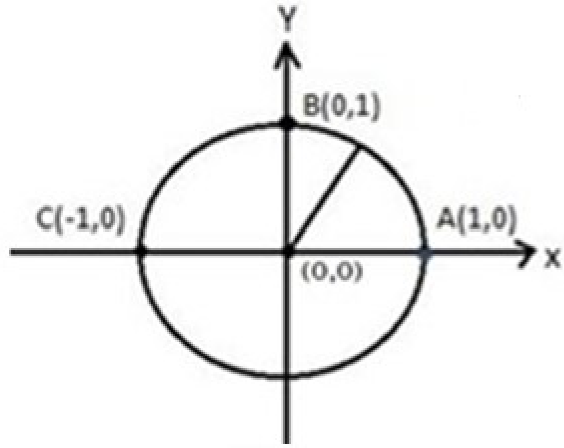


Figure 34

Which is the equation of a circle whose Center (0, 0), Radius=1



Z-Plane

Figure 35

From (i)

$$(x-0)^2+(y-0)^2=1^2$$

$$x^2+y^2=1$$

$$x^2+y^2-1=0 \qquad \text{--------------------------------(ii)}$$

Given,

$$w=\frac{1}{z-2}$$

$$z-2=\frac{1}{w}$$

$$\therefore z=\frac{1}{w}+2$$

i.e.

$$x+jy=\frac{1}{w}+2 \qquad [z=x+jy]$$

$$x+jy=\frac{1}{u+jv}+2 \qquad [w=u+jv]$$

$$x+jy=\frac{u-jv}{(u+jv)(u-jv)}+2 \qquad \text{[Multiplying by } u-jv\text{]}$$

$$x+jy=\frac{u-jv}{u^2-(jv)^2}+2$$

$$x+jy=\frac{u-jv}{u^2-j^2v^2}+2$$

$$x+jy=\frac{u-jv}{u^2+v^2}+2 \qquad [j^2=-1]$$

$$\therefore x-2+jy=\frac{u-jv}{u^2+v^2}$$

$$\therefore x - 2 + jy = \frac{u}{u^2 + v^2} - j\frac{v}{u^2 + v^2} \text{-------------------------(iii)}$$

Equating the co-efficient of real and imaginary part on both sides, we get,

$$x - 2 = \frac{u}{u^2 + v^2};\ y = \frac{-v}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2} + 2\ ;\ y = \frac{-v}{u^2 + v^2} \text{ ------------------------------(iv)}$$

Substituting these values x and y in equation (ii);

$$x^2 + y^2 - 1 = 0$$

$$\left(\frac{u}{u^2 + v^2} + 2\right)^2 + \left(\frac{-v}{u^2 + v^2}\right)^2 - 1 = 0$$

$$\left\{\frac{u + 2(u^2 + v^2)}{u^2 + v^2}\right\}^2 + \frac{v^2}{(u^2 + v^2)^2} = 1$$

$$\frac{\{u + 2(u^2 + v^2)\}^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} = 1$$

$$\frac{\{u + 2(u^2 + v^2)\}^2 + v^2}{(u^2 + v^2)^2} = 1$$

$$\{u + 2(u^2 + v^2)\}^2 + v^2 = (u^2 + v^2)^2$$

$$u^2 + 2 \times u \times 2(u^2 + v^2) + \{2(u^2 + v^2)\}^2 + v^2 = (u^2 + v^2)^2$$

$$u^2 + 4u(u^2 + v^2) + \{2(u^2 + v^2)\}^2 + v^2 = (u^2 + v^2)^2$$

$$u^2 + v^2 + 4u(u^2 + v^2) + \{2(u^2 + v^2)\}^2 = (u^2 + v^2)^2$$

$$(u^2 + v^2) + 4u(u^2 + v^2) + \{2(u^2 + v^2)\}^2 = (u^2 + v^2)^2$$

$$(u^2 + v^2) + 4u(u^2 + v^2) + 4(u^2 + v^2)^2 = (u^2 + v^2)^2$$

$$1 + 4u + 4(u^2 + v^2) = u^2 + v^2 \qquad \text{[Dividing by } u^2 + v^2\text{]}$$

$$1 + 4u + 4(u^2 + v^2) - (u^2 + v^2) = 0$$

$$1 + 4u + 3(u^2 + v^2) = 0$$

$$3(u^2 + v^2) + 4u + 1 = 0$$

$$(u^2 + v^2) + \frac{4}{3}u + \frac{1}{3} = 0$$

$$u^2 + \frac{4}{3}u + v^2 + \frac{1}{3} = 0$$

$$u^2 + \frac{4}{3}u + v^2 + \frac{1}{3} = 0$$

$$u^2 + v^2 + \frac{4}{3}u + \frac{1}{3} = 0$$

$$u^2 + v^2 + 2.\frac{2}{3}.u + 2.0.v + \frac{1}{3} = 0 \text{ -----------------------------(v)}$$

We know the general equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Whose centre is $(-g,-f)$ and radius is $\sqrt{g^2 + f^2 - c}$

Hence from (v)

Here $g = \frac{2}{3}$, $f = 0$ and $c = \frac{1}{3}$

The centre of new circle in w-plane is $(-g,-f) = ((-\frac{2}{3}),-0) = (-\frac{2}{3},0)$

Radius is $\sqrt{g^2 + f^2 - c} = \sqrt{(\frac{2}{3})^2 + 0^2 - \frac{1}{3}}$

$$= \sqrt{\frac{4}{9} - \frac{1}{3}}$$

$$= \sqrt{\frac{4-3}{9}}$$

$$= \sqrt{\frac{1}{9}}$$

$$= \frac{1}{3}$$

Now, we have to find out the image point A', B' *and* C'

Taking three sample points from figure 35, $A(1,0)$, $B(0,1)$ & $C(-1,0)$

Given,

$$w = \frac{1}{z-2}$$

$$w = \frac{1}{x + jy - 2} \qquad [z = x + jy]$$

$$= \frac{1}{x - 2 + jy}$$

$$= \frac{x - 2 - jy}{(x - 2 + jy)(x - 2 - jy)} \qquad \text{[Multiplying by } x - 2 - jy \text{]}$$

$$= \frac{x - 2 - jy}{(x-2)^2 - (jy)^2}$$

$$= \frac{x - 2 - jy}{(x-2)^2 + y^2} \qquad [[j^2 = -1;\ a^2 - b^2 = (a+b)(a-b)]]$$

$$w = \frac{x - 2 - jy}{(x-2)^2 + y^2}$$

$$w = \frac{x-2}{(x-2)^2 + y^2} - j\frac{y}{(x-2)^2 + y^2}$$

$$u + jv = \frac{x-2}{(x-2)^2 + y^2} - j\frac{y}{(x-2)^2 + y^2} \qquad [w = u + jv]$$

$$u + jv = \frac{x-2}{(x-2)^2 + y^2} - j\frac{y}{(x-2)^2 + y^2} \qquad \text{-----------------------(vi)}$$

Equating the co-efficient of real and imaginary part from (vi), we get,

$$u = \frac{x-2}{(x-2)^2 + y^2} \qquad \text{-----------------------------(vii)}$$

$$v = \frac{-y}{(x-2)^2 + y^2} \qquad \text{-----------------------------(viii)}$$

For $A(1,0)$: we get from vii & viii

$$u = \frac{x-2}{(x-2)^2 + y^2} = \frac{1-2}{(1-2)^2 + 0^2} = -\frac{1}{1} = -1$$

$$v = \frac{-y}{(x-2)^2 + y^2} = \frac{-0}{(1-2)^2 + 0^2} = 0$$

$\therefore A'(w = u + jv = -1 + j.0)$ is the image of A.

That is $A'(-1,0)$ -----------------------------(ix)

For $B(0,1)$: we get from vii & viii

$$u = \frac{x-2}{(x-2)^2 + y^2} = \frac{0-2}{(0-2)^2 + 1^2} = -\frac{2}{5}$$

$$v = \frac{-y}{(x-2)^2 + y^2} = \frac{-1}{(0-2)^2 + 1^2} = -\frac{1}{5}$$

$\therefore B'(w = u + jv = -\frac{2}{5} - \frac{1}{5}j)$ is the image of B.

That is $B'(-\frac{2}{5}, -\frac{1}{5})$ -----------------------------(x)

For $C(-1,0)$: we get from vii & viii

$$u = \frac{x-2}{(x-2)^2 + y^2} = \frac{-1-2}{(-1-2)^2 + 0^2} = -\frac{3}{9} = -\frac{1}{3}$$

$$v = \frac{-y}{(x-2)^2 + y^2} = \frac{-0}{(-1-2)^2 + 0^2} = 0$$

$\therefore C'(w = u + jv = -\frac{1}{3} + j.0)$ is the image of C.

That is $C'(-\frac{1}{3}, 0)$ -----------------------------(xi)

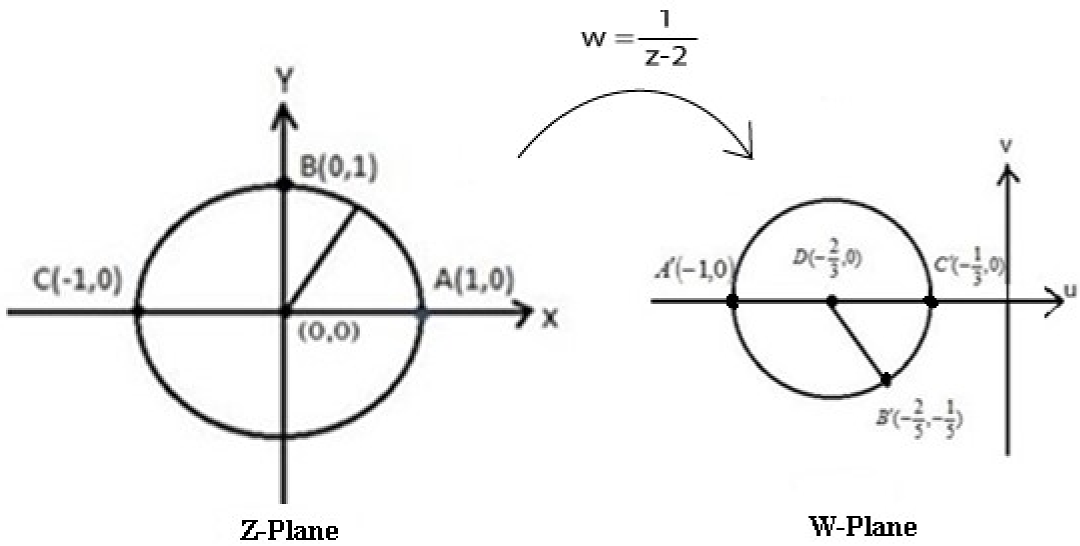


Figure 36

Justification:

We have, $A'(-1,0)$ $B'(-\frac{2}{5},-\frac{1}{5})$ $C'(-\frac{1}{3},0)$ and the radius & centre of the new circle (v) is Radius $=\frac{1}{3}$ and centre $D(-\frac{2}{3},0)$

