Problem 29: Unit Step Function

The unit function, also called Heaviside's unit function, $\mathbf{u(t)}$ is defined as

$$\mathbf{y = u(t) = 1 \qquad t \geq 0}$$

$$\mathbf{= 0 \qquad t < 0}$$

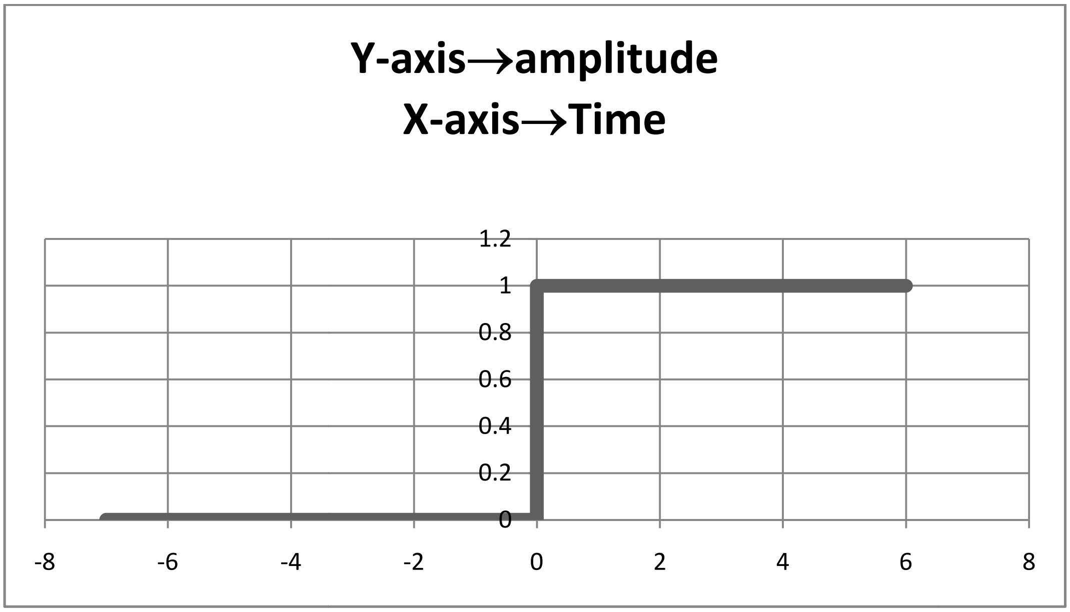


Figure 96

Example 77:

2u(t)

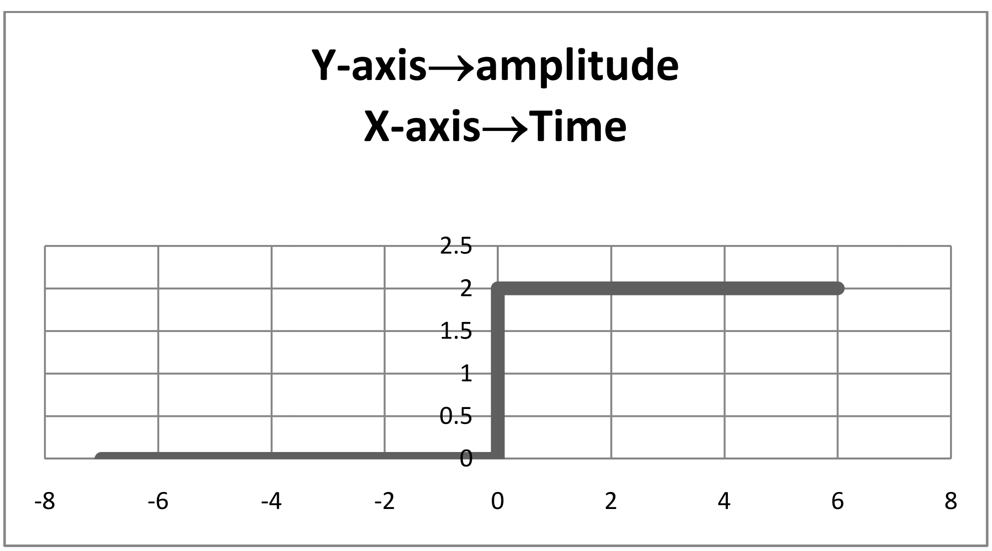


Figure 97

Example 78:

u(t-2)

Here,

$t-2=0$

$\therefore t=2$

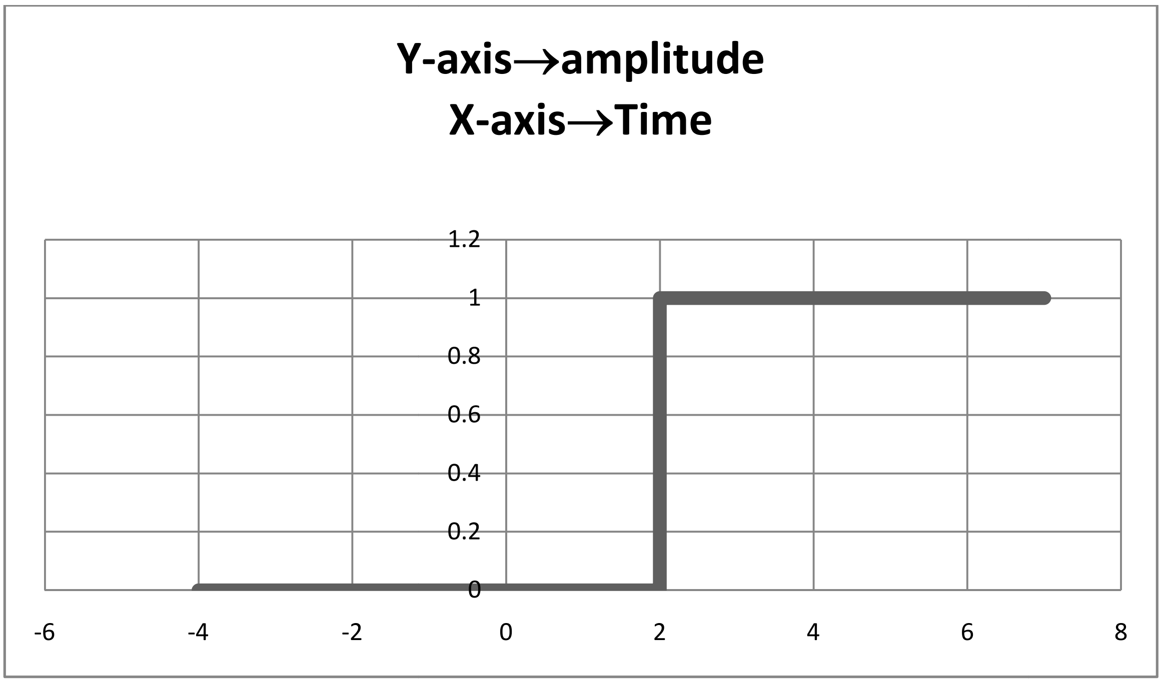


Figure 98

Example 79:

u(t-1)

Here,

$t-1=0$

$\therefore t=1$

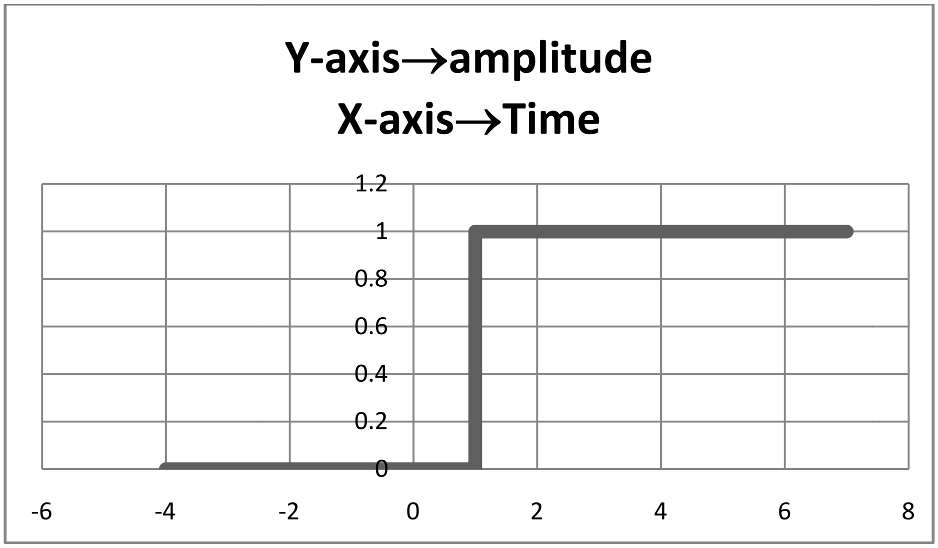


Figure 99

Example 80:

u(t+1)

Here,

$$t+1=0$$
$$\therefore t=-1$$

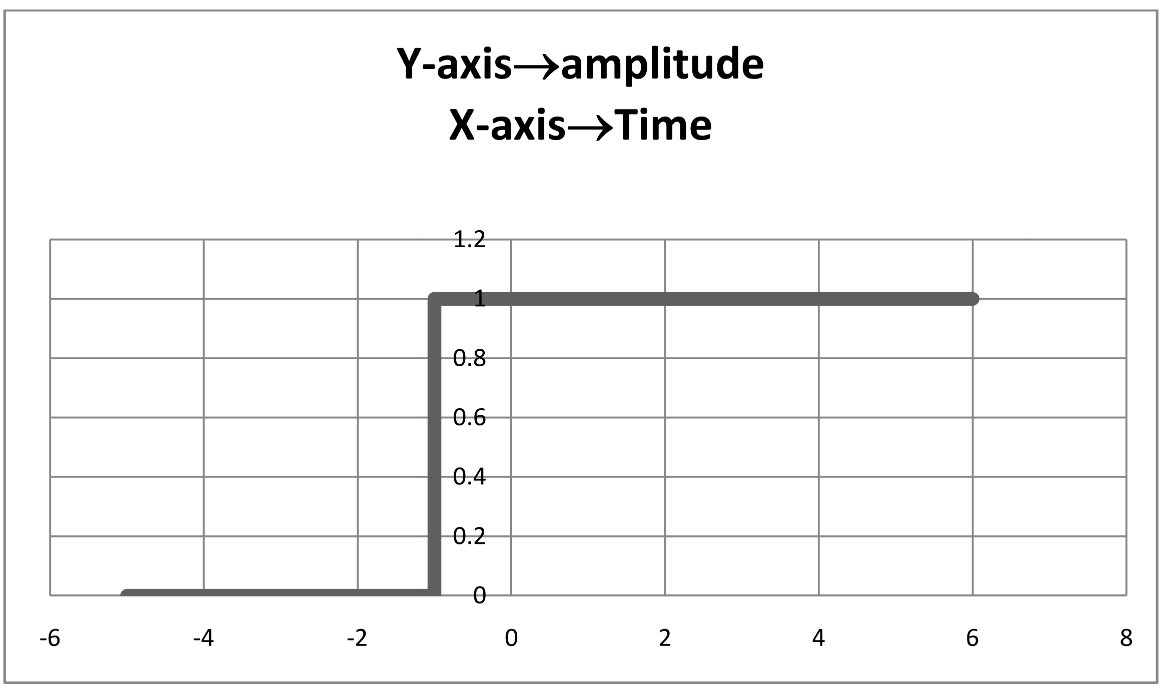


Figure 100

Example 81: u(t+2)

Here,

$$t + 2 = 0$$
$$\therefore t = -2$$

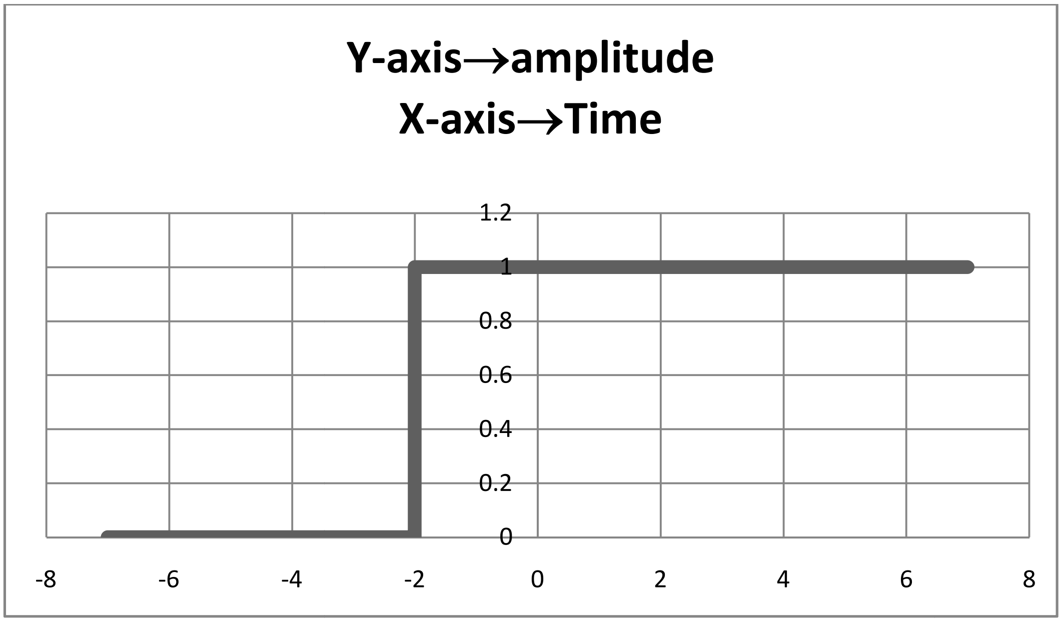


Figure 101

Example 82:-u(t)

