CHANGE OF AXES 4

Art. 31. Transformation of Co-ordinates.

The co-ordinates of a point or the equation of a curve are alway given with reference to a fixed origin and a set of axes of co-ordinates. The above co-ordinates of the equation of the curve changes when the origin is changed or the direction of axes changed or both. The process of changing the co-ordinate of a point or the equation of a curve is called **transformation of co-ordinates.** Now we have to investigate the mode of the change of the co-ordinates or the equation of the curve according to the transfer from one set to another set of co-ordinate axes.

Art. 32. Change of origin (Translation of axes)

To find the change in the co-ordinates of a point when the origin is shifted to another point O (α, β) where the direction of axes remains unalterad.

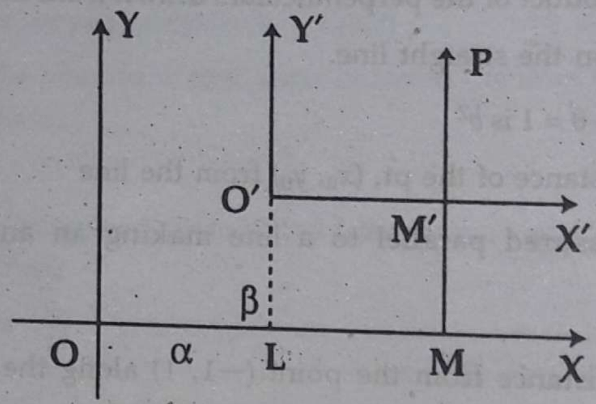


Fig. 10

Let us take a new pair of axes O′ X′ and O′Y′ parallel to the old pair OX and OY; O′ being the new origin whose co-ordinate are (α, β) referred to O′X′ and O′Y′.

let (x', y') be the co-ordinates referred to the axes O′X′ and O′Y′ or a point P, whose co-ordinate referred to the old axes are (x, y).

It is required to transform the co-ordinates (x, y) in terms or (x', y').

From O′ and P draw O′L and PM perpendiculars to OX. Let PM meet O′X′ in M′.

Then $OL = \alpha, LO' = \beta, OM = x, MP = y$

Also $O'M' = x'$ and $M'P = y'$

Therefore, $OM = OL + LM = OL + O'M'$

$\therefore\ x = \alpha + x'$ ... ... (i)

Similarly, $MP = MM' + M'P = LO' + M'P$

$\therefore\ y = \beta + y'$ ... ... (ii)

The transformed co-ordinates are

and $\left.\begin{aligned} x' &= x - \alpha \\ y' &= y - \beta \end{aligned}\right\}$ ... ... (2)

Rule : *In order to shift the origin to* (α, β) *the transformation is obtained by replacing* x *by* $x + a$ *and* y *by* $y + \beta$. *If from the transformed equation we want to get the old equations then replace* x *by* $x - \alpha$ *and* y *by* $y - \beta$. *This is known as shifting the origin back.*

Art. 33. Rotation of axes (origin fixed)

To find the change in the co-ordinates of a point when the direction of axes is turned through an angle θ *where as the origin of co-ordinates remains the same.*

Let OX and OY be the old axes and OX′ and OY′ set the new axes. O is the common origin for the two sets of axes. Let the angle X′ OX through which the axes have rotated be represented by θ.

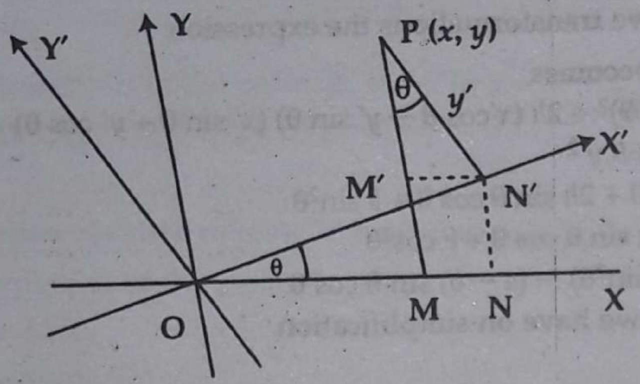


Fig. 11

Let P be any point in the plane and let its co-ordinates referred to the old axes be (x, y), and referred to the new axes be (x', y').

