Q # 95: Evaluate $\mathbf{I = \oint_C (3x^2y^2dx + 2x^3ydy)}$ between $\mathbf{O(0,0)}$ and $\mathbf{A(2,4)}$

a) along $\mathbf{c_1}$ i.e. $\mathbf{y = x^2}$

b) along $\mathbf{c_2}$ i.e. $\mathbf{y = 2x}$

c) along $\mathbf{c_3}$ i.e. $\mathbf{x = 0}$ from $\mathbf{(0,0)}$ to $\mathbf{(0,4)}$ and $\mathbf{y = 4}$ from $\mathbf{(0,4)}$ to $\mathbf{(2,4)}$

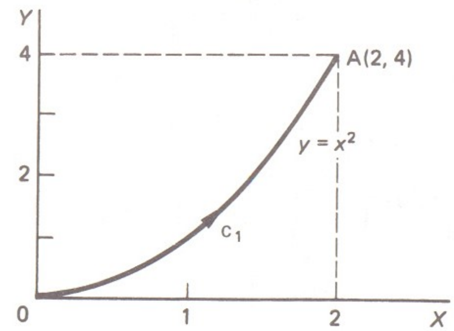

Figure # 120

$$\mathbf{I = \oint_C (3x^2y^2dx + 2x^3ydy)}$$

The path $\mathbf{c_1}$ is $\mathbf{y = x^2}$

$$\mathbf{\Rightarrow dy = 2xdx}$$

$$\mathbf{I_1 = \int_0^2 (3x^2y^2dx + 2x^3ydy)}$$

$$\mathbf{I_1 = \int_0^2 (3x^2(x^2)^2dx + 2x^3 \times x^2 \times 2xdx)}$$

$$\mathbf{I_1 = \int_0^2 (3x^6 + 4x^6)dx}$$

$$\mathbf{I_1 = \int_0^2 7x^6dx} = \left[7\frac{\mathbf{x}^7}{7}\right]_0^2$$

$$\mathbf{I_1 = 128}$$

a) In (b), the path of integration changes to $\mathbf{c_2}$ i.e. $\mathbf{y = 2x}$

