CHAPTER FIVE

FUNDAMENTALS OF PROBABILITY

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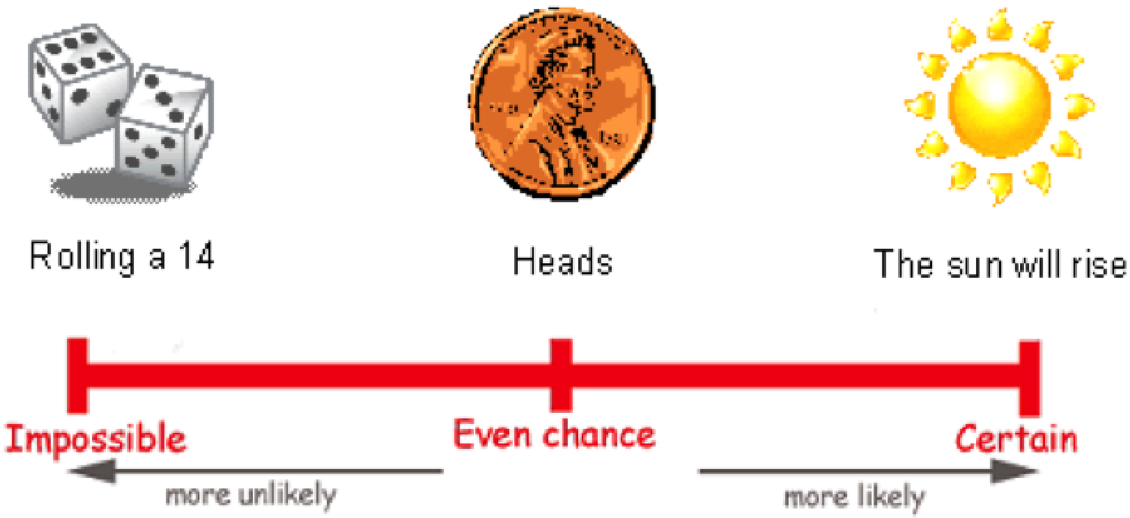


Concept related to probability

Define the followings with examples:

Experiment: Experiment is an act that can be repeated under given conditions.

Tossing of a coin or throwing of a dice or the drawing of a cards etc, are the example of experiment.

Outcomes: The results of an experiment are called outcomes.

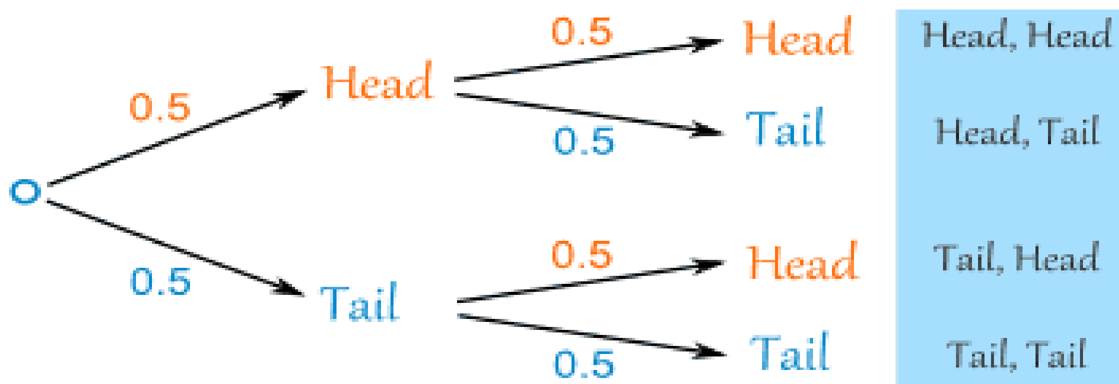
Random experiment: Experiments are called random experiments if the outcomes depend on chance and cannot be predicted with certainty.

Example: Tossing of a fair coin, throwing of dice etc are the examples of random experiments.