Let us try to determine x and y in terms of x', y' and θ, Draw PM perpendicular to OX, PN′ perpendicular to OX′, and N′N perpendicular to OX, and N′M′ parallel to OX.

Then $x = OM = ON - MN = ON - M'N'$

$= ON' \cos\theta - PN' \sin\theta = x' \cos\theta - y' \sin\theta$

$y = MP = MM' + M'P = NN' + M'P$

$= ON' \sin\theta + PN' \cos\theta = x' \sin\theta + y' \cos\theta$

Hence the formula for the rotation of the axes through an angle θ are.

and $\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta \end{aligned}\right\}$ ... ... (3)

Art. 34. Change of origin with the change of the direction of axes.

It is really the combination of Art. 32 and Art. 33. The best method is to apply Art. 32 first and then Art. 33. Of course the two transformations may also be made simultaneously.

Let us suppose that the system of axes be rectangular. The origin is shifted to the point (α, β) and then the axes are rotated through an angle θ. If the co-ordinates of any point be (x, y) in the old system, and (x', y') in the new system, from Art. 32, and Art. 33.

we have.

$$\text{and } \left.\begin{array}{l} x = \alpha + x' \cos\theta - y' \sin\theta \\ y = \beta + x' \sin\theta + y' \cos\theta \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Art. 35. Invariants. [R.U. 1983]

If by the rotation of the rectangular co-ordinate axes about the origin, the expression $ax^2 + 2hxy + by^2$ changes to

$$a'x'^2 + 2h'x'y' + b'y'^2,\ a + b = a' + b' \text{ and } ab - h^2 = a'b' - h'^2$$

Let (x, y) be the co-ordinates of a point P referred to a set of rectangular axes. If the axes are roated through an angle θ about the origin, let the co-ordinates of the same point P be (x', y') referred to the new system of rectangular axes. Then we have by

(3) Art. 33. $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.

By using the above transformations the expression

$ax^2 + 2hxy + by^2$ becomes

$$a(x' \cos\theta - y' \sin\theta)^2 + 2h\,(x' \cos\theta - y' \sin\theta)\cdot(x' \sin\theta + y' \cos\theta) + b\,(x' \sin\theta + y' \cos\theta)^2$$
$$= a'x'^2 + 2h'x'y' + b'y'^2$$

where $a' = a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta \quad \ldots \quad \ldots \quad \ldots \quad (1)$

$b' = a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta \quad \ldots \quad \ldots \quad \ldots \quad (2)$

$h' = h\,(\cos^2\theta - \sin^2\theta) - (a - b)\sin\theta\cos\theta \quad \ldots \quad \ldots \quad \ldots \quad (3)$

From (1) and (2) we have on simplification

$$a' + b' = a + b$$

Again $2a' = 2a\cos^2\theta + 4h\sin\theta \text{ cis }\theta + 2b\sin^2\theta$

$$= a\,(1 + \cos 2\theta) + 2h\sin 2\theta + b(1 - \cos 2\theta)$$
$$= a + b + 2h\sin 2\theta + (a - b)\cos 2\theta \quad \ldots \quad \ldots \quad (4)$$

Similarly. $2b' = a + b - 2h\sin 2\theta - (a - b)\cos 2\theta. \quad \ldots \quad \ldots \quad (5)$

From (4), (5) and (3) we have

$$4(a'b' - h'^2) = (a + b)^2 - \{2h\sin 2\theta + (a - b)\cos 2\theta\}^2$$
$$- \{2h\cos 2\theta - (a - b)\sin 2\theta\}^2$$
$$= (a + b) - 4h^2 - (a - b)^2 = 4ab - 4h^2$$
$$\text{or, } a'b' - h'^2 = ab - h^2$$

The two quantities $a + b$ and $ab - h^2$ for the expression $ax^2 + 2hxy + by^2$ are called **invariants** of transformation from one system of rectangular axes (Without change of the origin), because their values remain unchanged by the transformation.

