04. Quadrature

Example 180: Find the area common to the two parabola $\mathbf{x^2 = 4ay}$ and $\mathbf{y^2 = 4ax}$

Solution: Given curve are

$\mathbf{x^2 = 4ay}$ ------------------------------ (i)

$\mathbf{y^2 = 4ax}$ ------------------------------ (ii)

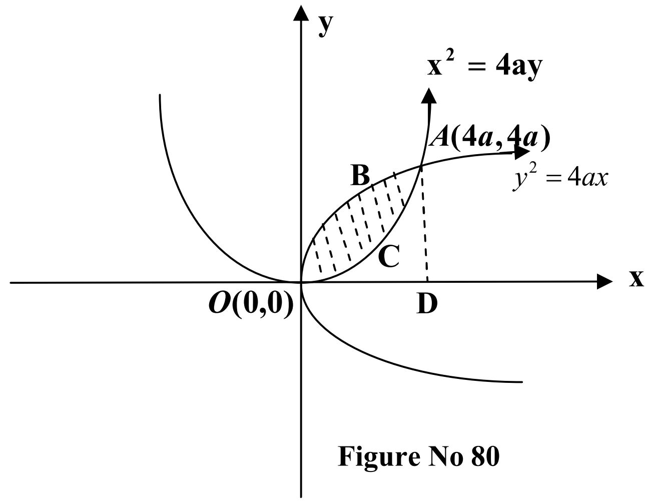

Figure No 80

Since equation (i) contain only even power of x. So this curve is symmetrical about $y-$axis and also curve (ii) is symmetrical about $x-$axis.

From (i) and (ii) we have

$$\mathbf{x^2 = 4ay}$$
$$\Rightarrow \mathbf{(x^2)^2 = (4ay)^2}$$
$$\Rightarrow \mathbf{x^4 = 16a^2y^2}$$
$$\Rightarrow \mathbf{x^4 = 16a^2 4ax} \ [\because \mathbf{y^2 = 4ax}]$$
$$\Rightarrow \mathbf{x^4 - 16 \cdot 4a^3x = 0}$$
$$\Rightarrow \mathbf{x(x^3 - 64a^3) = 0}$$
$$\therefore \quad \Rightarrow \mathbf{x = 0}\ ,\ \mathbf{x^3 = 64a^3}\ ;\text{ i.e. } \mathbf{x = 4a}$$

Putting the value of x in (ii),

$$\mathbf{y^2 = 4ax}$$
$$\Rightarrow \mathbf{y^2 = 4a.0\,[x = 0]}$$
$$\Rightarrow \mathbf{y^2 = 0}$$
$$\Rightarrow \mathbf{y = 0}$$

Again,

$$\mathbf{y^2 = 4ax}$$
$$\Rightarrow \mathbf{y^2 = 4a.4a\,[x = 4a]}$$
$$\Rightarrow \mathbf{y^2 = 16a^2}$$
$$\Rightarrow \mathbf{y = 4a}$$

Hence the coordinates of O (0, 0) and A (4a, 4a)

Now

$\because \boldsymbol{y^2 = 4ax;} \therefore \boldsymbol{y = \sqrt{4ax}}$ and

$\because \boldsymbol{x^2 = 4ay;} \therefore \boldsymbol{y = \dfrac{x^2}{4a}}$

Let $\boldsymbol{y_1 = f_1(x) = \sqrt{4ax}}$ ---(iii)

$\boldsymbol{\&\ y_2 = f_2(x) = \dfrac{x^2}{4a}}$ --(iv)

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We are to find the area of OBACO. It can be written as,

Area of OBACO $=$ area of OBADO $-$ area of OCADO [Upper Curve-Lower Curve]

$$= \int_0^{4a} f_1(x)dx - \int_0^{4a} f_2(x)dx$$

$$= \int_0^{4a} y_1 dx - \int_0^{4a} y_2 dx$$

$$= \int_0^{4a} \sqrt{4ax}\, dx - \int_0^{4a} \frac{x^2}{4a} \cdot dx \quad [From\ iii\ \&\ iv]$$

$$= 2\sqrt{a}\left[\frac{2}{3} \cdot x^{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}(4a)^{\frac{3}{2}} - 0 - \frac{1}{4a}\frac{(4a)^3}{3} - 0$$

$$= \frac{4\sqrt{a}}{3}(4)^{\frac{3}{2}} a^{\frac{3}{2}} - \frac{1}{4a}\frac{4^3 a^3}{3} = \frac{4\sqrt{a}}{3}\left(2^2\right)^{\frac{3}{2}} a^{\frac{3}{2}} - \frac{1}{4a}\frac{4^3 a^3}{3}$$

$$= \frac{4\sqrt{a}}{3} 2^3 a^{\frac{3}{2}} - \frac{1}{4a}\frac{4^3 a^3}{3} = \frac{4\sqrt{a}}{3}.8.a^{\frac{3}{2}} - \frac{4^2 a^2}{3}$$

$$= \frac{4a^{\frac{1}{2}}}{3}.8.a^{\frac{3}{2}} - \frac{16a^2}{3} = \frac{4.8}{3}.a^{\frac{1}{2}}..a^{\frac{3}{2}} - \frac{16a^2}{3} = \frac{4.8}{3}.a^{\frac{1}{2}+\frac{3}{2}} - \frac{16a^2}{3}$$

$$= \frac{4.8}{3}.a^{\frac{1+3}{2}} - \frac{16a^2}{3} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{32-16}{3}a^2 = \frac{16}{3}a^2$$

Therefore the required area is $\frac{16}{3}a^2$ square unit

OR

Given,

$y^2 = 4ax$ ------------------------------ (i)

$x^2 = 4ay$ ------------------------------ (ii)

From (i) and (ii) we have

$$x^2 = 4ay$$
$$\Rightarrow (x^2)^2 = (4ay)^2$$
$$\Rightarrow x^4 = 16a^2 y^2$$
$$\Rightarrow x^4 = 16a^2 4ax\ [\because y^2 = 4ax]$$

$$\Rightarrow \mathbf{x^4 - 16 \cdot 4a^3 x = 0}$$
$$\Rightarrow \mathbf{x(x^3 - 64a^3) = 0}$$
$$\therefore \quad \Rightarrow \mathbf{x = 0}\ ,\ \mathbf{x^3 = 64a^3}\ ;\text{ i.e. } \mathbf{x = 4a}$$

Putting the value of x in (i),

$$\mathbf{y^2 = 4ax}$$
$$\Rightarrow \mathbf{y^2 = 4a.0\,[x = 0]}$$
$$\Rightarrow \mathbf{y^2 = 0}$$
$$\Rightarrow \mathbf{y = 0}$$

Again,

$$\mathbf{y^2 = 4ax}$$
$$\Rightarrow \mathbf{y^2 = 4a.4a\,[x = 4a]}$$
$$\Rightarrow \mathbf{y^2 = 16a^2}$$
$$\Rightarrow \mathbf{y = 4a}$$

Hence the coordinates of O (0, 0) and A (4a, 4a)

Now

$\because y^2 = 4ax;\ \therefore y = \sqrt{4ax}$ and

$\because x^2 = 4ay;\ \therefore y = \dfrac{x^2}{4a}$

Let $y_1 = f_1(x) = \dfrac{x^2}{4a}$ ---(iii)

&

$y_2 = f_2(x) = \sqrt{4ax}$ -------------------------------------(iv)

We know,

$$\textbf{Area = upper curve − lower curve} = \int_a^b (y_2 - y_1)\,dx$$

$$= \int_0^{4a} (y_2 - y_1)dx$$

$$= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right)dx \quad [\textit{From iii \& iv}]$$

$$= \int_0^{4a} \sqrt{4ax}\,dx - \int_0^{4a} \frac{x^2}{4a} \cdot dx$$

$$= \int_0^{4a} 2\sqrt{a}\, x^{\frac{1}{2}} dx - \int_0^{4a} \frac{x^2}{4a} \cdot dx$$

$$= 2\sqrt{a}\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \cdot\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^{2+1}}{2+1}\right]_0^{4a}$$

$$= 2\sqrt{a}\left[\frac{2}{3} \cdot x^{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$=[\frac{4\sqrt{a}}{3}(4a)^{\frac{3}{2}}-0]-[\frac{1}{4a}\frac{(4a)^3}{3}-0]$$

$$=\frac{4\sqrt{a}}{3}(4)^{\frac{3}{2}}a^{\frac{3}{2}}-\frac{1}{4a}\frac{4^3a^3}{3}=\frac{4\sqrt{a}}{3}\left(2^2\right)^{\frac{3}{2}}a^{\frac{3}{2}}-\frac{1}{4a}\frac{4^3a^3}{3}$$

$$=\frac{4\sqrt{a}}{3}2^3a^{\frac{3}{2}}-\frac{1}{4a}\frac{4^3a^3}{3}=\frac{4\sqrt{a}}{3}.8.a^{\frac{3}{2}}-\frac{4^2a^2}{3}$$

$$=\frac{4a^{\frac{1}{2}}}{3}.8.a^{\frac{3}{2}}-\frac{16a^2}{3}=\frac{4.8}{3}.a^{\frac{1}{2}}.a^{\frac{3}{2}}-\frac{16a^2}{3}=\frac{4.8}{3}.a^{\frac{1}{2}+\frac{3}{2}}-\frac{16a^2}{3}$$

$$=\frac{4.8}{3}.a^{\frac{1+3}{2}}-\frac{16a^2}{3}=\frac{32}{3}a^2-\frac{16}{3}a^2=\frac{32-16}{3}a^2=\frac{16}{3}a^2$$

Therefore the required area is $\frac{16}{3}a^2$ square unit

Example 181: Find the area of the astroid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

Solution: Given Equation $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ ------------------------------(i)

If we replace x for x and –y for y then equation (i) is unchanged and if we replace x for y and y for x then equation is unchanged. So this curve in symmetric about the both axis and the line $y=x$ and it meets the axis at points $A(a,0)$, $B(0,a)$, $C(-a,0)$, and $D(0,-a)$. Draw the curve.

From equation (i) we have, $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

$$\Rightarrow y^{\frac{2}{3}}=a^{\frac{2}{3}}-x^{\frac{2}{3}}$$

$$\Rightarrow \left(y^{\frac{2}{3}}\right)^{\frac{1}{2}}=\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

$$\Rightarrow \left(y^{\frac{1}{3}}\right)=\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{1}{2}}$$

$$\Rightarrow \left(y^{\frac{1}{3}}\right)^3=\left(\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^{\frac{1}{2}}\right)^3$$

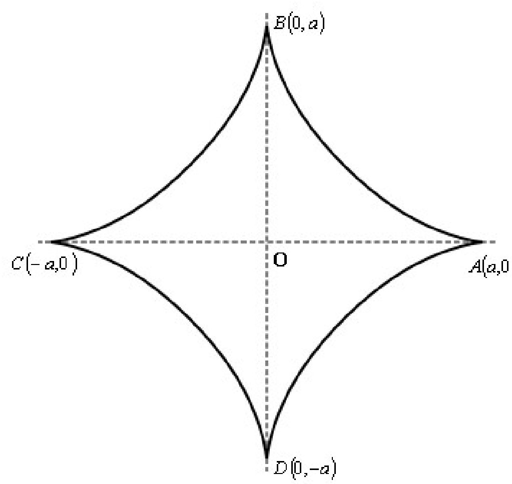

Figure No 81

$$\Rightarrow y = \left(\left(\left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{1}{2}}\right)^{3}\right) = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} \text{-------------------------------------(ii)}$$

Now we are to find the area of ABCDA. It can be written as

Area of ABCDA $= \mathbf{4 \times (area\ of\ ABOA)}$

$$= 4\int_0^a y\,dx = 4\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx \qquad [\because y = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}}]$$

$$= 4\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx \text{ --------------------------(iii)}$$

Let,

$$x = a\sin^3\theta$$

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}\left(a\sin^3\theta\right)$$

$$\Rightarrow \frac{dx}{d\theta} = a\frac{d}{d\theta}\left(\sin^3\theta\right)$$

$$\Rightarrow \frac{dx}{d\theta} = a \times 3\sin^2\theta\frac{d}{d\theta}(\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a \times 3\sin^2\theta\cos\theta$$

$$\Rightarrow \therefore dx = 3a\sin^2\theta\cos\theta d\theta$$

$x = a\sin^3\theta$	0	a
θ	$x = a\sin^3\theta$ $0 = a\sin^3\theta$ $0 = \sin^3\theta$ $0 = \sin\theta$ $\sin 0 = \sin\theta$ $0 = \theta$ $\theta = 0$	$x = a\sin^3\theta$ $a = a\sin^3\theta$ $1 = \sin^3\theta$ $1 = \sin\theta$ $\sin\frac{\pi}{2} = \sin\theta$ $\frac{\pi}{2} = \theta$ $\theta = \frac{\pi}{2}$

From (iii),

Area of ABCDA $= \mathbf{4 \times (area\ of\ ABOA)}$

$$= 4\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^{\frac{3}{2}} dx$$

$$= 4\int_0^{\pi/2} \left(a^{\frac{2}{3}} - (a\sin^3\theta)^{\frac{2}{3}}\right)^{\frac{3}{2}} 3a\sin^2\theta\cos\theta d\theta = 4\int_0^{\pi/2} \left(a^{\frac{2}{3}} - a^{\frac{3}{2}}(\sin^3\theta)^{\frac{2}{3}}\right)^{\frac{3}{2}} 3a\sin^2\theta\cos\theta d\theta$$

$$= 4\int_0^{\pi/2} \left(a^{\frac{2}{3}}\left(1 - (\sin^3\theta)^{\frac{2}{3}}\right)\right)^{\frac{3}{2}} 3a\sin^2\theta\cos\theta d\theta = 4\int_0^{\pi/2} \left(a^{\frac{2}{3}}\left(1 - (\sin\theta)^2\right)\right)^{\frac{3}{2}} 3a\sin^2\theta\cos\theta d\theta$$

$$= 4\int_0^{\pi/2}\left(a^{\frac{2}{3}}\left(1-\sin^2\theta\right)\right)^{\frac{3}{2}} 3a\sin^2\theta\cos\theta d\theta = 4\int_0^{\pi/2}\left(a^{\frac{2}{3}}\cos^2\theta\right)^{\frac{3}{2}} 3a\sin^2\theta\cos\theta d\theta$$

$$= 4\int_0^{\pi/2}\left(a^{\frac{2}{3}}\right)^{\frac{3}{2}}(\cos^2\theta)^{\frac{3}{2}}3a\sin^2\theta\cos\theta d\theta = 4\int_0^{\pi/2} a(\cos\theta)^3 3a\sin^2\theta\cos\theta d\theta$$

$$= 4\int_0^{\pi/2} a\cos^3\theta 3a\sin^2\theta\cos\theta d\theta = 4a\times 3a\int_0^{\pi/2}\cos^3\theta\sin^2\theta\cos\theta d\theta$$

$$= 12a^2\int_0^{\pi/2}\cos^3\theta\sin^2\theta\cos\theta d\theta = 12a^2\int_0^{\pi/2}\cos^4\theta\sin^2\theta d\theta$$

$$= 6\times a^2\times 2\int_0^{\pi/2}\cos^4\theta\sin^2\theta d\theta = 6\times a^2\times 2\int_0^{\pi/2}\sin^2\theta\cos^4\theta d\theta \text{------------------------------(iv)}$$

