Chapter Two

01. Formulae

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + c \; ; n \neq -1$ 2. $\int \dfrac{dx}{x} = \ln|x| + c$ 3. $\int e^x dx = e^x + c$

4. $\int \sin x dx = -\cos x$ 5. $\int \cos x dx = \sin x$ 6. $\int \tan x dx = \ln|\sec x| = -\ln|\cos x|$

7. $\int \cot x dx = \ln|\sin x| = -\ln|\operatorname{cosec} x|$ 8. $\int \sec x dx = \ln|\sec x + \tan x| = \ln\left|\tan(\dfrac{\pi}{4} + \dfrac{x}{2})\right|$

9. $\int \operatorname{cosec} x dx = \ln|\operatorname{cosec} x - \cot x| = \ln\left|\tan \dfrac{x}{2}\right|$ 10. $\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a} \tan^{-1} \dfrac{x}{a}$

11. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \dfrac{x}{a}$

12. $\int (ax+b)^n dx = \dfrac{1}{a(n+1)} (ax+b)^{n+1} \quad ; \; n \neq -1$

13. $\int \dfrac{dx}{ax+b} = \dfrac{1}{a} \ln|ax+b|$

14. $\int \dfrac{dx}{(ax+b)^2} = -\dfrac{1}{a(ax+b)}$

15. $\int \dfrac{xdx}{(ax+b)^2} = \dfrac{b}{a^2(ax+b)} + \dfrac{1}{a^2} \ln|ax+b|$

16. $\int \dfrac{dx}{x(ax+b)} = \dfrac{1}{b} \ln\left|\dfrac{x}{ax+b}\right|$

17. $\int \dfrac{dx}{x^2(ax+b)} = -\dfrac{1}{bx} + \dfrac{a}{b^2} \ln\left|\dfrac{ax+b}{x}\right|$

18. $\int \sqrt{(ax+b)} dx = \dfrac{2}{3a} \sqrt{(ax+b)^3}$

19. $\int x\sqrt{(ax+b)} dx = \dfrac{2(3ax-2b)}{15a^2} \sqrt{(ax+b)^3}$

20. $\int \dfrac{\sqrt{ax+b}}{x} dx = 2\sqrt{ax+b} + b\int \dfrac{dx}{x\sqrt{ax+b}}$

21. $\int \dfrac{dx}{\sqrt{ax+b}} = \dfrac{2\sqrt{ax+b}}{a}$

22. $\int \dfrac{dx}{x\sqrt{ax+b}} = \dfrac{1}{\sqrt{b}} \ln\left|\dfrac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}}\right| \quad ; b > 0$

23. $\int \dfrac{dx}{x\sqrt{ax+b}} = \dfrac{2}{\sqrt{-b}} \tan^{-1} \sqrt{\dfrac{ax+b}{-b}} \quad ; b < 0$

24. $\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$

25. $\int \sqrt{\frac{cx+d}{ax+b}}dx = \frac{\sqrt{ax+b}\sqrt{cx+d}}{a} + \frac{ad-bc}{2a}\int \frac{dx}{\sqrt{ax+b}\sqrt{cx+d}}$

26. $\int \frac{dx}{p^2-x^2} = \frac{1}{2p}\ln\left|\frac{p+x}{p-x}\right|$

27. $\int \frac{dx}{x^2-p^2} = \frac{1}{2p}\ln\left|\frac{x-p}{x+p}\right|$

28. $\int \frac{dx}{ax^2+c} = \frac{1}{\sqrt{ac}}\tan^{-1}(x\sqrt{\frac{a}{c}}) \;\; ; \;\; a,c>0$

29. $\int \frac{x}{ax^2+c}dx = \frac{1}{2a}\ln\left|ax^2+c\right|$

30. $\int \sqrt{x^2+p^2}dx = \frac{1}{2}[x\sqrt{x^2+p^2} + p^2\ln\left|x+\sqrt{x^2+p^2}\right|]$

31. $\int \sqrt{x^2-p^2}dx = \frac{1}{2}[x\sqrt{x^2-p^2} - p^2\ln\left|x+\sqrt{x^2-p^2}\right|]$

32. $\int \sqrt{p^2-x^2}dx = \frac{1}{2}(x\sqrt{p^2-x^2} + p^2\sin^{-1}\frac{x}{p})$

33. $\int \frac{dx}{\sqrt{x^2+p^2}} = \ln\left|x+\sqrt{x^2+p^2}\right|$

34. $\int \frac{dx}{\sqrt{x^2-p^2}} = \ln\left|x+\sqrt{x^2-p^2}\right|$

35. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + c$

36. $\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$

37. $\int \sin^3 ax\, dx = -\frac{1}{a}\cos ax + \frac{1}{3a}\cos^3 ax$

38. $\int \sin^n ax dx = -\frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int \sin^{n-2} ax dx$

39. $\int \cos^2 ax\, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$

40. $\int \cos^3 ax dx = \frac{1}{a}\sin ax - \frac{1}{3a}\sin^3 ax$

41. $\int \cos^n ax dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\, dx$

$$42. \int \sec^2 ax\,dx = \frac{1}{a}\tan ax$$

$$43. \int \sec^n ax\,dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax dx \;\; ; n \neq 1$$

$$44. \int x \sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{1}{a}x\cos x$$

$$45. \int x \cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{1}{a}x\sin ax$$

$$46. \int x^n \sin ax\,dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1}\cos ax dx \;\; n \text{ is positive}$$

$$47. \int x^n \cos ax\,dx = \frac{1}{a}x^n \sin ax - \frac{n}{a}\int x^{n-1}\sin ax\,dx \;\; n \text{ is positive}$$

$$48. \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$49. \int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx$$

$$50. \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$51. \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$52. \int x^n \ln ax\,dx = x^{n+1}\left[\frac{\ln ax}{n+1} - \frac{1}{(n+1)^2}\right] \;\; ; n \neq -1$$

$$53. \int \sin^{-1} ax\,dx = x\sin^{-1} ax + \frac{1}{a}\sqrt{1-a^2x^2}$$

$$54. \int \cos^{-1} ax\,dx = x\cos^{-1} ax - \frac{1}{a}\sqrt{1-a^2x^2}$$

$$55. \int \tan^{-1} ax\,dx = x\tan^{-1} ax - \frac{1}{2a}\ln(1+a^2x^2)$$

$$56. \int \sin mx \;\cos nx\,dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + c$$

$$57. \int \sin^m x\cos^n x\,dx = -\frac{\sin^{m-1} x\cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x\cos^n x\,dx$$

$$58. \int \sin^m x\cos^n x\,dx = \frac{\sin^{m+1} x\cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x\cos^{n-2} x\,dx$$

$$59. \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \{\frac{1.3.5.7.---------------(n-1)}{2.4.6.8.----------------(n)}\frac{\pi}{2} \;\; ; n \text{ even}$$

$$= \frac{2.4.6.8 - (n-1)}{1.3.5.7. - - - - - - - - - - - - - - - - - - - (n)}; \qquad n \text{ odd}$$

60. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

61. $a + ar + ar^2 + ar^3 + - - - - - - - - - - - - - ar^{n-1} = \frac{a(1-r^n)}{1-r} \quad ; r \neq 1$

62. $\int \sqrt{x^2 + a^2} dx = \frac{x\sqrt{x^2 + a^2}}{2} \pm \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

63. $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ *[নিচের ফাংশনকে ডিফারেন্সিয়েট করলে যদি উপরের ফাংশন পাওয়া যায় তাহলে তার ইন্টিগ্রেশন হল লগ অফ নিচের ফাংশন :]*

64. $[\int_0^{\pi/2} \sin^m x \cos^n x dx = \beta(m,n) = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}]$

65. $[\int_0^{\pi/2} \cos^m x dx = \int_0^{\pi/2} \sin^m x dx = \frac{1}{2} \times \sqrt{\pi} \times \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}]$

66. $\int_0^{2a} f(x) dx = 2\int_0^{a} f(x) dx$

67. $[\int_{-a}^{+a} f(t) dt = \int_0^{2a} f(t) dt]$

02. Some technique to integrate the functions for indefinite integral

Method # 01: linear factor

$$\text{i)} \int \frac{dx}{(ax+b)} \quad \text{ii)} \int \frac{dx}{(ax+b)^n}$$

[যদি হরে (Denominator) linear factor থাকে অর্থাৎ x এর power যদি একঘাত(one) হয় তবে ঐ factor সমান যে কোন variable ধরতে হবে]

Substitution $u = ax + b$

Let, $u = ax + b$

$\Rightarrow \frac{du}{dx} = a.1$

Therefore $dx = \frac{du}{a}$

$$\text{i)} \int \frac{dx}{ax+b}$$

$$\text{Thus } \int \frac{dx}{ax+b} = \int \frac{\frac{du}{a}}{u} = \frac{1}{a}\int \frac{du}{u} = \frac{1}{a}\ln|u| + c = \frac{1}{a}\ln|ax+b| + c$$

$$\text{ii)} \int \frac{dx}{(ax+b)^n}$$

$$\int \frac{dx}{(ax+b)^n} = \int \frac{\frac{du}{a}}{u^n} = \frac{1}{a}\int u^{-n}du = \frac{1}{a}\frac{u^{-n+1}}{(-n+1)} + c = \frac{(ax+b)^{-n+1}}{a(-n+1)} + c = \frac{1}{a(-n+1)(ax+b)^{n-1}} + c$$

Example 41:

Find a) $\int \frac{dx}{(3x+2)^5}$ b) $\int \frac{dx}{3x+2}$

Let, $u = 3x + 2.$

$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(3x+2)$

$\Rightarrow \frac{du}{dx} = 3\frac{d}{dx}(x) + \frac{d}{dx}(2)$

$\Rightarrow \frac{du}{dx} = 3.1 + 0$

$\Rightarrow \frac{du}{dx} = 3 \quad \therefore dx = \frac{1}{3}du$

$$\text{a)} \int \frac{dx}{(3x+2)^5} = \int \frac{1}{u^5}\frac{1}{3}du = \int u^{-5}\frac{1}{3}du$$

$$= \frac{1}{3}.\frac{u^{-5+1}}{-5+1} = \frac{1}{3}.\frac{u^{-4}}{-4} + c \quad [\because \int x^n dx = \frac{x^{n+1}}{n+1}]$$

$$= \frac{-1}{12}(3x+2)^{-4} + c$$

$$= \frac{-1}{12(3x+2)^4} + c$$

b) $\int \frac{dx}{3x+2} = \int \frac{1}{u}.\frac{1}{3}du = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|3x+2| + c$ Answer $[\because \int \frac{dx}{x} = \ln|x| + c]$

Method # 02: Quadratic Function

$$\int \frac{dx}{ax^2+bx+c}$$

It will be assumed that the polynomial $ax^2 + bx + c$ is irreducible; that is, it cannot be factored into two first-degree polynomials. As for Example

We can write: $ax^2 + bx + c = a[x^2 + \frac{b}{a}x + \frac{c}{a}]$

$$= a[x^2 + 2.x.\frac{b}{2a} + (\frac{b}{2a})^2 - (\frac{b}{2a})^2 + \frac{c}{a}]$$

$$= a[\{x^2 + 2.x.\frac{b}{2a} + (\frac{b}{2a})^2\} + \frac{c}{a} - (\frac{b}{2a})^2]$$

$$= a[(x + \frac{b}{2a})^2 + \frac{c}{a} - (\frac{b}{2a})^2]$$

নিয়ম:

01. যদি হরে (Denominator) দ্বিঘাত ফাংশন (**Quadratic Function**) $\mathbf{ax^2 + bx + c}$ থাকে অর্থাৎ x এর power যদি দ্বিঘাত(two)/ দুই হয় তবে প্রথমে $\mathbf{x^2}$ সহগ common নিতে হবে।
02. এরপর $\mathbf{(a+b)^2}$ অথবা $\mathbf{(a-b)^2}$ formula বানাতে হবে। এক্ষেত্রে মনে রাখতে হবে $\mathbf{(a+b)^2}$ অথবা $\mathbf{(a-b)^2}$ formula বানাতে গিয়ে অতিরিক্ত constant যোগ অথবা বিয়োগ করা যাবে কিন্তু variable বাড়ানো যাবেনা ।]

Example 42: Find $\int \frac{dx}{x^2+4x+13}$

Solution: $x^2 + 4x + 13 = x^2 + 2.x.2 + 2^2 + 13 - 2^2 = (x+2)^2 + 9$

Thus $\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(x+2)^2+9} = \int \frac{dx}{(x+2)^2+3^2} = \frac{1}{3}\tan^{-1}\frac{x+2}{3}$ *[Formula # 10]*

Example 43: Find $\int \frac{dx}{4x^2+8x+13}$

Solution:

01. $4x^2 + 8x + 13 = 4\left[x^2 + 2x + \frac{13}{4}\right]$ [First taking common 4]
02. $4x^2 + 8x + 13 = 4\left[x^2 + 2x + \frac{13}{4}\right] = 4\left[x^2 + 2x + 1 + \frac{13}{4} - 1\right]$
03. $4x^2 + 8x + 13 = 4\left[x^2 + 2x + \frac{13}{4}\right] = 4\left[(x+1)^2 + \frac{13}{4} - 1\right]$
04. $4x^2 + 8x + 13 = 4\left[x^2 + 2x + \frac{13}{4}\right] = 4\left[(x+1)^2 + \frac{13-4}{4}\right]$
05. $4x^2 + 8x + 13 = 4\left[x^2 + 2x + \frac{13}{4}\right] = 4\left[(x+1)^2 + \frac{9}{4}\right]$
06. $4x^2 + 8x + 13 = 4\left[x^2 + 2x + \frac{13}{4}\right] = 4\left[(x+1)^2 + \left(\frac{3}{2}\right)^2\right]$

$$\therefore \int \frac{dx}{4x^2+8x+13} = \int \frac{dx}{4\left[(x+1)^2 + \left(\frac{3}{2}\right)^2\right]} = \frac{1}{4}\int \frac{dx}{(x+1)^2 + (\frac{3}{2})^2}$$

$$= \frac{1}{4} \times \frac{1}{\frac{3}{2}} \tan^{-1} \frac{x+1}{\frac{3}{2}} + c \qquad \textit{[Formula \# 10]}$$

$$= \frac{1}{4} \times \frac{2}{3} \tan^{-1} \frac{x+1}{\frac{3}{2}} + c$$

$$= \frac{1}{6} \tan^{-1} \frac{2(x+1)}{3} + c \quad \text{Answer}$$

Example 44: $\int \frac{dv}{v^2 - 6v + 5} = \int \frac{dv}{v^2 - 2.v.3 + 3^2 - 4} = \int \frac{dv}{(v-3)^2 - 4} = \int \frac{dv}{(v-3)^2 - 2^2}$

$$= \frac{1}{2.2} \log \frac{v-3-2}{v-3+2} \quad \textit{[Formula \#27]}$$

$$= \frac{1}{4} \log \frac{v-5}{v-1} \textit{ Answer}$$

Example 45: Find $\int \frac{dx}{4x^2 + 1}$

01. $4x^2 + 1 = 4\left[x^2 + \frac{1}{4}\right]$

02. $4x^2 + 1 = 4\left[x^2 + \frac{1}{2^2}\right]$

03. $4x^2 + 1 = 4\left[x^2 + \left(\frac{1}{2}\right)^2\right]$

Then $\int \frac{dx}{4x^2+1} = \int \frac{dx}{4\left[x^2 + \left(\frac{1}{2}\right)^2\right]} = \frac{1}{4} \int \frac{dx}{x^2 + (\frac{1}{2})^2} = \frac{1}{4} \times \frac{1}{\frac{1}{2}} \tan^{-1} \frac{x}{\frac{1}{2}} + c = \frac{1}{4} \times \frac{2}{1} \tan^{-1} \frac{2x}{1} + c$

$$= \frac{1}{2} \tan^{-1} 2x + c$$

Example 46: Integrate the function $\frac{1}{4x^2 + 9}$

Solution: *Try yourself*: $= \frac{1}{6} \tan^{-1} \frac{2x}{3} + c$ Answer

Example 47: $\int \frac{dx}{16x^2 - 9}$

01. $16x^2 - 9 = 16\left[x^2 - \frac{9}{16}\right]$

02. $16x^2 - 9 = 16\left[x^2 - \left(\frac{3}{4}\right)^2\right]$

$$\therefore \int \frac{dx}{16x^2 - 9} = \int \frac{dx}{16\left[x^2 - \left(\frac{3}{4}\right)^2\right]} = \frac{1}{16}\int \frac{dx}{\left[x^2 - \left(\frac{3}{4}\right)^2\right]}$$

$$= \frac{1}{16}\cdot\frac{1}{2.\frac{3}{4}}\log\frac{x-\frac{3}{4}}{x+\frac{3}{4}} = \frac{1}{16}\cdot\frac{4}{6}\log\frac{x-\frac{3}{4}}{x+\frac{3}{4}} = \frac{1}{24}\log\frac{x-\frac{3}{4}}{x+\frac{3}{4}}$$ *[Formula #27]*

Method # 03:

We have, $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

[নিচের ফাংশনকে ডিফারেন্সিয়েট করলে যদি উপরের ফাংশন পাওয়া যায় তাহলে তার ইন্টিগ্রেশন হল লগ অফ নিচের ফাংশন]

নিয়ম:

- নিচের ফাংশনকে ডিফারেন্সিয়েট করলে যদি উপরের ফাংশন পাওয়া যায় তাহলে তার ইন্টিগ্রেশন হল লগ অফ নিচের ফাংশন। যেমন **Example 48** এ নীচের ফাংশনকে $(x^2 + 1)$ ডিফারেন্সিয়েট করলে উপরের ফাংশন $(2x)$ পাওয়া যায়, সেজন্য **Example 48** এর ইন্টিগ্রেশন ফলাফল হল: লগ অফ নিচের ফাংশন অর্থাৎ $\log(x^2 + 1)$]

Example 48: $\int \frac{2x}{x^2+1}dx = \ln(x^2+1) + c$

Here, $f(x) = x^2 + 1 \;\therefore f'(x) = 2x$

Proof:

$\int \frac{2x}{x^2+1}dx$ --------------------------------(i)

Let, $z = x^2 + 1$

$\therefore \frac{dz}{dx} = 2x$

$\Rightarrow dz = 2xdx$

$\therefore \int \frac{2x}{x^2+1}dx = \int \frac{dz}{z} = \ln z + c$ $\quad$ [$\because$ **Formulla 2 :** $\int \frac{dx}{x} = \ln|x| + c$]

$= \ln(x^2 + 1)$ [$\because z = x^2 + 1$] $\quad$ [$\because \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$]

Method # 04:

$\int \frac{px+q}{ax^2+bx+c}dx$

$$\therefore \frac{d}{dx}(ax^2+bx+c)=2ax+b$$

$$\int \frac{px+q}{ax^2+bx+c}dx$$

$$=\int \frac{\frac{p}{2a}(2ax+b)-\frac{pb}{2a}+q}{ax^2+bx+c}dx$$

$$=\int \frac{\frac{p}{2a}(2ax+b)}{ax^2+bx+c}dx+\int \frac{-\frac{pb}{2a}+q}{ax^2+bx+c}dx$$

$$=\int \frac{\frac{p}{2a}(2ax+b)}{ax^2+bx+c}dx+\int \frac{q-\frac{pb}{2a}}{ax^2+bx+c}dx$$

$$=\frac{p}{2a}\int \frac{(2ax+b)}{ax^2+bx+c}dx+(q-\frac{pb}{2a})\int \frac{1}{ax^2+bx+c}dx$$

$$=\underbrace{\frac{p}{2a}\int \frac{(2ax+b)}{ax^2+bx+c}dx}_{Method-03}++\underbrace{(q-\frac{pb}{2a})\int \frac{1}{ax^2+bx+c}dx}_{Method-02}$$

নিয়ম:

- যদি হরে (Denominator) দ্বিঘাত ফাংশন ax^2+bx+c থাকে অর্থাৎ x এর power যদি দ্বিঘাত(two)/ দুই হয় এবং লবে (numerator) এ যদি একঘাত ফাংশন থাকে অর্থাৎ x এর power যদি এক(one) হয় তখন নিচের ফাংশনকে ডিফারেন্সিয়েট করে উপরে (লবে)আগে লিখে ফেলতে হবে।
- এরপর উপরের x এর সাথে balance করার জন্য অতিরিক্ত constant দিয়ে গুন, ভাগ এর পর যোগ অথবা বিয়োগ করা যাবে কিন্তু variable বাড়ানো যাবেনা। মনে রাখতে হবে x এর সহগের সাথে যত গুন থাকবে তত দ্বারা ভাগ করতে হবে।
- এরপর লবের দুটি ফাংশনকে আলাদা করে ফেলতে হবে।]

Example 49: Find $\int \frac{xdx}{4x^2+8x+13}$

Solution:

01. Here denominator is $4x^2+8x+13$

$$\therefore \frac{d}{dx}(4x^2+8x+13)=8x+8$$

02. $8x+8$ কে প্রথমে লবে (Numerator) লিখে ফেলতে হবে।

03. কিন্তু উপরে (Numerator) আছে x, এই x এর সাথে balance করার জন্য $8x+8$ এর সাথে $\frac{1}{8}$ দ্বারা গুন করতে হবে। মনে রাখতে হবে x এর সহগের সাথে যত গুন থাকবে তত দ্বারা ভাগ করতে হবে। এখানে $8x+8$ এ x এর সাথে 8 গুনন আছে এজন্য 8 দ্বারা ভাগ করতে হবে।

04. এর পর x সাথে balance করার জন্য -1 বিয়োগ করতে হবে।

05. অর্থাৎ $x=\frac{1}{8}(8x+8)-1=\frac{1}{8}\times 8x+\frac{1}{8}\times 8-1=x+1-1=x$

$$\therefore \int \frac{xdx}{4x^2+8x+13} = \int \frac{\frac{1}{8}(8x+8)-1}{4x^2+8x+13}dx$$

$$= \frac{1}{8}\int \frac{8x+8}{4x^2+8x+13}dx - \int \frac{1}{4x^2+8x+13}dx$$

$$= \frac{1}{8}\int \frac{8x+8}{4x^2+8x+13}dx - \int \frac{1}{4(x^2+2x+\frac{13}{4})}dx$$

$$= \frac{1}{8}\int \frac{8x+8}{4x^2+8x+13}dx - \int \frac{1}{4(x^2+2x+\frac{13}{4})}dx$$

$$= \frac{1}{8}\int \frac{8x+8}{4x^2+8x+13}dx - \frac{1}{4}\int \frac{1}{(x^2+2x+1+\frac{13}{4}-1)}dx$$

$$= \frac{1}{8}\int \frac{8x+8}{4x^2+8x+13}dx - \frac{1}{4}\int \frac{1}{(x^2+2x+1)+\frac{9}{4})}dx$$

$$= \frac{1}{8}\int \frac{8x+8}{4x^2+8x+13}dx - \frac{1}{4}\int \frac{1}{(x+1)^2+(\frac{3}{2})^2}dx$$

$$= \underbrace{\frac{1}{8}\int \frac{8x+8}{4x^2+8x+13}dx}_{Method-03} + \underbrace{-\frac{1}{4}\int \frac{1}{(x+1)^2+(\frac{3}{2})^2}dx}_{Method-02}$$

[মনে রাখুন, এখানে দুটি integration এর প্রথমটি Method # 03 এবং পরেরটি Method # 02অনুসারে]

$$= \frac{1}{8}\ln(4x^2+8x+13) - \frac{1}{4}\times\frac{1}{\frac{3}{2}}\tan^{-1}\frac{x+1}{\frac{3}{2}}$$

[নিচের ফাংশনকে ডিফারেন্সিয়েট করলে যদি উপরের ফাংশন পাওয়া যায় তাহলে তার ইন্টিগ্রেশন হল লগ অফ নিচের ফাংশন : $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$ and ***Formula # 10]***

$$= \frac{1}{8}\ln(4x^2+8x+13) - \frac{1}{6}\tan^{-1}\frac{2(x+1)}{3} \quad \textit{Answer}$$

Example 50: $\int \frac{3xdx}{x^2 - x - 2}$

Solution:

01. Here denominator $x^2 - x - 2$

$\therefore \frac{d}{dx}(x^2 - x - 2) = 2x - 1$

02. $2x - 1$ কে প্রথমে লবে (Numerator) লিখে ফেলতে হবে।

03. কিন্তু উপরে (Numerator) আছে $\mathbf{3x}$, এই $\mathbf{3x}$ এর সাথে balance করার জন্য $\mathbf{2x-1}$ এর সাথে $\frac{3}{2}$ দ্বারা গুন করতে হবে। মনে রাখতে হবে x এর সহগের সাথে যত গুন থাকবে তত দ্বারা ভাগ করতে হবে। এখানে $\mathbf{2x-1}$ এ x এর সাথে ২ গুনন আছে এজন্য 2 দ্বারা ভাগ করতে হবে।

04. এর পর $\mathbf{3x}$ সাথে balance করার জন্য $\frac{3}{2}$ যোগ করতে হবে।

05. অর্থাৎ $3x = \frac{3}{2}(2x-1)+\frac{3}{2} = \frac{3}{2}\times 2x - \frac{3}{2}+\frac{3}{2} = 3x - \frac{3}{2}+\frac{3}{2} = 3x$

$$\therefore \int \frac{3x\,dx}{x^2-x-2} = \int \frac{\frac{3}{2}(2x-1)+\frac{3}{2}}{x^2-x-2}dx \quad \text{[Method \# 04]}$$

$$= \frac{3}{2}\int \frac{2x-1}{x^2-x-2}dx + \frac{3}{2}\int \frac{dx}{x^2-x-2}$$

[মনে রাখুন, এখানে দুটি integration এর প্রথমটি Method # 03 এবং পরেরটি Method # 02অনুসারে]

$$= \frac{3}{2}\log(x^2-x-2) + \frac{3}{2}\int \frac{dx}{x^2-x-2} \quad [\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c \ \text{\& Method-03]}$$

