Chapter Four

The gamma and Beta function

Method # 20:

The gamma function:

The gamma function $\mathbf{\Gamma(n)} = \int_0^\infty \mathbf{x^{n-1}e^{-x}dx}$ -------------------------------- (i)

and is convergent for x > 0.

Example 121:

Prove that $\mathbf{\Gamma(1) = 1}$

Solution Put $\mathbf{n = 1}$ in (i)

$$\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx$$

$$\Rightarrow \Gamma(1) = \int_0^\infty x^0 e^{-x}dx$$

$$\Rightarrow \Gamma(1) = \int_0^\infty 1.e^{-x}dx \qquad [\because x^0 = 1]$$

$$\Rightarrow \Gamma(1) = \int_0^\infty e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = -\left[e^{-x}\right]_0^\infty \quad [\because \int e^{ax}dx = \frac{1}{a}e^{ax}]$$

$$\Rightarrow \Gamma(1) = -\left[e^{-\infty} - e^{-0}\right] = -\left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = -\left[\frac{1}{\infty} - \frac{1}{1}\right] \quad [\because e^\infty = \infty;\ e^0 = 1]$$

$$\Rightarrow \Gamma(1) = -\left[0 - 1\right] \qquad [\because \frac{1}{\infty} = 0]$$

$$\Rightarrow \Gamma(1) = 1 \ \therefore \Gamma(1) = 1 \ \textit{(Proved)}$$

Example 122: Γ (n+1) = n Γ (n)

Solution We have, $\mathbf{\Gamma(n)} = \int_0^\infty \mathbf{x^{n-1}e^{-x}dx}$ -----------------------------------(i)

Put $\mathbf{n = n + 1}$ in (i)

$$\Gamma(n+1) = \int_0^\infty x^{n+1-1}e^{-x}dx$$

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}dx \text{ ---(ii)}$$

Now, we find $\int \mathbf{x^n e^{-x}dx}$

We have, $\int \mathbf{uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx}$

$$\therefore \int x^n e^{-x}dx = x^n \int e^{-x}dx - \int\{\frac{d}{dx}(x^n)\int e^{-x}dx\}dx$$

$$\Rightarrow \int x^n e^{-x} dx = x^n \left[\frac{e^{-x}}{-1}\right] - \int \left\{ n x^{n-1} \left[\frac{e^{-x}}{-1}\right] \right\} dx$$

$$\Rightarrow \int x^n e^{-x} dx = -x^n e^{-x} + \int n x^{n-1} e^{-x} dx$$

$$\Rightarrow \int x^n e^{-x} dx = -x^n e^{-x} + n\int x^{n-1} e^{-x} dx \text{ -----------------------------(iii)}$$

Putting the value of $\int x^n e^{-x} dx$ in (ii)

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx$$

$$= \left[-x^n e^{-x}\right]_0^\infty + n\int_0^\infty x^{n-1} e^{-x} dx \quad \text{[From (iii)]}$$

$$= \left[-x^n e^{-x}\right]_0^\infty + n\,\Gamma(n) \qquad [\because \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx]$$

$$= \left[-\infty^n e^{-\infty} + 0^n e^{-0}\right] + n\,\Gamma(n) = \left[-\infty^n \frac{1}{e^\infty} + 0^n \frac{1}{e^0}\right] + n\,\Gamma(n)$$

$$= \left[-\infty^n \frac{1}{\infty} + 0^n \frac{1}{1}\right] + n\,\Gamma(n) = \left[-\infty^n .0 + 0.1\right] + n\,\Gamma(n)$$

$\therefore \Gamma(n+1) = [0.1] + n\,\Gamma(n) = n\,\Gamma(n) = n\,\Gamma(n)$ ***(Proved)***

Example 123: $\Gamma(n) = (n-1)!$

Solution We have,

$\Gamma(n+1) = n\,\Gamma(n)$ **---(i)**

Put, $n = n-1$ in (i)

$$\Gamma(n-1+1) = (n-1)\,\Gamma(n-1)$$
$$\Gamma(n) = (n-1)\,\Gamma(n-1) \text{ ----------------------(ii)}$$

Put, $n = n-2$ in (i)

$$\Gamma(n-2+1) = (n-2)\,\Gamma(n-2)$$
$$\Gamma(n-1) = (n-2)\,\Gamma(n-2) \text{ -----------------(iii)}$$

Put, $n = n-3$ in (i)

$$\Gamma(n-3+1) = (n-3)\,\Gamma(n-3)$$
$$\Gamma(n-2) = (n-3)\,\Gamma(n-3) \text{ -----------------(iv)}$$

Put, $n = 1$ in (i)

$$\Gamma 1 + 1 = 1\,\Gamma 1$$
$$\Gamma 2 = 1\,\Gamma 1 \text{ --------------------------------------(v)}$$

Putting all the values of $\Gamma(n)$, $\Gamma(n-1)$, $\Gamma(n-2)$, $\Gamma(n-3)$ .... $\Gamma 2$ **in (i)**

$$\Gamma(n+1) = n\,\Gamma(n)$$

$= n(n-1)\Gamma(n-1)$ $\quad$ [$\because$ from (ii) $\Gamma(n) = (n-1)\Gamma(n-1)$]

$= n(n-1)(n-2)\Gamma(n-2)$ $\quad$ [$\because$ from (iii) $\Gamma(n-1) = (n-2)\Gamma(n-2)$]

$= n(n-1)(n-2)(n-3)\Gamma(n-3)$ $\quad$ [$\because$ from (iv) $\Gamma(n-2) = (n-3)\Gamma(n-3)$]

....

....

$= n(n-1)(n-2)(n-3)\ (n-4)$------------------------ $2\Gamma 2$

$= n(n-1)(n-2)(n-3)\ (n-4)$------------------------$2.1\Gamma 1$ [$\because$ from (v) $\Gamma 2 = 1\,\Gamma 1$]

$= n(n-1)(n-2)(n-3)\ (n-4)$-------------------------$2.1.1$ $\quad$ [$\because \Gamma(1) = 1$]

$= n(n-1)(n-2)(n-3)\ (n-4)$---------------------------2.1

$= n!$ $\quad$ [$\because n! = n(n-1)(n-2)(n-3)\ (n-4)$ ----------- 2.1]

$\therefore \Gamma(n+1) = n!$--(vi)

Put, $n = n-1$ in (vi),

$\Gamma(n+1) = n!$

$\therefore \Gamma(n-1+1) = (n-1)!$

$\therefore \Gamma(n) = (n-1)!$ --(vii) (***Proved***)

As for example

Find the values of $\Gamma(5)$, $\Gamma(6)$, $\Gamma(7)$, $\Gamma(8)$

Putting the value of $n = 5, 6, 7, 8$.................. in (vii)

$\therefore \Gamma(5) = (5-1)! = 4! = 4\times 3\times 2\times 1 = 24$

$\therefore \Gamma(6) = (6-1)! = 5! = 5\times 4\times 3\times 2\times 1 = 120$

$\therefore \Gamma(7) = (7-1)! = 6! = 6\times 5\times 4\times 3\times 2\times 1 = 720$

$\therefore \Gamma(8) = (8-1)! = 7! = 7\times 6\times 5\times 4\times 3\times 2\times 1 = 5040$ **Answer**

Example 124:

Draw Graph of y $= \Gamma(x)$

Values of $\Gamma(x)$ for a range of positive values of x are available in tabulated form in various sets of mathematical tables. These, together with the results established above, enable us to draw the graph of y $= \Gamma(x)$.

X	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$\Gamma(x)$	α	1.772	1.000	0.886	1.000	1.329	2.000	3.323	6.000

X	-0.5	-1.5	-2.5	-3.5
$\Gamma(x)$	-3.545	2.363	-0.945	0.270

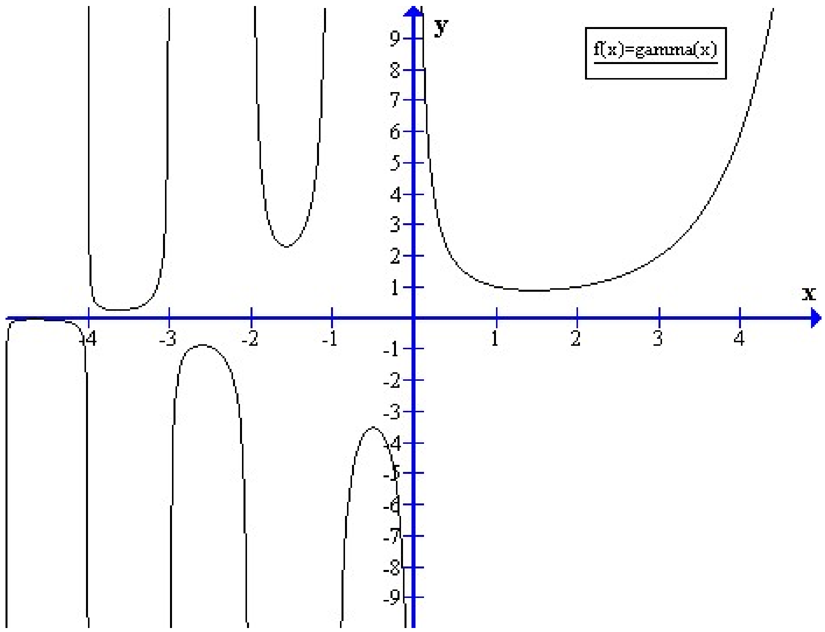

Figure # 29

Example 125: $\int_0^\infty x^7 e^{-x} dx$

Solution: We can write the above function

$$\int_0^\infty x^7 e^{-x} dx = \int_0^\infty x^{8-1} e^{-x} dx \text{ ------------------------------(i)}$$

We have,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

$$\therefore \Gamma(8) = \int_0^\infty x^{8-1} e^{-x} dx \text{ -------------------------------------(ii)}$$

