WAVES

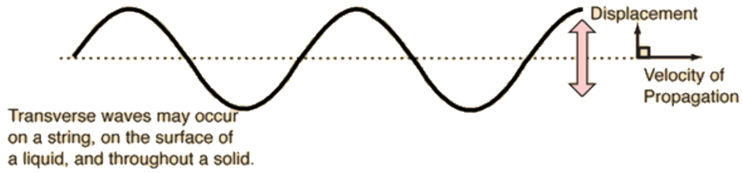
Wave: *If a vibratory disturbance occurs at any point in an elastic medium, this disturbance will be transmitted from one layer to the next through the medium, because of the elastic forces on adjacent layers, but this medium itself does not move as a whole. This is called wave.*

Types of wave:

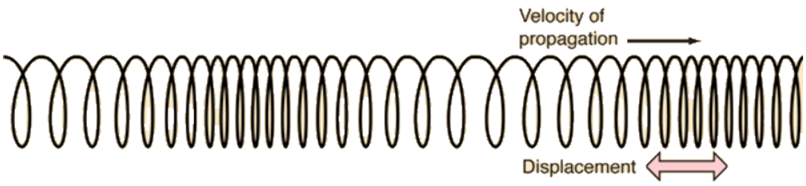
Generally, there are two types of wave, such as-

- Transverse Wave
- Longitudinal Wave

Transverse wave: *When the displacement of the medium is perpendicular to the propagation of the wave, then the wave is termed as transverse wave.*



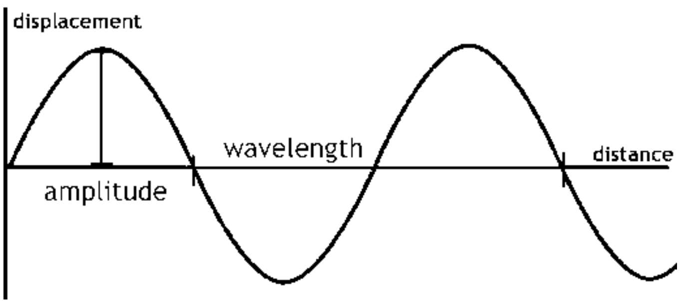
Longitudinal wave: *When the displacement of the medium is parallel to the propagation of the wave, then the wave is termed as longitudinal wave.*



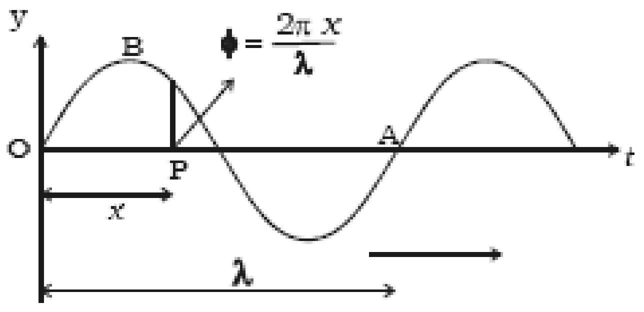
There are another two types of waves. They are,

- Progressive Wave
- Stationary of Standing Wave

Progressive wave: *Progressive wave is defined as the onward transmission of the vibratory motion of a body in an elastic medium from one particle to the successive particle.*



Differential Equation of Progressive Wave:



Plane Progressive wave

Let us assume that a progressive wave travels from the origin O along the positive direction of X axis, from left to right. The displacement of a particle at a given instant is

$$y = a \sin \omega t \qquad \ldots\ldots (1)$$

where, *a* is the amplitude of the vibration of the particle and $\omega = 2\pi n$.