We have, $[\beta(m,n) = 2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta = \dfrac{\Gamma m\Gamma n}{\Gamma(m+n)}$

Here, From (iv),

$2m-1=2$ & $2n-1=4$

$\Rightarrow 2m = 2+1$ $\Rightarrow 2n = 1+4$

$\Rightarrow 2m = 3$ $\Rightarrow 2n = 5$

$\Rightarrow m = \dfrac{3}{2}$ $\Rightarrow n = \dfrac{5}{2}$

From (iv), Area of ABCDA $= 6\times a^2\times 2\int_0^{\pi/2}\sin^2\theta\cos^4\theta d\theta$

$$= 6\times a^2\times\beta(m,n) = 6\times a^2\times\beta\left(\frac{3}{2},\frac{5}{2}\right) \qquad [m=\frac{3}{2}\ \&\ n=\frac{5}{2}]$$

$$= 6a^2\frac{\Gamma\frac{3}{2}\Gamma\frac{5}{2}}{\Gamma\left(\frac{3}{2}+\frac{5}{2}\right)} = 6a^2\frac{\Gamma\left(\frac{1}{2}+1\right)\Gamma\frac{5}{2}}{\Gamma\frac{3+5}{2}} = 6a^2\frac{\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{5}{2}}{\Gamma\frac{8}{2}} \quad [\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}]\ [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= 6a^2\frac{\frac{1}{2}\sqrt{\pi}\,\Gamma\frac{5}{2}}{\Gamma\frac{8}{2}} = 6a^2\frac{\frac{1}{2}\sqrt{\pi}\,\Gamma\frac{5}{2}}{\Gamma 4} = 6a^2\frac{\frac{1}{2}\sqrt{\pi}\,\Gamma\frac{5}{2}}{(4-1)!} \quad [\because \Gamma\frac{1}{2}=\sqrt{\pi}]\ [\because \Gamma n = (n-1)!]$$

$$= 6a^2\frac{\frac{1}{2}\sqrt{\pi}\,\Gamma\frac{5}{2}}{3!} = 6a^2\frac{\frac{1}{2}\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+1\right)}{3!} = 6a^2\frac{\frac{1}{2}\sqrt{\pi}\,\frac{3}{2}\Gamma\frac{3}{2}}{3!} \quad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= 6a^2 \frac{\frac{1}{2}\sqrt{\pi}\,\overline{\frac{3}{2}}\,\overline{\frac{1}{2}+1}}{3!} \quad = 6a^2 \frac{\frac{1}{2}\sqrt{\pi}\,\frac{3}{2}\times\frac{1}{2}\,\overline{\frac{1}{2}}}{3!} \qquad [\because \Gamma\,(n+1) = n\,\Gamma\,(n)]$$

$$= 6a^2 \frac{\frac{1}{2}\sqrt{\pi}\,\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{3!} \quad = 6a^2 \frac{\frac{1}{2}\sqrt{\pi}\,\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{3\times2\times1} \quad = 6a^2 \frac{\frac{1}{2}\pi\,\frac{3}{2}\times\frac{1}{2}}{3\times2\times1}$$

$$= 6a^2 \frac{\frac{1}{2}\pi\,\frac{1}{2}\times\frac{1}{2}}{2\times1} \quad = 6a^2 \frac{\pi}{2\times2\times2\times2\times1} \quad = 6a^2 \frac{\pi}{2\times2\times2\times2\times1}$$

$$= 3a^2 \frac{\pi}{2\times2\times2\times1} \quad = 3a^2 \frac{\pi}{8}$$

Therefore, the area of astroid is $\frac{3}{8}a^2\pi$ square unit.

Example 182: Find the area common on the circle $\mathbf{r = a}$ and the cardiode $\mathbf{r = a(1+\cos\theta)}$.

Solution: Given Equations are

$\mathbf{r = a}$ ----------------------------(i)

$\mathbf{r = a(1+\cos\theta)}$ ----------------(ii)

From (i)

$$\mathbf{r = a}$$

$$\mathbf{or}\quad \mathbf{r^2 = a^2}$$

$$\mathbf{or}\quad \mathbf{x^2 + y^2 = a^2}$$

Therefore, the center of the circle (i) is (0, 0) and radius **a**

From (i) and (ii), we have,

$$\mathbf{a = a(1 + \cos\theta)}$$

$$\Rightarrow \mathbf{1 = 1 + \cos\theta}$$

$$\Rightarrow \mathbf{1 - 1 = \cos\theta}$$

$$\mathbf{or}\quad \mathbf{\cos\theta = 0}$$

$$\mathbf{or}\quad \mathbf{\cos\theta = \cos(\pm\frac{\pi}{2})}$$

$$\mathbf{i,e.}\ \ \theta = \pm\frac{\pi}{2}$$

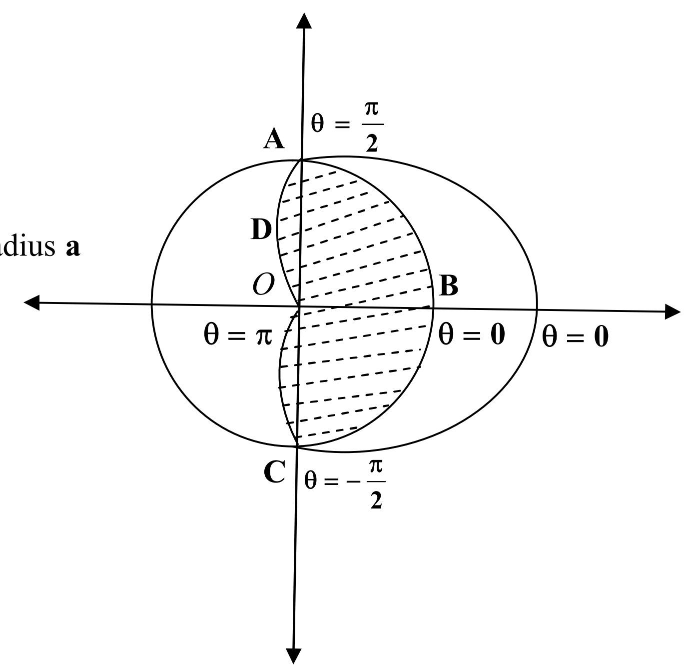

Figure No 82

Now we find the area of OABCO

Since its symmetric the initial line

Therefore the required area OABCO can be written as

Area of OABCO = 2 × area of OABO

= 2 {area of ODAO + area of OABO} -----------------------(iii)

Note Sheet

We know

01. Area of Sector $\mathbf{OAB} = \frac{1}{2}r^2\theta$

Proof: When $\theta = 2\pi^c$ then Area of the circle $= \pi r^2$

$$\theta = 1^c \qquad = \frac{\pi r^2}{2\pi}$$

$$\theta = \theta^c \qquad = \frac{\pi r^2}{2\pi} \times \theta$$

$$= \frac{1}{2} r^2 \theta$$

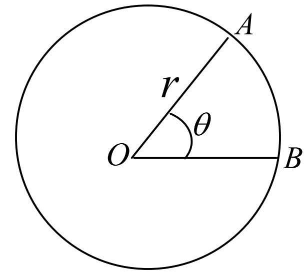

Figure No 83

$\therefore$ Area of Sector $\mathbf{OAB} = \frac{1}{2} r^2 \theta$

02. Arc length of AB $= r\theta$

$\theta = 360^\circ$ Then circumference length $= 2\pi r$

$$\theta = 1^\circ \quad \rule{5cm}{0.4pt} \quad = \frac{2\pi r}{360}$$

$$\theta = \theta^\circ \quad \rule{5cm}{0.4pt} \quad = \frac{2\pi r \times \theta}{360}$$

i.e. Arc length, $\quad AB = \frac{\theta}{360} \times 2\pi r = r\theta$

03. Another Method to find the sector area of OAB

When $\theta \to 0$ i.e. $\theta \to d\theta$

Then Arc length $AB = r \times d\theta$

$\theta \to 0$ Then sector area OAB will be like a triangle.

The area of this triangle $\mathbf{OAB} = \frac{1}{2} \times \text{base} \times \text{height}$

$$\Rightarrow OAB = \frac{1}{2} \times AB \times \text{height}$$

$$\Rightarrow OAB = \frac{1}{2} \times AB \times r = \frac{1}{2} \times r\theta \times r$$

$$\Rightarrow OAB = \frac{1}{2} r^2 \theta = \frac{1}{2} r^2 d\theta \qquad [\because \theta \to d\theta]$$

From (iii), **Area of OABCO = 2 × area of OABO**

$$= 2\{\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} r^2 d\theta + \frac{1}{2}\int_0^{\pi/2} r^2 d\theta\}$$

$$= 2\{\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} a^2(1+\cos\theta)^2 d\theta + \frac{1}{2}\int_0^{\pi/2} a^2 d\theta)\} = 2 \times \frac{1}{2}[a^2\{\int_{\frac{\pi}{2}}^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta + [\theta]_0^{\frac{\pi}{2}}\}]$$

$$= a^2\{\int_{\frac{\pi}{2}}^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta + [\frac{\pi}{2} - 0]\} = a^2\int_{\frac{\pi}{2}}^{\pi}(1+2\cos\theta+\cos^2\theta)d\theta + a^2[\frac{\pi}{2} - 0]\}$$

$$= a^2 \int_{\frac{\pi}{2}}^{\pi} (1 + 2\cos\theta + \cos^2\theta) d\theta + a^2 \times \frac{\pi}{2} = a^2 \times \frac{\pi}{2} + a^2 \int_{\frac{\pi}{2}}^{\pi} \{1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\} d\theta$$

$$= a^2 \times \frac{\pi}{2} + a^2 \int_{\frac{\pi}{2}}^{\pi} (1 + 2\cos\theta + \tfrac{1}{2} + \frac{1}{2}\cos 2\theta) d\theta = a^2 \times \frac{\pi}{2} + a^2 \int_{\frac{\pi}{2}}^{\pi} (\frac{3}{2} + 2\cos\theta + + \frac{1}{2}\cos 2\theta) d\theta$$

$$= a^2 \times \frac{\pi}{2} + a^2 \int_{\frac{\pi}{2}}^{\pi} \frac{3}{2} d\theta + a^2 \int_{\frac{\pi}{2}}^{\pi} 2\cos\theta \, d\theta + a^2 \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \cos 2\theta \, d\theta$$

$$= a^2 \times \frac{\pi}{2} + a^2 \int_{\frac{\pi}{2}}^{\pi} \frac{3}{2} d\theta + a^2 \int_{\frac{\pi}{2}}^{\pi} 2\cos\theta \, d\theta + a^2 \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} \cos 2\theta \, d\theta$$

$$= a^2 \times \frac{\pi}{2} + a^2 \left[\frac{3}{2}\theta\right]_{\frac{\pi}{2}}^{\pi} + 2a^2 [\sin\theta]_{\frac{\pi}{2}}^{\pi} + a^2 \frac{1}{2} \times \frac{1}{2} [\sin 2\theta]_{\frac{\pi}{2}}^{\pi}$$

$$= a^2 \times \frac{\pi}{2} + a^2 \left[\frac{3}{2}\pi - \frac{3}{2} \times \frac{\pi}{2}\right] + 2a^2 \left[\sin\pi - \sin\frac{\pi}{2}\right] + a^2 \times \frac{1}{4}\left[\sin 2\pi - \sin 2 \times \frac{\pi}{2}\right]$$

$$= a^2 \times \frac{\pi}{2} + a^2 \times \frac{3\pi}{2}\left[1 - \frac{1}{2}\right] + 2a^2 \left[\sin\pi - \sin\frac{\pi}{2}\right] + a^2 \times \frac{1}{4}[\sin 2\pi - \sin\pi]$$

$$= a^2 \times \frac{\pi}{2} + a^2 \times \frac{3\pi}{2} \times \frac{1}{2} + 2a^2[0 - 1] + a^2 \times \frac{1}{4}[0 - 0]$$

$$= a^2 \times \frac{\pi}{2} + a^2 \times \frac{3\pi}{4} - 2a^2 + 0 = a^2\left(\frac{\pi}{2} + \frac{3\pi}{4}\right) - 2a^2$$

$$= a^2\left(\frac{2\pi + 3\pi}{4}\right) - 2a^2 = a^2\left(\frac{5\pi}{4}\right) - 2a^2 = a^2\left(\frac{5\pi}{4} - 2\right)$$

Therefore, the required area is $a^2\left(\frac{5\pi}{4} - 2\right)$ square unit.

