General equation of second degree

Formula:

The general equation of second degree
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will represent

i) a pair of straight lines if the determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

i.e $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

ii) two parallel lines if $\Delta = 0$ and $h^2 - ab = 0$

iii) Two perpendicular lines if $\Delta = 0$ and $a + b = 0$

iv) a circle if $a = b$ and $h = 0$

v) a parabola if $\Delta \neq 0$ and $ab - h^2 = 0$

vi) an ellipse if $\Delta \neq 0$ and $ab - h^2 > 0$

vii) a hyperbola if $\Delta \neq 0$ and $ab - h^2 < 0$

viii) a rectangular hyperbola if $\Delta \neq 0$, $ab - h^2 < 0$ and $a+b$

* Equation of

i) a Circle $x^2 + y^2 = a^2$ or, $x^2 + y^2 + 2gx + 2fy + c = 0$

ii) a Parabola is $y^2 = 4ax$

iii) a Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

iv) a Hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Ex: Reduce the eqn $8x^2+4xy+5y^2-16x-14y+13=0$ to the standard form.

Soln: The given eqn is

$$8x^2+4xy+5y^2-16x-14y+13=0 \longrightarrow (1)$$

Comparing this eqn with the general eqn of second degree i.e $ax^2+2hxy+by^2+2gx+2fy+c=0$, we have

$a=8,\ h=2,\ b=5,\ g=-8,\ f=-7,\ c=13.$

$$\therefore \Delta = abc+2fgh-af^2-bg^2-ch^2$$
$$= 8.5.13+2(-7)(-8).2-8.(-7)^2-5(-8)^2-13.2^2$$
$$= 520+224-392-320-52$$
$$= 744-764$$
$$= -20$$
$$\therefore \Delta \neq 0$$

Again $ab-h^2 = 8.5-2^2$
$$= 40-4$$
$$= 36>0$$

$$\therefore ab-h^2>0$$

So the given eqn represent an ellipse.

let the eqn to the conic be

$$F(x,y) \equiv 8x^2+4xy+5y^2-16x-14y+13=0$$

$$\therefore \frac{\partial f}{\partial x} = 16x+4y-16=0$$
$$= 4x+y-4=0 \text{ -----(i)}$$
$$\frac{\partial f}{\partial y} = 4x+10y-14=0$$
$$= 2x+5y-7=0 \text{ -----(ii)}$$

If (x_1, y_1) be the co-ordinates of the centre of the conic, then these co-ordinates are obtained by solving eq^ns

$$4x_1 + y_1 - 4 = 0 \quad \text{---(iii)}$$

and $$2x_1 + 5y_1 - 7 = 0 \quad \text{---(iv)}$$

By cross multiplication (iii) and (iv), we get

$$\frac{x_1}{-7+20} = \frac{y_1}{-8+28} = \frac{1}{20-2}$$

or, $$\frac{x_1}{13} = \frac{y_1}{20} = \frac{1}{18}$$

$$\therefore x_1 = \frac{13}{18}, \quad y_1 = \frac{10}{9}.$$

$\therefore$ the centre of the given eq^n is $\left(\frac{13}{18}, \frac{10}{9}\right)$.

Again the eq^n (1) can be written as

$$8x^2 + 4xy + 5y^2 + c_1 = 0$$

where $$c_1 = gx_1 + fy_1 + c$$

$$= (-8)\cdot\frac{13}{18} + (-7)\cdot\frac{10}{9} + 13$$

$$= -\frac{52}{9} - \frac{70}{9} + 13$$

$$= \frac{-52-70+117}{9} = \frac{-5}{9}.$$

$\therefore$ The reduce eq^n is $8x^2 + 4xy + 5y^2 - 5/9 = 0$

or, $72x^2 + 36xy + 45y^2 = 5$

or, $\frac{72}{5}x^2 + \frac{36}{5}xy + 9y^2 = 1 \rightarrow (2$

Here $A = \frac{72}{5}$, $H = \frac{18}{5}$, $B = 9$.

length of the axes are given by

$$\frac{1}{r^4} - (A+B)\frac{1}{r^2} + AB - H^2 = 0$$

or, $$\frac{1}{r^4} - \left(\frac{72}{5} + 9\right)\frac{1}{r^2} + \frac{72}{5}\cdot 9 - \left(\frac{18}{5}\right)^2 = 0$$

or, $$\frac{1}{r^4} - \frac{117}{5}\frac{1}{r^2} + \frac{2916}{25} = 0$$

or $$\frac{1}{r^4} - \frac{117}{5}\frac{1}{r^2} + \frac{36\times 81}{5^2} = 0$$

or, $$\frac{1}{r^4} - \frac{36}{5}\frac{1}{r^2} - \frac{81}{5}\frac{1}{r^2} + \frac{36\times 81}{5^2} = 0$$

or $$\left(\frac{1}{r^2} - \frac{36}{5}\right)\left(\frac{1}{r^2} - \frac{81}{5}\right) = 0$$

Either $\frac{1}{r^2} - \frac{36}{5} = 0$ or $\frac{1}{r^2} - \frac{81}{5} = 0$

or $r = \frac{\sqrt{5}}{6}$ $\therefore r = \frac{\sqrt{5}}{9}$.

$\therefore r_1 = \sqrt{5}/6$, $r_2 = \sqrt{5}/9$.

$\therefore$ standard eq^n of an ellipse is

$$\frac{x^2}{(\sqrt{5/6})^2} + \frac{y^2}{(\sqrt{5}/9)^2} = 1 \quad \underline{\text{Ans.}}$$

or $36x^2 + 81y^2 = 5$.