The displacement of the particle P at a distance *x* from O at a given instant is given by,

$$y = a \sin (\omega t - \varphi) \qquad \ldots\ldots (2)$$

If the two particles are separated by a distance λ, they will differ by a phase of 2π. Therefore, the phase φ of the particle P at a distance *x* is,

$$\varphi = (2\pi/\lambda)\, x$$

Therefore, the displacement of the wave is,

$$y = a \sin (\omega t - 2\pi x/\lambda)$$

$$= a \sin (\omega t - kx) \quad \ldots\ldots (3)$$

where, wave number,

$$k = 2\pi/ \lambda$$

Since,

$$\omega = 2\pi n = 2\pi\, (v/\lambda),$$

the equation is given by,

$$y = a \sin [(2\pi vt/\lambda) - (2\pi x/\lambda)]$$

$$\text{or, } y = a \sin 2\pi/\lambda\, (vt - x) \qquad \ldots\ldots (4)$$

If the wave travels in opposite direction, the equation becomes,

$$y = a \sin 2\pi/\lambda\, (vt + x) \ldots\ldots (6)$$

Differentiatin equation (3) with respect to t,

$$dy/dt = a\omega \cos(\omega t - kx)$$

differentiating again with respect to t,

$$d^2 y/dt^2 = -a\omega^2 \sin(\omega t - kx) = -\omega^2 y \quad \ldots\ldots\ldots\ldots (7)$$

Differentiatin equation (3) with respect to x,

$$dy/dx = -ak \cos(\omega t - kx)$$

differentiating again with respect to t,

$$d^2 y/dx^2 = -ak^2 \sin(\omega t - kx) = -k^2 y \quad \ldots\ldots\ldots\ldots (8)$$

From equation (7) and (8),

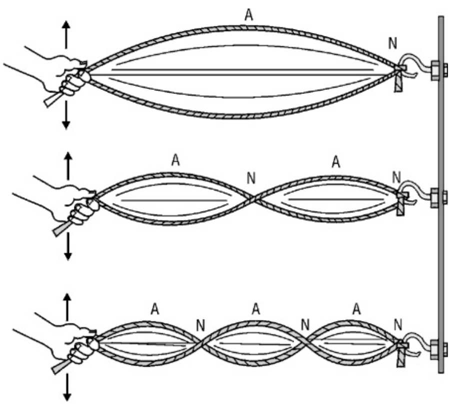
$$d^2 y/dt^2 = (\omega^2/ k^2)\, (d^2 y/dx^2)$$

So,

$$d^2 y/dt^2 = v^2\, (d^2 y/dx^2)$$

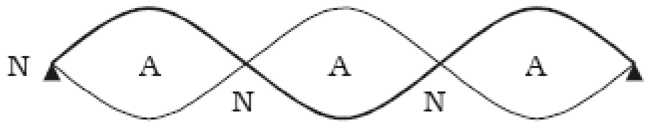
This is the required differential form of progressive wave.

Stationary wave: *Stationary waves are produced when two waves interfere in an invisible disturbance through the wave is moving throughout the material.*



Equation of Stationary Wave:

When two progressive waves of same amplitude and wavelength travelling along a straight line in opposite directions superimpose on each other, stationary waves are formed.



Stationary waves

Let us consider a progressive wave of amplitude *a* and wavelength λ travelling in the direction of X axis.

$$y_1 = a \sin 2\pi/\lambda\, [vt - x] \qquad \ldots\ldots (1)$$

This wave is reflected from a free end and it travels in the negative direction of X axis, then

$$y_2 = a \sin 2\pi/\lambda\, [vt + x] \ldots\ldots (2)$$

According to principle of superposition, the resultant displacement is,

$$y = y_1 + y_2$$

$$= a\,[\sin 2\pi/\lambda\, [vt - x] + \sin 2\pi/\lambda\, [vt + x]]$$

$$= a\,[2\sin (2\pi vt/\lambda) \cos (2\pi x/\lambda)]$$

So, $$y = 2a \cos (2\pi x/\lambda) \sin (2\pi vt/\lambda)$$

$$= A \sin (2\pi vt/\lambda) \qquad \ldots\ldots (3) \qquad \text{, where, } A = 2a \cos (2\pi x/\lambda)$$

This is the equation of a stationary wave.