Example 183: Find the area common to the circle $r = \sqrt{2}a$ and $r = 2a\cos\theta$

Solution: Given equations are

$r = \sqrt{2}a$ ----------------------------(i)

$r = 2a\cos\theta$ -----------------------(ii)

From (i) we have

$r = \sqrt{2}a$

$\Rightarrow r^2 = (\sqrt{2}a)^2$

$\Rightarrow r^2 = 2a^2$

$\Rightarrow x^2 + y^2 = 2a^2$

$\Rightarrow x^2 + y^2 = (\sqrt{2}a)^2$ ------------(iii)

From (iii),

The centre of the circle is $(0,0)$ and radius $\sqrt{2}a$

Also from (ii)
We have $\mathbf{x = r\cos\theta}$ and $\mathbf{y = r\sin\theta}$

So, $\frac{\mathbf{x}}{\mathbf{r}} = \mathbf{\cos\theta}$ and $\frac{\mathbf{y}}{\mathbf{r}} = \mathbf{\sin\theta}$

From (ii),

$$\mathbf{r = 2a\cos\theta}$$

$$\Rightarrow \mathbf{r = 2a.\frac{x}{r}\ r^2 = 2ax}$$

$$\Rightarrow \mathbf{r^2 = 2ax}$$

$$\Rightarrow \mathbf{x^2 + y^2 = 2ax}$$

$$\Rightarrow \mathbf{x^2 + y^2 - 2ax = 0}$$

$$\Rightarrow \mathbf{x^2 - 2ax + y^2 = 0}$$

$$\Rightarrow \mathbf{x^2 - 2ax + a^2 - a^2 + y^2 = 0}$$

$$\Rightarrow \mathbf{(x-a)^2 - a^2 + y^2 = 0}$$

$$\Rightarrow \mathbf{(x-a)^2 + y^2 = a^2} \text{------------------------(iv)}$$

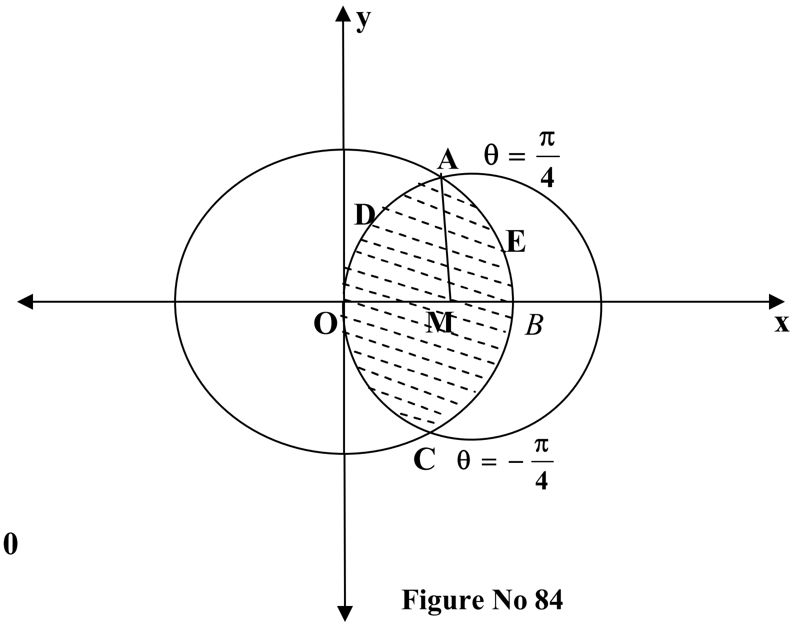

Figure No 84

From (iv), The centre of the circle is $\mathbf{(a,0)}$ and radius $\mathbf{a}$
Also from (i) and (ii)

$$\mathbf{\sqrt{2}a = 2a\cos\theta}$$

$$\Rightarrow \mathbf{\sqrt{2} = 2\cos\theta}$$

$$\Rightarrow \mathbf{1 = \frac{1}{\sqrt{2}}2\cos\theta}$$

$$\Rightarrow \mathbf{1 = \sqrt{2}\cos\theta}$$

$$\Rightarrow \mathbf{\frac{1}{\sqrt{2}} = \cos\theta}$$

$$\Rightarrow \mathbf{\cos\theta = \frac{1}{\sqrt{2}}}$$

$$\Rightarrow \mathbf{\cos\theta = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}}$$

$$\Rightarrow \mathbf{\theta = \frac{\pi}{4}}$$

We are to find the area of **OABCO**, which is symmetric about the x-axis. So we can write the area of **OABCO** is

Area of OABCO = 2 × (area of OABO)

Area of OABCO = 2 × (area of ODAMO + area of AEBM)

$$= \mathbf{2 \times \left\{ \int_{\pi/4}^{\pi/2} \frac{1}{2} 4a^2\cos^2\theta\, d\theta + \int_0^{\pi/4} \frac{1}{2} \times 2a^2\, d\theta \right\}}$$

$$= 2\times\left\{\frac{1}{2}\times 2a^2\int_{\pi/4}^{\pi/2} 2\cos^2\theta\, d\theta + \frac{1}{2}\times 2a^2\int_0^{\pi/4} d\theta\right\}$$

$$= 2\times\left\{a^2\int_{\pi/4}^{\pi/2}(1+\cos 2\theta)\, d\theta + \frac{1}{2}\times 2a^2\int_0^{\pi/4} d\theta\right\} [\because 2\cos^2\theta = 1+\cos 2\theta]$$

$$= 2\times\left\{a^2\int_{\pi/4}^{\pi/2}(1+\cos 2\theta)\, d\theta + a^2\int_0^{\pi/4} d\theta\right\} = 2\times\left\{a^2\int_{\pi/4}^{\pi/2} 1\, d\theta + a^2\int_{\pi/4}^{\pi/2}\cos 2\theta\, d\theta + a^2\int_0^{\pi/4} d\theta\right\}$$

$$= 2\times\left\{a^2[\theta]_{\pi/4}^{\pi/2} + a^2\left[\frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2} + a^2[\theta]_0^{\pi/4}\right\}$$

$$= 2\times\left\{a^2\left[\frac{\pi}{2}-\frac{\pi}{4}\right] + a^2\frac{1}{2}\left[\sin 2\times\frac{\pi}{2} - \sin 2\times\frac{\pi}{4}\right] + a^2\left[\frac{\pi}{4}-0\right]\right\}$$

$$= 2\times\left\{a^2\left[\frac{2\pi-\pi}{4}\right] + a^2\frac{1}{2}\left[\sin\pi - \sin\frac{\pi}{2}\right] + a^2\left[\frac{\pi}{4}-0\right]\right\}$$

$$= 2\times\left\{a^2\left[\frac{\pi}{4}\right] + a^2\frac{1}{2}[0-1] + a^2\left[\frac{\pi}{4}-0\right]\right\} = 2\times\left\{a^2\left[\frac{\pi}{4}\right] + a^2\frac{1}{2}[-1] + a^2\left[\frac{\pi}{4}\right]\right\}$$

$$= 2\times\left\{a^2\left[\frac{\pi}{4}\right] - a^2\frac{1}{2} + a^2\left[\frac{\pi}{4}\right]\right\} = 2\times a^2\left[\frac{\pi}{4}-\frac{1}{2}+\frac{\pi}{4}\right] = 2\times a^2\left[\frac{2\pi}{4}-\frac{1}{2}\right]$$

$$= 2\times a^2\left[\frac{\pi}{2}-\frac{1}{2}\right] = 2\times a^2\times\frac{1}{2}[\pi-1] = a^2[\pi-1]$$

Therefore the required area is $a^2[\pi-1]$ square unit.

Example 184: Find the area of the car diode $r = a(1-\cos\theta)$

Solution: Given the equation: $r = a(1-\cos\theta)$ ----------------------------(i)

If we replace $-\theta$ **for** θ then equation (i) unchanged. So the curve is symmetrical about the initial line.

If $r = 0$ then, $a(1-\cos\theta) = 0$

$\Rightarrow (1-\cos\theta) = 0$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \cos\theta = \cos 0$

$\Rightarrow \theta = 0$

Again, **If $r = 2a$ then, $a(1-\cos\theta) = 2a$**

$\Rightarrow a(1-\cos\theta) = 2a$

$\Rightarrow (1-\cos\theta) = 2$

$\Rightarrow -\cos\theta = 2-1$

$\Rightarrow -\cos\theta = 1$

$\Rightarrow \cos\theta = -1$

$\Rightarrow \cos\theta = \cos\pi$

$\Rightarrow \theta = \pi$

....

[We know Area of Sector $\mathbf{OAB} = \frac{1}{2}\mathbf{r}^2\theta$

When $\theta = 2\pi^c$ then Area of the circle $= \pi r^2$

$$\theta = 1^c \qquad = \frac{\pi r^2}{2\pi}$$

$$\theta = \theta^c \qquad = \frac{\pi r^2}{2\pi} \times \theta$$

$$= \frac{1}{2} r^2 \theta \text{]}$$

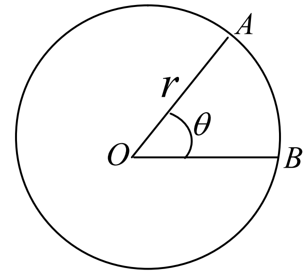

Figure No 85

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Therefore the required area of the cardioid is $= 2\int_0^{\pi} \frac{1}{2} r^2 d\theta$

I.e. Area $= 2 \times \frac{1}{2} \int_0^{\pi} a^2 (1 - \cos\theta)^2 d\theta$

$$= a^2 \int_0^{\pi} (2\sin^2 \frac{\theta}{2})^2 d\theta \qquad [\because 1 - \cos\theta = 2\sin^2 \frac{\theta}{2}]$$

$$= a^2 \int_0^{\pi} 4\sin^4 \frac{\theta}{2} d\theta = a^2 \times 4 \int_0^{\pi} \sin^4 \frac{\theta}{2} d\theta$$

$$= a^2 \times 4 \times 2 \int_0^{\pi/2} \sin^4 \frac{\theta}{2} d\theta$$

$$= a^2 \times 8 \int_0^{\pi/2} \sin^4 \frac{\theta}{2} d\theta$$

$$= a^2 \times 4 \times 2 \int_0^{\pi/2} \sin^4 \frac{\theta}{2} d\theta$$

$$= a^2 \times 4 \times 2 \int_0^{\pi/2} \cos^0 \frac{\theta}{2} \sin^4 \frac{\theta}{2} d\theta \text{ ------------------------------(ii)}$$

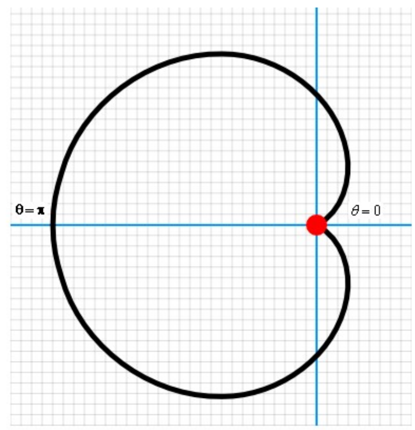


Figure No 86

We have,

$$[\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} [\because \Gamma n = (n-1)!]$$

Here, from (ii)

$2m - 1 = 0$	&	$2n - 1 = 4$
$\Rightarrow 2m = 1$		$\Rightarrow 2n = 1 + 4$
$\Rightarrow 2m = 1$		$\Rightarrow 2n = 5$
$\Rightarrow m = \frac{1}{2}$		$\Rightarrow n = \frac{5}{2}$

From (ii) $= \mathbf{a^2 \times 4 \times 2 \int_0^{\pi/2} \cos^0 \frac{\theta}{2} \sin^4 \frac{\theta}{2}\, d\theta}$

$$= 4 \times a^2 \times \beta(m,n) = 4 \times a^2 \times \beta\left(\frac{1}{2}, \frac{5}{2}\right) \qquad \left[m = \frac{1}{2} \ \& \ n = \frac{5}{2}\right]$$

$$= 4a^2 \frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{\Gamma\left(\frac{1}{2}+\frac{5}{2}\right)} = 4a^2 \frac{\sqrt{\pi}\,\Gamma\frac{5}{2}}{\Gamma\frac{1+5}{2}} \qquad \left[\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}\right] \left[\because \Gamma\frac{1}{2} = \sqrt{\pi}\right]$$

$$= 4a^2 \frac{\sqrt{\pi}\,\Gamma\frac{5}{2}}{\Gamma\frac{6}{2}} = 4a^2 \frac{\sqrt{\pi}\,\Gamma\frac{5}{2}}{\Gamma 3} = 4a^2 \frac{\Gamma\left(\frac{3}{2}+1\right).\sqrt{\pi}}{\Gamma 3}$$

$$= 4a^2 \frac{\frac{3}{2}\Gamma\frac{3}{2}.\sqrt{\pi}}{\Gamma 3} = 4a^2 \frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right).\sqrt{\pi}}{\Gamma 3} = 4a^2 \frac{\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}.\sqrt{\pi}}{\Gamma 3} \qquad [\because \Gamma\,(n+1) = n\,\Gamma\,(n)]$$

$$= 4a^2 \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{\Gamma 3} = 4a^2 \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{(3-1)!} \qquad \left[\because \Gamma\frac{1}{2} = \sqrt{\pi}\right] \ \& \ [\because \Gamma n = (n-1)!]$$

$$= 4a^2 \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{2!} = 4a^2 \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{2.1} = 4a^2 \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{2}$$

$= 4a^2 \dfrac{3.\sqrt{\pi}.\sqrt{\pi}}{2.2.2} = 4a^2 \dfrac{3\pi}{8} = a^2 \dfrac{3\pi}{2}$;This is the required area.

Example 185: Find the area of a loop of the curve $\mathbf{r^2 = a^2 \cos 2\theta}$

Solution: Given equation

$$\mathbf{r^2 = a^2 \cos 2\theta} \text{------------------------(i)}$$

If we replace $-\mathbf{r}$ for $\mathbf{r}$ and $-\theta$ for θ then equation (i) unchanged. So the curve is symmetrical about the pole also the curve is symmetrical about the initial line.

