Demoivre's theorem

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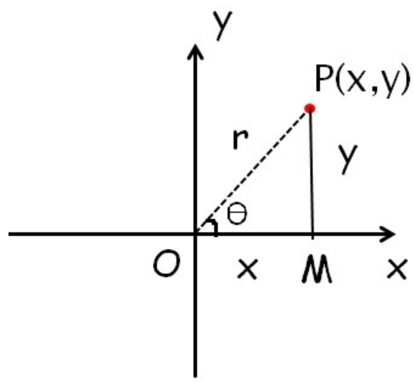
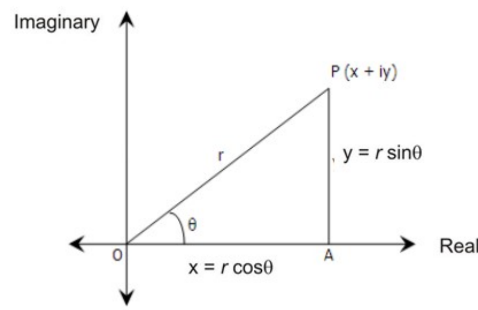


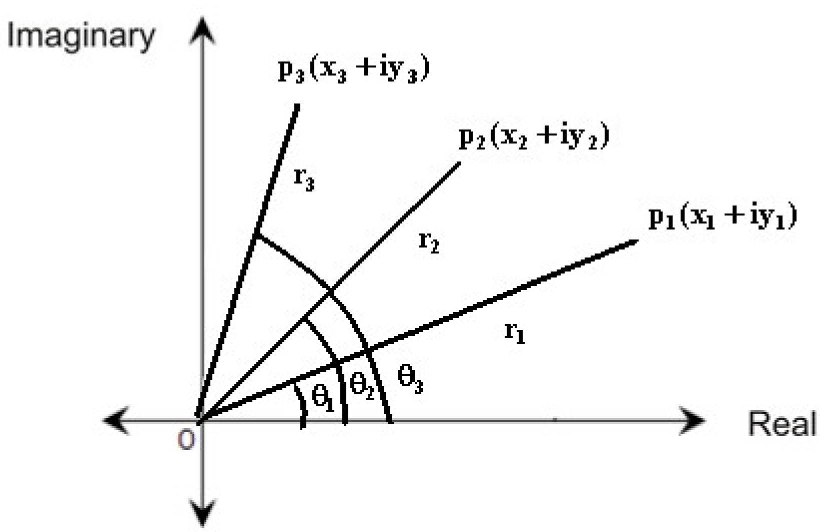


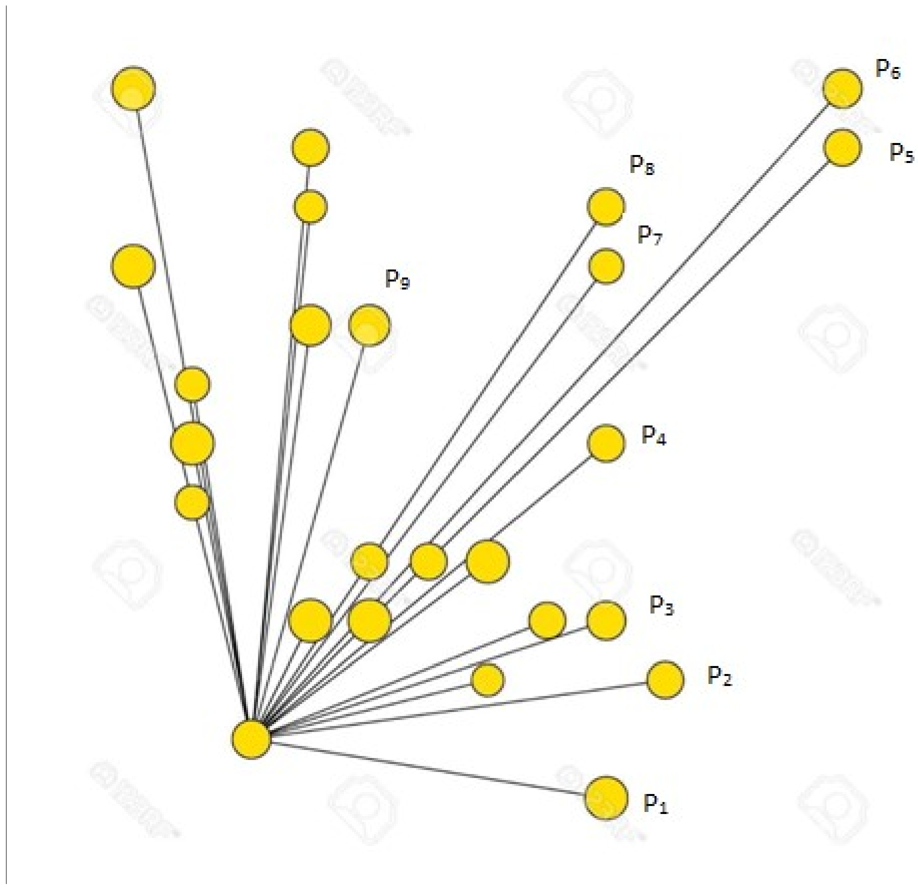
$$\frac{x}{r} = \cos\theta \qquad \text{and} \qquad \frac{y}{r} = \sin\theta$$