Nodes and Antinodes:

(a) At points where $x = 0, \lambda/2, \lambda, 3\lambda/2$, the values of $\cos 2\pi x/\lambda = \pm 1$

∴ $A = + 2a$. At these points the resultant amplitude is maximum. They are called *antinodes* as shown in figure.

(b) At points where $x = \lambda/4, 3\lambda/4, 5\lambda/4$..... the values of $\cos 2\pi x/\lambda = 0$.

∴ $A = 0$. The resultant amplitude is zero at these points. They are called *nodes.*

The distance between any two successive antinodes or nodes is equal to $\lambda/2$ and the distance between an antinode and a node is $\lambda/4$.

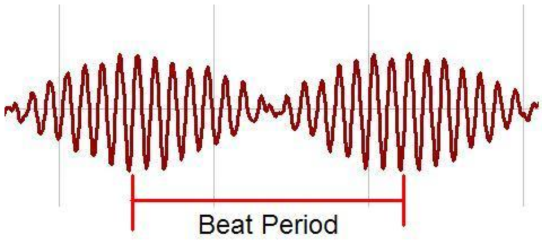
(c) When $t = 0, T/2, T, 3T/2, 2T$,.... then $\sin 2\pi t/T = 0$, the displacement is zero.

(d) When $t = T/4, 3T/4, 5T/4$ etc,....$\sin 2\pi t/T = \pm 1$, the displacement is maximum.

Difference between progressive waves and stationary waves:

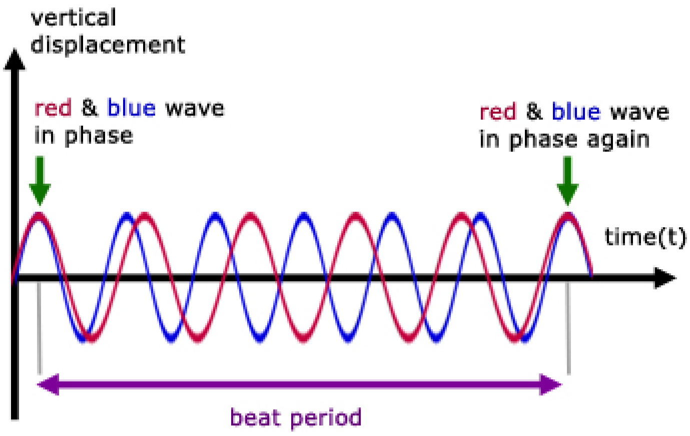
Progressive waves	Stationary waves
The disturbance produced in the medium travels onward, it being handed over from one particle to the next. Each particle executes the same type of vibration as the preceding one, though not at the same time.	There is no onward motion of the disturbance as no particle transfers its motion to the next. Each particle has its own characteristic vibration.
The amplitude of each partide is the same but the phase changes continuously,	The amplitudes of the different particles are different, ranging from zero at the nodes to maximum at the antinodes. All the particles in a given segment vibrate in phase but in opposite phase relative to the particles in the adjacent segment.
No particle is parmanently at rest. Different particles attain the state of momentary rest at different instants,	The particles at the nodes are permanently at rest but other particles attain their position of momentary rest simultaneously.
All the particles attain the same maximum velocity when they pass through their mean positions.	All the particles attain their own maximum velocity at the same time when they pass through their mean positions.
In the case of a longitudinal progressive wave all the parts of the medium undergo similar variation of density one after the other. At every point there will be a density variation.	In the case of a longitudinal stationary wave the variation of density is different at different points being maximum at the nodes and zero at the antinodes.
There is a flow of energy across every plane in the direction of propagation.	Energy is not transported across any plane.

Beat: *When two sound waves of different frequency approach the ear, the alternating constructive and destructive interference causes the sound to be alternately soft and loud due to the superposition of sound. This phenomenon is known as beat.*



Mathematical Analysis of Beat:

Let us consider two waves of slightly different frequencies n_1 and n_2 ($n_1 \sim n_2 < 10$) having equal amplitude travelling in a medium in the same direction.