From (i)

$$\therefore \int_0^\infty x^7 e^{-x} dx = \int_0^\infty x^{8-1} e^{-x} dx = \Gamma(8) = (8-1)! = 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

$$[\because \Gamma(n) = (n-1)!]$$

Example 126: $\int_0^\infty x^3 e^{-4x} dx$

Solution: Let, $y = 4x$

$\therefore dy = 4dx$

$$\Rightarrow dx = \frac{dy}{4}$$

x	0	∞
$y = 4x$	$y = 4x$ $y = 4.0 = 0$	$y = 4x$ $y = 4.\infty = \infty$

$$\therefore \int_0^\infty x^3 e^{-4x} dx$$

$$= \int_0^\infty (\frac{y}{4})^3 e^{-y} \frac{dy}{4} = \int_0^\infty (\frac{y}{4})^3 e^{-y} \frac{dy}{4} = \frac{1}{4}\int_0^\infty (\frac{y}{4})^3 e^{-y} dy$$

$$= \frac{1}{4}\int_0^\infty \frac{y^3}{4^3} e^{-y} dy = \frac{1}{4}\int_0^\infty \frac{y^3}{64} e^{-y} dy = \frac{1}{4} \times \frac{1}{64}\int_0^\infty y^3 e^{-y} dy$$

$$= \frac{1}{256}\int_0^\infty y^3 e^{-y} dy = \frac{1}{256}\int_0^\infty y^{4-1} e^{-y} dy$$

$$= \frac{1}{256} \times \Gamma(4) \quad [\because \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \; ; \; \therefore \Gamma(4) = \int_0^\infty x^{4-1} e^{-x} dx]$$

$$= \frac{3!}{256} = \frac{6}{256} = \frac{3}{128} \; \textit{Answer}$$

Example 127: $\int_0^\infty x^{\frac{1}{2}} e^{-x^2} dx$

Solution: Let, $y = x^2$

$$\Rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow dy = 2xdx$$

$$\Rightarrow dx = \frac{dy}{2x}$$

x	0	∞
$y = x^2$	$y = x^2$ $y = 0^2 = 0$	$y = x^2$ $y = \infty^2 = \infty$

Again

$y = x^2$

$$\therefore x = \sqrt{y}$$

$$\therefore x = y^{1/2}$$

$$\therefore x^{1/2} = (y^{\frac{1}{2}})^{1/2} = y^{1/4}$$

$$\int_0^\infty x^{\frac{1}{2}} e^{-x^2} dx$$

$$= \int_0^\infty y^{\frac{1}{4}} e^{-y} \frac{dy}{2x} = \int_0^\infty y^{\frac{1}{4}} e^{-y} \frac{dy}{2y^{1/2}} = \frac{1}{2}\int_0^\infty y^{\frac{1}{4}} e^{-y} \frac{dy}{y^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty y^{\frac{1}{4}} e^{-y} y^{-1/2} dy \quad = \frac{1}{2}\int_0^\infty y^{\frac{1}{4}-\frac{1}{2}} e^{-y} dy = \frac{1}{2}\int_0^\infty y^{-\frac{1}{4}} e^{-y} dy$$

$$= \frac{1}{2}\int_0^\infty y^{\frac{3}{4}-1} e^{-y} dy \; = \frac{1}{2}\Gamma(\frac{3}{4}) \; [\because \Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \; ; \; \therefore \Gamma(\frac{3}{4}) = \int_0^\infty x^{\frac{3}{4}-1} e^{-x} dx]$$

$$= \frac{1}{2} \times 1.2254 \ [\because \Gamma(\frac{3}{4}) = 1.2254] = 0.613$$ Answer

Example 128:

$$\Gamma\left(\frac{3}{2}\right) = ?, \ \Gamma\left(\frac{5}{2}\right) = ?, \Gamma\left(\frac{7}{2}\right) = ?$$

From this, using the recurrence relation $\Gamma(n+1) = n\Gamma n$ we can obtain the following:
We have,

$$\Gamma(n+1) = n\Gamma n \text{ --(i)}$$

Putting $n = \frac{1}{2}$ in (i),

$$\Rightarrow \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma\frac{1}{2} \text{ ----------------------------------(ii)}$$

$$\therefore \Gamma\left(\frac{3}{2}\right) = \Gamma(\frac{1}{2}+1) \quad [\because \frac{3}{2} = \frac{1}{2}+1]$$

$$= \frac{1}{2}\Gamma\frac{1}{2} \quad [\because \text{from (ii)}, \Gamma(\frac{1}{2}+1) = \frac{1}{2}\Gamma\frac{1}{2}]$$

$$= \frac{1}{2}(\sqrt{\pi}) \quad [\because \text{from example 138}, \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}]$$

$$\therefore \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\left(\sqrt{\pi}\right) \text{ --(iii)}$$

Again

$$\Gamma(n+1) = n\Gamma n$$

Putting $n = \frac{3}{2}$

$$\Gamma(n+1) = n\Gamma n$$

$$\Rightarrow \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\frac{3}{2} \text{ --(iv)}$$

$$\therefore \Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) \quad [\because \frac{5}{2} = \frac{3}{2}+1]$$

$$= \frac{3}{2}\Gamma\left(\frac{3}{2}\right) \quad [\because \text{from (iv)}, \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\frac{3}{2}]$$

$$= \frac{3}{2}.\frac{1}{2}\left(\sqrt{\pi}\right) \quad [\because \text{from (iii)}, \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\left(\sqrt{\pi}\right)]$$

$$\therefore \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right) = \frac{3\sqrt{\pi}}{4} \text{ ----------------------------------(v)}$$

Again,

$$\Gamma(n+1) = n\Gamma n$$

Putting $\mathbf{n} = \frac{5}{2}$

$$\Gamma(\mathbf{n}+1) = \mathbf{n}\Gamma\mathbf{n}$$

$$\Rightarrow \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) \text{------------------------------------}(vi)$$

For $\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) \qquad [\because \frac{7}{2} = \frac{5}{2}+1]$

$$= \frac{5}{2}\Gamma\left(\frac{5}{2}\right) \qquad [\because \textbf{from (vi)}, \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)]$$

$$= \frac{5}{2}\left(\frac{3\sqrt{\pi}}{4}\right) \qquad [\because \textbf{from (v)}, \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4}]$$

$$= \frac{15\sqrt{\pi}}{8}$$

Example 129:

$\Gamma\left(-\frac{3}{2}\right) = ?\ \Gamma\left(-\frac{1}{2}\right) = ?$

Using the recurrence relation in reverse

$$\Gamma(\mathbf{n}+1) = \mathbf{n}\Gamma\mathbf{n}$$

$$\Gamma\mathbf{n} = \frac{\Gamma(\mathbf{n}+1)}{\mathbf{n}} \text{--------------------------------------}(i)$$

We can also obtain

Putting $\mathbf{n} = -\frac{3}{2}$ in (1),

$$\Gamma\mathbf{n} = \frac{\Gamma(\mathbf{n}+1)}{\mathbf{n}}$$

$$\Rightarrow \Gamma(-\frac{3}{2}) = \frac{\Gamma(-\frac{3}{2}+1)}{-\frac{3}{2}}$$

$$\Rightarrow \Gamma(-\frac{3}{2}) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} \text{-------------------------------}(ii)$$

Again, Putting $\mathbf{n} = -\frac{1}{2}$ in (i),

$$\Gamma\mathbf{n} = \frac{\Gamma(\mathbf{n}+1)}{\mathbf{n}}$$

$$\Rightarrow \Gamma(-\frac{1}{2}) = \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}}$$

$$\Rightarrow \Gamma(-\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}}$$

$$\Rightarrow \Gamma(-\frac{1}{2}) = \frac{\sqrt{\pi}}{-\frac{1}{2}} \ [\because \Gamma\left(\frac{1}{2}\right) = \left(\sqrt{\pi}\right)]$$

$$\Rightarrow \Gamma(-\frac{1}{2}) = -2\sqrt{\pi} \text{ -------------------(iii)}$$

Putting the value of $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ in (ii),

$$\Rightarrow \Gamma(-\frac{3}{2}) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}}$$

$$\Rightarrow \Gamma(-\frac{3}{2}) = \frac{-2\sqrt{\pi}}{-\frac{3}{2}}$$

$$\Rightarrow \Gamma(-\frac{3}{2}) = -\frac{2}{3}(-2\sqrt{\pi})$$

$$\Rightarrow \Gamma(-\frac{3}{2}) = \frac{4}{3}(\sqrt{\pi})$$

Example 130:

$\Gamma(0) = ?\ \Gamma(-1) = ?$

Answer: We have,

$$\Gamma n = \frac{\Gamma(n+1)}{n} \text{ ------------------------------------(i)}$$

Putting $n = 0$ in (1),

$$\Gamma n = \frac{\Gamma(n+1)}{n}$$

$$\Rightarrow \Gamma 0 = \frac{\Gamma(0+1)}{0}$$

$$\Rightarrow \Gamma 0 = \frac{\Gamma(1)}{0}$$

$$\Rightarrow \Gamma 0 = \frac{1}{0} \ [\because \Gamma 1 = 1]$$

$$\Rightarrow \Gamma 0 = \infty$$

Putting $\mathbf{n=-1}$ in (i),

$$\Gamma \mathbf{n}=\frac{\Gamma(\mathbf{n}+1)}{\mathbf{n}}$$

$$\Gamma-1=\frac{\Gamma(-1+1)}{-1}$$

$$\Rightarrow \Gamma-1=\frac{\Gamma(0)}{-1}$$

$$\Rightarrow \Gamma-1=\frac{\infty}{-1}$$

$$\Rightarrow \Gamma-1=\infty$$

Similarly

$$\Rightarrow \Gamma-2=\infty$$
$$\Rightarrow \Gamma-3=\infty$$
$$\Rightarrow \Gamma-4=\infty$$

<u>Method # 21:</u>

The Beta function

The Beta function $\beta(\mathbf{m},\mathbf{n})$ is defined by

$$\beta(\mathbf{m},\mathbf{n})=\int_0^1 \mathbf{x}^{\mathbf{m}-1}(1-\mathbf{x})^{\mathbf{n}-1}\,\mathbf{dx} \text{ ------------------(i)}$$

Which, converges for m>0 and n>0.

Example 131:

Prove that $\beta(\mathbf{m},\mathbf{n})=\beta(\mathbf{n},\mathbf{m})$

Solution: We have,

$$\beta(\mathbf{m},\mathbf{n})=\int_0^1 \mathbf{x}^{\mathbf{m}-1}(1-\mathbf{x})^{\mathbf{n}-1}\,\mathbf{dx} \text{ ------------------------------------(i)}$$

Let, $(1-\mathbf{x})=\mathbf{u}$

$$\Rightarrow \mathbf{x}=1-\mathbf{u}$$

$$\Rightarrow \frac{\mathbf{dx}}{\mathbf{du}}=-1$$

$$\therefore \mathbf{dx}=-\mathbf{du}$$

x	0	1
$u=1-x$	$u=1-x$ $u=1-0=1$	$u=1-x$ $u=1-1=0$

From (i),

$$\beta(\mathbf{m},\mathbf{n})=\int_0^1 \mathbf{x}^{\mathbf{m}-1}(1-\mathbf{x})^{\mathbf{n}-1}\,\mathbf{dx}$$

$$\Rightarrow \beta(\mathbf{m},\mathbf{n})=\int_1^0 (1-\mathbf{u})^{\mathbf{m}-1}(\mathbf{u})^{\mathbf{n}-1}(-\mathbf{du})$$

$$\Rightarrow \beta(\mathbf{m},\mathbf{n})=-\int_1^0 (1-\mathbf{u})^{\mathbf{m}-1}(\mathbf{u})^{\mathbf{n}-1}\,\mathbf{du}$$

$$\Rightarrow \beta(m,n) = \int_0^1 (1-u)^{m-1}(u)^{n-1}du \qquad [\because \int_a^b f(x)dx = -\int_b^a f(x)dx]$$

$$\Rightarrow \beta(m,n) = \beta(n,m) \ \textit{(Proved)}$$

Example 132:

Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$

Solution: We **have,** $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ ----------------------------------(i)

Put $x = \sin^2\theta$, in (i)

