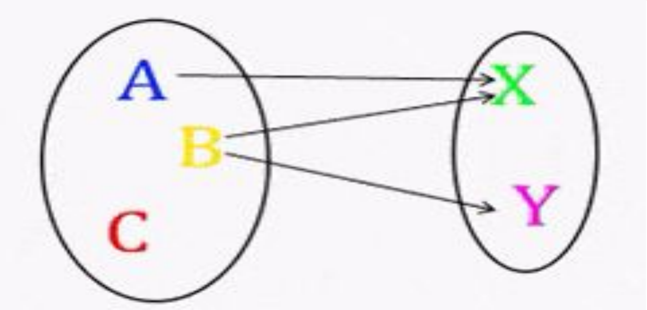
Discrete mathematics



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1.3 Predicates & Quantifiers

Why Predicates & Quantifiers?

- **Propositional Logic** cannot adequately express the meaning of all statements in mathematics and in Natural Language Processing (NLP)

Example:

- ❑ Every computer connected to the university network is functioning properly.
- ❑ MATH3 is functioning properly.
- ❑ CS2 is under attack by an intruder.
- ❑ There is a computer on the university network that is under attack by an intruder.

No rules of propositional logic to conclude

- A more powerful type of logic called "Predicate Logic"
- This logic is used to express the meaning of a wide range of statements in mathematics and computer science

How Predicates & Quantifiers work?

What is Predicate?

i. x is greater than 3
ii. x is greater than y
iii. x is greater than y and z
iv. x can speak English

These sentences are not propositions as values of x, y, z are not defined

To represent these sentences mathematically predicate logic is used

In the example (i):

Subject/Variable

"X is greater than 3"

Predicate

This statement has 2 parts

- The first part(subject) is the variable **x**
- The second part(predicate) is " is greater than 3"

denoted as ⇒ $P(x)$

Proposition Function

Proposition Function

Proposition Function:

Once a value has been assigned to x, the statement of P(x) becomes a proposition and has a truth value. P is called **“Proposition Function”.**

Example:

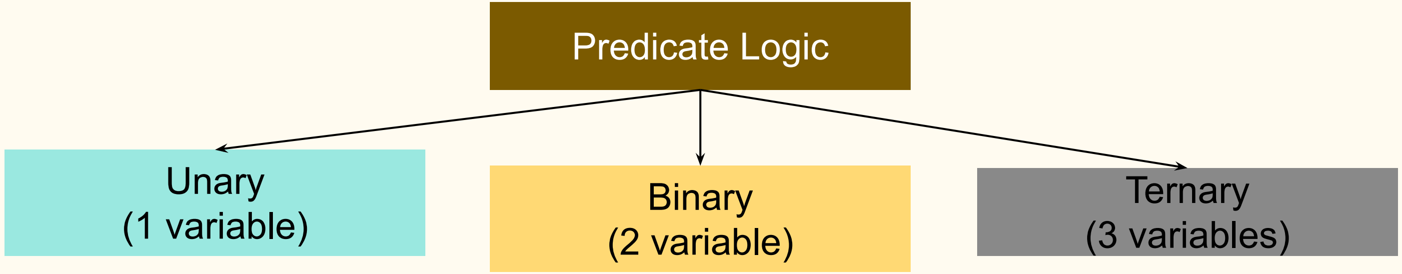
“x> 20”

This statement can be denoted as **P(x)**
Now once a value has been assigned to the variable x the statement P(x) becomes a proposition that has a truth value.
P(8) is the statement for **“x> 20” that “8>20”** which has a truth value of **“False”.**

Types of PREDICATE LOGIC

Types of Predicate Logic

- There are 3 types of predicate logic



Examples:

i. X is greater than 3 $\Rightarrow \boldsymbol{P(x)}$ [Unary]

ii. X is greater than y $\Rightarrow \boldsymbol{Q(x, y)}$ [Binary]

iii. X is greater than y and z $\Rightarrow \boldsymbol{R(x, y, z)}$ [Ternary]

iv. X can speak in English $\Rightarrow \boldsymbol{E(x)}$ [Unary]

Examples

- **Example-1:**

Let P(x) denote "x is greater than 3". What are the truth values of P(4) and P(2)?