Putting the value of $\mathbf{r = 0}$ in (i),

$\mathbf{r^2 = a^2 \cos 2\theta}$

$\Rightarrow \mathbf{0 = a^2 \cos 2\theta}$

$\Rightarrow \mathbf{a^2 \cos 2\theta = 0}$

$\Rightarrow \mathbf{\cos 2\theta = 0}$

$\Rightarrow \mathbf{\cos 2\theta = \cos \frac{\pi}{2}}$

Putting the value of $\mathbf{r = a}$ in (i),

$\mathbf{r^2 = a^2 \cos 2\theta}$

$\Rightarrow \mathbf{a^2 = a^2 \cos 2\theta}$

$\Rightarrow \mathbf{a^2 \cos 2\theta = a^2}$

$\Rightarrow \mathbf{\cos 2\theta = 1}$

$\Rightarrow \mathbf{\cos 2\theta = \cos 0}$

$\Rightarrow \mathbf{2\theta = 0}$

$\Rightarrow \mathbf{\theta = 0}$

$$\Rightarrow 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Therefore the required area is $= 2\int_0^{\pi/4} \frac{1}{2} r^2 d\theta$

$$= 2\int_0^{\pi/4} \frac{1}{2} a^2 \cos 2\theta \, d\theta$$

$$[\because r^2 = a^2 \cos 2\theta]$$

$$= \int_0^{\pi/4} a^2 \cos 2\theta \, d\theta = a^2 \int_0^{\pi/4} \cos 2\theta \, d\theta$$

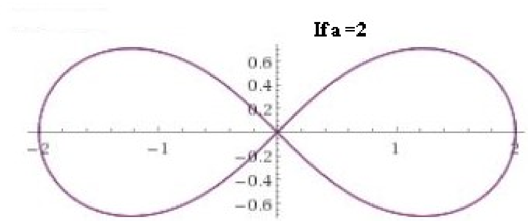


Figure No 87

$$= \frac{a^2}{2}[\sin 2\theta]_0^{\pi/4} = \frac{a^2}{2}[\sin 2\times\frac{\pi}{4} - \sin 2\times 0]$$

$$= \frac{a^2}{2}[\sin\frac{\pi}{2} - \sin 0] = \frac{a^2}{2}[1-0] = \frac{1}{2}a^2$$

Therefore the required area of a loop is $\frac{1}{2}a^2$ square unit.

Example 186: Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: Given equation,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ …-------------------……… (i)

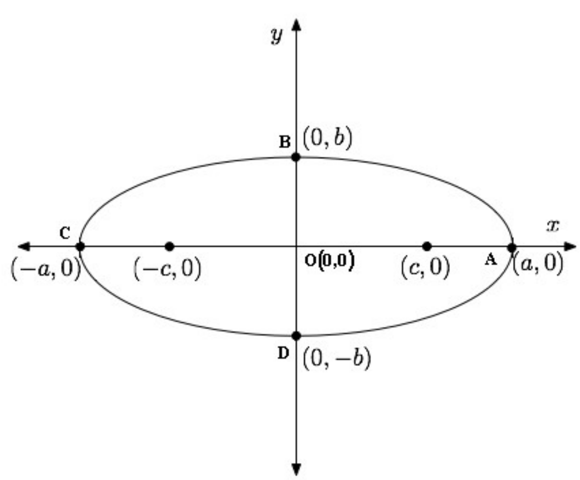
The curve is symmetrical about the both axis and it meets the axis at points **A(a,0), B(0,b), C(−a,0),** and **D(0,−b)**. Draw the curve,

From (i) we have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$\Rightarrow \frac{y^2}{b^2} = \frac{a^2 - x^2}{a^2}$$



$$\text{or, } y^2 = \frac{(a^2 - x^2)}{a^2} b^2$$

$$\text{or, } y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$

Now, the area of the ellipse is $= \mathbf{4 \times (area\ of\ ABOA)}$

$$= 4\int_0^a y\,dx$$

Figure No 88

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left\{0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0\right\} = \frac{4b}{a}\left\{0 + \frac{a^2}{4}\pi - 0 - 0\right\} = ab\pi$$

Therefore the area of the ellipse is, $\mathbf{ab\pi}$ per square unit.

Example 187: Find the area of the car diode $\mathbf{r = a(1 + \cos\theta)}$

Solution: Given the equation: $\mathbf{r = a(1 + \cos\theta)}$ ----------------------------(i)

If we replace $-\theta$ **for** θ then equation (i) unchanged. So the curve is symmetrical about the initial line.

If $\mathbf{r = 0}$ **then,** $\mathbf{a(1 + \cos\theta) = 0}$

$\Rightarrow (1 + \cos\theta) = 0$

$\Rightarrow \cos\theta = -1$

$\Rightarrow \cos\theta = \cos\pi$

$\Rightarrow \theta = \pi$

Again,

If $\mathbf{r = 2a}$ **then,** $\mathbf{a(1 + \cos\theta) = 2a}$

$\Rightarrow a(1 + \cos\theta) = 2a$

$\Rightarrow (1 + \cos\theta) = 2$

$\Rightarrow \cos\theta = 2 - 1$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \cos\theta = \cos 0$

$\Rightarrow \theta = 0$

....

[We know Area of Sector $\mathbf{OAB} = \frac{1}{2}r^2\theta$

When $\theta = 2\pi^c$ then Area of the circle $= \pi r^2$

$\theta = 1^c \qquad = \frac{\pi r^2}{2\pi}$

$\theta = \theta^c \qquad = \frac{\pi r^2}{2\pi} \times \theta$

$= \frac{1}{2}r^2\theta$]

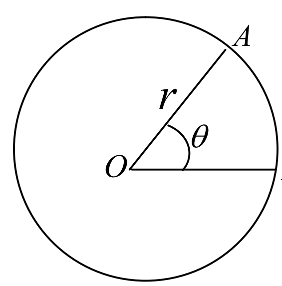


Figure No 89

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Therefore the required area of the cardioid is $=2\int_0^{\pi}\frac{1}{2}r^2 d\theta$

I.e. Area$=2\times\frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2 d\theta$

$$=a^2\int_0^{\pi}(2\cos^2\frac{\theta}{2})^2\ d\theta$$

$$=a^2\int_0^{\pi}4\cos^4\frac{\theta}{2}\ d\theta$$

put, $\frac{\theta}{2}=\phi,\ \therefore d\theta=2d\phi$

$$=4a^2\int_0^{\pi/2}\cos^4\phi\ 2d\phi$$

$$=8a^2\int_0^{\pi/2}\cos^4\phi\ d\phi \text{ ---------------------(ii)}$$

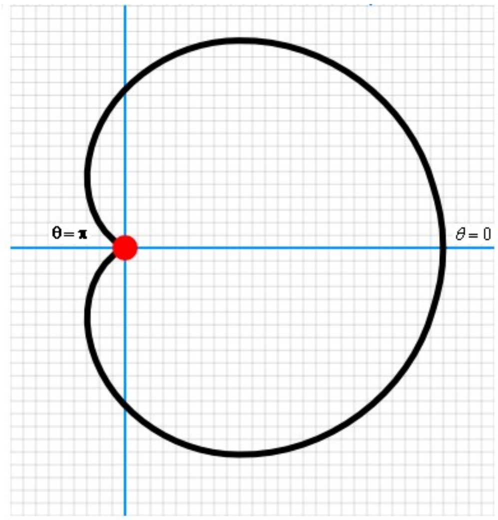


Figure No 90

We have,

$$I_n=\int_0^{\pi/2}\cos^n x dx=\frac{n-1}{n}\int_0^{\pi/2}\cos^{n-2}x dx$$

$$\therefore I_4=\int_0^{\pi/2}\cos^4 x dx=\frac{4-1}{4}\int_0^{\pi/2}\cos^{4-2}x dx$$

$$I_4=\int_0^{\pi/2}\cos^4 x dx=\frac{3}{4}\int_0^{\pi/2}\cos^2 x dx$$

$$I_4=\int_0^{\pi/2}\cos^4 x dx=\frac{3}{4}\times\frac{1}{2}\int_0^{\pi/2}2\cos^2 x dx$$

$$I_4=\int_0^{\pi/2}\cos^4 x dx=\frac{3}{8}\int_0^{\pi/2}(1+\cos 2x)dx\ [\because 2\cos^2 x=1+\cos 2x]$$

θ	0	π
$\varphi=\frac{\theta}{2}$	$\varphi=\frac{\theta}{2}$ $\varphi=\frac{0}{2}=0$	$\varphi=\frac{\theta}{2}$ $\varphi=\frac{\pi}{2}$

$$I_4=\int_0^{\pi/2}\cos^4 x dx=\frac{3}{8}\int_0^{\pi/2}1dx+\frac{3}{8}\int_0^{\pi/2}\cos 2x dx$$

$$I_4=\int_0^{\pi/2}\cos^4 x dx=\frac{3}{8}[x]_0^{\pi/2}+\frac{3}{8}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$I_4 = \int_0^{\pi/2} \cos^4 x dx = \frac{3}{8}\left[\frac{\pi}{2} - 0\right] + \frac{3}{8}\left[\frac{\sin 2 \times \frac{\pi}{2}}{2} - 0\right]$$

$$I_4 = \int_0^{\pi/2} \cos^4 x dx = \frac{3}{8}\left[\frac{\pi}{2} - 0\right] + \frac{3}{8}\left[\frac{\sin \pi}{2}\right]$$

$$I_4 = \int_0^{\pi/2} \cos^4 x dx = \frac{3}{8}\left[\frac{\pi}{2}\right] + \frac{3}{8}\left[\frac{\sin \pi}{2}\right] = \int_0^{\pi/2} \cos^4 x dx = \frac{3}{8}\left[\frac{\pi}{2}\right] + \frac{3}{8} \times 0 \qquad [\because \sin \pi = 0]$$

$$I_4 = \int_0^{\pi/2} \cos^4 x dx = \frac{3\pi}{16}$$

From (ii),

I.e. Area $= 8a^2 \int_0^{\pi/2} \cos^4 \phi \, d\phi = 8a^2 \times \frac{3\pi}{16} = \frac{3\pi}{2} a^2$

Therefore the required area is $\frac{3\pi}{2} a^2$ square unit.

Example 188: Find the area common to the curves $y^2 = ax$ and $x^2 + y^2 = 2ax$.

Solution: Given the equations

$$y^2 = ax \text{ ----------------------------(i)}$$
$$x^2 + y^2 = 2ax \text{ --------------------(ii)}$$

The curve (i) is symmetrical about the x axis and also the curve (ii) is symmetrical about both axes.

From equation (ii) we have,

$$x^2 + y^2 - 2ax = 0$$
$$\text{or}, (x-a)^2 + y^2 = a^2$$
$$\text{and}, y = \pm\sqrt{2ax - x^2}$$

Given Equation (ii),

$x^2 + y^2 = 2ax$

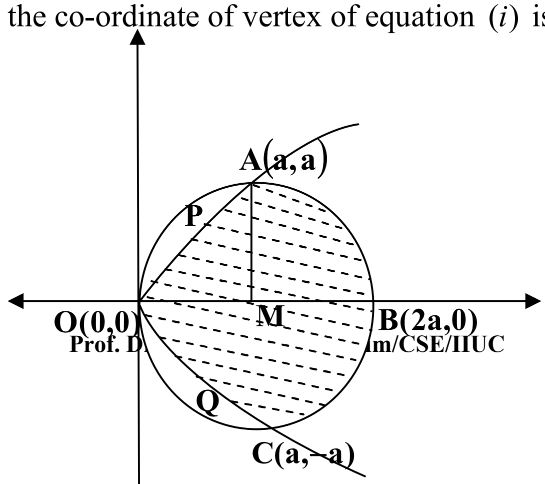
$\Rightarrow x^2 + y^2 - 2ax = 0$

$\text{or}, (x-a)^2 + y^2 = a^2$------------------------------(iii)

The center of the circle is $M(a,0)$ and radius is a also the co-ordinate of vertex of equation (i) is $O(0,0)$

From equation (i) and (ii) we have,

$$x^2 + y^2 = 2ax$$
$$\Rightarrow x^2 + ax = 2ax \quad [\because y^2 = ax]$$
$$\Rightarrow x^2 + ax - 2ax = 0$$



$$\Rightarrow x^2 - ax = 0$$
$$\Rightarrow x(x-a) = 0$$
$$\Rightarrow x = 0, a$$

Putting the value of x in (i)

Given, $y^2 = ax$

When, $x = 0$ then $y = 0$ and when $x = a$ then $y = \pm a$

Figure No 91

Therefore these two curves meet $O(0,0), A(a,a), C(a,-a),$ Draw the graph.

Therefore, the required area of OPABCQO *=2× (area of OPABMO)*

=2{area of OPAMO + area of MABM}

$$= 2\int_0^a y\,dx + 2\int_a^{2a} y dx$$

$$= 2\int_0^a \sqrt{ax}dx + 2\int_a^{2a} \sqrt{2ax - x^2}dx$$

$[\because y^2 = ax; \therefore y = \sqrt{ax}$ and $x^2 + y^2 = 2ax; \therefore y = \pm\sqrt{2ax - x^2}]$

$$= 2\int_0^a (ax)^{1/2} dx + 2\int_a^{2a} (2ax - x^2)^{1/2} dx$$

$$= 2a^{1/2}\int_0^a x^{1/2}dx + 2\int_a^{2a}\sqrt{a^2 - (x-a)^2}dx \qquad [\because a^2 - (x-a)^2 = 2ax - x^2]$$

$$= 2a^{1/2} \times \left[\frac{x^{1/2+1}}{1/2+1}\right]_0^a + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx = 2a^{1/2}\times\left[\frac{x^{3/2}}{3/2}\right]_0^a + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx$$

$$= 2a^{1/2}\times\frac{2}{3}\left[x^{3/2}\right]_0^a + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx = 2a^{1/2}\times\frac{2}{3}\left[a^{3/2} - 0\right] + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx$$

$$= 2a^{1/2}\times\frac{2}{3}\left[a^{3/2}\right] + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx = \frac{4}{3}a^{1/2}\left[a^{3/2}\right] + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx$$

$$= \frac{4}{3}a^{1/2+3/2} + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx = \frac{4}{3}a^{4/2} + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx$$

$$= \frac{4}{3}a^2 + 2\int_a^{2a}\sqrt{a^2-(x-a)^2}dx \qquad [\because \int\sqrt{p^2-x^2}dx = (\frac{x}{2}\sqrt{p^2-x^2} + \frac{p^2}{2}\sin^{-1}\frac{x}{p})]$$

$$= \frac{4}{3}a^2 + 2\left[\frac{(x-a)}{2}\sqrt{a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}\right]_a^{2a}$$

$$=\frac{4}{3}a^2+2\left[\frac{(2a-a)}{2}\sqrt{a^2-(2a-a)^2}+\frac{a^2}{2}\sin^{-1}\frac{2a-a}{a}-\frac{(a-a)}{2}\sqrt{a^2-(a-a)^2}-\frac{a^2}{2}\sin^{-1}\frac{a-a}{a}\right]$$

$$=\frac{4}{3}a^2+2\left[\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\frac{a}{a}-\frac{0}{2}\sqrt{a^2-0}-\frac{a^2}{2}\sin^{-1}\frac{0}{a}\right]$$

$$=\frac{4}{3}a^2+2\left[\frac{a}{2}\sqrt{0}+\frac{a^2}{2}\sin^{-1}.1-0.\sqrt{a^2}-\frac{a^2}{2}\sin^{-1}.0\right]$$

$$=\frac{4}{3}a^2+2\left[\frac{a}{2}.0+\frac{a^2}{2}\sin^{-1}.\sin\frac{\pi}{2}-0-0\right]$$

$$=\frac{4}{3}a^2+2\left[.0+\frac{a^2}{2}.\frac{\pi}{2}\right]=\frac{4}{3}a^2+2\left[\frac{a^2}{2}.\frac{\pi}{2}\right]=\frac{4}{3}a^2+2\left[\frac{a^2\pi}{4}\right]=\frac{4}{3}a^2+\left[\frac{a^2\pi}{2}\right]$$

Therefore required area is $=\frac{4}{3}a^2+\left[\frac{a^2\pi}{2}\right]$ square unit.

