# Chapter Three

## 01. Reduction Formulas

Integration by parts can be used to derive ***reduction formulas*** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function. For example, if *n* is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas

**Example 80:**
**Prove that:**

$$\textbf{01. } \int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$\textbf{02. } \int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

**Solution:** We first prove 01.

$$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx \text{ ---------------------------------(i)}$$

Let $u = \sin^{n-1} x$

$$\Rightarrow \frac{du}{dx} = (n-1)\sin^{n-1-1} x . \frac{d}{dx}(\sin x) \; [\because \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\Rightarrow \frac{du}{dx} = (n-1)\sin^{n-2} x . \cos x$$

$$\Rightarrow du = (n-1)\sin^{n-2} x . \cos x dx$$

Again,
Let, $dv = \sin x dx$

$$\Rightarrow \int dv = \int \sin x dx$$

$$\Rightarrow v = -\cos x$$

We have,

$$\int udv = uv - \int vdu \text{ ---------------------------------------------(ii)}$$

Putting the value of $u$, $dv$, $v$, $du$ in (ii)
**From (i)**

$$\int \sin^n x dx = \int \sin^{n-1} x \sin x dx$$

$$= \int \underbrace{\sin^{n-1} x}_{u} \; \underbrace{\sin x dx}_{dv} = \underbrace{\sin^{n-1} x}_{u} \; \underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v} \; \underbrace{(n-1)\sin^{n-2} x \cos x dx}_{du}$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)dx \quad [\therefore \cos^2 x = 1 - \sin^2 x]$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \{\sin^{n-2} x - \sin^{n-2} x . \sin^2 x\}dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\int \{\sin^{n-2} x - \sin^{n-2+2} x\}dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \{\sin^{n-2}x - \sin^n x\}dx$$

$$= -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}xdx - (n-1)\int \sin^n xdx$$

$$\therefore \int \sin^n xdx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}xdx - (n-1)\int \sin^n xdx$$

$$\Rightarrow \int \sin^n xdx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}xdx - n\int \sin^n xdx + \int \sin^n xdx$$

$$\Rightarrow \int \sin^n xdx + n\int \sin^n xdx - \int \sin^n xdx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}xdx$$

$$\Rightarrow n\int \sin^n xdx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2}xdx$$

$$\Rightarrow \int \sin^n xdx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{(n-1)}{n}\int \sin^{n-2}xdx + c \quad \textit{(Proved)}$$

**Now we will prove 02.**

$$\int \cos^n xdx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2}xdx$$

**Solution:**

$$\int \cos^n xdx = \int \cos^{n-1} x\cos xdx \text{------------------------------------------}(i)$$

Let

$$u = \cos^{n-1} x$$

$$\Rightarrow \frac{du}{dx} = (n-1)\cos^{n-1-1} x.\frac{d}{dx}(\cos x) \quad [\because \frac{d}{dx}(x^n) = nx^{n-1}]$$

$$\Rightarrow \frac{du}{dx} = (n-1)\cos^{n-2} x.(-\sin x)$$

$$\Rightarrow \frac{du}{dx} = -(n-1)\cos^{n-2} x.\sin x$$

$$\Rightarrow du = -(n-1)\cos^{n-2} x.\sin xdx$$

Let, $dv = \cos xdx$

$$\Rightarrow \int dv = \int \cos xdx$$

$$\Rightarrow v = \int \cos xdx = \sin x$$

We have,

$$\int udv = uv - \int vdu$$

So that $\int \cos^n xdx = \int \cos^{n-1} x\cos dx$

$$= \cos^{n-1} x.\sin x - \int \sin x\{-(n-1)\cos^{n-2} x.\sin x\}dx$$

$$= \cos^{n-1} x.\sin x + (n-1)\int \sin^2 x\cos^{n-2} xdx$$

$$= \cos^{n-1} x\sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} xdx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} xdx - (n-1)\int \cos^2 x.\cos^{n-2} xdx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} xdx - (n-1)\int \cos^{2+n-2} xdx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

$$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - n\int \cos^n x dx + \int \cos^n x dx$$

$$\therefore \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx - n\int \cos^n x dx + \int \cos^n x dx$$

$$\Rightarrow \int \cos^n x dx + n\int \cos^n x dx - \int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$\Rightarrow n\int \cos^n x dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x dx$$

$$\therefore \int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx + c \quad \textit{(Proved)}$$

**Example 81:**

**Prove that**

$$03. \int \sin^2 x dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int dx = \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C$$

$$04. \int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx = \frac{1}{2}x + \frac{1}{2}\sin x \cos x + C$$

