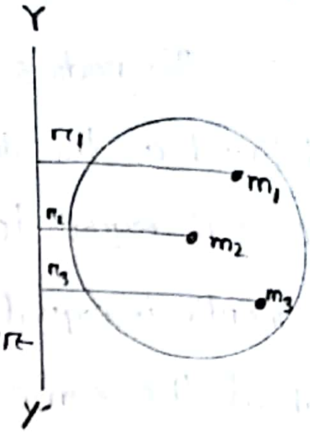
# Dynamics of Rigid Body

## Moment of inertia:




The measure of the inertia in the linear motion is called the mass of the system and its angular counter part is the so-called moment of inertia. The moment of inertia of a body is not only related to its mass but also the distributio of the mass throughout the body.

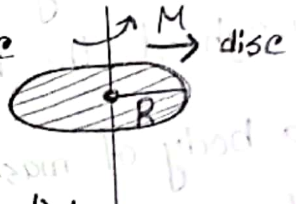
Let us consider a body of mass M and any axis YY'. Imagine the body to be composed of a large number of particles of masses $m_1, m_2, m_3$ etc. at distance $r_1, r_2, r_3 \ldots$ etc from the axis YY'. Then the moment of inertia of the particle $m_1$ about YY' is $m_1 r_1^2$, that of the particles $m_2$ i $m_2 r_2^2$ and so on. Therefore, the moment of inertia, I of the whole body, about the axis YY' is equal to the sum of $m_1 r_1^2, m_2 r_2^2, m_3 r_3^2 \ldots$ etc.

Thus, $I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots$

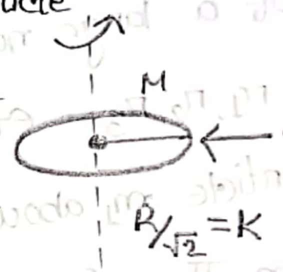
$$\therefore I = \sum_{i=1}^{n} m_i r_i^2$$

## Radius of gyration:

The radius of gyration of the body about an axis may be defined as the distance of a mass point from the same axis, whose mass is equal to the mass of the whole body and whose moment of inertia is equal to the moment of inertia of the body, if rotated about the same axis. The example is given below:




In the above diagram we have shown a disc of mass M and radius R, just below (to make you understand) we have shown a particle of mass M which is moving about the same axis in the circle of radius K.




So, for disc and particle,

$$I = \frac{MR^2}{2} \quad \text{—— (i)}$$

$$I = MK^2 \quad \text{—— (ii)}$$

Comparing (i) and (ii)

$K = \frac{R}{\sqrt{2}}$; Here K is the radius of gyration.

# Angular Momentum:

Consider a particle of mass m moving with a velocity v about an axis at a distance r.

The moment of the particle $= mv$

Moment of momentum of the particle $= mvr$

Moment of momentum is also called angular momentum and is denoted by L.

$\therefore$ Angular momentum,

$$L = mvr$$

$$\Rightarrow L = m(r\omega)r$$

$$\Rightarrow L = mr^2\omega$$

The moment of inertia of the particle about the axis of rotation,

$$I = mr^2$$

$$\Rightarrow L = I\omega$$

Angular momentum of rotatory motion is similar to linear momentum in translatory motion.

Angular momentum is a vector quantity. Its dimensions are $[ML^2T^{-1}]$

## Torque:

Consider a particle of mass ~~M~~ $m$ moving about an axis in a circular path of radius $r$. Let an external force $F$ act on the particle along the tangent to the circular path. The moment of the force $= Fr$. This moment of force is also called torque represented by the symbol $\tau$

$$\tau = Fr$$

But, $F = ma = mr\alpha$

$$\tau = (mr\alpha)r = mr^2\alpha$$

$$I = mr^2$$

$$\tau = I\alpha$$

Hence Torque is equal to the product of moment of inertia and angular acceleration. Torque is also defined as the rate of change of angular momentum.