Sol.

If x=2; P(2) $\Rightarrow 2\ is\ greater\ than\ 3$ (False value)

If x=4; P(4) $\Rightarrow 4\ is\ greater\ than\ 3$ (True value)

Example-2:

Let $Q(x, y)$ denote "$x = y + 3$" . What are the truth values of Q(1,2) and Q(3,0)?

Sol.

If x=1 and y=2; Q(1,2) $\Rightarrow 1 = 2 + 3$(False value)

If x=3 and y=0; Q(3,0) $\Rightarrow 3 = 0 + 3$(True value)

Examples(CONTD.)

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Example-3:

Let R(x, y, z) denote "$x + y = z$" . What are the truth values of R(1,2,3) and R(0,0,1)?

Sol.

If x=1 and y=2 and z=3; R(1,2,3) $\Rightarrow 1 + 2 = 3$(True value)

If x=0 and y=0 and z=1; R(0,0,1) $\Rightarrow 0 + 0 = 1$(False value)

Examples(CONTD.)

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Example-4:

Let P(x) be the statement " the word x contains the letter a". What are the truth values?

(a) P(Orange)

(b) P(True)

(c) P(Lemon)

(d) P(False)

Sol.

(a) If x=Orange ; P(Orange)⇒ "*the word Orange contains the letter a*"(True value)

(b) If x=True ; P(True)⇒ "*the word True contains the letter a*"(False value)

(c) If x=Lemon ; P(Lemon)⇒ "*the word Lemon contains the letter a*"(False value)

(d) If x=False ; P(False)⇒ "*the word False contains the letter a*"(True value)

Examples(CONTD.)

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Example-5:

Let $A(x)$ denote the statement " Computer x is under attack by an intruder". Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are the truth values of A(CS1), A(CS2) and A(MATH1)?

Sol.

(a) If x=CS1 ; A(CS1)$\Rightarrow$ "Computer CS1 is under attack by an intruder"(False value)

(b) If x=CS2 ; A(CS2)$\Rightarrow$ "Computer CS2 is under attack by an intruder"(True value)

(c) If x=MATH1 ; A(CS2)$\Rightarrow$ "Computer MATH1 is under attack by an intruder"(True value)

Quantifiers

Quantifiers

- When the variables in a propositional function are assigned values, the resulting statement becomes a proposition with a certain value. *"Quantification"* expresses the extent to which a predicate is true over a range of elements.

In English the words [all, Some, many, none, few, etc.....] are used in quantification.

Example:

"x can speak in English"

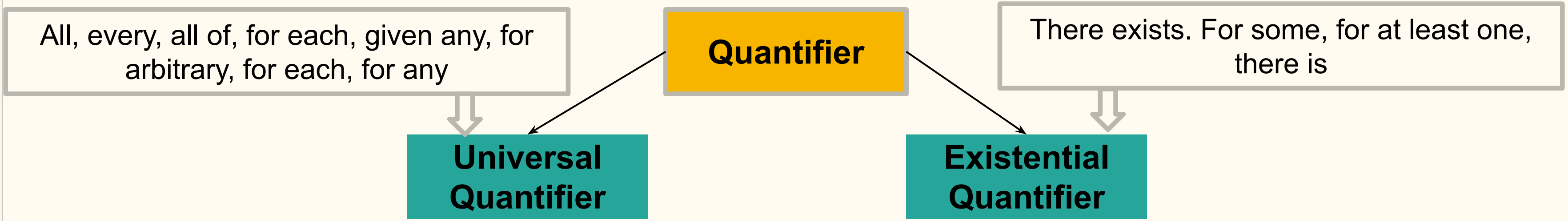
P(x)= x can speak English **[Possible values for x ⇒ Some, a student, All people, ...]**

Predicate Calculus (Definition):

The area of logic that deals with predicates and quantifiers is called **"Predicate Calculus".**

Types of QUANTIFIERS

Types of Quantifiers



Universal Quantification:

- Denoted by "∀"
- Which tells us that a predicate is true for every element under consideration.