We know the length of a line between two points (x_1,y_1) and (x_2,y_2) is:

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Here, in the w-plane

$$\therefore A'D=\sqrt{(u_1-u_2)^2+(v_1-v_2)^2}$$

$$=\sqrt{\left(-1-(-\frac{2}{3})\right)^2+(0-0)^2}$$

$$=\sqrt{\left(-1+\frac{2}{3})\right)^2+0}=\sqrt{\left(\frac{-3+2}{3}\right)^2}$$

$$=\sqrt{\left(\frac{-1}{3}\right)^2}=\sqrt{\frac{1}{9}}=\frac{1}{3}$$

$$\therefore B'D=\sqrt{(u_1-u_2)^2+(v_1-v_2)^2}$$

$$=\sqrt{\left(-\frac{2}{5}-(-\frac{2}{3})\right)^2+\left(-\frac{1}{5}+0\right)^2}$$

$$=\sqrt{\left(-\frac{2}{5}+\frac{2}{3}\right)^2+\left(-\frac{1}{5}+0\right)^2}$$

$$= \sqrt{\left(\frac{-6+10}{15}\right)^2 + \frac{1}{25}}$$

$$= \sqrt{\frac{16}{225} + \frac{1}{25}}$$

$$= \sqrt{\frac{16+9}{225}} = \sqrt{\frac{25}{225}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

$$\therefore C'D = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$$

$$= \sqrt{\left(-\frac{1}{3} - (-\frac{2}{3})\right)^2 + (0-0)^2}$$

$$= \sqrt{\left(-\frac{1}{3} + \frac{2}{3}\right)^2 + (0-0)^2}$$

$$= \sqrt{\left(\frac{-1+2}{3}\right)^2} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Example 14: Determine the image in w-plan of circle $|z| = 2$ in the z-plane under the transformation $w = \frac{1}{z-2}$ and show the region in w-plan onto which the region within the circle is mapped.

Answer: from figure 37

$OP = |z| = \sqrt{x^2 + y^2}$

Given,

$|z| = 2$

$\sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 2^2$ [Squaring]

$\therefore x^2 + y^2 = 4$

$(x-0)^2 + (y-0)^2 = 2^2$ ---------------------(i)

[We have, $(x-a)^2 + (y-b)^2 = r^2$]

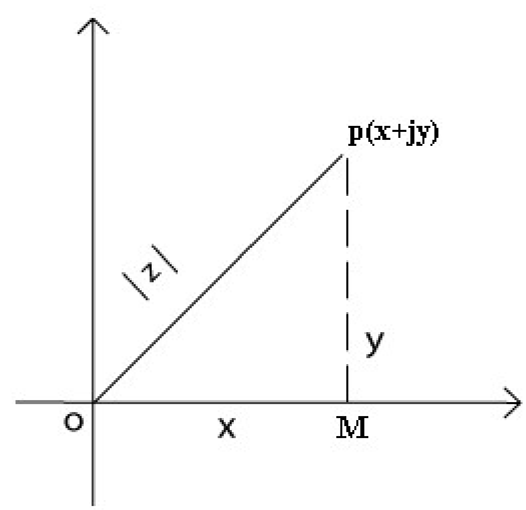


Figure no 37

The equation (i) represents a circle whose Center (0, 0) & radius = 2

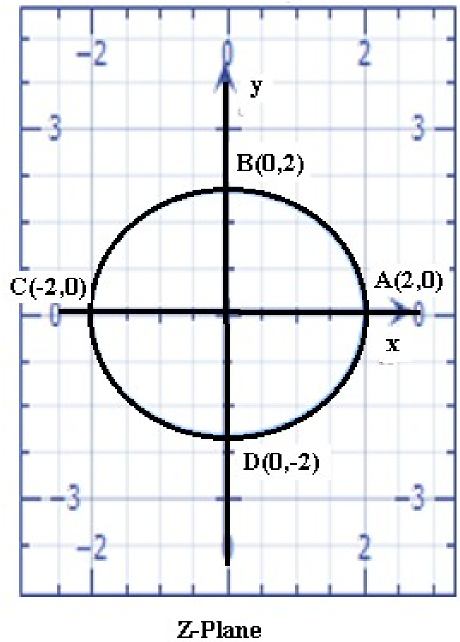


Figure no 38

Again, from (i)

$$(x-0)^2+(y-0)^2=2^2$$

$$x^2+y^2=4$$

$$x^2+y^2-4=0 \qquad \text{-----------------------(ii)}$$

Given,

$$w=\frac{1}{z-2}$$

$$\Rightarrow z-2=\frac{1}{w}$$

$$\Rightarrow \quad z=\frac{1}{w}+2$$

$$\Rightarrow x+jy=\frac{1}{u+jv}+2 \qquad [z=x+jy]\,\&\,[w=u+jv]$$

$$\Rightarrow x+jy=\frac{u-jv}{(u+jv)(u-jv)}+2$$

$$\Rightarrow x+jy=\frac{u-jv}{u^2-(jv)^2}+2$$

$$\Rightarrow x+jy=\frac{u-jv}{u^2-j^2v^2}+2$$

$$\Rightarrow x+jy=\frac{u-jv}{u^2+v^2}+2 \qquad [\therefore j^2=-1]$$

$$\Rightarrow x + jy - 2 = \frac{u - jv}{u^2 + v^2}$$

$$\Rightarrow x + jy - 2 = \frac{u}{u^2 + v^2} - j\frac{v}{u^2 + v^2}$$

$$\therefore x - 2 + jy = \frac{u}{u^2 + v^2} - j\frac{v}{u^2 + v^2} \qquad \text{-------------------------(iii)}$$

Now, equating real and imaginary part,

$$x - 2 = \frac{u}{u^2 + v^2}$$

$$\therefore x = \frac{u}{u^2 + v^2} + 2 \qquad \text{---------------------(iv)}$$

&

$$y = -\frac{v}{u^2 + v^2} \qquad \text{------------------------(v)}$$

Substituting these values x and y in equation (ii)

$$x^2 + y^2 - 4 = 0$$

$$\Rightarrow (\frac{u}{u^2 + v^2} + 2)^2 + (\frac{v}{u^2 + v^2})^2 - 4 = 0$$

$$\Rightarrow \left[\frac{u + 2(u^2 + v^2)}{u^2 + v^2}\right]^2 + \frac{v^2}{(u^2 + v^2)^2} - 4 = 0$$

$$\Rightarrow \frac{\{u + 2(u^2 + v^2)\}^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} = 4$$

$$\Rightarrow \frac{\{u + 2(u^2 + v^2)\}^2 + v^2}{(u^2 + v^2)^2} = 4$$

$$\Rightarrow \{u + 2(u^2 + v^2)\}^2 + v^2 = 4(u^2 + v^2)^2$$

$$\Rightarrow u^2 + 2 \times u \times 2(u^2 + v^2) + \{2(u^2 + v^2)\}^2 + v^2 = 4(u^2 + v^2)^2$$

$$\Rightarrow u^2 + 4u(u^2 + v^2) + 4(u^2 + v^2)^2 + v^2 = 4(u^2 + v^2)^2$$

$$\Rightarrow (u^2 + v^2) + 4u(u^2 + v^2) + 4(u^2 + v^2)^2 = 4(u^2 + v^2)^2$$

$$\Rightarrow 1 + 4u + 4(u^2 + v^2) = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4u + 4(u^2 + v^2) - 4(u^2 + v^2) = 0$$

$$\Rightarrow 1 + 4u = 0$$

$$\Rightarrow 4u = -1$$

$$\therefore u = -\frac{1}{4} \qquad ---------------(vi)$$

Which is the equation of a straight line parallel to the y-axis in the w-plane

Now, we have to find out the image point A', B', C' & D' for A (2, 0), B (0, 2), C (-2, 0) & D (0, -2) from figure no 38

We have, given

$$w = \frac{1}{z-2}$$

$$w = \frac{1}{x + jy - 2} \qquad [z = x + jy]$$

$$= \frac{1}{x-2+jy}$$

$$= \frac{x-2-jy}{(x-2+jy)(x-2-jy)} \qquad \text{[Multiplying by x-2-jy on numerator and denominator]}$$

$$= \frac{x-2-jy}{(x-2)^2-(jy)^2}$$

$$w = \frac{x-2-jy}{(x-2)^2-(jy)^2}$$

$$w = \frac{x-2-jy}{(x-2)^2+y^2} \qquad [\because j^2 = -1]$$

$$\Rightarrow u + jv = \frac{x-2-jy}{(x-2)^2+y^2} \qquad [w = u + jv]$$

$$\Rightarrow u + jv = \frac{x-2}{(x-2)^2+y^2} - j\frac{y}{(x-2)^2+y^2} \text{ ----------------------(vii)}$$

Now equating real and imaginary part

$$u = \frac{x-2}{(x-2)^2+y^2} \text{ ----------------------------(viii)}$$

$$v = \frac{-y}{(x-2)^2+y^2} \text{ ----------------------------(ix)}$$

For A (2, 0); we get from viii & ix

$$\Rightarrow u = \frac{2-2}{(2-2)^2+0^2} = \frac{0}{0} \quad \text{; undefined}$$

$$\Rightarrow v = -\frac{0}{(2-2)^2+0^2} = \frac{0}{0} \quad \text{; undefined}$$

So the image of A is A' is undefined.