Sample space: The collection or totality of all possible outcomes of a random experiment is called sample space. It is usually denoted by S or Ω.

If we toss a coin, the sample space is S = $\{H, T\}$ where H and T denote the head and tail of the coin respectively.

If we toss a coin two times, than the sample space is S = $\{HH, HT, TH, TT\}$



Sample point: Each element of a sample space is called sample point.

Event: Any subset of a sample space is called event. There are two types of event:

Simple event and Compound event

Simple event: An event is called simple event if it contains only one sample point.

Compound event: An event is called simple event if it contains more than one sample point.

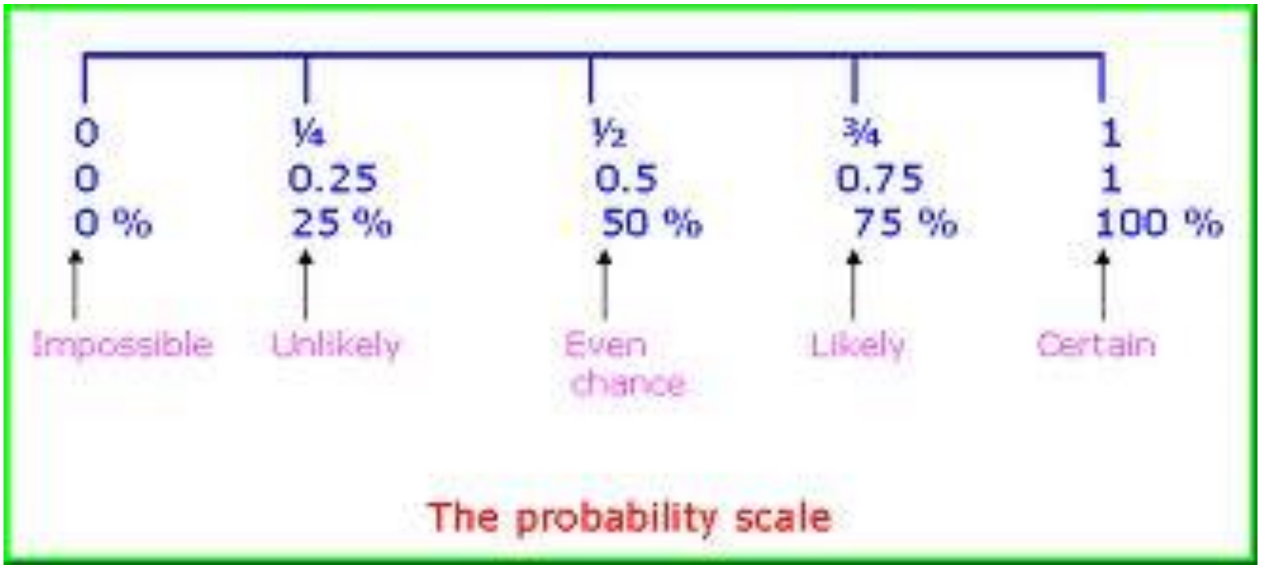
If we toss a coin two times, than the sample space is S = $\{HH, HT, TH, TT\}$

where H and T denote the head and tail of the coin respectively.

Here, A = $\{HH\}$ is a simple event and B = $\{HH, HT\}$ is a compound event.

Sure event: An event is called sure event when it always happens. The probability of a sure event is one.

Impossible event: An event is called impossible event when it never happens. The probability of an impossible event is always zero.



Mutually exclusive events: Two events are said to be mutually exclusive if they have no common points. If A and B are two mutually exclusive events, then AB $=$ ø.

Non mutually exclusive events: Two events are said to be not mutually exclusive event if they have common points. If A and B are two not mutually exclusive events, then AB $\neq$ ø.

Complementary event: Let A be any event defined on a sample space S or Ω then the complementary of A, denoted by $\bar{A}$ is the event consisting of all sample points in S but not in A.