[নিচের ফাংশনকে ডিফারেন্সিয়েট করলে যদি উপরের ফাংশন পাওয়া যায় তাহলে তার ইন্টিগ্রেশন হল লগ অফ নিচের ফাংশন :]

$$= \frac{3}{2}\log(x^2-x-2) + \frac{3}{2}\int \frac{dx}{x^2-2.x.\frac{1}{2}+(\frac{1}{2})^2-\frac{1}{4}-2} \quad \text{[Method \# 02]}$$

$$= \frac{3}{2}\log(x^2-x-2) + \frac{3}{2}\int \frac{dx}{(x-\frac{1}{2})^2-\frac{9}{4}}$$

$$= \frac{3}{2}\log(x^2-x-2) + \frac{3}{2}\int \frac{dx}{(x-\frac{1}{2})^2-(\frac{3}{2})^2}$$

$$= \frac{3}{2}\log(x^2-x-2) + \frac{3}{2}.\frac{1}{2.\frac{3}{2}}\log\frac{x-\frac{1}{2}-\frac{3}{2}}{x-\frac{1}{2}+\frac{3}{2}} \quad \textit{[Formula \#27]}$$

$$= \frac{3}{2}\log(x^2-x-2) + \frac{1}{2}\log\frac{x-2}{x+1} \quad \textit{Answer}$$

Example 51: $\int \frac{2x+3}{3x^2-x+1}dx$

01. Here denominator $\mathbf{3x^2-x+1}$

$$\therefore \frac{d}{dx}(3x^2-x+1) = 6x-1$$

02. $\mathbf{6x-1}$ কে প্রথমে লবে (Numerator) লিখে ফেলতে হবে।

03. কিন্তু উপরে (Numerator) আছে $\mathbf{2x+3}$, এই $\mathbf{2x+3}$ এর সাথে balance করার জন্য $\mathbf{6x-1}$ এর সাথে $\frac{1}{3}$ দ্বারা গুন করতে হবে। মনে রাখতে হবে x এর সহগের সাথে যত গুন থাকবে তত দ্বারা ভাগ করতে হবে। এখানে $\mathbf{6x-1}$ এ x এর সাথে 6 গুনন আছে এজন্য 6 দ্বারা ভাগ করতে হবে।

04. এর পর $\mathbf{2x+3}$ এর সাথে balance করার জন্য $\frac{1}{3}+3$ যোগ করতে হবে।

05. অর্থাৎ $2x+3=\frac{2}{6}(6x-1)+\frac{2}{6}+3=\frac{1}{3}\times 6x-\frac{1}{3}+\frac{1}{3}+3=2x+3$

$$\therefore \int \frac{2x+3}{3x^2-x+1}dx=\int \frac{\frac{1}{3}(6x-1)+\frac{1}{3}+3}{3x^2-x+1}dx \text{ [Method \# 04]}$$

$$=\int \frac{\frac{1}{3}(6x-1)+\frac{10}{3}}{3x^2-x+1}dx$$

$$=\int \frac{\frac{1}{3}(6x-1)+\frac{10}{3}}{3x^2-x+1}dx$$

$$=\frac{1}{3}\int \frac{6x-1}{3x^2-x+1}dx+\frac{10}{3}\int \frac{dx}{3x^2-x+1}$$

[মনে রাখুন, এখানে দুটি integration এর প্রথমটি Method # 03 এবং পরেরটি Method # 02অনুসারে]

$$=\frac{1}{3}\log(3x^2-x+1)+\frac{10}{3}.\frac{1}{3}\int \frac{dx}{x^2-\frac{x}{3}+\frac{1}{3}} \text{ [Formula \# 63]}$$

$$=\frac{1}{3}\log(3x^2-x+1)+\frac{10}{9}\int \frac{dx}{x^2-2.x.\frac{1}{6}+(\frac{1}{6})^2-(\frac{1}{6})^2+\frac{1}{3}}$$

$$=\frac{1}{3}\log(3x^2-x+1)+\frac{10}{9}\int \frac{dx}{x^2-2.x.\frac{1}{6}+(\frac{1}{6})^2-\frac{1}{36}+\frac{1}{3}}$$

$$=\frac{1}{3}\log(3x^2-x+1)+\frac{10}{9}\int \frac{dx}{x^2-2.x.\frac{1}{6}+(\frac{1}{6})^2+\frac{1}{3}-\frac{1}{36}}$$

$$=\frac{1}{3}\log(3x^2-x+1)+\frac{10}{9}\int \frac{dx}{(x-\frac{1}{6})^2+\frac{11}{36}}$$

$$=\frac{1}{3}\log(3x^2-x+1)+\frac{10}{9}\int \frac{dx}{(x-\frac{1}{6})^2+(\frac{\sqrt{11}}{6})^2}$$

$$=\frac{1}{3}\log(3x^2-x+1)+\frac{10}{9}.\frac{1}{\frac{\sqrt{11}}{6}}\tan^{-1}\frac{x-\frac{1}{6}}{\frac{\sqrt{11}}{6}} \text{ [Formula \# 10]}$$

$$=\frac{1}{3}\log(3x^2-x+1)+\frac{10}{9}.\frac{6}{\sqrt{11}}\tan^{-1}\frac{x-\frac{1}{6}}{\frac{\sqrt{11}}{6}}$$

$$=\frac{1}{3}\log(3x^2 - x+1)+\frac{20}{3\sqrt{11}}\tan^{-1}\frac{6x-1}{\sqrt{11}}\textit{ Answer}$$

Method # 05:

নিয়ম:

- যদি হরে (Denominator) দ্বিঘাত ফাংশন ax^2+bx+c থাকে অর্থাৎ x এর power যদি দ্বিঘাত(two)/ দুই হয় এবং লবেও (numerator) এ যদি দ্বিঘাত ফাংশন ax^2+bx+c থাকে অর্থাৎ x এর power যদি দ্বিঘাত(two)/ দুই হয় তখন নিচের ফাংশনকে হুবহু উপরে (লবে)আগে লিখে ফেলতে হবে।
- এরপর উপরের প্রদত্ত function এর সাথে balance করার জন্য অতিরিক্ত constant or variable যোগ অথবা বিয়োগ করা যাবে ।
- এরপর লবের দুটি বা ততোধিক ফাংশনকে আলাদা করে ফেলতে হবে।]

Example 52: $\int\frac{x^2 - x+1}{x^2+x+1}dx$

$$=\int\frac{(x^2+x+1)-2x}{x^2+x+1}dx$$

$$=\int\frac{(x^2+x+1)}{(x^2+x+1)}dx-\int\frac{2x}{x^2+x+1}dx=\int dx-\int\frac{2x}{x^2+x+1}dx$$ [Apply Method # 04 in 2nd Integral]

$$=x-\int\frac{(2x+1)-1}{x^2+x+1}dx=x-\int\frac{2x+1}{x^2+x+1}dx+\int\frac{1}{x^2+x+1}dx$$

[মনে রাখুন, এখানে দুটি integration এর প্রথমটি Method # 03 এবং পরেরটি Method # 02অনুসারে]

$$=x-\log(x^2+x+1)+\int\frac{dx}{x^2+2.x.\frac{1}{2}+(\frac{1}{2})^2-(\frac{1}{2})^2+1}$$

$$=x-\log(x^2+x+1)+\int\frac{dx}{x^2+2.x.\frac{1}{2}+(\frac{1}{2})^2+1-(\frac{1}{2})^2}$$

$$=x-\log(x^2+x+1)+\int\frac{dx}{x^2+2.x.\frac{1}{2}+(\frac{1}{2})^2+1-\frac{1}{4}}$$

$$=x-\log(x^2+x+1)+\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$=x-\log(x^2+x+1)+\int\frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$=x-\log(x^2+x+1)+\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\ \textit{[}\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+c\ \ \&\ \textbf{\textit{Formula \# 10]}}$$

$$= x - \log(x^2 + x + 1) + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= x - \log(x^2 + x + 1) + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}} \text{ Answer}$$

Example 53: $\int \frac{x^2}{x^2 - 4} dx$

$$= \int \frac{x^2 - 4 + 4}{x^2 - 4} dx = \int \frac{x^2 - 4}{x^2 - 4} dx + \int \frac{4}{x^2 - 4} dx$$

$$= \int dx + \int \frac{4}{x^2 - 4} dx = x + 4\int \frac{1}{x^2 - 2^2} dx$$

$$= x + 4.\frac{1}{2.2} \log \frac{x - 2}{x + 2} \qquad \textit{[Formula \#27]}$$

$$= x + \log \frac{x - 2}{x + 2} \text{ Answer}$$

Method # 06:

$$\int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx$$

মনে রাখুন

- যদি হরে (Denominator) দ্বিঘাত ফাংশন $\sqrt{ax^2 + bx + c}$ থাকে অর্থাৎ x এর power যদি দ্বিঘাত(two)/ দুই হয় এবং লবে (numerator) এ যদি একঘাত ফাংশন থাকে অর্থাৎ x এর power যদি এক(one) হয় তখন নিচের ফাংশন এর root এর ভিতরে যা থাকবে তাকে ডিফারেন্সিয়েট করে উপরে (লবে)আগে লিখে ফেলতে হবে।
- এরপর উপরের x এর সাথে balance করার জন্য অতিরিক্ত constant দিয়ে গুন, ভাগ এর পর যোগ অথবা বিয়োগ করা যাবে কিন্তু variable বাড়ানো যাবেনা।
- এরপর লবের দুটি ফাংশনকে আলাদা করে ফেলতে হবে।
- এরপর $ax^2 + bx + c = z$ ধরতে হবে।]

Example 54: $\int \frac{2x + 2}{\sqrt{x^2 + 2x + 1}} dx$

$$\text{Let}, z = x^2 + 2x + 1$$
$$\Rightarrow \frac{dz}{dx} = 2x + 2$$
$$\Rightarrow dz = (2x + 2)dx$$

$$\therefore \int \frac{2x + 2}{\sqrt{x^2 + 2x + 1}} dx$$

$$= \int \frac{dz}{\sqrt{z}} = \int z^{-\frac{1}{2}} dz = \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2} + 1} \quad [\because \int x^n dx = \frac{x^{n+1}}{n + 1}]$$

$$= \frac{z^{\frac{1}{2}}}{\frac{1}{2}} = 2\sqrt{z} = 2\sqrt{x^2 + 2x + 1} + c \quad \textit{Answer}$$

Example 55: $\int \frac{x-2}{\sqrt{2x^2-8x+5}}dx$

01. Here denominator: $2x^2-8x+5$

$\therefore \frac{d}{dx}(2x^2-8x+5)=4x-8$

02. $4x-8$ কে প্রথমে লবে (Numerator) লিখে ফেলতে হবে।

03. কিন্তু উপরে (Numerator) আছে $x-2$, এই $x-2$ এর সাথে balance করার জন্য $4x-8$ এর সাথে $\frac{1}{4}$ দ্বারা গুন করতে হবে।

05. অর্থাৎ $x-2=\frac{1}{4}(4x-8)=x-2$

$$\therefore \int \frac{x-2}{\sqrt{2x^2-8x+5}}dx$$

$$=\frac{1}{4}\int \frac{4x-8}{\sqrt{2x^2-8x+5}}dx$$

$$\therefore \frac{1}{4}\int \frac{4x-8}{\sqrt{2x^2-8x+5}}dx$$

Let, $z=2x^2-8x+5$

$\therefore \frac{dz}{dx}=4x-8$

$\Rightarrow dz=(4x-8)dx$

$$=\frac{1}{4}\int \frac{dz}{\sqrt{z}}=\frac{1}{4}\times\int z^{-1/2}dz=\frac{1}{4}\times\frac{z^{-1/2+1}}{-1/2+1} \qquad [\because \int x^n dx=\frac{x^{n+1}}{n+1}]$$

$$=\frac{1}{4}\times\frac{z^{1/2}}{1/2}=\frac{1}{4}\times 2\times\sqrt{z}=\frac{1}{2}\sqrt{z}=\frac{1}{2}\sqrt{2x^2-8x+5}$$ *Answer*

Method # 07:

$$\int \frac{dx}{(ax+b)\sqrt{cx+d}} \quad (a\neq 0, c\neq 0) \qquad ;\text{ put } cx+d=z^2$$

Example 56:

$$\frac{dx}{(2x+1)\sqrt{4x+3}}$$

Let, $z^2=4x+3$

$\Rightarrow 2zdz=4dx$

$\Rightarrow zdz=2dx$

$\Rightarrow dx=\frac{1}{2}zdz$

Again, $z^2=4x+3$

$\therefore x=\frac{z^{2.}-3}{4}$

$$\therefore \int \frac{dx}{(2x+1)\sqrt{4x+3}}$$

$$=\int \frac{\frac{1}{2}zdz}{(2.\frac{z^2-3}{4}+1)\sqrt{z^2}} \quad [\because x=\frac{z^{2.}-3}{4}\ \&\ z^2=4x+3]$$

$$=\frac{1}{2}\int \frac{zdz}{(\frac{z^2-3}{2}+1)z}$$

$$=\frac{1}{2}\int \frac{zdz}{\left(\frac{z^2-3+2}{2}\right)z}=\frac{1}{2}\int \frac{2zdz}{(z^2-3+2)z}=\frac{1}{2}\int \frac{2dz}{z^2-3+2}$$

$$=\int \frac{dz}{z^2-3+2}=\int \frac{dz}{z^2-1}=\int \frac{dz}{z^2-1^2}=\frac{1}{2.1}\log\frac{z-1}{z+1}$$ *[Formula #27]*

$$=\frac{1}{2}\log\frac{\sqrt{4x+3}-1}{\sqrt{4x+3}+1}$$ **Answer** $[z^2=4x+3;\ \therefore z=\sqrt{4x+3}]$