$$\tau = \frac{dL}{dt}$$

$$\Rightarrow \tau = \frac{d(I\omega)}{dt} = I\left(\frac{d\omega}{dt}\right)$$

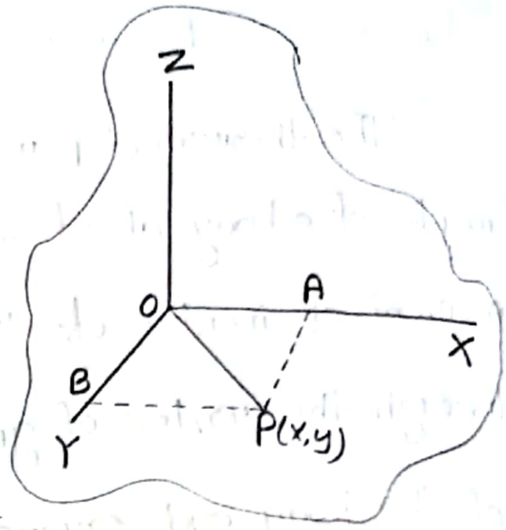
$$\tau = I\alpha$$

~~Hence torque plays the same part in rotatory motion of inertia and angular acceleration. Torque is also defined as the rate of change~~

- Hence, torque plays the same part in rotatory motion as force in translatory motion. It is a vector quantity. Its dimensions are $[ML^2T^{-2}]$. Its units are $kg\text{-}m^2s^{-2}$

## Theorem of perpendicular axis:

### (a) For a plane Lamina:




Consider a plane lamina having the axes OX and OY in the plane of the lamina. The axis OZ passes through O and is perpendicular to the plane of the lamina. Let the lamina be divided into a large number of particles, each of mass m. Let a particle of mass m be at P with coordinates $(x,y)$ and situated at a distance $r$ from the point of intersection of the axis.

$$\therefore r^2 = x^2 + y^2 \quad \text{—— (I)}$$

The moment of inertia of the particle P about the axis.

$$OZ = mr^2$$

The moment of inertia of the whole lamina about the axis OZ is given by,

$$I_Z = \sum mr^2 \quad \text{—— (II)}$$

The moment of inertia of the whole lamina about the axis OX, ~~is~~

$$I_x = \sum my^2 \quad \text{—— (III)}$$

Similarly, $I_y = \sum mx^2$ —— (IV)

From equation (II),

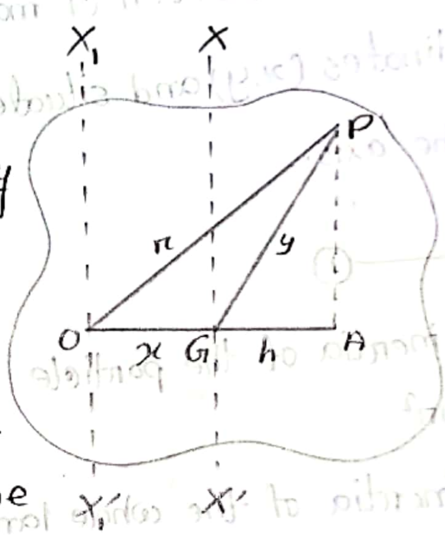
$$I_Z = \sum mr^2 = \sum m(x^2+y^2)$$

$$\Rightarrow I_Z = \sum mx^2 + \sum my^2 = I_y + I_Z$$

$$\therefore I_Z = I_x + I_y$$

# Theorem of Parallel Axes:

The theorem of parallel ~~axis~~ axes state that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through the centre of gravity and the product of the mass of the body and square of perpendicular distance between the two parallel axes.




Let us consider a plane laminar body having its center of gravity at G. The axis $XX'$ passes through the centre of gravity and its perpendicular to the plane of the laminar. The axis $X_1X_1'$ passes through the point O and is parallel to the axis $XX'$. The distance between the two parallel axes is $x$, which is shown in fig.

Let the lamina be divided into large number of particles each of mass m. The moment of inertia of the particle of mass m at P about $X_1X_1'$ is equal to $mr^2$. The moment of inertia of the whole lamina about the axis $X_1X_1'$ is given by,

$$I_0 = \sum mr^2 \text{ ——— } (1)$$

In the $\Delta OPA$ $\quad OP^2 = (OA)^2 + (AP)^2$

$$\Rightarrow r^2 = (x+h)^2 + (AP)^2$$

$$\Rightarrow r^2 = x^2 + 2xh + h^2 + (AP)^2$$

$$\Rightarrow r^2 = x^2 + 2xh + h^2 + y^2 - h^2$$

$$\Rightarrow r^2 = x^2 + y^2 + 2xh$$

putting the above value in eq ①

$$I_0 = \sum m(x^2 + y^2 + 2xh)$$

$$\Rightarrow I_0 = \sum mx^2 + \sum my^2 + \sum m(2xh)$$

$$\Rightarrow I_0 = Mx^2 + I_G + 2x\sum mh$$

Here, $\sum my^2 = I_G$ and $\sum mh = 0$

This is because the body balances about centre of mass at G. Therefore, the algebraic some of moments of all the particles abou the centre of gravity, i.e.