At time t = 0, both waves travel in same phase. The equations of the two waves are

$$y_1 = a \sin \omega_1 t$$

$$= a \sin (2\pi n_1)t \qquad \text{...... (1)}$$

$$y_2 = a \sin \omega_2 t$$

$$= a \sin (2\pi n_2)t \qquad \text{...... (2)}$$

When the two waves superimpose, the resultant displacement is given by

$$y = y_1 + y_2$$

$$\text{or, } y = a \sin (2\pi n_1)t + a \sin (2\pi n_2)t \qquad \text{...... (3)}$$

Therefore

$$y = 2a \sin 2\pi\{(n_1 + n_2)/2\}t \cos 2\pi\{(n_1 - n_2)/2\}t \qquad \text{...... (4)}$$

Substitute $A = 2a \cos 2\pi\{(n_1 - n_2)/2\}t$ and $n = (n_1 + n_2)/2$ in equation (4)

$$y = A \sin 2\pi nt$$

This represents a simple harmonic wave of frequency $n = (n_1 + n_2)/2$ and amplitude A which changes with time.

Conditions for loud (maxima) and soft (minima):

(i) The resultant amplitude is maximum (i.e) $\pm 2a$, if

$\cos 2\pi\,[(n_1\text{-}n_2)/2]\,t = \pm 1$

So, $2\pi\,[(n_1\text{-}n_2)/2]\,t = 0, \pi, 2\pi, \ldots\ldots\ldots, m\pi$

or $(n_1 - n_2)t = 0, 1, 2, \ldots\ldots, m$

The first maximum is obtained at $t_1 = 0$

The second maximum is obtained at,

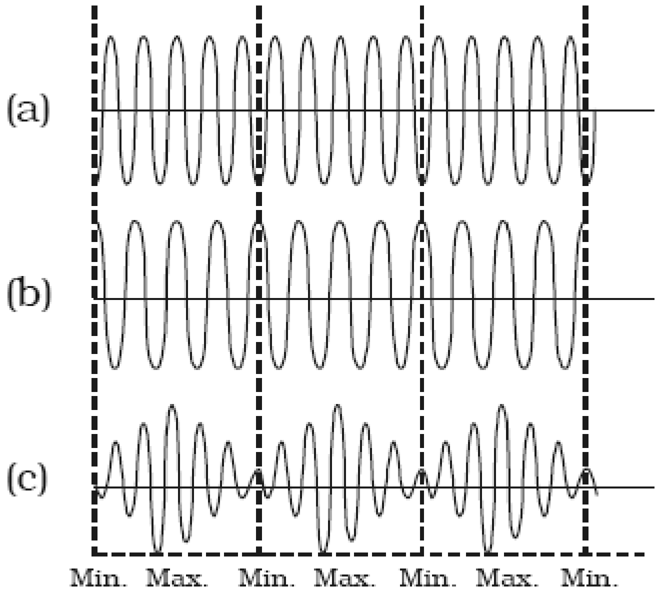
$t_2 = 1/n_1 - n_2$

The third maximum at $t_3 = 2/n_1 - n_2$ and so on.

The time interval between two successive maxima is,

$t_2 - t_1 = t_3 - t_2 = 1/n_1 - n_2$

Hence the number of beats produced per second is equal to the reciprocal of the time interval between two successive maxima.



(ii) The resultant amplitude is minimum (i.e) equal to zero, if

$\cos 2\pi\,[n_1\text{-}n_2/2]\,t = 0$

(i.e) $2\pi [n_1\text{-}n_2/2]\ t = \pi/2\ ,\ 3\pi/2,\ 5\ \pi/2\ ,\ldots\ldots\ldots\ldots\ldots,\ (2m+1)\pi/2$

Or, $[n_1\text{-}n_2/2]\ t = 1/2,\ 3/2,\ 5/2,\ \ldots\ldots\ldots\ldots\ldots,(2m+1)/2$

Where m = 0, 1, 2...