$$\Rightarrow x = r\cos\theta \qquad \Rightarrow y = r\sin\theta$$

$$\therefore (x + iy) = (r\cos\theta + i\,r\sin\theta)$$

$$\therefore (x + iy) = r(\cos\theta + i\sin\theta)$$





Let another complex number $\mathbf{p_1(x_1 + iy_1)}$

$$\therefore \mathbf{x_1 + iy_1 = r_1(\cos\theta_1 + i\sin\theta_1)}$$

$$\therefore \mathbf{x_2 + iy_2 = r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\therefore \mathbf{x_3 + iy_3 = r_3(\cos\theta_3 + i\sin\theta_3)}$$

$$\therefore \mathbf{x_4 + iy_4 = r_4(\cos\theta_4 + i\sin\theta_4)}$$

$$\therefore x_n + iy_n = r_n(\cos\theta_n + i\sin\theta_n)$$

State and Prove Demoivre's theorem

Statement: whatever be the value of n, positive or negative, integral or fractional, $\cos n\theta + i\sin n\theta$ is the value or one of the values of $(\cos\theta + i\sin\theta)^n$

Proof:

Case 1: when n is a positive integer,

By actual multiplication $(x_1 + iy_1)(x_2 + iy_2) = r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= r_1r_2\{\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2\}$

$= r_1r_2\{\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\}$ $[i^2 = -1]$

$= r_1r_2\{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2\}$

$[\cos A\cos B - \sin A\sin B = \cos(A+B)\ \&\ \sin A\cos B + \cos A\sin B = \sin(A+B)]$

$= r_1r_2\{\cos(\theta_1 + \theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$

$= r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\therefore r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

$\therefore (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$ -----------------(i)

Now, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$= \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}(\cos\theta_3 + i\sin\theta_3)$

$= \cos(\theta_1 + \theta_2)\cos\theta_3 + i\sin(\theta_1 + \theta_2)\cos\theta_3 + i\cos(\theta_1 + \theta_2)\sin\theta_3 + i^2\sin(\theta_1 + \theta_2)\sin\theta_3$

$= \cos(\theta_1 + \theta_2)\cos\theta_3 + i\sin(\theta_1 + \theta_2)\cos\theta_3 + i\cos(\theta_1 + \theta_2)\sin\theta_3 - \sin(\theta_1 + \theta_2)\sin\theta_3$