$$\sum mgh = 0$$

As $g$ is constant $\quad \sum mh = 0$

Hence, we can write, $\quad I_0 = I_G + Mx^2$

## Moment of Inertia of a Thin Uniform Bar (Rod)

⇒ Consider a thin uniform bar AB of mass M and length $l$ rotating about an axis passing through its centre and perpendicular to its length (axis $YY'$)

Mass of the bar $= M$

Length of bar $= l$

Mass per unit length $= \left(\frac{M}{l}\right)$

Take an element of length $dx$ at a distance $x$ from the axis.

Mass of the element $= \left(\frac{M}{l}\right)dx$

Moment of inertia of the element about ~~an~~ the axis $YY'$

$$= \left[\left(\frac{M}{l}\right)dx\right]x^2$$

Moment of inertia of the bar AB about the axis $YY'$.

$$I = 2\int_0^{l/2}\left(\frac{M}{l}\right)x^2dx$$

$$= \frac{2M}{l}\left[\frac{x^3}{3}\right]_0^{l/2}$$

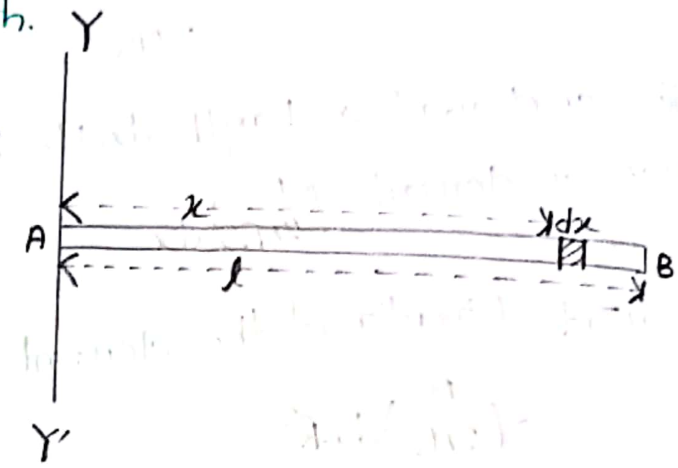
$$I = \frac{Ml^2}{12}$$

Also, $I = MK^2$

$$\therefore MK^2 = \frac{Ml^2}{12}$$

$$\Rightarrow K = \frac{l}{2\sqrt{3}}$$

## Moment of Inertia of a Bar AB about an axis passing through one end and perpendicular to its length.




Here, $I = \int_0^l \left(\frac{M}{l}\right) x^2 dx$

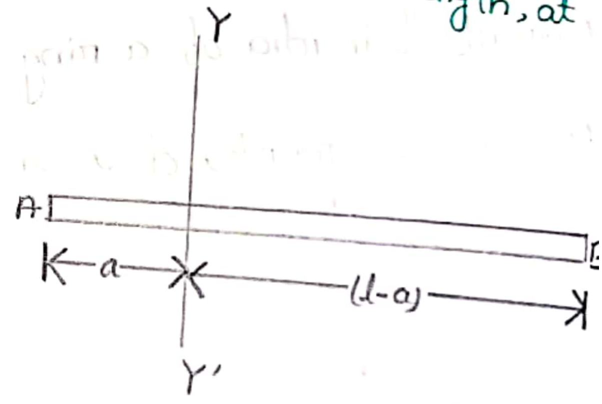
$$\Rightarrow I = \frac{Ml^2}{3}$$

But, $I = MK^2$

$$\Rightarrow MK^2 = \frac{Ml^2}{3}$$

$$\Rightarrow K = 1/\sqrt{3}$$

## Moment of Inertia of a Bar about an axis perpendicular to its Length, at a distance a from one end.

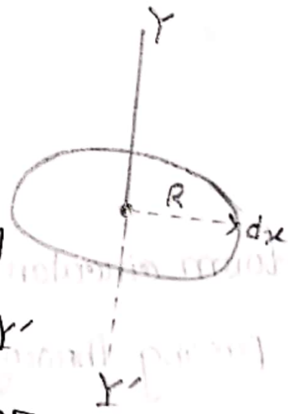



Here, $I = \int_{-a}^{l-a} \left(\frac{M}{l}\right) x^2 dx$

$$\Rightarrow I = \frac{M}{l}\left[\frac{x^3}{3}\right]_{-a}^{l-a}$$

$$\therefore I = M\left[\frac{2}{3} - la + a^2\right]$$

## Moment of Inertia of a Ring:




Consider a thin uniform ring of mass M and radius R. The ring rotates about an axis YY' passing through its centre and per-pendicular to its plane.