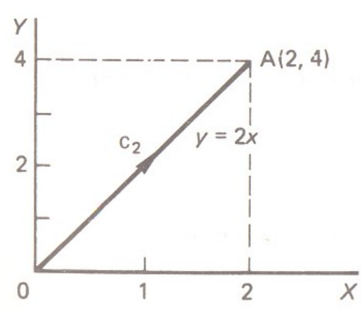

Figure # 121

$$I = \oint_C (3x^2y^2dx + 2x^3ydy)$$

The path c_2 is $y = 2x$

$$\Rightarrow dy = 2dx$$

$$I_2 = \int_0^2 (3x^2y^2dx + 2x^3ydy)$$

$$I_2 = \int_0^2 (3x^2(2x)^2dx + 2x^3 \times 2x \times 2dx)$$

$$I_2 = \int_0^2 (12x^4 + 8x^4)dx$$

$$I_2 = \int_0^2 20x^4dx$$

$$I_2 = \left[20\frac{x^5}{5}\right]_0^2$$

$$I_2 = 128$$

c) In the third case, the path c_3 is split

i. $x = 0$ from $(0,0)$ to $(0,4)$

ii. $y = 4$ from $(0,4)$ to $(2,4)$

Sketch the diagram and determine I_3

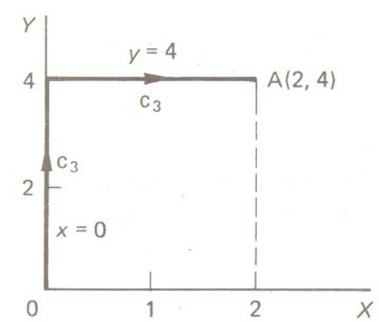

Figure # 122

i. From $(0,0)$ to $(0,4)$ $x = 0 \Rightarrow dx = 0$

$$I = \oint_C (3x^2y^2dx + 2x^3ydy)$$

The path c_3 is $x = 0$

$$\Rightarrow dx = 0$$

$$I_3 = \int_0^0 (3x^2y^2dx + 2x^3ydy)$$

$$I_3 = \int_0^4 (3.(0)^2 y^2 (0) + 2(0)^3 \times ydy)$$

$$\mathbf{I_3 = 0}$$

ii. From **(0,4)** to **(2,4)** $\quad$ $\mathbf{y = 4 \Rightarrow dy = 0}$

$$\mathbf{I = \oint_C (3x^2y^2dx + 2x^3ydy)}$$

The path $\mathbf{c_3}$ is $\mathbf{y = 4 \Rightarrow dy = 0}$

$$\mathbf{I_3 = \int_0^2 (3x^2.4^2dx + 2x^3.4.0)}$$

$$\mathbf{I_3 = \int_0^2 48x^2dx}$$

$$\mathbf{I_3 = 128}$$

In the above example, we have just worked through, we took three different paths and in each case , the line integral produced the same result. It appears, therefore, that in this case, the value of the integral is independent of the path of integration taken.

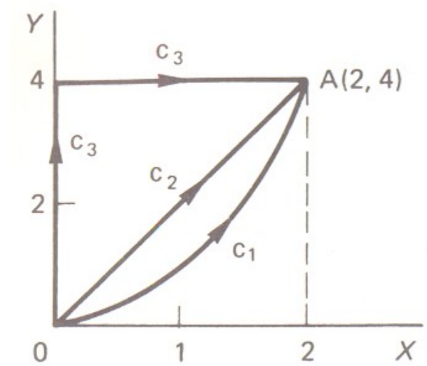

Figure # 123

We have been dealing with $\mathbf{I = \oint_C (3x^2y^2dx + 2x^3ydy)}$

On reflection, we see that the integrand $\mathbf{3x^2y^2dx + 2x^3ydy}$ is of the form $\mathbf{Pdx + Qdy}$ which we have met before and that it is, in fact, an exact differential of the function $\mathbf{z = x^3y^2}$, for

$$\mathbf{\frac{\delta z}{\delta x} = 3x^2y^2} \text{ and } \mathbf{\frac{\delta z}{\delta y} = 2x^3y}$$

This always happens. If the integrand of the given integral is seen to be an exact differential, then the value of the line integral is independent of the path taken and depends only on the coordinates of the two end points.

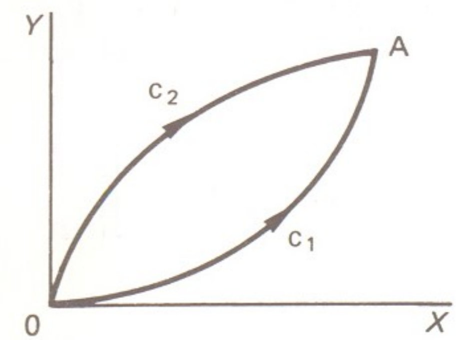

Figure # 124

If $\mathbf{I} = \oint_C \mathbf{(Pdx + Qdy)}$ and $\mathbf{(Pdx + Qdy)}$ is an exact differential, then $\mathbf{I_{C1} = I_{C2}}$

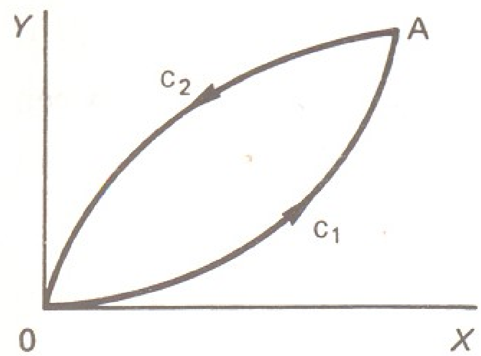

Figure # 125

If we reverse the direction of $\mathbf{c_2}$, then $\mathbf{I_{C1} = -I_{C2} \Rightarrow I_{C1} + I_{C2} = 0}$

Hence the integration taken round a closed curve is zero, provided $\mathbf{(Pdx + Qdy)}$ is an exact differential.

$\therefore$ If $\mathbf{(Pdx + Qdy)}$ is an exact differential, $\mathbf{I = \oint_C (Pdx + Qdy) = 0}$

Q# 96: Evaluate $\mathbf{I = \int_C \{3ydx + (3x + 2y)dy\}}$ from $\mathbf{A(1,2)}$ to $\mathbf{B(3,5)}$.