Cor. Removal of the-xy-term from the expression $ax^2 + 2hxy + by^2$

If we rotate the rectangular axes through an angle θ about the origin, we have transformations by Eq. 3, Art. 33.

$$x = x' \cos\theta - y' \sin\theta,\ y = x' \sin\theta + y' \cos\theta$$

Putting the values of x and y in $ax^2 + 2hxy + by^2$

we have , $a'x'^2 + 2h'x'y' + b'y'^2$

where the values of a'. b' and h' are available from (1), (2) and (3) of Art 35.

The co-efficient of $x'\,y'$ term in the new expression is $2h'$ where $h' = h\,(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta$. In order to remove the $x'y'$ term, the co-efficient h' is zero i.e.

$$h(\cos^2\theta - \sin^2\theta) - (a-b)\sin\theta\cos\theta = 0$$

$$\therefore h\cos 2\theta = \frac{1}{2}(a-b)\sin 2\theta \text{ or, } \tan 2\theta = \frac{2h}{a-b}$$

$$\therefore \theta = \tfrac{1}{2}\tan^{-1}\frac{2h}{a-b} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

Hence if the axes are rotated through an angle $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$ then xy term in the expression $ax^2 + 2hxy + by^2$ vanishes.

Ex. 1. Determine the equation of the curve $2x^2 + 3y^2 - 8x + 6y - 7 = 0$ when then origin is transferred to the point $(2, -1)$.

when then origin is transferred to the point $(2, -1)$ Put $x = x' + 2$, $y = y' - 1$ in the above equation. Then $2(x'+2)^2 + 3(y'-1)^2 - 8(x'+2) + 6(y'-1) - 7 = 0$

or, $2x'^2 + 3y'^2 = 18$

Now removing suffixes the equation referred to new axes is $2x^2 + 3y^2 = 18$

Ex. 2. Determine the equation of the parabola

$$x^2 - 2xy + y^2 + 2x - 4y + 3 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (i)$$

after rotating of axes through 45°

When the axes have been roated through an angle 45°, then

$x = x'\cos 45° - y'\sin 45° = (x' - y')/\sqrt{2}$

$y = x'\sin 45° + y'\cos 45° = (x' + y)/\sqrt{2}$

Put them in (1), then $\left(\frac{x'-y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x'-y'}{\sqrt{2}}\right)\frac{x'+y'}{\sqrt{2}} + \left(\frac{x'+y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x'-y'}{\sqrt{2}}\right) - 4\left(\frac{x'+y'}{\sqrt{2}}\right) + 3 = 0$

On simplification the equation reduces to

$2y'^2 - \sqrt{2}x' - 3\sqrt{2}y' + 3 = 0$

Now dropping the suffixes, the equation is

$2y^2 - \sqrt{2}x - 3\sqrt{2}y + 3 = 0$ Ans.

Ex. 3. Remove the first degree terms in

$3x^2 + 4y^2 - 12x + 4y + 13 = 0$

Complete the square,

$$\text{or, } 3(x^2 - 4x) + 4\left(y^2 + y + \frac{1}{4}\right) = 0$$

$$\text{or, } 3(x-2)^2 + 4\left(y + \frac{1}{2}\right)^2 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (ii)$$

Put $x - 2 = x'$ and $y + \frac{1}{2} = y'$ in (ii)

$\therefore\ 3x'^2 + 4y'^2 = 0$

This is satisfied only by $x' = 0, y' = 0$, which is the new origin.

By shifting origin at $\left(2, \frac{-1}{2}\right)$, the first degree terms can be removed.

Ex. 4. By transforming to parallel axes through a properly chosen point (h, k), prove that the equation.

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$$

can be reduced to one containing only the terms of the 2nd degree.