Example 189: Find the area common to the curve $y^2 = ax$ and $y^2 + x^2 = 4ax$

Solution: Given equations

$y^2 = ax$ ------------------------(i)

$y^2 + x^2 = 4ax$ ----------------(ii)

From equation (i) which (curve) is symmetrical about x-axis and it's vertex $A(0,0)$. Also from equation (ii)

$y^2 + x^2 = 4ax$

$\Rightarrow y^2 + x^2 - 4ax = 0$

$\Rightarrow x^2 + y^2 - 4ax = 0$

$\Rightarrow x^2 - 4ax + y^2 = 0$

$\Rightarrow x^2 - 2.x.2a + (2a)^2 - (2a)^2 + y^2 = 0$

$\Rightarrow (x-2a)^2 - (2a)^2 + y^2 = 0$

$\Rightarrow (x-2a)^2 + y^2 = (2a)^2$ --------------------(iiii)

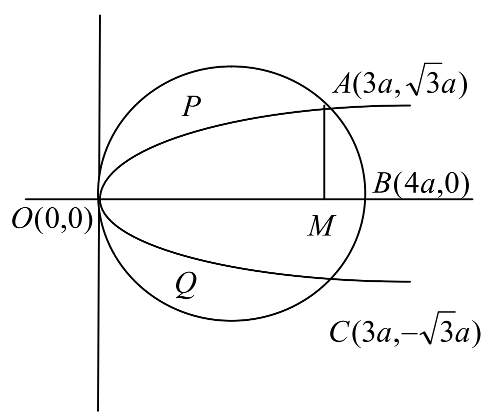


Figure No 92

From (iii)

The Center of the circle (ii) is $(2a,0)$ and radius $2a$

From (i) and (ii) we have,

$y^2 + x^2 = 4ax$

$\Rightarrow ax + x^2 = 4ax\ [\because \text{from (i)}: y^2 = ax]$

$\Rightarrow ax + x^2 - 4ax = 0$

$\Rightarrow x^2 - 4ax + ax = 0$

Putting the values of x in (i),

When $x = 0$

$y^2 = ax$

$y^2 = a \times 0$

$y = 0$

When $x = 3a$

$y^2 = ax$

$$\Rightarrow x^2 - 3ax = 0$$
$$\Rightarrow x(x-3a) = 0$$
$$\Rightarrow x = 0 \;\&\; (x-3a) = 0$$
$$\Rightarrow x = 0 \;\&\; x = 3a$$

This two curve intersect at $\mathbf{A(3a, \sqrt{3}a)}$ and $\mathbf{C(3a, -\sqrt{3}a)}$

Given From (i) & (ii),

$$y^2 = ax$$
$$\therefore y = \sqrt{ax} \text{ ------------------------(iv)}$$

and
$$y^2 + x^2 = 4ax$$
$$\Rightarrow y^2 = 4ax - x^2$$
$$\Rightarrow y = \sqrt{4ax - x^2} \text{ ----------------(v)}$$

Draw the graph. Now we are to find the area of OPABCQO. It can be written as,

Area of OPABCQO = 2(area of OPABMO)

= 2(area of OPAMO + area of ABMA)

$$= 2\left[\int_0^{3a} y\,dx + \int_{3a}^{4a} y\,dx\right]$$

$$= 2\left[\int_0^{3a} \sqrt{ax}\,dx + \int_{3a}^{4a} \sqrt{4ax - x^2}\,dx\right] \qquad [\because y = \sqrt{ax} \;\&\; \sqrt{4ax - x^2}\,]$$

$$= 2\int_0^{3a} \sqrt{ax}\,dx + 2\int_{3a}^{4a} \sqrt{4ax - x^2}\,dx = 2\int_0^{3a} (ax)^{\frac{1}{2}}\,dx + 2\int_{3a}^{4a} \sqrt{4ax - x^2}\,dx$$

$$= 2\int_0^{3a} (ax)^{\frac{1}{2}}\,dx + 2\int_{3a}^{4a} \sqrt{4a^2 - 4a^2 + 4ax - x^2}\,dx$$

$$= 2\int_0^{3a} (ax)^{\frac{1}{2}}\,dx + 2\int_{3a}^{4a} \sqrt{4a^2 - (4a^2 - 4ax + x^2)}\,dx$$

$$= 2\int_0^{3a} (ax)^{\frac{1}{2}}\,dx + 2\int_{3a}^{4a} \sqrt{4a^2 - \left\{(2a)^2 - 2.2a.x + x^2\right\}}\,dx$$

$$= 2\int_0^{3a} (ax)^{\frac{1}{2}}\,dx + 2\int_{3a}^{4a} \sqrt{(2a)^2 - \left\{(2a-x)^2\right\}}\,dx$$

$$= 2\int_0^{3a} (ax)^{\frac{1}{2}}\,dx + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\,dx$$

$$= 2 \times a^{\frac{1}{2}} \int_0^{3a} x^{\frac{1}{2}}\,dx + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\,dx$$

$$= 2 \times a^{\frac{1}{2}} \left[\frac{(x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^{3a} + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\,dx$$

$$= 2 \times a^{\frac{1}{2}} \left[\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^{3a} + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= 2 \times a^{\frac{1}{2}} \times \frac{2}{3} \left[(x)^{\frac{3}{2}} \right]_0^{3a} + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= 2 \times \frac{2}{3} \times a^{\frac{1}{2}} \left[(3a)^{\frac{3}{2}} - (0)^{\frac{3}{2}} \right] + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= 2 \times \frac{2}{3} \times a^{\frac{1}{2}} \left[(3a)^{\frac{3}{2}} - 0 \right] + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= 2 \times \frac{2}{3} \times a^{\frac{1}{2}} \times (3a)^{\frac{3}{2}} + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= 2 \times \frac{2}{3} \times 3^{\frac{3}{2}} \times a^{\frac{1}{2}} \times a^{\frac{3}{2}} + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= \frac{4}{3} \times 3^{\frac{3}{2}} \times a^{\frac{1}{2}+\frac{3}{2}} + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= \frac{4}{3} \times 3^{\frac{3}{2}} \times a^{\frac{4}{2}} + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= \frac{4}{3} \times 3^{\frac{3}{2}} \times a^2 + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= \frac{4}{3} \times 3^{\frac{3}{2}} a^2 + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (2a-x)^2}\, dx$$

$$= \frac{4}{3} \times 3^{\frac{3}{2}} a^2 + 2\int_{3a}^{4a} \sqrt{(2a)^2 - (x-2a)^2}\, dx$$

$$= \frac{4}{3} \times 3^{\frac{3}{2}} a^2 + 2\left[\frac{1}{2} \left\{ (x-2a)\sqrt{(2a)^2 - (x-2a)^2} + (2a)^2 \sin^{-1}\frac{x-2a}{2a} \right\} \right]_{3a}^{4a}$$

$$[\because \int \sqrt{p^2 - x^2}\, dx = \frac{1}{2}(x\sqrt{p^2 - x^2} + p^2 \sin^{-1}\frac{x}{p})]$$

$$= \frac{4}{3} \times 3^{\frac{3}{2}} a^2 + 2\left[\frac{1}{2} \left\{ \begin{array}{l} (4a-2a)\sqrt{(2a)^2 - (4a-2a)^2} + (2a)^2 \sin^{-1}\frac{4a-2a}{2a} \\ -(3a-2a)\sqrt{(2a)^2 - (3a-2a)^2} - (2a)^2 \sin^{-1}\frac{3a-2a}{2a} \end{array} \right\} \right]$$

$$= \frac{4}{3}\times 3^{\frac{3}{2}} a^2 + 2\left[\frac{1}{2}\left\{(2a)\sqrt{(2a)^2-(2a)^2} + (2a)^2 \sin^{-1}\frac{2a}{2a} - (a)\sqrt{(2a)^2-(a)^2} - (2a)^2 \sin^{-1}\frac{a}{2a}\right\}\right]$$

$$= \frac{4}{3}\times 3^{\frac{3}{2}} a^2 + 2\left[\frac{1}{2}\left\{(2a)\sqrt{0} + (2a)^2 \sin^{-1} 1 - (a)\sqrt{a^2} - (2a)^2 \sin^{-1}\frac{1}{2}\right\}\right]$$

$$= \frac{4}{3}\times 3^{\frac{3}{2}} a^2 + 2\left[\frac{1}{2}\left\{0 + (2a)^2 \sin^{-1}\sin\frac{\pi}{2} - (a)\times a - 4a^2 \sin^{-1}\sin\frac{\pi}{6}\right\}\right]$$

$$= \frac{4}{3}\times 3^{\frac{3}{2}} a^2 + 2\left[\frac{1}{2}\left\{4a^2\times\frac{\pi}{2} - a^2 - 4a^2\times\frac{\pi}{6}\right\}\right]$$

$$= \frac{4}{3}\times 3^{\frac{3}{2}} a^2 + 2\left[\frac{1}{2}\left\{2a^2\pi - a^2 - 4a^2\times\frac{\pi}{6}\right\}\right]$$

$$= \frac{4}{3}\times 3^{\frac{3}{2}} a^2 + 2a^2\pi - a^2 - 4a^2\times\frac{\pi}{6} \textit{ Answer}$$

Therefore the required are is $\frac{4}{3}\times 3^{\frac{3}{2}} a^2 + 2a^2\pi - a^2 - 4a^2\times\frac{\pi}{6}$ square unit.

Example 190: Find the area of the portion of the circle $x^2+y^2=1$ which lies inside the parabola $y^2=1-x$

Solution: Given equations

$$x^2+y^2=1\cdots\cdots\cdots(i)$$
$$y^2=1-x\cdots\cdots\cdots(ii)$$

The curve (i) is symmetrical about both axis and it radius 1 and centre $O(0.0)$ also curve (ii) is symmetrical about x-axis vertex $A(1,0)$. Draw graph, therefore

We get equation (i) and (ii),

$x^2+y^2=1\cdots\cdots\cdots(i)$

$\Rightarrow x^2+1-x=1\ [\because y^2=1-x]$

$\Rightarrow x^2+1-x-1=0$

$\Rightarrow x^2-x=0$

$\Rightarrow x(x-1)=0$

$\Rightarrow x=0,1$

Putting the values of x in (ii),

When $x=0$

$\Rightarrow y^2=1-x$

$\Rightarrow y^2=1-0$

$\Rightarrow y^2=1$

$\Rightarrow y=\pm 1$

When $x=1$

$\Rightarrow y^2=1-x$

$\Rightarrow y^2=1-1$

$\Rightarrow y^2=0$

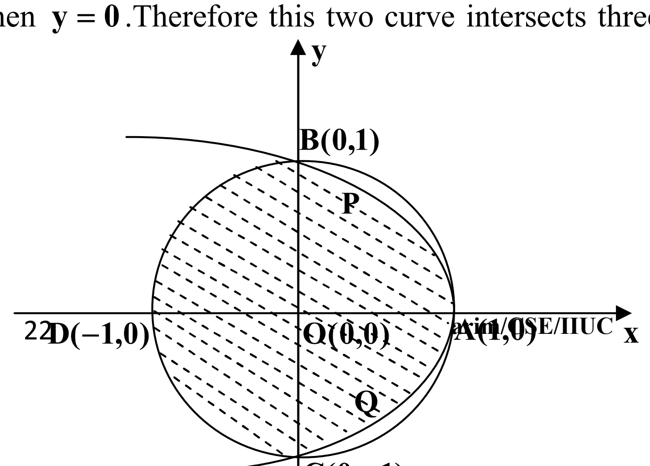
$\Rightarrow y=0$

When, $x=0$ then $y=\pm 1$ and when $x=1$ then $y=0$. Therefore this two curve intersects three points. This arc $A(1,0)\ B(0,1)\ C(0,-1)$.