For B (0, 2); we get from viii & ix

$$\Rightarrow u = \frac{0-2}{(0-2)^2+2^2} = \frac{-2}{4+4} = -\frac{2}{8} = -\frac{1}{4}$$

$$\Rightarrow v = -\frac{2}{(0-2)^2+2^2} = -\frac{2}{4+4} = -\frac{2}{8} = -\frac{1}{4}$$

So the image of B is $\mathbf{B'}(w = u + jv = -\frac{1}{4} - j\frac{1}{4})$

That is $\mathbf{B'}(-\frac{1}{4}, -\frac{1}{4})$ ----------------------(x)

For C (-2, 0); we get from viii & ix

$$\Rightarrow \mathbf{u} = \frac{-2-2}{(-2-2)^2+0^2} = -\frac{4}{16} = -\frac{1}{4}$$

$$\Rightarrow \mathbf{v} = -\frac{0}{(-2-2)^2+0^2} = -\frac{0}{16} = 0$$

So the image of C is $\mathbf{C'(w = u + jv = -\frac{1}{4} + j.0)}$

That is $\mathbf{C'(-\frac{1}{4},0)}$ -----------------(xi)

For D (0,-2) ; we get from viii & ix

$$\Rightarrow \mathbf{u} = \frac{0-2}{(0-2)^2+(-2)^2} = \frac{-2}{4+4} = \frac{-2}{8} = -\frac{1}{4}$$

$$\Rightarrow \mathbf{v} = -\frac{-2}{(0-2)^2+(-2)^2} = \frac{2}{4+4} = \frac{2}{8} = \frac{1}{4}$$

So the image of D is $\mathbf{D'(w = u + jv = -\frac{1}{4} + j\frac{1}{4})}$

That is $\mathbf{D'(-\frac{1}{4},\frac{1}{4})}$ ---------------------(xii)

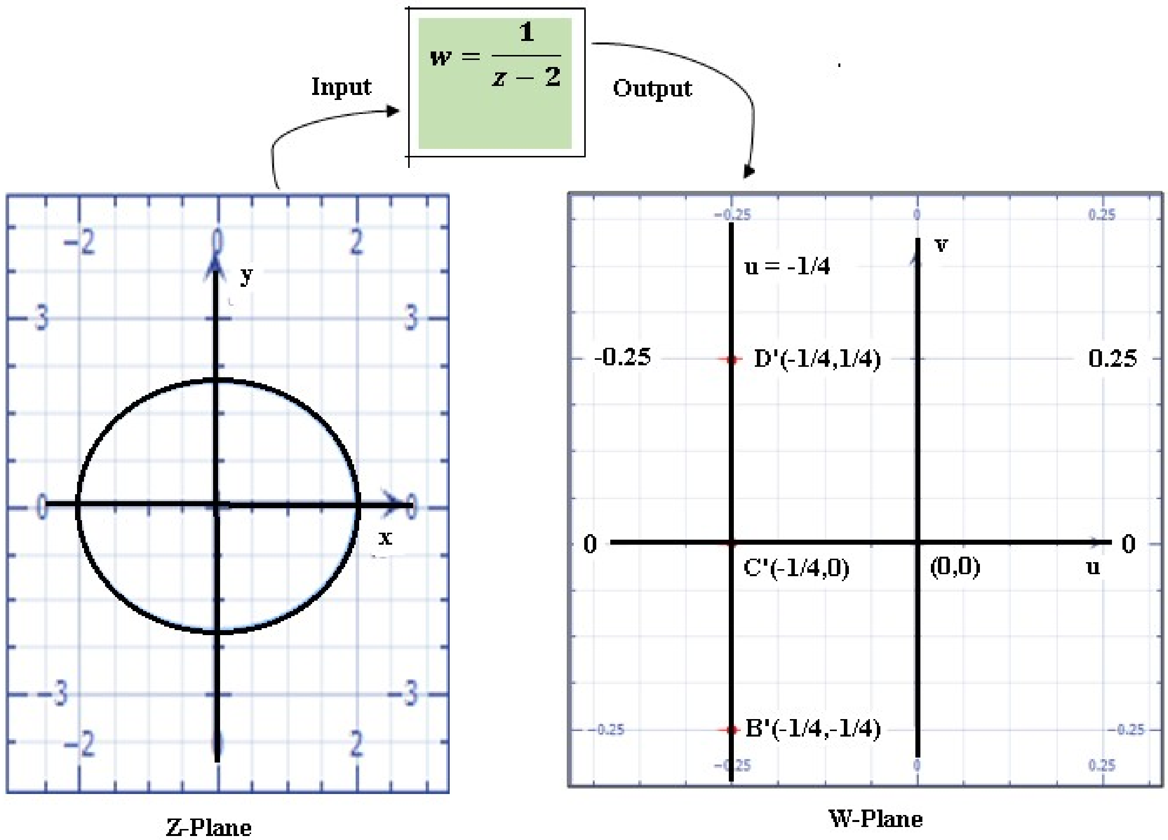
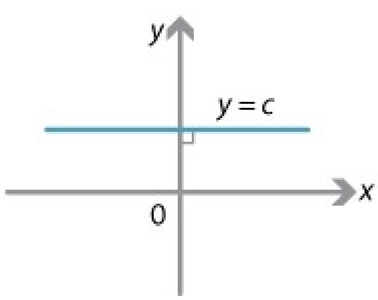
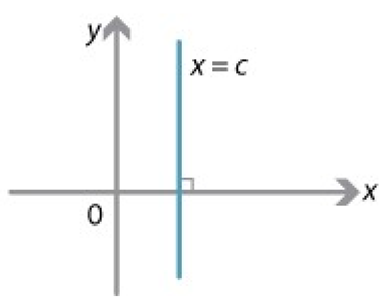


Figure 39

y = 0.x +1

x	-2	-3	-1	0	1	2	3	-4	4
y	1	1	1	1	1	1	1	1	1





Home Task:

01. Determine the image in the w-plane of the circle $|\mathbf{z}| = \mathbf{2}$ in the z-plane under the transformation $\mathbf{w} = \dfrac{\mathbf{z+j}}{\mathbf{z-j}}$ and show the region in the w-plane onto which the region within the circle is mapped.
02. Draw a graph of $y = \dfrac{1}{x}$

Answer:

x	-2	-1.5	-1	-0.5	-0.4	-0.2	0	0.2	0.4	0.5	1	1.5	2
$y = \dfrac{1}{x}$	-0.5	-0.6	-1	-2	-2.5	-5	inf	5	2.5	2	1	0.6	0.5

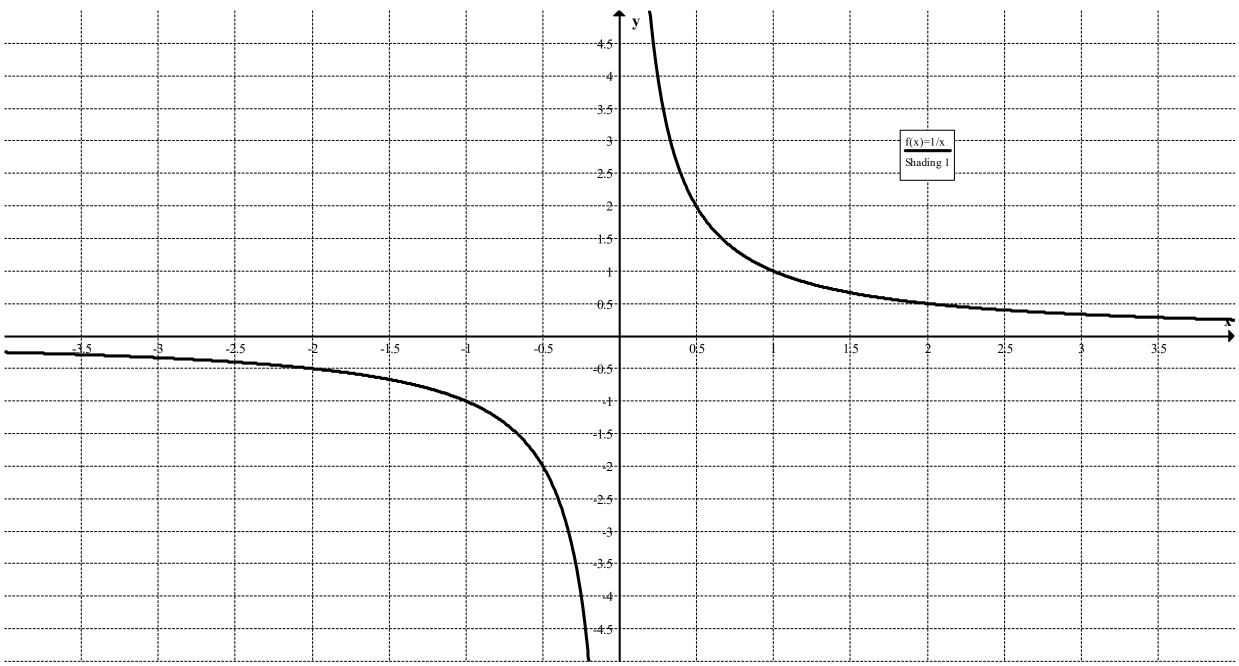


Figure 40: Graph of $y = \dfrac{1}{x}$

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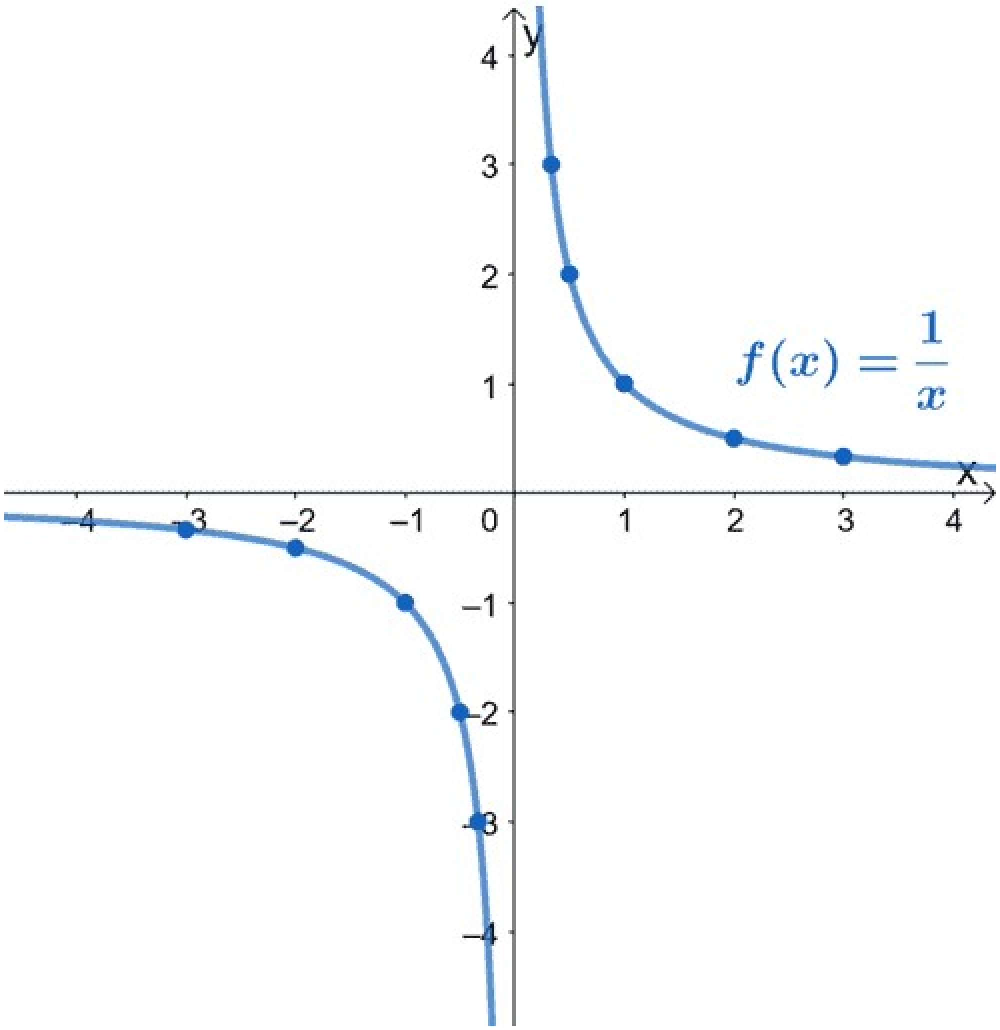

Figure 41

Regular Function/ Analytic Function:

A function $\mathbf{w = f(z)}$ is said to be regular or analytic at a point $\mathbf{z = z_0}$, if it is defined and single-valued and has a derivative (rate of change) at every point at and around $\mathbf{z = z_0}$

Singular Point:

A point at which **a function** $\mathbf{f(z)}$ **is not analytic** is known as a singular point or singularity of the function.