Equally likely outcomes: Outcomes are called equally likely if one does not occur more often than the other. In this case the sample points of a sample space are all equal probable.

Exhaustive outcomes: Outcomes of an experiment are said to be exhaustive if they include all possible outcomes.

In throwing a die exhaustive numbers of outcomes are 6.

Conditional Probability: If A and B are two events in S. Then the conditional probability of A for given value of B, denoted by P [A | B] is defined by

$$P\,[A \mid B] = \frac{P[AB]}{P[B]} \;;\; P\,[B]>0$$

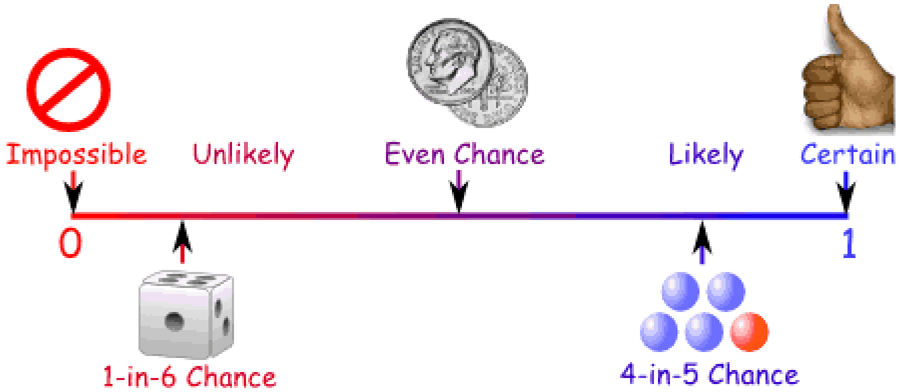
Similarly, $$P\,[B \mid A\,] = \frac{P[AB]}{P[A]} \;;\; P[A]>0$$

Independent Event: Two events A and B are said to be independent if and only if one of the following conditions holds:

(i) P[AB]=P[A]P[B]
(ii) P [A | B] = P[A]
(iii) P [B | A]= P[B]

Dependent Event: Two events A and B are said to be dependent if and only if one of the following conditions holds:

(i) P[AB]≠ P[A]P[B]
(ii) P [A | B] ≠ P[A]
(iii) P [B | A] ≠ P[B]



Definition of probability:

There are four approaches of defining probability. They are

(i) Classical or mathematical or priori definition of probability.
(ii) Empirical or statistical or posterior or frequency probability.
(iii) Subjective probability.
(iv) Axiomatic probability.

Classical or mathematical or priori definition of probability: If there are n mutually exclusive, equally likely and exhaustive outcomes of a random experiment and if m of these outcomes are favorable to an event A, then the probability of the event A, denoted by P[A] is defined as

$$P[A] = \frac{N[A]}{N[S]} = \frac{m}{n} \; ; \; 0 \le P[A] \le 1$$

This definition of probability is given by Laplace.

Axiomatic probability: Suppose S is a sample space and A is an event of this sample space. Then the probability of the event A, denoted by P[A] must satisfy the following four axioms:

(i) $P[A] \geq 0$
(ii) $P[S] = 1$
(iii) If A and B are mutually exclusive events, then $P[AUB]= P[A]+P[B]$
(iv) Let A_1, A_2,………………A_k be a sequence of K mutually exclusive events, then $P[A_1UA_2U\ldots\ldots\ldots UA_K]= P[A_1]+ P[A_2]+\ldots\ldots\ldots+p[A_k]$

Laws of Probability

There are two important rules or laws of probability;
(i) Addition laws of probability
(ii) Multiplication laws of probability.