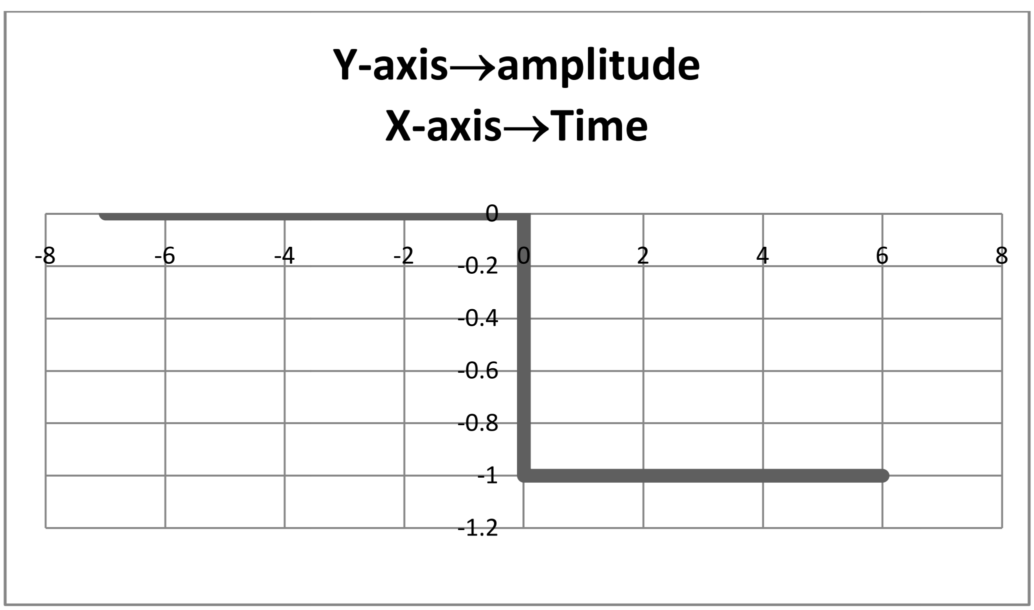


Figure 102

Example 83: -2u(t)

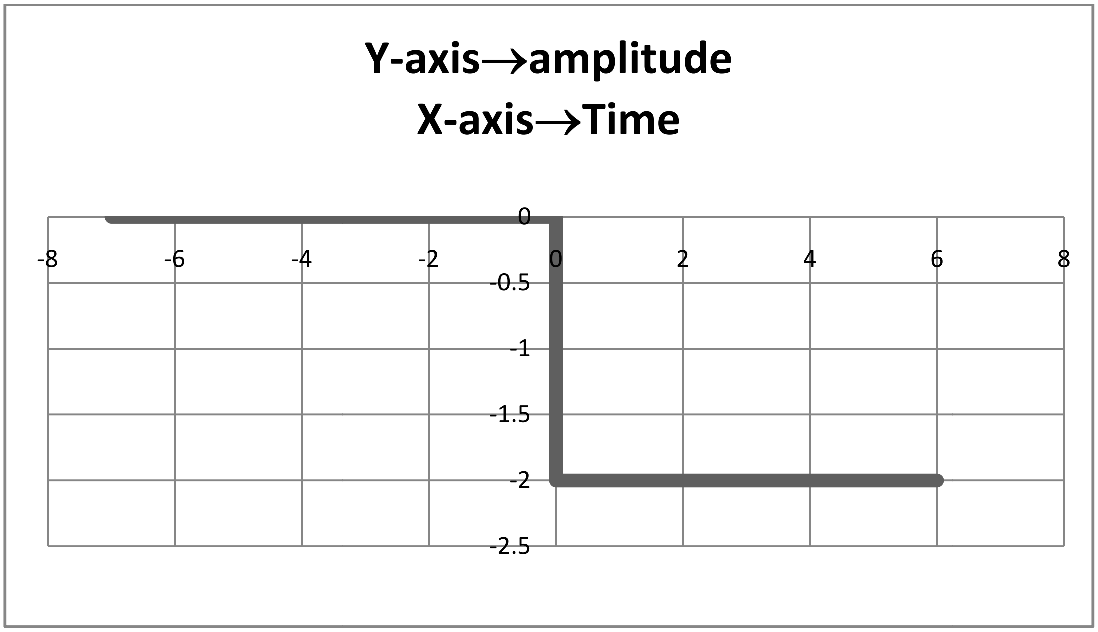


Figure 103

Example 84: -2u(t-1)

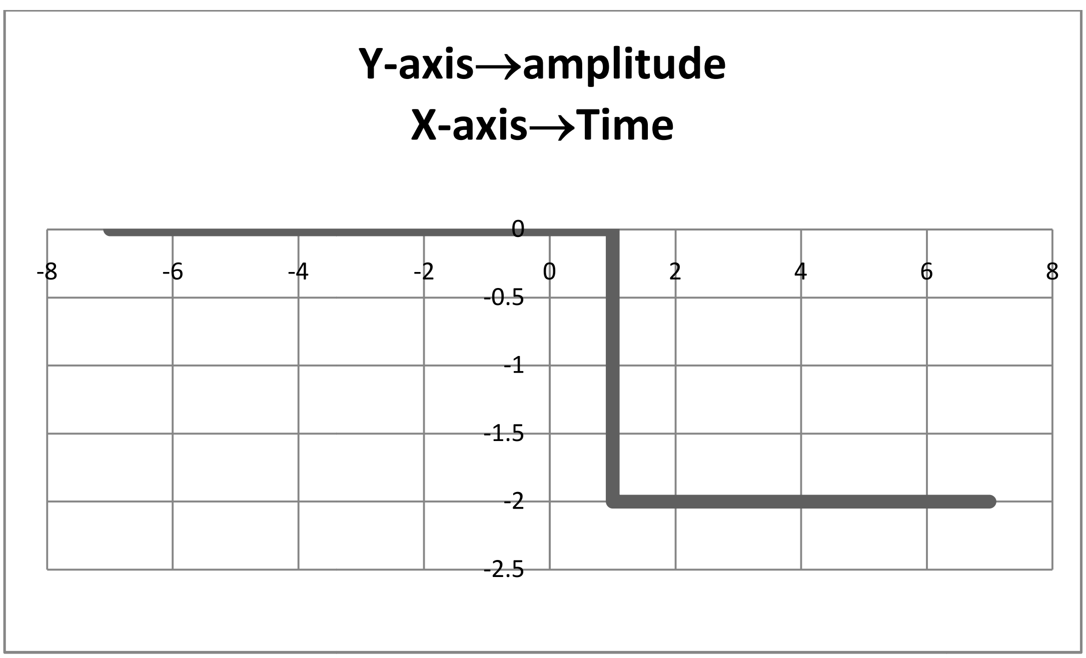


Figure 104

Example 85: u(t)

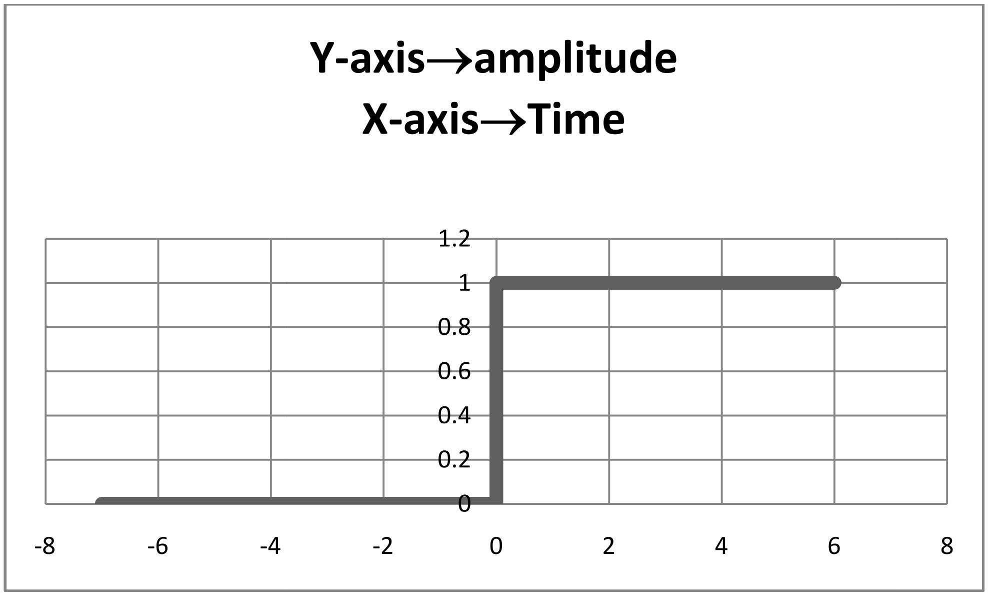


Figure 105

Example 86: -u(t-2)

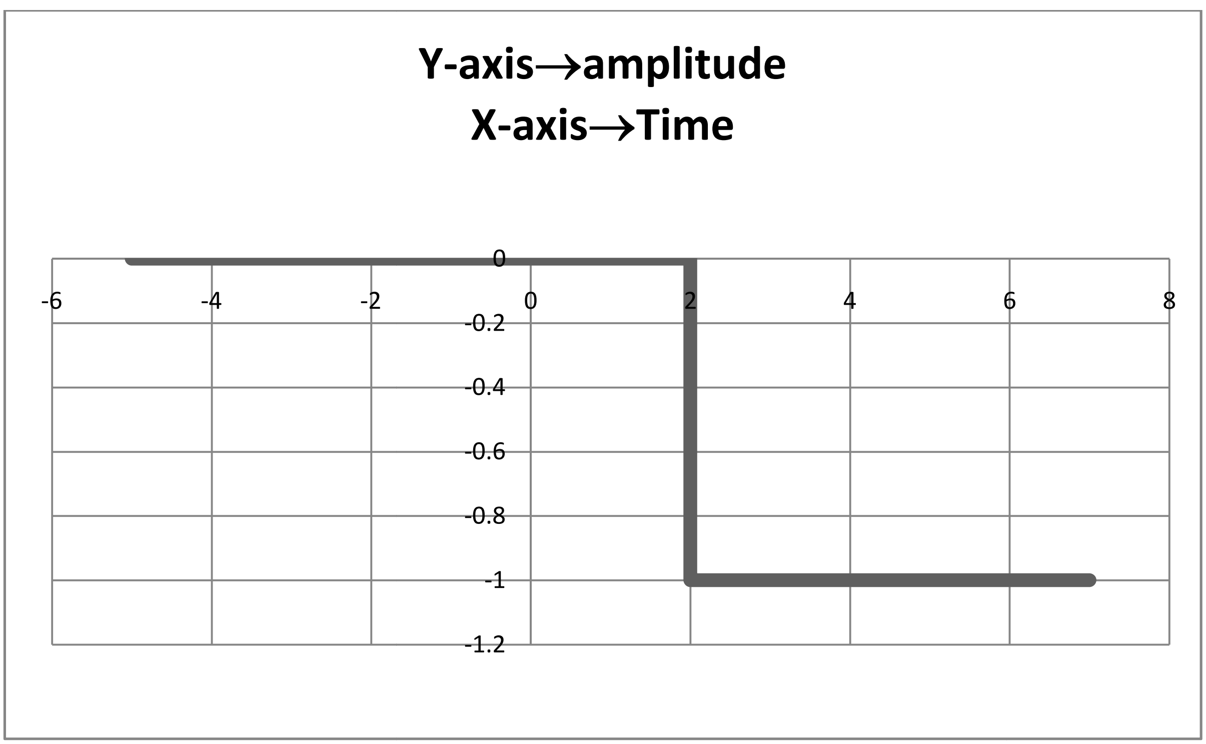


Figure 106

Example 87: u(t)-u(t-2)