The first minimum is obtained at,

$t_1' = 1/2(n_1 - n_2)$

The second minimum is obtained at,

$t_2' = 3/2(n_1 - n_2)$

The third minimum is obtained at,

$t_3' = 5/2(n_1 - n_2)$ and so on

Time interval between two successive mimima is

$t_2' - t_1' = t_3' - t_2' = 1/n_1 - n_2$

Hence, the number of beats produced per second is equal to the reciprocal of time interval between two successive minima.

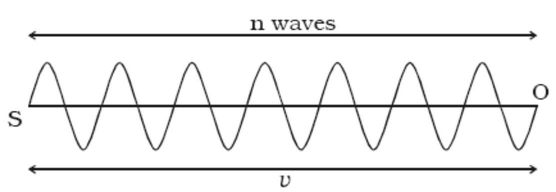
Doppler Effect

The whistle of a fast moving train appears to increase in pitch as it approaches a stationary observer and it appears to decrease as the train moves away from the observer. This apparent change in frequency was first observed and explained by Doppler in 1845.

The phenomenon of the apparent change in the frequency of sound due to the relative motion between the source of sound and the observer is called Doppler effect.

The apparent frequency due to Doppler effect for different cases can be deduced as follows.

(i) Both source and observer at rest



Both source and observer at rest

Suppose S and O are the positions of the source and the observer respectively. Let *n* be the frequency of the sound and *v* be the velocity of sound. In one second, *n* waves produced by the source travel a distance SO = *v*

The wavelength is $\lambda = n$

(ii) When the source moves towards the stationary observer

If the source moves with a velocity v_s towards the stationary observer, then after one second, the source will reach S', such that SS' = v_S. Now n waves emitted by the source will occupy a distance of $(v - v_S)$ only as shown in figure.

Therefore the apparent wavelength of the sound is

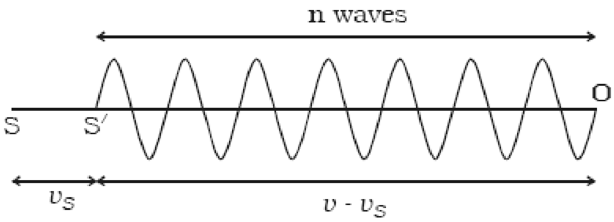
$\lambda' = (v - v_S)/n$

The apparent frequency

$n' = v/\lambda' = \{v/(v - v_S)\}n$ …... (1)

So,

$$n' = \left(\frac{v}{v - v_s}\right) n$$



Source moves towards observer at rest

As $n' > n$, the pitch of the sound appears to increase.

When the source moves away from the stationary observer

If the source moves away from the stationary observer with velocity v_s, the apparent frequency will be given by

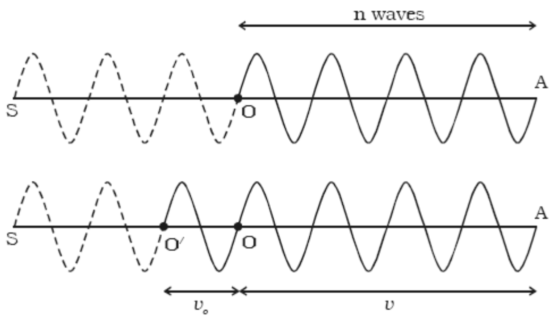
$n' = [v/\{v - (-v_S)\}]n = [v/(v + v_S)]\ n$ …... (2)

So,

$$n' = \left(\frac{v}{v - v_s}\right) n$$

As $n' < n$, the pitch of the sound appears to decrease.

(iii) Source is at rest and observer in motion



Observer is moving towards a source at rest

S and O represent the positions of source and observer respectively.

The source S emits n waves per second having a wavelength $\lambda = v/n$

Consider a point A such that OA contains n waves which crosses the ear of the observer in one second. (i.e) when the first wave is at the point A, the n^{th} wave will be at O, where the observer is situated.