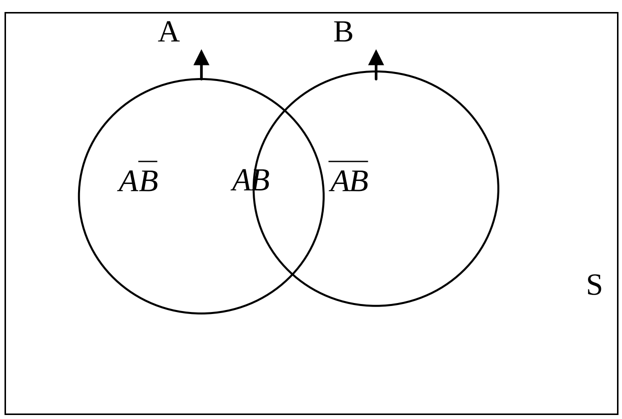
Addition laws of probability are two types

(i) **Mutually exclusive events (ii) Non Mutually exclusive events**

Theorem: State and prove additive laws of probability for two non mutually exclusive events.

Statement: If A and B are two events, then $P[AUB]= P[A]+P[B] – P[AB]$

Proof:



$$AUB = A\bar{B}UABU\bar{A}B$$

It is obvious from the Venn-diagram $A = A\bar{B} \cup AB$

Therefore, $P[A] = P[A\bar{B} \cup AB]$

$$P\left[A\right] \quad = P\,[\,A\ \overline{B}\,] + P\,[\,A\ B\,]\ \ldots\ldots\ldots\ldots\ldots..(i);$$

Since $A\ B$ and $A\ \overline{B}$ are mutually exclusive.

$$\text{Similarly,} \qquad B = A\ B \cup \overline{A}\ B$$

$$\text{Therefore, } P\left[B\right] = P\,[\ A\ B \cup \overline{A}\ B\ ]$$

$$P\left[B\right] \quad = P\,[\,A\ B\,] + P\,[\,\overline{A}\ B\,]\ \ldots\ldots\ldots\ldots\ldots..(ii)\ ;$$

Since $A\ B$ and $A\ \overline{B}$ are mutually exclusive.

$$\text{Now, } A\cup B = A\ \overline{B} \cup A\ B \cup \overline{A}\ B$$

$$\text{Therefore, } P\,[\,A\cup B\,] = P\,[\,A\ \overline{B} \cup A\ B \cup \overline{A}\ B\,]\ldots\ldots\ldots\ldots\ldots..(iii);$$

Since $A\ \overline{B}$, $A\ B$ and $\overline{A}\ B$ are mutually exclusive.

Now adding (i) and (ii) we get,

$$P\left[A\right] + P\left[B\right] = P\,[\,A\ \overline{B}\,] + P\,[\,A\ B\,] + P\,[\,A\ B\,] + P\,[\,\overline{A}\ B\,]$$

$$= P\,[\,A\ \overline{B}\,] + P\,[\,A\ B\,] + P\,[\,\overline{A}\ B\,] + P\,[\,A\ B\,]$$

$$= P\,[\,A\cup B\,] + P\,[\,A\ B\,]$$

$$\text{Therefore, } P\,[\,A\cup B\,] = P\left[A\right] + P\left[B\right] - P\,[\,A\ B\,]$$

(This completes the proof of the theorem)

But if A and B are mutually exclusive, then $P\,[\,A\ B\,] = 0$

$$\text{In that case,} \qquad P\,[\,A\cup B\,] = P\left[A\right] + P\left[B\right].$$

Theorem: State and prove additive laws of probability for three non-mutually exclusive events.

Statement: If A, B, C are three non-**mutually exclusive** events then,

P(A U B U C) = P(A) + P(B)+ P(C)- P(AB)- P(AC)- P(BC)+ P(ABC)

Proof: Let, B U C = D; then

P(A U D) = P(A)+ P(D)- P(AD)

Or, P(A U B U C) = P(A)+P(B U C)-P[A(B U C)]

= P (A) + P(B)+ P(C)- P(BC)-P(AB U AC)

= P(A) + P(B)+ P(C)- P(BC)-[P(AB)+ P(AC)-P(ABAC)]

= P(A) + P(B)+ P(C)- P(AB)- P(AC)- P(BC)+ P(ABC);

[Since, ABAC =ABC]

(This completes the proof of the theorem)

But if A and B are mutually exclusive, then P(AB) = 0, P(BC)= 0, P(AC)=0 and P(ABC) = 0. In that case, P (A U B U C) = P (A) + P (B) + P(C).

