$\therefore \quad 6 = w\sqrt{a^2 - 64}$ (i)

Again v = 8 cm/sec when y = 6 cm

$\therefore \quad 8 = w\sqrt{a^2 - 64}$ (ii)

Dividing (ii) by (i) and squaring

$$\frac{64}{36} = \frac{a^2 - 36}{a^2 - 64}$$

or, a = **10 cm.**

Substituting a = 10 cm in eqn. (i)

$\therefore \quad 6 = w\sqrt{100 - 64}$

or, w = **1 rad/sec.**

Hence frequency,

$$n = \frac{w}{2\pi} = \frac{1}{2\pi} \text{ Hz.}$$

time period

$$T = \frac{1}{n} = \mathbf{2\pi \text{ seconds.}}$$

Example 1.7. *A simple harmonic motion is represented by*

$$y = 10 \sin (10t - \frac{\pi}{6})$$

where y is measured in metres, t in seconds and the phase angle in radians. Calculate (i) the frequency, (ii) the time period, (iii) the maximum displacement, (iv) the maximum, velocity and (v) the maximum acceleration and (vi) displacement, velocity and acceleration at time t = 0 and t = 1 second.

Soln.

Here $y = 10 \sin (10t - \frac{\pi}{6})$ (1)

Comparing with the displacement equation

$y = a \sin (wt + \delta)$ (2)

we get, (i) $w = 2\pi n = 10$

or, $n = \frac{10}{2\pi}$ = **1.6 Hz.**

(ii) time period, $T = \frac{1}{n} = \frac{2\pi}{10}$ = **0.63 sec.**

(iii) maximum displacement (amplitude) a = **10 m.**

(iv) maximum velocity,

$v_{max} = w.a = 10 \times 10$ = **100 m/sec.**

(v) $(\text{accln.})_{max} = -w^2a = -(10)2 \times 10$

$= \mathbf{-1000 \text{ m/sec}^2}$.

minus sign shows that the acceleration is directed towards the mean position.

(vi) From eqn. (1)

(a) at t = 0

$y = 10 \sin (-\frac{\pi}{6}) = -5$ ***m.***

velocity, $\frac{dy}{dt} = a\,w \cos \delta$

$= 10 \times 10 \cos (-\frac{\pi}{6})$

$= 100 \times 0.866$ = **86.6 m/sec.**

Acceleration, $\frac{d^2y}{dt^2} = -aw^2 \sin \delta$

$= -10 \times 10^2 \times \sin (-\frac{\pi}{6})$

$= -10 \times 100 \times 0.5$

$= \mathbf{-500 \text{ m/sec}^2}$.

(b) From eqn. (1), at t = 1,

displacement

$y = 10 \sin (10 - \frac{\pi}{6})$

$$= 10 \sin \left(\frac{60 - 3.142}{6}\right)$$

$$= 10 \sin \left(\frac{56.858}{6}\right)$$

$= 10 \sin (3\pi)$ approximately

$$= 10 \sin \pi$$

$$= 0$$

velocity, $\frac{dy}{dt} = aw \cos \left(10 - \frac{\pi}{6}\right)$

$= aw \cos (\pi)$ approximately

$$= 10 \times 10 \times (-1)$$

$= -100$ m/sec.

Accln., $\frac{d^2y}{dt^2} = -a\, w^2 \sin \left(10 - \frac{\pi}{6}\right)$

$= -aw^2 \sin (\pi)$ approximately.

$= 0.$

Example 1.8. *A particle performs simple harmonic motion given by the equation*

$$y = 20 \sin (wt + \alpha)$$

If the time period is 30 seconds and the particle has a displacement of 10 cm at $t = 0$, find (i) epoch, (ii) the phase angle at $t = 5$ seconds and (iii) the phase difference between two positions of the particle 15 seconds apart.

Soln.

Here

$$y = 20 \sin (wt + \alpha)$$

$T = 30$ secs.

