1. Define Fourier series in the interval $(-L, L)$. Sketch the following function for four cycles, CO3 U

$$y = f(t) = \begin{cases} 0; & -4 \leq t < 0 \\ 4; & 0 \leq t < 4 \end{cases}$$

Also find the Fourier series for the function.

OR

Derive the complex form of Fourier series.

بِسْمِ اللّٰهِ الرَّحْمٰنِ الرَّحِيْمِ

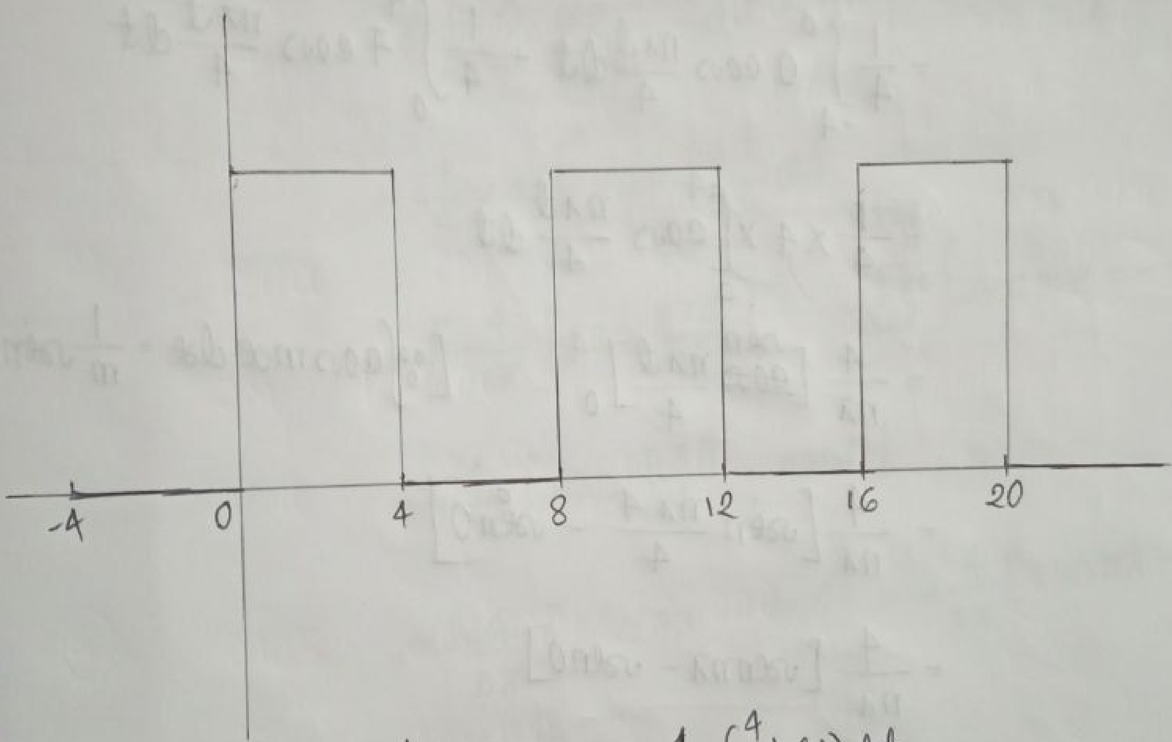
Ikram

Autumn - 22

Answer to the Q.no-1

$$y = f(t) = \begin{cases} 0; & -4 \le t \le 0 \\ 4; & 0 \le t < 4 \end{cases}$$

$$f(t) = f(t+8) \qquad T = 2L = 8 \quad \therefore L = 4$$



$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{1}{4}\int_{-4}^{4} f(t)\,dt$$

$$= \frac{1}{4}\int_{-4}^{0} 0.\,dt + \frac{1}{4}\int_{0}^{4} 4\,dt$$

$$= \frac{1}{4}\left[4t\right]_0^4 = \frac{1}{4} \times 16 = 4$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt$$

$$= \frac{1}{4}\int_{-4}^{4} f(t)\cos\frac{n\pi t}{4}\,dt$$

$$= \frac{1}{4}\int_{-4}^{0} 0\cos\frac{n\pi t}{4}\,dt + \frac{1}{4}\int_{0}^{4} 4\cos\frac{n\pi t}{4}\,dt$$

$$= \frac{1}{4}\times 4\times\int_{0}^{4}\cos\frac{n\pi t}{4}\,dt$$

$$= \frac{4}{n\pi}\left[\sin\frac{n\pi t}{4}\right]_0^4 \qquad \left[\because \int\cos mx\,dx = \frac{1}{m}\sin mx\right]$$

$$= \frac{4}{n\pi}\left[\sin\frac{n\pi 4}{4} - \sin 0\right]$$

$$= \frac{4}{n\pi}\left[\sin n\pi - \sin 0\right]$$

$$= \frac{4}{n\pi}\sin n\pi.$$

$$= 0 \qquad [\because \sin\pi = \sin 2\pi \ldots \sin n\pi = 0]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\, \sin \frac{n\pi t}{L}\, dt$$

$$= \frac{1}{4}\int_{-4}^{4} f(t)\, \sin \frac{n\pi t}{4}\, dt$$

$$= \frac{1}{4}\int_{-4}^{0} 0 \cdot \sin \frac{n\pi t}{4}\, dt + \frac{1}{4}\int_{0}^{4} 4 \cdot \sin \frac{n\pi t}{4}\, dt$$

$$= \frac{1}{4} \times 4 \int_{0}^{4} \sin \frac{n\pi t}{4}\, dt$$

$$= \left\{ \frac{4}{n\pi} \left[-\cos \frac{n\pi t}{4}\right]_0^4 \right. \qquad \left[\because \int \sin mx = -\frac{1}{m}\cos mx\right]$$

$$= \frac{-4}{n\pi}\left[\cos \frac{n\pi 4}{4} - \cos 0\right]$$

$$= -\frac{4}{n\pi}\left[\cos n\pi - \cos 0\right] = -\frac{4}{n\pi}\left[\cos n\pi - 1\right]$$

The fourier series:

$$f(t) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi t}{L} + \sum b_n \sin \frac{n\pi t}{L}$$

$$= \frac{\cancel{5}4}{2} + \sum 0 \cdot \cos \frac{n\pi t}{L} + \sum -\frac{4}{n\pi}\left[\cos n\pi - 1\right] \sin \frac{n\pi t}{L}$$

$$= 2 \cancel{*} - \frac{4}{n\pi} \sum (\cos n\pi - 1) \sin \frac{n\pi t}{L}$$

(Ans.),

Problem 19: Derive Complex form of Fourier series

OR

We have, the trigonometric form of Fourier series is:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\omega t)+\sum_{n=1}^{\infty}b_n\sin(n\omega t)\text{ ------------------(iv)}$$

We have,

$$e^x=1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\text{-----------------}$$