For example, the function $\mathbf{\frac{1}{z-2}}$ has a singular point at $\mathbf{z = 2}$.

$[\because \frac{1}{z-2} = \frac{1}{2-2} = \frac{1}{0} = \infty]$

Points in a region where $\mathbf{f(z)}$ ceases (বিরত থাকা) to be regular (disjoint/discontinuous/disconnected/irregular) are called singular points or singularities.

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The point at which the function is not differentiable is called a singular point of the function.

Necessary Condition for $f(z)$ to be Analytic: The necessary and sufficient condition for a function $f(z) = u + jv$ to be analytic at all the points in a region R are:

Cauchy-Riemann Test

We have Cauchy-Riemann Equations are

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \qquad \textit{and} \qquad \frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$$

We said earlier that where a function fails to be regular, a singular point or singularity occurs. i.e. where $w = f(z)$ is not continuous or where the **Cauchy-Riemann Test fails.**

Determine where each of the following functions fails to be regular, i.e. where singularities occur.

As for example

$$w = f(z) = \frac{1}{(z-2)(z-3)}$$

Singularities at $z = 2$ and $z = 3$ *Answer*

Example 15

Determine the function $w = f(z) = z^2 - 4$ is analytic or not.

Answer:

Given,

$$w = f(z) = z^2 - 4$$

$$\Rightarrow u + iv = (x + iy)^2 - 4 \qquad [\because w = u + iv \ \& \ z = x + iy]$$

$$\Rightarrow u + iv = x^2 + 2xiy + i^2y^2 - 4$$

$$\Rightarrow u + iv = x^2 - y^2 - 4 + 2xiy \qquad [i^2 = -1] \text{--------------------------(i)}$$

Equating real and imaginary part,

$$u = x^2 - y^2 - 4 \text{-------------------------------------(ii)}$$

$$v = 2xy \qquad \text{------------------------------------(iii)}$$

From (ii)

$$u = x^2 - y^2 - 4$$

$$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x}(x^2 - y^2 - 4)$$

$$\frac{\delta u}{\delta x} = 2x - 0 - 0$$

$$\frac{\delta u}{\delta x} = 2x \qquad \text{------------------------------------(iv)}$$

Again

$\mathbf{u = x^2 - y^2 - 4}$

$$\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}(x^2 - y^2 - 4)$$

$$\frac{\delta u}{\delta y} = 0 - 2y - 0$$

$$\frac{\delta u}{\delta y} = -2y \text{ ---(v)}$$

From (iii)

$\mathbf{v = 2xy}$

$$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}(2xy)$$

$$\frac{\delta v}{\delta x} = 2y \text{ ---(vi)}$$

Again

$\mathbf{v = 2xy}$

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}(2xy)$$

$$\frac{\delta v}{\delta y} = 2x \text{ ---(vii)}$$

We have Cauchy-Riemann Equations are

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \qquad \& \qquad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \text{ ------(viii)}$$

Putting the values of $\frac{\delta u}{\delta x}, \frac{\delta v}{\delta y}, \frac{\delta u}{\delta y}, \frac{\delta v}{\delta x}$ in (viii)

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \qquad \& \qquad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

$$\Rightarrow 2x = 2x \qquad \Rightarrow -2y = -2y$$

L.H.S = R.H.S   **L.H.S = R.H.S**

Since the Cauchy-Riemann Equations are satisfied by the function $\mathbf{w = f(z) = z^2 - 4}$.
Hence the function $\mathbf{w = f(z) = z^2 - 4}$ is analytic.

Example 16

Determine the function $\mathbf{w = f(z) = z\bar{z}}$ is analytic or not.

Answer:

Given,

$$\mathbf{w = f(z) = z\bar{z}}$$

$\mathbf{u + iv = (x + iy)(x - iy)}$ $\qquad [\because \mathbf{w = u + iv \ \& \ z = x + iy \ \& \ \bar{z} = x - iy}]$

$\mathbf{u + iv = x^2 - xiy + xiy - i^2y^2}$

$\mathbf{u + iv = x^2 + y^2}$ $\qquad \mathbf{[i^2 = -1]}$ --------------------(i)

Equating real and imaginary part,

$\mathbf{u = x^2 + y^2}$ --(ii)

$\mathbf{v = 0}$ --(iii)

From (ii)

$\mathbf{u = x^2 + y^2}$

$$\frac{\delta \mathbf{u}}{\delta \mathbf{x}} = \frac{\delta}{\delta \mathbf{x}}(\mathbf{x^2 + y^2})$$

$\dfrac{\delta \mathbf{u}}{\delta \mathbf{x}} = \mathbf{2x}$ --(iv)

From (iii)

$\mathbf{v = 0}$

$\dfrac{\delta \mathbf{v}}{\delta \mathbf{x}} = \mathbf{0}$ --(v)

Again,

$\mathbf{u = x^2 + y^2}$

$$\frac{\delta \mathbf{u}}{\delta \mathbf{y}} = \frac{\delta}{\delta \mathbf{y}}(\mathbf{x^2 + y^2})$$

$\dfrac{\delta \mathbf{u}}{\delta \mathbf{y}} = \mathbf{2y}$ --(vi)

and

$\mathbf{v = 0}$

$\dfrac{\delta \mathbf{v}}{\delta \mathbf{y}} = \mathbf{0}$ --(vii)

We have Cauchy-Riemann Equations are

$\dfrac{\delta \mathbf{u}}{\delta \mathbf{x}} = \dfrac{\delta \mathbf{v}}{\delta \mathbf{y}}$ $\qquad \& \qquad$ $\dfrac{\delta \mathbf{u}}{\delta \mathbf{y}} = -\dfrac{\delta \mathbf{v}}{\delta \mathbf{x}}$ ---------------------------(viii)

Putting the values of $\dfrac{\delta \mathbf{u}}{\delta \mathbf{x}}, \dfrac{\delta \mathbf{v}}{\delta \mathbf{y}}, \dfrac{\delta \mathbf{u}}{\delta \mathbf{y}}, \dfrac{\delta \mathbf{v}}{\delta \mathbf{x}}$ in (viii)

$\dfrac{\delta \mathbf{u}}{\delta \mathbf{x}} = \dfrac{\delta \mathbf{v}}{\delta \mathbf{y}}$ $\qquad \& \qquad$ $\dfrac{\delta \mathbf{u}}{\delta \mathbf{y}} = -\dfrac{\delta \mathbf{v}}{\delta \mathbf{x}}$

$\Rightarrow \mathbf{2x = 0}$ $\qquad$ $\Rightarrow \mathbf{2y = -0}$

L.H.S ≠ R.H.S $\qquad$ **L.H.S ≠ R.H.S**

Since the Cauchy-Riemann Equations are not satisfied by the function $\mathbf{w = f(z) = z\bar{z}}$.

Hence the function $\mathbf{w = f(z) = z\bar{z}}$ is not analytic.

Example 17

Determine the function $w = f(z) = e^{z} = e^{x}(\cos y + i\sin y)$ is analytic or not. ***Also find its derivative that is*** $f'(z) = ?$

Answer:

We have,

$$e^{x} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + ----------------$$

Put $x = ix$,

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + ---------$$

$[i^2 = -1;\ i^3 = i^2.i = -i;\ i^4 = i^2.i^2 = (-1).(-1) = +1;\ i^5 = i^4.i = i]$

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \frac{-ix^7}{7!} + ---------$$

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ---- + (\frac{ix^1}{1!} - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + ---------)$$

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ---- + i(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ---------)$$

$[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + -----;\quad \sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} ----------]$

$\therefore e^{ix} = \cos x + i\sin x$

$\therefore e^{iy} = \cos y + i\sin y$ --------------------------------(i)

We have,

$w = f(z) = e^{z} = e^{x+iy}$ $\qquad [z = x + iy]$

$w = f(z) = e^{z} = e^{x}.e^{iy}$

$w = f(z) = e^{x}(\cos y + i\sin y)$ $\qquad$ [From (i): $e^{iy} = \cos y + i\sin y$]

$w = u + iv = e^{x}\cos y + ie^{x}\sin y$ $\qquad [\because w = u + iv]$---------------------------(ii)

Equating real and imaginary part,

$u = e^{x}\cos y$ ------------------------------(iii)

$v = e^{x}\sin y$ ------------------------------(iv)

From (iii)

$u = e^{x}\cos y$

Differentiating (iii) partially with respect to x

$$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x}(e^{x}\cos y)$$

$$\frac{\delta u}{\delta x} = \cos y \frac{\delta}{\delta x}(e^x)$$

$$\frac{\delta u}{\delta x} = \cos y e^x$$

$$\frac{\delta u}{\delta x} = e^x \cos y \qquad \text{------------------------------(v)}$$

Again, Differentiating (iii) partially with respect to y

$$u = e^x \cos y$$

$$\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}(e^x \cos y)$$

$$\frac{\delta u}{\delta y} = e^x \frac{\delta}{\delta y}(\cos y)$$

$$\frac{\delta u}{\delta y} = -e^x \sin y \qquad \text{------------------------------(vi)}$$

From (iv),

$$v = e^x \sin y$$

Differentiating (iv) partially with respect to x

$$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}(e^x \sin y)$$

$$\frac{\delta v}{\delta x} = \sin y \frac{\delta}{\delta x}(e^x)$$

$$\frac{\delta v}{\delta x} = \sin y e^x$$

$$\frac{\delta v}{\delta x} = e^x \sin y \qquad \text{-------------------------------(vii)}$$

Again, Differentiating (iv) partially with respect to y

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}(e^x \sin y)$$

$$\frac{\delta v}{\delta y} = e^x \frac{\delta}{\delta y}(\sin y)$$

$$\frac{\delta v}{\delta y} = e^x \cos y \qquad \text{-------------------------------(viii)}$$

We have Cauchy-Riemann Equations are

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \qquad \& \qquad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \text{ ------(ix)}$$

Putting the values of $\frac{\delta u}{\delta x}, \frac{\delta v}{\delta y}, \frac{\delta u}{\delta y}, \frac{\delta v}{\delta x}$ in (ix)

$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$ & $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

$\Rightarrow e^x \cos y = e^x \cos y$ $\Rightarrow -e^x \sin y = -e^x \sin y$

L.H.S = R.H.S **L.H.S = R.H.S**

Since the Cauchy-Riemann Equations are satisfied by the function $w = f(z) = e^x(\cos y + i \sin y)$. Hence the function $w = f(z) = e^x(\cos y + i \sin y)$ is analytic.