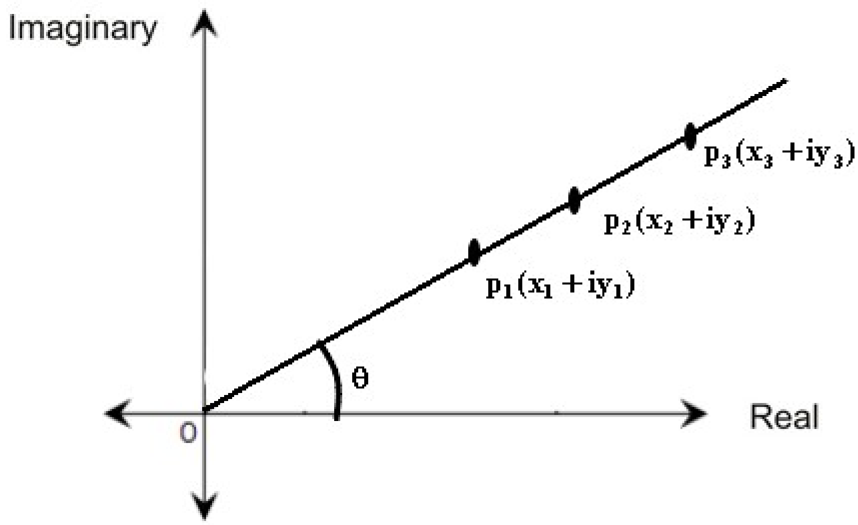
$= \cos(\theta_1 + \theta_2)\cos\theta_3 - \sin(\theta_1 + \theta_2)\sin\theta_3 + i\{\sin(\theta_1 + \theta_2)\cos\theta_3 + \cos(\theta_1 + \theta_2)\sin\theta_3\}$

$= \cos(\theta_1 + \theta_2 + \theta_3) + i\{\sin(\theta_1 + \theta_2 + \theta_3\}$

So, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots\ldots\ldots\ldots\ldots(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots\ldots + \theta_n) + i\{\sin(\theta_1 + \theta_2 + \theta_3 + \theta_4\ldots\ldots\ldots + \theta_n\}$

If $\theta_1 = \theta_2 = \theta_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = \theta_n = \theta$



Then $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)........................(\cos\theta_n + i\sin\theta_n)$

$= \cos(\theta_1 + \theta_2 + \theta_3 + ... + \theta_n) + i\{\sin(\theta_1 + \theta_2 + \theta_3 + \theta_3 .... + \theta_n\}$

$\Rightarrow (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)........................(\cos\theta + i\sin\theta)$

$= \cos(\theta + \theta + \theta + ... + \theta) + i\{\sin(\theta + \theta + \theta + \theta................. + \theta\}$

$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Case 2: when n is a negative integer

Let us suppose, $n = -m$, where m is a positive integer.

$\Rightarrow (\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{1}{\cos m\theta + i\sin m\theta}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{(\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{(\cos m\theta - i\sin m\theta)}{\cos^2 m\theta - i^2\sin^2 m\theta}$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{(\cos m\theta - i\sin m\theta)}{\cos^2 m\theta + \sin^2 m\theta} \qquad [i^2 = -1]$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \frac{(\cos m\theta - i\sin m\theta)}{1} \qquad [\cos^2\theta + \sin^2\theta = 1]$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = (\cos m\theta - i\sin m\theta)$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos(-m\theta) + i\sin(-m\theta) \qquad [\cos(-\theta) = \cos\theta;\ \sin(-\theta) = -\sin\theta]$$

$$\Rightarrow (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad [\because n = -m]$$

Case 3: when n is a fraction, positive or negative

Let us suppose, $n = \frac{p}{q}$, where q is a positive integer and p is any integer, positive or negative.

From case 1:

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q = \cos q\frac{\theta}{q} + i\sin q\frac{\theta}{q} \qquad [(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$$

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q = \cos\theta + i\sin\theta$$

Taking the q-th roots on both sides,

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q = \cos\theta + i\sin\theta$$

$$\{(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})^q\}^{\frac{1}{q}} = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}) = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

So, $\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}$ is one of the values of $(\cos\theta + i\sin\theta)^{\frac{1}{q}}$