Example 57: $\int \frac{dx}{(2+x)\sqrt{1+x}}$

$$\therefore \int \frac{dx}{(2+x)\sqrt{1+x}}$$

$$=\int \frac{2zdz}{(2+z^2-1)\sqrt{z^2}} \quad [\because x=z^2-1\ \&\ z^2=1+x]$$

$$=2\int \frac{dz}{z^2+1}=2\tan^{-1}\frac{z}{1}$$ *[Formula # 10]*

$$=2\tan^{-1}\sqrt{1+x}$$ *Answer* $[\because z^2=1+x;\ \therefore z=\sqrt{1+x}]$

Let, $z^2=1+x$
$\Rightarrow 2zdz=1dx$
Again, $z^2=1+x$
$\Rightarrow x=z^2-1$

<u>**Method # 08:**</u> $\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}(a\neq 0, p\neq 0)$

Let, $px+q=\frac{1}{z}$

Then Find $x=?$

And $dx=?$

Example 58:

$$\int \frac{dx}{(2x+3)\sqrt{x^2+3x+2}}$$

$$\therefore \int \frac{dx}{(2x+3)\sqrt{x^2+3x+2}}$$

$$=\int \frac{-\frac{1}{2z^2}dz}{\frac{1}{z}\sqrt{\frac{1}{4}(\frac{1}{z}-3)^2+3.\frac{1}{2}(\frac{1}{z}-3)+2}}$$

$$=\frac{1}{2}\int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{1}{4}(\frac{1}{z}-3)^2+3.\frac{1}{2}(\frac{1}{z}-3)+2}}$$

Let, $2x+3=\frac{1}{z}$
$\Rightarrow 2dx=-\frac{1}{z^2}dz$
$\Rightarrow dx=-\frac{1}{2z^2}dz$
Again, $2x+3=\frac{1}{z}$
$\Rightarrow 2x=\frac{1}{z}-3$
$\therefore\ x=\frac{1}{2}(\frac{1}{z}-3)=\frac{1-3z}{2z}$

$$= \frac{1}{2}\int \frac{-z\,dz}{z^2\sqrt{\frac{1-6z+9z^2}{4z^2}+\frac{3-9z}{2z}+2}}$$

$$= \frac{1}{2}\int \frac{-dz}{z\sqrt{\frac{1-6z+9z^2+6z-18z^2+8z^2}{4z^2}}} = \frac{1}{2}\int \frac{-dz}{z\sqrt{\frac{1-z^2}{4z^2}}} = \frac{1}{2}\int \frac{-dz}{z.\frac{1}{2z}\sqrt{1-z^2}}$$

$$= \frac{2}{2}\int \frac{-dz}{\sqrt{1-z^2}} = \int \frac{-dz}{\sqrt{1-z^2}} = -\sin^{-1} z$$

$$= -\sin^{-1}\left(\frac{1}{2x+3}\right) \quad [\because 2x+3 = \frac{1}{z};\therefore z = \frac{1}{2x+3}] \quad \text{[Formula \#11]}$$

Method # 09:

$\int (px+q)\sqrt{(ax^2+bx+c)}\,dx (a \neq 0)$

মনে রাখুন

- যদি একঘাত ফাংশন $px+q$ (অর্থাৎ x এর power যদি এক(one) হয়) এবং দ্বিঘাত ফাংশন $\sqrt{ax^2+bx+c}$ (অর্থাৎ x এর power যদি দ্বিঘাত(two)/ দুই হয়) পাশাপাশি থাকে তখন দ্বিঘাত ফাংশন ax^2+bx+c কে ডিফারেন্সিয়েট করে একঘাত ফাংশন $px+q$ এর জায়গায় আগে লিখে ফেলতে হবে।
- এরপর একঘাত ফাংশন $px+q$ এর x এর সাথে balance করার জন্য অতিরিক্ত constant দিয়ে গুন, ভাগ এর পর যোগ অথবা বিয়োগ করা যাবে কিন্তু variable বাড়ানো যাবেনা।
- এরপর দুটি ফাংশনকে আলাদা করে ফেলতে হবে।
- এরপর ***দুটি integration এর প্রথমটিতে*** $ax^2+bx+c=z$ ***ধরতে হবে এবং পরেরটিতে*** ax^2+bx+c ***কে*** $(a+b)^2$ অথবা $(a-b)^2$ formula বানাতে হবে। এক্ষেত্রে মনে রাখতে হবে $(a+b)^2$ অথবা $(a-b)^2$ formula বানাতে গিয়ে অতিরিক্ত constant যোগ অথবা বিয়োগ করা যাবে কিন্তু variable বাড়ানো যাবেনা।

Example 59: $\int(x-1)\sqrt{x^2-x+1}\,dx$

[Here, $px+q = x-1$ and $ax^2+bx+c = x^2-x+1$]

$[\because \frac{d}{dx}(x^2-x+1) = 2x-1 \text{ and } x-1 = \frac{1}{2}(2x-1)+\frac{1}{2}-1]$

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$[\therefore \frac{d}{dx}(x^2-x+1) = 2x-1\ ]$

02. $2x-1$ কে প্রথমে $x-1$ এর জায়গায় লিখে ফেলতে হবে।

03.এরপর $x-1$ এর সাথে balance করার জন্য $2x-1$ এর সাথে $\frac{1}{2}$ দ্বারা গুন করতে হবে।

04. এর পর $2x-1$ এর সাথে balance করার জন্য $\frac{1}{2}-1$ যোগ করতে হবে।

05. অর্থাৎ $x-1=\frac{1}{2}(2x-1)+\frac{1}{2}-1=\frac{1}{2}\times 2x-\frac{1}{2}+\frac{1}{2}-1=x-1$

$\therefore \int (x-1)\sqrt{x^2-x+1}\,dx$

$=\int \{\frac{1}{2}(2x-1)+\frac{1}{2}-1\}\sqrt{x^2-x+1}dx=\int \{\frac{1}{2}(2x-1)-\frac{1}{2}\}\sqrt{x^2-x+1}dx$

$\therefore \int (x-1)\sqrt{x^2-x+1}\,dx=\frac{1}{2}\int (2x-1)\sqrt{x^2-x+1}\,dx-\frac{1}{2}\int \sqrt{x^2-x+1}\,dx$ --------------(i)

মনে রাখুন

- *এখানে দুটি* ***integration*** *এর প্রথমটিতে* $x^2-x+1=z$ *ধরতে হবে এবং পরেরটিতে* x^2-x+1 *কে* $(a+b)^2$ অথবা $(a-b)^2$ formula বানাতে হবে। এক্ষেত্রে মনে রাখতে হবে $(a+b)^2$ অথবা $(a-b)^2$ formula বানাতে গিয়ে অতিরিক্ত constant যোগ অথবা বিয়োগ করা যাবে কিন্তু variable বাড়ানো যাবেনা।]

Now, $\int (2x-1)\sqrt{x^2-x+1}\,dx$

Let, $z=x^2-x+1$

$\Rightarrow \frac{dz}{dx}=2x-1$

$\Rightarrow dz=(2x-1)dx$

$\therefore \int (2x-1)\sqrt{x^2-x+1}\,dx$

$=\int \sqrt{z}dz=\int z^{1/2}dz$

$=\frac{z^{1/2+1}}{1/2+1}$ $\qquad [\because \int x^n dx=\frac{x^{n+1}}{n+1}]$

$=\frac{z^{3/2}}{3/2}=\frac{2}{3}z^{3/2}=\frac{2}{3}(x^2-x+1)^{3/2}$

Now, $\int (x^2-x+1)dx$

$=\int \{x^2-2.x.\frac{1}{2}+(\frac{1}{2})^2-(\frac{1}{2})^2+1\}dx$

$=\int \{(x-\frac{1}{2})^2-\frac{1}{4}+1\}dx=\int \{(x-\frac{1}{2})^2+1-\frac{1}{4}\}dx$

$=\int \{(x-\frac{1}{2})^2+\frac{4-1}{4}\}dx=\int \{(x-\frac{1}{2})^2+\frac{3}{4}\}dx$

$=\int \{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2\}dx$

$\therefore \int \sqrt{x^2-x+1}\,dx=\int \sqrt{\{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2\}}dx$

$$=\{\frac{(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}{2}\pm\frac{(\frac{\sqrt{3}}{2})^2}{2}\sin^{-1}\frac{(x-\frac{1}{2})}{\frac{\sqrt{3}}{2}}\}$$

[$\because$ *Formula # 62* $\int\sqrt{x^2+a^2}\,dx=\frac{x\sqrt{x^2+a^2}}{2}\pm\frac{a^2}{2}\sin^{-1}\frac{x}{a}$]

Putting the value of $\int(2x-1)\sqrt{x^2-x+1}\,dx$ and $\int\sqrt{x^2-x+1}\,dx$ in (i)

$$\therefore\int(x-1)\sqrt{x^2-x+1}\,dx=\frac{1}{2}\int(2x-1)\sqrt{x^2-x+1}\,dx-\frac{1}{2}\int\sqrt{x^2-x+1}\,dx$$

$$=\frac{1}{2}\times\frac{2}{3}(x^2-x+1)^{3/2}-\frac{1}{2}\{\frac{(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}{2}\pm\frac{(\frac{\sqrt{3}}{2})^2}{2}\sin^{-1}\frac{(x-\frac{1}{2})}{\frac{\sqrt{3}}{2}}\}$$

$$=\frac{1}{3}(x^2-x+1)^{3/2}-\frac{1}{2}\{\frac{(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}{2}\pm\frac{(\frac{\sqrt{3}}{2})^2}{2}\sin^{-1}\frac{(x-\frac{1}{2})}{\frac{\sqrt{3}}{2}}\}\textit{Answer}$$

Example 60: $\int(3x-2)\sqrt{(x^2-x+1)}dx$

[$\because\frac{d}{dx}(x^2-x+1)=2x-1$ **and** $3x-2=\frac{3}{2}(2x-1)+\frac{3}{2}-2$]

[Here, $px+q=3x-2$ and $ax^2+bx+c=x^2-x+1$]

[$\therefore\frac{d}{dx}(x^2-x+1)=2x-1$ **and** $x-1=\frac{1}{2}(2x-1)+\frac{1}{2}-1$]

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[$\therefore\frac{d}{dx}(x^2-x+1)=2x-1$]

02. $2x-1$ কে প্রথমে $3x-2$ এর জায়গায় লিখে ফেলতে হবে।

03.এরপর $3x-2$ এর সাথে balance করার জন্য $2x-1$ এর সাথে $\frac{3}{2}$ দ্বারা গুন করতে হবে।

04. এর পর $2x-1$ এর সাথে balance করার জন্য $\frac{3}{2}-2$ যোগ করতে হবে।

05. অর্থাৎ $\frac{3}{2}(2x-1)+\frac{3}{2}-2=\frac{3}{2}\times2x-\frac{3}{2}+\frac{3}{2}-2=3x-2$

$$\therefore\int(3x-2)\sqrt{(x^2-x+1)}dx$$

$$=\int\{\frac{3}{2}(2x-1)+\frac{3}{2}-2\}\sqrt{x^2-x+1}dx\ [\because\frac{d}{dx}(x^2-x+1)=2x-1]$$

$$= \int \{\frac{3}{2}(2x-1) - \frac{1}{2}\}\sqrt{x^2 - x + 1}dx$$

$$= \frac{3}{2}\int (2x-1)\sqrt{x^2 - x + 1}dx - \frac{1}{2}\int \sqrt{x^2 - x + 1}dx \text{ -----------------------(i)}$$

মনে রাখুন

- ***এখানে দুটি integration এর প্রথমটিতে*** $x^2 - x + 1 = z$ ***ধরতে হবে এবং পরেরটিতে*** $x^2 - x + 1$ ***কে*** $(a+b)^2$ অথবা $(a-b)^2$ formula বানাতে হবে। এক্ষেত্রে মনে রাখতে হবে $(a+b)^2$ অথবা $(a-b)^2$ formula বানাতে গিয়ে অতিরিক্ত constant যোগ অথবা বিয়োগ করা যাবে কিন্তু variable বাড়ানো যাবেনা ।]

Now, $\int (2x-1)\sqrt{x^2 - x + 1}dx$

Let, $z = x^2 - x + 1$

$\therefore \frac{dz}{dx} = 2x - 1$

$\therefore dz = (2x-1)dx$

$$\int (2x-1)\sqrt{x^2 - x + 1}dx$$

$$= \int \sqrt{z}dz = \int z^{1/2}dz$$

$$= \frac{z^{1/2+1}}{1/2 + 1} \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1}]$$

$$= \frac{z^{3/2}}{3/2} = \frac{2}{3}z^{3/2} = \frac{2}{3}(x^2 - x + 1)^{3/2} \text{ -------------------------(ii)}$$

And, $\int \sqrt{x^2 - x + 1}dx$

$$= \int \sqrt{x^2 - 2.x.\tfrac{1}{2} + (\frac{1}{2})^2 + 1 - (\frac{1}{2})^2}dx$$

$$= \int \sqrt{x^2 - 2.x.\tfrac{1}{2} + \frac{1}{4} + 1 - \frac{1}{4}}dx = \int \sqrt{(x - \tfrac{1}{2})^2 + \frac{3}{4}}\,dx$$

$$= \int \sqrt{(x - \tfrac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\,dx$$

$$= \{\frac{(x - \frac{1}{2})\sqrt{(x - \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}}{2} \pm \frac{(\frac{\sqrt{3}}{2})^2}{2}\sin^{-1}\frac{(x - \frac{1}{2})}{\frac{\sqrt{3}}{2}}\} \text{ ------------------(iii)}$$

$$[\because \textit{Formula \# 62} \int \sqrt{x^2 + a^2}dx = \frac{x\sqrt{x^2 + a^2}}{2} \pm \frac{a^2}{2}\sin^{-1}\frac{x}{a}]$$

Putting the value of $\int (2x-1)\sqrt{x^2 - x + 1}\,dx$ and $\int \sqrt{x^2 - x + 1}dx$ in (i)

$$\int (3x-2)\sqrt{(x^2-x+1)}dx = \frac{3}{2}\int (2x-1)\sqrt{x^2-x+1}dx - \frac{1}{2}\int \sqrt{x^2-x+1}dx$$

$$= \frac{3}{2}.\frac{2}{3}(x^2-x+1)^{3/2} - \frac{1}{2}[\{\frac{(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}{2} \pm \frac{(\frac{\sqrt{3}}{2})^2}{2} \sin^{-1}\frac{(x-\frac{1}{2})}{\frac{\sqrt{3}}{2}}\}]$$

$$= (x^2-x+1)^{3/2} - \frac{1}{2}[\{\frac{(x-\frac{1}{2})\sqrt{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}}{2} \pm \frac{(\frac{\sqrt{3}}{2})^2}{2} \sin^{-1}\frac{(x-\frac{1}{2})}{\frac{\sqrt{3}}{2}}\}] \textit{ Answer}$$

Method # 10: Partial Fraction

Fractions

A Fraction (such as $\frac{7}{4}$) has two numbers: $\frac{\textbf{Numerator}}{\textbf{Deno min ator}}$

The top number is the Numerator.
The bottom number is the Denominator.