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}(\sin^2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 2\sin\theta\frac{d}{d\theta}(\sin\theta) \qquad [\frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\Rightarrow dx = 2\sin\theta\cos\theta d\theta$$

We have,

$\sin^2\theta = x$

$$\Rightarrow \sin\theta = \sqrt{x}$$

$$\Rightarrow \theta = \sin^{-1}\sqrt{x}$$

x	0	1
$x = \sin^2\theta$ $\sin\theta = \sqrt{x}$ $\theta = \sin^{-1}\sqrt{x}$	$\theta = \sin^{-1}\sqrt{x}$ $\theta = \sin^{-1}\sqrt{0}$ $\theta = \sin^{-1}0$ $\theta = \sin^{-1}\sin 0$ $\theta = 0$	$\theta = \sin^{-1}\sqrt{1}$ $\theta = \sin^{-1}1$ $= \sin^{-1}\sin\frac{\pi}{2}$ $= \frac{\pi}{2}$

From (i)

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$$\Rightarrow \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} \times 2\sin\theta\cos\theta d\theta$$

$$\Rightarrow \beta(m,n) = \int_0^{\pi/2} (\sin\theta)^{2m-2}(\cos^2\theta)^{n-1} \times 2\sin\theta\cos\theta d\theta$$

$$\Rightarrow \beta(m,n) = \int_0^{\pi/2} (\sin\theta)^{2m-2}(\cos\theta)^{2n-2} \times 2\sin\theta\cos\theta d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-2}\sin\theta.(\cos\theta)^{2n-2}\cos\theta d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-2+1}(\cos\theta)^{2n-2+1} d\theta$$

$$\Rightarrow \beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta \quad \textit{(Proved)}$$

Example 133:

Prove that, $\beta(m,n) = \dfrac{\Gamma m \Gamma n}{\Gamma(m+n)} = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

Solution: We have,

$$I_n = \int_0^{\pi/2} \sin^n x\,dx = [-\frac{1}{n}\sin^{n-1} x\cos x]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = [-\frac{1}{n}\sin^{n-1}(\frac{\pi}{2})\cos\frac{\pi}{2} + \frac{1}{n}\sin^{n-1}(0)\cos 0] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = [-\frac{1}{n}.1.0 + 0.1] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n} I_{n-2} \text{------------------(i)}$$

Again, We have,

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \left[\frac{1}{n}\cos^{n-1} x\sin x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \left[\frac{1}{n}\cos^{n-1}(\frac{\pi}{2})\sin(\frac{\pi}{2}) - \frac{1}{n}\cos^{n-1} 0\sin 0\right] + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \left[\frac{1}{n}.0.1 - \frac{1}{n}.1.0\right] + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx = \frac{n-1}{n} I_{n-2} \text{---------------------(ii)}$$

A third reduction formula for products of sines and cosines is

$$I_{m,n} = \int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}(x)\cos^n(x)dx \text{------------------(iii)}$$

$$\Rightarrow I_{m,n} = \int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \frac{m-1}{m+n} I_{m-2,n}$$

$$\Rightarrow I_{m,n} = \frac{m-1}{m+n} I_{m-2,n} \text{-- (iv)}$$

Alternatively,

$$\int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m(x)\cos^{n-2}(x)dx \text{---------------------------(v)}$$

$$\Rightarrow I_{m,n} = \frac{n-1}{m+n} I_{m,n-2} \text{--(vi)}$$

Again, we have

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1} d\theta$$

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} (\sin x)^{2m-1}(\cos x)^{2n-1} dx$$

According to (iii), we can write

$$\int_0^{\pi/2} (\sin x)^{2m-1}(\cos x)^{2n-1} dx = \frac{(2m-1)-1}{(2m-1)+(2n-1)}\int_0^{\pi/2} \sin^{2m-1-2}(x)\cos^{2n-1}(x)dx$$

$$[\because I_{m,n} = \int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}(x)\cos^n(x)dx]$$

$$\Rightarrow \int_0^{\pi/2} (\sin x)^{2m-1}(\cos x)^{2n-1} dx = \frac{2m-2}{2m+2n-2}\int_0^{\pi/2} \sin^{2m-3}(x)\cos^{2n-1}(x)dx$$

$$\Rightarrow \int_0^{\pi/2} (\sin x)^{2m-1}(\cos x)^{2n-1} dx = \frac{2(m-1)}{2(m+n-1)}\int_0^{\pi/2} \sin^{2m-3}(x)\cos^{2n-1}(x)dx$$

$$\Rightarrow \int_0^{\pi/2} (\sin x)^{2m-1} (\cos x)^{2n-1}\, dx = \frac{(m-1)}{(m+n-1)} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\, dx = \frac{(m-1)}{(m+n-1)} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx \text{ --------------(vii)}$$

Again from (v), we have

$$\int_0^{\pi/2} \sin^m(x) \cos^n(x) dx = \frac{n-1}{m+n} \int_0^{\pi/2} \sin^m(x) \cos^{n-2}(x) dx$$

Hence we can write

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx = \frac{(2n-1)-1}{2m-3+2n-1} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1-2}(x) dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx = \frac{2n-2}{2m+2n-4} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-3}(x) dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx = \frac{2(n-1)}{2(m+n-2)} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-3}(x) dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx = \frac{(n-1)}{(m+n-2)} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-3}(x) dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx = \frac{n-1}{m+n-2} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-3}(x) dx \text{ -------------------(viii)}$$

Putting the value of $\int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx$ in (vii)

From (vii),

$$\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\, dx = \frac{(m-1)}{(m+n-1)} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-1}(x) dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\, dx = \frac{(m-1)}{(m+n-1)} \cdot \frac{n-1}{m+n-2} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-3}(x) dx \text{ [From viii]}$$

$$\Rightarrow \int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\, dx = \frac{m-1}{m+n-1} \cdot \frac{n-1}{m+n-2} \int_0^{\pi/2} \sin^{2m-3}(x) \cos^{2n-3}(x) dx$$

$$\therefore 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\,dx = \frac{m-1}{m+n-1}\cdot\frac{n-1}{m+n-2}.2\int_0^{\pi/2} \sin^{2m-3}(x)\cos^{2n-3}(x)dx$$

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\,dx = \frac{m-1}{m+n-1}\cdot\frac{n-1}{m+n-2}.2\int_0^{\pi/2} \sin^{2m-3}(x)\cos^{2n-3}(x)dx$$

--------------------------(ix)

$$\because \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\,dx$$

$$\therefore \beta(m-1,n-1) = 2\int_0^{\pi/2} \sin^{2(m-1)-1} x \cos^{2(n-1)-1} x\,dx$$

$$\therefore \beta(m-1,n-1) = 2\int_0^{\pi/2} \sin^{2m-2-1} x \cos^{2n-2-1} x\,dx$$

$$\therefore \beta(m-1,n-1) = 2\int_0^{\pi/2} \sin^{2m-3} x \cos^{2n-3} x\,dx \text{ --------------------------(x)}$$

$$\therefore 2\int_0^{\pi/2} \sin^{2m-3} x \cos^{2n-3} x\,dx = \beta(m-1,n-1) \text{ --------------------------(xi)}$$

Putting the value of $2\int_0^{\pi/2} \sin^{2m-3} x \cos^{2n-3} x\,dx = \beta(m-1,n-1)$ in (ix)

From (ix),

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\,dx = \frac{m-1}{m+n-1}\cdot\frac{n-1}{m+n-2}.2\int_0^{\pi/2} \sin^{2m-3}(x)\cos^{2n-3}(x)dx$$

$$\therefore \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\,dx = \frac{m-1}{m+n-1}\cdot\frac{n-1}{m+n-2}.\beta(m-1,n-1) \text{ -----------(xii)}$$

This is obviously a reduction formula for $\beta(m,n)$ and the process can be repeated as required.
For Example

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x\,dx = \frac{m-1}{m+n-1}\cdot\frac{n-1}{m+n-2}.\beta(m-1,n-1)$$

$$\Rightarrow \beta(m,n) = \frac{m-1}{m+n-1}\cdot\frac{n-1}{m+n-2}.\beta(m-1,n-1)$$

$$\therefore \beta(4,3) = \frac{4-1}{4+3-1}\cdot\frac{3-1}{4+3-2}.\beta(4-1,3-1)$$

$\therefore \beta(4,3)=\frac{3}{6}.\frac{2}{5}.\beta(3,2)$ --(xiii)

Similarly,

$$\beta(m,n)=\frac{m-1}{m+n-1}.\frac{n-1}{m+n-2}.\beta(m-1,n-1)$$

$$\Rightarrow \beta(3,2)=\frac{3-1}{3+2-1}.\frac{2-1}{3+2-2}.\beta(3-1,2-1)$$

$\Rightarrow \beta(3,2)=\frac{2}{4}.\frac{1}{3}.\beta(2,1)$ --(xiv)

We have,

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}x\cos^{2n-1}x\,dx$$

$$\Rightarrow \beta(2,1)=2\int_0^{\pi/2}\sin^{2.2-1}x\cos^{2.1-1}x\,dx$$

$\Rightarrow \beta(2,1)=2\int_0^{\pi/2}\sin^3 x\cos^1 x\,dx$ --(xv)

Let $\sin x = z$

$\Rightarrow z=\sin x$

$\Rightarrow \frac{dz}{dx}=\cos x$

$\Rightarrow dz=\cos x\,dx$

x	$\frac{\pi}{2}$	0
$z=\sin x$	$z=\sin x=\sin\frac{\pi}{2}=1$	$z=\sin x=\sin 0=0$

From (xv), we get

$$\beta(2,1)=2\int_0^{\pi/2}\sin^3 x\cos^1 x\,dx$$

$$\Rightarrow \beta(2,1)=2\int_0^1 z^3\,dz=2\left[\frac{z^{3+1}}{3+1}\right]_0^1=2\left[\frac{z^4}{4}\right]_0^1$$

$$\Rightarrow \beta(2,1)=2\left[\frac{1^4}{4}-\frac{0}{4}\right]=2\left[\frac{1}{4}-0\right]=2\left[\frac{1}{4}\right]$$

$\Rightarrow \beta(2,1)=\left[\frac{1}{2}\right]=\frac{1}{2}$ ---(xvi)