2nd Part: We have,
From (ii)

$w = f(z) = u + iv = e^x \cos y + ie^x \sin y$

$w = f(z) = u + iv$ ----------------------------(x)

Differentiating (x) with respect to x

$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$

$f'(z) = e^x \cos y + ie^x \sin y$ [From (v) & (vii); $\frac{\delta u}{\delta x} = e^x \cos y, \frac{\delta v}{\delta x} = e^x \sin y$]

$f'(z) = e^x(\cos y + i \sin y)$

$f'(z) = e^x e^{iy}$ [From (i): $e^{iy} = \cos y + i \sin y$]

$f'(z) = e^{x+iy}$

$f'(z) = e^z$ Answer [$z = x + iy$]

Example 18
Determine the function $w = f(z) = z^3$ is analytic or not.
Answer:
Given,

$w = f(z) = z^3$

$\Rightarrow u + iv = (x + iy)^3$ [$\because w = u + iv$ & $z = x + iy$]

$\Rightarrow u + iv = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$ [$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$]

$\Rightarrow u + iv = x^3 + i3x^2y - 3xy^2 - iy^3$ [$i^2 = -1$]

$\Rightarrow u + iv = x^3 - 3xy^2 + i(3x^2y - y^3)$ ---------------------------(i)

Equating real and imaginary part

$u = x^3 - 3xy^2$ -----------------------------(ii)

$v = 3x^2y - y^3$ -----------------------------(iii)

Differentiating (ii) partially with respect to x

$u = x^3 - 3xy^2$

$$\frac{\delta u}{\delta x} = \frac{\delta}{\delta x}(x^3 - 3xy^2)$$

$$\frac{\delta u}{\delta x} = 3x^2 - 3y^2 \qquad \text{-----------------------------(iv)}$$

Again, Differentiating (ii) partially with respect to y

$$u = x^3 - 3xy^2$$

$$\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}(x^3 - 3xy^2)$$

$$\frac{\delta u}{\delta y} = \frac{\delta}{\delta y}(x^3) - 3x\frac{\delta}{\delta y}(y^2)$$

$$\frac{\delta u}{\delta y} = 0 - 6xy$$

$$\frac{\delta u}{\delta y} = -6xy \qquad \text{-----------------------------(v)}$$

Differentiating (iii) partially with respect to x

$$v = 3x^2y - y^3$$

$$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}(3x^2y - y^3)$$

$$\frac{\delta v}{\delta x} = \frac{\delta}{\delta x}(3x^2y) - \frac{\delta}{\delta x}(y^3)$$

$$\frac{\delta v}{\delta x} = y\frac{\delta}{\delta x}(3x^2) - \frac{\delta}{\delta x}(y^3)$$

$$\frac{\delta v}{\delta x} = 6xy - 0$$

$$\frac{\delta v}{\delta x} = 6xy \qquad \text{-----------------------------(vi)}$$

Again, Differentiating (iii) partially with respect to y

$$v = 3x^2y - y^3$$

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}(3x^2y - y^3)$$

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}(3x^2y) - \frac{\delta}{\delta y}(y^3)$$

$$\frac{\delta v}{\delta y} = 3x^2\frac{\delta}{\delta y}(y) - \frac{\delta}{\delta y}(y^3)$$

$$\frac{\delta v}{\delta y} = 3x^2.1 - 3y^2$$

$$\frac{\delta v}{\delta y} = 3x^2 - 3y^2 \qquad \text{-----------------------------(vii)}$$

We have Cauchy-Riemann Equations are

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \qquad \& \qquad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x} \text{----------------(viii)}$$

Putting the values of $\frac{\delta u}{\delta x}, \frac{\delta v}{\delta y}, \frac{\delta u}{\delta y}, \frac{\delta v}{\delta x}$ in (viii)

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \qquad \& \qquad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

$$\Rightarrow 3x^2 - 3y^2 = 3x^2 - 3y^2 \qquad \Rightarrow -6xy = -6xy$$

L.H.S = R.H.S **L.H.S = R.H.S**

Since the Cauchy-Riemann Equations are satisfied by the function $\mathbf{w = f(z) = z^3}$. Hence the function $\mathbf{w = f(z) = z^3}$ is analytic.

Home Task:

$\mathbf{f(z) = f(x+iy) = (x^3 - 3xy^2 - 2x) + i(3x^2y - y^3 - 2y)}$ is analytic or not.

Example 19

Derive Laplace's Equation from Cauchy-Riemann Equations

Answer:

We have Cauchy-Riemann Equations are

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \qquad \textit{and} \qquad \frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$$

That is

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \qquad \text{-----------------------------(i)}$$

$$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y} \qquad \text{-----------------------------(ii)}$$

Differentiating (i) with respect to x we get,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$$

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta x}\left(\frac{\delta v}{\delta y}\right)$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{\delta^2 v}{\delta x \delta y} \qquad \text{-----------------------------(iii)}$$

Differentiating (i) with respect to y we get,

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta x}\right) = \frac{\delta}{\delta y}\left(\frac{\delta v}{\delta y}\right)$$

$$\frac{\delta^2 u}{\delta y \delta x} = \frac{\delta^2 v}{\delta y^2}$$

$$\frac{\delta^2 v}{\delta y^2} = \frac{\delta^2 u}{\delta y \delta x} \qquad \text{-----------------------------(iv)}$$

Again

Differentiating (ii) with respect to x we get,

$$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$$

$$\frac{\delta}{\delta x}\left(\frac{\delta v}{\delta x}\right) = -\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta y}\right)$$

$$\frac{\delta^2 v}{\delta x^2} = -\frac{\delta^2 u}{\delta x \delta y} \qquad \text{-----------------------------(v)}$$

Differentiating (ii) with respect to y we get,

$$\frac{\delta v}{\delta x} = -\frac{\delta u}{\delta y}$$

$$\frac{\delta}{\delta y}\left(\frac{\delta v}{\delta x}\right) = -\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right)$$

$$\frac{\delta^2 v}{\delta y \delta x} = -\frac{\delta^2 u}{\delta y^2}$$

$$\frac{\delta^2 u}{\delta y^2} = -\frac{\delta^2 v}{\delta y \delta x} \qquad \text{-----------------------------(vi)}$$

Adding (iii) & (vi)

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \cancel{\frac{\delta^2 v}{\delta y \delta x}} - \cancel{\frac{\delta^2 v}{\delta x \delta y}}$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \qquad \text{-----------------------------(vii)}$$

Adding (iv) & (v)

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = \cancel{\frac{\delta^2 u}{\delta y \delta x}} - \cancel{\frac{\delta^2 u}{\delta x \delta y}}$$

$$\frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2} = 0 \qquad \text{-----------------------------(viii)}$$

The Equation no (vii) & (viii) are called Laplace's Equation

What is Harmonic Function?

Any function which satisfies the Laplace's Equation is known as a harmonic function. If $f(z) = u + jv$ is an analytic function, then u *and* v *are* both *harmonic function..*

A function $\mathbf{f(x,y,z)}$ is called a harmonic function if its second-order partial derivatives exist and if it satisfies Laplace's equation: $\frac{\delta^2 f}{\delta x^2}+\frac{\delta^2 f}{\delta y^2}+\frac{\delta^2 f}{\delta z^2}=0$

Example 20:

Prove that $u = x^2 - y^2$ and $v = \frac{y}{x^2+y^2}$ are harmonic functions of (x,y)

Answer: Given

$u = x^2 - y^2$ -------------------------------(i)

Differentiating (i) with respect to x

$$\frac{\delta u}{\delta x}=\frac{\delta}{\delta x}(x^2-y^2)$$

$$\frac{\delta u}{\delta x}=2x+0$$

$$\frac{\delta u}{\delta x}=2x \qquad \text{-------------------------------(ii)}$$

Again differentiating (ii) with respect to x

$$\frac{\delta u}{\delta x}=2x$$

$$\frac{\delta}{\delta x}\left(\frac{\delta u}{\delta x}\right)=\frac{\delta}{\delta x}(2x)$$

$$\frac{\delta^2 u}{\delta x^2}=2 \qquad \text{-------------------------------(iii)}$$

Now differentiating (i) with respect to y

$$u = x^2 - y^2$$

$$\frac{\delta u}{\delta y}=\frac{\delta}{\delta y}(x^2-y^2)$$

$$\frac{\delta u}{\delta y}=0-2y$$

$$\frac{\delta u}{\delta y}=0-2y \qquad \text{-------------------------------(iv)}$$

Again differentiating (iv) with respect to y

$$\frac{\delta u}{\delta y}=-2y$$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right)=-\frac{\delta}{\delta y}(2y)$$

$$\frac{\delta^2 u}{\delta y^2}=-2 \qquad \text{-------------------------------(v)}$$

Adding (iii) and (v)

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=2-2=0$$

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0$$

Since $u(x,y)=x^2-y^2$ satisfies Laplace's equation. Hence $u(x,y)=x^2-y^2$ is a harmonic function.

Now, given

$$v=\frac{y}{x^2+y^2} \quad \text{------------------------------(vi)}$$

Differentiating (vi) with respect to x

$$\frac{\delta v}{\delta x}=\frac{\delta}{\delta x}\left(\frac{y}{x^2+y^2}\right)$$

$$\frac{\delta v}{\delta x}=\frac{(x^2+y^2)\frac{\delta}{\delta x}(y)-y\frac{\delta}{\delta x}(x^2+y^2)}{(x^2+y^2)^2} \qquad [\because \frac{d}{dx}(\frac{u}{v})=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}]$$

$$=\frac{(x^2+y^2).0-y(2x+0)}{(x^2+y^2)^2}$$

$$=\frac{0-2xy}{(x^2+y^2)^2}$$

$$\frac{\delta v}{\delta x}=\frac{-2xy}{(x^2+y^2)^2} \quad \text{-------------------------------(vii)}$$

Again differentiating (vii) with respect to x

$$\frac{\delta}{\delta x}\left(\frac{\delta v}{\delta x}\right)=-\frac{\delta}{\delta x}\left\{\frac{2xy}{(x^2+y^2)^2}\right\}$$

$$\frac{\delta^2 v}{\delta x^2}=-\left[\frac{(x^2+y^2)^2\frac{\delta}{\delta x}(2xy)-2xy\frac{\delta}{\delta x}(x^2+y^2)^2}{\{(x^2+y^2)^2\}^2}\right]$$

$$\frac{\delta^2 v}{\delta x^2}=\frac{-(x^2+y^2)^2 2y\frac{\delta}{\delta x}(x)+2xy\times 2(x^2+y^2)^{2-1}\frac{\delta}{\delta x}(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\delta^2 v}{\delta x^2}=\frac{-(x^2+y^2)^2 2y.1+4xy(x^2+y^2)(2x+0)}{(x^2+y^2)^4}$$

$$\frac{\delta^2 v}{\delta x^2}=\frac{-(x^2+y^2)^2 2y+8x^2y(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\delta^2 v}{\delta x^2} = \frac{(x^2+y^2)[-(x^2+y^2)2y+8x^2y]}{(x^2+y^2)^3(x^2+y^2)}$$

$$\frac{\delta^2 v}{\delta x^2} = \frac{[-2x^2y-2y^3+8x^2y]}{(x^2+y^2)^3}$$

$$\frac{\delta^2 v}{\delta x^2} = \frac{6x^2y-2y^3}{(x^2+y^2)^3} \qquad \text{-------------------------------(viii)}$$