Example: $\frac{7}{4}$

Proper Fractions: The numerator is less than the denominator Examples: $\frac{1}{3}, \frac{3}{4}, \frac{2}{7}$

Improper Fractions: The numerator is greater than (or equal to) the denominator Examples: $\frac{4}{3}, \frac{11}{4}, \frac{7}{7}$

A proper fraction may be written as the sum of partial fractions according to the following rules:

Procedure#01: Linear Factors: None of which are repeated

Example: $\frac{x+4}{(x+7)(2x-1)} \equiv \frac{A}{x+7} + \frac{B}{2x-1}$

Procedure#02: Linear Factors: Some of which are repeated:

Example: $\frac{3x-1}{(x+4)^2} \equiv \frac{A}{x+4} + \frac{B}{(x+4)^2}$ $[\because (x+4)^2 = (x+4)(x+4)]$

Procedure#03: Quadratic Factors: None of which are repeated

Example: $\frac{x^2-3}{(x-2)(x^2+4)} \equiv \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$

Procedure#04: Quadratic Factors: Some of which are repeated

Example: $\frac{x^2-4x+1}{(x^2+1)^2(x^2+x+1)} \equiv \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{x^2+x+1}$

Example 61: $\int \frac{dx}{x^2+x-2}$

Solution: The integrand is a proper rational function that can be written as

$$\frac{1}{x^2+x-2}=\frac{1}{x^2+2x-x-2}=\frac{1}{x(x+2)-1(x+2)}=\frac{1}{(x+2)(x-1)} \text{ ------------------(i)}$$

Let, $\frac{1}{(x+2)(x-1)}=\frac{A}{x+2}+\frac{B}{x-1}$ **[Procedure#01]----------------------------------(ii)**

Multiplying (ii) by denominator $(x+2)(x-1)$

$$\frac{1}{(x+2)(x-1)}\times(x+2)(x-1)=\frac{A}{(x+2)}\times(x+2)(x-1)+\frac{B}{(x-1)}\times(x+2)(x-1)$$

$\Rightarrow 1=A(x-1)+B(x+2)$ --(iii)

Let, $x+2=0$

$\therefore \quad x=-2$

Putting $x=-2$ in (iii)

$1=A(-2-1)+B(-2+2)$

$\Rightarrow 1=A(-2-1)+B.0$

$\Rightarrow 1=-3A$

$\therefore A=-\frac{1}{3}$

Again, Let, $x-1=0$

$\therefore \quad x=1$

Putting $x=1$ in (iii)

$1=A(1-1)+B(1+2)$

$\Rightarrow 1=A.0+B(1+2)$

$\Rightarrow 1=3B$

$\Rightarrow B=\frac{1}{3}$

Substituting these values $A=-\frac{1}{3}$ and $B=\frac{1}{3}$ in (ii)

$$\frac{1}{(x+2)(x-1)}=\frac{-1/3}{x+2}+\frac{1/3}{x-1}$$

$$\therefore \frac{1}{x^2+x-2}=\frac{1}{(x+2)(x-1)}=\frac{-1/3}{x+2}+\frac{1/3}{x-1}$$

Now,

$$\int\frac{dx}{x^2+x-2}=\int\frac{-1/3}{x+2}dx+\int\frac{1/3}{x-1}dx$$

$$=-\frac{1}{3}\int\frac{1}{x+2}dx+\frac{1}{3}\int\frac{1}{x-1}dx$$

$=-\frac{1}{3}\ln(x+2)+\frac{1}{3}\ln(x-1)$ ***Answer*** **[Formula-63:** $\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+c$ **]**

Example 62: $\int \frac{3x-1}{(x+4)^2}dx$

Solution: Let, $\frac{3x-1}{(x+4)^2} \equiv \frac{A}{x+4} + \frac{B}{(x+4)^2}$ **[Procedure#02] ----------------(i)**

Multiplying (i) by the denominator $(x+4)^2$

$$\frac{3x-1}{(x+4)^2} \times (x+4)^2 = \frac{A}{(x+4)} \times (x+4)^2 + \frac{B}{(x+4)^2} \times (x+4)^2$$

$$(3x-1) = A \times (x+4) + B$$

$$(3x-1) = A(x+4) + B \text{ --(ii)}$$

Let, $x+4=0$

$\therefore \quad x=-4$

Putting $x=-4$ in (ii)

$$(3x-1) = A(x+4) + B$$

$$\Rightarrow (3.(-4)-1) = A(-4+4) + B$$

$$\Rightarrow -12-1 = B$$

$$\Rightarrow -13 = B$$

$$\therefore B = -13$$

Equation (ii) can be written as

$$(3x-1) = A(x+4) + B$$

$$\Rightarrow (3x-1) = Ax + 4A + B$$

$$\Rightarrow 3x-1 = Ax + 4A + B \text{ ---(iii)}$$

Equating the coefficient (সহগ) of x and constant terms on both sides in (iii)

$$3 = A$$

$$-1 = 4A + B$$

Putting the value of A and B in (i), we get

$$\frac{3x-1}{(x+4)^2} \equiv \frac{A}{x+4} + \frac{B}{(x+4)^2}$$

$$\Rightarrow \frac{3x-1}{(x+4)^2} \equiv \frac{3}{x+4} + \frac{-13}{(x+4)^2}$$

$$\therefore \int \frac{3x-1}{(x+4)^2}dx = \int \frac{3}{x+4}dx + \int \frac{-13}{(x+4)^2}dx \text{ ----------------------(iv)}$$

From (iv),

$$\therefore \int \frac{3x-1}{(x+4)^2}dx = \int \frac{3}{x+4}dx + \int \frac{-13}{(x+4)^2}dx$$

$$= \int \frac{3}{z}dz + \int \frac{-13}{z^2}dz$$

$$= \int \frac{3}{z}dz - 13\int z^{-2}dz$$

$$= 3\int \frac{1}{z}dz - 13\int z^{-2}dz$$

Let,
$x+4=z$ [Method-01]
$\Rightarrow z = x+4$
$\Rightarrow \frac{dz}{dx} = 1+0$
$\Rightarrow dz = dx$

$$= 3\ln z - 13\frac{z^{-2+1}}{-2+1} \text{[Formula-1\&2, } \int x^n dx = \frac{x^{n+1}}{n+1}; n \neq -1;\ \int \frac{dx}{x} = \ln x \text{]}$$

$$= 3\ln z - 13\frac{z^{-1}}{-1} = 3\ln z + 13z^{-1} = 3\ln z + 13\frac{1}{z}$$

$$= 3\ln(x+4) + 13\frac{1}{(x+4)} = 3\ln(x+4) + \frac{13}{(x+4)}$$ ***Answer***

Example 63: $\int \frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} dx.$

$$\frac{x^2 + x - 2}{3x^3 - x^2 + 3x - 1} = \frac{x^2 + x - 2}{x^2(3x-1) + (3x-1)} = \frac{x^2 + x - 2}{(3x-1)(x^2+1)}$$

Let, $\frac{x^2 + x - 2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx + C}{x^2+1}$ [**Procedure#03**]----------------------(i)

Multiplying (i) by the denominator $(3x-1)(x^2+1)$ yields

$$\frac{x^2 + x - 2}{(3x-1)(x^2+1)} \times (3x-1)(x^2+1) = \frac{A}{3x-1} \times (3x-1)(x^2+1) + \frac{Bx + C}{x^2+1} \times (3x-1)(x^2+1)$$

$x^2 + x - 2 = A(x^2+1) + (Bx + C)(3x-1)$ --(ii)

Putting $3x - 1 = 0$ in (ii)

$$\Rightarrow 3x = 1$$

$$\Rightarrow x = \frac{1}{3}$$

Substituting $x = \frac{1}{3}$ in (ii)

$$x^2 + x - 2 = A(x^2+1) + (Bx + C)(3x-1)$$

$$\Rightarrow (\frac{1}{3})^2 + \frac{1}{3} - 2 = A((\frac{1}{3})^2 + 1) + (B \times \frac{1}{3} + C)(3 \times \frac{1}{3} - 1)$$

$$\Rightarrow (\frac{1}{3})^2 + \frac{1}{3} - 2 = A((\frac{1}{3})^2 + 1) + (B \times \frac{1}{3} + C)(1 - 1)$$

$$\Rightarrow \frac{1}{9} + \frac{1}{3} - 2 = A(\frac{1}{9} + 1) + 0$$

$$\Rightarrow \frac{1 + 3 - 18}{9} = \frac{A + 9A}{9}$$

$$\Rightarrow 10A = -14$$

$\Rightarrow A = \frac{-14}{10} = -\frac{7}{5}$ --(iii)

Equation (ii) can be written as

$$x^2 + x - 2 = A(x^2+1) + (Bx + C)(3x-1)$$

$$\Rightarrow x^2 + x - 2 = Ax^2 + A + 3Bx^2 - Bx + 3Cx - C)$$

$$\Rightarrow x^2 + x - 2 = Ax^2 + 3Bx^2 - Bx + 3Cx - C + A)$$

$$\Rightarrow x^2.1 + x.1 - 2 = (A+3B)x^2 + (-B+3C)x + (A-C) \text{ --------(iv)}$$

Equating corresponding coefficient (সহগ) of x^2, x and constant term on both sides from (iv)

$$1 = (A+3B) \text{ ---------------------------------(v)}$$

$$1 = (-B+3C) \text{ -------------------------------(vi)}$$

$$-2 = A - C \text{ -------------------------------------(vii)}$$

Putting the value of $A = -\frac{7}{5}$ in (vii), we get

$$-2 = A - C$$

$$\Rightarrow -2 = -\frac{7}{5} - C$$

$$\Rightarrow -2 + \frac{7}{5} = -C$$

$$\Rightarrow \frac{-10+7}{5} = -C$$

$$\Rightarrow \frac{-3}{5} = -C$$

$$\Rightarrow \frac{3}{5} = C$$

$$\therefore C = \frac{3}{5}$$

Putting the value of $A = -\frac{7}{5}$ in (v), we get

$$1 = (A+3B)$$

$$\Rightarrow 1 = -\frac{7}{5} + 3B$$

$$\Rightarrow 1 + \frac{7}{5} = 3B$$

$$\Rightarrow \frac{12}{5} = 3B$$

$$\Rightarrow \frac{4}{5} = B$$

$$\therefore B = \frac{4}{5}$$

So, Putting the values of $A = -\frac{7}{5}$, $B = \frac{4}{5}$, $C = \frac{3}{5}$ in (i) becomes

$$\frac{x^2 + x - 2}{(3x-1)(x^2+1)} = \frac{A}{3x-1} + \frac{Bx+C}{x^2+1}$$

$$\frac{x^2 + x - 2}{(3x-1)(x^2+1)} = \frac{-\frac{7}{5}}{3x-1} + \frac{\frac{4}{5}x + \frac{3}{5}}{x^2+1}$$

And

$$\int \frac{x^2+x-2}{(3x-1)(x^2+1)}dx = -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\int \frac{x}{x^2+1}dx + \frac{3}{5}\int \frac{dx}{x^2+1}$$

$$\int \frac{x^2+x-2}{(3x-1)(x^2+1)}dx = -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{4}{5}\times\frac{1}{2}\int \frac{2x}{x^2+1}dx + \frac{3}{5}\int \frac{dx}{x^2+1}$$

$$\int \frac{x^2+x-2}{(3x-1)(x^2+1)}dx = -\frac{7}{5}\int \frac{dx}{3x-1} + \frac{2}{5}\int \frac{2x}{x^2+1}dx + \frac{3}{5}\int \frac{dx}{x^2+1}$$

$$= -\frac{7}{15}\ln|3x-1| + \frac{2}{5}\ln\left(x^2+1\right) + \frac{3}{5}\tan^{-1}x + C \textit{ Answer}$$

[Formula-63: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$; ***Formula # 10]***

Example 64: $\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}dx$.