$\therefore \quad w = \frac{2\pi}{T} = \frac{2\pi}{30} = \frac{\pi}{15}$ rad/sec.

(i) at $t = 0$, $y = 10$ cm.

$\therefore \quad 10 = 20 \sin \left(\frac{\pi}{15} \times 0 + \alpha\right)$

or, $\sin \alpha = \frac{10}{20} = 0.5$

or, $\alpha = \frac{\pi}{6}$ **radian.**

(ii) at $t = 5$ sec.

the phase angle $= (wt + \alpha)$

$$= \left(\frac{\pi}{15} \times 5 + \frac{\pi}{6}\right)$$

$$= \frac{\pi}{2}$$

(iii) at $t = 0$ the phase angle

$\theta_1 = \left(\frac{\pi}{15} \times 0 + \frac{\pi}{6}\right) = \frac{\pi}{6}$ radian

at $t = 15$ sec. the phase angle

$$\theta_2 = \left(\frac{\pi}{15} \times 15 + \frac{\pi}{6}\right)$$

$= \frac{7\pi}{6}$ radian.

$\therefore$ the phase difference,

$\theta_2 - \theta_1 = \frac{7\pi}{6} - \frac{\pi}{6} = \pi$ **radian.**

Example 1.9. *A body describing SHM has a maximum acceleration of 8π m/s^2 and a maximum speed of 1.6 m/s. Find the period T and the amplitude a.*

Soln :

$a_{max} = w^2 a$ (ignoring the minus sign)

$$= (2\pi n)^2 a = \left(\frac{2\pi}{T}\right)^2 a = \frac{4\pi^2}{T^2}\, a = 8\pi \text{ m/s}^2.$$

$$= \frac{1}{T}\cdot\frac{m w^2 a^2}{4}\int_0^T 2\sin^2(wt+\phi)\,dt$$

$$= \frac{m w^2 a^2}{4T}\int_0^T [1-\cos 2(wt+\phi)]\,dt$$

$$= \frac{m w^2 a^2}{4T}\left[\int_0^T dt - \int_0^T \cos 2(wt+\phi)\right] dt$$

The average value of both a sine and a cosine function for a complete cycle or a whole time period T is zero. We, therefore, have

average P.E. of the particle

$$= \frac{1}{4T} m w^2 a^2 [\,t\,]_0^T - 0$$

$$= \frac{1}{4T} m w^2 a^2 T$$

$$= \frac{1}{4} m w^2 a^2$$

$$= \frac{1}{4} ka^2 \qquad [\because mw^2 = K] \qquad (1.18)$$

The kinetic energy (K.E.) of the particle at displacement y is given by

$$= \frac{1}{2} m \left(\frac{dy}{dt}\right)^2$$

$$= \frac{1}{2} m \left[\frac{d}{dt} a\,sin(wt+\phi)\right]^2$$

$$= \frac{1}{2} m\, w^2 a^2 \cos^2(wt+\phi)$$

The average K.E. of the particle over a complete cycle or a whole time period T, as in the case of P.E., is given by

$$\frac{1}{T}\int_0^T \frac{1}{2} m\, w^2 a^2 \cos^2(wt+\phi)\,dt$$

$$= \frac{m w^2 a^2}{4T}\int_0^T 2\cos^2(wt+\phi)\,dt$$

$$= \frac{m w^2 a^2}{4T}\int_0^T [1+\cos 2(wt+\phi)]\,dt$$

$$= \frac{m w^2 a^2}{4T}\left[\int_0^T dt + \int_0^T cos\, 2(wt+\phi)\,dt\right]$$

Again, the average value of a sine or cosine function over a complete cycle or a whole time period is zero. Hence

average K.E. of the particle

$$= \frac{m w^2 a^2}{4T}[\,t\,]_0^T$$

$$= \frac{m w^2 a^2}{4T}.T$$

$$= \frac{1}{4} m w^2 a^2 = \frac{1}{4} ka^2. \qquad (1.19)$$

Thus,

average value of P.E. of the particle

= average value of K.E. of the particle

$$= \frac{1}{4} m w^2 a^2 = \frac{1}{4} ka^2$$

= half the total energy.