From (xiii),

$$\therefore \beta(4,3)=\frac{3}{6}.\frac{2}{5}.\beta(3,2)$$

$$\therefore \beta(4,3)=\frac{3}{6}.\frac{2}{5}.\beta(3,2)$$

$$\therefore \beta(4,3)=\frac{3}{6}.\frac{2}{5}.\frac{2}{4}.\frac{1}{3}.\beta(2,1) \qquad \text{[From xiv, } \beta(3,2)=\frac{2}{4}.\frac{1}{3}.\beta(2,1)]$$

$$\therefore \beta(4,3)=\frac{3}{6}.\frac{2}{5}.\frac{2}{4}.\frac{1}{3}.\frac{1}{2} \qquad \text{[From xvi, } \beta(2,1)=\frac{1}{2}]$$

$$\therefore \beta(4,3)=\frac{3}{6}.\frac{2}{5}.\frac{2}{4}.\frac{1}{3}.\frac{1}{2}=\frac{3\times 2}{6\times 5}.\frac{2\times 1}{4\times 3}.\beta(2,1)=\frac{(3\times 2)}{(6\times 5)}\frac{(2\times 1)}{(4\times 3)}\times\frac{1}{2}=\frac{(3!)(2!)}{6!}$$

$$\therefore \beta(4,3)=\frac{3}{6}.\frac{2}{5}.\frac{2}{4}.\frac{1}{3}.\frac{1}{2}=\frac{(3!)(2!)}{6!} \text{-------------------------------------(xvii)}$$

$$\therefore \beta(4,3)=\frac{3}{6}.\frac{2}{5}.\frac{2}{4}.\frac{1}{3}.\frac{1}{2}=\frac{(3!)(2!)}{6!}=\frac{(4-1)!)(3-1)!}{(4+3-1)!} \text{--------------(xviii)}$$

Since, $\beta(4,3)=\frac{3}{6}.\frac{2}{5}.\frac{2}{4}.\frac{1}{3}.\frac{1}{2}=\frac{3\times 2}{6\times 5}.\frac{2\times 1}{4\times 3}.\beta(2,1)$

Similarly,

$$\therefore \beta(5,3)=\frac{(4)(2)}{(7)(6)}\frac{(3)(1)}{(5)(4)}\beta(3,1) \text{---(xix)}$$

Now $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}x\cos^{2n-1}x\,dx$

$$\Rightarrow \beta(3,1)=2\int_0^{\pi/2}\sin^{2.3-1}x\cos^{2.1-1}x\,dx$$

$$\Rightarrow \beta(3,1)=2\int_0^{\pi/2}\sin^{5}x\cos^{1}x\,dx$$

Let $\sin x = z$

$\Rightarrow z=\sin x$

$\Rightarrow \frac{dz}{dx}=\cos x$

$\Rightarrow dz=\cos x\,dx$

x	$\frac{\pi}{2}$	0
$z=\sin x$	$z=\sin x=\sin\frac{\pi}{2}=1$	$z=\sin x=\sin 0=0$

$$\beta(3,1)=2\int_0^{\pi/2}\sin^{5}x\cos^{1}x\,dx$$

$$\Rightarrow \beta(3,1)=2\int_0^{1}z^5\,dz$$

$$\Rightarrow \beta(3,1)=2\left[\frac{z^{5+1}}{5+1}\right]_0^1=2\left[\frac{z^6}{6}\right]_0^1=2\left[\frac{1^4}{6}-\frac{0}{4}\right]=2\left[\frac{1}{6}-0\right]$$

$$\Rightarrow \beta(3,1)=2\left[\frac{1}{6}\right]=\left[\frac{1}{3}\right]=\frac{1}{3}$$

From (xix),

$$\therefore \beta(5,3) = \frac{(4)(2)}{(7)(6)}\frac{(3)(1)}{(5)(4)}\beta(3,1)$$

$$\therefore \beta(5,3) = \frac{(4)(2)}{(7)(6)}\frac{(3)(1)}{(5)(4)}\frac{1}{3}$$

$$\therefore \beta(5,3) = \frac{(4)(2)}{(7)(6)}\frac{(3)(1)}{(5)(4)}\frac{1}{3}.\frac{2}{2}$$

$$\therefore \beta(5,3) = \frac{(4)(3)}{(7)(6)}\frac{(2)(1)}{(5)(4)}\frac{2}{3}.\frac{1}{2}$$

$$\therefore \beta(5,3) = \frac{(4)(3)(2)(1)}{(7)(6)(5)(4)(3)(2)}.\frac{(2)(1)}{1}$$

$$\therefore \beta(5,3) = \frac{(4!)(2!)}{(7!)}$$

$$\therefore \beta(5,3) = \frac{(4!)(2!)}{(7!)} = \frac{(5-1)!)(3-1)!}{(5+3-1)!}$$

In general, $\beta(m,n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$ (***Proved***)

$$\beta(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} [\because \Gamma n = (n-1)!] \text{------------------(xx)}$$

Another Way:

Prove that $\beta(m,n) = \dfrac{\Gamma m \Gamma n}{\Gamma(m+n)} = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

We have

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \text{ --(i)}$$

Let,

$x = zy$ [Where z is not a function of x]

$$\frac{dx}{dy} = z.1$$

$$dx = zdy$$

From (i), We get

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$$

$$\Gamma(n) = \int_0^\infty (zy)^{n-1} e^{-zy} zdy$$

$$\Gamma\text{ (n)} = \int_0^\infty z^{n-1} y^{n-1} e^{-zy} z dy$$

$$\Gamma\text{ (n)} = \int_0^\infty z^{n} . z^{-1} . y^{n-1} e^{-zy} z dy$$

$$\Gamma\text{ (n)} = \int_0^\infty z^{n} . \frac{1}{z} . y^{n-1} e^{-zy} z dy$$

$$\Gamma\text{ (n)} = \int_0^\infty z^{n} y^{n-1} e^{-zy} dy$$

$$\Gamma\text{ (n)} = z^{n} \int_0^\infty y^{n-1} e^{-zy} dy$$

$$\frac{1}{z^n} \Gamma n = \int_0^\infty y^{n-1} e^{-zy} dy \qquad \text{--------------------------------------(ii)}$$

Multiplying (ii) on both sides by $z^{m-1} e^{-z}$

$$\frac{1}{z^n} \Gamma n = \int_0^\infty y^{n-1} e^{-zy} dy$$

$$z^{m-1} e^{-z} \frac{1}{z^n} \Gamma n = z^{m-1} e^{-z} \int_0^\infty y^{n-1} e^{-zy} dy \text{ --------------------------------------(iii)}$$

Integrating both sides with respect to z from $\mathbf{0}$ to ∞

$$z^{m-1} e^{-z} \frac{1}{z^n} \Gamma n = z^{m-1} e^{-z} \int_0^\infty y^{n-1} e^{-zy} dy$$

$$\int_0^\infty z^{m-1} e^{-z} \frac{1}{z^n} \Gamma n dz = \int_0^\infty [z^{m-1} e^{-z} \int_0^\infty y^{n-1} e^{-zy} dy] dz$$

$$\int_0^\infty z^{m-1} e^{-z} \Gamma n dz = \int_0^\infty [z^{m-1} . z^{n} e^{-z} \int_0^\infty y^{n-1} e^{-zy} dy] dz$$

$$\int_0^\infty z^{m-1} e^{-z} \Gamma n dz = \int_0^\infty [\int_0^\infty z^{m-1} . z^{n} e^{-z} y^{n-1} e^{-zy} dy] dz$$

$$\int_0^\infty z^{m-1} e^{-z} \Gamma n dz = \int_0^\infty [\int_0^\infty z^{m+n-1} e^{-z} e^{-zy} dz] y^{n-1} dy$$

$$\int_0^\infty z^{m-1} e^{-z} \Gamma n dz = \int_0^\infty [\int_0^\infty z^{m+n-1} e^{-z-zy} dz] y^{n-1} dy$$

$$\int_0^\infty z^{m-1} e^{-z} \Gamma n dz = \int_0^\infty [\int_0^\infty z^{m+n-1} e^{-(1+y)z} dz] y^{n-1} dy$$

$$\int_0^\infty z^{m-1} e^{-z} \Gamma n dz = \int_0^\infty \frac{\Gamma(m+n)}{(1+y)^{m+n}} y^{n-1} dy \qquad [\frac{\Gamma(m+n)}{(1+y)^{m+n}} = \int_0^\infty z^{m+n-1} e^{-(1+y)z} dz]$$

$$\int_0^\infty z^{m-1}e^{-z}\Gamma n dz = \Gamma(m+n)\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$$

$$\int_0^\infty z^{m-1}e^{-z}\Gamma n dz = \Gamma(m+n)\beta(m,n) \qquad [\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy = \beta(m,n)]$$

$$\beta(m,n) = \frac{1}{\Gamma(m+n)}\int_0^\infty z^{m-1}e^{-z}\Gamma n dz$$

$$\beta(m,n) = \frac{\Gamma n}{\Gamma(m+n)}\int_0^\infty z^{m-1}e^{-z}dz$$

$$\beta(m,n) = \frac{\Gamma n}{\Gamma(m+n)}\Gamma m \qquad [\Gamma m = \int_0^\infty z^{m-1}e^{-z}dz]$$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} \qquad \text{(Proved)}$$

Example 134: $\int_0^1 x^5(1-x)^4\,dx$

Solution: We have, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ ---------------------------(i)

Given, $\int_0^1 x^5(1-x)^4\,dx$

$= \int_0^1 x^{6-1}(1-x)^{5-1}dx$ --(ii)

Compare (i) and (ii),

Then $m-1 = 6-1$ and $n-1 = 5-1$

$\Rightarrow m = 6$ $\Rightarrow n = 5$

We have,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m,n) = \frac{\Gamma m\Gamma n}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}dx = \beta(6,5) = \frac{(6-1)!(5-1)!}{(6+5-1)!} = \frac{\Gamma 6\Gamma 5}{\Gamma(6+5)}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}dx = \beta(6,5) = \frac{(6-1)!(5-1)!}{(6+5-1)!}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}dx = \beta(6,5) = \frac{5!4!}{10!}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}dx = \beta(6,5) = \frac{(5\times4\times3\times2\times1)\times(4\times3\times2\times1)}{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}\,dx = \beta(6,5) = \frac{(4\times3\times2\times1)}{10\times9\times8\times7\times6}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}\,dx = \beta(6,5) = \frac{4}{10\times9\times8\times7}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}\,dx = \beta(6,5) = \frac{1}{10\times9\times2\times7}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}\,dx = \beta(6,5) = \frac{1}{90\times2\times7}$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{5-1}\,dx = \beta(6,5) = \frac{1}{180\times7}$$

$$\therefore \int_0^1 x^{6-1}(1-x)^{5-1}\,dx = \beta(6,5) = \frac{1}{1260}\ \textit{Answer}$$

Example 135: $\int_0^1 x^4\sqrt{1-x^2}\,dx$

Solution: We have, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ -------------------------(i)