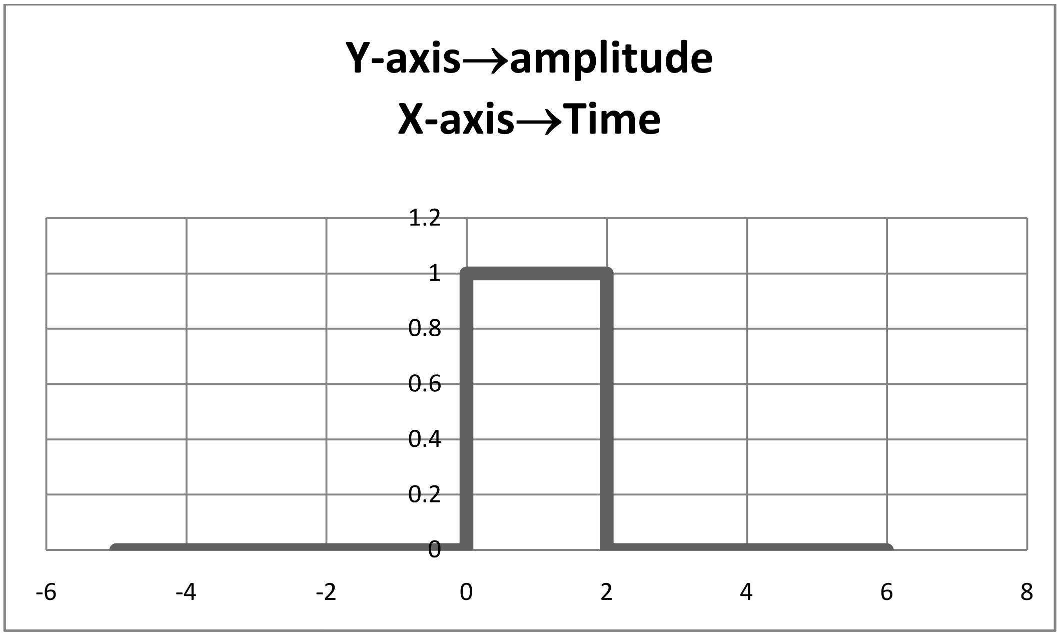


Figure 107

Example 88: u(t-pi/2)

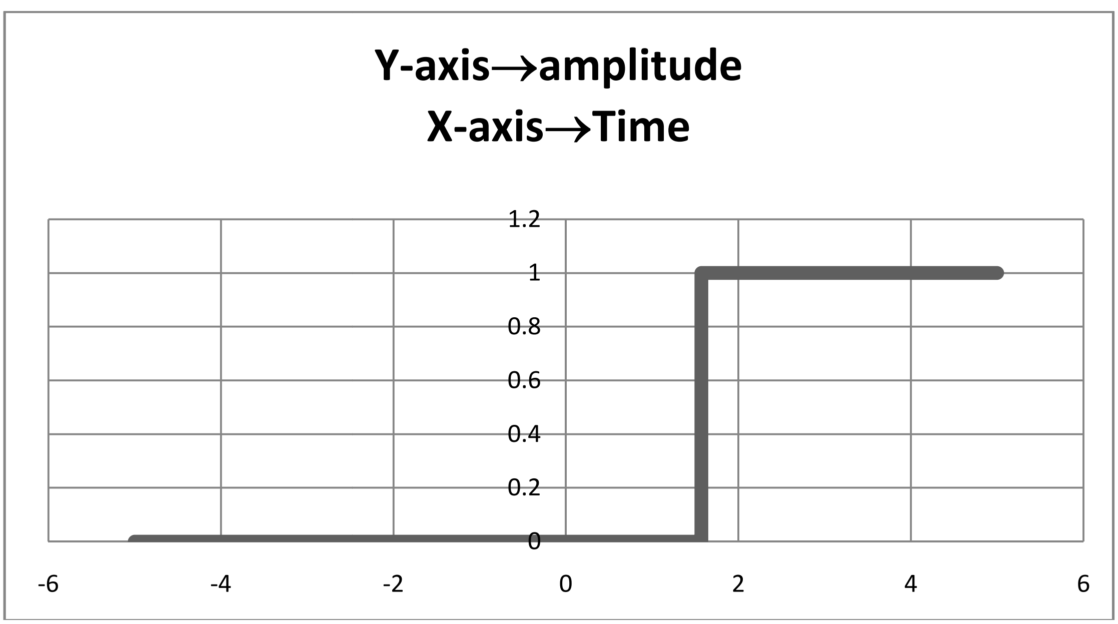


Figure 108

Problem 30: Ramp Function

The ramp function $\mathbf{r(t)}$ is defined as

$$\mathbf{r(t)} = \begin{cases} \mathbf{t} & \mathbf{t \geq 0} \\ \mathbf{0} & \mathbf{t < 0} \end{cases}$$

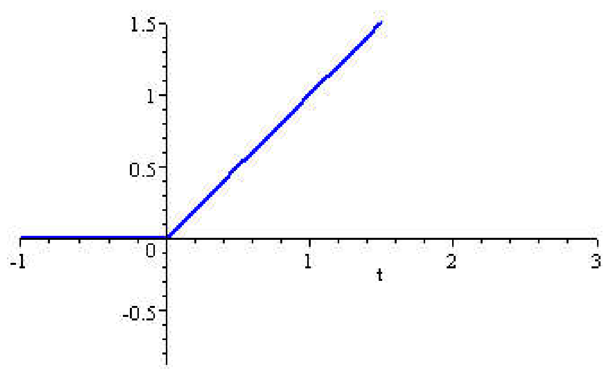


Figure 109

Example 89: Given that, $\boldsymbol{x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)}$

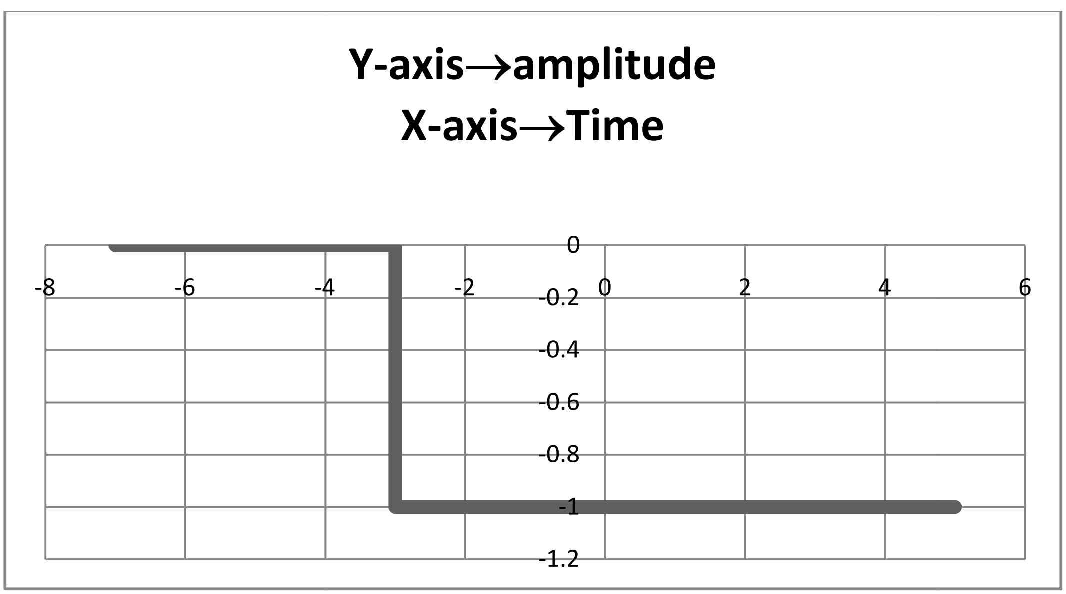
Answer:

01. $\boldsymbol{-u(t+3)} =>$

So,

$-u(t+3) = -1\ ;\ t \geq -3;$ $\qquad$ $here, t+3 = 0$

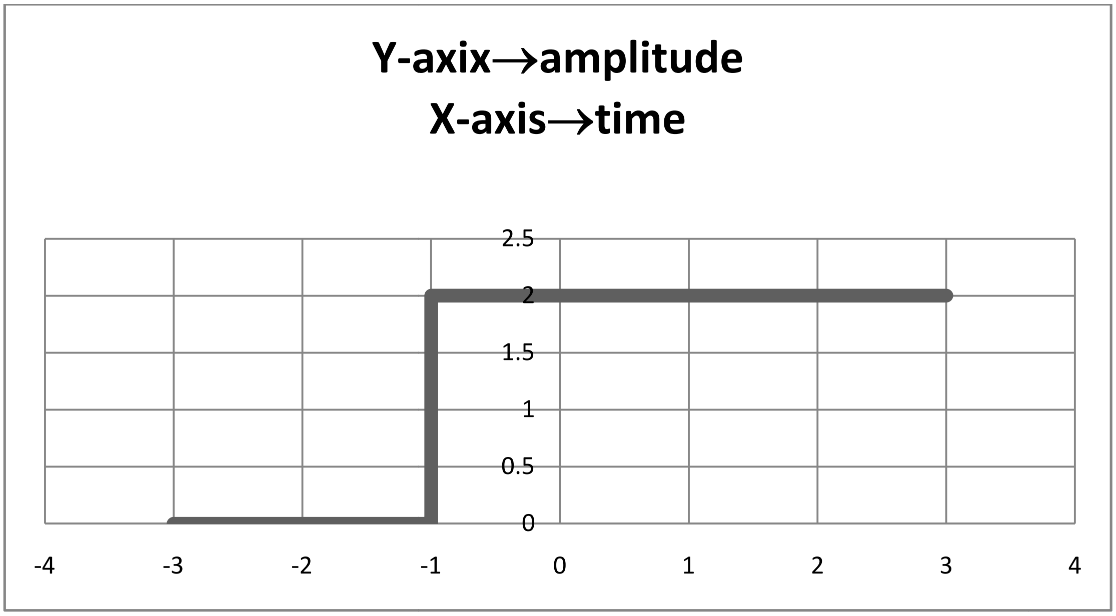
$\qquad = 0;\ \ t < -3$ $\qquad$ $\therefore t = -3$



02. $\boldsymbol{2u(t+1)}$

$2u(t+1) = 2; t \geq -1$ $\qquad$ $here, t+1 = 0$

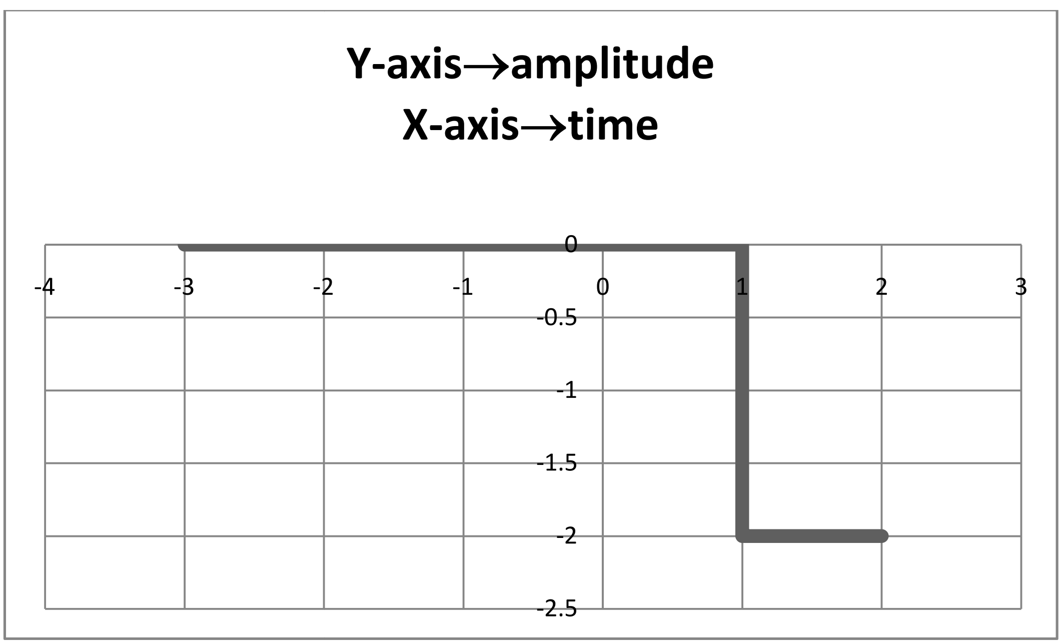
$\qquad = 0; t < -1$ $\qquad$ $t = -1$



03. $-2u\,(t-1)$

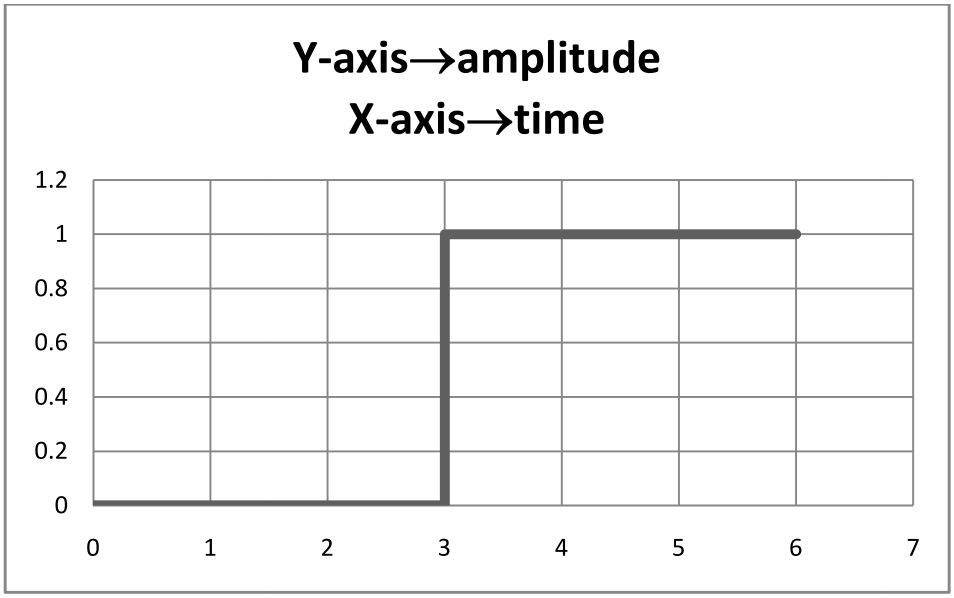
$\therefore\ -2u(t-1) = -2; t \geq 1$ $\qquad$ $here, t-1 = 0$