When the observer moves towards the stationary source,

Suppose the observer is moving towards the stationary source with velocity v_o. After one second the observer will reach the point O′ such that OO′ = v_o. The number of waves crossing the observer will be n waves in the distance OA in addition to the number of waves in the distance OO′ which is equal to v_0/λ .

Therefore, the apparent frequency of sound is

$n' = n + v_0/\lambda = n + (v_0/v)\, n$

$= \{(v + v_o)/v\}\, n$ …... (3)

So,

$$n' = \left(\frac{v - v_0}{v}\right) n$$

As n’ > n, the pitch of the sound appears to increase.

When the observer moves away from the stationary source,

$n' = [\{v + (-v_0)\}/v]\, n$

So,

$$n' = \left(\frac{v - v_0}{v}\right) n$$

…... (4)

As $n' < n$, the pitch of sound appears to decrease.

(iv) Both observer and source are in motion:

When source and observer are facing to each other

Then, the aparent frequency,

$n' = \{(v + v_o) / (v-v_s)\}n$ ....(5)

and after crossing each other,

$n' = \{(v - v_o) / (v + v_s)\}n$ ....(6)

Group velocity:

The group velocity of a wave is the velocity with which the overall shape of the wave's amplitudes propagates through space.

The group velocity v_g is defined by the equation:

$$v_g = d\omega/dk$$

where

ω is the wave's angular frequency (usually expressed in radians per second), and k is the angular wavenumber (usually expressed in radians per meter).

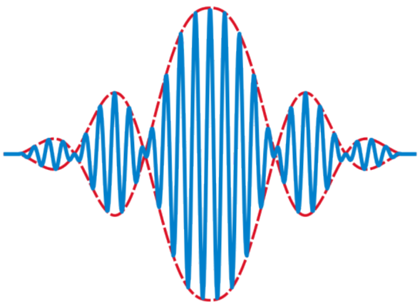


Figure 1 :Group velocity

Phase velocity:

The phase velocity of a wave is the rate at which the phase of the wave propagates in space. This is the velocity at which the phase of any one frequency component of the wave travels. For such a component, any given phase of the wave (for example, the crest) will appear to travel at the phase velocity.

In terms of the wave's angular frequency ω, which specifies angular change per unit of time, and wavenumber (or angular wave number) k, which represents the proportionality between the angular frequency ω and the linear speed (speed of propagation) v_p,

$$v_p = \omega / k$$

Relation between Group Velocity and Phase Velocity:

We know that,

Phase velocity,

$$v_p = \omega/ k \text{ ....}(1)$$

Again,

energy of the wave is,

$$E = hn = (h/ 2\pi)\ (2\pi n) = \hbar\ \omega$$

and momentum,

$$p = h/\lambda = (h/ 2\pi)(2\pi/\lambda) = \hbar\ k$$

So, equation (1) becomes,

$$v_p = \hbar\omega/ \hbar p = E/p \text{ .... } (2)$$

For relativistic particle,

$$E = mc^2$$

$$p= mv$$

Now, equation (2) becomes,

$$v_p = mc^2/ mv = c^2/v \text{ ....}(3)$$

Group velociy,

$$V_g = d\omega/dk$$

$$= d(\hbar\omega)/d(\hbar p)$$

$$= dE/dp \text{....................... } (4)$$

Again, for relativistic particle,

$$E^2 = m_0{}^2c^4 + p^2c^2 \text{ .... } (5)$$

differentiation above equation ,

$$2EdE = 0 + 2pc^2dp$$

$$\text{or, } dE/dp = pc^2/E = v$$

hence, equation (4) becomes,

$$v_g = v$$

Therefore, from equation (4) we get,

$$v_p = c^2 / v_g$$

$$\text{or, } \mathbf{v_g\ v_p = c^2}$$

This is the relation between group velocity and phase velocity.