Put $x=ix$,

$$e^{ix}=1+\frac{ix}{1!}+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\frac{(ix)^6}{6!}+\frac{(ix)^7}{7!}+\text{---------}$$

$$[i^2=-1;\ i^3=i^2.i=-i;\ i^4=i^2.i^2=(-1).(-1)=+1;\ i^5=i^4.i=i]$$

$$e^{ix}=1+\frac{ix}{1!}+\frac{-x^2}{2!}+\frac{-ix^3}{3!}+\frac{x^4}{4!}+\frac{ix^5}{5!}+\frac{-x^6}{6!}+\frac{-ix^7}{7!}+\text{---------}$$

$$e^{ix}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\text{----}+\left(\frac{ix^1}{1!}-\frac{ix^3}{3!}+\frac{ix^5}{5!}-\frac{ix^7}{7!}+\text{--------}\right)$$

$$e^{ix}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\text{----}+i\left(\frac{x^1}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\text{--------}\right)$$

$$\left[\because \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\text{-----};\quad \sin x=\frac{x^1}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\text{----------}\right]$$

$$\therefore e^{ix}=\cos x+i\sin x\text{ ------------------------------------(v)}$$

Similarly,

$$e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + ----------------$$

Put $x = -ix$,

$$e^{-ix} = 1 + \frac{-ix^1}{1!} + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \frac{(-ix)^5}{5!} + \frac{(-ix)^6}{6!} + \frac{(-ix)^7}{7!} + ----$$

$[(-i)^2 = -1; (-i)^3 = (-i)^2.(-i) = i; (-i)^4 = (-i)^2.(-i)^2 = (-1).(-1) = +1;$

$(-i)^5 = (-i)^4.(-i) = (+1).(-i) = -i]$

$$e^{-ix} = 1 + \frac{-ix^1}{1!} + \frac{-x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{-ix^5}{5!} + \frac{-x^6}{6!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{ix^1}{1!} - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ----------- \frac{ix^1}{1!} + \frac{ix^3}{3!} - \frac{ix^5}{5!} + \frac{ix^7}{7!} + ----$$

$$e^{-ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + -------------- i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ---\right)$$

$$[\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + -----; \quad \sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} --------]$$

$\therefore e^{-ix} = \cos x - i \sin x$ --(vi)

Adding (v) and (vi),

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

$e^{ix} + e^{-ix} = 2\cos x$

$\therefore \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ --(vii)

Again Subtracting (v) and (vi)

$e^{ix} = \cos x + i \sin x$

$e^{-ix} = \cos x - i \sin x$

$e^{ix} - e^{-ix} = 2i \sin x$

$\therefore \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ --(viii)

From equation (vii)

$\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$

Hence From equation (vii), we can write,

$$\therefore \cos(n\omega t) = \frac{1}{2}(e^{in\omega t} + e^{-in\omega t}) \text{ ------------------------------(ix)}$$

And from equation (vii)

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

Hence from equation (vii), we can write

$$\therefore \sin(n\omega t) = \frac{1}{2i}(e^{in\omega t} - e^{-in\omega t}) \text{ ---------------------------(x)}$$

Putting the values of (ix) and (x) in (iv), we get,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \{\frac{1}{2}(e^{in\omega t} + e^{-in\omega t})\} + \sum_{n=1}^{\infty} b_n \{\frac{1}{2i}(e^{in\omega t} - e^{-in\omega t})\}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \{\frac{1}{2}(e^{in\omega t} + e^{-in\omega t})\} + b_n \{\frac{1}{2i}(e^{in\omega t} - e^{-in\omega t})\}]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\{\frac{1}{2}(a_n e^{in\omega t} + a_n e^{-in\omega t})\} + \{\frac{1}{2i}(b_n e^{in\omega t} - b_n e^{-in\omega t})\}]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\frac{1}{2}(a_n e^{in\omega t} + a_n e^{-in\omega t}) + \frac{1}{2i}(b_n e^{in\omega t} - b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\frac{1}{2}(a_n e^{in\omega t}) + \frac{1}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{1}{2i}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(-1)}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(-1)}{2i}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(i^2)}{2i}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(i^2)}{2i}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\frac{1}{2}(a_n e^{in\omega t}) + \frac{(-1)(i\ )}{2}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{(-1)(i\ )}{2}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\frac{1}{2}(a_n e^{in\omega t}) + \frac{-i}{2}(b_n e^{in\omega t}) + \frac{1}{2}(a_n e^{-in\omega t}) + \frac{-i}{2}(-b_n e^{-in\omega t})]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\{\frac{1}{2}(a_n e^{in\omega t}) - \frac{i}{2}(b_n e^{in\omega t})\} + \{\frac{1}{2}(a_n e^{-in\omega t}) + \frac{i}{2}(b_n e^{-in\omega t})\}]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [\frac{1}{2}(a_n - ib_n)e^{in\omega t}) + \frac{1}{2}(a_n + ib_n)e^{-in\omega t})] \text{ ----------------------(xi)}$$

Let, $\mathbf{c_0 = \frac{a_0}{2}}$ --------------------------------------(a)

$c_n = \frac{1}{2}(a_n - ib_n)$ --------------------------------(b)

$c^*_n = \frac{1}{2}(a_n + ib_n)$ ------------------------------(c)

Then from (viii), we get the series is:

$$f(t) = c_0 + \sum_{n=1}^{\infty}[c_n e^{in\omega t} + c^*_n e^{-in\omega t}]$$

$$f(t) = c_0 + \sum_{n=1}^{\infty}[c_n e^{in\omega t} + c_{-n} e^{-in\omega t}] \text{ [Say } c^*_n = c_{-n}]$$

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega t} + \sum_{n=1}^{\infty} c_{-n} e^{-in\omega t} \text{ -------(d)= ---+------+.......}$$

$$f(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega t} + \sum_{n=-1}^{-\infty} c_n e^{in\omega t} \text{ ---------(e)=----------+-------+-----}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \qquad \text{--------------(f)}$$

[We have, $\mathbf{c_n e^{in\omega t}}$ -1 0 1

Put n = 0, then

$\mathbf{c_n e^{in\omega t}}$

$= c_0 e^{i\times 0\times\omega t}$

$= c_0 e^0 = c_0.1 = c_0$]

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$ ---------------------------------------(xii) ***Which is referred to as the complex or exponential form of the Fourier Series expansion of the function f(t)***

Where

$$\because e^{ix} = \cos x + i\sin x \quad \text{[from (v)]}$$

$$\therefore e^{in\omega t} = \cos n\omega t + i\sin n\omega t \text{ ------------------------------(xiii)}$$

$$c_0 = \frac{a_0}{2} = \frac{1}{2}a_0 = \frac{1}{2}\frac{1}{L}\int_{-L}^{L} f(t)dt$$

$$c_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt$$

$$c_0 = \frac{1}{T}\int_{-L}^{L} f(t)dt \text{ [Where Period T= 2L]----------------(xiv)}$$

and

$$c_n = \frac{1}{2}(a_n - ib_n) \text{ --(xv)}$$

$$c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n) \text{ --------------------------------(xvi)}$$