The partial fraction decomposition of the integrand is

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2} \text{ --------------------------(i)}$$

[Procedure#04]

Multiplying (1) by the denominator $(x+2)(x^2+3)^2$ yields on both sides

$$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\times(x+2)(x^2+3)^2 = \frac{A}{x+2}\times(x+2)(x^2+3)^2 + \frac{Bx+C}{x^2+3}\times(x+2)(x^2+3)^2 + \frac{Dx+E}{(x^2+3)^2}\times(x+2)(x^2+3)^2$$

$\Rightarrow 3x^4+4x^3+16x^2+20x+9$

$= A(x^2+3)^2 + (Bx+C)(x^2+3)(x+2) + (Dx+E)(x+2)$ -----------------------(ii)

$\Rightarrow 3x^4+4x^3+16x^2+20x+9 = A(x^4+2x^2.3+3^2) + (Bx+C)(x^3+2x^2+3x+6)$

$+ (Dx^2+2Dx+Ex+2E)$

$\Rightarrow 3x^4+4x^3+16x^2+20x+9 = A(x^4+6x^2+9) + (Bx+C)(x^3+2x^2+3x+6)$

$+ (Dx^2+2Dx+Ex+2E)$

$\Rightarrow 3x^4+4x^3+16x^2+20x+9 = Ax^4+6Ax^2+9A + (Bx^4+2Bx^3+3Bx^2+6Bx$

$+ Cx^3+2Cx^2+3Cx+6C) + (Dx^2+2Dx+Ex+2E)$

$\Rightarrow 3x^4+4x^3+16x^2+20x+9 = Ax^4+6Ax^2+9A+Bx^4+2Bx^3+3Bx^2+6Bx$

$+ Cx^3+2Cx^2+3Cx+6C+Dx^2+2Dx+Ex+2E$

$\Rightarrow 3x^4+4x^3+16x^2+20x+9 = Ax^4+Bx^4+2Bx^3$

$+ 6Bx+3Cx+2Dx+Ex+9A+6C+2E$

$\Rightarrow 3x^4+4x^3+16x^2+20x+9 = (A+B)x^4+(2B+C)x^3+(6A+3B+2C+D)x^2$

$+ (6B+3C+2D+E)x + (9A+6C+2E)$ ..-..--……......-------------------------------. (iii)

Equating the coefficient (সহগ) of $\mathbf{x^4, x^3, x^2, x}$ and constant term from (iii)

$\mathbf{A + B = 3}$ ---(iv)

$\mathbf{6A + 3B + 2C + D = 16}$ --(v)

$\mathbf{6B + 3C + 2D + E = 20}$ --(vi)

$\mathbf{9A + 6C + 2E = 9}$ ---(vii)

Let, $\mathbf{x + 2 = 0}$ in (ii),

$\Rightarrow \mathbf{x = -2}$

Substituting $\Rightarrow \mathbf{x = -2}$ in (ii),

$$\mathbf{3(-2)^4 + 4(-2)^3 + 16(-2)^2 + 20(-2) + 9 = A((-2)^2 + 3)^2 + (B(-2) + C)((-2)^2 + 3)(-2 + 2)}$$
$$\mathbf{+ (D(-2) + E)(-2 + 2)}$$

$$\Rightarrow \mathbf{3.16 - 4.8 + 16.4 - 40 + 9 = A(4 + 3)^2 + (-2B + C)(4 + 3).0 + (-2D + E).0}$$
$$\Rightarrow \mathbf{48 - 32 + 64 - 40 + 9 = 49A + 0 + 0}$$
$$\Rightarrow \mathbf{16 + 24 + 9 = 49A + 0 + 0}$$
$$\Rightarrow \mathbf{40 + 9 = 49A}$$
$$\Rightarrow \mathbf{49 = 49A}$$
$$\Rightarrow \mathbf{A = 1}$$

Put the value of $\mathbf{A = 1}$ in (iv), we get

$$\mathbf{A + B = 3}$$
$$\mathbf{1 + B = 3}$$
$$\mathbf{B = 3 - 1}$$
$$\mathbf{B = 2}$$

Putting the value of $\mathbf{A, B}$ in (v),

$$\mathbf{6A + 3B + 2C + D = 16}$$
$$\Rightarrow \mathbf{6.1 + 3.2 + 2C + D = 16}$$
$$\Rightarrow \mathbf{6 + 6 + 2C + D = 16}$$
$$\Rightarrow \mathbf{12 + 2C + D = 16}$$
$$\Rightarrow \mathbf{2C + D = 16 - 12}$$
$$\Rightarrow \mathbf{2C + D = 4}$$

$\Rightarrow \mathbf{D = 4 - 2C}$ --------------------------------(viii)

Putting the value of $\mathbf{B, D}$ in (vi),

$$\mathbf{6B + 3C + 2D + E = 20}$$
$$\Rightarrow \mathbf{6.2 + 3C + 2(4 - 2C) + E = 20}$$
$$\Rightarrow \mathbf{12 + 3C + 8 - 4C + E = 20}$$
$$\Rightarrow \mathbf{20 + 3C - 4C + E = 20}$$
$$\Rightarrow \mathbf{20 - C + E = 20}$$
$$\Rightarrow \mathbf{-C + E = 0}$$
$$\Rightarrow \mathbf{-C = -E}$$

$\Rightarrow \mathbf{C = E}$ ---(ix)

Putting the value of $\mathbf{A, C}$ in (vii),

$$\mathbf{9A + 6C + 2E = 9}$$
$$\Rightarrow \mathbf{9.1 + 6.E + 2E = 9}$$
$$\Rightarrow \mathbf{9.1 + 8E = 9}$$
$$\Rightarrow \mathbf{9 + 8E = 9}$$
$$\Rightarrow \mathbf{8E = 9 - 9}$$

$\Rightarrow 8E = 0$

$\Rightarrow E = 0$ --(x)

Putting the value of $\mathbf{E}$ in (ix),

$C = E$

$\Rightarrow C = 0$ --(xi)

Putting the value of $\mathbf{C}$ in (viii),

$\Rightarrow D = 4 - 2C$

$\Rightarrow D = 4 - 0$

$\Rightarrow D = 4$

Substituting $\mathbf{A, B, C, D, E}$ in (i), we get

$$\frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx + C}{x^2+3} + \frac{Dx + E}{(x^2+3)^2}$$

$$\Rightarrow \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x + 0}{x^2+3} + \frac{4x + 0}{(x^2+3)^2}$$

$$\Rightarrow \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} = \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

$$\therefore \int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} dx = \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3} dx + 4\int \frac{x}{(x^2+3)^2} dx$$

$$\therefore \int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} dx = \int \frac{dx}{x+2} + \int \frac{2x}{x^2+3} dx + 4\int \frac{x}{(x^2+3)^2} dx$$

$$= \ln|x+2| + \ln(x^2+3) + 4\int \frac{x}{(x^2+3)^2} dx$$

[Formula-63: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$; *Formula # 10]*

Now, $\int \frac{x}{(x^2+3)^2} dx$

Let, $x^2 + 3 = z$

$\frac{dz}{dx} = 2x$

$2x\,dx = dz$

$x\,dx = \frac{dz}{2}$

$$= \frac{1}{2}\int \frac{dz}{z^2} = \frac{1}{2}\int z^{-2} dz = \frac{1}{2} \times \frac{z^{-2+1}}{-2+1} = -\frac{1}{2} \times \frac{1}{z} = -\frac{1}{2(x^2+3)}$$

$$\therefore 4\int \frac{x}{(x^2+3)^2} dx = -4 \times \frac{1}{2(x^2+3)} = -\frac{2}{(x^2+3)}$$

$$\therefore \int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} dx = \ln|x+2| + \ln(x^2+3) + 4\int \frac{x}{(x^2+3)^2} dx$$

$$\therefore \int \frac{3x^4 + 4x^3 + 16x^2 + 20x + 9}{(x+2)(x^2+3)^2} dx = \ln|x+2| + \ln(x^2+3) - \frac{2}{(x^2+3)}$$ *Answer*

Example 65: $\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx$

Solution: The integrand is an improper rational function since the numerator has degree 4 and the denominator has degree 2.

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}$$

$$= \quad x^2+x-2 \,\big|\, 3x^4+3x^3-5x^2+x-1 \,\big|\, 3x^2+1$$

$$\underline{\pm 3x^4 \pm 3x^3 \mp 6x^2}$$

$$x^2+x-1$$

$$\underline{\pm x^2 \pm x \mp 2}$$

$$1$$

$$\therefore \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \frac{1}{x^2+x-2}$$

And hence

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = \int (3x^2+1)dx + \int \frac{dx}{x^2+x-2}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = \int 3x^2dx + \int 1dx + \int \frac{dx}{x^2+x-2}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = \int 3x^2dx + \int 1dx + \int \frac{dx}{x^2+2.x.\frac{1}{2}+(\frac{1}{2})^2-(\frac{1}{2})^2-2}$$

[Method # 02]

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = \int 3x^2dx + \int 1dx + \int \frac{dx}{(x+\frac{1}{2})^2-\frac{1}{4}-2}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = \int 3x^2dx + \int 1dx + \int \frac{dx}{(x+\frac{1}{2})^2-\frac{(1+8)}{4}}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = \int 3x^2dx + \int 1dx + \int \frac{dx}{(x+\frac{1}{2})^2-\frac{9}{4}}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = \int 3x^2dx + \int 1dx + \int \frac{dx}{(x+\frac{1}{2})^2-(\frac{3}{2})^2}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = 3.\frac{x^{2+1}}{2+1} + x + \frac{1}{2.\frac{3}{2}}\ln\frac{x+\frac{1}{2}-\frac{3}{2}}{x+\frac{1}{2}+\frac{3}{2}}$$ *[Formula #27]*

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = 3.\frac{x^3}{3}+x+\frac{1}{3}\ln\frac{\frac{2x+1-3}{2}}{\frac{2x+1+3}{2}}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = 3.\frac{x^3}{3}+x+\frac{1}{3}\ln\frac{\frac{2x-2}{2}}{\frac{2x+4}{2}}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = 3.\frac{x^3}{3}+x+\frac{1}{3}\ln\frac{\frac{2(x-1)}{2}}{\frac{2(x+2)}{2}}$$

$$\Rightarrow \int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}dx = 3.\frac{x^3}{3}+x+\frac{1}{3}\ln\frac{(x-1)}{(x+2)}$$

Example 66: $\int \frac{x^4+2x+6}{x^3+x^2-2x}dx$

Now,

$$\frac{x^4+2x+6}{x^3+x^2-2x} = x^3+x^2-2x\left|\begin{array}{l} x^4+2x+6 \\ \underline{\pm x^4 \pm x^3 \mp 2x^2} \end{array}\right| x-1$$

$$\begin{array}{c} -x^3+2x^2+2x+6 \\ \underline{\mp x^3 \mp x^2 \pm 2x} \\ 3x^2+6 \end{array}$$

$$\therefore \quad \frac{x^4+2x+6}{x^3+x^2-2x} = (x-1)+\frac{3x^2+6}{x^3+x^2-2x}$$

$$\int \frac{x^4+2x+6}{x^3+x^2-2x}dx = \int \{(x-1)+\frac{3x^2+6}{x^3+x^2-2x}\}\, dx$$

$$= \int \{(x-1)+\frac{3x^2+6}{x(x^2+x-2)}\}dx = \int \{(x-1)+\frac{3x^2+6}{x(x^2+2x-x-2)}\}dx$$

$$= \int \{(x-1)+\frac{3x^2+6}{x\{x(x+2)-1(x+2)\}}dx = \int \{(x-1)+\frac{3x^2+6}{x(x-1)(x+2)}\}dx$$

$$\int \frac{x^4+2x+6}{x^3+x^2-2x}dx = \int (x-1)dx + \int \frac{3x^2+6}{x(x-1)(x+2)}dx \text{ ----------------------------(i)}$$

Let, $\frac{3x^2+6}{x(x-1)(x+2)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+2}$ --(ii)

[Procedure#01]

Multiplying (ii) by the denominator $\mathbf{x(x-1)(x+2)}$ yields on both sides

$$\frac{3x^2+6}{x(x-1)(x+2)}\times x(x-1)(x+2) = \frac{A}{x}\times x(x-1)(x+2)+\frac{B}{x-1}\times x(x-1)(x+2)+\frac{C}{x+2}\times x(x-1)(x+2)$$

$3x^2+6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$ ----------------------------------(iii)

Let, $x+2=0$ in (3),

$\Rightarrow x=-2$

Substituting $\Rightarrow x=-2$ in (iii),

$$3x^2+6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$
$$\Rightarrow 3(-2)^2+6 = A(-2-1)(-2+2) + B(-2)(-2+2) + C(-2)(-2-1)$$
$$\Rightarrow 3.4+6 = A(-2-1).0 + B(-2).0 + C(-2)(-3)$$
$$\Rightarrow 18 = 0+0+6C$$
$$\Rightarrow 18 = 6C$$
$$\Rightarrow 6C = 18$$
$$\Rightarrow C = 3$$

Let, $x-1=0$ in (iii),

$\Rightarrow x=1$

Substituting $x=1$ in (iii),

$$3x^2+6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$
$$\Rightarrow 3(1)^2+6 = A(1-1)(1+2) + B(1)(1+2) + C(1)(1-1)$$
$$\Rightarrow 3.1+6 = A.0.(1+2) + B(1)(3) + C(1).0$$
$$\Rightarrow 9 = 0+3B+0$$
$$\Rightarrow 9 = 3B$$
$$\Rightarrow 3B = 9$$
$$\Rightarrow B = 3$$