Transforming to parallel axes through (h, k) we have

$$12(x' + h)^2 - 10(x' + h)(y' + k) + 2(y' + k)^2 + 11(x' + h) - 5(y' + k) + 2 = 0$$

or, $12x'^2 - 10x'y' + 2y'^2 + (24h - 10k + 11)x' + (-10h + 4k - 5)y' + (12h^2 - 10hk + 2k^2 + 11h - 5k + 2) = 0$ ... ... (i)

Now equate the co-efficients of x', y' to zero then

$24h - 10k + 11 = 0$ also $-10h + 4k - 5 = 0$

and $12h^2 - 10hk + 2k^2 + 11h - 5k + 2 = 0$

Solve the first two equations $h = -\frac{3}{2}$, $k = -\frac{5}{2}$ and it is clear that these values of h, k satisfy the equation (ii). Hence the equation. (i) becomes $12x'^2 - 10x'y' + 2y'^2 = 0$

5. The clirection cosine of axes remaining the same, choose a new origin such that the new ordinates of the pair of points whose old co-ordinate are (5, –13) and (–3,11) may be of forms (h, k) and (–h, –k). [N.U.H-1...]

Solution : Let the co-ordinate of new origin be (α, β).

Given that the old co-ordinates of the two points are (5, –13) and (–3, 11) whose new ordinates are (h, k) and (–h, –k).

So, $(5 - \alpha, -13 - \beta) = (h, k)$ and

$(-3 - \alpha, 11 - \beta) = (-h, k)$

Equating corresponding Components,

$5 - \alpha = h, \quad -13 - \beta = -k$

$-3 - \alpha = -h \quad 11 - \beta = k$

$\therefore 2 - 2\alpha = 0 \quad -2 - 2\beta = 0$

$\therefore \alpha = 1 \quad \therefore \beta = -1$

Thus, The co-ordinate of new origin (1, – 1)

6. Transfer the equation $11x^2 - 4xy + 14y^2 - 58x - 44y + 126 = 0$ to new axes of X and Y whose equation are $x - 2y + 1 = 0$ and $2x + y - 8 = 0$ respectively. [N.U.H-2000; 2008]

Solution : Given that

$$11x^2 - 4xy + 14y^2 - 58x - 44y + 126 = 0 \ldots \quad \ldots \quad \ldots \quad (1)$$

Let $x - 2y + 1 = 0$ and $2x + y - 8 = 0$ be The new axes of X and Y.

New co-ordinate of (x, y) be (x_1, y_1).

So, $x_1 = \dfrac{2x + y - 8}{\sqrt{2^2 + 1^2}} = \dfrac{2x + y - 8}{\sqrt{5}}$.

$$\Rightarrow 2x + y - 8 = \sqrt{5}x_1 \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$y_1 = \dfrac{x - 2y + 1}{\sqrt{1^2 + 2^2}} = \dfrac{x - 2y + 1}{\sqrt{5}}$ and

$$\Rightarrow x - 2y + 1 = \sqrt{5}y_1 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

From (2) and (3)

$$4x + 2y - 16 = 2\sqrt{5}\,x_1$$

$$x - 2y + 1 = \sqrt{5}y_1$$

$$(+) \qquad 5x - 15 = \sqrt{5}y_1 + 2\sqrt{5}x_1$$

$$5x = 15 + \sqrt{5}y_1 + 2\sqrt{5}x_1$$

$$x = \frac{2\sqrt{5}x_1 + \sqrt{5}y_1 + 15}{5}$$

$$= \frac{2x_1 + y_1}{\sqrt{5}} + 3$$

From (2) ; $y = -2x + \sqrt{5}x_1 + 8$

$$= \frac{-4x_1 - 2y_1}{\sqrt{5}} - 6 + \sqrt{5}x_1 + 8$$

$$= \frac{-4x_1 - 2y_1 + 5x_1}{\sqrt{5}} + 2$$

$$= \frac{x_1 - 2y_1}{\sqrt{5}} + 2$$

$\therefore \ x = h + 3$

Where $h = \dfrac{2x_1 + y_2}{\sqrt{5}} + 3$

$y = k + 2 \ k = \dfrac{x_1 - 2y_1}{\sqrt{5}} + 2$

Putting the values of x and y in (1)

we get

$11(h+3)^2 - 4(h+3)(k+2) + 14(k+2)^2 - 58(h+3) - 44(k+2) + 126 = 0$

$\Rightarrow 11(h^2 + 6h + 9) - 4(hk + 2h + 3k + 6) + 14(k^2 + 4k + 4) - 58h - 174 - 44k - 88 + 126 = 0$