Again,

$$v = \frac{y}{x^2+y^2}$$

Differentiating with respect to y

$$\frac{\delta v}{\delta y} = \frac{\delta}{\delta y}\left(\frac{y}{x^2+y^2}\right)$$

$$\frac{\delta v}{\delta y} = \frac{(x^2+y^2)\frac{\delta}{\delta y}(y) - y\frac{\delta}{\delta y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\delta v}{\delta y} = \frac{(x^2+y^2).1 - y(0+2y)}{(x^2+y^2)^2}$$

$$\frac{\delta v}{\delta y} = \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}$$

$$\frac{\delta v}{\delta y} = \frac{(x^2-y^2)}{(x^2+y^2)^2} \qquad \text{-------------------------------(ix)}$$

Again, differentiating (ix) with respect to y

$$\frac{\delta}{\delta y}\left(\frac{\delta v}{\delta y}\right) = \frac{\delta}{\delta y}\left\{\frac{(x^2-y^2)}{(x^2+y^2)^2}\right\}$$

$$\frac{\delta^2 v}{\delta y^2} = \frac{(x^2+y^2)^2\frac{\delta}{\delta y}(x^2-y^2) - (x^2-y^2)\frac{\delta}{\delta y}(x^2+y^2)^2}{\{(x^2+y^2)^2\}^2}$$

$$\frac{\delta^2 v}{\delta y^2} = \frac{(x^2+y^2)^2(0-2y) - (x^2-y^2)\times 2(x^2+y^2)^{2-1}\frac{\delta}{\delta y}(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\delta^2 v}{\delta y^2} = \frac{(x^2+y^2)^2(0-2y) - (x^2-y^2)\times 2(x^2+y^2)(0+2y)}{(x^2+y^2)^4}$$

$$\frac{\delta^2 v}{\delta y^2}=\frac{(x^2+y^2)^2(-2y)-2(x^2-y^2)(x^2+y^2)(2y)}{(x^2+y^2)^4}$$

$$\frac{\delta^2 v}{\delta y^2}=\frac{(x^2+y^2)^2(-2y)-4y(x^2-y^2)(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\delta^2 v}{\delta y^2}=\frac{(x^2+y^2)[(x^2+y^2)(-2y)-4y(x^2-y^2)]}{(x^2+y^2)(x^2+y^2)^3}$$

$$\frac{\delta^2 v}{\delta y^2}=\frac{(x^2+y^2)(-2y)-4y(x^2-y^2)}{(x^2+y^2)^3}$$

$$\frac{\delta^2 v}{\delta y^2}=\frac{-2x^2y-2y^3-4x^2y+4y^3}{(x^2+y^2)^3}$$

$$\frac{\delta^2 v}{\delta y^2}=\frac{-6x^2y+2y^3}{(x^2+y^2)^3} \quad \text{-------------------------------(x)}$$

Adding (viii) and(x),

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}=\frac{6x^2y-2y^3}{(x^2+y^2)^3}+\frac{-6x^2y+2y^3}{(x^2+y^2)^3}$$

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}=\frac{6x^2y-2y^3-6x^2y+2y^3}{(x^2+y^2)^3}$$

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}=0$$

Since $\mathbf{v(x,y)=\frac{y}{x^2+y^2}}$ satisfies Laplace's equation. Hence $\mathbf{v(x,y)=\frac{y}{x^2+y^2}}$ is a harmonic function. ***(Proved)***

Example 21:

Prove that, $\mathbf{u=e^{-x}(x\sin y-y\cos y)}$ is a harmonic function.

Answer:

Given, $\mathbf{u=e^{-x}(x\sin y-y\cos y)}$**....................................(i)**

Differentiating (i) with respect to x,

$$\frac{\delta u}{\delta x}=\frac{\delta}{\delta x}[e^{-x}(x\sin y-y\cos y)]$$

$$\frac{\delta u}{\delta x}=e^{-x}\frac{\delta}{\delta x}(x\sin y-y\cos y)+(x\sin y-y\cos y)\frac{\delta}{\delta x}(e^{-x})\ [\frac{d}{dx}(uv)=u\frac{d}{dx}v+v\frac{d}{dx}u]$$

$$\frac{\delta u}{\delta x}=e^{-x}[\frac{\delta}{\delta x}(x\sin y)-\frac{\delta}{\delta x}(y\cos y)]-(x\sin y-y\cos y)e^{-x}\ \ [\because \frac{d}{dx}(e^{-x})=-e^{-x}]$$

$$\frac{\delta u}{\delta x}=e^{-x}[\{x\frac{\delta}{\delta x}\sin y+\sin y\frac{\delta}{\delta x}x\}-\{y\frac{\delta}{\delta x}\cos y+\cos y\frac{\delta}{\delta x}y\}]-e^{-x}(x\sin y-y\cos y)$$

$$\frac{\delta u}{\delta x}=e^{-x}[\{x(0)+\sin y.1\}-\{y(0)+\cos y.(0)\}]-e^{-x}(x\sin y-y\cos y)$$
$$\frac{\delta u}{\delta x}=e^{-x}[0+\sin y-0-0]-e^{-x}(x\sin y-y\cos y)$$
$$\frac{\delta u}{\delta x}=e^{-x}\sin y-e^{-x}(x\sin y-y\cos y)$$
$$\frac{\delta u}{\delta x}=e^{-x}\sin y-xe^{-x}\sin y+ye^{-x}\cos y$$
$$\frac{\delta u}{\delta x}=e^{-x}(\sin y-x\sin y+y\cos y)\text{....................................}(ii)$$
Again Differentiating (ii) with respect to x,
$$\frac{\delta u}{\delta x}=e^{-x}(\sin y-x\sin y+y\cos y)$$
$$\frac{\delta}{\delta x}(\frac{\delta u}{\delta x})=e^{-x}\frac{\delta}{\delta x}(\sin y-x\sin y+y\cos y)+(\sin y-x\sin y+y\cos y)\frac{\delta}{\delta x}e^{-x}$$
$$\frac{\delta^2 u}{\delta x^2}=e^{-x}\frac{\delta}{\delta x}(\sin y-x\sin y+y\cos y)+(-)e^{-x}(\sin y-x\sin y+y\cos y)$$
$$\frac{\delta^2 u}{\delta x^2}=e^{-x}[\frac{\delta}{\delta x}(\sin y)-\frac{\delta}{\delta x}(x\sin y)+\frac{\delta}{\delta x}(y\cos y)]-e^{-x}(\sin y-x\sin y+y\cos y)$$
$$\frac{\delta^2 u}{\delta x^2}=e^{-x}[0-\{x\frac{\delta}{\delta x}(\sin y)+\sin y\frac{\delta x}{\delta x}\}+y\frac{\delta}{\delta x}(\cos y)+\cos y\frac{\delta}{\delta x}y]-(\sin y-x\sin y+y\cos y)e^{-x}$$
$$\frac{\delta^2 u}{\delta x^2}=e^{-x}[0-x.0-\sin y.1+y.0+\cos y.0]-e^{-x}(\sin y-x\sin y+y\cos y)$$
$$\frac{\delta^2 u}{\delta x^2}=e^{-x}[0-0-\sin y+0+0]-e^{-x}(\sin y-x\sin y+y\cos y)$$
$$\frac{\delta^2 u}{\delta x^2}=e^{-x}(-\sin y-\sin y+x\sin y-y\cos y)$$
$$\frac{\delta^2 u}{\delta x^2}=e^{-x}(-2\sin y+x\sin y-y\cos y)\text{....................................}(iii)$$
Now differentiating (i) with respect to y,
$$u=e^{-x}(x\sin y-y\cos y)$$
$$\frac{\delta u}{\delta y}=e^{-x}\frac{\delta}{\delta y}(x\sin y-y\cos y)+(x\sin y-y\cos y)\frac{\delta}{\delta y}(e^{-x})$$
$$\frac{\delta u}{\delta y}=e^{-x}\{x\frac{\delta}{\delta y}(\sin y)+\sin y\frac{\delta x}{\delta y}-(y\frac{\delta}{\delta y}\cos y+\cos y\frac{\delta}{\delta y}(y))\}+(x\sin y-y\cos y).0$$
$$\frac{\delta u}{\delta y}=e^{-x}\{x\cos y+\sin y.0-(-y\sin y+\cos y.1)\}+0$$

$$\frac{\delta u}{\delta y} = e^{-x}(x\cos y + y\sin y - \cos y)\text{..................................(iv)}$$

Again differentiating (iv) with respect to y

$$\frac{\delta u}{\delta y} = e^{-x}(x\cos y + y\sin y - \cos y)$$

$$\frac{\delta}{\delta y}\left(\frac{\delta u}{\delta y}\right) = \frac{\delta}{\delta y}[e^{-x}(x\cos y + y\sin y - \cos y)]$$

$$\frac{\delta^2 u}{\delta y^2} = e^{-x}\frac{\delta}{\delta y}(x\cos y + y\sin y - \cos y) + (x\cos y + y\sin y - \cos y)\frac{\delta}{\delta y}(e^{-x})$$

$$\frac{\delta^2 u}{\delta y^2} = e^{-x}\left[\frac{\delta}{\delta y}(x\cos y) + \frac{\delta}{\delta y}(y\sin y) - \frac{\delta}{\delta y}(\cos y)\right] + (x\cos y + y\sin y - \cos y)\frac{\delta}{\delta y}(e^{-x})$$

$$\frac{\delta^2 u}{\delta y^2} = e^{-x}\left[x\frac{\delta}{\delta y}(\cos y) + \cos y\frac{\delta}{\delta y}x + y\frac{\delta}{\delta y}(\sin y) + \sin y\frac{\delta}{\delta y}y - \frac{\delta}{\delta y}(\cos y)\right] + (x\cos y + y\sin y - \cos y).0$$

$$\frac{\delta^2 u}{\delta y^2} = e^{-x}[-x\sin y + \cos y.0 + y\cos y + \sin y + \sin y] + 0$$

$$\frac{\delta^2 u}{\delta y^2} = e^{-x}[-x\sin y + y\cos y + 2\sin y]\text{......................................(v)}$$

Adding (iii) and (v),

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = e^{-x}[-2\sin y + x\sin y - y\cos y] + e^{-x}[2\sin y - x\sin y + y\cos y]$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = e^{-x}[-2\sin y + x\sin y - y\cos y + 2\sin y - x\sin y + y\cos y]$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

Since $u(x,y) = e^{-x}(x\sin y - y\cos y)$ satisfies Laplace's equation,

hence $u(x,y) = e^{-x}(x\sin y - y\cos y)$ is a Harmonic function.