Mass of the ring = M

Length of the ring $= 2\pi R$

Mass per unit of length $= \frac{M}{2\pi R}$

Take an element of length $dx$. Its distance from the axis is $R$.

Mass of element $= \left(\frac{M}{2\pi R}\right) dx$

Moment of inertia of the element about the axis.

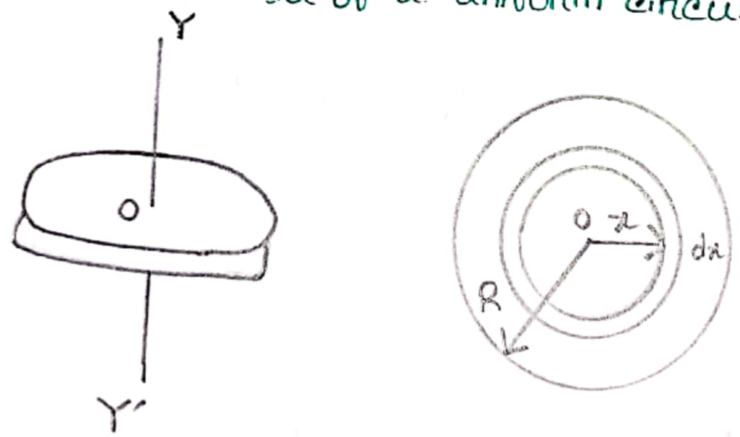
$$= \left(\frac{M}{2\pi R}\right) dx \cdot R^2$$

Moment of inertia of the ring,

$$I = \frac{MR^2}{2\pi R} \int_0^{2\pi R} dx = MR^2$$

Moment of inertia of a ring about its diameter $= \frac{1}{2} MR^2$

## Moment of inertia of a uniform circular disc




consider a uniform circular disc of mass $M$ and radius $R$ rotating about an axis passing through its centre and perpendicular to its plane.

Mass of the disc $= M$

Area of disc $= \pi R^2$

Mass per unit area $= \left(\frac{M}{\pi R^2}\right)$

Consider a thin element of the disc of radius $x$ and radial thickness $dx$.

Area of the element $= 2\pi x \, dx$

Mass of the element $= \left(\frac{M}{\pi R^2}\right) 2\pi x \, dx$

$$= \left(\frac{2M}{R^2}\right) x \, dx$$

Moment of inertia of the element about the axis of rotation.

$$= (\text{Mass})(x^2)$$

$$= \left[\left(\frac{2M}{R^2}\right) x \, dx\right] x^2 = \left(\frac{2M}{R^2}\right) x^3 dx$$

Moment of inertia of the whole disc about the axis of rotation,

$$I = \int_0^R \left(\frac{2M}{R^2}\right) x^3 dx$$

$$\Rightarrow I = \frac{2M}{R^2} \int_0^R x^3 dx$$

$$\Rightarrow I = \frac{2M}{R^2} \left[\frac{x^4}{4}\right]_0^R$$

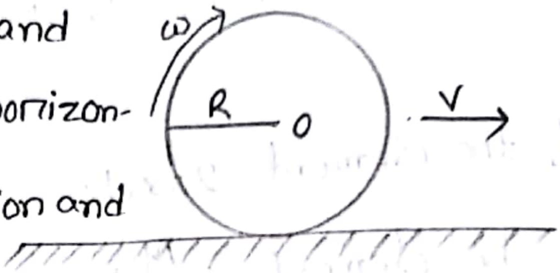
$$\Rightarrow I = \frac{2M}{R^2} \left[\frac{R^4}{4}\right]$$

$$= \frac{MR^2}{2}$$

Moment of inertia of a circular disc about its diameter $= \frac{MR^2}{4}$

## Kinetic energy of a body Rolling on a Horizontal plane.

Consider body of mass M, Radius R and moment of Inertia I rolling on a horizontal plane. It has motion of a rotation and motion of translation. Its angular velocity is $\omega$ and linear velocity is v.