Let, $x=0$ in (iii),

Substituting $x=0$ in (iii),

$$3x^2+6 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1)$$
$$\Rightarrow 3(0)^2+6 = A(0-1)(0+2) + B(0)(0+2) + C(0)(0-1)$$
$$\Rightarrow 3.0+6 = A(-1)(2) + B.0.(2) + C.0.(0-1)$$
$$\Rightarrow 6 = -2A+0+0$$
$$\Rightarrow 6 = -2A$$
$$\Rightarrow -2A = 6$$
$$\Rightarrow A = -3$$

Putting the values of $A=-3$, $B=3$, $C=3$ in (ii) becomes

$$\frac{3x^2+6}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$
$$\Rightarrow \frac{3x^2+6}{x(x-1)(x+2)} = \frac{-3}{x} + \frac{3}{x-1} + \frac{3}{x+2}$$

From (i),

$$\int \frac{x^4+2x+6}{x^3+x^2-2x}dx = \int (x-1)dx + \int \frac{3x^2+6}{x(x-1)(x+2)}dx$$

$$\therefore \int \frac{x^4+2x+6}{x^3+x^2+2x} = \int (x-1)dx + \int \{-\frac{3}{x} + \frac{3}{x-1} + \frac{3}{x+2}\}dx$$

$$\therefore \int \frac{x^4+2x+6}{x^3+x^2+2x} = \int (x-1)dx - \int \frac{3}{x}dx + \int \frac{3}{x-1}dx + \int \frac{3}{x+2}dx$$

$$\therefore \int \frac{x^4+2x+6}{x^3+x^2+2x} = \frac{x^2}{2} - x - 3\ln x + 3\ln(x-1) + 3\ln(x+2)$$ ***Answer***

[Formula-63: $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$; ***Formula # 10]***

Example 67: $\int \frac{2x+4}{x^3-2x^2}dx$

Answer: The integrand can be rewritten as

$$\frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)}$$

Let,

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} \text{ ---(i)}$$

Multiplying (i) by the denominator $x^2(x-2)$ yields on both sides

$$\frac{2x+4}{x^2(x-2)} \times x^2(x-2) = \frac{A}{x} \times x^2(x-2) + \frac{B}{x^2} \times x^2(x-2) + \frac{C}{x-2} \times x^2(x-2)$$

$$\Rightarrow 2x+4 = Ax(x-2) + B(x-2) + Cx^2 \text{ -----------------------(ii)}$$

$$\Rightarrow 2x+4 = Ax^2 - 2Ax + Bx - 2B + Cx^2$$

$$\Rightarrow 2x+4 = Ax^2 + Cx^2 + Bx - 2Ax - 2B$$

$$\Rightarrow 2x+4 = (A+C)x^2 + (B-2A)x - 2B \text{ ----------------------(iii)}$$

Equating the coefficient (সহগ) of x^2, x and constant term

$$A + C = 0 \text{ --(iv)}$$
$$B - 2A = 2 \text{ --(v)}$$
$$-2B = 4 \text{ ---(vi)}$$

Let, $x - 2 = 0$ in (ii),

$\Rightarrow x = 2$

Substituting $x = 2$ in (ii),

$$\Rightarrow 2x+4 = Ax(x-2) + B(x-2) + Cx^2$$
$$\Rightarrow 2.2+4 = A.2(2-2) + B(2-2) + C.2^2$$
$$\Rightarrow 4+4 = A.2.0 + B.0 + C.2^2$$
$$\Rightarrow 4+4 = 0+0+C.2^2$$
$$\Rightarrow 8 = 4C$$
$$\Rightarrow 4C = 8$$
$$\Rightarrow C = 2$$

Let, $x = 0$ in (ii),

Substituting $x = 0$ in (ii)

$$\Rightarrow 2x+4 = Ax(x-2)+B(x-2)+Cx^2$$
$$\Rightarrow 2.0+4 = A.0(0-2)+B(0-2)+C.0^2$$
$$\Rightarrow 4 = 0+B(-2)+0$$
$$\Rightarrow 4 = -2B$$
$$\Rightarrow -2B = 4$$
$$\Rightarrow B = -2$$

Putting the value of $\mathbf{C = 2}$ in (iv),

$$A + C = 0$$
$$A + 2 = 0$$
$$A = -2$$

Putting the values of $\mathbf{A = -2,\ \ B = -2,\ \ C = 2}$ in (i) becomes

$$\frac{2x+4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$
$$\frac{2x+4}{x^2(x-2)} = \frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2}$$
$$\therefore \int \frac{2x+4}{x^2(x-2)}dx = -2\int\frac{dx}{x} - 2\int\frac{dx}{x^2} + 2\int\frac{dx}{x-2}$$
$$\therefore \int \frac{2x+4}{x^2(x-2)}dx = -2\int\frac{dx}{x} - 2\int x^{-2}dx + 2\int\frac{dx}{x-2}$$
$$\therefore \int \frac{2x+4}{x^2(x-2)}dx = -2\ln x - 2\frac{x^{-2+1}}{-2+1} + 2\ln(x-2) + C\ [\because \int x^n dx = \frac{x^{n+1}}{n+1}]$$
$$\therefore \int \frac{2x+4}{x^2(x-2)}dx = -2\ln x - 2\frac{x^{-1}}{-1} + 2\ln(x-2) + C$$
$$\therefore \int \frac{2x+4}{x^2(x-2)}dx = -2\ln x + 2\frac{1}{x} + 2\ln(x-2) + C \textit{ Answer}$$

Method # 11:

$$\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}dx$$

Let, $\mathbf{a\cos x + b\sin x = l}$ **(Differentiate of denominator) + m (denominator)**

Example 68: $\int \frac{3\cos x + 4\sin x}{4\cos x + 5\sin x}dx$

Let, $\mathbf{I} = \int \frac{3\cos x + 4\sin x}{4\cos x + 5\sin x}dx$

Let, $3\cos x + 4\sin x = l\{\frac{d}{dx}(4\cos x + 5\sin x)\} + m(4\cos x + 5\sin x)$ ----------------------(i)

$$\Rightarrow 3\cos x + 4\sin x = l(-4\sin x + 5\cos x) + m(4\cos x + 5\sin x)$$

$$\Rightarrow 3\cos x + 4\sin x = -4l\sin x + 5l\cos x + 4m\cos x + 5m\sin x$$
$$\Rightarrow 3\cos x + 4\sin x = \sin x(-4l + 5m) + \cos x(5l + 4m)$$
$$\Rightarrow 3\cos x + 4\sin x = \cos x(5l + 4m) + \sin x(-4l + 5m)$$

Equating the co-efficient (সহগ) **of** $\cos x$ **and** $\sin x$ **we get**

$$\therefore 3\cos x = \cos x(5l + 4m)$$
$$\Rightarrow 3 = 5l + 4m \text{.....................................(ii)}$$

$$\therefore 4\sin x = \sin x(-4l + 5m)$$
$$\Rightarrow 4 = -4l + 5m \text{....................................(iii)}$$

$(ii)\times 4 + (iii)\times 5 \Rightarrow$

$$\Rightarrow 12 + 20 = 20l + 16m - 20l + 25m$$
$$\Rightarrow 32 = 41m$$
$$\Rightarrow m = {}^{32}\!/_{41}$$

Putting the value of $m = {}^{32}\!/_{41}$ in **(ii)**

$$\Rightarrow 3 = 5l + 4\cdot\frac{32}{41}$$
$$\Rightarrow 5l = 3 - \frac{128}{41} = \frac{123 - 128}{41}$$
$$\therefore 5l = -\frac{5}{41}$$
$$\Rightarrow l = -\frac{1}{41}$$

Putting the value of l, m in (i)

$$\therefore 3\cos x + 4\sin x = l\{\frac{d}{dx}(4\cos x + 5\sin x)\} + m(4\cos x + 5\sin x)$$
$$\therefore 3\cos x + 4\sin x = -\frac{1}{41}(-4\sin x + 5\cos x) + \frac{32}{41}(4\cos x + 5\sin x) \text{---------------(iv)}$$

$$\therefore I = \int \frac{3\cos x + 4\sin x}{4\cos x + 5\sin x}dx$$
$$\Rightarrow I = \int \frac{-{}^{1}\!/_{41}(-4\sin x + 5\cos x) + {}^{32}\!/_{41}(4\cos x + 5\sin x)}{4\cos x + 5\sin x}dx \quad \textit{[From (iv)]}$$
$$= \int \{-{}^{1}\!/_{41}\frac{(-4\sin x + 5\cos x)}{4\cos x + 5\sin x} + \frac{32}{41}\frac{(4\cos x + 5\sin x)}{(4\cos x + 5\sin x)}\}dx$$
$$= \int \{-{}^{1}\!/_{41}\frac{(-4\sin x + 5\cos x)}{4\cos x + 5\sin x}dx + \int \frac{32}{41}dx$$
$$= -\frac{1}{41}\ln(4\cos x + 5\sin x) + \frac{32}{41}x + c \text{ Answer } \textit{[Formula\#63]}$$

[নিচের ফাংশনকে ডিফারেন্সিয়েট করলে যদি উপরের ফাংশন পাওয়া যায় তাহলে তার ইন্টিগ্রেশন হল লগ অফ নিচের ফাংশন :]

Method # 12:

$$\int \frac{a\cos x + b\sin x + e}{c\cos x + d\sin x + f}$$

Let, $a\cos x + b\sin x + e = l$ (Differentiate of denominator) + m (denominator) +n

Example 69: $\int \frac{2 + 3\sin x - \cos x}{1 + \cos x + \sin x}dx$

Let, $2 + 3\sin x - \cos x = l \times \frac{d}{dx}(1 + \cos x + \sin x) + m(1 + \cos x + \sin x) + n$

$\Rightarrow 2 + 3\sin x - \cos x = l(0 - \sin x + \cos x) + m(1 + \cos x + \sin x) + n$ ---------------(A)

$\Rightarrow 2 + 3\sin x - \cos x = -l\sin x + l\cos x + m + m\cos x + m\sin x + n$

$\Rightarrow 2 + 3\sin x - \cos x = (-l + m)\sin x + (l + m)\cos x + m + n$

$\Rightarrow -\cos x + 3\sin x + 2 = (l + m)\cos x + (-l + m)\sin x + m + n$

Equating the co-efficient (সহগ) of $\cos x$, $\sin x$ and constant term we get

$l + m = -1$ --------------------------------(i)

$-l + m = 3$ ------------------------------(ii)

$m + n = 2$ ---------------------------------(iii)

(i)+ (ii) =>

$l + m = -1$

$-l + m = 3$

$2m = 2$

$\therefore m = 1$

Putting the value of $m = 1$ in (iii)

$m + n = 2$

$\Rightarrow 1 + n = 2$

$\Rightarrow n = 1$

Putting the value of $m = 1$ in (i)

$l + m = -1$

$\Rightarrow l + 1 = -1$

$\Rightarrow l = -2$

Putting the value of l, m, n in (A)

$2 + 3\sin x - \cos x = l(0 - \sin x + \cos x) + m(1 + \cos x + \sin x) + n$

$\Rightarrow 2 + 3\sin x - \cos x = -2(0 - \sin x + \cos x) + 1(1 + \cos x + \sin x) + 1$

$\Rightarrow 2 + 3\sin x - \cos x = 2\sin x - 2\cos x + 1 + \cos x + \sin x + 1$

$\Rightarrow 2 + 3\sin x - \cos x = -2(-\sin x + \cos x) + (1 + \cos x + \sin x) + 1$

$$\therefore \int \frac{2 + 3\sin x - \cos x}{1 + \cos x + \sin x}dx$$

$$= \int \frac{-2(-\sin x + \cos x) + (1 + \cos x + \sin x) + 1}{1 + \cos x + \sin x}dx$$

$$= -2\int \frac{-\sin x + \cos x}{1 + \cos x + \sin x}dx + \int \frac{(1 + \cos x + \sin x)}{(1 + \cos x + \sin x)}dx + \int \frac{1}{1 + \cos x + \sin x}dx$$

$$= -2\log(1+\cos x+\sin x)+x+\int \frac{1}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}dx$$ *[Formula#63]*

$$[\because 1+\cos x = 2\cos^2\frac{x}{2};\ \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}]$$

$$= -2\log(1+\cos x+\sin x)+x+\int \frac{\frac{1}{\cos^2\frac{x}{2}}}{\frac{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}dx$$

[Dividing by $\cos^2 x/2$]

$$= -2\log(1+\cos x+\sin x)+x+\int \frac{\sec^2\frac{x}{2}}{\frac{2\cos^2\frac{x}{2}}{\cos^2\frac{x}{2}}+\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}}dx$$

$$= -2\log(1+\cos x+\sin x)+x+\int \frac{\sec^2\frac{x}{2}}{2+\frac{2\sin\frac{x}{2}}{\cos\frac{x}{2}}}dx$$

$$= -2\log(1+\cos x+\sin x)+x+\int \frac{\sec^2\frac{x}{2}}{2+2\tan\frac{x}{2}}dx$$

$$= -2\log(1+\cos x+\sin x)+x+\ln(2+2\tan\frac{x}{2})$$ *[Formula#63]*

[নিচের ফাংশনকে ডিফারেন্সিয়েট করলে যদি উপরের ফাংশন পাওয়া যায় তাহলে তার ইন্টিগ্রেশন হল লগ অফ নিচের ফাংশন]