Raising to the p-th power,

$$(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q}) = (\cos\theta + i\sin\theta)^{\frac{1}{q}}$$

$$\{(\cos\frac{\theta}{q} + i\sin\frac{\theta}{q})\}^p = \{(\cos\theta + i\sin\theta)^{\frac{1}{q}}\}^p$$

$$\{(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q})\}^{p}=\{(\cos\theta+i\sin\theta)^{\frac{p}{q}}\}$$

$$\{(\cos\frac{\theta}{q}+i\sin\frac{\theta}{q})\}^{p}=\{(\cos\theta+i\sin\theta)^{\frac{p}{q}}\}$$

$$\Rightarrow \cos p\frac{\theta}{q}+i\sin p\frac{\theta}{q}=\{(\cos\theta+i\sin\theta)^{\frac{p}{q}}\}$$

$$\Rightarrow \cos\frac{p}{q}\theta+i\sin\frac{p}{q}\theta=\{(\cos\theta+i\sin\theta)^{\frac{p}{q}}\}$$

$$\Rightarrow (\cos\theta+i\sin\theta)^{\frac{p}{q}}=\cos\frac{p}{q}\theta+i\sin\frac{p}{q}\theta$$

$$\Rightarrow (\cos\theta+i\sin\theta)^{n}=\cos n\theta+i\sin n\theta \qquad [n=\frac{p}{q}]$$

Q-01: If $x_r=\cos\frac{\pi}{3^r}+i\sin\frac{\pi}{3^r}$, prove that $x_1x_2x_3\ldots\ldots\ldots\ldots\inf=i$

Answer: Given,

$$x_r=\cos\frac{\pi}{3^r}+i\sin\frac{\pi}{3^r}$$

Putting r = 1, 2, 3, 4, …………..

$$x_1=\cos\frac{\pi}{3^1}+i\sin\frac{\pi}{3^1}$$

$$x_2=\cos\frac{\pi}{3^2}+i\sin\frac{\pi}{3^2}$$

$$x_3=\cos\frac{\pi}{3^3}+i\sin\frac{\pi}{3^3}$$

$$x_4=\cos\frac{\pi}{3^4}+i\sin\frac{\pi}{3^4}$$

……………………………………

……………………………………

……………………………………..

$$\therefore x_1x_2x_3\ldots\ldots\ldots\ldots$$

$$(\cos\frac{\pi}{3^1}+i\sin\frac{\pi}{3^1})\ (\cos\frac{\pi}{3^2}+i\sin\frac{\pi}{3^2})\ (\cos\frac{\pi}{3^3}+i\sin\frac{\pi}{3^3})\ (\cos\frac{\pi}{3^4}+i\sin\frac{\pi}{3^4})............$$

$$=\cos(\frac{\pi}{3^1}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\frac{\pi}{3^4}+............)+i\sin(\frac{\pi}{3^1}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\frac{\pi}{3^4}+............)$$

$$=\cos\{\frac{\pi}{3^1}(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+............)\}+i\sin\{\frac{\pi}{3^1}(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+............)\}$$

$$[\because (1-x)^{-1}=1+x+x^2+x^3+x^4+......................]$$

$$=\cos\{\frac{\pi}{3^1}(1-\frac{1}{3})^{-1}\}+i\sin\{\frac{\pi}{3^1}(1-\frac{1}{3})^{-1}\}$$

$$=\cos\{\frac{\pi}{3^1}(\frac{3-1}{3})^{-1}\}+i\sin\{\frac{\pi}{3^1}(\frac{3-1}{3})^{-1}\}$$

$$=\cos\{\frac{\pi}{3^1}(\frac{2}{3})^{-1}\}+i\sin\{\frac{\pi}{3^1}(\frac{2}{3})^{-1}\}$$

$$=\cos\{\frac{\pi}{3^1}(\frac{1}{\frac{2}{3}})\}+i\sin\{\frac{\pi}{3^1}(\frac{1}{\frac{2}{3}})\}$$

$$=\cos\{\frac{\pi}{3^1}(\frac{3}{2})\}+i\sin\{\frac{\pi}{3^1}(\frac{3}{2})\}$$

$$=\cos\{\frac{\pi}{2}\}+i\sin\{\frac{\pi}{2}\}$$

$$=0+i.1$$

$$=i$$

Q-02: If $(1+i\frac{x}{a})(1+i\frac{x}{b})(1+i\frac{x}{c}).........=A+iB$, Then prove that

$$\text{(i)}\ (1+\frac{x^2}{a^2})(1+\frac{x^2}{b^2})(1+\frac{x^2}{c^2})..............=A^2+B^2$$

$$\text{(ii)}\ \tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}+............=\tan^{-1}\frac{B}{A}$$