Home Task

i. Prove that $f(z) = |z|^2$ is not harmonic functions but $f(z) = \ln(|z|^2)$ is harmonic.

ii. **Is** $f(x,y,z) = x^2 + y^2 - 2z^2$ harmonic? What about $f(x,y,z) = x^2 - y^2 + z^2$ **?**

iii. **Show that the function** $u(x,y) = 3x^3 - 9xy^2$ **is harmonic**

iv. **Verify** $u(x,y) = x^3 - 3xy^2 - 5y$ **is harmonic**

v. **Test the following functions harmonic or not**

   a) $u = x^3 - 3xy^2$

b) $u = e^{-y} \sin x$

c) $u = e^{x} \cos y$

d) $u(x,y) = 3x^2y + 2x^2 - y^3 - 2y^2$

e) $u(x,y) = e^{x}(x \cos y - y \sin y)$

f) $u = y^3 - 3x^2y$

g) $u = 2x(1-y)$

h) $u(x,y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

i) $u(x,y) = x^2 - y^2 - 2xy - 2x + 3y$

Example 22:

Find the value of integral $\int_0^{1+i} (x - y + ix^2)dz$

(a) Along straight line from $z = 0$ to $z = 1 + i$

(b) Along real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$.

Answer:

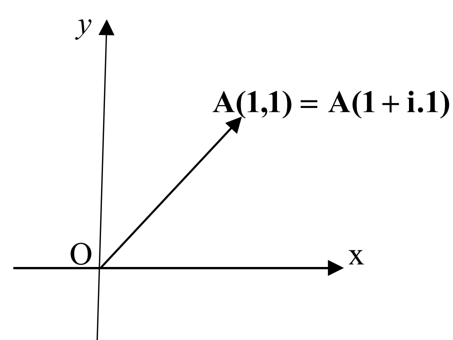

Figure 42

a) Along **OA** line:

Given,

$z = 0$

$x + iy = 0$ $\quad [z = x + iy]$

$x + iy = 0 + i.0$

Equating real and imaginary part,

$x = 0$ & $y = 0$

That is coordinate of O (0, 0)

Again

Given,

$z = 1 + i$

$x + iy = 1 + i.1$ $\quad [z = x + iy]$

Equating real and imaginary part,

$x = 1$ & $y = 1$

That is coordinate of A (1,1)

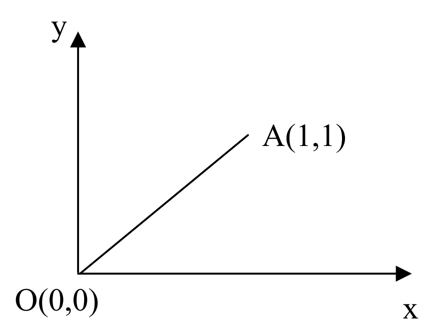

Figure 43

Now, The Equation of a straight line OA passing through $O(0,0)$ and $A(1,1)$ is

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$$

$$\Rightarrow \frac{y-0}{0-1}=\frac{x-0}{0-1}$$

$$\Rightarrow \frac{y}{-1}=\frac{x}{-1}$$

$$\Rightarrow y=x \qquad \text{....................}(i)$$

We have

$$z=x+iy$$

$$z=x+ix \qquad [\because y=x \qquad \text{from}(i)]$$

$$\frac{dz}{dx}=\frac{d}{dx}(x+ix)$$

$$\frac{dz}{dx}=1+i.1$$

$$dz=(1+i)dx \qquad \text{.................}(ii)$$

Now,

$$\int_{OA}(x-y+ix^2)dz$$

$$=\int_0^1(x-x+ix^2)(1+i)dx \qquad [\text{from(i)and(ii)}\ ; y=x \quad dz=(1+i)dx\,]$$

$$=\int_0^1(ix^2)(1+i)dx$$

$$=(1+i)\int_0^1 ix^2dx$$

$$=(1+i)i\int_0^1 x^2dx$$

$$=(1+i).i.[\frac{x^3}{3}]_0^1$$

$$=i(1+i)[\frac{1}{3}-\frac{0}{3}]$$

$$=i(1+i)[\frac{1}{3}]$$

$$=\frac{1}{3}(i-1) \qquad [i^2=-1]$$

b) Along OB and then along BA; Along OB from $z=0$ to $z=1$ and then along BA, from $z=1$ to $z=i+1$

Solution:
Along line OB
Given,
$z = 0$
$x + iy = 0$ $\quad [z = x + iy]$
$x + iy = 0 + i.0$
Equating real and imaginary part,
$x = 0 \ \& \ y = 0$
That is coordinate of O (0, 0)

Again,
Given,
$z = 1$
$x + iy = 1$ $\quad [z = x + iy]$
$x + iy = 1 + 0$
$x + iy = 1 + i.0$
Equating real and imaginary part,
$x = 1 \ \& \ y = 0$
That is coordinate of B (1,0)

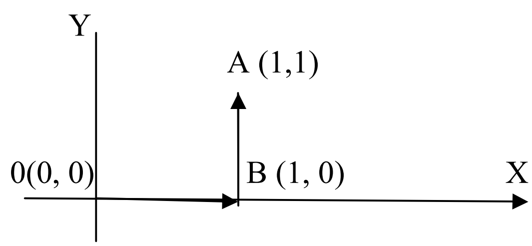

Figure 44

Required Integral,

$$\int_{OB} (x - y + ix^2)dz + \int_{BA} (x - y + ix^2)dz \quad \text{........................(iii)}$$

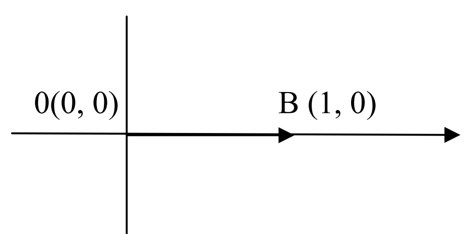

Figure 45

The Equation of OB is:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$$

$$\frac{y-0}{0-0}=\frac{x-0}{0-1}$$

$$-y=0$$

$$\therefore y=0 \quad \text{..........................(iv)}$$

We have,

$$z=x+iy$$

$$z=x+i.0 \qquad [\therefore y=0\ ]$$

$$z=x$$

$$\frac{dz}{dx}=\frac{d}{dx}(x)$$

$$\frac{dz}{dx}=1$$

$$dz=dx \quad \text{..........................(v)}$$

Now first part of (iii),
Hence,

$$\int_{OB}(x-y+ix^2)dz$$

$$=\int_0^1(x-0+ix^2)dx \qquad \text{[From (iv) and (v);}\ y=0;\ dz=dx]$$

$$=\int_0^1(x+ix^2)dx$$

$$=\left[\frac{x^2}{2}+i\frac{x^3}{3}\right]_0^1$$

$$=\frac{1}{2}+i\left(\frac{1}{3}\right)-\frac{0}{2}-i\frac{0}{3}$$

$$=\frac{1}{2}+i.\frac{1}{3}$$

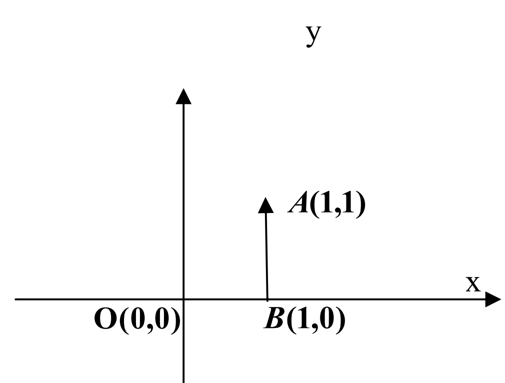

Figure 46

Along line BA:
The Equation of BA is:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}$$

$$\frac{y-0}{0-1}=\frac{x-1}{1-1}$$

$$\frac{y}{-1}=\frac{x-1}{0}$$

$$y.0=(x-1)(-1)$$

$$0=-x+1$$

$$-x=-1$$

$$x=1 \text{ ------------(vi)}$$

$$\therefore dx=0 \text{ ---------(vii)}$$ [Differentiating vi]

We have,

$$z=x+iy$$

$$z=1+iy \qquad [\text{from(vi); } x=1\ ]$$

$$z=1+iy$$

$$\frac{dz}{dy}=\frac{d}{dy}(1+iy)$$

$$\frac{dz}{dy}=0+i\frac{dy}{dy}$$

$$\frac{dz}{dy}=0+i.1$$

$$dz=idy\text{.......................................(viii)}$$

Second part of (iii)
Now,

$$\int_{BA}(x-y+ix^2)\,dz$$

$$=\int_0^1(1-y+i.1^2)\ idy \qquad [\text{from (vi),(viii); } x=1;\ dz=idy]$$

$$=\int_0^1(1-y+i)\ idy$$

$$=i\int_0^1(1-y+i)\,dy$$

$$= i\left[y - \frac{y^2}{2} + iy\right]_0^1$$

$$= i\left[y - \frac{y^2}{2} + iy\right]_0^1$$

$$= i\left[1 - \frac{1}{2} + i - (0 - \frac{0}{2} + 0)\right]$$

$$= i\left[\frac{1}{2} + i\right]$$

$$= \frac{i}{2} - 1 \text{...} \textit{(ix)}$$

putting result in (iii),

$$\int_{OB} (x - y + ix^2)\,dz + \int_{BA} (x - y + ix^2)\,dz$$

$$= \left(\frac{1}{2} + i\frac{1}{3}\right) + \left(\frac{i}{2} - 1\right)$$

$$= \frac{1}{2} + \frac{i}{3} + \frac{i}{2} - 1$$

$$= -\frac{1}{2} + \frac{5}{6}i \qquad \textbf{Answer}$$

What is pole: pole is a certain type of singularity of a function can be found by substituting the denominator of the function equal to zero. Roots of denominator indicates poles. That is, Poles represents the points where a complex function cease to be analytic.

Cauchy's Theorem

The theorem states that if $f(z)$ is analytic everywhere within a simply-connected region then $\oint_c f(z)dz = 0$ for every simple closed path C lying in the region.