$\Rightarrow 11h^2 - 4hk + 14k^2 - 5 = 0$

$$\Rightarrow 11\left(\frac{2x_1 + y_1}{\sqrt{5}}\right)^2 - 4\left(\frac{2x_1 + y_1}{\sqrt{5}}\right)\left(\frac{x_1 - 2y_1}{\sqrt{5}}\right) + 14\left(\frac{x_1 - 2y_1}{\sqrt{5}}\right)^2 - 5 = 0$$

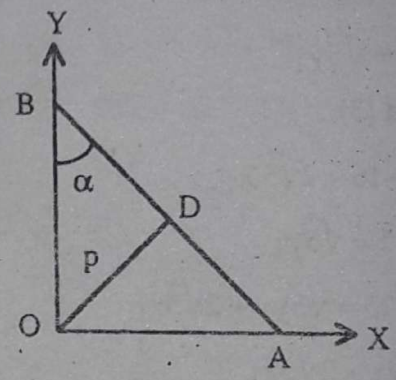
$\Rightarrow 11(4x_1^2 + 4x_1y_1 + y_1^2) - 4(2x_1^2 - 4x_1y_1 - 2y_1^2) + 14(x_1^2 - 4x_1y_1 + 4y_1^2) - 25 = 0$

$= 50x_1^2 - 75y_1^2 = 25$

$= 2x_1^2 + 3y_1^2 = 1$ which is required equation.

7. Reduce the equation of a straight line in the polar form $r\cos(\theta - \alpha) = p$. [N.U.H. 200]

Solution :



Let the line AB makes an angle α with Y-axis and OD= p. Form the figure, equation of AB is $\frac{x}{OA} + \frac{y}{OB} = 1$ ... ... (1)

Sinec OD perpendicular to AB and OD = P

From the Δ OBD ; $\quad \sin\alpha = \frac{OD}{OB} = \frac{P}{OB}$

$\therefore OB = \frac{P}{\sin\alpha}$

Again, from the Δ AOD,

$\sin(90° - \alpha) = \frac{OD}{OA} = \frac{P}{OA}$

$\therefore OA = \frac{P}{\cos\alpha}$

Putting the values of OA and OB in (1) we get $\frac{x}{\frac{P}{\cos\alpha}} + \frac{y}{\frac{P}{\sin\alpha}} = 1$

$\Rightarrow x\cos\alpha + y\sin\alpha = p$ ... ...

Relation between cartesian and polar Co-ordinates we can write,

$x = r\cos\theta, \quad y = r\sin\theta$

Equation (2) becomes

$r\cos\theta\cos\alpha + r\sin\theta\sin\alpha = p$

$\Rightarrow r\cos(\theta - \alpha) = p$ which is required polar equation.

8. The equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ is transformed to $4x^2 + 2y^2 = 1$ when refered to rectangular axes through the point (2, 3). Find the inclination of the later axes to the former. [N.U.H. 2004]

Solution : Given that

$$3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0 \quad \ldots \quad \ldots \quad \ldots (1)$$

Transfer the origin to a point (2, 3). Then putting $x = x_1 + 2$, $y = y_1 + 3$ in (1), we get (1) $\Rightarrow 3(x_1 + 2)^2 + 2(x_1 + 2)(y_1 + 3) + 3(y_1 + 3)^2 - 18(x_1 + 2) - 22(y_1 + 3) + 50 = 0$

$$\Rightarrow 3(x_1^2 + 4x_1 + 4) + 2(x_1y_1 + 3x_1 + 2y_1 + 6) + 3(y_1^2 + 6y_1 + 9) - 18x_1 - 36 - 22y_1 - 66 + 50 = 0$$

$$= 3x_1^2 + 2x_1y_1 + 3y_1^2 = 1$$

Removing suffixes, we get

$$3x^2 + 2xy + 3y^2 = 1 \quad \ldots \quad \ldots \quad \ldots (2)$$

Let the axes rotated be an angle θ, Then we know, $\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$; where $a = 3, b = 3, h = 1$