Now path is given, so the integrand is doubtless an exact differential of some function $\mathbf{z = f(x, y)}$.

Here, $\mathbf{(Pdx + Qdy) = 3ydx + (3x + 2y)dy}$

$\therefore \mathbf{P = 3y}$ and $\mathbf{Q = 3x + 2y}$

In fact $\mathbf{\frac{\delta P}{\delta y} = 3}$ and $\mathbf{\frac{\delta Q}{\delta x} = 3}$

We have already dealt with the integration of exact differentials, so there is no difficulty.

Compare with $\mathbf{I = \int_C \{Pdx + Qdy\}}$.

$\mathbf{P = \frac{\delta z}{\delta x} = 3y}$ $\qquad \therefore \mathbf{z = \int 3ydx = 3xy + f(y)}$----------------------------(i)

$$Q = \frac{\delta z}{\delta y} = 3x + 2y \qquad \therefore z = \int (3x + 2y)dy = 3xy + y^2 + F(x) \text{-------------(ii)}$$

For (i) and (ii) to agree

$\mathbf{f(y) = y^2}$ and $\mathbf{F(x) = 0}$

Hence $\mathbf{z = 3xy + y^2}$

$$I = \int_C \{3ydx + (3x + 2y)dy\}$$

$$I = \int_C \{3ydx + 3xdy + 2ydy\}$$

$$I = \int_C \{3xdy + 3ydx + 2ydy\}$$

$$I = \int_C \{d(3xy) + d(y^2)\}$$

$$I = \int_{(1,2)}^{(3,5)} d(3xy + y^2)$$

$$I = \left[3xy + y^2\right]_{(1,2)}^{(3,5)}$$

$$I = \left[3 \times 3 \times 5 + 5^2 - 3 \times 1 \times 2 - 2^2\right]$$

$$I = [45 + 25 - 6 - 4] = [70 - 10] = 60$$

<u>Exact differentials in three independent variables</u>

A line integral in space naturally involves three independent variables, but the method is very much like that for two independent variables.

$\mathbf{dz = Pdx + Qdy + Rdw}$ is an exact differential of $\mathbf{z = f(x, y, w)}$

if $\frac{\delta P}{\delta y} = \frac{\delta Q}{\delta x}; \frac{\delta P}{\delta w} = \frac{\delta R}{\delta x}; \frac{\delta R}{\delta y} = \frac{\delta Q}{\delta w};$

If the test is successful, then

a) $\mathbf{I = \oint_C (Pdx + Qdy + Rdw)}$ is independent of the path of integration

b) $\mathbf{I = \oint_C (Pdx + Qdy + Rdw)}$ is zero

Green's Theorem:

Let **P** and Q be two functions of x and y that are finite and continuous inside and on the boundary c of a region R in the xy-plane.

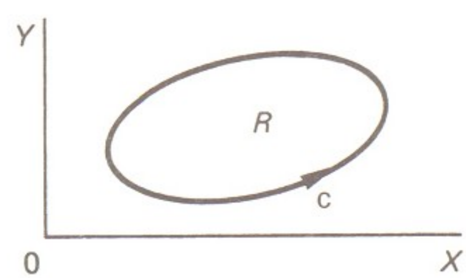

Figure # 126

If the first partial derivatives are continuous within the region and on the boundary, then Green's theorem states that

$$\iint_R \left(\frac{\delta P}{\delta y} - \frac{\delta Q}{\delta x}\right) dxdy = -\oint (Pdx + Qdy)$$

That is, a double integral over the plane region R cane be transformed into a line integral over the boundary c of a region and the action is reversible.

Green's Theorem

Green's Theorem enables an integral over a plane area to be expressed in terms of a line integral round its boundary curve.