Given, $\int_0^1 x^4\sqrt{1-x^2}\,dx$

$$= \int_0^1 x^4(1-x^2)^{1/2}\,dx$$

$$= \int_0^1 x^4(1-x^2)^{1/2}\,dx \text{ ---(ii)}$$

Let, $y = x^2$

$$\Rightarrow \frac{dy}{dx} = 2x$$

$$\Rightarrow dy = 2x\,dx$$

$$\Rightarrow dx = \frac{dy}{2x} \text{ ---(iii)}$$

Again

$y = x^2$

x	0	1
$y = x^2$	$y = x^2$ $y = 0^2 = 0$	$y = x^2$ $y = 1^2 = 1$

$$\therefore x = \sqrt{y}$$

$$\therefore x = y^{1/2}$$

$$\therefore x^{1/2} = (y^{\frac{1}{2}})^{1/2} = y^{1/4}$$

From (iii),

$$\mathbf{dx = \frac{dy}{2x}} \text{---(iv)}$$

$$\Rightarrow \mathbf{dx = \frac{dy}{2y^{1/2}}} \; [\because \mathbf{x = y^{1/2}}]$$

$$\Rightarrow \mathbf{dx = \frac{1}{2} y^{-1/2} dy}$$

$$\therefore \int_0^1 \mathbf{x^4 \sqrt{1-x^2}\, dx} = \int_0^1 \mathbf{x^4 (1-x^2)^{1/2}\, dx}$$

$$= \int_0^1 \mathbf{(x^2)^2 (1-x^2)^{1/2}\, dx} = \int_0^1 \mathbf{y^2 (1-y)^{1/2} \frac{1}{2} y^{-\frac{1}{2}} dy}$$

$$= \mathbf{\frac{1}{2}} \int_0^1 \mathbf{y^2 . y^{-1/2} (1-y)^{\frac{1}{2}} dy} = \mathbf{\frac{1}{2}} \int_0^1 \mathbf{y^{2-\frac{1}{2}} (1-y)^{\frac{1}{2}} dy}$$

$$= \mathbf{\frac{1}{2}} \int_0^1 \mathbf{y^{\frac{3}{2}} (1-y)^{\frac{1}{2}} dy} \text{--(v)}$$

Compare (i) and (v),

$$\mathbf{m-1 = \frac{3}{2}} \qquad \& \qquad \mathbf{n-1 = \frac{1}{2}}$$

$$\Rightarrow \mathbf{m = \frac{3}{2}+1} \qquad \mathbf{n = \frac{1}{2}+1}$$

$$\Rightarrow \mathbf{m = \frac{3+2}{2}} \qquad \mathbf{n = \frac{1+2}{2}}$$

$$\Rightarrow \mathbf{m = \frac{5}{2}} \qquad \mathbf{n = \frac{3}{2}}$$

We have,

$$\int_0^1 \mathbf{x^{m-1}(1-x)^{n-1} dx} = \beta(\boldsymbol{m,n}) = \frac{\Gamma \boldsymbol{m} \Gamma \boldsymbol{n}}{\Gamma(\boldsymbol{m+n})} = \frac{(\boldsymbol{m}-1)!(\boldsymbol{n}-1)!}{(\boldsymbol{m+n}-1)!}$$

From (v),

$$\therefore \int_0^1 \mathbf{x^4 \sqrt{1-x^2}\, dx} = \mathbf{\frac{1}{2}} \int_0^1 \mathbf{y^{\frac{3}{2}} (1-y)^{\frac{1}{2}} dy}$$

$$= \mathbf{\frac{1}{2}} \int_0^1 \mathbf{y^{\frac{5}{2}-1} (1-y)^{\frac{3}{2}-1} dy} = \mathbf{\frac{1}{2}} \beta(\mathbf{\frac{5}{2}, \frac{3}{2}}) \qquad [\because \int_0^1 \mathbf{x^{m-1}(1-x)^{n-1} dx = \beta(m,n)}]$$

$$= \mathbf{\frac{1}{2}} \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2}+\frac{3}{2})} \qquad [\because \beta\mathbf{(m,n)} = \frac{\Gamma \mathbf{m} \Gamma \mathbf{n}}{\Gamma\mathbf{(m+n)}}]$$

$$=\frac{1}{2}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{8}{2})}=\frac{1}{2}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}=\frac{1}{2}\frac{(\frac{3\sqrt{\pi}}{4})(\frac{\sqrt{\pi}}{2})}{(4-1)!}$$

$$=\frac{1}{2}\frac{(\frac{3\sqrt{\pi}}{4})(\frac{\sqrt{\pi}}{2})}{3!}=\frac{1}{2}\frac{(\frac{3\sqrt{\pi}}{4})(\frac{\sqrt{\pi}}{2})}{3\times 2}=\frac{1}{2}\frac{(\frac{3\sqrt{\pi}}{4})(\frac{\sqrt{\pi}}{2})}{6}$$

$$=\frac{1}{12}(\frac{3\sqrt{\pi}}{4})(\frac{\sqrt{\pi}}{2})=\frac{1}{12}(\frac{3\pi}{8})=\frac{1}{4}(\frac{\pi}{8})=\frac{\pi}{32}\ \textit{Answer}$$

Example 136: $\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta$ -------------------------------------(i)

The equation (i) can be written as

$$=\frac{1}{2}\times 2\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta \text{ -------------------------------------(ii)}$$

We have, $\beta(m,n)=2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$ ----------------(iii)

Compare (ii) and (iii), we get

$2m-1=5$ & $2n-1=4$

$\Rightarrow 2m=5+1$ $\Rightarrow 2n=4+1$

$\Rightarrow 2m=6$ $\Rightarrow 2n=5$

$\Rightarrow m=3$ $\Rightarrow n=\frac{5}{2}$

We have, $2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta=\beta(m,n)$

$$\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta=\frac{1}{2}\times 2\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta$$

$$=\frac{1}{2}\times 2\int_0^{\pi/2}\sin^{2.3-1}\theta\cos^{2.\frac{5}{2}-1}\theta\,d\theta$$

$$=\frac{1}{2}\times\beta(3,\frac{5}{2})\ [\because 2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta=\beta(m,n)]\text{-------(iv)}$$

Again,

$$\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}=\frac{(m-1)!(n-1)!}{(m+n-1)!}$$

From (iv),

$$\therefore\int_0^{\pi/2}\sin^5\theta\cos^4\theta\,d\theta=\frac{1}{2}\times\beta(3,\frac{5}{2})$$

$$=\frac{1}{2}\frac{(3-1)!(\frac{5}{2}-1)!}{(3+\frac{5}{2}-1)!}=\frac{1}{2}\frac{\Gamma 3\Gamma\frac{5}{2}}{\Gamma(3+\frac{5}{2})}=\frac{1}{2}\frac{(3-1)!(\frac{5}{2}-1)!}{(3+\frac{5}{2}-1)!}=\frac{1}{2}\frac{\Gamma 3\Gamma\frac{5}{2}}{\Gamma(\frac{11}{2})}$$

$$=\frac{1}{2}\frac{(3-1)!\,\Gamma\frac{5}{2}}{\Gamma(\frac{11}{2})}=\frac{1}{2}\frac{2!\,\Gamma\frac{5}{2}}{\Gamma(\frac{11}{2})}=\frac{1}{2}\frac{2\,\Gamma\frac{5}{2}}{\Gamma(\frac{11}{2})}=\frac{\Gamma\frac{5}{2}}{\Gamma(\frac{11}{2})}=\frac{\frac{3\sqrt{\pi}}{4}}{\frac{945\sqrt{\pi}}{32}}$$

$$=\frac{3\sqrt{\pi}}{4}\times\frac{32}{945\sqrt{\pi}}=\frac{3}{4}\times\frac{32}{945}=\frac{3}{1}\times\frac{8}{945}=\frac{8}{315}\quad \textit{Answer}$$

Example 137: $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

Solution:

$$I=\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$$

$$I=\int_0^{\pi/2}\tan^{1/2}\theta\,d\theta=\int_0^{\pi/2}\frac{\sin^{\frac{1}{2}}\theta}{\cos^{\frac{1}{2}}\theta}d\theta=\int_0^{\pi/2}\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\theta=\frac{1}{2}\times 2\int_0^{\pi/2}\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\theta$$

$$=\frac{1}{2}\times 2\int_0^{\pi/2}\sin^{2\times\frac{3}{4}-1}\theta\cos^{2\times\frac{1}{4}-1}\theta\,d\theta$$

$$=\frac{1}{2}\beta(\frac{3}{4},\frac{1}{4})\ [\because 2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta=\beta(m,n)\,]$$

Again,

$$\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}=\frac{(m-1)!(n-1)!}{(m+n-1)!}$$

$$\therefore I=\int_0^{\pi/2}\tan^{1/2}\theta\,d\theta=\frac{1}{2}\beta(\frac{3}{4},\frac{1}{4})$$

$$=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(\frac{3}{4}+\frac{1}{4})}=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}=\frac{1}{2}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{1}$$

$$=\frac{1}{2}\cdot\frac{(1.2254)(3.6256)}{1.0000}=2.2214\quad \textit{Answer}$$

Example 138:

Prove that $\Gamma\frac{1}{2} = \sqrt{\pi}$

Solution: We have,

$$\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \frac{(m-1)!(n-1)!}{(m+n-1)!} \text{------------------------------(i)}$$

Putting the value of $m = n = \frac{1}{2}$ in (i)

$$\therefore \ \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$$

$$\therefore \ \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma(1)}$$

$$\therefore \ \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{(\Gamma\frac{1}{2})^2}{\Gamma(1)}$$

$$\therefore \ \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{(\Gamma\frac{1}{2})^2}{1} [\because \Gamma 1 = 1]$$

$$\therefore \ \beta\left(\frac{1}{2}, \frac{1}{2}\right) = (\Gamma\frac{1}{2})^2$$

$$\therefore \ (\Gamma\frac{1}{2})^2 = \beta\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\therefore \ (\Gamma\frac{1}{2}) = \sqrt{\beta\left(\frac{1}{2}, \frac{1}{2}\right)} \text{-------------------------------------(ii)}$$

Again, we have,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \text{ -----------------------------(iii)}$$

From (iii)

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Putting the value of $m = n = \frac{1}{2}$ in (iii)

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\Rightarrow \beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}\,dx$$

$$\Rightarrow \beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^1 x^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}}\,dx$$

$$\therefore \beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^1 \frac{1}{x^{1/2}(1-x)^{1/2}}\,dx \text{ ----------------------------(iv)}$$

Put $x = \sin^2\theta$, in (iv)

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}(\sin^2\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 2\sin^{2-1}\theta\frac{d}{d\theta}(\sin\theta)\ [\because \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\Rightarrow \frac{dx}{d\theta} = 2\sin\theta\frac{d}{d\theta}(\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$$

We have,

$\sin^2\theta = x$

$\Rightarrow \sin\theta = \sqrt{x}$

$\Rightarrow \theta = \sin^{-1}\sqrt{x}$

x	0	1
$x = \sin^2\theta$ $\sin\theta = \sqrt{x}$ $\theta = \sin^{-1}\sqrt{x}$	$\theta = \sin^{-1}\sqrt{x}$ $\theta = \sin^{-1}\sqrt{0}$ $\theta = \sin^{-1} 0$ $\theta = \sin^{-1}\sin 0$ $\theta = 0$	$\theta = \sin^{-1}\sqrt{1}$ $\theta = \sin^{-1} 1$ $= \sin^{-1}\sin\frac{\pi}{2}$ $= \frac{\pi}{2}$