Multiplication law of probability: If A and B are two events, then

P [AB] = **P** [A] P [B | A] = P [B] P [A | B]

Proof: From the definition of conditional probability, we have

P [A | B] = $\frac{P[AB]}{P[B]}$; P[B]>0

It follows that P [AB] = P [B] P [A | B]………………….(i)

Similarly, P [B | A] = $\frac{P[AB]}{P[A]}$; P[A]>0

It follows that P [AB] = **P** [A] P [B | A]………………….(ii)

From (i) and (ii) we have, P [AB] = **P** [A] P [B | A] = P [B] P [A | B]

But if A and B are independent events, then **P** [A] = P [A | B] and P [B] = P [B | A]

Hence, P [AB] = **P** [A] P [B].

Theorem: Show that $0 \leq P(A) \leq 1$ or Show that the value of the probability lies between 0 to 1.

Proof: Let A be the event in S (sample space)

Also let, N(S) = n

And N (A) = m

By the definition of probability, P [A] = $\frac{N(A)}{N(S)} = \frac{m}{n}$

Clearly, $0 \le m \le n$

Or, $\frac{0}{n} \le \frac{m}{n} \le \frac{n}{n}$ [Dividing bots side by n]

Or, $0 \le P(A) \le 1$ (Proved)

Theorem: If A and $\overline{A}$ are mutually and exhaustive, then show that $P(A) + P(\overline{A}) = 1$

Proof: From the complementary laws of events, we have $A \cup \overline{A} = S$

Therefore, $P(A \cup \overline{A}) = P(S)$

Or, $P(A \cup \overline{A}) = 1$ [By axiomatic definition P(S) = 1]

Or, $P(A) + P(\overline{A}) = 1$ [since A and $\overline{A}$ are mutually exclusive]

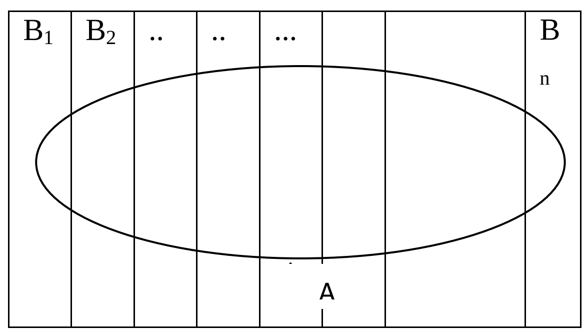
Therefore, $P(A) + P(\overline{A}) = 1$ (Proved)

Theorem: State and Prove Bayes' theorem

Statement of Bayes theorem: Let B_1, B_2,………………,B_n be n mutually exclusive and exhaustive events in a random experiment and A be any event in S. Then Bayes theorem state that

$$P[B_i|A] = \frac{P[B_i].P[A/B_i]}{\sum_{i=1}^{n} P[B_i]P[A/B_i]}; i = 1,2,\ldots\ldots\ldots,n$$

Proof: From the definition of conditional probability, we have



A = AB_1 U AB_2 U ………………U AB_n

$$P[B_i | A] = \frac{P[AB_i]}{P[A]} \text{ …………… (i)}$$

$$= \frac{P[B_i].P[A/B_i]}{P[A]} \text{ ……… (ii) (Since, P [AB] = P [B] P [A | B])}$$

It is obvious from the venn diagram

A = AB_1 U AB_2 U ………………U AB_n

Therefore, $P(A) = P(AB_1 \cup AB_2 \cup \ldots\ldots\ldots\ldots\ldots\ldots \cup AB_n)$

$= P(AB_1) + P(AB_2) + \ldots\ldots\ldots\ldots\ldots\ldots + P(AB_n)$

Since, AB_1 , AB_2 …..and AB_n are mutually exclusive.