Example 1.11. *(i) What is the mechanical (total) energy of the oscillating system of Example 1.1 ?*

Soln.

Total energy = P.E. + K.E. $\frac{1}{4} ka^2$

Now k = 200N/m and a = 0.04m [Ex. 1.1 (i) and (iv)]

$\therefore$ Total energy $= \frac{1}{2}(200)(0.04)^2 =$ **0.16 joules**

Also total energy,

$\frac{1}{4}ka^2 = 2\pi^2 m\, a^2 n^2$

$= 2(3.14)^2(2)(0.04)^2(159)$

$=$ **0.16 joules**

From Ex. 1.1

a = 0.04 m

m = 2 Kg

n = 159 Hz

(ii) Compute the velocity, the acceleration and the kinetic and potential energies of the body when it has moved in half-way from its initial position toward the centre of motion.

$v = \pm w\sqrt{a^2 - y^2}$

At half-way,

$= \pm(10\ sec^{-1})\sqrt{(0.04m)^2 - (0.02m)^2}$

$y = \frac{a}{2} = \frac{0.04}{2} m = 0.02m$

$= \pm\frac{2\sqrt{3}}{10}\ m/sec^{-1}$

$w = 10\ sec^{-1}$

$= \pm$ **0.346 m sec⁻¹**

acceleration $= -w^2 y = -(10\ sec^{-1}).(0.02\ m)$

$=$ **– 2.0 m sec⁻²**

K. E. $= \frac{1}{2}mv^2 = \frac{1}{2}(2)(0.346)^2 \simeq$ **0.12 joules**

P.E. $= \frac{1}{2}ky^2 = \frac{1}{2}(200)(0.02)^2 \simeq$ **0.04 joules**

$\therefore$ Total energy = P.E. + K.E. = (0.04 + 0.12) joules

= **0.16 joules.**

Note that total energy is constant.

Example 1.12. *A block whose mass is 680 gm is at rest on a table top and is fastened to an anchored horizontal spring. The*

spring constant of the spring is 65 N/m. There is negligible friction between the block and the table top. The block is pulled a distance x = 11 cm from its equilibrium position at x = 0 and released from rest at t = 0.

(i) What force does the spring exert on the block just before the block is released ?

from Hooke's law

$F = -ky = -(65\ N/m)(0.11\ m)$

$=$ **7.2 N**

minus sign simply indicates that force and displacement are oppositely directed.

(ii) What are the angular frequency, the frequency, and the period of the resulting oscillation ?

angular frequency, $w = \sqrt{\frac{k}{m}} = \sqrt{\frac{65\ N/m}{0.68\ Kg}} = 9.78$ rad/s.

frequency, $n = \frac{w}{2\pi} = \frac{9.78\ N/m}{2 \times 3.14} \simeq$ **1.56 Hz**

and the period, $T = \frac{1}{n} = \frac{1}{1.56\ Hz} =$ **0.64 s**

(iii) What is the amplitude of the oscillation ?

since the block is released from rest 11 cm from its equilibrium position, its kinetic energy will be zero whenever it is again 11 cm from that position.

Hence its maximum displacement, *i.e.*, amplitude is zero.

or, a = 11 cm = 0.11 m.

(iv) What are the maximum velocity and acceleration of the oscillating block ?

$v_{max} = wa = (9.78\ rad/s)(0.11\ m) =$ **1.1 m/s.**

$a_{max} = -w^2 a = (9.78\ rad/s)^2(0.11\ m) =$ **– 11 m/s².**

(v) What is the phase constant φ for the motion ?