**Solution:**

We have

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

In the case where $n = 2$, these formulas yields

$$\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$\Rightarrow \int \sin^2 x dx = -\frac{1}{2}\sin^{2-1} x \cos x + \frac{2-1}{2}\int \sin^{2-2} x dx \; [n = 2]$$

$$\Rightarrow \int \sin^2 x dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int \sin^0 x dx$$

$$\Rightarrow \int \sin^2 x dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int 1.dx$$

$$\Rightarrow \int \sin^2 x dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int dx$$

$$\Rightarrow \int \sin^2 x dx = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x$$

$$\therefore \int \sin^2 x dx = \frac{1}{2}x - \frac{1}{2}\sin x \cos x + C \quad \textbf{\textit{(Proved)}}$$

*Again, we have*

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

$$\Rightarrow \int \cos^2 x dx = \frac{1}{2}\cos^{2-1} x \sin x + \frac{2-1}{2}\int \cos^{2-2} x dx \; [n=2]$$

$$\Rightarrow \int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int \cos^0 x dx$$

$$\Rightarrow \int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}\int 1.dx$$

$$\Rightarrow \int \cos^2 x dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x$$

$$\therefore \int \cos^2 x dx = \frac{1}{2}x + \frac{1}{2}\cos x \sin x + c$$ ***(Proved)***

***Alternative forms of these integration formulas can be derived from the trigonometric identities***

$\sin^2 x = \frac{1}{2}(1 - \cos x2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Which follow from the double angle formulas?

$\cos 2x = 1 - 2\sin^2 x$ And $\cos 2x = 2\cos^2 x - 1$

These identities yield

$$\int \sin^2 x dx = \frac{1}{2}\int (1 - \cos 2x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$$\int \cos^2 x dx = \frac{1}{2}\int (1 + \cos 2x) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \qquad [\because \sin 2x = 2\sin x \cos x]$$

**Example 82:**

**Prove that**

05. $\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}\int \sin x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$

06. $\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$

***Answer***

We have, $\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{(n-1)}{n}\int \sin^{n-2} x dx$

$n = 3$, we get

$$\int \sin^3 x dx = -\frac{1}{3}\sin^2 x \cos x + \frac{2}{3}\int \sin x dx = -\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x + C$$

Again we have,

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

$n = 3$, we get

$$\int \cos^3 x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\int \cos x dx = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C$$

By using the identity $\sin^2 x = 1 - \cos^2 x$, $\cos^2 x = 1 - \sin^2 x$.

$$\int \sin^3 x dx = \frac{1}{3}\cos^3 x - \cos x + C$$

$$\int \cos^3 x dx = \sin x - \frac{1}{3}\sin^3 x + C$$

**Example 83: Prove that** $I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n} I_{n-2}$

***Answer:***

$$I_n = \int_0^{\pi/2} \sin^n x\, dx = [-\frac{1}{n}\sin^{n-1} x \cos x]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = [-\frac{1}{n}\sin^{n-1}(\frac{\pi}{2})\cos\frac{\pi}{2} + \frac{1}{n}\sin^{n-1}(0)\cos 0] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = [-\frac{1}{n}.1.0 + 0.1] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \qquad [\because I_n = \int_0^{\pi/2} \sin^n x\, dx \therefore I_{n-2} = \int_0^{\pi/2} \sin^{n-2} x\, dx]$$

**Example 84: Prove that** $I_n = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n} I_{n-2}$

***Answer***

$$\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \left[\frac{1}{n}\cos^{n-1} x \sin x\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \left[\frac{1}{n}\cos^{n-1}(\frac{\pi}{2})\sin(\frac{\pi}{2}) - \frac{1}{n}\cos^{n-1} 0 \sin 0\right] + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \left[\frac{1}{n}.0.1 - \frac{1}{n}.1.0\right] + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \cos^n x dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x dx$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \qquad [\because I_n = \int_0^{\pi/2} \cos^n x\, dx \therefore I_{n-2} = \int_0^{\pi/2} \cos^{n-2} x\, dx]$$

**Example 85: Prove that**

$$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.--------\frac{4}{5}.\frac{2}{3}.1; \text{when n is odd}$$