$\qquad = 0;\ t < 1$ $\qquad$ $t = 1$

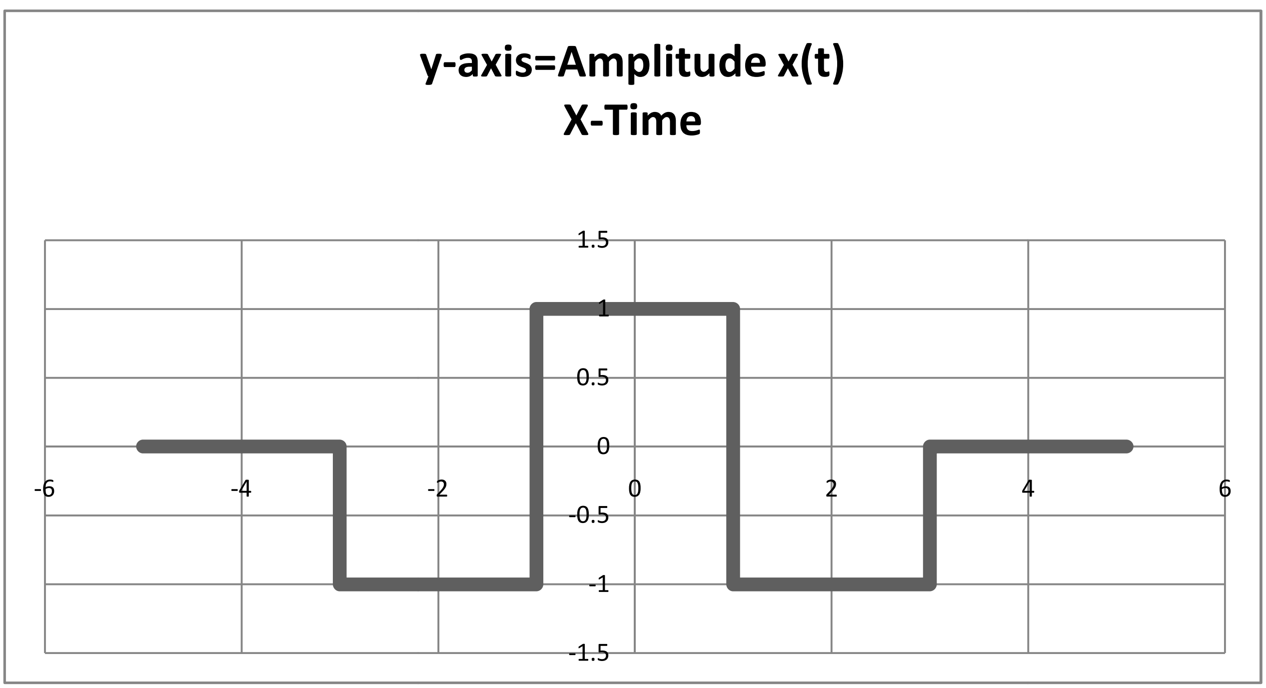


04. $\boldsymbol{u(t-3)} =>$ $here, t-3=0$

$t=3$



$$\boldsymbol{x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)}$$



Example 90:

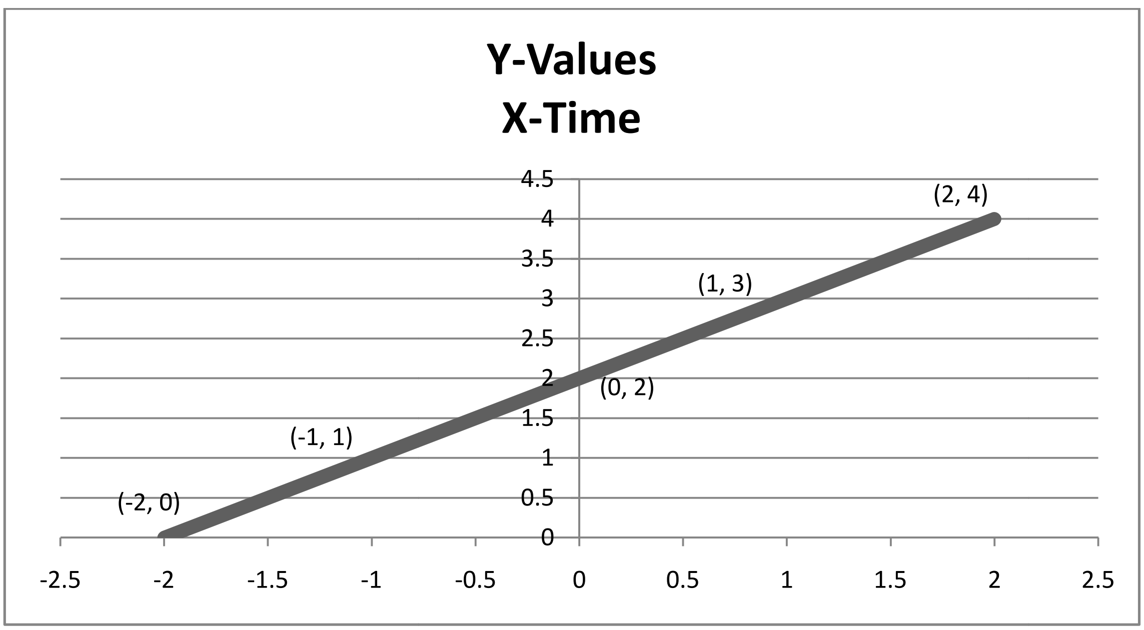
$$\boldsymbol{x(t) = r(t+2) - r(t+1) - r(t-1) + r(t-2)}$$

Solve:

$r(t+2) = t+2;\ t \geq -2$ $\qquad$ $Here,\ t+2=0$

$\qquad = 0;\ \ t < -2$ $\qquad$ $\therefore t = -2$

t	-2	-1	0	1	2	3
$r(t+2) = t+2$	0	1	2	3	4	5

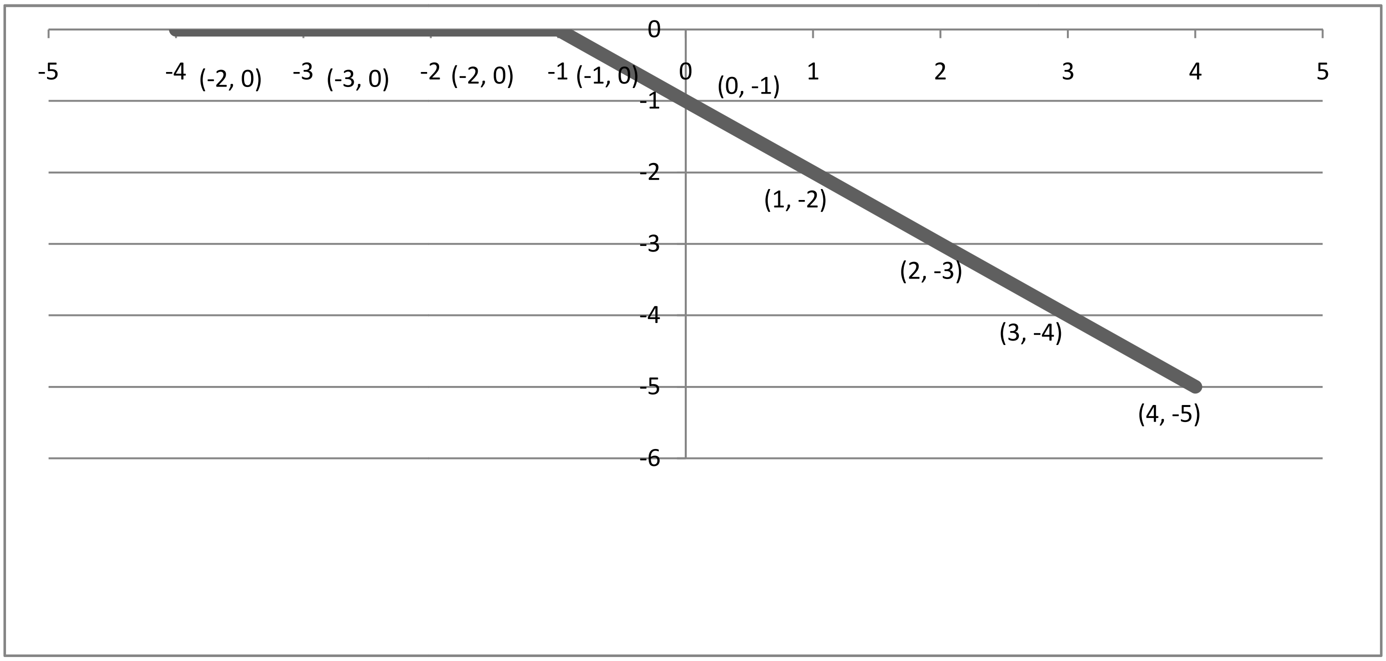


Again,

$-r(t+1) = -(t+1); t \geq -1$ $\qquad$ $Here, t+1=0$

$\qquad = 0;\ \ t < -1$ $\qquad$ $\therefore t = -1$

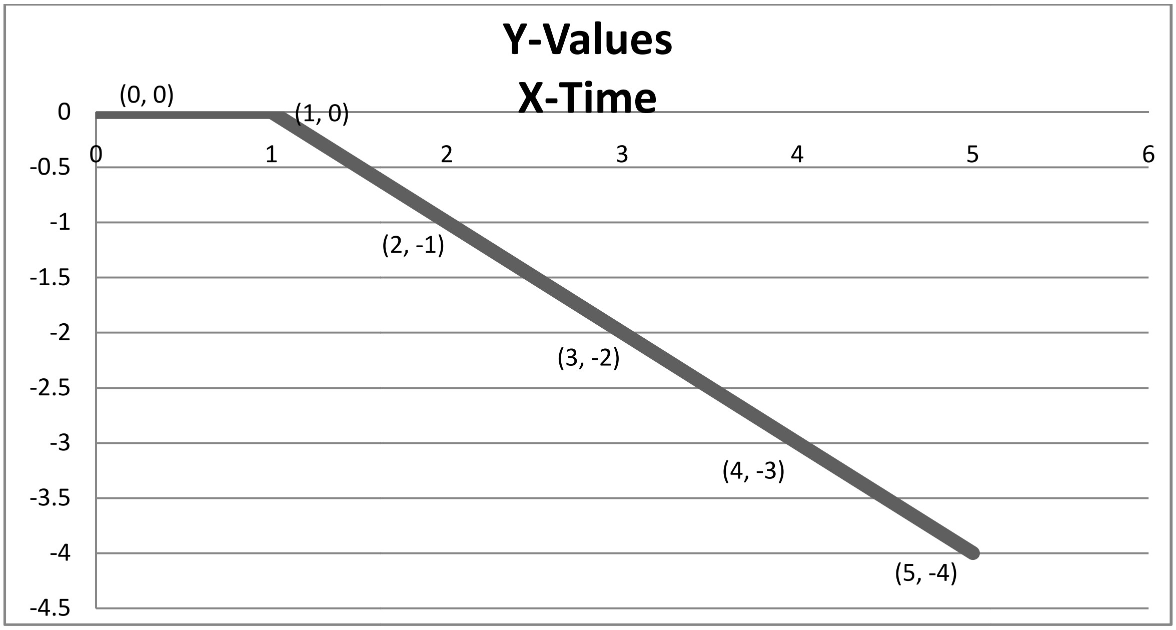
t	-1	0	1	2	3	4
$-r(t+1) = -(t+1)$	0	-1	-2	-3	-4	-5



$$\therefore -r(t-1) = -(t-1) \qquad ; t \geq 1$$
$$= 0 \qquad ; t < 1$$

$$Here,\ t - 1 = 0$$
$$\therefore t = 1$$

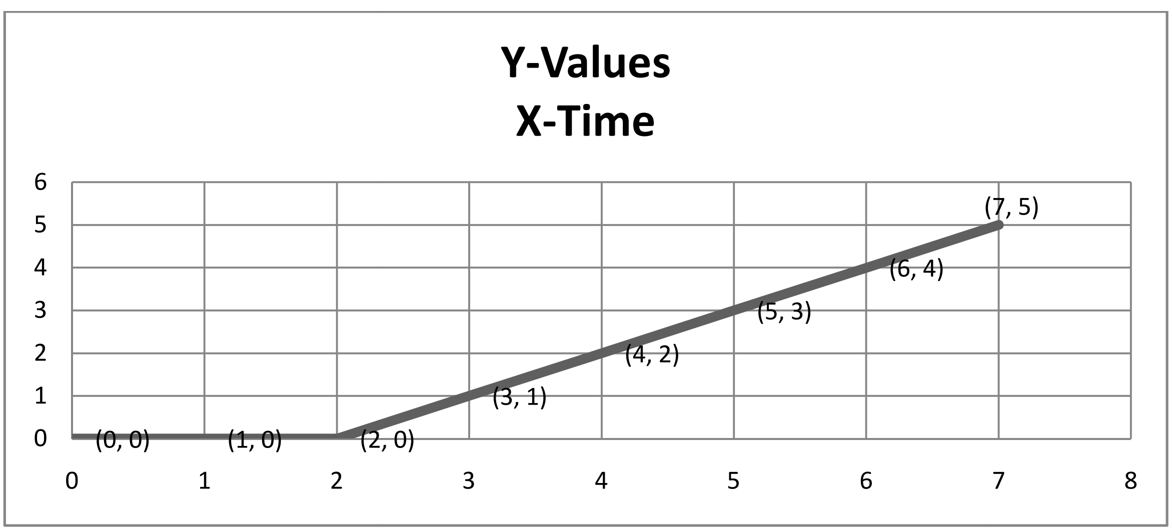
t	1	2	3	4	5
$-r(t-1) = -(t-1)$	0	-1	-2	-3	-4