From (iv), we get

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^1 \frac{1}{x^{1/2}(1-x)^{1/2}}\,dx$$

$$\Rightarrow \beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^{\pi/2} \frac{1}{(\sin^2\theta)^{1/2}(1-\sin^2\theta)^{1/2}}\,2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow \beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^{\pi/2} \frac{1}{(\sin^2\theta)^{1/2}(\cos^2\theta)^{1/2}}\,2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow \beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^{\pi/2} \frac{1}{\{(\sin\theta)^2\}^{1/2}\{(\cos\theta)^2\}^{1/2}}\,2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow \beta(\tfrac{1}{2},\tfrac{1}{2}) = \int_0^{\pi/2} \frac{1}{(\sin\theta)(\cos\theta)}\,2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow \beta(\frac{1}{2},\frac{1}{2}) = \int_0^{\pi/2} 2\,d\theta = 2[\theta]_0^{\pi/2} = 2\left[\frac{\pi}{2} - 0\right] = 2\times\frac{\pi}{2}$$

$$\Rightarrow \beta(\frac{1}{2},\frac{1}{2}) = \pi \text{ ---(v)}$$

Putting the value of $\beta(\frac{1}{2},\frac{1}{2}) = \pi$ in (ii),

$$\therefore\ (\Gamma\frac{1}{2}) = \sqrt{\beta\,(\frac{1}{2},\frac{1}{2})}$$

$$\therefore\ (\Gamma\frac{1}{2}) = \sqrt{\pi}\ \text{(Proved)}$$

Example 139:

Show that $\int_0^1 \frac{x^5}{\sqrt{(1-x^2)}}dx$

Let $I = \int_0^1 \frac{x^5}{\sqrt{(1-x^2)}}dx$

Put $x = \sin\theta$

$\Rightarrow \frac{dx}{d\theta} = \cos\theta$

$\Rightarrow dx = \cos\theta\, d\theta$

x	0	1
$x = \sin\theta$ $\therefore \theta = \sin^{-1} x$	$\therefore \theta = \sin^{-1} x$ $\theta = \sin^{-1} 0$ $\theta = \sin^{-1}\sin 0$ $\theta = 0$	$\theta = \sin^{-1} 1 = \sin^{-1} 1$ $= \sin^{-1}\sin\frac{\pi}{2}$ $= \frac{\pi}{2}$

$$I = \int_0^1 \frac{x^5}{\sqrt{(1-x^2)}}dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^5\theta}{\sqrt{(1-\sin^2\theta)}}\cos\theta\, d\theta = \int_0^{\pi/2} \frac{\sin^5\theta}{\sqrt{(\cos^2\theta)}}\cos\theta\, d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^5\theta}{\cos\theta}\cos\theta\, d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \sin^5\theta\, d\theta$$

$$\Rightarrow I = \frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{5+1}{2})}{\Gamma(\frac{5+2}{2})} \qquad [\int_0^{\pi/2}\cos^m x dx = \int_0^{\pi/2}\sin^m x dx = \frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}]$$

$$\Rightarrow I=\frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{6}{2})}{\Gamma(\frac{7}{2})}=\frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(3)}{\Gamma(\frac{7}{2})}$$

$$\Rightarrow I=\frac{1}{2}\times\sqrt{\pi}\times\frac{(3-1)!}{\Gamma(\frac{7}{2})} \qquad [\because \Gamma n=(n-1)!]$$

$$\Rightarrow I=\frac{1}{2}\times\sqrt{\pi}\times\frac{2!}{\Gamma(\frac{5}{2}+1)}=\frac{1}{2}\times\sqrt{\pi}\times\frac{2!}{\frac{5}{2}\Gamma(\frac{5}{2})} \qquad [\because \Gamma\,(n+1)=n\,\Gamma\,(n)]$$

$$\Rightarrow I=\frac{1}{2}\times\sqrt{\pi}\times\frac{2.1}{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}\ I=\frac{1}{2}\times\sqrt{\pi}\times\frac{2.1}{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}} \qquad [\because \Gamma\frac{1}{2}=\sqrt{\pi}]$$

$$\Rightarrow I=\frac{1}{2}\times\frac{2.1}{\frac{5}{2}\frac{3}{2}\frac{1}{2}}=\frac{2.2.2}{5.3.1}=\frac{8}{15}\ \textit{Answer}$$

Example 140:

Show that $\int_0^{2\pi}\cos^4 x\,dx=\frac{3\pi}{4}$

Solution: Let, $I=\int_0^{2\pi}\cos^4 x\,dx$

$$\Rightarrow I=4\int_0^{\pi/2}\cos^4 x\,dx \qquad [\because \int_0^{2a}f(x)dx=2\int_0^{a}f(x)dx]\ [\text{That is } 2\pi=4\times\frac{\pi}{2}]$$

$$\Rightarrow I=4\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{4+1}{2})}{\Gamma(\frac{4+2}{2})} \qquad [\int_0^{\pi/2}\cos^m x dx=\frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}]$$

$$\Rightarrow I=4\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{6}{2})}=4\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{3}{2}+1)}{\Gamma(3)}$$

$$\Rightarrow I=4\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\frac{3}{2}\Gamma(\frac{3}{2})}{(3-1)!}=4\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\frac{3}{2}\Gamma(\frac{1}{2}+1)}{(2)!}$$

$$\Rightarrow I=4\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{2.1}=4\times\frac{1}{2}\times\sqrt{\pi}\times\frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{2.1}$$

$$\Rightarrow I=4\times\frac{3\pi}{16}=\frac{3\pi}{4}\ \textit{Answer}$$

Example 141:

Show that $\int_0^{\pi/2} \cos^3 x \cos 2x dx = \frac{2}{5}$

Solution: Let $I = \int_0^{\pi/2} \cos^3 x \cos 2x dx$

$$\Rightarrow I = \int_0^{\pi/2} \cos^3 x(2\cos^2 x - 1)dx \qquad [\because 2\cos^2 x - 1 = \cos 2x]$$

$$\Rightarrow I = \int_0^{\pi/2} 2\cos^5 x dx - \int_0^{\pi/2} \cos^3 x dx = 2\int_0^{\pi/2} \cos^5 x dx - \int_0^{\pi/2} \cos^3 x dx$$

$$\Rightarrow I = 2 \times \frac{1}{2} \times \sqrt{\pi} \times \frac{\Gamma(\frac{5+1}{2})}{\Gamma(\frac{5+2}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{\Gamma(\frac{3+1}{2})}{\Gamma(\frac{3+2}{2})} \; [\int_0^{\pi/2} \cos^m x dx = \frac{1}{2} \times \sqrt{\pi} \times \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}]$$

$$\Rightarrow I = 2 \times \frac{1}{2} \times \sqrt{\pi} \times \frac{\Gamma(\frac{6}{2})}{\Gamma(\frac{7}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{\Gamma(\frac{4}{2})}{\Gamma(\frac{5}{2})}$$

$$\Rightarrow I = \sqrt{\pi} \times \frac{\Gamma(3)}{\Gamma(\frac{7}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{\Gamma(2)}{\Gamma(\frac{5}{2})}$$

$$\Rightarrow I = \sqrt{\pi} \times \frac{(3-1)!}{\Gamma(\frac{7}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{(2-1)!}{\Gamma(\frac{5}{2})} \qquad [\because \Gamma n = (n-1)!]$$

$$\Rightarrow I = \sqrt{\pi} \times \frac{2!}{\Gamma(\frac{7}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1!}{\Gamma(\frac{5}{2})}$$

$$\Rightarrow I = \sqrt{\pi} \times \frac{2.1}{\Gamma(\frac{7}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\Gamma(\frac{5}{2})}$$

$$\Rightarrow I = \sqrt{\pi} \times \frac{2.1}{\Gamma(\frac{5}{2}+1)} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\Gamma(\frac{3}{2}+1)} \qquad [\because \Gamma\,(n+1) = n\,\Gamma\,(n)]$$

$$\Rightarrow I = \sqrt{\pi} \times \frac{2.1}{\frac{5}{2}\Gamma(\frac{5}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\frac{3}{2}\Gamma(\frac{3}{2})}$$

$$\Rightarrow \mathbf{I} = \sqrt{\pi} \times \frac{2.1}{\frac{5}{2}\Gamma(\frac{3}{2}+1)} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\frac{3}{2}\Gamma(\frac{1}{2}+1)}$$

$$\Rightarrow \mathbf{I} = \sqrt{\pi} \times \frac{2.1}{\frac{5}{2}\frac{3}{2}\Gamma(\frac{3}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}$$

$$\Rightarrow \mathbf{I} = \sqrt{\pi} \times \frac{2.1}{\frac{5}{2}\frac{3}{2}\Gamma(\frac{1}{2}+1)} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}$$

$$\Rightarrow \mathbf{I} = \sqrt{\pi} \times \frac{2.1}{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}$$

$$\Rightarrow \mathbf{I} = \sqrt{\pi} \times \frac{2.1}{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}} - \frac{1}{2} \times \sqrt{\pi} \times \frac{1}{\frac{3}{2}\frac{1}{2}\sqrt{\pi}} \qquad [\because \Gamma\tfrac{1}{2} = \sqrt{\pi}]$$

$$\Rightarrow \mathbf{I} = \frac{2.1}{\frac{5}{2}\frac{3}{2}\frac{1}{2}} - \frac{1}{2}\frac{1}{\frac{3}{2}\frac{1}{2}} = \frac{16}{15} - \frac{4}{6} = \frac{32-20}{30} = \frac{12}{30} = \frac{2}{5} \quad (\textit{Proved})$$

Example 142:

Show that $\int_0^{\pi/2} \cos^8 x \sin^6 x dx = \frac{5\pi}{4096}$

Solution: Let $\mathbf{I} = \int_0^{\pi/2} \cos^8 x \sin^6 x dx$

$$\mathbf{I} = \int_0^{\pi/2} \cos^8 x \sin^6 x dx$$

We have, $\int_0^{\pi/2} \sin^m x \cos^n x dx = \beta(m,n) = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

$$\Rightarrow \mathbf{I} = \beta(8,6)$$

$$\Rightarrow \mathbf{I} = \frac{\Gamma(\frac{8+1}{2})\Gamma(\frac{6+1}{2})}{2\Gamma(\frac{8+6+2}{2})} = \frac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{2})}{2\Gamma(\frac{16}{2})} = \frac{\Gamma(\frac{9}{2})\Gamma(\frac{7}{2})}{2\Gamma(8)}$$

$$\Rightarrow \mathbf{I} = \frac{\Gamma(\frac{7}{2}+1)\Gamma(\frac{5}{2}+1)}{2\times(8-1)!} \qquad [\because \Gamma n = (n-1)!]$$

$$\Rightarrow I = \frac{\frac{7}{2}\Gamma(\frac{7}{2}).\frac{5}{2}\Gamma(\frac{5}{2})}{2\times(8-1)!} \qquad [\because \Gamma\,(n+1) = n\,\Gamma\,(n)]$$

$$\Rightarrow I = \frac{\frac{7}{2}\Gamma(\frac{7}{2}).\frac{5}{2}\Gamma(\frac{5}{2})}{2\times 7!} = \frac{\frac{7}{2}\Gamma(\frac{7}{2}).\frac{5}{2}\Gamma(\frac{5}{2})}{2\times 7!}$$

$$\Rightarrow I = \frac{\frac{7}{2}\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma(\frac{1}{2}).\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma(\frac{1}{2})}{2\times 7!}$$

$$\Rightarrow I = \frac{\frac{7}{2}\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}}{2\times 7!} \qquad [\because \left\lceil\frac{1}{2} = \sqrt{\pi}\right]$$

$$\Rightarrow I = \frac{7.5.3.5.3.1.\pi}{2.2.2.2.2.2.2.2\times 7!} = \frac{7.5.3.5.3.1.\pi}{2.2.2.2.2.2.2.2\times 7.6.5.4.3.2.1}$$

$$\Rightarrow I = \frac{5\pi}{2.2.2.2.2.2.2.2\times 2.4.2.1} = \frac{5\pi}{2.2.2.2.2.2.2.2\,2.2.2.2} = \frac{5\pi}{4096} \quad \textit{Answer}$$