We have,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt$$

$$\therefore c_n = \frac{1}{2}(a_n - ib_n)$$

$$= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt - i\frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\int_{-L}^{L}\{f(t)\cos(n\omega t) - if(t)\sin(nwt)\}dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)\{\cos(n\omega t) - i\sin(nwt)\}dt$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\omega t}dt \text{ [Since } e^{-ix} = \cos x - i\sin x \text{ from (vi)]}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\omega t}dt$$

$$c_n = \frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt \text{ [} T = 2L\text{] ---------------------(xvii)}$$

and

$$c^*_n = c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$= \frac{1}{2}\left(\frac{1}{L}\int_{-L}^{L} f(t)\cos(n\omega t)dt + i\frac{1}{L}\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\left(\int_{-L}^{L} f(t)\cos(n\omega t)dt + i.\int_{-L}^{L} f(t)\sin(nwt)dt\right)$$

$$= \frac{1}{2L}\left(\int_{-L}^{L} f(t)\{\cos(n\omega t)dt + i\sin(nwt)dt\ )\}$$

$$= \frac{1}{2L}\int_{-L}^{L} f(t)e^{in\omega t}dt \ [\because e^{ix} = \cos x + i\sin x \text{ from (2)]}$$

$$\therefore c^*_n = c_{-n} = \frac{1}{2L}\int_{-L}^{L} f(t)e^{in\omega t}dt$$

$$\therefore c^*_n = c_{-n} = \frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega t}dt \text{ [Period } T = 2L\text{]-------(xviii)}$$

In summary, the complex form of the Fourier series expansion of a periodic function f (t), of period T, is:

$$\mathbf{f(t)=\sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$$

Where,

$$\mathbf{c_0=\frac{1}{T}\int_{-L}^{L} f(t)dt}$$

$$\mathbf{c_n=\frac{1}{T}\int_{-L}^{L} f(t)e^{-in\omega t}dt}$$

$$\mathbf{c^*{}_n=c_{-n}=\frac{1}{T}\int_{-L}^{L} f(t)e^{in\omega t}dt}$$

2. a) Find Harmonic analysis of the given Fourier series CO4

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n}\sin\frac{n\pi t}{3}$$

b) Plot the line (at least 6) spectrum (discrete frequency spectra) for the Fourier series CO4

$$\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{5}_{\text{DC value}} + \underbrace{\sum_{n=1}^{\infty}\frac{4}{n}.\sin 2n\pi t}_{\text{AC value}}$$

2(a)

Given,

$$f(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+2}}{n} \sin \frac{n\pi t}{3}$$

$a_n = 0 \qquad b_n = \frac{4}{\pi} \frac{(-1)^{n+2}}{n}$

$R = \sqrt{a^2 + b^2} \qquad R_n = C_n = \sqrt{a_n^2 + b_n^2}$

$a_n = 0 \qquad b_n = \frac{4}{\pi} \frac{(-1)^{n+2}}{n} \qquad R_n = C_n = \sqrt{a_n^2 + b_n^2}$

$a_1 = 0 \qquad b_1 = \frac{4}{\pi} \frac{(-1)}{1} \qquad C_1 = \sqrt{\left(-\frac{4}{\pi}\right)^2} = 1.27$

$a_2 = 0 \qquad b_2 = \frac{\cancel{4}^{2}}{\pi} \frac{1}{\cancel{2}} = \frac{2}{\pi} \qquad C_2 = \sqrt{\left(\frac{2}{\pi}\right)^2} = 0.63$

$a_3 = 0 \qquad b_3 = \frac{4}{\pi} \frac{(-1)}{3} = -\frac{4}{3\pi} \qquad C_3 = \sqrt{\left(-\frac{4}{3\pi}\right)^2} = 0.42$

$a_4 = 0 \qquad b_4 = \frac{\cancel{4}}{\pi} \frac{1}{\cancel{4}} = \frac{1}{\pi} \qquad C_4 = \sqrt{\left(\frac{1}{\pi}\right)^2} = 0.32$

$a_5 = 0 \qquad b_5 = \frac{4}{\pi} \cdot \frac{(-1)}{5} = -\frac{4}{5\pi} \qquad C_5 = \sqrt{\left(-\frac{4}{5\pi}\right)^2} = 0.25$

$a_6 = 0 \qquad b_6 = \frac{\cancel{4}^{2}}{\pi} \cdot \frac{1}{\cancel{6}_3} = \frac{2}{3\pi} \qquad C_6 = \sqrt{\left(\frac{2}{3\pi}\right)} = 0.21$

Here

$$n\omega = \frac{n\pi}{3}$$

$n=1$ 1st Harmonic $\omega = \frac{\pi}{3} = 1.05$

$n=2$ 2nd " $\omega = \frac{2\pi}{3} = 2.09$

$n=3$ 3rd " $\omega = \frac{3\pi}{3} = 3.14$

$n=4$ 4th " $\omega = \frac{4\pi}{3} = 4.19$

$n=5$ 5th " $\omega = \frac{5\pi}{3} = 5.23$

$n=6$ 6th " $\omega = \frac{\cancel{6}^{2}\pi}{\cancel{3}} = 6.28.$

(b)

