Planes

Defⁿ: A plane is a surface such that if any two points are taken on it, the straight line joining them lies wholly on the surface.

1. Prove that the general equation of first degree in x, y, z i.e. $ax+by+cz+d=0$ represents a plane.

Proof: Given eqⁿ. is $ax+by+cz+d=0 \longrightarrow (1)$

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) are two points taken on the surface (1).

$$\therefore \quad ax_1+by_1+cz_1+d=0 \longrightarrow (2)$$

$$ax_2+by_2+cz_2+d=0 \longrightarrow (3)$$

Multiplying (3) by λ and adding with (2) we get

$$a(\lambda x_2+x_1)+b(\lambda y_2+y_1)+c(\lambda z_2+z_1)+d(\lambda+1)=0$$

$$\text{i.e. } a\left(\frac{\lambda x_2+x_1}{\lambda+1}\right)+b\left(\frac{\lambda y_2+y_1}{\lambda+1}\right)+c\left(\frac{\lambda z_2+z_1}{\lambda+1}\right)+d=0 \longrightarrow (4)$$

Relation (4) shows that the point $\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right)$ lies on the surface (1). But the point lies on the line joining the points (x_1, y_1, z_1) & (x_2, y_2, z_2) which divides them in the ratio $\lambda : 1$. For different values of λ we get different points of the line joining the points (x_1, y_1, z_1) & (x_2, y_2, z_2) and all of them lie on the surface (1). $\therefore$ The st. line wholly lies on the surface.

$\therefore$ The surface (1) is a plane.

2. General eqⁿ. of a plane through one given point:

Let the equation of the plane be

$$ax+by+cz+d=0 \longrightarrow (1)$$

and (x_1, y_1, z_1) be any point lie on the plane (1).

$$\therefore \quad ax_1+by_1+cz_1+d=0 \longrightarrow (2)$$

Subtracting (2) from (1),

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \longrightarrow (3)$$

Eqⁿ. (3) is any plane through one given point (x_1, y_1, z_1).

3. To find the equation of a plane passing through three given points:

Sol: Let the given point are (x_1, y_1, z_1), (x_2, y_2, z_2) & $(x_3, \ldots$

As the points lies on the plane (1) so we have

$$ax_1 + by_1 + cz_1 + d = 0 \longrightarrow (2)$$
$$ax_2 + by_2 + cz_2 + d = 0 \longrightarrow (3)$$
$$ax_3 + by_3 + cz_3 + d = 0 \longrightarrow (4)$$

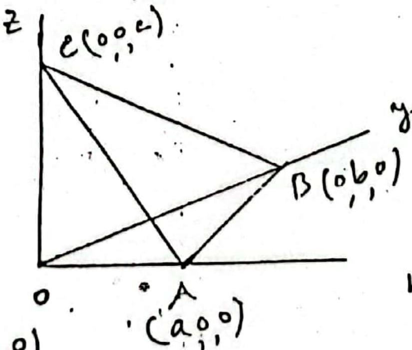
As a, b, c, d are unknowns so eliminating a, b, c, d from (1), (2), (3) & (4),

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \longrightarrow (5)$$

Eqn. (5) is the required plane.

3. Intercept form of the plane: To find the equation of a plane which intersect the axes with intercepts a, b & c.

Sol. Given that the plane intersect on the axes at A, B & C such that $OA = a$, $OB = b$ & $OC = c$ so the coordinates of A, B, C are $A(a,0,0)$ $B(0,b,0)$ and $C(0,0,c)$.



Let the eqn. of plane through $A(a,0,0)$ be

$$\alpha(x-a) + \beta(y-0) + \gamma(z-0) = 0$$

or, $\alpha x + \beta y + \gamma z - a\alpha = 0 \longrightarrow (1)$

If the points $(0,b,0)$ & $(0,0,c)$ also lies on the plane (1)

$$\therefore \quad 0\cdot\alpha + b\beta + 0\cdot\gamma - a\alpha = 0$$

or, $-a\alpha + b\beta + 0\cdot\gamma = 0 \longrightarrow (2)$

& $-a\alpha + 0\cdot\beta + c\gamma = 0 \longrightarrow (3)$

From (2) & (3), $\dfrac{\alpha}{bc} = \dfrac{\beta}{ca} = \dfrac{\gamma}{ab} = K$ (say)

or, $\alpha = Kbc$, $\beta = Kca$, $\gamma = Kab$.