Example 143:

Show that $\int_0^{\pi} x\sin^6 x\cos^4 x dx = \frac{3\pi^2}{512}$

Solution: Let, $I = \int_0^{\pi} x\sin^6 x\cos^4 x dx$

$$\Rightarrow I = \int_0^{\pi}(\pi - x)\sin^6(\pi - x)\cos^4(\pi - x)dx$$

$$\Rightarrow I = \int_0^{\pi}\pi\sin^6(\pi - x)\cos^4(\pi - x)dx - \int_0^{\pi}x\sin^6(\pi - x)\cos^4(\pi - x)dx$$

$$\Rightarrow I = \pi\int_0^{\pi}\{\sin(\pi - x)\}^6\{\cos(\pi - x)\}^4 dx - \int_0^{\pi}x\{\sin(\pi - x)\}^6\{\cos(\pi - x)\}^4 dx$$

$$\Rightarrow I = \pi\int_0^{\pi}(\sin x)^6(-\cos x)^4 dx - \int_0^{\pi}x(\sin x)^6(-\cos x)^4 dx$$

$[\because \sin(\pi - x) = \sin x \quad \& \cos(\pi - x) = -\cos x]$

$$\Rightarrow I = \pi\int_0^{\pi}\sin^6 x.\cos^4 x dx - \int_0^{\pi}x\sin^6 x\cos^4 x dx$$

$$\Rightarrow I = \pi\int_0^{\pi}\sin^6 x.\cos^4 x dx - I$$

$$\Rightarrow I + I = \pi\int_0^{\pi}\sin^6 x.\cos^4 x dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \sin^6 x.\cos^4 x dx = 2\pi\int_0^{\pi/2} \sin^6 x.\cos^4 x dx$$

$$\Rightarrow I = \pi\int_0^{\pi/2} \sin^6 x.\cos^4 x dx \text{ --------------------------(i)}$$

We have, $\displaystyle\int_0^{\pi/2} \sin^m x\cos^n x dx = \beta(m,n) = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

From (i),

$$I = \pi\int_0^{\pi/2} \sin^6 x.\cos^4 x dx$$

$$\Rightarrow I = \pi\beta(6,4)$$

$$\Rightarrow I = \pi\frac{\Gamma(\frac{6+1}{2})\Gamma(\frac{4+1}{2})}{2\Gamma(\frac{6+4+2}{2})}$$

$$\Rightarrow I = \pi\frac{\Gamma(\frac{7}{2})\Gamma(\frac{5}{2})}{2\Gamma(\frac{12}{2})} = \pi\frac{\Gamma(\frac{7}{2})\Gamma(\frac{5}{2})}{2\Gamma(\frac{12}{2})} = \frac{\pi}{2}\frac{\Gamma(\frac{7}{2})\Gamma(\frac{5}{2})}{\Gamma(6)}$$

$$\Rightarrow I = \frac{\pi}{2}\frac{\Gamma(\frac{5}{2}+1)\Gamma(\frac{3}{2}+1)}{(6-1)!} \qquad [\because \Gamma n = (n-1)!]$$

$$\Rightarrow I = \frac{\pi}{2}\frac{\frac{5}{2}\Gamma(\frac{5}{2})\frac{3}{2}\Gamma(\frac{3}{2})}{(6-1)!} \qquad [\because \Gamma\ (n+1) = n\,\Gamma\ (n)]$$

$$\Rightarrow I = \frac{\pi}{2}\frac{\frac{5}{2}\Gamma(\frac{5}{2})\frac{3}{2}\Gamma(\frac{3}{2})}{5!} = \frac{\pi}{2}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{5.4.3.2.1}$$

$$\Rightarrow I = \frac{\pi}{2}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{5.4.3.2.1} \qquad [\because \Gamma\frac{1}{2} = \sqrt{\pi}]$$

$$\Rightarrow I = \frac{5.3.3.\pi^2}{2.2.2.2.2.2.5.4.3.2.1} = \frac{3\pi^2}{2.2.2.2.2.2.4.2.1} = \frac{3\pi^2}{512} \ \textit{Answer}$$

Example 144:

Show that $\displaystyle\int_0^1 x^2(1-x^2)^{5/2} dx = \frac{5\pi}{256}$

Solution: Let $\mathbf{I = \int_0^1 x^2(1-x^2)^{5/2}\,dx = \frac{5\pi}{256}}$

Put $\mathbf{x = \sin\theta}$

$\Rightarrow \mathbf{\frac{dx}{d\theta} = \cos\theta}$

$\Rightarrow \mathbf{dx = \cos\theta\, d\theta}$

x	0	1
$x = \sin\theta$ $\therefore \theta = \sin^{-1} x$	$\therefore \theta = \sin^{-1} x$ $\theta = \sin^{-1} 0$ $\theta = \sin^{-1} \sin 0$ $\theta = 0$	$\theta = \sin^{-1} 1 = \sin^{-1} 1$ $= \sin^{-1} \sin\frac{\pi}{2}$ $= \frac{\pi}{2}$

$$\mathbf{I = \int_0^1 x^2(1-x^2)^{5/2}\,dx = \frac{5\pi}{256}}$$

$$\Rightarrow \mathbf{I = \int_0^{\pi/2} \sin^2\theta(1-\sin^2\theta)^{5/2}\cos\theta\, d\theta}$$

$$\Rightarrow \mathbf{I = \int_0^{\pi/2} \sin^2\theta(\cos^2\theta)^{5/2}\cos\theta\, d\theta}$$

$$\Rightarrow \mathbf{I = \int_0^{\pi/2} \sin^2\theta\cos^5\theta\cos\theta\, d\theta}$$

$$\Rightarrow \mathbf{I = \int_0^{\pi/2} \sin^2\theta\cos^6\theta\, d\theta} \text{ ------------------------(i)}$$

We have, $\mathbf{\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \beta(m,n) = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}}$

$$\mathbf{I = \int_0^{\pi/2} \sin^2\theta\cos^6\theta\, d\theta}$$

$$\Rightarrow \mathbf{I = \beta(2,6)}$$

$$\Rightarrow \mathbf{I = \frac{\Gamma(\frac{2+1}{2})\Gamma(\frac{6+1}{2})}{2\Gamma(\frac{2+6+2}{2})} = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}{2\Gamma(\frac{10}{2})} = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{7}{2})}{2\Gamma(5)}}$$

$$\Rightarrow \mathbf{I = \frac{\Gamma(\frac{1}{2}+1)\Gamma(\frac{5}{2}+1)}{2\Gamma(5)}}$$

$$\Rightarrow \mathbf{I = \frac{\frac{1}{2}\Gamma(\frac{1}{2})\frac{5}{2}\Gamma(\frac{5}{2})}{2(5-1)!}}$$ $[\because \mathbf{\Gamma\ (n+1) = n\,\Gamma\ (n)}]$ and $[\because \mathbf{\Gamma n = (n-1)!}]$

$$\Rightarrow \mathbf{I = \frac{\frac{1}{2}\Gamma(\frac{1}{2})\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})}{2(4)!} = \frac{\frac{1}{2}\sqrt{\pi}\,\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{2(4)!} = \frac{\frac{1}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\pi}{2.4.3.2.1}}$$

$$\Rightarrow I = \frac{5.3.\pi}{2.2.2.22.4.3.2.1} = \frac{5\pi}{2.2.2.2.2.4.2.1} = \frac{5\pi}{2.2.2.2.2.2.2.2}$$

$$\Rightarrow I = \frac{5\pi}{256} \textit{ Answer}$$

Example 145:

Show that $\int_0^1 x^6\sqrt{(1-x^2)}\,dx = \frac{5\pi}{256}$

Solution: $I = \int_0^1 x^6\sqrt{(1-x^2)}\,dx$

Put $x = \sin\theta$

$\Rightarrow \frac{dx}{d\theta} = \cos\theta$

$\Rightarrow dx = \cos\theta\,d\theta$

x	0	1
$x = \sin\theta$ $\therefore \theta = \sin^{-1}x$	$\therefore \theta = \sin^{-1}x$ $\theta = \sin^{-1}0$ $\theta = \sin^{-1}\sin 0$ $\theta = 0$	$\theta = \sin^{-1}1 = \sin^{-1}1$ $= \sin^{-1}\sin\frac{\pi}{2}$ $= \frac{\pi}{2}$

$$I = \int_0^1 x^6\sqrt{(1-x^2)}\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} (\sin\theta)^6\sqrt{(1-\sin^2\theta)}\,\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \sin^6\theta\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \sin^6\theta\cos\theta\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \sin^6\theta\cos^2\theta\,d\theta \text{ ------------------------(i)}$$

We have, $\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \beta(m,n) = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

$$\Rightarrow I = \int_0^{\pi/2} \sin^6\theta\cos^2\theta\,d\theta$$

$$\Rightarrow I = \beta(6,2)$$

$$\Rightarrow I = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})} = \frac{\Gamma(\frac{6+1}{2})\Gamma(\frac{2+1}{2})}{2\Gamma(\frac{6+2+2}{2})} = \frac{\Gamma(\frac{7}{2})\Gamma(\frac{3}{2})}{2\Gamma(\frac{10}{2})}$$

$$I = \frac{\Gamma(\frac{7}{2})\Gamma(\frac{3}{2})}{2\Gamma(5)}$$ Same as previous problem

$$I = \frac{5\pi}{256}$$ ***Answer***

Example 146:

Show that $\int_0^a x^4\sqrt{(a^2 - x^2)}\,dx$

Solution: $I = \int_0^a x^4\sqrt{(a^2 - x^2)}\,dx$

Put $x = a\sin\theta$

$\Rightarrow \frac{dx}{d\theta} = a\cos\theta$

$\Rightarrow dx = a\cos\theta\, d\theta$

x	0	a
$x = a\sin\theta$ $\sin\theta = \frac{x}{a}$ $\therefore \theta = \sin^{-1}\frac{x}{a}$	$\therefore \theta = \sin^{-1}\frac{x}{a}$ $\therefore \theta = \sin^{-1}\frac{0}{a}$ $\therefore \theta = \sin^{-1} 0$ $\therefore \theta = \sin^{-1}\sin 0$ $\therefore \theta = 0$	$\therefore \theta = \sin^{-1}\frac{x}{a}$ $\therefore \theta = \sin^{-1}\frac{a}{a}$ $\theta = \sin^{-1} 1$ $= \sin^{-1}\sin\frac{\pi}{2}$ $= \frac{\pi}{2}$

$$I = \int_0^a x^4\sqrt{(a^2 - x^2)}\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} (a\sin\theta)^4\sqrt{(a^2 - (a\sin\theta)^2)}\,a\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} a^4\sin^4\theta\sqrt{(a^2 - a^2\sin^2\theta)}\,a\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} a^4\sin^4\theta\sqrt{(a^2(1 - \sin^2\theta)}\,a\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} a^4\sin^4\theta\sqrt{a^2\cos^2\theta}\,a\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} a^4\sin^4\theta\, a\cos\theta\, a\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} a^6\sin^4\theta\cos^2\theta\,d\theta$$

$$\Rightarrow I = a^6\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta \text{ ----------------------------(i)}$$

We have, $\int_0^{\pi/2} \sin^m x\cos^n x\,dx = \beta(m,n) = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

$$I = a^6 \int_0^{\pi/2} \sin^4\theta \cos^2\theta \, d\theta$$

$$\Rightarrow I = a^6 \beta(4,2)$$

$$\Rightarrow I = a^6 \frac{\Gamma(\frac{4+1}{2})\Gamma(\frac{2+1}{2})}{2\Gamma(\frac{4+2+2}{2})}$$

$$\Rightarrow I = a^6 \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{2\Gamma(\frac{8}{2})} = a^6 \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{2\Gamma(4)} = a^6 \frac{\Gamma(\frac{3}{2}+1)\Gamma(\frac{1}{2}+1)}{2\Gamma(4)}$$

$$\Rightarrow I = a^6 \frac{\frac{3}{2}\Gamma(\frac{3}{2})\frac{1}{2}\Gamma(\frac{1}{2})}{2\Gamma(4)} \qquad [\because \Gamma\,(n+1) = n\,\Gamma\,(n)]$$

$$\Rightarrow I = a^6 \frac{\frac{3}{2}\frac{3}{2}\Gamma(\frac{1}{2})\frac{1}{2}\Gamma(\frac{1}{2})}{2(4-1)!} \qquad [\because \Gamma n = (n-1)!]$$

$$\Rightarrow I = a^6 \frac{\frac{3}{2}\frac{3}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2\,3!} = a^6 \frac{\frac{3}{2}\frac{3}{2}\frac{1}{2}\pi}{2\,3!} = a^6 \frac{\frac{3}{2}\frac{3}{2}\frac{1}{2}\pi}{2\,3.2.1}$$

$$\Rightarrow I = a^6 \frac{3.3.\pi}{2.2.2.2\,3.2.1} = a^6 \frac{3\pi}{2.2.2.2.2.1} = a^6 \frac{3\pi}{32}$$ *Answer*

Example 147:

Show that $\int_0^1 x^4(1-x)^{3/2}\,dx = \frac{256}{15015}$

Solution: Let $I = \int_0^1 x^4(1-x)^{3/2}\,dx$

Put $x = \sin^2\theta$

$$\Rightarrow \frac{dx}{d\theta} = 2\sin\theta\frac{d}{d\theta}(\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 2\sin\theta\cos\theta$$

$$\Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$$

x	0	1
$x = \sin^2\theta$ $\sin\theta = \sqrt{x}$ $\theta = \sin^{-1}\sqrt{x}$	$\theta = \sin^{-1}\sqrt{x}$ $\theta = \sin^{-1}\sqrt{0}$ $\theta = \sin^{-1} 0$ $\theta = \sin^{-1}\sin 0$ $\theta = 0$	$\theta = \sin^{-1}\sqrt{1}$ $\theta = \sin^{-1} 1$ $= \sin^{-1}\sin\frac{\pi}{2}$ $= \frac{\pi}{2}$

$$I = \int_0^1 x^4(1-x)^{3/2}\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} (\sin^2\theta)^4(1-\sin^2\theta)^{3/2}\,2\sin\theta\cos\theta\,d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \sin^8\theta.(\cos^2\theta)^{3/2}\, 2\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \sin^8\theta.(\cos\theta)^3\, 2\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \sin^8\theta.\cos^3\theta\, 2\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow I = 2\int_0^{\pi/2} \sin^9\theta.\cos^4\theta\, d\theta \text{-------------------------------(i)}$$

We have, $\int_0^{\pi/2} \sin^m x \cos^n x dx = \beta(m,n) = \dfrac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

From (i)

$$I = 2\int_0^{\pi/2} \sin^9\theta.\cos^4\theta\, d\theta$$

$$\Rightarrow I = 2\beta(9,4)$$

$$\Rightarrow I = 2\frac{\Gamma(\frac{9+1}{2})\Gamma(\frac{4+1}{2})}{2\Gamma(\frac{9+4+2}{2})} = 2\frac{\Gamma(\frac{10}{2})\Gamma(\frac{5}{2})}{2\Gamma(\frac{15}{2})} = 2\frac{\Gamma(5)\Gamma(\frac{5}{2})}{2\Gamma(\frac{15}{2})}$$

$$\Rightarrow I = \frac{\Gamma(5)\Gamma(\frac{5}{2})}{\Gamma(\frac{15}{2})} = \frac{(5-1)!\Gamma(\frac{5}{2})}{\Gamma(\frac{15}{2})} \qquad [\because \Gamma n = (n-1)!]$$

$$\Rightarrow I = \frac{4!\Gamma(\frac{5}{2})}{\Gamma(\frac{15}{2})} = \frac{4!\Gamma(\frac{3}{2}+1)}{\Gamma(\frac{13}{2}+1)} = \frac{4.3.2.1.\frac{3}{2}\Gamma(\frac{3}{2})}{\frac{13}{2}\Gamma(\frac{13}{2})}$$

$$\Rightarrow I = \frac{4.3.2.1.\frac{3}{2}.\frac{1}{2}\Gamma(\frac{1}{2})}{\frac{13}{2}\frac{11}{2}\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})} = \frac{4.3.2.1.}{\frac{13}{2}\frac{11}{2}\frac{9}{2}\frac{7}{2}\frac{5}{2}}$$

$$\Rightarrow I = \frac{4.3.2.1.2.2.2.2.2}{13.11.9.7.5} = \frac{4.2.1.2.2.2.2.2}{13.11.3.7.5}$$

$$\Rightarrow I = \frac{2.2.2.2.2.2.2.2}{13.11.3.7.5} = \frac{256}{15015} \quad \textit{Answer}$$

Example 148:

Show that $\int_0^{\pi} x\cos^4 x dx = \frac{3\pi^2}{16}$

Solution: Let $I = \int_0^{\pi} x\cos^4 xdx$

$$I = \int_0^{\pi} (\pi - x)\cos^4(\pi - x)dx \qquad [\int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx]$$

$$I = \int_0^{\pi} \pi\cos^4(\pi - x)dx - \int_0^{\pi} x\cos^4(\pi - x)dx$$

$$I = \int_0^{\pi} \pi\{\cos(\pi - x)\}^4 dx - \int_0^{\pi} x\{\cos(\pi - x)\}^4 dx$$

$$I = \int_0^{\pi} \pi\{\cos(2.90 - x)\}^4 dx - \int_0^{\pi} x\{\cos(2.90 - x)\}^4 dx$$

$$I = \int_0^{\pi} \pi(-\cos x)^4 dx - \int_0^{\pi} x(-\cos x)^4 dx$$

$$I = \int_0^{\pi} \pi(\cos x)^4 dx - \int_0^{\pi} x(\cos x)^4 dx$$

$$I = \int_0^{\pi} \pi\cos^4 xdx - \int_0^{\pi} x\cos^4 xdx$$

$$I = \int_0^{\pi} \pi\cos^4 xdx - I \qquad [I = \int_0^{\pi} x\cos^4 xdx]$$

$$I + I = \int_0^{\pi} \pi\cos^4 xdx$$

$$2I = \int_0^{\pi} \pi\cos^4 xdx$$

$$2I = 2\int_0^{\pi/2} \pi\cos^4 xdx$$

$$I = \pi\int_0^{\pi/2} \cos^4 xdx$$

$$I = \pi \times \frac{3\pi}{16} \qquad \text{[from example 140 : } \int_0^{\pi/2} \cos^4 x dx = \frac{3\pi}{16}\text{]}$$

$$I = \frac{3\pi^2}{16} \quad \textit{(Proved)}$$

Example 149:

Show that $\int_0^{\infty} \frac{t^4}{(1+t^2)^4} dt = \frac{\pi}{32}$

Let $t = \tan\theta$

$\frac{dt}{d\theta} = \sec^2\theta$

$dt = \sec^2\theta d\theta$

Solution: Let, $I = \int_0^{\infty} \frac{t^4}{(1+t^2)^4} dt$

t	0	∞
$t = \tan\theta$ $\therefore \theta = \tan^{-1} t$	$\theta = \tan^{-1} t$ $\theta = \tan^{-1} 0$ $= \tan^{-1}\tan 0 = 0$	$\theta = \tan^{-1}\infty$ $= \tan^{-1}\tan\frac{\pi}{2}$ $= \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \frac{\tan^4\theta}{(1+\tan^2\theta)^4} \sec^2\theta d\theta$$

$$I = \int_0^{\pi/2} \frac{\tan^4\theta}{(\sec^2\theta)^4} \sec^2\theta d\theta$$

$$I = \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^8\theta} \sec^2\theta d\theta$$

$$I = \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^6\theta} d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin^4\theta}{\cos^4\theta} \frac{1}{\sec^6\theta} d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin^4\theta}{\cos^4\theta} \frac{\cos^6\theta}{1} d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin^4\theta}{1} \frac{\cos^2\theta}{1} d\theta$$

$$I = \int_0^{\pi/2} \sin^4\theta \cos^2\theta d\theta \text{ ---------------------------(i)}$$

We have, $\int_0^{\pi/2} \sin^m x \cos^n x dx = \beta(m,n) = \dfrac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}$

From (i),

$$I = \int_0^{\pi/2} \sin^4\theta\cos^2\theta d\theta = \frac{\Gamma(\frac{4+1}{2})\Gamma(\frac{2+1}{2})}{2\Gamma(\frac{4+2+2}{2})} = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{2\Gamma(\frac{8}{2})} = \frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{2\Gamma(4)}$$

$$= \frac{\frac{3}{2}\frac{1}{2}\Gamma(\frac{1}{2})\frac{1}{2}\Gamma(\frac{1}{2})}{2(4-1)!} = \frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}\frac{1}{2}\sqrt{\pi}}{2.3!} = \frac{\frac{3}{2}\frac{1}{2}\frac{1}{2}\pi}{2.3!} = \frac{3\pi}{2.2.2.2.3.2.1}$$

$$= \frac{\pi}{2.2.2.2.2.1} = \frac{\pi}{32}$$ *(Proved)*