Answer:

Given, $(1+i\frac{x}{a})(1+i\frac{x}{b})(1+i\frac{x}{c}).........=A+iB$ ------------------------------------(i)

Let, $1 = r\cos\alpha$, $\frac{x}{a} = r\sin\alpha$

So, $\frac{\frac{x}{a}}{1} = \frac{r\sin\alpha}{r\cos\alpha}$

$\therefore \frac{x}{a} = \tan\alpha$

$\therefore \tan\alpha = \frac{x}{a}$

$\therefore \alpha = \tan^{-1}\frac{x}{a}$

Again, Let, $1 = r\cos\beta$, $\frac{x}{b} = r\sin\beta$

So, $\frac{\frac{x}{b}}{1} = \frac{r\sin\beta}{r\cos\beta}$

$\therefore \frac{x}{b} = \tan\beta$

$\therefore \tan\beta = \frac{x}{b}$

$\therefore \beta = \tan^{-1}\frac{x}{b}$

Again, Let, $1 = r\cos\gamma$, $\frac{x}{c} = r\sin\gamma$

So, $\frac{\frac{x}{c}}{1} = \frac{r\sin\gamma}{r\cos\gamma}$

$\therefore \frac{x}{c} = \tan\gamma$ etc. ,

$\therefore \tan\gamma = \frac{x}{c}$

$\therefore \gamma = \tan^{-1}\frac{x}{c}$

From (i), we get

Given, $(1+i\frac{x}{a})(1+i\frac{x}{b})(1+i\frac{x}{c})\text{.........} = A+iB$

$$\Rightarrow (1+i\tan\alpha)(1+i\tan\beta)(1+i\tan\gamma)\text{.........}= A+iB$$

$$\Rightarrow (1+i\frac{\sin\alpha}{\cos\alpha})(1+i\frac{\sin\beta}{\cos\beta})(1+i\frac{\sin\gamma}{\cos\gamma})\text{.........} = A+iB$$

$$\Rightarrow (\frac{\cos\alpha+i\sin\alpha}{\cos\alpha})(\frac{\cos\beta+i\sin\beta}{\cos\beta})(\frac{\cos\gamma+i\sin\gamma}{\cos\gamma})\text{.........} = A+iB$$

$$\Rightarrow (\frac{1}{\cos\alpha})(\frac{1}{\cos\beta})(\frac{1}{\cos\gamma})\text{.........}(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)\text{.........} = A+iB$$

$$\Rightarrow (\sec\alpha)(\sec\beta)(\sec\gamma)\text{.........}(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)\text{.........}= A+iB$$

$$\Rightarrow (\sec\alpha)(\sec\beta)(\sec\gamma)\text{.........}[\cos(\alpha+\beta+\gamma+\text{.........})+i\sin(\alpha+\beta+\gamma+\text{.........})] = A+iB$$

Equating the real and imaginary part on both sides, we get,

$(\sec\alpha)(\sec\beta)(\sec\gamma)\text{.........}[\cos(\alpha+\beta+\gamma+\text{.........})] = A$ --------------------(ii)

$(\sec\alpha)(\sec\beta)(\sec\gamma)\text{.........}[\sin(\alpha+\beta+\gamma+\text{.........})] = B$ --------------------(iii)

Squaring (ii) and (iii), we get,

$(\sec^2\alpha)(\sec^2\beta)(\sec^2\gamma)\text{.........}[\cos^2(\alpha+\beta+\gamma+\text{.........})] = A^2$ --------------------(iv)

$(\sec^2\alpha)(\sec^2\beta)(\sec^2\gamma)\text{.........}[\sin^2(\alpha+\beta+\gamma+\text{.........})] = B^2$ --------------------(v)