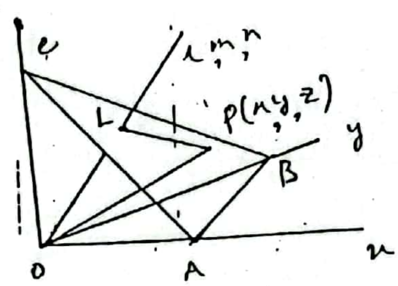
Putting these in eqn. (1)

$$\therefore K\{bc(x-a) + cay + abz\} = 0$$

or, $bcx + cay + abz = abc \quad [K \neq 0]$

or, $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1 \longrightarrow (4)$

4. **Normal form of a plane:** To find the equation of a plane in terms of p the length of perpendicular from the origin to the plane and the direction cosines are l, m, n.



Sol. Let ABC be the required plane. Let OL be the normal to the plane such that $OL = p$ and the d.c's of OL are l, m, n. Taking a point $P(x, y, z)$ on the plane and PL is drawn perpendicular on OL & joining OP.

Taking the projection of OP on OL then we get

$$OL = (x-0)l + (y-0)m + (z-0)n$$

$$\text{or, } p = lx + my + nz$$

$$\text{or, } lx + my + nz = p \rightarrow (1)$$

which is the eq^n. of the plane in the normal form. Here the co-efficients of x, y, z i.e. l, m, n are the d.c's of the normal to the plane.

5. **Reduction of the general equation of a plane into the normal form:**

Sol. Let the eq^n of the plane in the general form be

$$ax + by + cz + d = 0 \rightarrow (1)$$

and its normal form be $lx + my + nz = p. \rightarrow (2)$

By the condition eq^n. (1) & (2) represent the same plane so that their co-efficients are proportional.

$$\therefore \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{p}{-d}$$

$$\text{or, } l = -ap/d, \quad m = -bp/d, \quad n = -cp/d.$$

As $l^2 + m^2 + n^2 = 1$ so $\frac{p^2}{d^2}(a^2 + b^2 + c^2) = 1$

$$\text{or, } p^2 = d^2/(a^2+b^2+c^2) = \frac{d^2}{\Sigma a^2}$$

$$\text{or, } p = \pm \frac{d}{\sqrt{\Sigma a^2}}$$

$$\therefore l = -\frac{a}{d}\left(\frac{-d}{\sqrt{\Sigma a^2}}\right) = \frac{a}{\sqrt{\Sigma a^2}}, \quad m = -\frac{b}{d}\left(\frac{-d}{\sqrt{\Sigma a^2}}\right) = \frac{b}{\sqrt{\Sigma a^2}}$$

$$\& \; n = -\frac{c}{d}\left(-\frac{d}{\sqrt{\Sigma a^2}}\right) = \frac{c}{\sqrt{\Sigma a^2}}$$

i.e. the d.c's of the normal are $\frac{a}{\sqrt{\Sigma a^2}}, \frac{b}{\sqrt{\Sigma a^2}}, \frac{c}{\sqrt{\Sigma a^2}}$ the d.c's of the normal to the

... are the d.r's of the ...

(4)

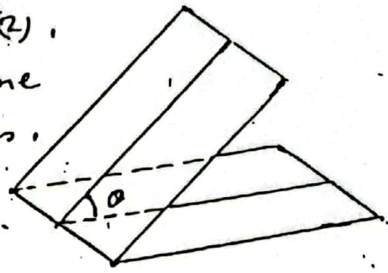
6. Angle between two planes:

Let the eqn. of two planes are

$$a_1x+b_1y+c_1z+d_1=0 \longrightarrow (1)$$

&

$$a_2x+b_2y+c_2z+d_2=0, \longrightarrow (2)$$



Let θ be the angle between the planes (1) & (2). We have angle betn two planes is the same as the angle betn the normals to the planes. Here a_1, b_1, c_1 and a_2, b_2, c_2 are the d.r's of the normals to the planes (1) & (2). Angle betn them must be

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\Sigma a_1^2 \, \Sigma a_2^2}} \longrightarrow (3)$$

and

$$\sin\theta = \left\{\frac{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}{\Sigma a_1^2 \, \Sigma a_2^2}\right\}^{1/2} \longrightarrow (4)$$

i.e. Either (3) or (4) gives the angle between the planes (1) & (2).

Corl-1 Condition of perpendicularity of two planes:

If the planes (1) & (2) are perpendicular then $\theta=\pi/2$ so that $\cos\theta=\cos\pi/2=0$.

$$\therefore \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{\Sigma a_1^2 \, \Sigma a_2^2}}=0$$

or, $a_1.a_2+b_1b_2+c_1c_2=0$.

which is the condition of perpendicularity.

Corl-2 condition of parallelism:

If the planes are parallel then $\theta=0$ or π so that $\sin\theta=0$.

$$\therefore (b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2=0$$

or, $(b_1c_2-b_2c_1)^2=0$, $(c_1a_2-c_2a_1)^2=0$, $(a_1b_2-a_2b_1)^2=0$

or, $b_1c_2-b_2c_1=0$, $c_1a_2-c_2a_1=0$, $a_1b_2-a_2b_1=0$

or, $\frac{b_1}{b_2}=\frac{c_1}{c_2}$, $\frac{c_1}{c_2}=\frac{a_1}{a_2}$, $\frac{a_1}{a_2}=\frac{b_1}{b_2}$

$$\therefore \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}.$$

which is the condition of parallelism.