$= P[B_1]P[A|B_1] + P[B_2]P[A|B_2] + \ldots\ldots\ldots\ldots\ldots\ldots + P[B_n]P[A|B_n]$

(Since, P [AB] = P [B] P [A | B])

$$P(A) = \sum_{i=1}^{n} P[B_i]P[A/B_i] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

Putting the value of P(A) in equation (ii), we have

$$P[B_i|A] = \frac{P[B_i].P[A/B_i]}{\sum_{i=1}^{n} P[B_i]P[A/B_i]}; i = 1,2,\ldots\ldots\ldots\ldots, n$$

(This completes the proof)

Mathematical problem of Bayes' theorem: In a factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the product. Of the total of their output 5, 4 and 2 percent are defective bolts respectively. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machine (i) A and (ii) C.

Solution: Let

B_1 : An item produced by machine A

B_2: An item produced by machine B

B_3: An item produced by machine C

A: Defective item produced by the machines.

We have, $P[B_1] = 25\% = 0.25$, $P[B_2] = 35\% = 0.35$, $P[B_3] = 40\% = 0.40$,

And $P(A|B_1) = 5\% = 0.05$, $P(A|B_2) = 4\% = 0.04$, $P(A|B_3) = 2\% = 0.02$,

We have to find, $P(B_1|A)$ and $P(B_3|A)$

According to Bayes' theorem, $P[B_i|A] = \dfrac{P[B_i].P[A/B_i]}{\sum_{i=1}^{n} P[B_i]P[A/B_i]}; i = 1,2,\ldots\ldots\ldots\ldots, n$

$$P(B_1|A) = \frac{P(B_1)P(A/B_1)}{P(B_1)P(A/B_1)+P(B_2)P(A/B_2)+P(B_3)P(A/B_3)}$$

$$= \frac{(0.25)(0.05)}{(0.25)(0.05)+(0.35)(0.04)+(0.40)(0.02)}$$

$$= 0.3623$$

Similarly,

$$P(B_3|A) = \frac{P(B_3)P(A/B_3)}{P(B_1)P(A/B_1)+P(B_2)P(A/B_2)+P(B_3)P(A/B_3)}$$

$$= \frac{(0.40)(0.02)}{(0.25)(0.05)+(0.35)(0.04)+(0.40)(0.02)}$$

$$= 0.2319$$

Problem: If P[A] = 0.6, P[B]= 0.8 and P[AB]= 0.50 Find (i) P[$\overline{A}$]; (ii) P[A U B] ;(iii) P[A |B] ; iv) P[B |A] (v) P[$A \ \overline{B}$]; (vi) P[$\overline{A} \ B$]; (vii) P[$\overline{A} \ \overline{B}$] ; (viii) P[$\overline{A \cup B}$] (xi) Are the events A and B independent? (x) Are A and B mutually exclusive?

Ans: Solution: (i) We know: $P(A)+P(\overline{A})=1$

Here P[A] = 0.6. Hence, P[$\overline{A}$]=1-0.6=0.40

(ii)By additive law of probability, we know P[AUB]= P[A]+P[B] – P[AB]

= 0.60+0.80-.50=0.90

(iii)From the definition of conditional probability, we have

$$P[A | B] = \frac{P[AB]}{P[B]} \text{ ; } P[B]>0$$

$$= \frac{0.50}{0.80} = 0.625$$

(iv) Similarly, $P[B | A] = \frac{P[AB]}{P[A]}$; P[A]>0

$$= \frac{0.50}{0.60} = 0.833$$

(v)P[$A \ \overline{B}$]= $P[A \cap \overline{B}]$=P[A]-P[AB]

= 0.60-0.50=0.10

(vi)P[$\overline{A}\ \ B$]= $P\left[\overline{A}\cap B\right]$== P[B]-P[AB]

= 0.8-0.50=0.30

(vii) P[$\overline{A}\ \overline{B}$]

According to De Morgan's law, $P[\ \overline{A}\ \overline{B}\ ] = P[\overline{A}\cap\overline{B}]$] = P[$\overline{A\cup B}$]=1- P[A U B]

= 1-0.90=0.10

(viii)P**[** $\overline{A\cup B}$]=1- P[A U B]= 1-0.90=0.10

(xi) **Are the events A and B independent?**

Ans: The events A and B are independent if P [AB] = P [A] P [B].