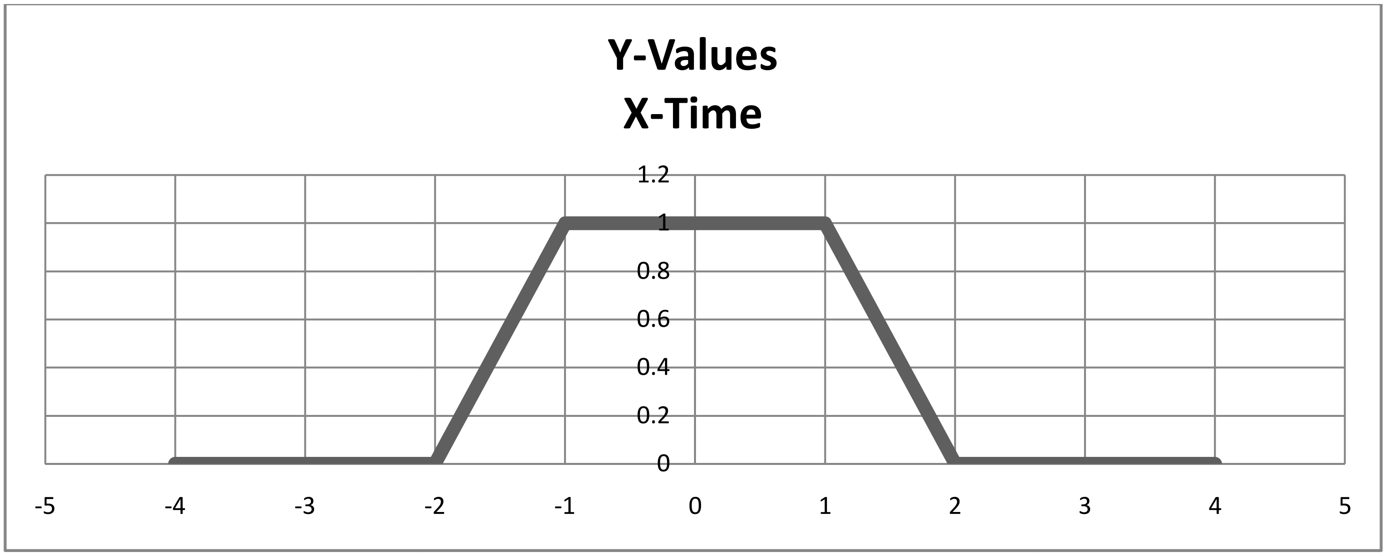
$$r(t-2) = (t-2) \quad ; \; t \geq 2 \qquad\qquad Here, t-2 = 0$$

$$= 0 \quad ; t < 2 \qquad\qquad \therefore t = 2$$

t	2	3	4	5	6	7
$r(t-2) = t-2$	0	1	2	3	4	5



$$\therefore x(t) = r(t+2) - r(t+1) - r(t-1) + r(t-2)$$



Problem 31: Impulse Function

The Impulse function $\delta(t)$ is defined as

$\delta(t) = 1$	$t = 0$
$= 0$	**Otherwise**

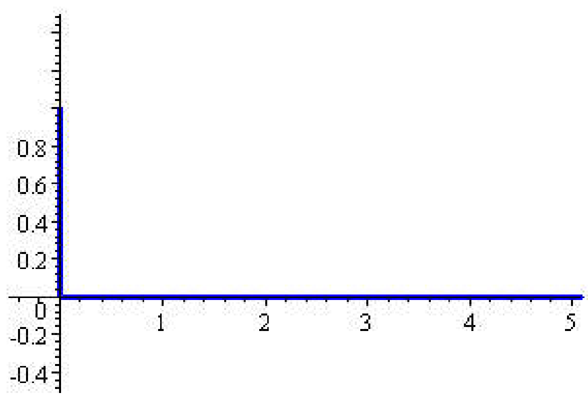
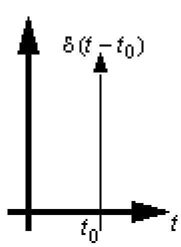


Figure 110



Delta (Impulse) Function

Example 91: Find $\mathbf{L[u(t-a)] = \frac{e^{-as}}{s}}$

Proof: We have

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

Here, $\mathbf{f(t) = u(t-a)}$

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$

$$\mathbf{L(u(t-a)) = \int_0^\infty u(t-a)e^{-st}dt}$$

$$\mathbf{L(u(t-a)) = \int_0^a 0.e^{-st}dt + \int_a^\infty 1.e^{-st}dt}$$

[The unit function $\mathbf{u(t-a)}$ is defined as

$\mathbf{u(t) = 1 \qquad t \geq a}$

$\mathbf{\quad = 0 \qquad t < a}$

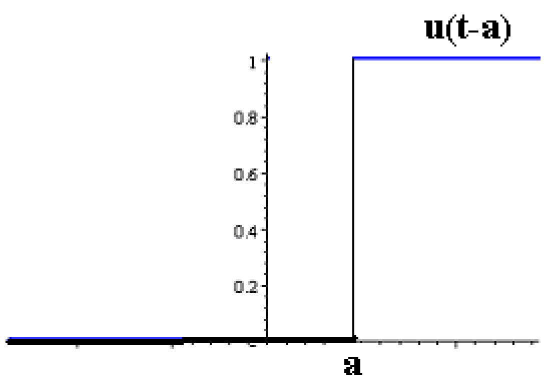


Figure 111

$$\mathbf{L(u(t-a)) = \int_0^a 0.e^{-st}dt + \int_a^\infty 1.e^{-st}dt}$$

$$\mathbf{= 0 + \int_a^\infty 1.e^{-st}dt}$$

$$\mathbf{= \int_a^\infty e^{-st}dt}$$

$$\mathbf{= \left[\frac{e^{-st}}{-s}\right]_a^\infty}$$

$$= -\frac{1}{s}\left[e^{-s\times\infty} - e^{-s\times a}\right]$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-as}\right]$$

$$= -\frac{1}{s}\left[\frac{1}{e^{\infty}} - e^{-as}\right]$$

$$= -\frac{1}{s}\left[\frac{1}{\infty} - e^{-as}\right]$$

$$= -\frac{1}{s}\left[0 - e^{-as}\right]$$

$$= +\frac{1}{s}\left[e^{-as}\right]$$

$$= \frac{e^{-as}}{s}$$

$$\therefore L(u(t-a)) = \frac{e^{-as}}{s} \text{ (Proved)}$$

$$\therefore L(u(t-2)) = \frac{e^{-2s}}{s}$$

Example 92: Express the following function in terms of unit step functions and find its Laplace transform:

$$f(t) = \begin{cases} 8\ ; & t < 2 \\ 6\ ; & t > 2 \end{cases}$$

Solution:

We have

$$L(u(t-a)) = \frac{e^{-as}}{s}$$

$$\therefore L(u(t-2)) = \frac{e^{-2s}}{s} \text{--------------} (i)$$

Given,

$$f(t) = \begin{cases} 8\ ; & t < 2 \\ 6\ ; & t > 2 \end{cases}$$

$$\mathrm{f(t)} = \begin{cases} 8+0; & \mathrm{t} < 2 \\ 8-2\ ; & \mathrm{t} > 2 \end{cases}$$

$$\mathrm{f(t)} = 8 + \begin{cases} 0\ ; & \mathrm{t} < 2 \\ -2\ ; & \mathrm{t} > 2 \end{cases}$$

$$\mathrm{f(t)} = 8 + (-2)\begin{cases} 0\ ; & \mathrm{t} < 2 \\ 1\ ; & \mathrm{t} > 2 \end{cases}$$

$$\mathrm{f(t)} = 8 + (-2)\begin{cases} 1\ ; & \mathrm{t} > 2 \\ 0\ ; & \mathrm{t} < 2 \end{cases}$$

$$\mathbf{f(t) = 8 + (-2)u(t-2)}$$
$$\mathbf{f(t) = 8 - 2u(t-2)}$$
$$\mathbf{L\{f(t)\} = L\{8 - 2u(t-2)\}}$$
$$\mathbf{L\{f(t)\} = L\{8\} - 2Lu(t-2)\}}$$
$$\mathbf{L\{f(t)\} = 8L\{1\} - 2Lu(t-2)\}}$$
$$\mathbf{L\{f(t)\} = 8 \times \frac{1}{s} - 2 \times \frac{e^{-2s}}{s}}$$
$$[\because L(1) = \frac{1}{S}(\textit{from example } 55)] \ \& \ \mathrm{L[u(t-2)]} = \frac{e^{-2s}}{s} (\text{from example } 91)]$$

Problem 32:

Product of $u(t)$ vs. Shifting the Function Along the *t*-axis

1. $f(t).u(t)$

The $\mathbf{f(t)}$ part begins at $\mathbf{t = 0}$

2. $f(t).u(t-a)$

The $\mathbf{f(t)}$ part begins at $\mathbf{t = a}$

3. $f(t-a).u(t)$

The $\mathbf{f(t)}$ part has been **shifted** to the right by *a* units and begins at $\mathbf{t = 0}$

4. $f(t-a).u(t-a)$

The $\mathbf{f(t)}$ part has been **shifted** to the right by *a* units and begins at $\mathbf{t = a}$

Example 93: $g(t) = \sin t.u(t)$

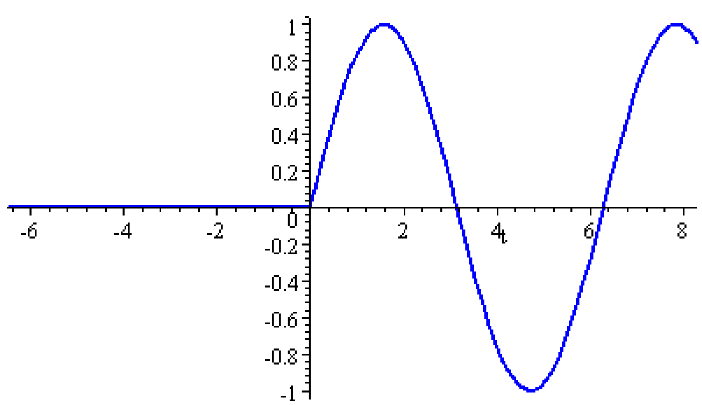


Figure 112: the $\mathbf{\sin t}$ part starts at $\mathbf{t = 0}$

Example 94: $g(t) = \sin t.u(t - 0.7)$

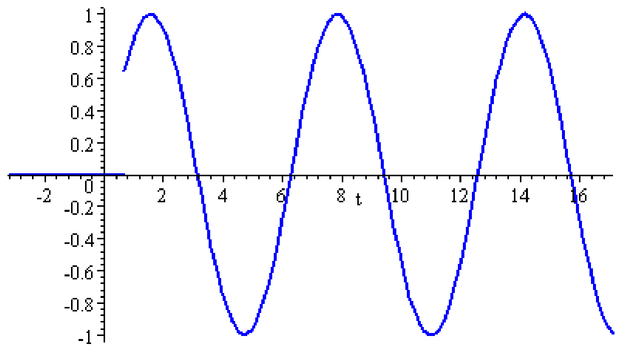


Figure 113: the $\sin t$ part starts at $t = 0.7$

Example 95: $g(t) = \sin(t - 0.7).u(t)$

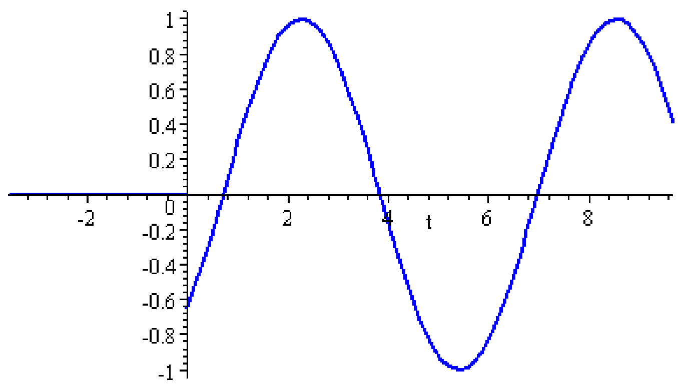


Figure 114: the $\sin t$ part has been shifted 0.7 units to the right, and it starts at $t = 0$