$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\frac{2.1}{3-3}$$

$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\infty$$

$$\Rightarrow \theta = \frac{1}{2}\cdot\frac{\pi}{2} = \pi/4 = 45°$$

We know that the axes rotated be an angle θ, then

$x = x_1\cos\theta - y_1\sin\theta$, $\qquad y = x_1\sin\theta + y_1\cos\theta$

$x = x_1\cos 45° - y_1\sin 45°$ $\qquad y = x_1\sin 45° + y_1\cos 45°$

$x = \dfrac{x_1 - y_1}{\sqrt{2}}$ $\quad x = \dfrac{x_1 + y_1}{\sqrt{2}}$

Putting the values of x and y in (2) we get

$$(2) \Rightarrow 3\left(\frac{x_1 - y_1}{\sqrt{2}}\right)^2 + 2\left(\frac{x_1 - y_1}{\sqrt{2}}\right)\left(\frac{x_1 + y_1}{\sqrt{2}}\right) + 3\left(\frac{x1 + y1}{\sqrt{2}}\right)^2 = 1$$

$$\Rightarrow 3(x_1 - y_1)^2 + 2(x_1^2 - y_1^2) + 3(x_1 + y_1)^2 = 2$$

$$\Rightarrow 3(x_1^2 - 2x_1y_1 + y_1^2) + 2x_1^2 - 2y_1^2 + 3(x_1^2 + 2x_1y_1 + y_1^2) = 2$$

$$\Rightarrow 8x_1^2 + 4y_1^2 = 2$$

$$\Rightarrow 4x_1^2 + 2y_1^2 = 1$$

Remoing suffixes,

$$\therefore 4x^2 + 2y^2 = 1$$

Which is the required transformed equation

9. If the direction of axes is turned through an angle 30° and the origin remains unchanged th find the transformation equation of $x^2 + 2\sqrt{3}xy - y^2 - 2a^2 = 0$. Idntify and sketch it. [N.U.H. 2009, 199

Solution : given that,

$x^2 + 2\sqrt{3}xy - y^2 - 2a^2 = 0$ ... ...

Since the axes rotated be an angle 30° and origin be unchanged.

So, $x = x_1\cos30^\circ - y_1\sin30^\circ = \dfrac{\sqrt{3}x_1 - y_1}{\sqrt{2}}$

$y = x_1\sin30^\circ + y_1\cos30^\circ = \dfrac{x_1 + \sqrt{3}y_1}{\sqrt{2}}$

Putting this valus in (1) we get

$$(1) = \left(\frac{\sqrt{3}x_1 - y_1}{\sqrt{2}}\right)^2 + 2\sqrt{3}\left(\frac{\sqrt{3}x_1 - y_1}{\sqrt{2}}\right)$$

$$\left(\frac{x_1 + \sqrt{3}y_1}{2}\right) - \left(\frac{x_1 + \sqrt{3}y_1}{2}\right)^2 - 2a^2 = 0$$

$$\Rightarrow \frac{1}{4}(3x_1^2 - 2\sqrt{3}x_1y_1 + y_1^2) + \frac{2\sqrt{3}}{4}(\sqrt{3}x_1^2 + 3x_1y_1 - x_1y_1 + \sqrt{3}y_1^2) - \frac{1}{4}(x_1^2 + 2\sqrt{3}x_1y_1 + 3y_1^2) - 2a^2 = 0$$

$$\Rightarrow 3x_1^2 - 2\sqrt{3}x_1y_1 + y_1^2 + 6x_1^2 + 4\sqrt{3}x_1y_1 - 6y_1^2 - x_1^2 - 2\sqrt{3}x_1y_1 - 3y_1^2 - 8a^2 = 0$$

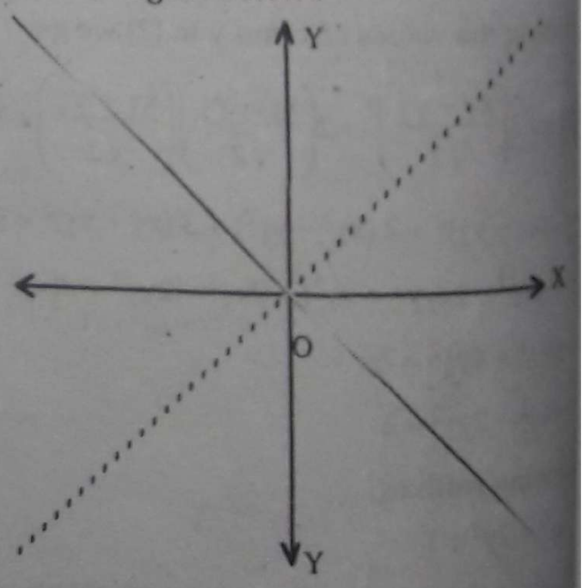
$$\Rightarrow 8x_1^2 - 8y_1^2 - 8a^2 = 0$$