Existential Quantification:

- Denoted by "∃"
- Which tells us that a predicate is true for one or more element under consideration.

Quantifiers

- **Universal Quantification:**

The universal quantification of P(x) is the statement "P(x) for all values of x in the domain"

"$\forall_x P(x)$" read as "for all x P(x)" / "for every x P(x)"

Example:

P(x) = x can speak in English.

$\Rightarrow \underbrace{\textbf{Everyone}}_{\forall_x}\ \underbrace{\textbf{can speak in English.}}_{P(x)}$

Existential Quantification:

The existential quantification of P(x) is the statement "there exists an element x in the domain such that P(x)"

"$\exists_x P(x)$" read as

"There is an x such that P(x)"/"There is at least one x such that P(x)"/ "for some x P(x)'

Example:

P(x) = x can speak in English.

$\Rightarrow \underbrace{\textbf{There is a student}}_{\exists_x}\ \underbrace{\textbf{can speak in English.}}_{P(x)}$

Examples

-

Example-1:

Let Q(x) be the statement "x<2". What is/are the truth values of the quantification $\forall_x Q(x)$, where the domain consists of all real numbers?

Sol.

Here the quantification is: $\forall_x Q(x)$

Domain ={real numbers}={…….,-1,0,1,2}

If x= -1; $\boldsymbol{x < 2 \Rightarrow -1 < 2}$; true

If x= 0; $\boldsymbol{x < 2 \Rightarrow 0 < 2}$; true

If x= 1; $\boldsymbol{x < 2 \Rightarrow 1 < 2}$; true

If x= 2; $\boldsymbol{x < 2 \Rightarrow 2 < 2}$; false

Therefore as in case of all values the Q(x) is not true

so the truth value of the quantification is **false**.

Examples(CONTD.)

- **Example-2:**

What is the truth value of the quantification $\forall_x P(x)$, where Let P(x) be the statement **"$x^2 < 10$"** and the domain consists of the positive integers not exceeding 4?

Sol.

Given that, P(x) be the statement **"$x^2 < 10$"**
Here the quantification is: $\forall_x P(x)$
Domain , x={0,1,2,3,4}

If x= 0; $x^2 < 10 \Rightarrow 0^2 < 10$; true
If x= 1; $x^2 < 10 \Rightarrow 1^2 < 10$; true
If x= 2; $x^2 < 10 \Rightarrow 2^2 < 10$; true
If x= 3; $x^2 < 10 \Rightarrow 3^2 < 10$; true
If x= 4; $x^2 < 10 \Rightarrow 4^2 < 10$; false

Therefore as in case of all values the P(x) is not true
so the truth value of the quantification is **false**.

Examples(CONTD.)

Example-3:

What is the truth value of the quantification $\exists_x P(x)$, where Let P(x) be the statement **"$x^2 > 10$"** and the domain consists of the positive integers not exceeding 4?

Sol.

Given that, P(x) be the statement **"$x^2 > 10$"**
Here the quantification is: $\exists_x P(x)$
Domain , x={0,1,2,3,4}

If x= 0; $\boldsymbol{x^2 < 10 \Rightarrow 0^2 > 10}$; false
If x= 1; $\boldsymbol{x^2 < 10 \Rightarrow 1^2 > 10}$; false
If x= 2; $\boldsymbol{x^2 < 10 \Rightarrow 2^2 > 10}$; false
If x= 3; $\boldsymbol{x^2 < 10 \Rightarrow 3^2 > 10}$; false
If x= 4; $\boldsymbol{x^2 < 10 \Rightarrow 4^2 > 10}$; true

Therefore as in case of all values the P(x) is not false but for some values it is true so the truth value of the quantification is **TRUE**.

Examples(CONTD.)

- **Example-4:**

Let P(x) be the statement "$\mathbf{x} = \boldsymbol{x^2}$" . If the domain consists of the integers, what are the truth values?

a) P(0)
b) P(1)
c) P(2)
d) P(-1)
e) $\exists_x P(x)$
f) $\forall_x P(x)$

Sol.