If P and Q are two single-valued functions of x and y, continuous over a plane surface S, and c is its boundary curve, then

$$\therefore \oint_c (Pdx + Qdy) = -\iint_R \left(\frac{\delta P}{\delta y} - \frac{\delta Q}{\delta x}\right) dxdy \text{ -----------------------------(i)}$$

$$\therefore \oint_c (Pdx + Qdy) = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dxdy \text{ -------------------------------(ii)}$$

Where the line integral is taken round c in an anticlockwise manner. In vector terms, this becomes:

S is a two-dimensional space enclosed by a simple closed curve c.

$$\left|\vec{dS}\right| = dS = dxdy$$

$$\vec{dS} = \hat{\eta}\, dS = \hat{k}\, dxdy$$

If $\vec{F} = P\hat{i} + Q\hat{j}$ Where $P = P(x,y)$ and $Q = Q(x,y)$ then

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}\right) \times (P\hat{i} + Q\hat{j}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ P & Q & 0 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\delta}{\delta y}(0) - \frac{\delta}{\delta z}(Q)\right] - \hat{j}\left[\frac{\delta}{\delta x}(0) - \frac{\delta}{\delta z}(P)\right] + \hat{k}\left[\frac{\delta}{\delta x}(Q) - \frac{\delta}{\delta y}(P)\right]$$

$$= \hat{i}[0] - \hat{j}[0 - 0] + \hat{k}\left[\frac{\delta}{\delta x}(Q) - \frac{\delta}{\delta y}(P)\right]$$

$$= \hat{k}\left[\frac{\delta}{\delta x}(Q) - \frac{\delta}{\delta y}(P)\right]$$

Since P and Q are functions of x and y, that is in the xy-plane $\frac{\delta Q}{\delta z} = \frac{\delta P}{\delta z} = 0$

So $\int \text{curl}\,\vec{F}.\vec{dS} = \int \text{curl}\,\vec{F}.\hat{\eta}\, dS$ and in the xy plane $\hat{\eta} = \hat{k}$

$$\int_S \mathbf{curl}\,\vec{F}.\vec{dS} = \int_S \hat{k}\left(\frac{\delta Q}{\delta x}-\frac{\delta P}{\delta y}\right).\hat{\eta}\,dS == \int_S \hat{k}\left(\frac{\delta Q}{\delta x}-\frac{\delta P}{\delta y}\right).\hat{k}\,dS = \iint_S \left(\frac{\delta Q}{\delta x}-\frac{\delta P}{\delta y}\right)dxdy\ [\because \hat{k}.\hat{k}=1]$$

$$\therefore \int_S \mathbf{curl}\,\vec{F}.\vec{dS} = \iint_S \left(\frac{\delta Q}{\delta x}-\frac{\delta P}{\delta y}\right)dxdy \text{ -------------------------------------(i)}$$

We have, from Stoke's theorem:

$$\therefore \int_S \mathbf{curl}\,\vec{F}.\vec{dS} = \oint_c \vec{F}.\vec{dr}$$

$$= \oint_c (P\hat{i}+Q\hat{j}).(dx\hat{i}+dy\hat{j}+dz\hat{k})\ [\because \vec{r} = x\hat{i}+y\hat{j}+z\hat{k}]$$

$$= \oint_c (Pdx+Qdy) \text{ --(ii)}$$

From (i) and (ii)

$$\therefore \int_S \mathbf{curl}\,\vec{F}.\vec{dS} = \iint_S \left(\frac{\delta Q}{\delta x}-\frac{\delta P}{\delta y}\right)dxdy = \oint_c (Pdx+Qdy)$$

$$\therefore \iint_S \left(\frac{\delta Q}{\delta x}-\frac{\delta P}{\delta y}\right)dxdy = \oint_c (Pdx+Qdy)$$

Q# 97: Evaluate $\mathbf{I} = \oint_C \{(2x+y)dx+(3x-2y)dy\}$ taken in an anticlockwise manner round the triangle with vertices at **O(0,0), A(1,0), B(1,2)**