Adding (iv) & (v)

$$(\sec^2\alpha\sec^2\beta\sec^2\gamma....)\cos^2(\alpha+\beta+\gamma+....)+(\sec^2\alpha\sec^2\beta\sec^2\gamma...)\sin^2(\alpha+\beta+\gamma+...)=A^2+B^2$$

$$\Rightarrow(\sec^2\alpha\sec^2\beta\sec^2\gamma...........)\{\cos^2(\alpha+\beta+\gamma+......)+\sin^2(\alpha+\beta+\gamma+....)\}=A^2+B^2$$

$$\Rightarrow(\sec^2\alpha\sec^2\beta\sec^2\gamma...........)1=A^2+B^2$$

$$\Rightarrow\sec^2\alpha\sec^2\beta\sec^2\gamma...........=A^2+B^2$$

$$\Rightarrow(1+\tan^2\alpha)(1+\tan^2\beta)(1+\tan^2\gamma)...........=A^2+B^2$$

$\therefore (1+\frac{x^2}{a^2})(1+\frac{x^2}{b^2})(1+\frac{x^2}{c^2})............=A^2+B^2$ proved (i)

Now, (iii) $\div$ (ii)

$$\frac{(\sec\alpha)(\sec\beta)(\sec\gamma)..........[\sin(\alpha+\beta+\gamma+.........)]}{(\sec\alpha)(\sec\beta)(\sec\gamma)..........[\cos(\alpha+\beta+\gamma+.........)]}=\frac{B}{A}$$

$$\Rightarrow\frac{[\sin(\alpha+\beta+\gamma+.........)]}{[\cos(\alpha+\beta+\gamma+.........)]}=\frac{B}{A}$$

$$\Rightarrow\tan(\alpha+\beta+\gamma+.........)=\frac{B}{A}$$

$$\Rightarrow\alpha+\beta+\gamma+.........=\tan^{-1}\frac{B}{A}$$

$\therefore (\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}+...........)=\tan^{-1}\frac{B}{A}$ proved (ii)

Q-03: Using Demoivre's theorem, solve the equation $x^6+x^5+x^4+x^3+x^2+x^1+1=0$

Answer: We have, $x^6+x^5+x^4+x^3+x^2+x^1+1=0$

Multiplying the given equation by (x-1), we get,

$$(x-1)(x^6+x^5+x^4+x^3+x^2+x^1+1)=0$$

$$\Rightarrow x^7-1=0$$

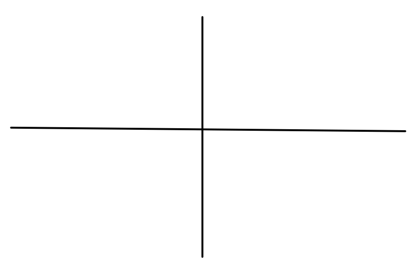
$$\Rightarrow x^7=1$$

cos(4.pi+0)

=cos(8.90 +0)

=cos0

=1



$$\Rightarrow \mathbf{x}^7 = \mathbf{1}$$

$$\Rightarrow x = (1)^{1/7}$$

$$\Rightarrow x = (\cos 0 + i \sin 0)^{1/7}$$

$$\Rightarrow x = \{\cos(2n\pi + 0) + i \sin(2n\pi + 0)\}^{1/7}$$

$$\Rightarrow \mathbf{x = \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}} \qquad [(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$$

Putting n =0, 1, 2, 3, 4, 5 and 6, we get roots of equation as

$$\Rightarrow \mathbf{x = \cos 0 + i\sin 0}$$

$$\Rightarrow \mathbf{x = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}}$$

$$\Rightarrow \mathbf{x = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}}$$

$$\Rightarrow \mathbf{x = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7}}$$

$$\Rightarrow \mathbf{x = \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}}$$

$$\Rightarrow \mathbf{x = \cos\frac{10\pi}{7} + i\sin\frac{10\pi}{7}}$$

$$\Rightarrow \mathbf{x = \cos\frac{12\pi}{7} + i\sin\frac{12\pi}{7}}$$

Q-04: Using Demoivre's theorem, find the values of

$i.(\sqrt{3} + i.1)^{1/5}$

$\mathbf{ii.(8i)^{1/3}}$

$\mathbf{iii.(32)^{1/5}}$

Answer:

$\mathbf{i.(\sqrt{3} + i.1)^{1/5}}$

Let, $\mathbf{\sqrt{3} = r\cos\theta}$ --------------------(i)