Given from (i) and (ii),

$x^2+y^2=1$

$\Rightarrow y^2=1-x^2$



$\Rightarrow y = \sqrt{1-x^2}$ -------------------------(iii)

and

$y^2 = 1 - x$

$\Rightarrow y = \sqrt{1-x}$ --------------------------(iv)

Figure No 93

We are to find the area of APBDCQA which is symmetrical about x-axis, so we can write,
Area of APBDCQA=2{area of APBOA + area of OBDO}

$$= 2\left[\int_0^1 y\,dx + \int_{-1}^0 y\,dx\right] = 2\int_0^1 y\,dx + 2\int_{-1}^0 y\,dx$$

$$= 2\int_0^1 \sqrt{1-x}\,dx + 2\int_{-1}^0 \sqrt{1-x^2}\,dx = 2\int_0^1 (1-x)^{1/2}\,dx + 2\int_{-1}^0 (1-x^2)^{1/2}\,dx$$

$$= 2\left[\frac{(1-x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}(-1)\right]_0^1 + 2\int_{-1}^0 (1-x^2)^{1/2}\,dx = -2\left[\frac{(1-x)^{\frac{1}{2}+1}}{\frac{3}{2}}(-1)\right]_0^1 + 2\int_{-1}^0 (1-x^2)^{1/2}\,dx$$

$$= -2\times\frac{2}{3}\left[(1-x)^{\frac{3}{2}}\right]_0^1 + 2\int_{-1}^0 (1-x^2)^{1/2}\,dx$$

$$= -\frac{4}{3}\left[(1-x)^{\frac{3}{2}}\right]_0^1 + 2\left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_{-1}^0$$

$$= -\frac{4}{3}\left[(1-1)^{\frac{3}{2}} - (1-0)^{\frac{3}{2}}\right] + 2\left[\frac{1}{2}\times 0\times\sqrt{1-0^2} + \frac{1}{2}\sin^{-1}0 - \frac{1}{2}(-1)\sqrt{1-(-1)^2} - \frac{1}{2}\sin^{-1}(-1)\right]$$

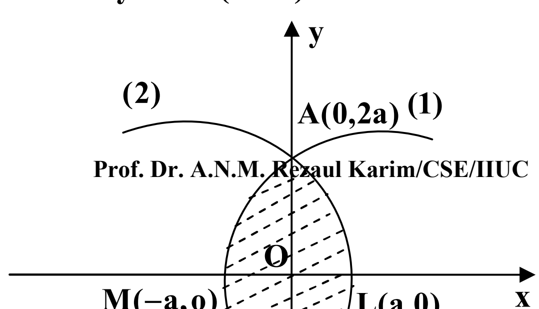
$$= -\frac{4}{3}\left[(0)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] + 2\left[0 + \frac{1}{2}\sin^{-1}\sin 0 + \frac{1}{2}\sqrt{1-1} + \frac{1}{2}\sin^{-1}(1)\right]$$

$$= -\frac{4}{3}[0-1] + 2\left[\frac{1}{2}\times 0 + \frac{1}{2}\sqrt{0} + \frac{1}{2}\sin^{-1}\sin\frac{\pi}{2}\right] = -\frac{4}{3}[-1] + 2\left[0 + \frac{1}{2}\times 0 + \frac{1}{2}\times\frac{\pi}{2}\right]$$

$$= +\frac{4}{3} + 2\left[0 + 0 + \frac{1}{2}\times\frac{\pi}{2}\right] = +\frac{4}{3} + 2\left[\frac{\pi}{4}\right] = \frac{4}{3} + \left[\frac{\pi}{2}\right] = \frac{4}{3} + \frac{\pi}{2}$$

Therefore the required area is $\left(\frac{4}{3} + \frac{\pi}{2}\right)$ square unit.

Example 191: Show that the area enclosed between the parabolas $y^2 = 4a(x+a)$ and $y^2 = -4a(x-a)$ is $16\frac{a^2}{3}$



Solution: Given equation,

$$y^2 = 4a(x+a)$$

$$y^2 = 4ax + 4a^2 \text{--------------------(i)}$$

and

$$y^2 = -4a(x-a)$$

$$y^2 = -4ax + 4a^2 \text{-------------------(ii)}$$

Figure No 94

The curve (i) and (ii) are symmetrical about x-axis and vertex of (i) at $(-a,0)$ and (ii) at $(a,0)$

From (i) and (ii) we have,

$$4ax + 4a^2 = -4ax + 4a^2$$
$$\Rightarrow 4ax + 4a^2 + 4ax - 4a^2 = 0$$
$$\Rightarrow 8ax = 0$$
$$\Rightarrow x = 0$$

Putting the value of $x = 0$ in (i),

$$\Rightarrow y^2 = 4ax + 4a^2$$
$$\Rightarrow y^2 = 4a.0 + 4a^2 [\because x = 0]$$
$$\Rightarrow y^2 = 4a^2$$
$$\Rightarrow y = \pm 2a$$

Therefore this two curve cut at $A(0,2a)$ and $B(0,-2a)$.Draw the graph. We are to find the area of ALBMA which is symmetrical about the x axis.

Required area $\quad ALBMA = 2\{\text{area of } AMOA + \text{area of } ALOA\}$

$$\Rightarrow ALBMA = 2\int_{-a}^{0} \sqrt{4ax+4a^2}\,dx + 2\int_{0}^{a} \sqrt{4a^2 - 4ax}\,dx$$

$$\Rightarrow ALBMA = 2\int_{-a}^{0} \sqrt{4a(x+a)}\,dx + 2\int_{0}^{a} \sqrt{4a(a-x)}\,dx$$

$$\Rightarrow ALBMA = 2\int_{-a}^{0} \sqrt{4a}\sqrt{(x+a)}\,dx + 2\int_{0}^{a} \sqrt{4a}\sqrt{(a-x)}\,dx$$

$$\Rightarrow ALBMA = 2\int_{-a}^{0} 2\sqrt{a}\sqrt{(x+a)}\,dx + 2\int_{0}^{a} 2\sqrt{a}\sqrt{(a-x)}\,dx$$

$$\Rightarrow ALBMA = 4\int_{-a}^{0} \sqrt{a}\sqrt{(x+a)}\,dx + 4\int_{0}^{a} \sqrt{a}\sqrt{(a-x)}\,dx$$

$$\Rightarrow ALBMA = 4\sqrt{a}\int_{-a}^{0} \sqrt{(x+a)}\,dx + 4\sqrt{a}\int_{0}^{a} \sqrt{(a-x)}\,dx$$

$$\Rightarrow ALBMA = 4\sqrt{a}\int_{-a}^{0} (x+a)^{1/2}\,dx + 4\sqrt{a}\int_{0}^{a} (a-x)^{1/2}\,dx$$

$$\Rightarrow ALBMA = 4\sqrt{a}\left[\frac{(x+a)^{1/2+1}}{1/2+1}\right]_{-a}^{0} + 4\sqrt{a}\left[\frac{(a-x)^{1/2+1}}{1/2+1}.(-1)\right]_{0}^{a}$$

$$\Rightarrow \text{ALBMA} = 4\sqrt{a}\left[\frac{(x+a)^{3/2}}{3/2}\right]_{-a}^{0} - 4\sqrt{a}\left[\frac{(a-x)^{3/2}}{3/2}\right]_{0}^{a}$$

$$\Rightarrow \text{ALBMA} = 4\sqrt{a}\times\frac{2}{3}\left[(x+a)^{3/2}\right]_{-a}^{0} - 4\sqrt{a}\times\frac{2}{3}\times\left[(a-x)^{3/2}\right]_{0}^{a}$$

$$\Rightarrow \text{ALBMA} = 4\sqrt{a}\times\frac{2}{3}\left[(0+a)^{3/2} - (-a+a)^{3/2}\right] - 4\sqrt{a}\times\frac{2}{3}\times\left[(a-a)^{3/2} - (a-0)^{3/2}\right]$$

$$\Rightarrow \text{ALBMA} = 4\sqrt{a}\times\frac{2}{3}\left[(a)^{3/2} - (0)^{3/2}\right] - 4\sqrt{a}\times\frac{2}{3}\times\left[(0)^{3/2} - (a)^{3/2}\right]$$

$$\Rightarrow \text{ALBMA} = 4\sqrt{a}\times\frac{2}{3}\left[(a)^{3/2}\right] - 4\sqrt{a}\times\frac{2}{3}\times\left[-(a)^{3/2}\right]$$

$$\Rightarrow \text{ALBMA} = 4\sqrt{a}\times\frac{2}{3}\left[(a)^{3/2}\right] + 4\sqrt{a}\times\frac{2}{3}\times\left[(a)^{3/2}\right]$$

$$\Rightarrow \text{ALBMA} = 4\sqrt{a}\times\frac{2}{3}\left[(a)^{3/2} + (a)^{3/2}\right]$$

$$\Rightarrow \text{ALBMA} = 4\sqrt{a}\times\frac{2}{3}\left[2a^{3/2}\right] = 4\sqrt{a}\times\frac{4}{3}\left[a^{3/2}\right] = 4a^{1/2}\times\frac{4}{3}\left[a^{3/2}\right]$$

$$\Rightarrow \text{ALBMA} = 4a^{1/2}\times\frac{4}{3}\times a^{3/2} = \frac{16}{3}\times a^{3/2}\times a^{1/2} = \frac{16}{3}\times a^{3/2+1/2}$$

$$\Rightarrow \text{ALBMA} = \frac{16}{3}\times a^{4/2} = \frac{16}{3}\times a^{2}$$

Therefore the required area is $\frac{16}{3}\times a^2$ square unit.

Example 192: Find the area of the segment cut off from the parabola $y^2 = 2x$ by the straight line $y = 4x - 1$.

Solution: Given equations

$y^2 = 2x$ ------------------------(i)

$y = 4x - 1$ ---------------------(ii)

The curve (i) is symmetrical about x-axis. From equation (i) and (ii) we have,

$$y^2 = 2x$$
$$\Rightarrow (4x-1)^2 = 2x \; [\because y = 4x - 1]$$
$$\Rightarrow (4x-1)^2 = 2x$$
$$\Rightarrow (4x)^2 - 2\times 4x\times 1 + 1^2 = 2x$$
$$\Rightarrow 16x^2 - 8x - 2x + 1 = 0$$

Putting the value of $x = \frac{1}{8}$ in (ii),

$$\Rightarrow y = 4x - 1$$
$$\Rightarrow y = 4\times\frac{1}{8} - 1$$
$$\Rightarrow y = \frac{1}{2} - 1$$
$$\Rightarrow y = -\frac{1}{2}$$

Putting the value of $x = \frac{1}{2}$ in (ii),

$\Rightarrow \mathbf{8x(2x-1)-1(2x-1)}$

$\Rightarrow \mathbf{(8x-1)(2x-1)=0}$

$\Rightarrow \mathbf{8x-1=0}$ and $\mathbf{2x-1=0}$

$\Rightarrow \mathbf{8x=1}$ and $\mathbf{2x=1}$

i,e $\Rightarrow \mathbf{x=\frac{1}{8},\frac{1}{2}}$

When $\mathbf{x=\frac{1}{8}}$ then $\mathbf{y=-\frac{1}{2}}$ and when $\mathbf{x=\frac{1}{2}}$ then $\mathbf{y=1}$ therefore the line (ii) cut the parabola (i) at points $\mathbf{A\left(\frac{1}{2},1\right)}$, $\mathbf{B\left(\frac{1}{8},-\frac{1}{2}\right)}$.Drawn figure. Now we are to find the area of **OABO** .

Given

$\mathbf{y=4x-1}$

$\Rightarrow \mathbf{4x=y+1}$

$\mathbf{x=\frac{1}{4}(y+1)}$ ------------(iii)

and

$\mathbf{y^2=2x}$

$\Rightarrow \mathbf{x=\frac{1}{2}y^2}$ ---------------(iv)

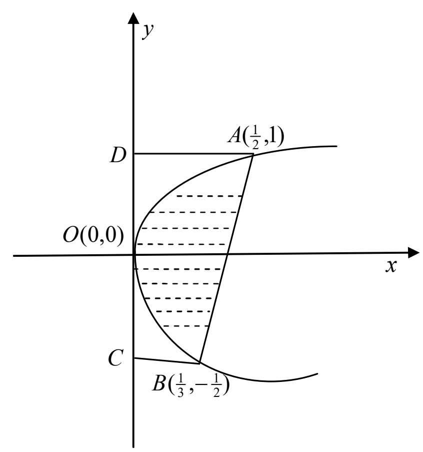


Figure No 95

The area of **OABO** = area of **ADCBA** − area $\mathbf{[OADO+0BCO]}$

$$= \int_{-\frac{1}{2}}^{1} \mathbf{x\,dy} - \int_{-\frac{1}{2}}^{1} \mathbf{x\,dy}$$

$$= \frac{1}{4}\int_{-\frac{1}{2}}^{1} \mathbf{(y+1)dy} - \frac{1}{2}\int_{-\frac{1}{2}}^{1} \mathbf{y^2 dy} \quad \textit{[From (iii) and (iv)]}$$

$$= \frac{1}{4}\left[\frac{y^2}{2}+y\right]_{-\frac{1}{2}}^{1} - \frac{1}{2}\left[\frac{y^3}{3}\right]_{-\frac{1}{2}}^{1}$$

$$= \frac{1}{4}\left[\frac{1^2}{2} + 1 - \frac{(-\frac{1}{2})^2}{2} - (-\frac{1}{2})\right] - \frac{1}{2}\left[\frac{1^3}{3} - \frac{(-\frac{1}{2})^3}{3}\right]$$

$$= \frac{1}{4}\left[\frac{1}{2} + 1 - \frac{\frac{1}{4}}{2} + \frac{1}{2}\right] - \frac{1}{2}\left[\frac{1}{3} - \frac{-\frac{1}{8}}{3}\right] = \frac{1}{4}\{\frac{1}{2} + 1 - \frac{1}{8} + \frac{1}{2}\} - \frac{1}{2}\{\frac{1}{3} + \frac{1}{24}\}$$

$$= \frac{1}{4}\{1 + 1 - \frac{1}{8}\} - \frac{1}{2}\{\frac{1}{3} + \frac{1}{24}\} = \frac{1}{4}\{2 - \frac{1}{8}\} - \frac{1}{2}\{\frac{1}{3} + \frac{1}{24}\} = \frac{1}{4}\{\frac{16-1}{8}\} - \frac{1}{2}\{\frac{8+1}{24}\}$$

$$= \frac{1}{4}\{\frac{15}{8}\} - \frac{1}{2}\{\frac{9}{24}\} = \frac{15}{32} - \frac{3}{16} = \frac{15-6}{32} = \frac{9}{32}$$

Therefore the required area is $\frac{9}{32}$ square unit.

Example 193: Find the area of the parabola $y^2 = 4ax$ cut off by the lotus rectum.