Given that, Let P(x) be the statement "$\mathbf{x} = \boldsymbol{x^2}$"

(a) Domain , x={0}

If x= 0; $\boldsymbol{x = x^2 \Rightarrow 0 = 0}$; true

Therefore the truth value of the quantification is **TRUE**.

Examples(CONTD.)

- **Example-4:**

Let P(x) be the statement **"$x = x^2$"** . If the domain consists of the integers, what are the truth values?

a) P(0)
b) P(1)
c) P(2)
d) P(-1)
e) $\exists_x P(x)$
f) $\forall_x P(x)$

Sol.

Given that, Let P(x) be the statement **"$x = x^2$"**

(b) Domain , x={1}

If x= 1; $x = x^2 \Rightarrow 1 = 1$; true

Therefore the truth value of the quantification is **TRUE**.

Examples(CONTD.)

- **Example-4:**

Let P(x) be the statement "**x** $= \boldsymbol{x^2}$" . If the domain consists of the integers, what are the truth values?

a) P(0)
b) P(1)
c) P(2)
d) P(-1)
e) $\exists_x P(x)$
f) $\forall_x P(x)$

Sol.

Given that, Let P(x) be the statement "**x** $= \boldsymbol{x^2}$"

(c) Domain , x={2}

If x= 2; $\boldsymbol{x = x^2 \Rightarrow 2 = 4}$; false

Therefore the truth value of the quantification is **FALSE**.

Examples(CONTD.)

- **Example-4:**

Let P(x) be the statement **"x = x^2"** . If the domain consists of the integers, what are the truth values?

a) P(0)
b) P(1)
c) P(2)
d) P(-1)
e) $\exists_x P(x)$
f) $\forall_x P(x)$

Sol.

Given that, Let P(x) be the statement **"x = x^2"**

(d) Domain , x={-1}

If x= -1; $\boldsymbol{x = x^2 \Rightarrow -1 = 1}$; false

Therefore the truth value of the quantification is **FALSE**.

Examples(CONTD.)

Example-4:

Let P(x) be the statement **"x = x^2"** . If the domain consists of the integers, what are the truth values?

a) P(0)

b) P(1)

c) P(2)

d) P(-1)

e) $\exists_x P(x)$

f) $\forall_x P(x)$

Sol.

Given that, Let P(x) be the statement **"x = x^2" and $\exists_x P(x)$**

Domain , x={..........,-3,-2,-1,0,1,2,3,........}

(e) If x= -2; $x = x^2 \Rightarrow -2 = 4$; false

If x= -1; $x = x^2 \Rightarrow -1 = 1$; false

If x= 0; $x = x^2 \Rightarrow 0 = 0$; true

If x= 1; $x = x^2 \Rightarrow 1 = 1$; true

If x= 2; $x = x^2 \Rightarrow 2 = 4$; false

Therefore as in case of all values the P(x) is not false but for some values it is true so the truth value of the quantification is **TRUE**.

Examples(CONTD.)

Example-4:

Let P(x) be the statement **“x = x^2”** . If the domain consists of the integers, what are the truth values?

a) P(0)
b) P(1)
c) P(2)
d) P(-1)
e) $\exists_x P(x)$
f) $\forall_x P(x)$

Sol.

a) Given that, Let P(x) be the statement **“x = x^2” and $\forall_x P(x)$**

Domain , x={..........,-3,-2,-1,0,1,2,3,........}

(f) If x= -2; $x = x^2 \Rightarrow -2 = 4$; false

If x= -1; $x = x^2 \Rightarrow -1 = 1$; false

If x= 0; $x = x^2 \Rightarrow 0 = 0$; true

If x= 1; $x = x^2 \Rightarrow 1 = 1$; true

If x= 2; $x = x^2 \Rightarrow 2 = 4$; false

Therefore as in case of all values the P(x) is not true so the truth value of the quantification is **FALSE**.

Translating from English into Logical Expression using Predicates & Quantifiers

Exercises

Translate the following sentences from English into logical connectives using predicate & quantifiers:

(a) Every student in the class has studied calculus.