$\mathbf{1 = r\sin\theta}$ ----------------------(ii)

Squaring (i) & (ii),

$$3 = r^2 \cos^2\theta \quad \text{-----------------------(iii)}$$

$$1^2 = r^2 \sin^2\theta \quad \text{----------------------(iv)}$$

Adding (iii) & (iv),

$$3 + 1 = r^2\cos^2\theta + r^2\sin^2\theta$$

$$4 = r^2(\cos^2\theta + \sin^2\theta)$$

$$4 = r^2.1$$

$$r = 2 \quad \text{----------------------(v)}$$

(ii) ÷ (i),

$$\frac{1}{\sqrt{3}} = \frac{r\sin\theta}{r\cos\theta}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \tan\theta$$

$$\Rightarrow \tan\frac{\pi}{6} = \tan\theta$$

$$\Rightarrow \tan\theta = \tan\frac{\pi}{6}$$

$$\Rightarrow \theta = \frac{\pi}{6} \qquad \text{----------------------(vi)}$$

Given

$$(\sqrt{3} + i.1)^{1/5}$$

$$= (r\cos\theta + i.r\sin\theta)^{1/5}$$

$$= \{r(\cos\theta + i\sin\theta)\}^{1/5}$$

$$= \{2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})\}^{1/5}$$

$$=\{2^{1/5}(\cos(2n\pi+\frac{\pi}{6})+i\sin(2n\pi+\frac{\pi}{6})\}^{1/5}$$

$$=\{2^{1/5}\{\cos(\frac{12n\pi+\pi}{6})+i\sin(\frac{12n\pi+\pi}{6})\}^{1/5}$$

$$=\{2^{1/5}\{\cos(\frac{12n\pi+\pi}{30})+i\sin(\frac{12n\pi+\pi}{30})\}\ [(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta]$$

Putting n = 0, 1 , 2 , 3, 4

II) $(8i)^{1/3}$

$$8i=8(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$$

$$(8i)^{1/3}=\{8(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})\}^{\frac{1}{3}}$$

$$(8i)^{1/3}=8^{1/3}\{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\}^{\frac{1}{3}}$$

$$(8i)^{1/3}=(2^3)^{1/3}\{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\}^{\frac{1}{3}}$$

$$(8i)^{1/3}=2\{\cos(2n\pi+\frac{\pi}{2})+i\sin(2n\pi+\frac{\pi}{2})\}^{\frac{1}{3}}$$

$$(8i)^{1/3}=2\{\cos(\frac{4n\pi+\pi}{2})+i\sin(\frac{4n\pi+\pi}{2})\}^{\frac{1}{3}}\ [(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta]$$

$$(8i)^{1/3}=2\{\cos\frac{1}{3}(\frac{4n\pi+\pi}{2})+i\sin\frac{1}{3}(\frac{4n\pi+\pi}{2})\}$$

$$(8i)^{1/3}=2\{\cos(\frac{4n\pi+\pi}{6})+i\sin(\frac{4n\pi+\pi}{6})\}$$

Putting n = 0, 1 , 2

$$(8i)^{1/3}=2\{\cos(\frac{4.0\pi+\pi}{6})+i\sin(\frac{4.0\pi+\pi}{6})\}=2\{\cos(\frac{\pi}{6})+i\sin(\frac{\pi}{6})\}$$

$$(8i)^{1/3}=2\{\cos(\frac{4.1.\pi+\pi}{6})+i\sin(\frac{4.1.\pi+\pi}{6})\}=2\{\cos(\frac{5\pi}{6})+i\sin(\frac{5\pi}{6})\}$$

$$(8i)^{1/3}=2\{\cos(\frac{4.2.\pi+\pi}{6})+i\sin(\frac{4.2.\pi+\pi}{6})\}=2\{\cos(\frac{9\pi}{6})+i\sin(\frac{9\pi}{6})\}$$