Given,

$$v(t) = 5 + \sum_{n=1}^{\infty} \frac{4}{n} \sin 2n\pi t$$

$a_n = 0$ $b_n = \frac{4}{n}$ $C_n = \sqrt{a_n^2 + b_n^2}$

$a_1 = 0$ $b_1 = \frac{4}{1}$ $C_1 = \sqrt{(4)^2} = 4$

$a_2 = 0$ $b_2 = 2$ $C_2 = \sqrt{(2)^2} = 2$

$a_3 = 0$ $b_3 = \frac{4}{3}$ $C_3 = \sqrt{(\frac{4}{3})^2} = 1.33$

$a_4 = 0$ $b_4 = \frac{4}{4}$ $C_4 = 1.$

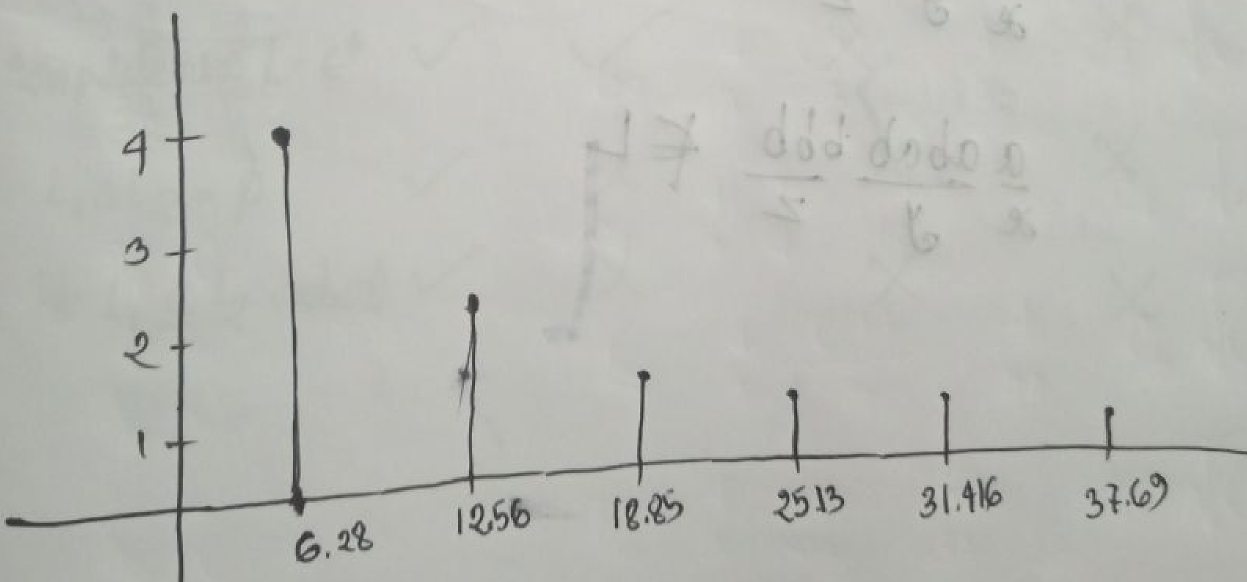
$a_5 = 0$ $\qquad b_5 = \frac{4}{5}$ $\qquad c_5 = 0.8$

$a_6 = 0$ $\qquad b_6 = \frac{\cancel{4}^{2}}{\cancel{6}_{3}}$ $\qquad c_6 = 0.67$

Here

$n\omega = 2n\pi$

$n=1$	1st Harmonic	$\omega = 2\pi = 6.28$
$n=2$	2nd "	$\omega = 4\pi = 12.56$
$n=3$	3rd "	$\omega = 6\pi = 18.85$
$n=4$	4th "	$\omega = 8\pi = 25.13$
$n=5$	5th "	$\omega = 10\pi = 31.416$
$n=6$	6th "	$\omega = 12\pi = 37.69$



Line spectrum.

3	a)	Prove that $L(f''(t)) = s^2 L\{f(t)\} - s\,f(0) - f'(0)$	CO3
		OR	
		Find the inverse Laplace transform of $\frac{s+4}{s(s-1)(s-2)}$	

Example 76: Prove that $L(f''(t)) = s^2 L\{f(t)\} - s\,f(0) - f'(0)$

We have

$$L(f(t)) = \int_0^\infty f(t)e^{-st}\,dt \quad \text{---(i)}$$

$$\therefore L(f'(t)) = \int_0^\infty f'(t)e^{-st}\,dt \quad [f(t) = f'(t)] \text{ -------------------------(ii)}$$

Now, $\int f'(t)e^{-st}\,dt$

$$\int f'(t)e^{-st}dt = e^{-st}\int f'(t)\,dt - \int\{\frac{d}{dt}(e^{-st})\int f'(t)\,dt\}dt$$

$$[\because \int uvdx = u\int vdx - \int\{\frac{d}{dx}(u)\int vdx\}dx]$$

$$\int f'(t)e^{-st}dt = e^{-st}\int \frac{d}{dt}(f(t))\,dt - \int\{\frac{d}{dt}(e^{-st})\int \frac{d}{dt}(f(t))\,dt\}dt \qquad [\because \frac{d}{dt}(f(t)) = f'(t)]$$

$$\int f'(t)e^{-st}dt = e^{-st}f(t) - \int\{\frac{d}{dt}(e^{-st})f(t)\}dt$$

$$\int f'(t)e^{-st}dt = e^{-st}f(t) - \int -s(e^{-st})f(t)dt$$

$$[\because \frac{d}{dx}(e^{mx}) = e^{mx}.\frac{d}{dx}(mx) = e^{mx}.(m) = me^{mx}]$$

$$\int f'(t)e^{-st}dt = e^{-st}f(t) + \int s(e^{-st})f(t)dt$$

$$\int f'(t)e^{-st}dt = e^{-st}f(t) + s\int e^{-st}f(t)dt$$

From (ii),

$$\therefore L(f'(t)) = \int_0^\infty f'(t)e^{-st}dt = \left[e^{-st}f(t)\right]_0^\infty + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[e^{-s\times\infty}f(\infty) - e^{-s\times 0}f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[e^{-\infty}f(\infty) - e^{0}f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{e^\infty}f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{e^\infty}f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[\frac{1}{\infty}f(\infty) - 1.f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = \left[0.f(\infty) - f(0)\right] + s\int_0^\infty e^{-st}f(t)dt$$

$$\therefore L(f'(t)) = -f(0) + s\int_0^\infty e^{-st} f(t)dt$$

$$\therefore L(f'(t)) = -f(0) + sL\{f(t)\} \qquad [\because L(f(t)) = \int_0^\infty f(t)e^{-st}dt]$$

$$\therefore L(f'(t)) = -f(0) + sL\{f(t)\}$$

$$\therefore L(f'(t)) = sL\{f(t)\} - f(0) \qquad \text{--(iii)}$$

Now replacing $f(t)$ by $f'(t)$ and $f'(t)$ by $f''(t)$ in (iii), we get

$$\therefore L(f'(t)) = sL\{f(t)\} - f(0)$$

$$\therefore L(f''(t)) = sL\{f'(t)\} - f'(0) \qquad \text{--(iv)}$$

Putting the value of $L(f'(t))$ from (iii) in (iv), we get

$$\therefore L(f''(t)) = sL\{f'(t)\} - f'(0)$$

$$\therefore L(f''(t)) = s[sL\{f(t)\} - f(0)] - f'(0) \qquad [L(f'(t)) = sL\{f(t)\} - f(0)]$$

$$\therefore L(f''(t)) = s^2L\{f(t)\} - s\,f(0) - f'(0) \qquad \text{--(v)}$$

$$\therefore L(f''(t)) = s^2L\{f(t)\} - s\,f(0) - f'(0) \qquad \text{(Proved)}$$

Find inverse Laplace transform of: $\dfrac{s+4}{s(s-1)(s-2)}$

OR

Solution:

Let,

$$\frac{s+4}{s(s-1)(s-2)}=\frac{A}{s}+\frac{B}{(s-1)}+\frac{C}{(s-2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Multiplying by $s(s-1)(s-2)$ in both sides

$$\Rightarrow \frac{s+4}{s(s-1)(s-2)}\times s(s-1)(s-2)=A\frac{s(s-1)(s-2)}{s}+B\frac{s(s-1)(s-2)}{(s-1)}+C\frac{s(s-1)(s-2)}{(s-2)}$$

$$\Rightarrow s+4=A(s-1)(s-2)+Bs(s-2)+Cs(s-1) \quad \ldots\ldots\ldots\ldots (ii)$$

