Application

Question → 3.17 : Bacteria in a certain culture increase at a rate proportional to the number present. If the number doubles in one hour, how long does it take for the number to triple?

Solution :

Let, $y$ be the number of Bacteria at the present time $t$.

Then, the function will be,

$$y = f(t) \quad \text{——— (i)}$$

Now, according to the question,

$$\frac{dy}{dt} \propto y$$

$$\text{or, } \frac{dy}{dt} = ky$$

$$\text{or, } \frac{dy}{dt} - ky = 0 \quad \text{——— (ii)}$$

Which is the linear differential equation and satisfies equation - (i)

Now, integrating factor,

$$I.F. = e^{\int -K dt}$$

$$= e^{-Kt}$$

Multiplying eq$^n$ - (ii) by I.F.

$$e^{-Kt} \frac{dy}{dt} - Ky e^{-Kt} = 0$$

or, $$\frac{d}{dt}\left(e^{-Kt} \cdot y\right) = 0$$

Now, integrating,

$$e^{-Kt} \cdot y = c$$

or, $$y = c e^{Kt} \quad \text{——— (iii)}$$

Applying initial condition,

when, $t = 0$ then $y = y_0$

Putting in equation - (iii), we get,

$$c = y_0$$

Hence, equation - (iii) becomes,

$$y = y_0 e^{Kt} \quad \text{——— (iv)}$$

Now, applying the additional condition to find the constant of proportionality K,

when, $t = 1$ hr then, $y = 2y_0$.

Putting in equation -(iv), we get,

$$2y_0 = y_0 e^{K \cdot 1}$$

or, $e^K = 2$

or, $K = \log 2$

Hence, equation -(iv) becomes,

$$y = y_0 e^{t \log 2}$$

Now, for final requirement, substituting $y = 3y_0$

$$3y_0 = y_0 e^{t \log 2}$$

or, $3 = e^{t \log 2}$

or, $t \log 2 = \log 3$

or, $t = \dfrac{\log 3}{\log 2} = 1.58$ hr

Hence, the number of Bacteria will be triple in 1.58 hr.

Ans

Question → 3.18: When a cake is removed from an oven, its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take to cool off to a room temperature of 70°F?

Solution:

Let, $T$ be the temperature of the cake at the time $t$ minutes.

Now, according to Newton's law of cooling, the differential equation governing the present situation is -

$$-\frac{dT}{dt} \propto (T - T_0) \quad \text{——} \ (1)$$

where, $T_0$ denotes the room temperature. $= 70°F$

Now, equation (1) can be written,

$$\frac{dT}{dt} = K\,(T - T_0)$$

where, $K$ is the constant of proportionality

Separating the variable,

$$\frac{dT}{T - T_0} = K \cdot dt$$

Now, integrating,

$$\log (T-T_0) = Kt + A$$

or, $T - T_0 = e^{Kt+A}$

or, $T - T_0 = e^{Kt} . e^{A}$

or, $T = T_0 + C e^{Kt}$ —— (11)

Applying initial condition,

when, $t = 0$, $T = 300°F$

Putting in equation - (11), we get,

$$300 = 70 + C e^{K \cdot 0}$$

or, $C = 230$

$\therefore$ $C = 230$

Applying additional condition,

when, $t = 3$, $T = 200°F$

Putting in equation - (11), we get,

$$200 = 70 + 230 e^{K \cdot 3}$$

or, $130 = 230 e^{3K}$

or, $e^{3K} = \frac{13}{23}$

or, $3K = \log \frac{13}{23}$

or, $K = \frac{1}{3} \log \frac{13}{23}$

or, $K = -0.19$

Hence, equation - (ii) becomes,

$$T = 70 + 230 e^{-0.19t} \quad \text{——— (iii)}$$

For finding the final cooling time we've to substitute $T = 70°F$. However, it will not give any finite solution. Yet, we can put $T = 70.5°F$ and get an approximate time period.

$$70.5 = 70 + 230 e^{-0.19t}$$

or, $70.5 - 70 = 230 e^{-0.19t}$

or, $e^{-0.19t} = \frac{0.5}{230}$

or, $-0.19t = \log \frac{0.5}{230}$

or, $t = \frac{-6.13}{-0.19}$

or, $t \simeq 32.3$

Hence, the cake will approximately be at room temperature in about half an hour.

Ans

Question → 3·19 : According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 300K and the substance cools from 370K to 340K in 15 minutes, find when the temperature will be 310K.

Solution:

Let $T$ be the temperature of the substance at the time $t$ minutes.

Now, according to Newton's law of cooling,

$-\frac{dT}{dt} \propto (T - T_0)$ ; where $T_0$ is the temperature of moving air. $(T_0 = 300K)$

or, $\frac{dT}{dt} = K (T - T_0)$

or, $\frac{dT}{T - T_0} = K\,dt$ [ Separating the variables.]

Now, integrating,

$\log (T - T_0) = Kt + A$

or, $T - T_0 = e^{Kt + A}$

or, $T - T_0 = e^{Kt} \cdot e^{A}$

or. $T = T_0 + c\, e^{Kt}$ ——— (1)

Applying initial condition.

when, $t = 0$, $T = 370$

Putting in equation -(1), we get,

$370 = 300 + c\, e^{K \cdot 0}$

or, $c = 70$

Applying additional condition,

when, $t = 15$, $T = 340$

putting in equation -(1), we get

$340 = 300 + 70\, e^{K \cdot 15}$

or, $e^{15K} = \frac{40}{70}$

or, $15K = \log \frac{4}{7}$

or, $K \simeq -0.0162$

Hence, equation -(1) becomes -

$$T = 300 + 70\, e^{-0.0162t} \quad \text{——— (ii)}$$

Now, for finding the time required for the substance to cool off to 310 K, substituting T = 310 in equation (ii)

$$310 = 300 + 70\, e^{-0.0162t}$$

$$\text{or, } e^{-0.0162t} = \frac{1}{7}$$

$$\text{or, } -0.0162t = \log \frac{1}{7}$$

$$\text{or, } t \simeq 52$$

Hence, the required time is 52 minutes approximately.