Solution: Given equation,

$$y^2 = 4ax \text{ --------------(i)}$$

The curve is symmetrical about the x-axis. Whose vertex 0(0, 0) and co-ordinate of focus (a, 0) and the equation of latus rectum $x = a$

Putting the value of x in (i),

$$y^2 = 4ax$$
$$\Rightarrow y^2 = 4a.a[x = a]$$
$$\Rightarrow y^2 = 4a^2$$
$$\Rightarrow y = \pm 2a$$

When $x = a$ then $y = \pm 2a$

Therefore the curve cut off the line $x = a$ at A (a, 2a) and B (a, -2a). Draw graph, we are to find the area of OABO. We can write it

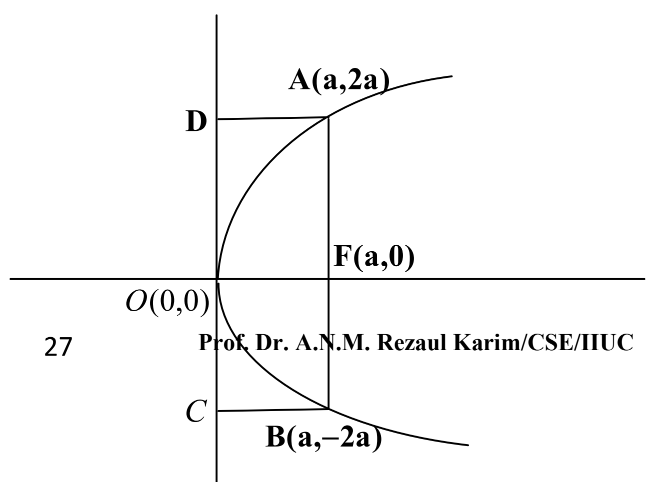
Area of OABO = area of ABCD – area (OADO+OBCO)

= 2 area of DAFOD – 2 area of OADO

$$= 2\int_0^{2a} xdy - 2\int_0^{2a} xdy$$

$$= 2\int_0^{2a} ady - 2\int_0^{2a} \frac{1}{4a} y^2 dy$$

$$[\because x = a \text{ \& } y^2 = 4ax; \therefore x = \frac{y^2}{4a}]$$



$$=2a\left[y\right]_0^{2a}-2\frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{2a}$$

$$=2a\left[2a-0\right]-\frac{2}{4a}\left[\frac{(2a)^3}{3}-\frac{0^3}{3}\right]$$

$$=2a\left[2a\right]-\frac{2}{4a}\left[\frac{8a^3}{3}-0\right]$$

Figure No 96

$$=2a\left[2a\right]-\frac{1}{2a}\left[\frac{8a^3}{3}-0\right]=4a^2-\frac{1}{6a}8a^3=4a^2-\frac{4}{3}a^2$$

$$=(\frac{12-4}{3})a^2=\frac{8}{3}a^2$$

Therefore the required area is $\frac{8}{3}a^2$ square unit.

Example 194: Find the area between the curve $y^2(2a-x)=x^3$ and its asymptotes.

Solution*:* *give equation,*

$$y^2(2a-x)=x^3$$

$$\Rightarrow y^2=\frac{x^3}{(2a-x)} \text{ -------------------------(i)}$$

$$\Rightarrow y=\left[\frac{x^3}{(2a-x)}\right]^{\frac{1}{2}}=\left[\frac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}\right]=\frac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}$$

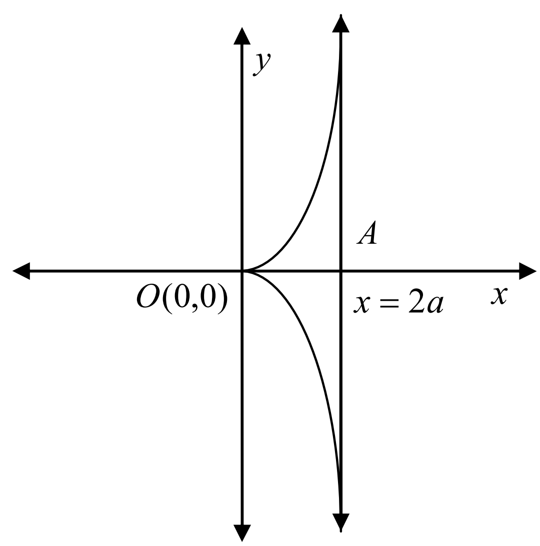


Figure No 97

The curve is symmetrical about x-axis passes through origin. If $x>2a$ *and* $x<0$ *then y is imaginary and if* $x=2a$ *then y is infinite, i.e. the curves lies between* $x=0$ *and* $x=2a$ *where* $x=2a$ *is an asymptotes.*

Required area $=2\int_0^{2a} y\,dx$

$$=2\int_0^{2a}\frac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}dx \text{ ------------------------(ii)}$$

Putting

$x = 2a\sin^2\theta$

$\Rightarrow \dfrac{dx}{d\theta} = \dfrac{d}{d\theta}(2a\sin^2\theta)$

$\Rightarrow \dfrac{dx}{d\theta} = 2a\dfrac{d}{d\theta}(\sin^2\theta)$

$\Rightarrow \dfrac{dx}{d\theta} = 2a\times 2\sin\theta\dfrac{d}{d\theta}(\sin\theta)$

$\Rightarrow \dfrac{dx}{d\theta} = 2a\times 2\sin\theta\times\cos\theta$

$\Rightarrow dx = 4a\sin\theta\cos\theta\, d\theta$

$x = 2a\sin^2\theta$	0	$2a$
θ	$x = 2a\sin^2\theta$ $\Rightarrow 0 = 2a\sin^2\theta$ $\Rightarrow 0 = \sin^2\theta$ $\Rightarrow 0 = \sin\theta$ $\Rightarrow \sin 0 = \sin\theta$ $\Rightarrow 0 = \theta$ $\Rightarrow \theta = 0$	$x = 2a\sin^2\theta$ $\Rightarrow 2a = 2a\sin^2\theta$ $\Rightarrow 1 = \sin^2\theta$ $\Rightarrow 1 = \sin\theta$ $\Rightarrow \sin\frac{\pi}{2} = \sin\theta$ $\Rightarrow \frac{\pi}{2} = \theta$ $\Rightarrow \theta = \frac{\pi}{2}$

From (ii),

Required area $= 2\int_0^{2a} \dfrac{x^{\frac{3}{2}}}{(2a-x)^{\frac{1}{2}}}dx$

$= 2\int_0^{\frac{\pi}{2}} \dfrac{(2a\sin^2\theta)^{\frac{3}{2}}}{(2a-2a\sin^2\theta)^{\frac{1}{2}}}4a\sin\theta\cos\theta\, d\theta$

$$= 2\int_0^{\frac{\pi}{2}} \frac{(2a)^{\frac{3}{2}}(\sin^2\theta)^{\frac{3}{2}}}{(2a)^{\frac{1}{2}}(1-\sin^2\theta)^{\frac{1}{2}}}4a\sin\theta\cos\theta\, d\theta = 8a\int_0^{\frac{\pi}{2}} \frac{(2a)^{\frac{3}{2}}\sin^3\theta}{(2a)^{\frac{1}{2}}(\cos^2\theta)^{\frac{1}{2}}}\sin\theta\cos\theta\, d\theta$$

$$= 8a\int_0^{\frac{\pi}{2}} \frac{(2a)^{\frac{3}{2}}\sin^3\theta}{(2a)^{\frac{1}{2}}\cos\theta}\sin\theta\cos\theta d\theta \; = 8a\int_0^{\frac{\pi}{2}} \frac{(2a)^{\frac{1}{2}+1}\sin^3\theta}{(2a)^{\frac{1}{2}}\cos\theta}\sin\theta\cos\theta\, d\theta$$

$$= 8a\int_0^{\frac{\pi}{2}} \frac{(2a)^{\frac{1}{2}}.2a.\sin^3\theta}{(2a)^{\frac{1}{2}}\cos\theta}\sin\theta\cos\theta\, d\theta \; = 8a\int_0^{\frac{\pi}{2}} \frac{2a.\sin^3\theta}{\cos\theta}\sin\theta\cos\theta\, d\theta$$

$$= 16a^2\int_0^{\frac{\pi}{2}} \sin^4\theta d\theta \; = 16a^2\int_0^{\frac{\pi}{2}} \sin^4\theta.1d\theta \; = 16a^2\int_0^{\frac{\pi}{2}} \sin^4\theta.(\cos\theta)^0 d\theta$$

$$= 16a^2\int_0^{\frac{\pi}{2}} \sin^4\theta.\cos^0\theta d\theta \text{ ---------------------------------(iii)}$$

We have,

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}\,(\cos\theta)^{2n-1}\,d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m\Gamma n}{\Gamma(m+n)}[\because \Gamma n = (n-1)!]$$

Here, from (v)

$$\begin{array}{lll} \mathbf{2m-1=4} & \& & \mathbf{2n-1=0} \\ \Rightarrow \mathbf{2m=4+1} & & \Rightarrow \mathbf{2n=1} \\ \Rightarrow \mathbf{2m=5} & & \Rightarrow \mathbf{2n=1} \\ \Rightarrow \mathbf{m=\frac{5}{2}} & & \Rightarrow \mathbf{n=\frac{1}{2}} \end{array}$$

From (iii), Required area

$$= 16a^2\int_0^{\frac{\pi}{2}} \sin^4\theta.\cos^0\theta d\theta$$

$$= 8a^2 \times 2\int_0^{\frac{\pi}{2}} \sin^4\theta.\cos^0\theta d\theta = 8a^2 \times \beta(m,n) = 8a^2 \times \beta(\frac{5}{2},\frac{1}{2}) \qquad [\because m=\frac{3}{2} \text{ and } n=\frac{1}{2}]$$

$$= 8a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{1}{2}\right)} \qquad [\because \beta(m,n) = = \frac{\Gamma m\Gamma n}{\Gamma(m+n)}]$$

$$= 8a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5+1}{2}\right)} = 8a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{6}{2}\right)} = 8a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} = 8a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{(3-1)!} \qquad [\because \Gamma n = (n-1)!]$$

$$= 8a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2!} = 8a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\times 1} = 8a^2 \frac{\Gamma\left(\frac{5}{2}\right).\sqrt{\pi}}{2\times 1} = 8a^2 \frac{\Gamma\left(\frac{3}{2}+1\right).\sqrt{\pi}}{2\times 1} = 8a^2 \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right).\sqrt{\pi}}{2}$$

$$= 8a^2 \frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right).\sqrt{\pi}}{2} = 8a^2 \frac{\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\sqrt{\pi}}{2} = 8a^2 \frac{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}.\sqrt{\pi}}{2} = 8a^2 \frac{\frac{3}{2}\times\frac{1}{2}\times\pi}{2}$$

$$= 8a^2 \frac{3\pi}{2\times 4} = 8a^2 \frac{3\pi}{8} = 3\pi a^2$$

Therefore the required area is $3a^2\pi$ *square .unit*

Example 195: Find the area of the segment cut off from the parabola $y^2 = 2x$ by the straight line $y = 4x - 1$.

Solution: Given the equations

$$y^2 = 2x \text{ ----------- (i)}$$
$$y = 4x - 1 \text{ ---------- (ii)}$$

The carve (i) is symmetrical about x axis and from (i) p (ii) we have

$$\mathbf{y^2 = 2x}$$
$$\Rightarrow \mathbf{(4x-1)^2 = 2x \ [\because y = 4x-1 \text{ from (ii)}]}$$
$$\Rightarrow \mathbf{16x^2 - 8x + 1 = 2x}$$
$$\Rightarrow \mathbf{16x^2 - 8x - 2x + 1 = 0}$$
$$\Rightarrow \mathbf{8x(2x-1) - 1(2x-1) = 0}$$
$$\Rightarrow \mathbf{(2x-1)(8x-1) = 0}$$
$$\Rightarrow \mathbf{(2x-1) = 0 \ \text{ and } \ (8x-1) = 0}$$
$$\Rightarrow \mathbf{2x = 1 \ \text{ and } \ 8x = 1}$$

i.e. $x = \mathbf{\frac{1}{2}, \frac{1}{8}}$

Putting the value of x in (ii),

$$\mathbf{y = 4x - 1}$$
$$\Rightarrow \mathbf{y = 4.\frac{1}{2} - 1 \ [\because x = \frac{1}{2}]}$$
$$\Rightarrow \mathbf{y = 2 - 1}$$
$$\Rightarrow \mathbf{y = 1}$$

Again, $\mathbf{y = 4x - 1}$

$$\Rightarrow \mathbf{y = 4.\frac{1}{8} - 1 \ [\because x = \frac{1}{8}]}$$
$$\Rightarrow \mathbf{y = \frac{1}{2} - 1}$$
$$\Rightarrow \mathbf{y = -\frac{1}{2}}$$

Therefore line cut the parabola at $\mathbf{A\left(\frac{1}{2}, 1\right), B\left(\frac{1}{8}, -\frac{1}{2}\right)}$. Draw figure. We are to find the area of OABO.

The area of OABO = trapezium ABCD – (area OADO + area OBCO)

$$= \frac{1}{2}\mathbf{(AD + BC) \times DC - \int_{-\frac{1}{2}}^{1} f(y)\,dy}$$

$$= \frac{1}{2}\mathbf{(AD + BC) \times DC - \int_{-\frac{1}{2}}^{1} x\,dy}$$

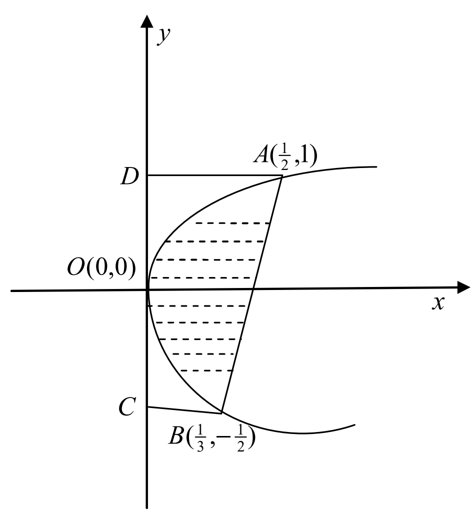


Figure No 98

Given, $\mathbf{y^2 = 2x}$

$$\therefore \mathbf{x = \frac{y^2}{2}}$$

$$\therefore \mathbf{x = f(y) = \frac{y^2}{2}}$$

$$= \frac{1}{2}\mathbf{(AD + BC) \times DC - \int_{-\frac{1}{2}}^{1} \frac{1}{2} y^2\,dy}$$

$$= \mathbf{\frac{1}{2}\left(\frac{1}{2} + \frac{1}{8}\right) \times \frac{3}{2} - \frac{1}{2} \times \frac{1}{3}\left[y^3\right]_{-\frac{1}{2}}^{1}}$$

$$= \mathbf{\frac{1}{2}\left(\frac{4+1}{8}\right) \times \frac{3}{2} - \frac{1}{2} \times \frac{1}{3}\left[y^3\right]_{-\frac{1}{2}}^{1} = \frac{1}{2}\left(\frac{5}{8}\right) \times \frac{3}{2} - \frac{1}{2} \times \frac{1}{3}\left[1^3 - (-\frac{1}{2})^3\right]}$$

$$= \mathbf{\frac{15}{32} - \frac{1}{6}\left[1 + \frac{1}{8}\right] = \frac{15}{32} - \frac{1}{6}\left[\frac{8+1}{8}\right] = \frac{15}{32} - \frac{1}{6}\left[\frac{9}{8}\right] = \frac{15}{32} - \frac{1}{2}\left[\frac{3}{8}\right] = \frac{15}{32} - \frac{3}{16}}$$

$$=\frac{15}{32}-\frac{3}{16}=\frac{9}{32}$$

Therefore the required area is $\frac{9}{32}$ square unit.