Put $s=0$ in equation (ii),

$$\Rightarrow 0+4=A(0-1)(0-2)+B\times 0(0-2)+C\times 0(0-1)$$

$$\Rightarrow 4=2A$$

$\therefore A = 2$

Put $s-1=0$, i.e. $s=1$ in equation (ii),

$\Rightarrow 1+4 = A(1-1)(1-2) + B\times 1(1-2) + C\times 1(1-1)$

$\Rightarrow 5 = 0 - B + 0$

$\therefore B = -5$

Put $s-2=0$, i.e. $s=2$ in equation (ii),

$\Rightarrow 2+4 = A(2-1)(2-2) + B\times 2(2-2) + C\times 2(2-1)$

$\Rightarrow 6 = 0+0+C(4-2)$

$\Rightarrow 6 = 0+0+2C$

$\therefore C = 3$

Putting the value of A, B, C in equation (i), we get,

$$\frac{s+4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-2)}$$

$$\frac{s+4}{s(s-1)(s-2)} = \frac{2}{s} + \frac{-5}{(s-1)} + \frac{3}{(s-2)}$$

$$\therefore\ L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right) = L^{-1}\left(\frac{2}{s}\right) + L^{-1}\left(\frac{-5}{s-1}\right) + L^{-1}\left(\frac{3}{s-2}\right)$$

$$L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right) = 2L^{-1}\left(\frac{1}{s}\right) - 5L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{s-2}\right) \text{---------} (iii)$$

Since

Since

01. We have $\therefore L(f(t)) = L(1) = \frac{1}{s}$ [Example 55]

$$\therefore 1 = L^{-1}(\frac{1}{s})$$

$$\therefore L^{-1}(\frac{1}{s}) = 1$$

02. We have, $\therefore L(f(t)) = L(e^{at}) = \frac{1}{s-a}$ [Example 58]

$$\therefore e^{at} = L^{-1}(\frac{1}{s-a})$$

$$\therefore e^{t} = L^{-1}(\frac{1}{s-1})$$

$$\therefore L^{-1}(\frac{1}{s-1}) = e^{t}$$

03. $\therefore L(f(t)) = L(e^{at}) = \frac{1}{s-a}$ [Example 58]

$$\therefore e^{at} = L^{-1}(\frac{1}{s-a})$$

$$\therefore e^{2t} = L^{-1}(\frac{1}{s-2})$$

$$\therefore L^{-1}(\frac{1}{s-2}) = e^{2t} \text{ Answer}$$

Putting these values in (iii), we get

$$L^{-1}\left(\frac{s+4}{s(s-1)(s-2)}\right) = 2L^{-1}\left(\frac{1}{s}\right) - 5L^{-1}\left(\frac{1}{s-1}\right) + 3L^{-1}\left(\frac{1}{s-2}\right)$$

$$= 2.1 - 5e^{t} + 3e^{2t} \text{ Answer}$$

b) Express the following function in terms of unit step functions and find its Laplace transform CO3

$$f(t)=\begin{cases}10\,; & t<3\\ 8\,; & t>3\end{cases}$$

OR

Find Fourier Transform of

$$\begin{aligned} f(t) &= 1 \quad &;0\le t<1\\ &= -1 \quad &;-1\le t<0\\ &= 0 \quad &;|t|>1 \end{aligned}$$

3 (b) $f(t) = \begin{cases} 10 ; & t < 3 \\ 8 ; & t > 3 \end{cases}$

→ We have,

$$L(u(t-a)) = \frac{e^{-as}}{s}$$

$$\therefore L(u(t-3)) = \frac{e^{-3s}}{s} \quad \text{--- (1)}$$

Given,

$$f(t) = \begin{cases} 10 ; & t < 3 \\ 8 ; & t > 3 \end{cases}$$

$$f(t) = \begin{cases} 10+0 ; & t < 3 \\ 10-2 ; & t > 3 \end{cases}$$

$$f(t) = 10 + \begin{cases} 0 ; & t < 3 \\ -2 ; & t > 3 \end{cases}$$

$$f(t) = 10 + (-2) \begin{cases} 0 ; & t < 3 \\ 1 ; & t > 3 \end{cases}$$

$$f(t) = 10 + (-2) \begin{cases} 1 ; & t > 3 \\ 0 ; & t < 3 \end{cases}$$

$$f(t) = 10 + (-2)\, u(t-3)$$

$$L(f(t)) = L\{10 - 2u(t-3)\}$$

$$L(f(t)) = L\{10\} - 2L\{u(t-3)\}$$

$L\{f\}$

$$L\{f(t)\} = 10\, L(1) - 2L\,u(t-3)$$

$$L\{f(t)\} = 10 \times \frac{1}{s} - 2\,\frac{e^{-3s}}{s}$$

[from ex-55 and ex-91]

$$\therefore L\{f(t)\} = \frac{10}{s} - 2\frac{e^{-3s}}{s}.$$

Ans

Example 41: Find Fourier Transform of

$$f(t) = 1 \qquad ; 0 \le t < 1$$
$$= -1 \qquad ; -1 \le t < 0$$
$$= 0 \qquad ; |t| > 1$$

We have $g(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$

$$g(\omega) = \int_{-\infty}^{-1} f(t)e^{-i\omega t}dt + \int_{-1}^{0} f(t)e^{-i\omega t}dt + \int_{0}^{1} f(t)e^{-i\omega t}dt + \int_{1}^{\infty} f(t)e^{-i\omega t}dt$$

$$g(\omega) = \int_{-\infty}^{-1} 0.e^{-i\omega t}dt + \int_{-1}^{0} (-1)e^{-i\omega t}dt + \int_{0}^{1} 1.e^{-i\omega t}dt + \int_{1}^{\infty} 0.e^{-i\omega t}dt$$

$$g(\omega) = -\int_{-1}^{0} e^{-i\omega t}dt + \int_{0}^{1} e^{-i\omega t}dt$$

$$g(\omega) = -\left[\frac{e^{-i\omega t}}{-i\omega}\right]_{-1}^{0} + \left[\frac{e^{-i\omega t}}{-i\omega}\right]_{0}^{1} \qquad [\because \int e^{-mx}dx = \frac{e^{-mx}}{-m}]$$

$$g(\omega) = -\left[\frac{e^{-i\omega.0}}{-i\omega} - \frac{e^{-i\omega(-1)}}{-i\omega}\right] + \left[\frac{e^{-i\omega.1}}{-i\omega} - \frac{e^{-i\omega.0}}{-i\omega}\right]$$

$$g(\omega) = -\left[\frac{e^{-0}}{-i\omega} - \frac{e^{i\omega}}{-i\omega}\right] + \left[\frac{e^{-i\omega}}{-i\omega} - \frac{e^{-0}}{-i\omega}\right]$$

$$g(\omega) = -\left[\frac{\frac{1}{e^{0}}}{-i\omega} - \frac{e^{i\omega}}{-i\omega}\right] + \left[\frac{e^{-i\omega}}{-i\omega} - \frac{\frac{1}{e^{0}}}{-i\omega}\right]$$