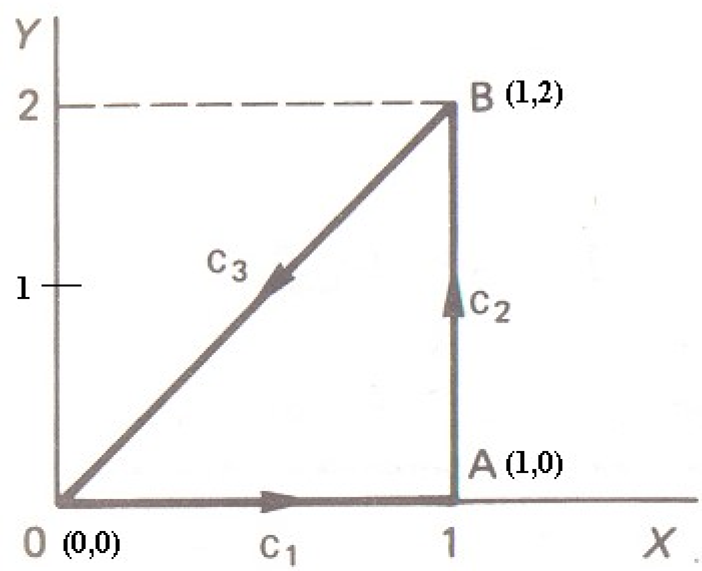

Figure # 127

$$\therefore \oint_c (Pdx+Qdy) = \iint_R \left(\frac{\delta Q}{\delta x}-\frac{\delta P}{\delta y}\right)dxdy$$

$$\mathbf{I} = \oint_C \{(2x+y)dx+(3x-2y)dy\}$$

a) There are clearly three stages with $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$. Work through the complete evaluation to determine the value of $\mathbf{I}$. It will be good revision.

i) The equation of OA is:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}\Rightarrow\frac{y-0}{0-0}=\frac{x-0}{0-1}\Rightarrow\frac{y}{0}=\frac{x}{-1}\Rightarrow -y=x.0\Rightarrow y=0$$

$\mathbf{c}_1$ is $y=0\ \therefore dy=0$

$$I=\oint_C\{(2x+y)dx+(3x-2y)dy\}$$

$$\Rightarrow I=\oint_C\{(2x+0)dx+(3x-2.0).0\}$$

$$\therefore I_1=\int_0^1 2x\,dx=\left[\frac{2x^2}{2}\right]_0^1=1$$

ii) The equation of AB is:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}\Rightarrow\frac{y-0}{0-2}=\frac{x-1}{1-1}\Rightarrow\frac{y}{-2}=\frac{x-1}{0}\Rightarrow -2x+2=0\Rightarrow -2x=-2\Rightarrow x=1$$

$\mathbf{c}_2$ is $x=1\ \therefore dx=0$

$$I=\oint_C\{(2x+y)dx+(3x-2y)dy\}$$

$$\Rightarrow I=\oint_C\{(2.1+y).0+(3.1-2y)dy\}$$

$$\therefore I_2=\int_0^2(3-2y)dy=\left[3y-\frac{2y^2}{2}\right]_0^2=2$$

iii) The equation of OB or BO is:

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2}\Rightarrow\frac{y-0}{0-2}=\frac{x-0}{0-1}\Rightarrow\frac{y}{-2}=\frac{x}{-1}\Rightarrow -y=-2x\Rightarrow y=2x$$

$\mathbf{c}_3$ is $y=2x$

Then, Given $y=2x$

$$\frac{dy}{dx}=\frac{d}{dx}(2x)$$

$$\frac{dy}{dx}=2$$

$\therefore dy=2dx$

$$I=\oint_C\{(2x+y)dx+(3x-2y)dy\}$$

$$I=\oint_C\{(2x+2x)dx+(3x-2\times 2x)2dx\}$$

$$\Rightarrow I=\oint_C\{4x\,dx+(3x-4x)2dx\}$$

$$\Rightarrow I=\oint_C\{4x+6x-8x\}dx$$

$$\therefore I_3 = \int_1^0 2x\,dx = \left[\frac{2x^2}{2}\right]_1^0 = -1$$

Finally $\mathbf{I = I_1 + I_2 + I_3 = 1 + 2 + (-1) = 2}$

b) By Green's theorem

$$\therefore \oint_c (\mathbf{Pdx + Qdy}) = \iint_R (\frac{\delta \mathbf{Q}}{\delta \mathbf{x}} - \frac{\delta \mathbf{P}}{\delta \mathbf{y}})\mathbf{dxdy}$$

$$\mathbf{I} = \oint_C \{(\mathbf{2x + y})\mathbf{dx} + (\mathbf{3x - 2y})\mathbf{dy}\}$$