Here, P [A] P [B]=(0.60) (0.80)=0.48≠ P [AB]=0.50

Hence the events A and B are not independent.

(x) **Are A and B mutually exclusive?**

Ans: A and B mutually exclusive if P [AB]=0

Here, P[AB]= 0.50≠0

Hence A and B are not mutually exclusive.

Assignment: If P(A)= $\frac{1}{3}$, P(B)= $\frac{3}{4}$ and P(A U B)= $\frac{11}{12}$. Find (i) P($A|B$) (iii) P $\left(\overline{A}\ \ |B\ \ \right)$.

Application Problem: Suppose A and B are two mutually exclusive events with P[A]=.35 and P[B]=.15. Find (i) P[A $\cup$ B] (ii) P[$\overline{A}$] (iii) P[A $\cap$ B] (iv) P[$\overline{A}\cup\overline{B}$]

Solution:

(i) According to axiom 3.

P[AUB]= P[A]+P[B]= 0.35+0.15=0.50

(ii) P[$\overline{A}$]=1-0.35=0.65

(iii) P[A $\cap$ B]=0; Since A and B are mutually exclusive.

(iv) According to De Morgan's law, $\overline{A}\cup\overline{B} = \left(\overline{A\cap B}\right)$

$P\ (\ \overline{A}\cup\overline{B}) = P\left(\overline{A\cap B}\right)$=1- P[A $\cap$ B]=1-0=1

Application problem: The distribution of number of stores according to size in 3 areas is given in following table:

Area	Store size		
	Large(L)	Medium(M)	Small(S)
A	30	45	75
B	150	125	275
C	20	130	150

Find the probabilities (i) P(M); (ii) P[BM] (iii) P(B U L) and (iv) P(A | M) (v) Are the events A and L independent?

Solution:

Area	Store size			
	Large(L)	Medium(M)	Small(S)	**Total**
A	30	45	75	**150**
B	150	125	275	**550**
C	20	130	150	**300**
Total	**200**	**300**	**500**	**1000**

Here total number of stores is 1000.That is N(S)=1000.

(i) Here M is the event of medium size store. Then N(M)= 300

Therefore P[M] = $\frac{N(M)}{N(S)} = \frac{300}{1000}$ = 0.300

(ii) B ∩ M = BM.

P[BM]= $\frac{N(BM)}{N(S)} = \frac{125}{1000}$ = 0.125

(iii) P(B U L)=P[B]+P[L]-P[BL]

$$= \frac{N(B)}{N(S)} + \frac{N(L)}{N(S)} \quad \frac{N(BL)}{N(S)}$$

$$= \frac{550}{1000} + \frac{200}{1000} \quad \frac{150}{1000}$$

=0.60

(iv) P(A | M)= $\frac{p(AM)}{P(M)} = \frac{\frac{N(AM)}{N(S)}}{\frac{N(M)}{N(S)}} = \frac{N(AM)}{N(S)} = \frac{45}{300}$ = 0.15

(v) Are the events A and L independent?

Solution: The events A and L are independent if P [AL] = P [A] P [L].

Here, P [AL] = $\frac{N(AL)}{N(S)} = \frac{30}{1000}$

And P [A] P [L]= $\frac{N(A)}{N(S)} \times \frac{N(\ L)}{N(S)} = \frac{150}{1000} \times \frac{200}{1000} = \frac{30}{1000} = P[AL]$

Hence the events A and L are independent.