$$g(\omega) = -\left[\frac{\frac{1}{1}}{-i\omega} - \frac{e^{i\omega}}{-i\omega}\right] + \left[\frac{e^{-i\omega}}{-i\omega} - \frac{\frac{1}{1}}{-i\omega}\right]$$

$$g(\omega) = -\left[\frac{1}{-i\omega} - \frac{e^{i\omega}}{-i\omega}\right] + \left[\frac{e^{-i\omega}}{-i\omega} - \frac{1}{-i\omega}\right]$$

$$g(\omega) = \left[\frac{1}{i\omega} - \frac{e^{i\omega}}{i\omega}\right] + \left[-\frac{e^{-i\omega}}{i\omega} + \frac{1}{i\omega}\right]$$

$$g(\omega) = \frac{1}{i\omega} + \frac{1}{i\omega} - \frac{e^{i\omega}}{i\omega} - \frac{e^{-i\omega}}{i\omega}$$

$$g(\omega)=\frac{2}{i\omega}-\frac{1}{i\omega}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{1}{i\omega}\frac{2}{2}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{2}{i\omega}\frac{1}{2}(e^{i\omega}+e^{-i\omega})$$

$$g(\omega)=\frac{2}{i\omega}-\frac{2}{i\omega}\cos\omega \qquad [\because \cos x=\frac{1}{2}(e^{ix}+e^{-ix})]$$

$$g(\omega)=\frac{2}{i\omega}(1-\cos\omega)\ \textit{Answer}$$

4. ~~a)~~ State first shift theorem. Using the theorem evaluate $\mathcal{L}\{e^{-4t}t^2\}$ CO1 U

The First Shift Theorem

We have seen that a Laplace transform of $\mathbf{f(t)}$ is a function of $\mathbf{s}$ only, i.e.

$$\mathbf{L\{f(t)\} = f(s)}$$

The first shift theorem states that,

If $\mathbf{L\{f(t)\} = f(s)}$ ------------------------------(i)

Then $\mathbf{L\{e^{-at} f(t)\} = f(s+a)}$ ------------------------------(ii)

Example 69: Find $L\{e^{-4t}t^2\} = ?$

Answer:

We have,

If $f(t) = t^n$

Then $L(f(t)) = L(t^n) = \frac{n!}{s^{n+1}}$

For $n = 2$;

If $f(t) = t^2$

Then $L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$

We are to find $L\{e^{-4t}t^2\} = ?$

Here, $f(t) = t^2$

The first shift theorem states that,

If $L\{f(t)\} = f(s)$ ---------------------------(i)

Then $L\{e^{-4t}t^2\} = f(s+4)$ ---------------------------(ii)

We have, according to equation no (i), $L(f(t)) = L(t^2) = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$ [Here $f(s) = \frac{2!}{s^3}$]

If $f(s) = \frac{2!}{s^3}$]

$\therefore f(s+4) = \frac{2!}{(s+4)^3}$

Hence, according to equation no (ii), we can write

$L\{e^{-4t} * t^2\} = f(s+4) = \frac{2!}{(s+4)^3}$ *Answer*

CO1 U

b) Define unit step function. Sketch the following function,

$$x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)$$

Example 89: Given that, $x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)$

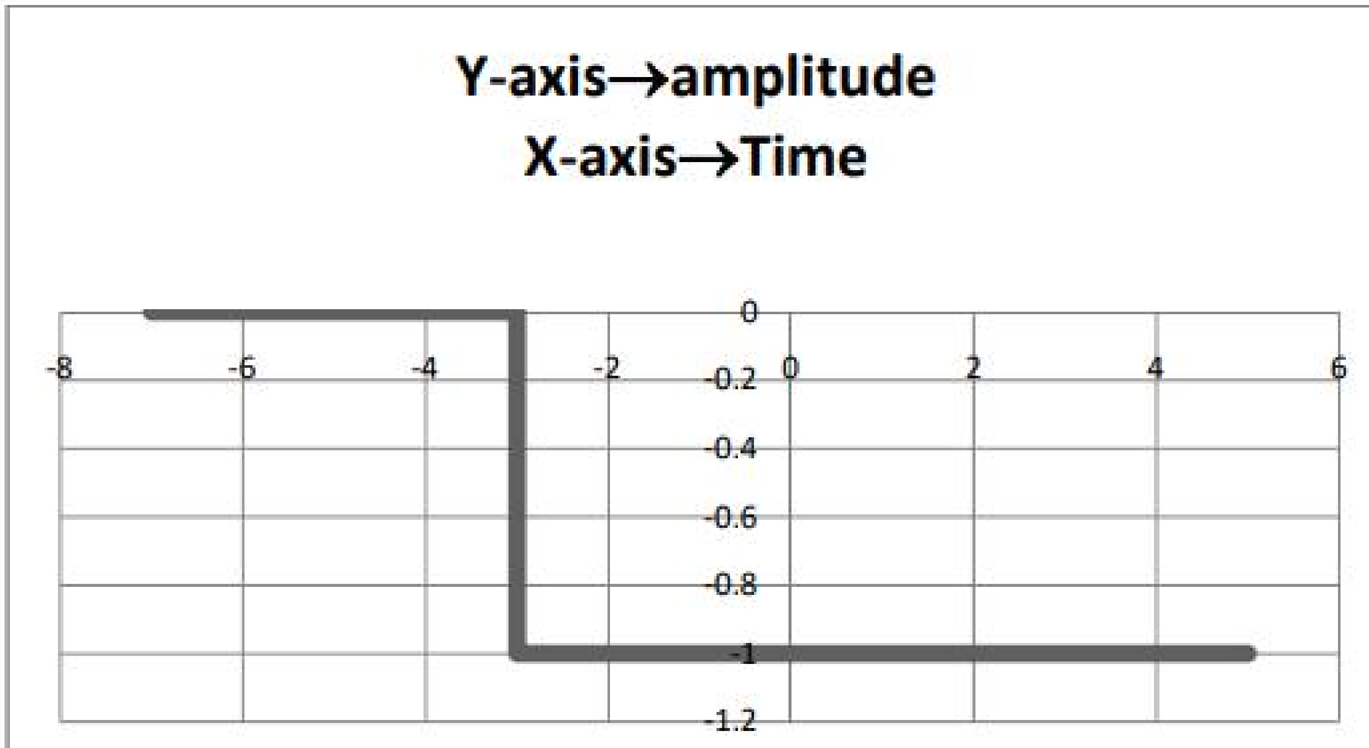
Answer:

01. $-u(t+3) =>$

So,

$-u(t+3) = -1\,;\ t \geq -3;$ $\qquad$ $here, t+3 = 0$

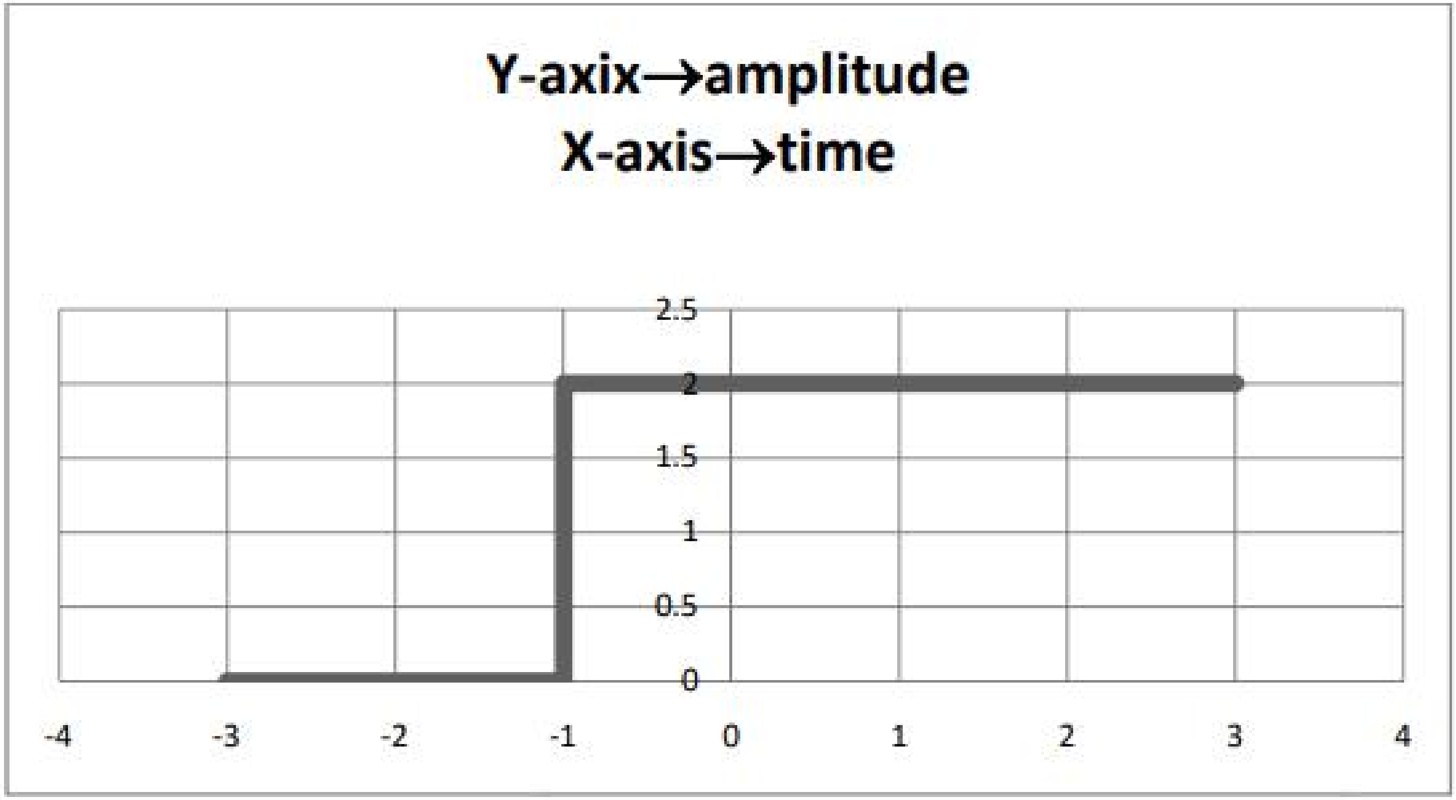
$\quad = 0;\ \ t < -3$ $\qquad$ $\therefore t = -3$