7. Plane parallel to a given plane and passing through a given point:

Let the given plane be

$$ax+by+cz+d=0 \longrightarrow (1)$$

and the given point be (x_1, y_1, z_1).

We have eq^n. of a plane parallel to (1) is

$$ax+by+cz+K=0. \longrightarrow (2)$$

If (x_1, y_1, z_1) lies on the plane (2) then

$$ax_1+by_1+cz_1+K=0. \longrightarrow (3)$$

Eliminating K from (2) & (3) we get

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0. \longrightarrow (4)$$

Eq^n. (4) is the required plane parallel to eq^n. (1).

8. Plane perpendicular to a given plane and passing through two given points:

Let the given points are (x_1, y_1, z_1), (x_2, y_2, z_2) and the given plane be

$$ax+by+cz+d=0. \longrightarrow (1)$$

We have any plane through (x_1, y_1, z_1) be

$$a_1(x-x_1)+b_1(y-y_1)+c_1(z-z_1)=0, \longrightarrow (2)$$

since (x_2, y_2, z_2) lies on the plane (2) so we have

$$a_1(x_2-x_1)+b_1(y_2-y_1)+c_1(z_2-z_1)=0. \longrightarrow (3)$$

Also if the plane (2) is perpendicular to (1) so we have

$$a_1a+b_1b+c_1c=0. \longrightarrow (4)$$

Eliminating a_1, b_1, c_1 from (2), (3) & (4)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0. \longrightarrow (5)$$

Eq^n. (5) is the required plane.

9. Planes perpendicular to coordinate planes:

We have equations of coordinate planes are $x=0$, $y=0$ and $z=0$. Let $ax+by+cz+d=0$ be the plane perpendicular to yz-plane i.e. $x=0$ or, $x+0\cdot y+0\cdot z+0=0$.

If these planes are perpendicular to each other then

$a\cdot1+b\cdot0+c\cdot0=0$ or, $a=0$ so that the plane perpendicular to yz-plane is $by+cz+d=0$ which is parallel to x-axis. Similarly the planes $ax+cz+d=0$ [illegible] are $\perp^r$ to zx & xy-planes [illegible]

Plane through the line of intersection of two planes:

Let the given planes are

$$a_1x+b_1y+c_1z+d=0 \longrightarrow (1)$$

& $$a_2x+b_2y+c_2z+d_2=0. \longrightarrow (2)$$

Any plane through the intersection of (1) & (2) is

$$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2)=0. \longrightarrow (3)$$

If the plane (3) is parallel to or perpendicular to a given plane or if the plane (3) passes through a given point then λ can be determined.

11. To find the condition that three planes may have a common line of intersection:

Let the equation of the planes are

$$u_1 \equiv a_1x+b_1y+c_1z+d_1=0 \longrightarrow (1)$$
$$u_2 \equiv a_2x+b_2y+c_2z+d_2=0 \longrightarrow (2)$$
$$u_3 \equiv a_3x+b_3y+c_3z+d_3=0 \longrightarrow (3)$$

We have any plane through the line of intersection of (1) & (2) is, $u_1+\lambda u_2=0. \longrightarrow (4)$

If the planes (1), (2) & (3) passes through one line then the planes $u_1+\lambda u_2=0$ & $u_3=0$ represent the same plane.

$$\therefore u_1+\lambda u_2=-\mu u_3 \quad \text{or, } u_1+\lambda u_2+\mu u_3=0$$

$$\text{or, } (a_1+\lambda a_2+\mu a_3)x+(b+\lambda b_2+\mu b_3)y+(c_1+\lambda c_2+\mu c_3)z + d_1+\lambda d_2+\mu d_3=0$$

Now, the co-efficients are zero seperately,

$$\therefore a_1+\lambda a_2+\mu a_3=0,\ b_1+\lambda b_2+\mu b_3=0,\ c_1+\lambda c_2+\mu c_3=0$$
$$\& \ d_1+\lambda d_2+\mu d_3=0.$$

Eliminating λ & μ from the above relations in the matrix form as

$$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}=0 \longrightarrow (4)$$

In the matrix (4) at least two of the 3rd order determinant must be zero.

12. Length of perpendicular from a point to a plane:

Let the equation of the plane be

$$lx+my+nz=p \longrightarrow (1)$$

and the point be (x_1, y_1, z_1).

We have any plane through the point $P(x_1, y_1, z_1)$ and parall-