$$= x_1^2 - y_1^2 = a^2$$

Removing suffixes, we get

$x^2 - y^2 = a^2$ which is the equation of rectangular hyperbola.

2nd part : we have. $x^2 - y^2 = a^2$. It is rectangular hyperbola whose asymptotes are $y = \pm$ Asymptotes makes an angles with x – axis are $\pm 45^\circ$. Its figure below :



10. Transform the equation $17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$ to one in which there is no term involuing x, y and xy both sets of axes being rectangular. [N.U.H. 2003]

Solution : Given equation

$17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$ ... ... (1)

Let transfer the origin to a point (4, k)

Then $x = x_1 + h,\ y = y_1 + k$

Putting these values in (1) we get, $17(x_1 + h)^2 + 18(x_1 + h)(y_1 + k) - 7(y_1 + k)^2 - 16(x_1 + 4h - 32(y_1 + k) - 18 = 0$

$\Rightarrow$ $17(x_1^2 + 2x_1h + h^2) + 18(x_1y_1 + x_1k + y_1h + hk) - 7(y_1^2 + 2y_1k + k^2) - 16x_1 - 16h - 32y_1 - 32k - 18 = 0$

$\Rightarrow$ $17x_1^2 + 18x_1y_1 - 74y_1^2 + (34h + 18k - 16)x_1 + (18h - 14x - 32)y_1 + 17h^2 + 18hk - 7k^2 - 16h - 32k - 18 = 0$ ... ... (2)

Since there is no terms of x, y. So equating the coefficient of x_1, y_1 from both sides.

$\therefore$ $34h + 18k - 16 = 0$

$18h - 14k - 32 = 0$

By the rule of cross multiplication,

$$\therefore \quad \frac{h}{-576 - 224} = \frac{k}{-288 + 1088} = \frac{1}{476 - 324}$$

$$= \frac{h}{-800} = \frac{k}{800} = \frac{1}{-800}$$

$\therefore$ $h = 1,\quad k = -1$

Putting the values of h and k in (2) we get $17x_1^2 + 18x_1y_1 - 7y_1^2 + 0 + 0 + 17 + 18(1)(-1) - 7 - 16 - 32(-1) - 18 = 0$

$17x_1^2 + 18x_1y_1 - 7y_1^2 - 10 = 0$

Removing Suffixes,

$\therefore$ $17x^2 + 18xy - 7y^2 - 10 = 0$ ... ... (3)

If we rotated the axes through an angle θ

$$\text{Then } \theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$$

$$\therefore \quad 2\theta = \tan^{-1}\frac{2.9}{17+7}$$

$$\Rightarrow \tan 2\theta = \frac{18}{24} = \frac{3}{4}$$

$$\therefore \quad \sin 2\theta = \frac{3}{5}$$

$$\therefore \quad 2\sin\theta\cos\theta = \frac{3}{5} \qquad \text{Cos}2\theta = \frac{4}{5}$$

$$\therefore \quad \sin\theta = \frac{3}{10\cos\theta} \qquad 2\cos^2\theta - 1 = \frac{4}{5}$$

$$= \frac{3\sqrt{10}}{10.3} = \frac{1}{\sqrt{10}} \qquad \text{Cos}^2\theta = \frac{9}{10} \quad \therefore \cos\theta = \frac{3}{\sqrt{10}}$$