(b) For every person x, if person x is a student in the class then x has studied calculus.

(c) Some student in this class has visited Mexico.

(d) There is a person x, having the properties that x is a student in the class and x has visited Mexico.

(e) For every person x, if x is a student in the class, then x has visited Mexico or x has visited Canada.

Exercise(CONTD.)

- (a) Every student in the class has studied calculus.

Sol.

Domain, x= student

Every student in the class has studied calculus.

$\boxed{\forall}$ x $\boxed{C(x)}$

Ans: $\forall_x C(x)$

(b) For every person x, if person x is a student in the class then x has studied calculus.

Sol.

Domain, x= person

For every person x, if person x is a student in the class then x has studied calculus.

$\boxed{\forall}$ x $\boxed{S(x)}$ $\rightarrow$ $\boxed{C(x)}$

Ans: $\forall_x (S(x) \rightarrow C(x))$

Exercise(CONTD.)

(c) Some student in this class has visited Mexico.

<u>Sol.</u>

Domain, x= student

Some student in this class has visited Mexico.

$\boxed{\exists}$ x $\boxed{M(x)}$

Ans: $\exists_x M(x)$

(d) There is a person x, having the properties that x is a student in the class and x has visited Mexico.

<u>Sol.</u>

Domain, x= person

There is a person x, having the properties that x is a student in the class and x has visited Mexico.

$\boxed{\exists}$ x $\boxed{S(x)}$ $\wedge$ $\boxed{M(x)}$

Ans: $\exists_x (S(x) \wedge M(x))$

Exercise(CONTD.)

-

(e) For every person x, if x is a student in the class, then x has visited Mexico or x has visited Canada.

Sol.

Domain, x= person

For every person x, if x is a student in the class, then x has visited Mexico or x has visited Canada.

$\boxed{\forall}$ x $\boxed{S(x)}$ $\rightarrow$ $\boxed{M(x)}$ $\vee$ $\boxed{C(x)}$

$$\text{Ans: } \forall_x \quad (S(x) \rightarrow M(x) \vee C(x))$$

Translating from English into Logical Expression using Predicates & Quantifiers

Exercises

Let **P(x)** be the statement **"x can speak Russian"** and
Let **Q(x)** be the statement **"x knows the computer language C++"**
Express each of these statements in terms of P(x), Q(x), quantifiers and logical connectives.
Here domain is ***"all the students at your school"***.

(a) There is a student at your school who can speak Russian and who knows C++.
(b) There is a student at your school who can speak Russian but who doesn't know C++.
(c) No student at your school can speak Russian or knows C++.
(d) Every student at your school either can speak Russian or knows C++.

Exercises(Sol.)

Let **P(x)** be the statement **"x can speak Russian"** and

Let **Q(x)** be the statement **"x knows the computer language C++"**

Express each of these statements in terms of P(x), Q(x), quantifiers and logical connectives.

Here domain is ***"all the students at your school"***.

(a) There is a student at your school who can speak Russian and who knows C++.

Sol.

There is a student at your school who can speak Russian and who knows C++.

$\exists$ x P(x) $\land$ Q(x)

Ans. $\exists_x (P(x) \land Q(x))$

(b) There is a student at your school who can speak Russian but who doesn't know C++.

Sol.

There is a student at your school who can speak Russian but who doesn't know C++.

$\exists$ x P(x) $\land$ $\neg$Q(x)

Ans. $\exists_x (P(x) \land \neg Q(x))$

Exercises(Sol.)

Let **P(x)** be the statement **"x can speak Russian"** and

Let **Q(x)** be the statement **"x knows the computer language C++"**

Express each of these statements in terms of P(x), Q(x), quantifiers and logical connectives.

Here domain is ***"all the students at your school"***.

(c)No student at your school can speak Russian or knows C++.

Sol.

No student at your school can speak Russian or knows C++.

$\forall$ x $P(x)$ $\vee$ $Q(x)$

Ans. $\neg\forall_x \ (P(x) \vee Q(x))$

(d) Every student at your school either can speak Russian or knows C++.