Ans

Question → 3.20 (Estimation of time of murder) The body of a murder victim was discovered at 11.00 p.m. The doctor took the temperature of the body at 11.30 p.m., which was 94.6°F. He again took the temperature after one hour when showed 93.4°F and noticed that the temperature of the room was 70°F. Estimate the time of death. (Normal temperature of human body = 98.6°F).

Solution:

Let, T be the temperature of the body at the time t.

Now, according to Newton's law of cooling, the differential equation governing the present situation is —

$$-\frac{dT}{dt} \propto (T - T_0) \quad \text{—①}$$

~~where~~, and, $T_0$ denotes the room temperature. [∴ $T_0 = 70°F$]

Now, equation —① can be written,

$$\frac{dT}{dt} = K(T - T_0) \; ; \quad \text{where } K \text{ is a constant}$$

$$\text{or, } \frac{dT}{T - T_0} = K \cdot dt \qquad \text{[separating the variables]}$$

Now, integrating,

$$\log (T-T_0) = Kt + A$$

or, $T - T_0 = e^{Kt+A}$

or, $T - T_0 = e^{Kt} \cdot e^{A}$

or, $T = T_0 + c e^{Kt}$ —— (11)

Applying initial condition,

when temperature measured first,

$t = 0$, $T = 94.6°F$

Putting in equation - (11), we get,

$$94.6 = 70 + c e^{K \cdot 0}$$

or, $c = 24.6$

$\therefore$ $c = 24.6$

Applying additional condition,

$t = 1$ hr, $T = 93.4°F$

Putting in equation – (11), we get

$$93.4 = 70 + 24.6 e^{K \cdot 1}$$

or, $e^{K} = \frac{23.4}{24.6}$

or, $K = \log \frac{23.4}{24.6}$

$\therefore$ $K = -0.05$

Hence, equation - (ii) becomes,

$$T = 70 + 24.6 e^{-0.05t} \quad \text{--- (iii)}$$

For finding the time period since the death, substituting, $T = 98.6°F$ in equation - (iii)

$$98.6 = 70 + 24.6 e^{-0.05t}$$

or, $e^{-0.05t} = \frac{28.6}{24.6}$

or, $-0.05t = \log \frac{28.6}{24.6}$

or, $t = \frac{0.15}{-0.05}$

or, $t = -3$

$\therefore$ $t = 3$ hrs before

Therefore, the estimated time of death is,

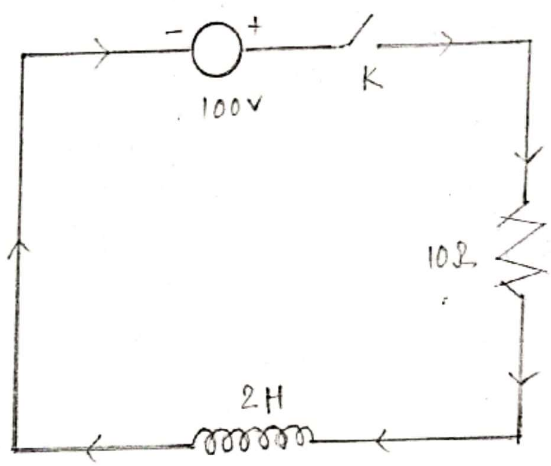
11.30 - 3.00 = 8.30 p.m. (approximate)

Ans

Question → 3.26: A generator having emf 100V is connected in series with a 10Ω resistor and an inductor of 2H. If the switch K is closed at time t=0, obtain a differential equation for the current and determine the current at time t.

Solution:

Let, i be the current in amperes flowing as shown in the figure;




Now, Voltage applied = 100V

Voltage drop across resistance $(Ri) = 10.i$

Voltage drop across inductor $\left(L\frac{di}{dt}\right) = 2\frac{di}{dt}$

Thus, Applying Kirchhoff's voltage law, we have,

$$100 = 10i + 2\frac{di}{dt}$$

or, $\frac{di}{dt} + 5i = 50$ ——(i)

which is required eqnation for current.

Again, equation -(i) is linear differential equation.

so, integrating factor,

$I.F. = e^{\int 5dt}$

$= e^{5t}$

Now, multiplying equation-(i) by $e^{5t}$, we get

$e^{5t}\frac{di}{dt} + 5i\,e^{5t} = 50\,e^{5t}$

or, $\frac{d}{dt}(e^{5t}.\,i) = 50\,e^{5t}$

Now, integrating,

$e^{5t}.\,i = 50\,e^{5t}\,\frac{1}{5}$

or, $i\,e^{5t} = 10\,e^{5t} + c$ ——(ii)

Since, the switch is closed at time $t=0$, we must have $i=0$ at $t=0$ and hence,

$0.\,e^{5.0} = 10\,e^{5.0} + c$

or, $0 = 10 + c$

or, $c = -10$

Putting $c = -10$ in equation - (11), we get

$$i \cdot e^{5t} = 10 e^{5t} - 10$$

or, $i = 10 - 10 e^{-5t}$

or, $i = 10(1 - e^{-5t})$

which is the equation for determining current at time $t$.

Ans.