Example 196: Find the whole area of the cycloid $\mathbf{x = a(\theta + \sin\theta)}$ and $\mathbf{y = a(1-\cos\theta)}$) bounded by its base.

Solution: Given equations,

$\mathbf{x = a(\theta + \sin\theta)}$ -------------------------(i)

$\mathbf{y = a(1-\cos\theta)}$ -------------------------(ii)

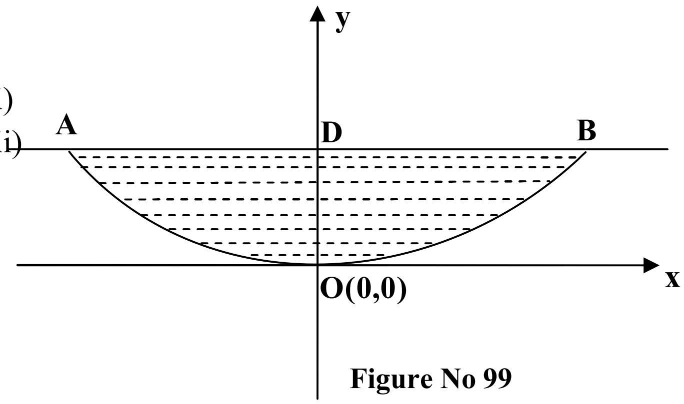


Figure No 99

When $y = 0$ then from (ii),

$$\mathbf{y = a(1-\cos\theta)}$$
$$\Rightarrow \mathbf{0 = a(1-\cos\theta)\ [\because y = 0]}$$
$$\Rightarrow \mathbf{0 = (1-\cos\theta)}$$
$$\Rightarrow \mathbf{-1 = -\cos\theta}$$
$$\Rightarrow \mathbf{1 = \cos\theta}$$
$$\Rightarrow \mathbf{\cos 0 = \cos\theta}$$
$$\Rightarrow \mathbf{0 = \theta}$$
$$\Rightarrow \mathbf{\theta = 0}$$

When $\mathbf{y = 2a}$ then from (ii),

$$\mathbf{y = a(1-\cos\theta)}$$
$$\Rightarrow \mathbf{2a = a(1-\cos\theta)\ [\because y = 2a]}$$
$$\Rightarrow \mathbf{2 = (1-\cos\theta)}$$
$$\Rightarrow \mathbf{2-1 = -\cos\theta}$$
$$\Rightarrow \mathbf{1 = -\cos\theta}$$
$$\Rightarrow \mathbf{-1 = \cos\theta}$$
$$\Rightarrow \mathbf{\cos\pi = \cos\theta}$$
$$\Rightarrow \mathbf{\pi = \theta}$$
$$\Rightarrow \mathbf{\theta = \pi}$$

Draw the graph. We are to find the area of AOBDA.

From (ii),

$$\mathbf{y = a(1-\cos\theta)}$$
$$\Rightarrow \frac{dy}{d\theta} = \frac{d}{d\theta}\{a(1-\cos\theta)\}$$
$$\Rightarrow \mathbf{\frac{dy}{d\theta} = \frac{d}{d\theta}(a - a\cos\theta)}$$
$$\Rightarrow \mathbf{\frac{dy}{d\theta} = \{0 - a(-\sin\theta)\}}$$
$$\Rightarrow \mathbf{\frac{dy}{d\theta} = a\sin\theta}$$
$$\Rightarrow \mathbf{dy = a\sin\theta\, d\theta}$$

Now the required area: $AOBDA = 2 \times area\ of\ ODBO$

$$= 2\int_0^{\pi} xdy$$

$$= 2\int_0^{\pi} a(\theta + \sin\theta)\, a\sin\theta\, d\theta \qquad [\because x = a(\theta + \sin\theta) \text{ and } dy = a\sin\theta\, d\theta]$$

$$= 2\int_0^{\pi} a^2(\theta + \sin\theta)\sin\theta\, d\theta = 2a^2\int_0^{\pi}(\theta + \sin\theta)\sin\theta\, d\theta$$

$$= 2a^2\int_0^{\pi}(\theta\sin\theta + \sin^2\theta)\, d\theta = 2a^2\int_0^{\pi}\theta\sin\theta\, d\theta + 2a^2\int_0^{\pi}\sin^2\theta\, d\theta$$

$$= 2a^2\int_0^{\pi}\theta\sin\theta\, d\theta + \frac{1}{2}\times 2a^2\int_0^{\pi}2\sin^2\theta\, d\theta$$

$$= 2a^2\int_0^{\pi}\theta\sin\theta\, d\theta + \frac{1}{2}\times 2a^2\int_0^{\pi}(1 - \cos 2\theta)\, d\theta \text{ ------------------------(iii)}$$

Now, $\int \theta\sin\theta\, d\theta$

$$= \theta\int\sin\theta\, d\theta - \int\{\frac{d}{d\theta}(\theta)\int\sin\theta\, d\theta\}d\theta \qquad [\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$= \theta\int\sin\theta\, d\theta - \int 1.(-\cos\theta)\, d\theta$$

$$= \theta(-\cos\theta) - \int 1.(-\cos\theta)\, d\theta$$

$$= \theta(-\cos\theta) + \int 1.(\cos\theta)\, d\theta$$

$$= \theta(-\cos\theta) + \sin\theta$$

$$= -\theta\cos\theta + \sin\theta$$

and

$\int(1 - \cos 2\theta)\, d\theta$

$$= \int 1d\theta - \int\cos 2\theta\, d\theta$$

$$= \theta - \frac{1}{2}\sin 2\theta$$

From (iii),

Now the required area: $AOBDA = 2 \times area\ of\ ODBO$

$$= 2a^2\int_0^{\pi}\theta\sin\theta\, d\theta + \frac{1}{2}\times 2a^2\int_0^{\pi}(1 - \cos 2\theta)\, d\theta$$

$$= 2a^2\left[-\theta\cos\theta + \sin\theta\right]_0^{\pi} + \frac{1}{2}\times 2a^2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= 2a^2\left[-\pi\cos\pi + \sin\pi - (-0\cos 0 + \sin 0)\right] + \frac{1}{2}\times 2a^2\left[\pi - \frac{1}{2}\sin 2\pi - (0 - \frac{1}{2}\sin 2\times 0)\right]$$

$$= 2a^2\left[-\pi(-1) + 0 - (-0.1 + 0)\right] + \frac{1}{2}\times 2a^2\left[\pi - \frac{1}{2}\times 0 - (0 - \frac{1}{2}\times 0)\right]$$

$$= 2a^2[\pi]+\frac{1}{2}\times 2a^2[\pi] = 2a^2[\pi]+a^2[\pi] = 2a^2\pi + a^2\pi = 3a^2\pi$$

Therefore the required area is $3a^2\pi$ square unit.

Example 197: Find the area of the loop of the curve $r = a\theta\cos\theta$ between 0 and $\frac{\pi}{2}$

Solution: Given, $r = a\theta\cos\theta$ -------------------------(i)

Therefore the required area $= \int_0^{\pi/2}\frac{1}{2}r^2 d\theta$

$$= \int_0^{\pi/2}\frac{1}{2}(a\theta\cos\theta)^2 d\theta \; [\because r = a\theta\cos\theta]$$

$$= \int_0^{\pi/2}\frac{1}{2}a^2\theta^2(\cos\theta)^2 d\theta = \int_0^{\pi/2}\frac{1}{2}a^2\theta^2\cos^2\theta d\theta$$

$$= \frac{1}{2}a^2\int_0^{\pi/2}\theta^2\cos^2\theta d\theta = \frac{1}{2}a^2\times\frac{1}{2}\times 2\int_0^{\pi/2}\theta^2\cos^2\theta d\theta$$

$$= \frac{1}{2}a^2\times\frac{1}{2}\int_0^{\pi/2}\theta^2 2\cos^2\theta d\theta = \frac{1}{2}a^2\times\frac{1}{2}\int_0^{\pi/2}\theta^2(1+\cos 2\theta)d\theta \quad [\because 2\cos^2\theta = 1+\cos 2\theta]$$

$$= \frac{1}{2}a^2\times\frac{1}{2}\int_0^{\pi/2}(\theta^2+\theta^2\cos 2\theta)d\theta \quad = \frac{1}{2}a^2\times\frac{1}{2}\int_0^{\pi/2}\theta^2 d\theta + \frac{1}{2}a^2\times\frac{1}{2}\int_0^{\pi/2}\theta^2\cos 2\theta d\theta$$

$$= \frac{1}{4}a^2\int_0^{\pi/2}\theta^2 d\theta + \frac{1}{4}a^2\int_0^{\pi/2}\theta^2\cos 2\theta d\theta$$

$$= \frac{1}{4}a^2\left[\frac{\theta^{2+1}}{2+1}\right]_0^{\pi/2} + \frac{1}{4}a^2\left[\theta^2\int\cos 2\theta d\theta - \int\left\{\frac{d}{d\theta}(\theta^2)\int\cos 2\theta d\theta\right\}d\theta\right]_0^{\pi/2}$$

$$[\because \int uv dx = u\int v dx - \int\{\frac{d}{dx}(u)\int v dx\}dx]$$

$$= \frac{1}{4}a^2\left[\frac{\theta^3}{3}\right]_0^{\pi/2} + \frac{1}{4}a^2\left[\theta^2\int\cos 2\theta - \int\left\{2\theta\frac{\sin 2\theta}{2}\right\}d\theta\right]_0^{\pi/2}$$

$$= \frac{1}{4}a^2\left[\frac{\theta^3}{3}\right]_0^{\pi/2} + \frac{1}{4}a^2\left[\theta^2\times\frac{\sin 2\theta}{2} - \int\{\theta\sin 2\theta\}d\theta\right]_0^{\pi/2}$$

$$= \frac{1}{4}a^2\left[\frac{\theta^3}{3}\right]_0^{\pi/2} + \frac{1}{4}a^2\left[\frac{1}{2}\theta^2\sin 2\theta - \int\theta\sin 2\theta d\theta\right]_0^{\pi/2}$$

$$=\frac{1}{4}a^2\left[\frac{\theta^3}{3}\right]_0^{\pi/2}+\left[\frac{1}{4}a^2\times\frac{1}{2}\theta^2\sin 2\theta\right]_0^{\pi/2}-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta$$

$$=\frac{1}{4}a^2\left[\frac{(\frac{\pi}{2})^3}{3}-\frac{0^3}{3}\right]+\frac{1}{4}a^2\times\frac{1}{2}\left[(\frac{\pi}{2})^2\sin 2\times\frac{\pi}{2}-0^2\sin 2\times 0\right]-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta$$

$$=\frac{1}{4}a^2\left[\frac{\frac{\pi^3}{8}}{3}-0\right]+\frac{1}{4}a^2\times\frac{1}{2}\left[\frac{\pi^2}{4}\sin\pi-0\right]-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta$$

$$=\frac{1}{4}a^2\left[\frac{\pi^3}{24}\right]+\frac{1}{4}a^2\times\frac{1}{2}\left[\frac{\pi^2}{4}\times 0-0\right]-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta \quad [\because \sin\pi=0]$$

$$=\frac{1}{4}a^2\left[\frac{\pi^3}{24}\right]+\frac{1}{4}a^2\times\frac{1}{2}[0-0]-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta$$

$$=\frac{1}{4}a^2\left[\frac{\pi^3}{24}\right]+\frac{1}{4}a^2\times\frac{1}{2}\times 0-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta$$

$$=\frac{1}{4}a^2\left[\frac{\pi^3}{24}\right]+0-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta \quad =\frac{1}{4}a^2\left[\frac{\pi^3}{24}\right]-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\int_0^{\pi/2}\theta\sin 2\theta d\theta$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[\theta\int\sin 2\theta d\theta-\int\left\{\frac{d}{d\theta}(\theta)\int\sin 2\theta d\theta\right\}d\theta\right]_0^{\pi/2}$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[\theta\left(-\frac{\cos 2\theta}{2}\right)-\int\left\{1.\left(-\frac{\cos 2\theta}{2}\right)\right\}d\theta\right]_0^{\pi/2}$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[\theta\left(-\frac{\cos 2\theta}{2}\right)+\frac{1}{2}\int\cos 2\theta d\theta\right]_0^{\pi/2}$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[-\frac{1}{2}\theta\cos 2\theta+\frac{1}{2}\int\cos 2\theta d\theta\right]_0^{\pi/2}$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[-\frac{1}{2}\theta\cos 2\theta+\frac{1}{2}\frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[-\frac{1}{2}\theta\cos 2\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[-\frac{1}{2}\frac{\pi}{2}\cos 2\times\frac{\pi}{2}+\frac{1}{4}\sin 2\times\frac{\pi}{2}-\left(-\frac{1}{2}\times 0\cos 2\times 0+\frac{1}{4}\sin 2\times 0\right)\right]$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[-\frac{1}{2}\frac{\pi}{2}\cos\pi+\frac{1}{4}\sin\pi-\left(-\frac{1}{2}\times 0\cos 0+\frac{1}{4}\sin 0\right)\right]$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[-\frac{1}{2}\frac{\pi}{2}\cos\pi+\frac{1}{4}\sin\pi-\left(-\frac{1}{2}\times 0\times 1+\frac{1}{4}\times 0\right)\right]$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[-\frac{1}{2}\frac{\pi}{2}(-1)+\frac{1}{4}\times 0-(0+0)\right]\quad=\frac{1}{96}a^2\pi^3-\frac{1}{4}a^2\left[+\frac{1}{4}\pi+0\right]$$

$$=\frac{1}{96}a^2\pi^3-\frac{1}{16}a^2\pi\quad=\frac{a^2\pi}{16}\left(\frac{1}{6}\pi^2-1\right)\ \text{Answer}$$