Sol.

Every student at your school either can speak Russian or knows C++.

$\forall$ x $P(x)$ $\vee$ $Q(x)$

Ans. $\forall_x \ (P(x) \vee Q(x))$

Exercises(To-Do)

Let **C(x)** be the statement **“x has a cat”** and
Let **D(x)** be the statement **“x has a dog”** and
Let **F(x)** be the statement **“x has a ferret”**
Express each of these statements in terms of C(x), D(x), F(x), quantifiers and logical connectives.
Here domain is ***“all the students in your class”***.

(a) A student in your class has a cat, a dog and a ferret.
(b) All students in your class have a cat, a dog or a ferret.
(c) Some student in your class has a cat and a ferret, but not a dog .
(d) No student in your class has a cat, a dog and a ferret.
(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has one of these animals as a pet.

Translating from Logical Expression into English using Predicates & Quantifiers

Exercises

- Translate these statements into English where

C(x) be the statement **"x is a comedian"** and

F(x) be the statement **"x is funny"**

Here domain consists of *"all people"*.

(a) $\forall_x (C(x) \rightarrow F(x))$

(b) $\forall_x (C(x) \wedge F(x))$

(c) $\exists_x (C(x) \rightarrow F(x))$

(d) $\exists_x (C(x) \wedge F(x))$

Exercises(Sol.)

- Translate these statements into English where
C(x) be the statement **"x is a comedian"** and
F(x) be the statement **"x is funny"**
Here domain consists of *"all people"*.

Note:

- In case of translation to English for better language expression we have converted the sentence while finally writing the answer
- In this case answers can be different and all will be considered correct unless the meaning is ok

(a) $\forall_x (C(x) \rightarrow F(x))$

Sol.

$$\forall_x (C(x) \rightarrow F(x))$$

For all x (all people) if x is a comedian then x is funny.

Ans. Every comedian is funny.

Exercises(Sol.)

- Translate these statements into English where
C(x) be the statement **"x is a comedian"** and
F(x) be the statement **"x is funny"**
Here domain consists of *"all people"*.

(b) $\forall_x (C(x) \land F(x))$

Sol.

$$\forall_x (C(x) \land F(x))$$

$\underbrace{\text{For all x (all people),}}$ x is a comedian $\boxed{\text{and}}$ x is funny.

Ans. Every person is a funny comedian.

Exercises(Sol.)

- Translate these statements into English where
C(x) be the statement **"x is a comedian"** and
F(x) be the statement **"x is funny"**
Here domain consists of *"all people"*.

(c)$\exists_x (C(x) \rightarrow F(x))$

Sol.

$$\exists_x (C(x) \rightarrow F(x))$$

$\underbrace{\text{There is a x(people)}}$ $\boxed{\text{if}}$ x is a comedian $\boxed{\text{then}}$ x is funny.

Ans. There exists a person such that if he is a comedian, then he is funny.

Exercises(Sol.)

- Translate these statements into English where
C(x) be the statement **"x is a comedian"** and
F(x) be the statement **"x is funny"**
Here domain consists of *"all people"*.

(d) $\exists_x (C(x) \wedge F(x))$
Sol.

$$\exists_x (C(x) \wedge F(x))$$

$\underbrace{\text{There is a x (people),}}$ x is a comedian $\boxed{\text{and}}$ x is funny.

<u>**Ans.**</u> Some comedians are funny.

Negating Quantified Expression

Rules

De Morgan's law for negation of quantifiers:

Original Statement	**∀_x P(x)**	**∃_x P(x)**
Negation	〖¬∀〗_x P(x)	〖¬∃〗_x P(x)
After Negation Statement	∃_x¬P(x)	∀_x¬P(x)

Exercises

Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English.

(a) Every student in your class has taken a course in calculus.
(b) There is an honest politician.
(c) Some old dogs can learn new tricks.
(d) No rabbit knows calculus.
(e) Every bird can fly.

Exercises(Sol.)

-

(a) Every student in your class has taken a course in calculus.