Proof:

$$\because \frac{d}{dx}(2+2\tan x/2)$$

$$= 0+2\sec^2\frac{x}{2}\times\frac{d}{dx}(\frac{x}{2}) \quad [\because \frac{d}{dx}(\tan x)=\sec^2 x]$$

$$= 2\sec^2\frac{x}{2}\times\frac{1}{2} = 2\times\frac{1}{2}\sec^2\frac{x}{2} = \sec^2\frac{x}{2}$$

Method # 13: Integration by parts is based on the equation: $\int \mathbf{udv} = \mathbf{uv} - \int \mathbf{vdu}$

Where u and v are both differentiable functions of x. Prove that $\int \mathbf{udv} = \mathbf{uv} - \int \mathbf{vdu}$

Answer: Let $\mathbf{u(x)}$ and $\mathbf{v(x)}$ have continuous derivatives. Then

$$\frac{d}{dx}(u(x)v(x)) = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$$

$$\Rightarrow \frac{d}{dx}(u(x)v(x)) = u(x)v'(x) + v(x)u'(x)$$

$$\int \frac{d}{dx}\{u(x)v(x)\}dx = \int\{u(x)v'(x) + v(x)u'(x)\}dx$$

$$\Rightarrow u(x)v(x) = \int[(u(x)v'(x) + v(x)u'(x))]dx \text{ ---------------------- (i)}$$

$$\Rightarrow u(x)v(x) = \int(u(x)v'(x)dx + \int v(x)u'(x)dx$$

$$\Rightarrow \int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \text{ ----------------------- (ii)}$$

Rewriting (ii)

$$\Rightarrow \int u(x)\frac{d}{dx}(v)dx = u(x)v(x) - \int v(x)\frac{d}{dx}(u)dx$$

$$[\therefore v'(x) = \frac{d}{dx}(v)\ \&\ u'(x) = \frac{d}{dx}(u)]$$

$$\Rightarrow \int u(x)d(v) = u(x)v(x) - \int v(x)d(u)$$

$\Rightarrow \int udv = uv - \int vdu$ ***(Proved)***

Example 70: $\int x e^x dx$

Solution:

We have, $\int u\,dv = uv - \int v\,du$ -------------------------(i)

Given, $\int \underbrace{x}_{u}\underbrace{e^x dx}_{dv}$

Let,

$u = x$

$\Rightarrow \frac{du}{dx} = 1$

$\Rightarrow du = dx$

Let

$dv = e^x dx$.

$\Rightarrow \int dv = \int e^x dx$

$\Rightarrow v = e^x$

Putting the values of u, dv, v, du in (i)

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u}\underbrace{e^x dx}_{dv} = \underbrace{xe^x}_{uv} - \int \underbrace{e^x}_{v}\underbrace{dx}_{du}$$

$\int xe^x dx = xe^x - e^x + c$. ***Answer***

Example 71: $\int x \ln x dx.$

Solution:

We have, $\int u\,dv = uv - \int v\,du$ ------------------------(i)

Given, $\int \underbrace{(\ln x)}_{u}\underbrace{xdx}_{dv}$

Let, $\quad \mathbf{u = \ln x \;\Rightarrow \frac{du}{dx} = \frac{1}{x} \qquad \Rightarrow du = \frac{dx}{x}}$

Let, $\quad \mathbf{dv = xdx \;\Rightarrow \int dv = \int x\,dx \;\Rightarrow v = \frac{x^2}{2}}$

Putting the values of **u, dv, v, du** in (i)

$$\int u\,dv = uv - \int v\,du$$

$$\int x \ln x dx = \ln x \underbrace{\frac{x^2}{2}}_{v} - \int \underbrace{\frac{x^2}{2}}_{v}\underbrace{\frac{dx}{x}}_{du}$$

$$= \frac{x^2 \ln x}{2} - \int \frac{xdx}{2}$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c \qquad \text{[Formula-1: } \int x^n dx = \frac{x^{n+1}}{n+1} + c \; ; n \neq -1 \text{]}$$

Example 72: Find $\int \tan^{-1} x dx$

Solution:

We have, $\int u\,dv = uv - \int v\,du$ ---------------------------(i)

Given, $\int \underbrace{\tan^{-1} x}_{u}\, \underbrace{dx}_{dv}$

Let

$\mathbf{u = \tan^{-1} x}$

$\mathbf{\frac{du}{dx} = \frac{1}{1+x^2}}$

$\mathbf{du = \frac{dx}{1+x^2}}$

Let

$\mathbf{dv = dx}$

$\mathbf{\int dv = \int dx}$

$\mathbf{\therefore v = x}$

Putting the values of **u, dv, v, du** in (i)

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{\tan^{-1} x}_{u}\, \underbrace{dx}_{dv} = \underbrace{(\tan^{-1} x)}_{u} \underbrace{x}_{v} - \int \underbrace{x}_{v} \underbrace{\frac{dx}{1+x^2}}_{du}$$

$$= x \tan^{-1} x - \int \frac{x}{1+x^2} dx = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx.$$

$$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) \quad \textit{[Formula\#63] Answer}$$

Example 73: $\int x \sin x dx$

Solution: We have, $\int u\,dv = uv - \int v\,du$ -------------------------(i)

Given, $\int \underbrace{x}_{u}\, \underbrace{\sin x dx}_{dv}$

Let,

$\mathbf{u = x}$

$\mathbf{\Rightarrow \frac{du}{dx} = 1}$

$\mathbf{\Rightarrow du = dx}$

Let

$\mathbf{dv = \sin x dx}$

Prc $\mathbf{\int dv = \int \sin x\, dx}$

$\mathbf{\Rightarrow v = -\cos x}$

Putting the values of $\mathbf{u, dv, v, du}$ in (i)

$$\int \mathbf{u\,dv = uv - \int v\,du}$$

$$\int \underbrace{\mathbf{x}}_{\mathbf{u}}\ \underbrace{\mathbf{\sin x dx}}_{\mathbf{dv}} = \underbrace{\mathbf{x}}_{\mathbf{u}}\ \underbrace{\mathbf{(-\cos x)}}_{\mathbf{v}} - \int \underbrace{\mathbf{(-\cos x)}}_{\mathbf{v}}\ \underbrace{\mathbf{dx}}_{\mathbf{du}}$$

$$= \mathbf{-x\cos x + \int \cos x dx}$$

$$= \mathbf{-x\cos x + \sin x + c} \quad \textit{Answer}$$

Example 74: $\int \mathbf{x^2 e^x dx}$

Solution:

We have, $\int \mathbf{u\,dv = uv - \int v\,du}$ -------------------------------(i)

Given, $\int \underbrace{\mathbf{x^2}}_{\mathbf{u}}\ \underbrace{\mathbf{e^x dx}}_{\mathbf{dv}}$

Let

$\mathbf{u = x^2}$

$\Rightarrow \mathbf{\dfrac{du}{dx} = 2x}$

$\Rightarrow\ \mathbf{du = 2xdx}$.

Let,

$\mathbf{dv = e^x dx}$

$\Rightarrow \int \mathbf{dv = \int e^x dx}$

$\Rightarrow\ \mathbf{v = e^x}$

Putting the values of $\mathbf{u, dv, v, du}$ in (i)

$$\int \mathbf{u\,dv = uv - \int v\,du}$$

$$\int \underbrace{\mathbf{x^2}}_{\mathbf{u}}\ \underbrace{\mathbf{e^x dx}}_{\mathbf{dv}} = \underbrace{\mathbf{x^2}}_{\mathbf{u}}\ \underbrace{\mathbf{e^x}}_{\mathbf{v}} - \int \underbrace{\mathbf{e^x}}_{\mathbf{v}}\ \underbrace{\mathbf{2xdx}}_{\mathbf{du}}$$

$$= \mathbf{x^2 e^x - 2\int e^x x dx}$$

$$= \mathbf{x^2 e^x - 2(xe^x - e^x) + c} \quad \textit{[See example 51]}$$

<u>Another way:</u>

Show that: $\int \mathbf{uvdx = u\int vdx - \int \{\frac{d}{dx}(u)\int vdx\}dx}$

We have, $\mathbf{\dfrac{d}{dx}(uw) = u\dfrac{d}{dx}(w) + w\dfrac{d}{dx}(u)}$

$$\Rightarrow \int \mathbf{\frac{d}{dx}(uw)dx = \int \{u\frac{d}{dx}(w) + w\frac{d}{dx}(u)\}dx}$$

$$\Rightarrow \int \mathbf{\frac{d}{dx}(u,w)dx = \int u\frac{dw}{dx}dx + w\int \frac{du}{dx}dx}$$

$$=> \int u\frac{dw}{dx}dx = \int \frac{d}{dx}(uw)dx - w\int \frac{du}{dx}dx$$

$$= uw - \int w\frac{du}{dx}dx \text{ ---------------------------------------(i)}$$

Let,

$$\frac{dw}{dx} = v$$

$$=> dw = vdx$$

$$=> \int dw = \int vdx$$

$$=> w = \int vdx$$

Putting $w = \int vdx$ in (i)

$$\int uvdx = u\int vdx - \int \{\frac{d}{dx}(u)\int vdx\}dx \qquad \textit{(Proved)}$$

Example 75: $\int x \log x dx$

$$= \int \log x \; x\, dx$$

$$= \log x\int x\,dx - \int \{\frac{d}{dx}(\log x)\int xdx\}\,dx \quad [\because \int uvdx = u\int vdx - \int \{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= \log x \cdot \frac{x^2}{2} - \int \{\frac{1}{x}\cdot\frac{x^2}{2}\}dx = \log x \cdot \frac{x^2}{2} - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\cdot\frac{x^2}{2} \qquad [\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\ ]$$

$$= \frac{x^2}{2}\log x - \frac{1}{4}x^2 \textit{ Answer}$$

<u>Method # 14</u>:

Prove that $\int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + c$

We have, $\int e^x f(x)dx$

$$= f(x)\int e^x dx - \int \{\frac{d}{dx}f(x)\int e^x dx\}\,dx \quad [\because \int uvdx = u\int vdx - \int \{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= f(x)\,e^x - \int f'(x)\,e^x dx$$

$$\therefore \int e^x f(x)dx = e^x f(x) - \int f'(x)\,e^x dx$$

$$\therefore e^x\ f(x) = \int e^x f(x)dx + \int f'(x)\,e^x dx$$

$$\therefore e^x\ f(x) = \int e^x\{f(x) + f'(x)\}dx + c$$

$$\therefore \int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + c \ \textit{(Proved)}$$

Example 76: $\int e^x(\sin x + \cos x)\,dx = e^x \sin x + c$

[Here, $f(x) = \sin x$ and $f'(x) = \cos x$]

Example 77: $\int e^x(\sin x-\cos x)\,dx=\int e^x(\sin x+(-\cos x))\,dx$

$$=\int e^x(-\cos x+\sin x)\,dx$$

$$=e^x(-\cos x)+c$$

[Here, $f(x)=-\cos x$ and $f'(x)=-(-\sin x)=\sin x$]

Example 78: $\int \dfrac{xe^x}{(x+1)^2}dx$

$$=\int \frac{(x+1)-1}{(x+1)^2}e^x dx$$

$$=\int\left\{\frac{x+1}{(x+1)^2}-\frac{1}{(x+1)^2}\right\}e^x dx$$

$$=\int\left\{\frac{1}{x+1}-\frac{1}{(x+1)^2}\right\}e^x dx$$

$$=e^x\,\frac{1}{x+1}+c$$

[Here,

$f(x)=\dfrac{1}{x+1}$

$f(x)=(x+1)^{-1}$

$\therefore f'(x)=-1(x+1)^{-1-1}\dfrac{d}{dx}(x+1)$

$[\because \dfrac{d}{dx}(x^n)=nx^{n-1}]$

$\therefore f'(x)=-1(x+1)^{-2}(1+0)$

$\therefore f'(x)=-\dfrac{1}{(x+1)^2}$]

Example 79: $\int e^x\,\dfrac{1-\sin x}{1-\cos x}dx$

$$=\int e^x\,\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}dx \qquad [\because \sin x=2\sin\frac{x}{2}\cos\frac{x}{2};\ \ 1-\cos x=2\sin^2\frac{x}{2}]$$

$$=\int\left\{\frac{1}{2\sin^2\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right\}e^x dx=\int\left\{\frac{1}{2\sin^2\frac{x}{2}}-\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right\}e^x dx$$

$$=\int\left\{\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}-\cot\frac{x}{2}\right\}e^x dx=\int e^x\left\{(-\cot\frac{x}{2})+\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right\}dx$$

$$=e^x(-\cot\frac{x}{2})+c \qquad [\because \int e^x\{f(x)+f'(x)\}\,dx=e^x f(x)+c]$$

$$=-e^x\cot\frac{x}{2}+c \quad \textit{Answer}$$

[Here, $f(x)=-\cot\dfrac{x}{2}$

$$f'(x) = -(-\operatorname{cosec}^2 \frac{x}{2}).\frac{d}{dx}(\frac{x}{2}) \qquad [\because \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x]$$

$$\Rightarrow f'(x) = \operatorname{cosec}^2 \frac{x}{2}(\frac{1}{2})$$

$$\Rightarrow f'(x) = \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2}$$