Cauchy's Integral Formula:

If $f(z)$ is analytic inside and on the boundary C of a simply-connected region then for any point 'a' within the curve 'C': $\oint_c \frac{f(z)}{z - a}dz = 2\pi i \times f(a)$

Example 23:

Evaluate $\int_c \frac{z^2 - z + 1}{z - 1}dz$

Where c is the circle i) $|z| = 1$ ii) $|z| = \frac{1}{2}$

We have,

$$z = x + jy$$

$$\therefore |z| = \sqrt{x^2 + y^2}$$

Given,

i) $|z| = 1$

$$\sqrt{x^2 + y^2} = 1$$

$$\therefore x^2 + y^2 = 1$$

$$(x-0)^2 + (y-0)^2 = 1^2 \quad \text{----------------(i)}$$

[We have, $(x-a)^2 + (y-b)^2 = r^2$]

Which is the equation of a circle whose Center (0, 0), Radius =1

Let $f(z) = z^2 - z + 1$ and singularity point is $a = 1$

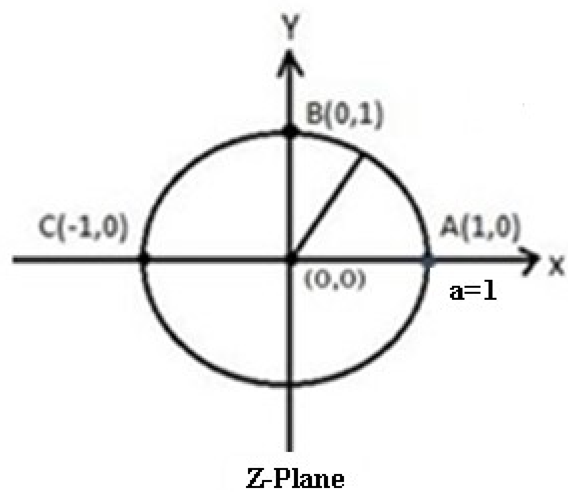

Figure 47

$\because$ $a = 1$ is on the circle c, then by Cauchy's Integral formula

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i \times f(a)$$

$$\Rightarrow \int_c \frac{z^2 - z + 1}{z-1} dz = 2\pi i \times f(1) \; [a = 1]$$

Here, $f(z) = z^2 - z + 1$

$$\therefore f(1) = 1^2 - 1 + 1 = 1$$

$$\Rightarrow \int_c \frac{z^2 - z + 1}{z-1} dz = 2\pi i \times f(1)$$

$$\Rightarrow \int_c \frac{z^2 - z + 1}{z - 1} dz = 2\pi i \times 1$$

$$\Rightarrow \int_c \frac{z^2 - z + 1}{z - 1} dz = 2\pi i \text{ } \textit{Answer}$$

Given,

i) $|z| = \frac{1}{2}$

$$\sqrt{x^2 + y^2} = \frac{1}{2}$$

$$\therefore x^2 + y^2 = \frac{1}{4}$$

$$(x - 0)^2 + (y - 0)^2 = (\frac{1}{2})^2 \qquad \text{----------------(i)}$$

[We have, $(x - a)^2 + (y - b)^2 = r^2$]

Which is the equation of a circle whose Center (0, 0), Radius $= \frac{1}{2}$

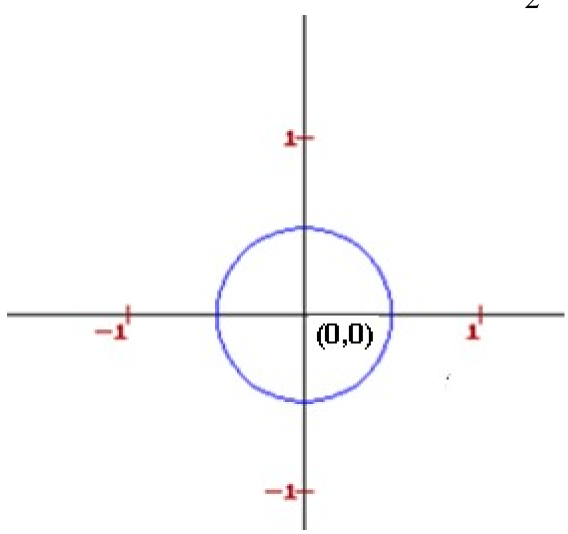

Figure 48

$\because$ $a = 1$ is outside the circle c, then by Cauchy's theorem

$$\int_c f(z) dz = 0$$

$$\therefore \int_c \frac{z^2 - z + 1}{z - 1} dz = 0$$

Example 24:

Evaluate $\int_c \frac{z}{z^2-3z+2}dz$

Where c is the circle $|z-2| = \frac{1}{2}$

We have,

$\mathbf{z = x + jy}$

$\mathbf{z - 2 = x + jy - 2}$

$\mathbf{z - 2 = x - 2 + jy}$

$\therefore |\mathbf{z-2}| = \sqrt{(\mathbf{x-2})^2 + \mathbf{y}^2}$

Given,

$|\mathbf{z-2}| = \frac{1}{2}$

$\therefore |\mathbf{z-2}| = \sqrt{(\mathbf{x-2})^2 + \mathbf{y}^2} = \frac{1}{2}$

$\therefore (\mathbf{x-2})^2 + \mathbf{y}^2 = \frac{1}{4}$

$\therefore (\mathbf{x-2})^2 + (\mathbf{y-0})^2 = (\frac{1}{2})^2$

Which is the equation of a circle whose Center (2, 0), Radius $= \frac{1}{2}$

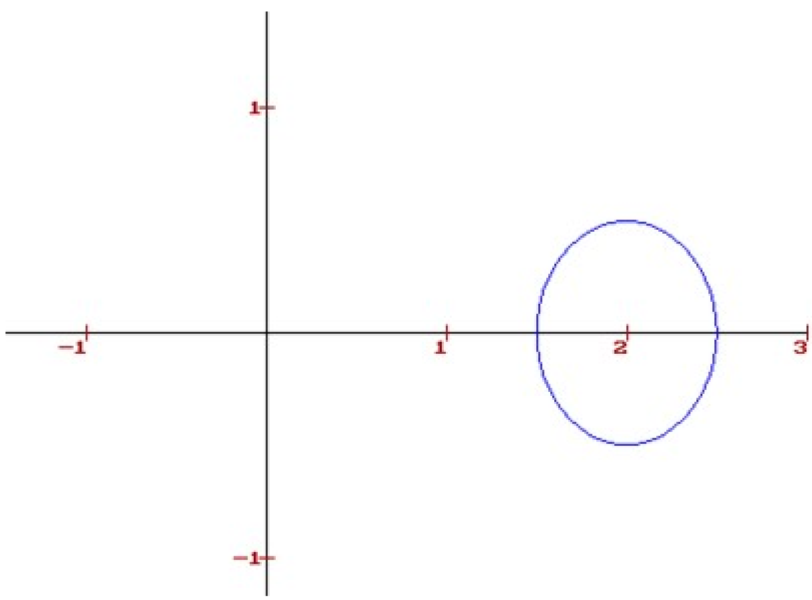

Figure 49

Poles: $\mathbf{z^2 - 3z + 2 = 0}$

$\mathbf{z^2 - 2z - z + 2 = 0}$

$$z(z-2)-1(z-2)=0$$
$$(z-2)(z-1)=0$$
That is $z=1,2$

There is only one pole at $z=2$ inside the given circle.

$$\int_c \frac{z}{z^2-3z+2}dz$$
$$=\int_c \frac{z}{z^2-2z-z+2}dz$$
$$=\int_c \frac{z}{z(z-2)-1(z-2)}dz$$
$$=\int_c \frac{z}{(z-1)(z-2)}dz$$
$$=\int_c \frac{\frac{z}{z-1}}{z-2}dz$$
$$\int_c \frac{f(z)}{z-a}dz = 2\pi i\times f(a)$$

Here, $f(z)=\dfrac{z}{z-1}$

$$\therefore f(2)=\frac{2}{2-1}=2$$

Hence, from Cauchy's Integral Formula:

$$\int_c \frac{f(z)}{z-a}dz = 2\pi i\times f(a)$$
$$\int_c \frac{f(z)}{z-2}dz = 2\pi i\times f(2) \qquad [a=2]$$
$$\int_c \frac{\frac{z}{z-1}}{z-2}dz = 2\pi i\times f(2)$$
$$\int_c \frac{\frac{z}{z-1}}{z-2}dz = 2\pi i\times 2$$
$$\int_c \frac{\frac{z}{z-1}}{z-2}dz = 4\pi i$$

Example 25:

Evaluate $\int_c \frac{2z+1}{z^2+z}dz$

Where c is the circle $|z| = \frac{1}{2}$

$\sqrt{x^2+y^2} = \frac{1}{2}$

$\therefore x^2+y^2 = \frac{1}{4}$

$(x-0)^2+(y-0)^2 = (\frac{1}{2})^2$ ----------------(i)

[We have, $(x-a)^2+(y-b)^2 = r^2$]

Which is the equation of a circle whose Center (0, 0), Radius = $\frac{1}{2}$

Poles: $z^2+z=0$

$z(z+1)=0$

That is $z=0, z=-1$

There is only one pole at $z=0$ inside the given circle.

$\int_c \frac{2z+1}{z^2+z}dz$

$\int_c \frac{2z+1}{z(z+1)}dz$

$= \int_c \frac{\frac{2z+1}{z+1}}{z}dz$

Here, $f(z) = \frac{2z+1}{z+1}$

$\therefore f(0) = \frac{1}{1} = 1$

Hence, from Cauchy's Integral Formula:

$\int_c \frac{f(z)}{z-a}dz = 2\pi i \times f(a)$

$\int_c \frac{f(z)}{z-0}dz = 2\pi i \times f(0)$ $[a=0]$

$\int_c \frac{f(z)}{z}dz = 2\pi i \times f(0)$

$\int_c \frac{f(z)}{z}dz = \int_c \frac{\frac{2z+1}{z+1}}{z}dz = 2\pi i \times f(0)$

$$\int_c \frac{f(z)}{z}dz = \int_c \frac{\frac{2z+1}{z+1}}{z}dz = 2\pi i \times 1$$

$$\int_c \frac{f(z)}{z}dz = \int_c \frac{\frac{2z+1}{z+1}}{z}dz = 2\pi i$$

Example 26:

Find the residue at pole of $\dfrac{1-2z}{z(z-1)(z-2)}$

Answer:

Let, $f(z) = \dfrac{1-2z}{z(z-1)(z-2)}$

Poles $z(z-1)(z-2) = 0$

$z = 0;\ (z-1) = 0;\ (z-2) = 0$

$z = 0;\ z = 1;\ z = 2$

Residue of $f(z)$ at $(z = 0)$ $= \lim_{z\to 0}(z-0)f(z)$

$$= \lim_{z\to 0} z\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 0} \frac{1-2z}{(z-1)(z-2)}$$

$$= \frac{1}{(-1)(-2)}$$

$$= \frac{1}{2}$$

Residue of $f(z)$ at $(z = 1)$ $= \lim_{z\to 1}(z-1)f(z)$

$$= \lim_{z\to 1} (z-1)\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z\to 1} \frac{1-2z}{z(z-2)}$$

$$= \frac{1-2}{(1)(1-2)}$$

$$= 1$$

Residue of $f(z)$ at $(z = 2)$ $= \lim_{z\to 2}(z-2)f(z)$

$$= \lim_{z\to 2} (z-2)\frac{1-2z}{z(z-1)(z-2)}$$

$$= \lim_{z \to 2} \frac{1-2z}{z(z-1)}$$

$$= \frac{1-2\times 2}{(2)(2-1)}$$

$$= -\frac{3}{2}$$

https://complex-analysis.com/content/complex_integration.html

https://www.academia.edu/39133549/Complex_Variables_with_Applications?email_work_card=title (vvvvvvvvvvvvvvv imp)