<u>Sol.</u>

Logical Expression of the statement: $\forall_x C(x)$

Negation of the statement: $\neg(\forall_x C(x))$

After Negation: $\exists_x \neg C(x)$

<u>Ans</u>: There is a student in your class who has not taken a course in calculus.

(b) There is an honest politician.

<u>Sol.</u>

Logical Expression of the statement: $\exists_x H(x)$

Negation of the statement: $\neg(\exists_x H(x))$

After Negation: $\forall_x \neg H(x)$

<u>Ans</u>: Every politician is not honest/ dishonest.

Not all politicians are honest.

Exercises(Sol.)

(c) Some old dogs can learn new tricks.

Sol.

Logical Expression of the statement: $\exists_x T(x)$

Negation of the statement: $\neg(\exists_x T(x))$

After Negation: $\forall_x \neg T(x)$

Ans: No old dogs can learn new tricks.

(d) No rabbit knows calculus.

(will be discussed in special cases)

(e) Every bird can fly.

Sol.

Logical Expression of the statement: $\forall_x F(x)$

Negation of the statement: $\neg(\forall_x F(x))$

After Negation: $\exists_x \neg F(x)$

Ans: There exists a bird that cannot fly.

Special Cases

Exercises

Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English.

(a) No rabbit knows calculus.

(b) No monkey can speak French.

(c) No one can keep a secret.

Examples(CONTD.)

- **A. No rabbit knows calculus**

<u>Sol.</u>

<u>1st Method:</u>

Rewriting the above given sentence :

All rabbits do not know calculus.

$$\forall_x \neg P(x)$$

Negation: $\neg(\forall_x \neg P(x)) \equiv \exists_x P(x)$

<u>Ans</u>. There is a rabbit that knows calculus.

<u>2nd Method:</u>

No rabbit knows calculus

$$\neg(\forall_x P(x))$$

Negation: $\neg(\neg(\forall_x P(x)) \equiv \forall_x P(x)$

<u>Ans</u>. All rabbits knows calculus.

Examples(CONTD.)

- **B. No monkey can speak French.**

<u>Sol.</u>

<u>1st Method:</u>

Rewriting the above given sentence :

Every monkey cannot speak French.

$$\forall_x \neg F(x)$$

Negation: $\neg(\forall_x \neg F(x)) \equiv \exists_x F(x)$

<u>Ans</u>. There is a monkey that can speak French.

<u>2nd Method:</u>

No monkey can speak French.

$$\neg(\forall_x F(x))$$

Negation: $\neg(\neg(\forall_x F(x)) \equiv \forall_x F(x)$

<u>Ans</u>. All monkeys can speak French.

Examples(CONTD.)

6. No one can keep a secret.

Sol.

1st Method:

Rewriting the above given sentence :

All persons cannot keep a secret.

$$\forall_x \neg S(x)$$

Negation: $\neg(\forall_x \neg S(x)) \equiv \exists_x S(x)$

Ans. Somebody can keep a secret.

2nd Method:

No one can keep a secret.

$$\neg(\forall_x S(x))$$

Negation: $\neg(\neg(\forall_x S(x)) \equiv \forall_x S(x)$

Ans. Everybody can keep a secret.

Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an inverse" is

  $\forall x \exists y (x + y = 0)$

  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

  $\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Thinking of Nested Quantification

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

    $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

    $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same truth value.
2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then $\forall x \exists y Q(x,y)$ is true, but $\exists y \forall x Q(x,y)$ is false.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y \, P(x,y)$

   Answer: True

4. $\exists x \, \exists \, y \, P(x,y)$

   Answer: True

Questions on Order of Quantifiers

Example 2: Let U be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: False

3. $\exists x \forall y\, P(x,y)$

   Answer: False

4. $\exists x \exists y\, P(x,y)$

   Answer: True

Quantifications of Two Variables

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	$P(x,y)$ is true for every pair x,y.	There is a pair x, y for which $P(x,y)$ is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which $P(x,y)$ is true.	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which $P(x,y)$ is true.	P(x,y) is false for every pair x,y

THANK YOU