02. $2u(t+1)$

$2u(t+1) = 2; t \geq -1$ $\qquad$ $here, t+1 = 0$

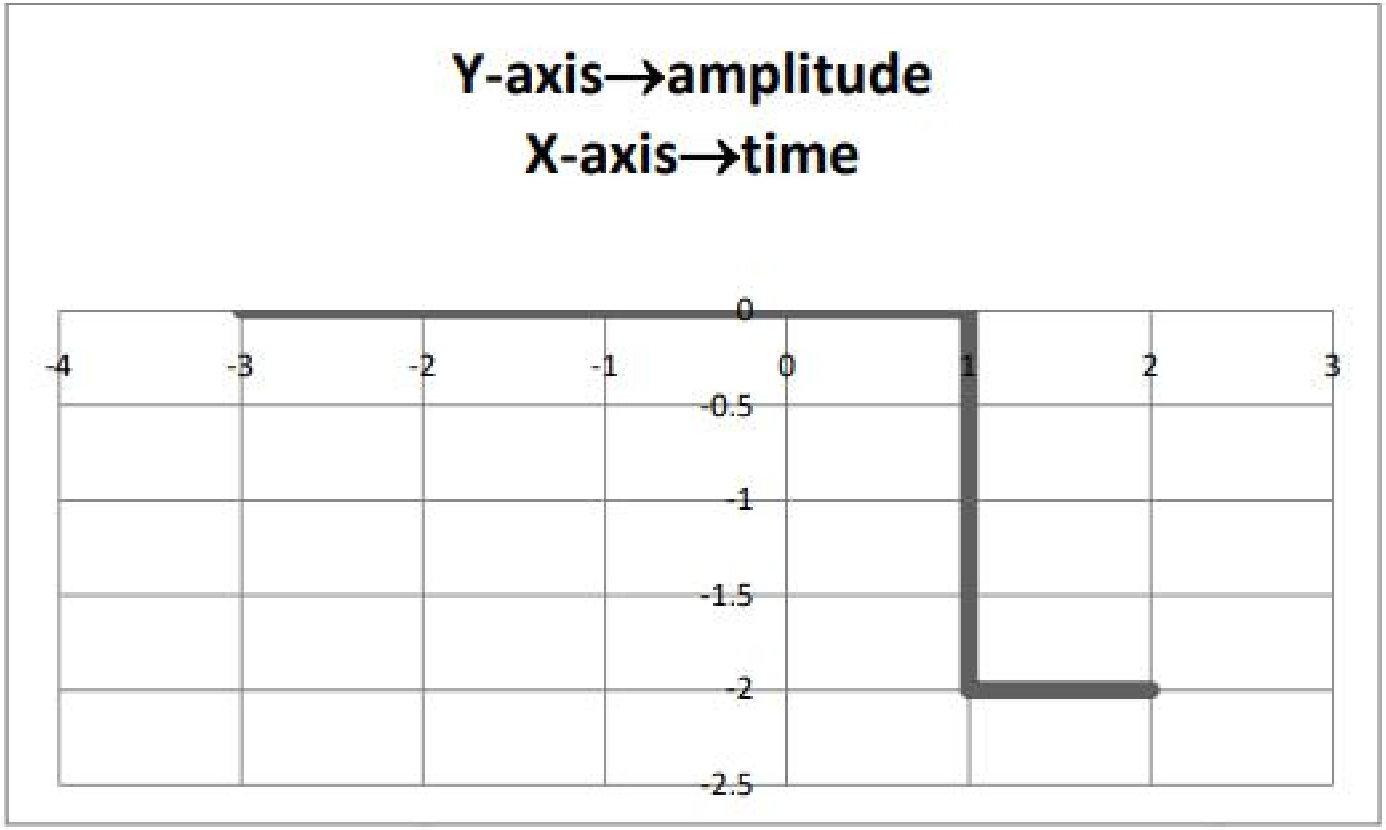
$= 0; t < -1$ $\qquad$ $t = -1$



03. $-2u\,(t-1)$

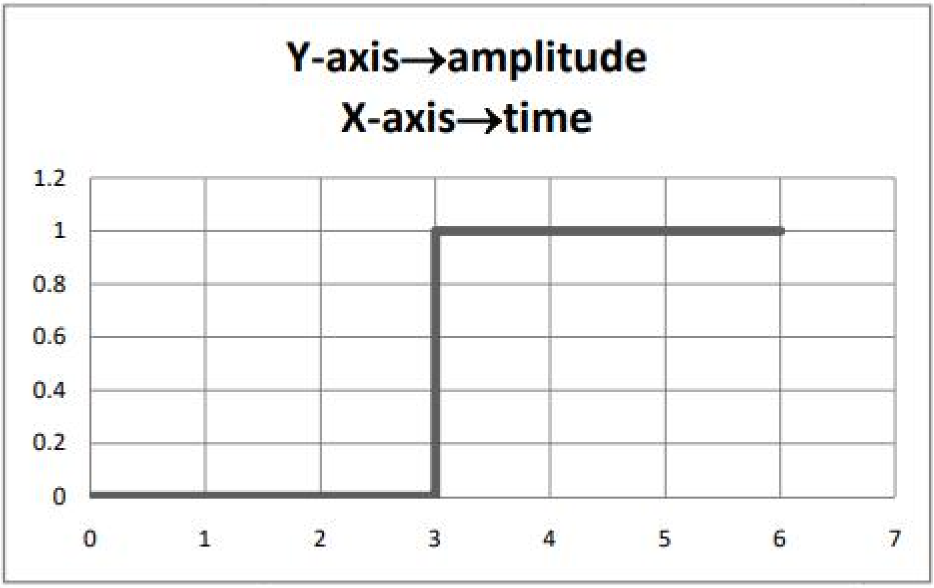
$\therefore\ -2u(t-1) = -2; t \geq 1$ $\qquad$ $here, t-1 = 0$

$= 0;\ t < 1$ $\qquad$ $t = 1$

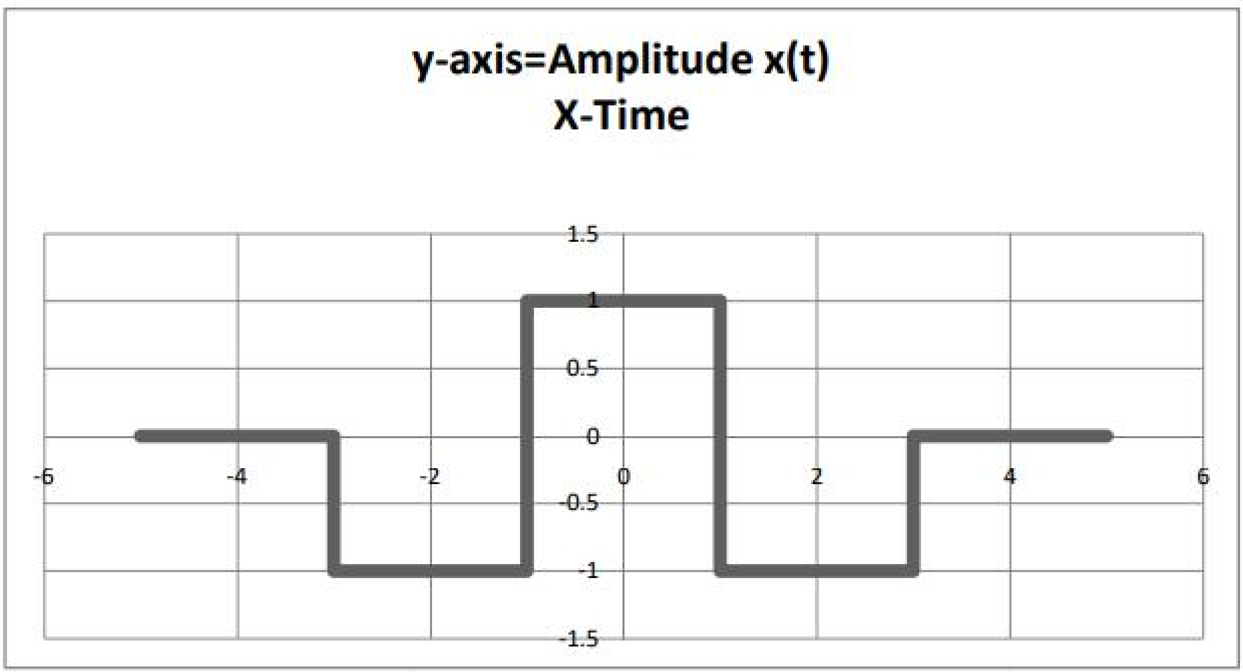

Y-axis→amplitude
X-axis→time
0
-0.5
-1
-1.5
-2
-2.5
-4
-3
-2
-1
0
1
2
3

04. $\boldsymbol{u(t-3)} =>$ $\qquad here, t-3=0$

$$t = 3$$



$$\boldsymbol{x(t) = -u(t+3) + 2u(t+1) - 2u(t-1) + u(t-3)}$$



5\. a) Write **MATLAB code** to sketch line spectrum (at least 6) for the following Fourier series $\underbrace{f(t)}_{\text{Complex wave}} = \underbrace{2}_{\text{DC value}} + \underbrace{[\sum_{n=1}^{\infty} (\cos n\pi + 1) \sin \frac{n\pi t}{3}]}_{\text{AC value}}$ CO5 App

b) Make **a function in MATLAB environment** to raise a complex wave $f(t)$ in the time interval of [-4, 20] for the following Fourier series: CO5 App

$$f(t) = 4\pi + \sum_{n=1}^{\infty} \frac{3}{n\pi} \cos n\pi t$$

c) If CO5 App

$$x[n] = 5 \;;\; n = 0 \qquad\qquad h[n] = 3 \;;\; n = 0$$
$$= 6 \;;\; n = 1 \quad \text{and} \quad = -2 \;;\; n = 1$$

Write MATLAB code to find the convolution sum of the above signals.

5 . a) $f(t) = 2 + \left[\sum_{n=1}^{\infty} (\cos n\pi + 1) \sin \frac{n\pi t}{3} \right]$

here $a = 0$

$b = \cos n\pi + 1$

$n = 1 : 6 ;$

$w = \frac{n\pi}{3}$

$a = 0$

$b = \cos n\pi + 1$

$r = \text{sqrt}\,(a.^2 + b.^2)$

stem (w, r)

5.c)

```
x = [5 3];
h = [6 -2];
y = conv(x,h);
```