Here, $\mathbf{P = 2x + y}$

$$\therefore \frac{\delta \mathbf{P}}{\delta \mathbf{y}} = \mathbf{1}$$

and

$$\mathbf{Q = 3x - 2y}$$

$$\therefore \frac{\delta \mathbf{Q}}{\delta \mathbf{x}} = \mathbf{3}$$

We have, Green's theorem

$$\therefore \oint_c (\mathbf{Pdx + Qdy}) = \iint_R (\frac{\delta \mathbf{Q}}{\delta \mathbf{x}} - \frac{\delta \mathbf{P}}{\delta \mathbf{y}})\mathbf{dxdy}$$

$$\oint_c (Pdx + Qdy) = -\iint_R (\frac{\delta P}{\delta y} - \frac{\delta Q}{\delta x})dxdy$$

Now,

$$\mathbf{I} = -\iint_R (\frac{\delta \mathbf{P}}{\delta \mathbf{y}} - \frac{\delta \mathbf{Q}}{\delta \mathbf{x}})\mathbf{dxdy}$$

$$\mathbf{I} = -\iint_R (\mathbf{1 - 3})\mathbf{dxdy}$$

$$I = -\iint_R -2dxdy$$

$$\mathbf{I} = \mathbf{2}\iint_R \mathbf{dxdy = 2A} \qquad \text{-----------------------------------(i)}$$

$$[\iint_R dxdy = A\,]$$

Now, The area of the triangle: $A = \frac{1}{2} \times base \times height$

$$A = \frac{1}{2} \times OA \times AB \qquad \text{[Figure 122]}$$

$$A = \frac{1}{2} \times 1 \times 2 = 1$$

From (i),

$I = 2\times A = 2\times 1 = 2$ $[\ A = 1]$

L.H.S = **R.H.S** (Proved)

Q# 98: Evaluate the line Integral $\mathbf{I = \oint_C \{xydx + (2x - y)dy\}}$ round the region bounded by the curves $\mathbf{y = x^2}$ and $\mathbf{x = y^2}$ by the use of Green's theorem.

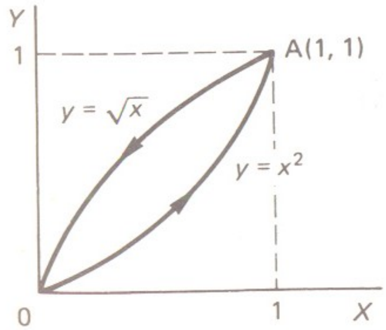

Figure # 128

Answer: Points of intersection are $\mathbf{O}(0,0)$ and $\mathbf{A}(1,1)$. P and Q are known, so there is no difficulty.

Now, By Green's theorem

$$\mathbf{I = \oint_C \{xydx + (2x - y)dy\}}$$

Here, $\mathbf{P = xy}$

$$\therefore \frac{\delta \mathbf{P}}{\delta \mathbf{y}} = \mathbf{x}$$

and $\mathbf{Q = 2x - y}$

$$\therefore \frac{\delta \mathbf{Q}}{\delta \mathbf{x}} = \mathbf{2}$$

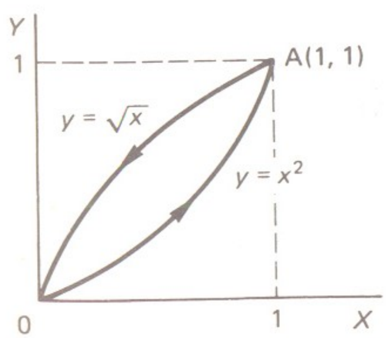

Figure # 129

We have, $\displaystyle \iint_R \left(\frac{\delta \mathbf{P}}{\delta \mathbf{y}} - \frac{\delta \mathbf{Q}}{\delta \mathbf{x}}\right)\mathbf{dxdy} = -\oint_c (\mathbf{Pdx + Qdy})$

$$\therefore -\iint_R \left(\frac{\delta \mathbf{P}}{\delta \mathbf{y}} - \frac{\delta \mathbf{Q}}{\delta \mathbf{x}}\right)\mathbf{dxdy} = \oint_c (\mathbf{Pdx + Qdy})$$

$$\therefore \oint_c (\mathbf{Pdx} + \mathbf{Qdy}) = -\iint_R (\frac{\delta \mathbf{P}}{\delta \mathbf{y}} - \frac{\delta \mathbf{Q}}{\delta \mathbf{x}})\mathbf{dxdy}$$