$$v = R\omega$$

The body possesses kinetic energy due to motion of rotation and motion of translation. The total kinetic energy of the body at any instant is given by

$$E = \frac{1}{2}I\omega^2 + \frac{1}{2}Mr^2 = \frac{1}{2}MK^2 + \frac{1}{2}MV^2$$

$$E = \frac{1/2\, MK^2 r^2}{R^2} + \frac{1}{2}Mr^2$$

$$= 1/2\, MV^2\left[\frac{K^2}{R^2} + 1\right]$$

Special cases:

① For a circular disc:

$K^2 = R^2/2$ , $K^2/R^2 = 1/2$

$\therefore E = \frac{1}{2}MV^2(1/2 + 1)$

$\therefore E = 2/4\, MV^2$

(2) For a sphere,

$$K^2 = \frac{2}{5}R^2$$

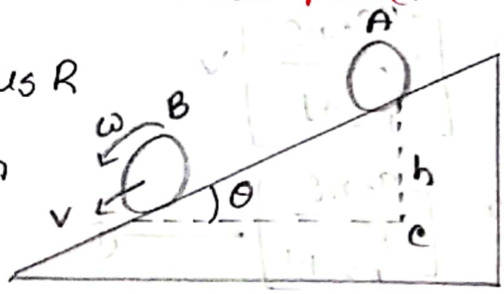
$$\Rightarrow K^2/R^2 = 2/5$$

$$\therefore E = \frac{1}{2}Mv^2\left[\frac{2}{5}+1\right]$$

$$\therefore E = \frac{7}{10}Mv^2$$

## Acceleration of a body Rolling down an Inclined plane.

~~Ex~~ consider a body of mass M, radius R and moment of inertia I rolling down an inclined plane. Suppose the body starts at A and reaches B after covering a distance AB = l. At B, its angular velocity is $\omega$ and linear velocity is v. The vertical distance through which the body has moved = h




Loss in potential energy $= Mgh = \cancel{mg}\ Mgl\sin\theta$

Gain in Kinetic energy $= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2$

$$= \frac{1}{2}\frac{Iv^2}{R^2} + \frac{1}{2}Mv^2$$

$$= \frac{MK^2v^2}{2R^2} + \frac{Mv^2}{2}$$

$$= \frac{Mv^2}{2}\left(\frac{K^2}{R^2}+1\right)$$

As the loss in potential energy is equal to the gain in kinetic energy

$$\frac{Mv^2}{2}\left(\frac{K^2}{R^2}+1\right)=Mg\,l\sin\theta$$

$$v^2=\left[\frac{2g\sin\theta}{(K^2/R^2+1)}\right]l \quad \text{——— (i)}$$

Differentiating eq (i) with respect to time,

$$2v\frac{dv}{dt}=\left[\frac{2g\sin\theta}{\frac{K^2}{R^2}+1}\right]\frac{dl}{dt}$$

But $\frac{dl}{dt}=v$ and $\frac{dv}{dt}=a$

$$\therefore 2va=\left[\frac{2g\sin\theta}{\frac{K^2}{R^2}+1}\right]v$$

$$\therefore a=\left[\frac{g\sin\theta}{\frac{K^2}{R^2}+1}\right] \quad \text{——— (ii)}$$

From eq (ii) it is clear that a is independent of the mass of the body. The value of a depends upon $\frac{K^2}{R^2}$.

If $\frac{K^2}{R^2}$ is large, a is small.

If $\frac{K^2}{R^2}$ is small, a will be large.

Special case,

In this case of a solid sphere,

$$\frac{K^2}{R^2}=2/5$$

and $a_1=\frac{5}{7}(g\sin\theta)$ ——— (iii)

In the case of a disc, $K^2/R^2=1/2$

$$\therefore a_2=2/3\,g\sin\theta \quad \text{——— (iv)}$$

In the case of the ring, $\frac{K^2}{R^2} = 1$

$$\therefore a_3 = \frac{g \sin\theta}{2}$$

In the case of the spherical shell, $\frac{K^2}{R^2} = \frac{2}{3}$

$$\therefore a_4 = \frac{3}{5} g \sin\theta \text{ --- (v)}$$

as $a_1 > a_4$ the solid sphere will roll down first than the hollow spher of the same radius. This principle is used in ~~spe~~ separating hollow and solid lead shots. The solid lead shots will reach the end of the inclined plane first and the hollow lead shots will reach later on. Similarly a hollow ring will reach later than a disc.