We know,

$x = x_1\cos\theta - y_1\sin\theta$

$$= \frac{3x_1 - y_1}{\sqrt{10}}$$

$y = x_1\sin\theta + y_1\cos\theta$

$$= \frac{x_1 + 3y_1}{\sqrt{10}}$$

Putting the values of x, y in (3) we get

$$(3) \Rightarrow 17\left(\frac{3x_1 - y_1}{\sqrt{10}}\right)^2 + 18\left(\frac{3x_1 - y_1}{\sqrt{10}}\right)\left(\frac{x_1 + 3y_1}{\sqrt{10}}\right)$$

$$-7\left(\frac{x_1 + 3y_1}{\sqrt{10}}\right) - 10 = 0$$

$$\Rightarrow 17(9x_1^2 - 6x_1y_1 + y_1^2) + 18(3x_1^2 + 9x_1y_1 - 3y_1^2)$$

$$-7(x_1^2 \Rightarrow 6x_1y_1 + 9y_1^2) - 100 = 0$$

$$\Rightarrow (153 + 54 - 7)\, x_1^2 = (-102 + 144 - 42)\, x_1y_1 + (17 - 54 - 63)\, y_1^2 \Rightarrow 100$$

$$\Rightarrow 200x_1^2 - 100y_1^2 = 100$$

$$\Rightarrow 2x_1^2 - y_1^2 = 1$$

Removing suffixes, we get

$$2x^2 - y^2 = 1$$

Which is the required transformation.

EXERCISE IV

1. Transform to parallel axes through the new origin of the equations.

(a) Origin $(1, -2)$, $2x^2 + y^2 - 4x + 4y = 0$

Ans. $2x^2 + y^2 = 6$

[D. U. 1982]

(b) Origin $(3, 1)$, $x^2 - 6x + 2y^2 + 7 = 0$ Ans. $x^2 + 2y^2 + 4y = 0$.

2. Transform to axes inclined at 45° to the original axes the equations.

(i) $x^2 - y^2 = a^2$

Ans. $2xy = a^2$

(ii) $x^2 - y^2 - 2\sqrt{2}x - 10\sqrt{2}y + 2 = 0$

Ans. $xy + 6x + 4y = 1$

3. Remove the first degree terms in each of the following equations.

(a) $3x^2 - 4y^2 - 6x - 8y - 10 = 0$

Ans. $5x^2 - 4y^2 = 9$

(b) $2x^2 + 5y^2 - 12x + 10y - 7 = 0$

Ans. $2x^2 + y^2 = 40$

(c) $3x^2 - 4y^2 + 6x + 24y - 135 = 0$

Ans. $3x^2 - 4y^2 = 102$

4. Prove that the value of $g^2 + f^2$ in the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ remains uneffected by orthogonal transformation without change of origin.

5. Transform the equation $11x^2 + 24xy + 4y^2 - 20x - 40y - 5 = 0$ to rectangular axes through the point $(2, -1)$ and inclined at an angle $\tan^{-1}\frac{4}{3}$

Ans. $x^2 - 4y^2 + 1 = 0$

6. Determine the angle through which the axes must be rotated to remove the xy term in the equation. $7x^2 - 6\sqrt{3}xy + 13y^2 = 16$ Ans. $\theta = 30°$, $x^2 + 4y^2 = 4$

7. By transforming to parallel axes through a properly chosen point (h, k) prove that the equation $2x^2 + y^2 - xy - 5x - 4y + 11 = 0$ can be reduced to one containing terms of the 2nd degree only. Ans. (2, 3)

8. The direction of axes remaining the same, choose a new origin such that the new co-ordinate of the pair of points whose old co-ordinates are (5,—13) and (—3, 11) may be the forms (h, k) and $(-h, -k)$ [N. U. H. 1997] Ans. (1, —1)

9. Transform the equation $17x^2 + 18xy - 7y^2 - 16x - 32y - 18 = 0$ to one in which there is no term involving x, y and xy, both sets of axes being rectangular.

[N. U. H. 2003] Ans. (1,—1), $\frac{1}{2}\tan^{-1}3/4$, $2x^2 - y^2 = 1$

10. Transform the axes inclined at 30° to the original axes the equation, $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$ [N. U. H. 2009, 1999] Ans. $x^2 - y^2 = a^2$

11. The equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ is transformed to $4x^2 + 2y^2 = 1$ when referred to rectangular axes through the point (2, 3). Find the inclination of the latter axes to the former. [N. U. H. 2004] [R. U. 1960]