iii) $(32)^{1/5}$

$$(32)^{1/5}=\{32(1)\}^{1/5}$$

$$(32)^{1/5} = 32^{1/5}(\cos 0 + i\sin 0)^{1/5}$$

$$(32)^{1/5} = (2^5)^{1/5}(\cos 0 + i\sin 0)^{1/5}$$

$$(32)^{1/5} = 2\{\cos(2n\pi + 0) + i\sin(2n\pi + 0)\}^{1/5} \; [(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$$

$$(32)^{1/5} = 2\{\cos(\frac{2n\pi}{5}) + i\sin(\frac{2n\pi}{5})\}$$

Putting n = 0, 1 , 2 , 3, 4

$$(32)^{1/5} = 2\{\cos(\frac{2.0.\pi}{5}) + i\sin(\frac{2.0.\pi}{5})\} = 2(\cos 0 + i\sin 0) = 2(1+0) = 2$$

$$(32)^{1/5} = 2\{\cos(\frac{2.1.\pi}{5}) + i\sin(\frac{2.1.\pi}{5})\} = 2\{\cos(\frac{2\pi}{5}) + i\sin(\frac{2\pi}{5})\}$$

$$(32)^{1/5} = 2\{\cos(\frac{2.2.\pi}{5}) + i\sin(\frac{2.2.\pi}{5})\} = 2\{\cos(\frac{4\pi}{5}) + i\sin(\frac{4\pi}{5})\}$$

$$(32)^{1/5} = 2\{\cos(\frac{2.3.\pi}{5}) + i\sin(\frac{2.3.\pi}{5})\} = 2\{\cos(\frac{6\pi}{5}) + i\sin(\frac{6\pi}{5})\}$$

$$(32)^{1/5} = 2\{\cos(\frac{2.4.\pi}{5}) + i\sin(\frac{2.4.\pi}{5})\} = 2\{\cos(\frac{8\pi}{5}) + i\sin(\frac{8\pi}{5})\}$$

Q-5: Using Demoivres theorem find the quadratic equation whose roots are the nth power of the roots of the equation, $x^2 - 2x\cos\theta + 1 = 0$

Answer:

The given equation is $x^2 - 2x\cos\theta + 1 = 0$

$$x = \frac{-(-2\cos\theta) \pm \sqrt{(-2\cos\theta)^2 - 4.1.1}}{2.1}$$

$$x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$x = \frac{2\cos\theta \pm \sqrt{-4 + 4\cos^2\theta}}{2}$$

$$x = \frac{2\cos\theta \pm \sqrt{-4(1-\cos^2\theta)}}{2}$$

$$x=\frac{2\cos\theta\pm\sqrt{-4\sin^2\theta}}{2}$$
$$x=\frac{2\cos\theta\pm\sqrt{4i^2\sin^2\theta}}{2}$$
$$x=\frac{2\cos\theta\pm2i\sqrt{\sin^2\theta}}{2}$$
$$x=\frac{2\cos\theta\pm2i\sin\theta}{2}$$
$$x=\frac{2(\cos\theta\pm i\sin\theta)}{2}$$
$$x=\cos\theta\pm i\sin\theta$$

Let α and β are the roots of the equation $x^2-2x\cos\theta+1=0$

$\therefore \alpha=\cos\theta+i\sin\theta$ and $\beta=\cos\theta-i\sin\theta$

We have to form a new equation whose roots are α^n and β^n

We know any equation is $x^2-(sum\,of\,the\,roots)x+product\,of\,the\,roots=0$

$$x^2-(\alpha^n+\beta^n)x+\alpha^n\beta^n=0$$
$$x^2-[(\cos\theta+i\sin\theta)^n+(\cos\theta-i\sin\theta)^n]x+(\cos\theta+i\sin\theta)^n(\cos\theta-i\sin\theta)^n=0$$
$$[\because(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta]\ [\because(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta]$$
$$x^2-[(\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)]x+(\cos n\theta+i\sin n\theta)(\cos n\theta-i\sin n\theta)=0$$
$$x^2-[\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta]x+[\cos^2 n\theta-(i)^2\sin^2 n\theta]=0$$
$$x^2-[\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta]x+[\cos^2 n\theta+\sin^2 n\theta]=0$$
$$x^2-[2\cos n\theta]x+1=0$$
$x^2-2x\cos n\theta+1=0$ Answer.