Now, $\mathbf{I} = -\iint_R (\frac{\delta \mathbf{P}}{\delta \mathbf{y}} - \frac{\delta \mathbf{Q}}{\delta \mathbf{x}})\mathbf{dxdy}$

$$\mathbf{I} = -\iint_R (\mathbf{x} - 2)\mathbf{dxdy}$$

$\mathbf{I} = -\int_0^1 \int_{y=x^2}^{y=\sqrt{x}} (\mathbf{x} - 2)\mathbf{dydx}$ [Here upper limit of $\mathbf{y}$ is $\mathbf{y} = \sqrt{\mathbf{x}}$ and the lower limit is $\mathbf{y} = \mathbf{x}^2$]

$$\mathbf{I} = -\int_0^1 (\mathbf{x} - 2)[\mathbf{y}]_{x^2}^{\sqrt{x}}\mathbf{dx}$$

$$\therefore \mathbf{I} = -\int_0^1 (\mathbf{x} - 2)\left[\sqrt{\mathbf{x}} - \mathbf{x}^2\right]\mathbf{dx}$$

$$\therefore \mathbf{I} = -\int_0^1 (\mathbf{x}^{3/2} - \mathbf{x}^3 - 2\mathbf{x}^{1/2} + 2\mathbf{x}^2)\mathbf{dx}$$

$$\therefore \mathbf{I} = -\left[\frac{2}{5}\mathbf{x}^{5/2} - \frac{1}{4}\mathbf{x}^4 - \frac{4}{3}\mathbf{x}^{3/2} + \frac{2}{3}\mathbf{x}^3\right]_0^1 = \frac{31}{60}$$

Now, L.H.S. $\oint \mathbf{Pdx} + \mathbf{Qdy}$

$$= \oint \mathbf{xydx} + (2\mathbf{x} - \mathbf{y})\mathbf{dy}$$

$$\mathbf{I}_1 = \int_0^1 \mathbf{x}.\mathbf{x}^2\mathbf{dx} + (2\mathbf{x} - \mathbf{x}^2)2\mathbf{xdx}$$

$$= \int_0^1 \mathbf{x}^3\mathbf{dx} + 4\mathbf{x}^2 - 2\mathbf{x}^3\mathbf{dx}$$

$$= \int_0^1 (-\mathbf{x}^3\mathbf{dx} + 4\mathbf{x}^2)\mathbf{dx}$$

$$= \left[\frac{-\mathbf{x}^4}{4} + 4\frac{\mathbf{x}^3}{3}\right]_0^1$$

$$= \left[\frac{-1}{4} + \frac{4}{3}\right] = \frac{13}{12}$$

and $\oint \mathbf{Pdx} + \mathbf{Qdy}$

$$= \oint \mathbf{xydx} + (2\mathbf{x} - \mathbf{y})\mathbf{dy}$$

$$\mathbf{I}_2 = \int_1^0 \mathbf{xydx} + (2\mathbf{x} - \mathbf{y})\mathbf{dy}$$

$$\mathbf{I_2 = \int_1^0 x.\sqrt{x}dx + (2x - \sqrt{x})\frac{1}{2}x^{-1/2}dx}$$

$$\mathbf{I_2 = \int_1^0 (x^{3/2} + x^{1/2} - \frac{1}{2})dx}$$

$$\mathbf{I_2 = \left[\frac{x^{5/2}}{\frac{5}{2}} + \frac{x^{3/2}}{\frac{3}{2}} - \frac{1}{2}x\right]_1^0}$$

$$\mathbf{I_2 = \left[0 - \frac{2}{5} - \frac{2}{3} + \frac{1}{2}\right] = -\frac{17}{30}}$$

Finally $\mathbf{I = I_1 + I_2 = \frac{13}{12} - \frac{17}{30} = \frac{31}{60}}$

$\mathbf{L.H.S = R.H.S}$ (Proved)