Example 96: $g(t) = \sin(t - 0.7).u(t - 0.7)$

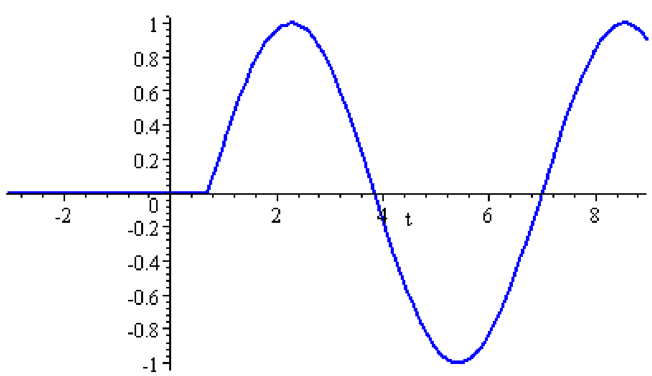


Figure 115: the $\sin t$ part has been shifted 0.7 units to the right, and it starts at $t = 0.7$

Example 97: If $\mathbf{f(t) = \sin t}$ then the graph of $\mathbf{g(t) = \sin t.u(t - 2\pi)}$ is

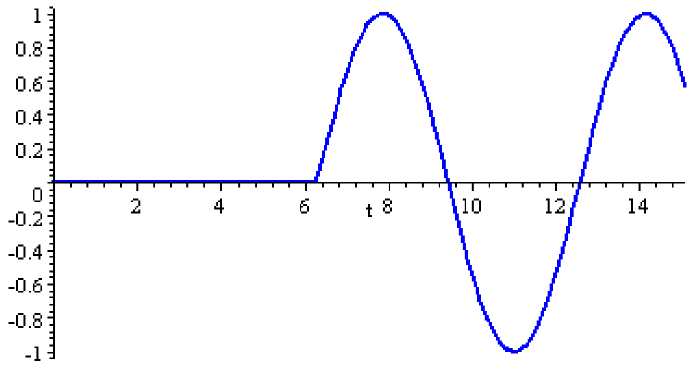


Figure 116: The $\mathbf{\sin t}$ portion starts at $\mathbf{t = 2\pi}$ because we have multiplied $\mathbf{\sin t}$ by $\mathbf{u(t - 2\pi)}$

Example 98: If $\mathbf{f(t) = 10e^{-2t}}$, then the graph of $\mathbf{g(t) = 10e^{-2t}.u(t - 5)}$ is

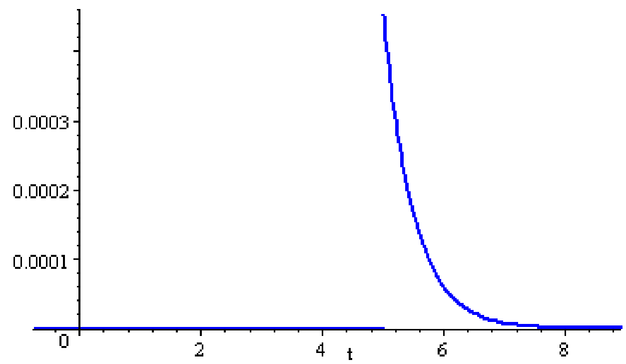


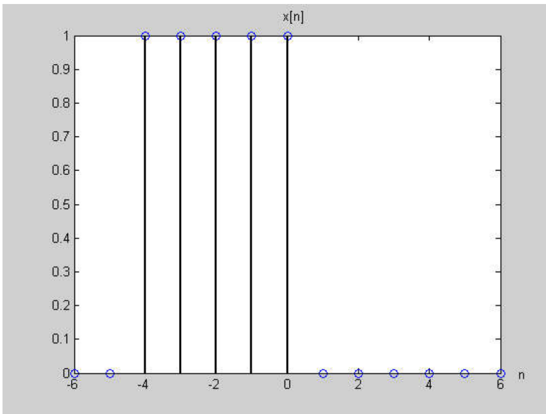
Figure 117: The portion $\mathbf{10e^{-2t}}$ starts at $\mathbf{t = 5}$

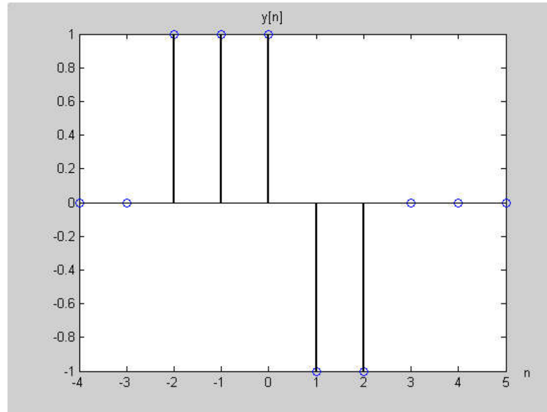
Home Task:

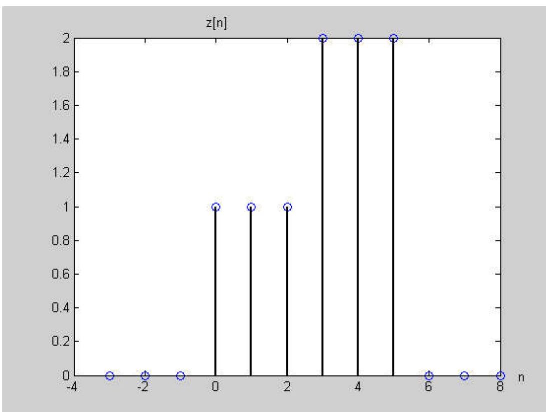
Draw the graph of $\mathbf{f(t) = 4u(t) - 8u(t - 1) + 4u(t - 2)}$

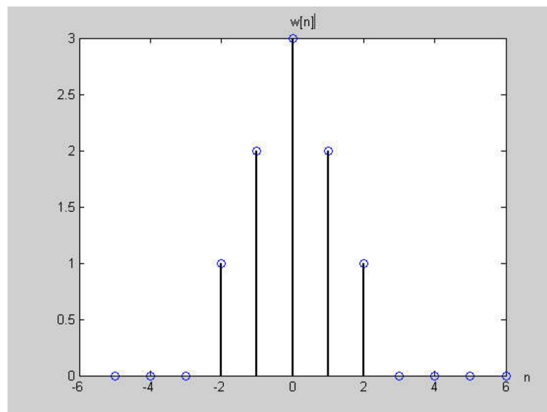
1. Consider the discrete-time signals depicted in the following figures. Evaluate the convolution sums indicated below:

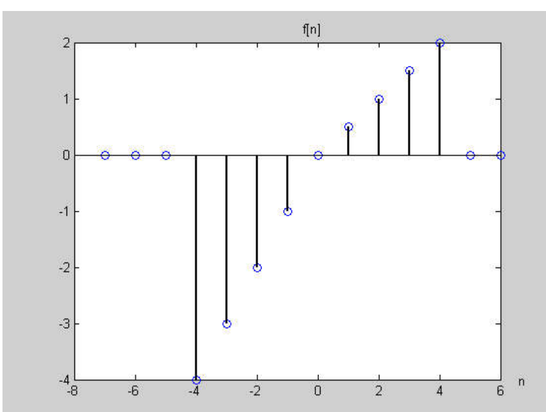
1. m[n] = x[n]*z[n]
2. m[n] = x[n]*y[n]
3. m[n] = x[n]*f[n]
4. m[n] = x[n]*g[n]
5. m[n] = y[n]*z[n]
6. m[n] = y[n]*g[n]
7. m[n] = y[n]*w[n]
8. m[n] = y[n]*f[n]
9. m[n] = z[n]*g[n]
10. m[n] = w[n]*g[n]
11. m[n] = f[n]*g[n]

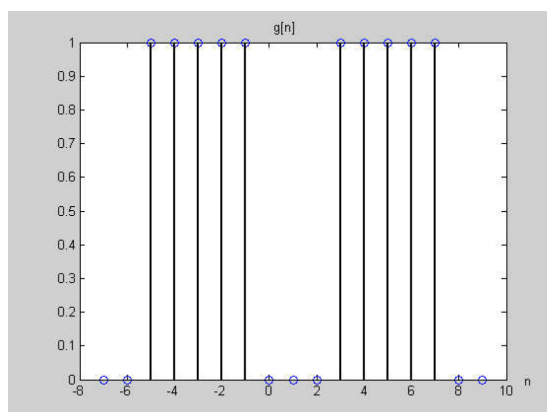






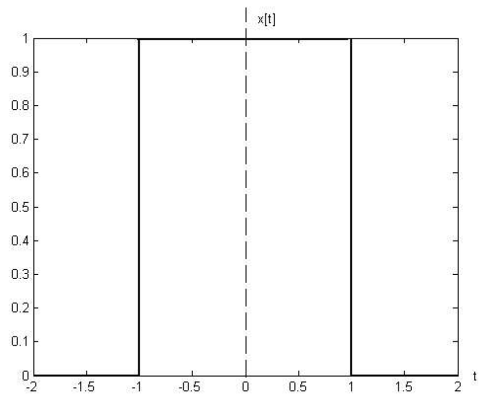


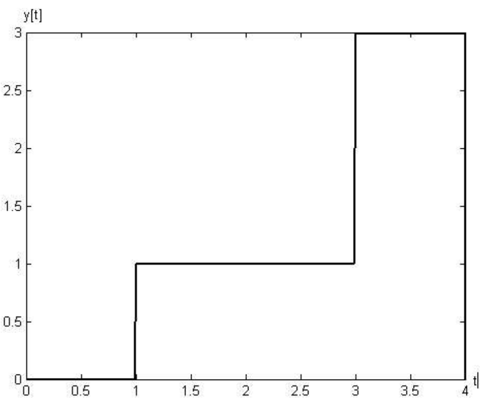


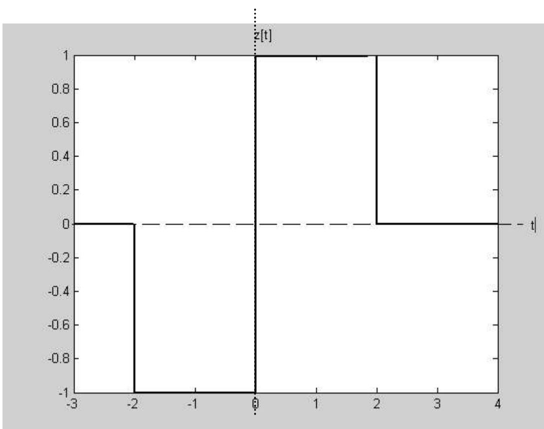


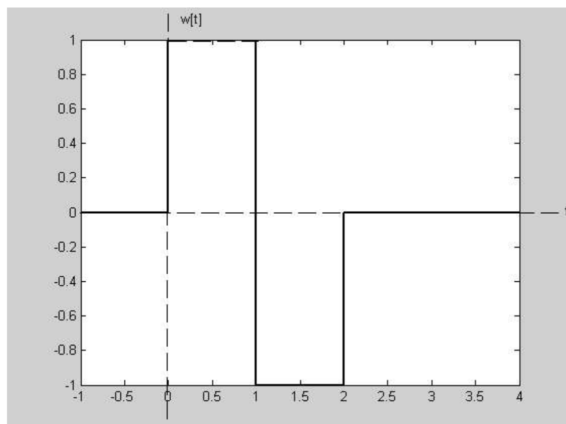
2. Consider the Continuous-time signals depicted in the following figures. Evaluate the convolution Integrals indicated below:

i) m(t)=x(t)*y(t)
ii) m(t)=x(t)*z(t)
iii) m(t)=y(t)*z(t)
iv) m(t)=y(t)*w(t)









3. Sketch the waveforms of the following signals:

a) x(t) = u(t)-u(t-2)

b) x(t) = u(t+1)-2u(t)+u(t-1)

c) $\boldsymbol{x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)}$

d) $\mathbf{x(t) = 4u(t-1) - 8u(t-4) + 4u(t-6)}$

e) $\mathbf{x(t) = r(t-1) + u(t) - u(t-1) - 2r(t-2) + r(t-3)}$

f) $\mathbf{x(t) = r(t+1) - r(t) + r(t-2)}$

g) $\mathbf{x(t) = r(t+2) - r(t+1) - r(t-1) + r(t-2)}$

Prblem 33: Inverese Laplaces transform

Example 99:

$If\ L[f(t)] = F(S), Then\ L^{-1}[F(S)] = f(t)$

$Where\ L^{-1}\ is\ called\ the\ inverse\ Laplace\ transform\ operator.$

$(1) L^{-1}\left(\frac{1}{S}\right) = 1 \left[\because (1) L[f(t)) = L(1) = \frac{1}{s}\right]$

$(2) L^{-1}\left(\frac{a}{s}\right) = a \left[\because L(f(t)) = L(a) = \frac{a}{s}\right]$

$(3) L^{-1}\left(\frac{1}{s^2}\right) = t \left[\because L(f(t)) = L(t) = \frac{1}{s^2}\right]$

$(4) L^{-1}\left(\frac{1}{s-a}\right) = e^{at} \left[\because L(f(t)) = L(e^{at}) = \frac{1}{s-a}\right]$

$(5) L^{-1}\left(\frac{1}{s-2}\right) = e^{2t} \left[\because L(f(t)) = L(e^{2t}) = \frac{1}{s-2}\right]$

$(6) L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at [\because L(f(t)) = L(\sin at) = \frac{a}{s^2+a^2}]$

$(7) L^{-1}\left(\frac{2}{S^2+2^2}\right) = \sin 2t [\because L(f(t)) = L(\sin 2t) = \frac{2}{s^2+2^2}]$

$(8) L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at [\because L(f(t)) = L(\cos at) = \frac{s}{s^2+a^2}]$

$(9) L^{-1}\left(\frac{s}{s^2+2^2}\right) = \cos 2t [\because L(f(t)) = L(\cos 2t) = \frac{s}{s^2+2^2}]$

$(10) L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sin hat [\because L(f(t)) = L(\sin hat) = \frac{a}{s^2-a^2}]$

$(11) L^{-1}\left(\frac{2}{S^2-2^2}\right) = \sin h2t [\because L(f(t)) = L(\sin h2t) = \frac{2}{s^2-2^2}]$

$(12) L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cos hat [\because L(f(t)) = L(\cos hat) = \frac{s}{s^2-a^2}]$

$(13) L^{-1}\left(\frac{s}{s^2-2^2}\right) = \cos h2t [\because L(f(t)) = L(\cos h2t) = \frac{s}{s^2-2^2}]$

Example 100 Find the inverse Laplace transforms of the following:

(i) $\frac{1}{s-2}$, (ii) $\frac{s}{s^2-16}$, (iii) $\frac{5}{s^2+25}$, (iv) $\frac{1}{(s+3)^2-4}$, (v) $\frac{1}{2s-7}$

Answers:

(i) Given $f(s) = \frac{1}{s-2}$

We have, $\therefore L(f(t)) = L(e^{at}) = \frac{1}{s-a}$ **[Example 58]**

$$\therefore e^{at} = L^{-1}\left(\frac{1}{s-a}\right)$$

$$\therefore e^{2t} = L^{-1}\left(\frac{1}{s-2}\right)$$

$$\therefore L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$$ Answer

(ii) $L^{-1}\left(\frac{s}{s^2-16}\right)=?$

We have, $\therefore L(f(t))=$ $\mathrm{L}(\cosh at)=\frac{s}{s^2-a^2}$ **[Example 63]**

$\therefore \cosh at = L^{-1}\left(\frac{s}{s^2-a^2}\right)$

$\therefore \cosh 4t = L^{-1}\left(\frac{s}{s^2-4^2}\right)$

$\therefore L^{-1}\left(\frac{s}{s^2-4^2}\right)=\cosh 4t$ Answer

(iii) We have, $\therefore L(f(t))=L\ (\sin at)=\frac{a}{s^2+a^2}$ **[Example 60]**

$\therefore \sin at = L^{-1}\left(\frac{a}{s^2+a^2}\right)$

$\therefore \sin 5t = L^{-1}\left(\frac{5}{s^2+5^2}\right)$

$\therefore L^{-1}\left(\frac{5}{s^2+5^2}\right)=\sin 5t$

$\therefore L^{-1}\left(\frac{5}{s^2+5^2}\right)=\sin 5t$ *Answer*

Example 101:

Find inverse Laplace transform of: $\frac{s+4}{s(s-1)(s-2)}$

Solution:

Let,

$$\frac{s+4}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{(s-1)}+\frac{C}{(s-2)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (i)$$

Multiplying by $s(s-1)(s-2)$ in both sides

$$\Rightarrow \frac{s+4}{s(s-1)(s-2)}\times s(s-1)(s-2)=A\frac{s(s-1)(s-2)}{s}+B\frac{s(s-1)(s-2)}{(s-1)}+C\frac{s(s-1)(s-2)}{(s-2)}$$

$$\Rightarrow s+4=A(s-1)(s-2)+Bs(s-2)+Cs(s-1) \quad \ldots\ldots\ldots\ldots (ii)$$

Put $s=0$ in equation (ii),

$\Rightarrow 0+4=A(0-1)(0-2)+B\times 0(0-2)+C\times 0(0-1)$

$\Rightarrow 4=2A$

$\therefore A = 2$

Put $s - 1 = 0$, i.e. $s = 1$ in equation (ii),

$\Rightarrow 1 + 4 = A(1-1)(1-2) + B \times 1(1-2) + C \times 1(1-1)$

$\Rightarrow 5 = 0 - B + 0$

$\therefore B = -5$

Put $s - 2 = 0$, i.e. $s = 2$ in equation (ii),

$\Rightarrow 2 + 4 = A(2-1)(2-2) + B \times 2(2-2) + C \times 2(2-1)$

$\Rightarrow 6 = 0 + 0 + C(4-2)$

$\Rightarrow 6 = 0 + 0 + 2C$

$\therefore C = 3$

Putting the value of A, B, C in equation (i), we get,

$$\frac{s+4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$$

$$\frac{s+4}{s(s-1)(s-2)} = \frac{2}{s} + \frac{-5}{(s-1)} + \frac{3}{(s-2)}$$

$$\therefore\ L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right) = L^{-1}\left(\frac{2}{s}\right) + L^{-1}\left(\frac{-5}{s-1}\right) + L^{-1}\left(\frac{3}{s-2}\right)$$

$$L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right) = 2L^{-1}\left(\frac{1}{s}\right) - 5L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{s-2}\right) - - - - - - - - - (iii)$$

Since

01. We have $\therefore L(f(t)) = L(1) = \frac{1}{s}$ [Example 55]

$\therefore 1 = L^{-1}(\frac{1}{s})$

$\therefore L^{-1}(\frac{1}{s}) = 1$

02. We have, $\therefore L(f(t)) = L(e^{at}) = \frac{1}{s-a}$ [Example 58]

$\therefore e^{at} = L^{-1}(\frac{1}{s-a})$

$\therefore e^{t} = L^{-1}(\frac{1}{s-1})$

$\therefore L^{-1}(\frac{1}{s-1}) = e^{t}$

03. $\therefore L(f(t)) = L(e^{at}) = \dfrac{1}{s-a}$ [Example 58]

$$\therefore e^{at} = L^{-1}(\frac{1}{s-a})$$

$$\therefore e^{2t} = L^{-1}(\frac{1}{s-2})$$

$$\therefore L^{-1}(\frac{1}{s-2}) = e^{2t} \text{ Answer}$$

Putting these values in (iii), we get

$$L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right) = 2L^{-1}\left(\frac{1}{s}\right) - 5L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{s-2}\right)$$

$$= 2.1 - 5e^{t} + 3e^{2t} \text{ Answer}$$

Convolution Sum

The following steps are to be taken

i. Folding
ii. Shifting
iii. Multiplication
iv. Summation

1st times:

i. Folding
ii. Multiplication
iii. Summation

2nd times and more

i. Shifting
ii. Multiplication
iii. Summation

Example 102:

Evaluate the convolution sums of y [n] = x [n]*h [n]

Where,

$$x[n] = \begin{cases} 1, & n = 0 \\ 3, & n = 1 \end{cases} \quad \text{and} \quad h[n] = \begin{cases} 2, & n = 0 \\ 1, & n = 1 \end{cases}$$; ***n represents the time index***

MATLAB

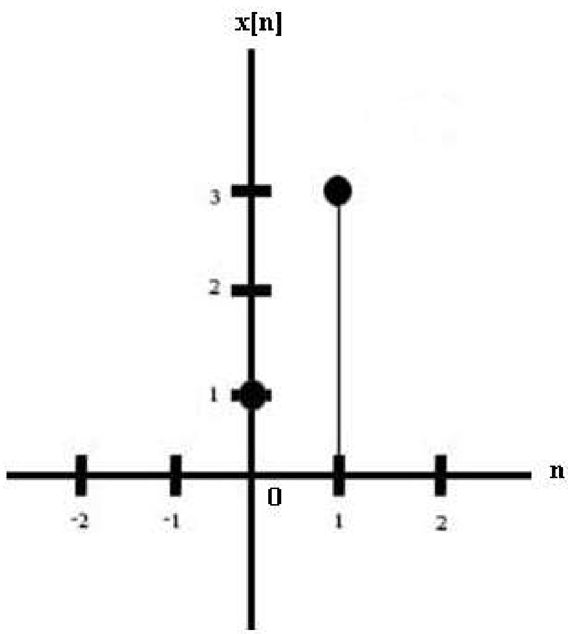
x=[1 3]

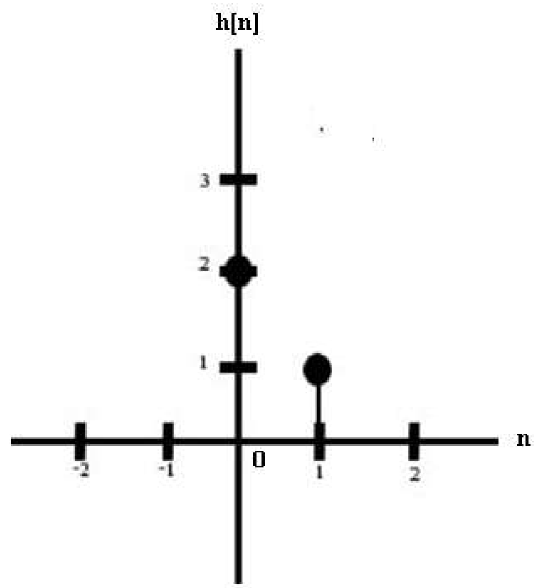
h=[2 1]

y =conv(x,h)

y= [2 7 3]

Solution:





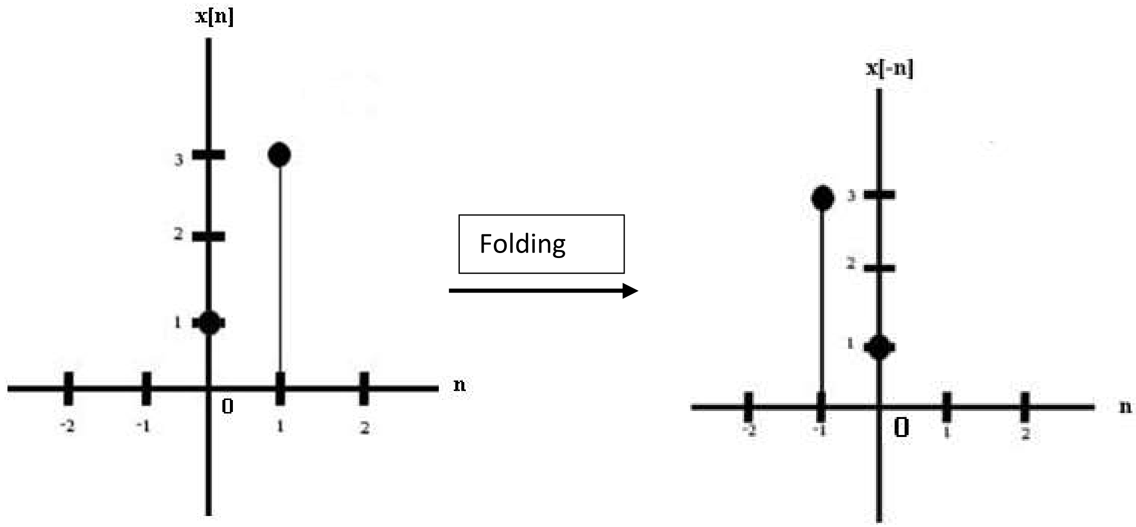
1st time:

(i). Folding:

$$x[n] = x[-n]$$

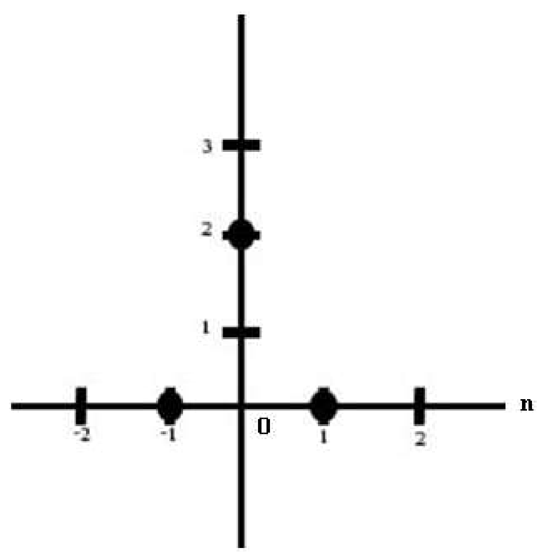
i.e. $x[0] = x[0]$

$x[1] = x[-1]$



(ii). Multiplication:

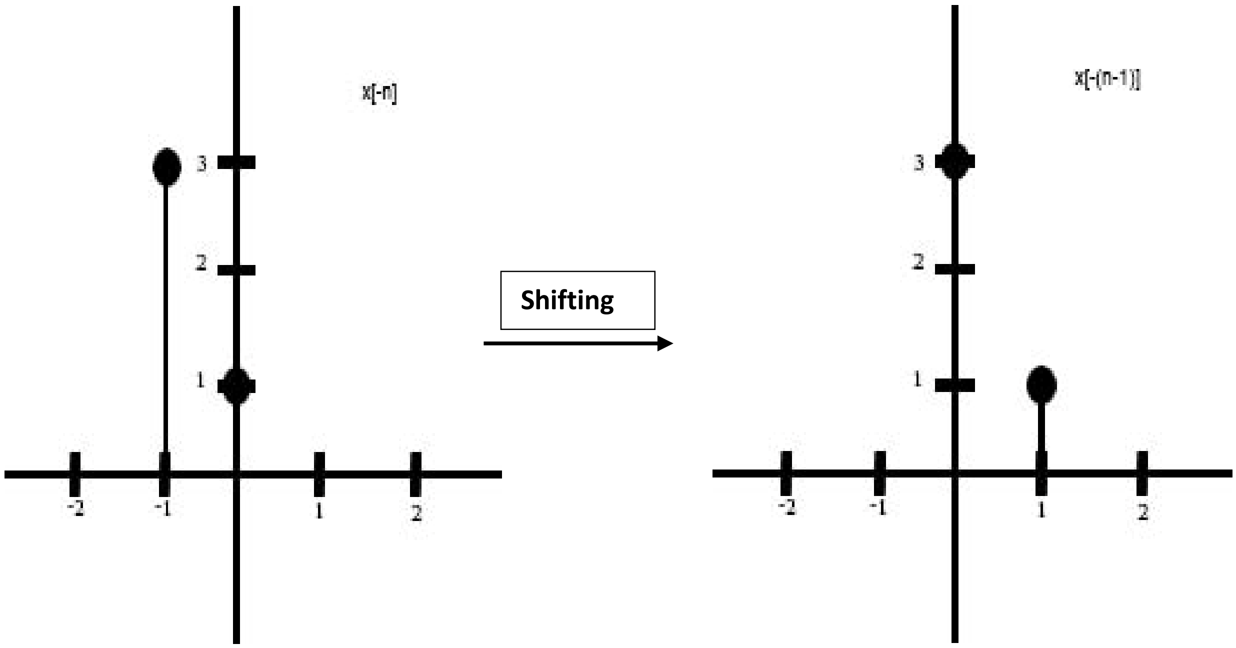
$$x[-n] * h[n]$$



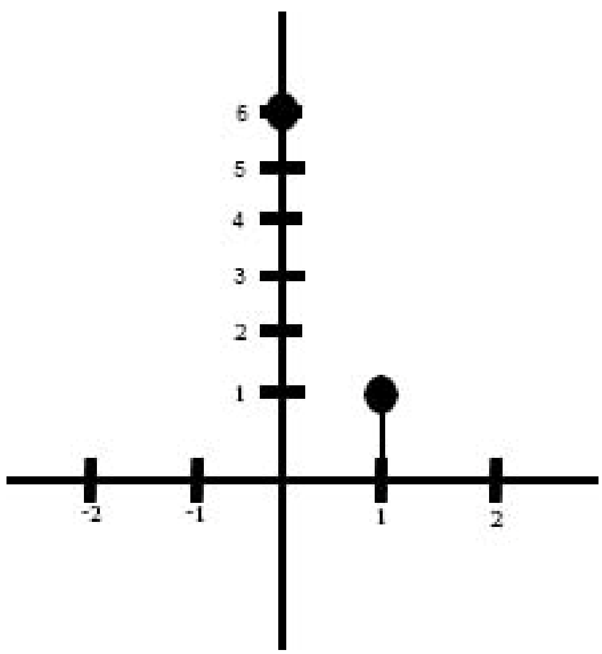
(iii). Summation: y[0] = 0 + 2 + 0 = 2

2nd time:

(i). Shifting:



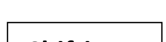
(ii) Multiplication: $x[-(n-1)] * h[n]$

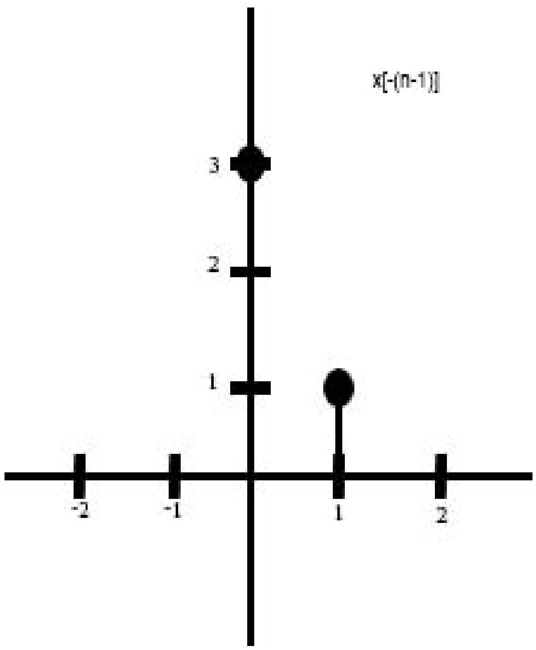


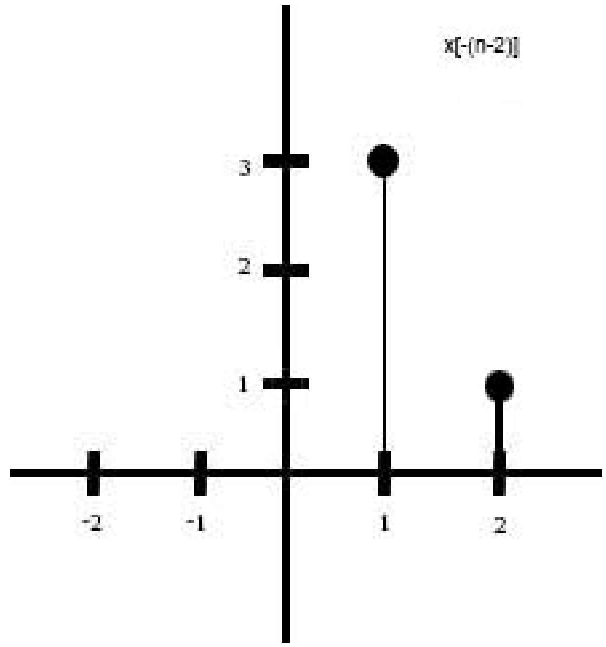
(iii). Summation: $y[1] = 0 + 6 + 1 + 0 = 7$

3rd time:

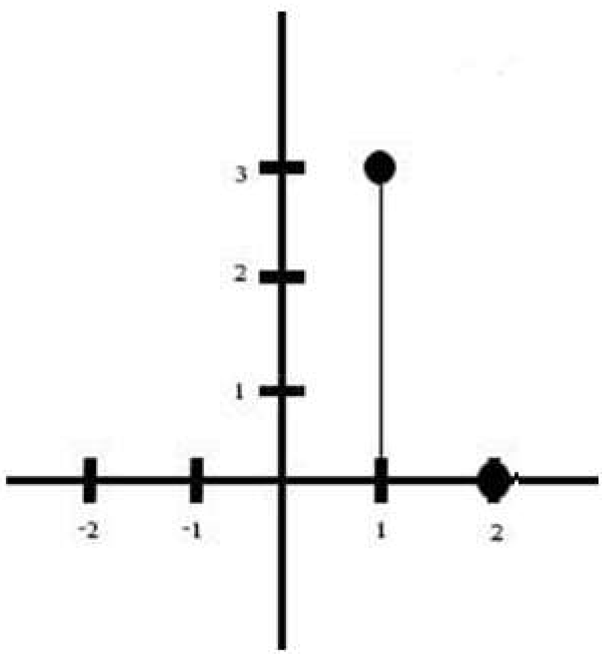
(i). Shifting:





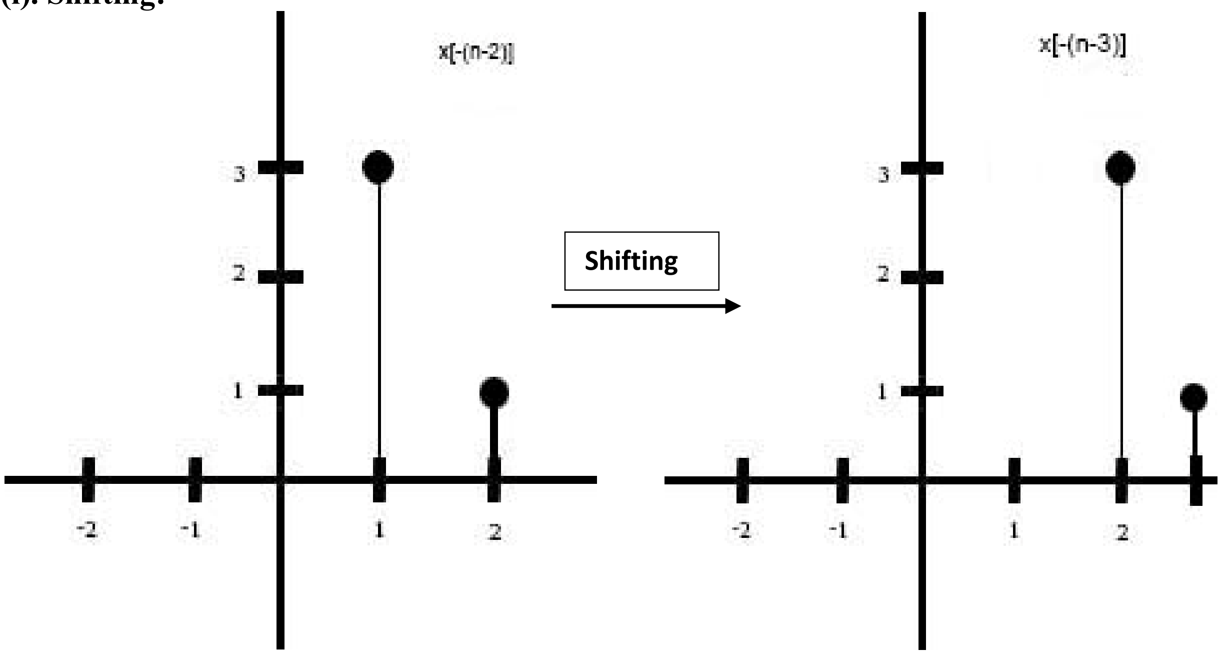


(ii). Multiplication: x[-(n-2)] * h[n]

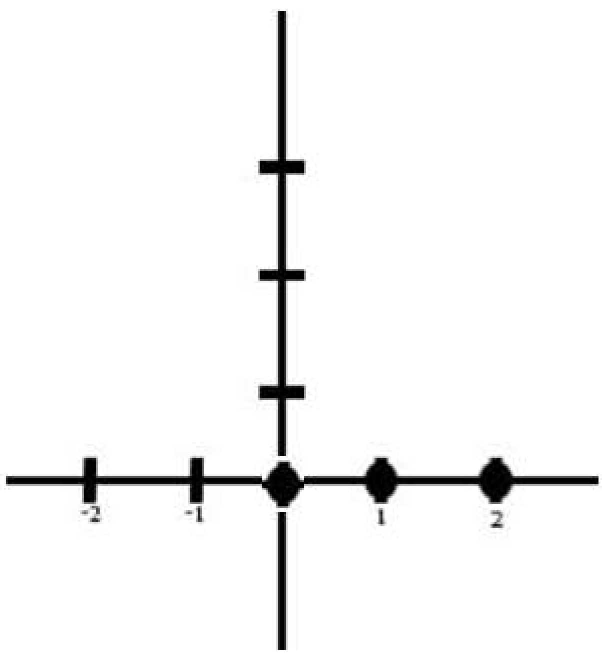


(iii). Summation: y [2] = 0 + 0 + 3 + 0 = 3

4^{th} time:
(i). Shifting:



(ii). Multiplication: x[-(n-3)] * h[n]



iii) Summation: y[3] = 0+0+0+0=0

Finally we get,

∴ $y[0] = 2$
$y[1] = 7$
$y[2] = 3$

∴ Convolution Sum: $y[n] = [2\ 7\ 3]$