**We have**

$$I_n = \int_0^{\pi/2} \sin^n x\, dx = [-\frac{1}{n}\sin^{n-1} x \cos x]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = [-\frac{1}{n}\sin^{n-1}(\frac{\pi}{2})\cos\frac{\pi}{2} + \frac{1}{n}\sin^{n-1}(0)\cos 0] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = [-\frac{1}{n}.1.0 + 0.1] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx \qquad = \frac{n-1}{n} I_{n-2}$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \text{--------------------------------------(i)}$$

Now replacing $\mathbf{n}$ successively by $\mathbf{n-2, n-4, n-6............3}$ etc. we can write as in (i)

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

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$$I_3 = \frac{3-1}{3} I_{3-2} = \frac{2}{3} I_1$$

Putting the values of $I_{n-2}, I_{n-4}, I_{n-6}, \ldots\ldots\ldots\ldots I_3$ in (i), we get

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4}$$

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6}$$

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$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{2}{3} I_1 \text{------------------(ii)}$$

We have,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx$$

$$\therefore I_1 = \int_0^{\pi/2} \sin^1 x \, dx = \int_0^{\pi/2} \sin x \, dx = -[\cos x]_0^{\pi/2} = -[\cos \frac{\pi}{2} - \cos 0] = -[0-1] = -[-1] = 1$$

From (ii), we get,

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{2}{3} I_1$$

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{2}{3} .1$$

$$\therefore I_n = \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot - - - - - - - - \frac{4}{5} \cdot \frac{2}{3} .1; \text{when } n \text{ is odd}$$

**Example 86: Prove that**

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot - - - - - - - - \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}; \quad \text{when } n \text{ is even}$$

Answer: **We have**

$$I_n = \int_0^{\pi/2} \sin^n x \, dx = [-\frac{1}{n} \sin^{n-1} x \cos x]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x \, dx = [-\frac{1}{n} \sin^{n-1} (\frac{\pi}{2}) \cos \frac{\pi}{2} + \frac{1}{n} \sin^{n-1} (0) \cos 0] + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x \, dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = [-\frac{1}{n}.1.0 + 0.1] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\Rightarrow I_n = \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \quad = \frac{n-1}{n} I_{n-2}$$

$$\Rightarrow I_n = \frac{n-1}{n} I_{n-2} \text{----------------------------------------(i)}$$

Now replacing $n$ successively by $n-2, n-4, n-6 ...........4,2$ etc. we can write as in (i)

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

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$$I_4 = \frac{4-1}{4} I_{4-2} = \frac{3}{4} I_2$$

$$I_2 = \frac{2-1}{2} I_{2-2} = \frac{1}{2} I_0$$

Putting the values of $I_{n-2}, I_{n-4}, I_{n-6}, ...........I_4, I_2$ in (i), we get

$$I_n = \frac{n-1}{n} I_{n-2}$$

$$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} I_{n-4}$$

$$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} I_{n-6}$$

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$$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} ...................... \frac{3}{4} I_2$$

$$I_n = \frac{n-1}{n} . \frac{n-3}{n-2} . \frac{n-5}{n-4} ...................... \frac{3}{4} . \frac{1}{2} I_0 \text{-------------------------(ii)}$$

We have, $I_n = \int_0^{\pi/2} \sin^n x\,dx$

$$I_0 = \int_0^{\pi/2} \sin^0 x\,dx = \int_0^{\pi/2} (\sin x)^0\,dx = \int_0^{\pi/2} 1\,dx = [x]_0^{\pi/2} = [\frac{\pi}{2} - 0] = \frac{\pi}{2}$$

From (ii), we get,

$$I_n = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.....................\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}$$

$$\therefore I_n = \int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}.\frac{n-3}{n-2}.\frac{n-5}{n-4}.--------\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}; \text{when } n \text{ is even}$$

**Example 87:** Prove that

$$\int \sin^m(x)\cos^n(x)dx = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}(x)\cos^n(x)dx$$

***Answer:***

$$\int \sin^m(x)\cos^n(x)dx = \int \sin^{m-1}(x)\sin^m(x)\cos^n(x)dx$$

$$\int \sin^m(x)\cos^n(x)dx = \int \sin^{m-1}(x)\cos^n(x)\sin(x)dx \text{ -----------------------(i)}$$

Let $u = \sin^{m-1}(x)\cos^n(x)$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\{\sin^{m-1}(x)\cos^n(x)\}$$

$$\Rightarrow \frac{du}{dx} = \sin^{m-1}(x)\frac{d}{dx}\{\cos^n(x)\} + \cos^n(x)\frac{d}{dx}\{\sin^{m-1}(x)\}$$

$$[\because \frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u]$$

$$\Rightarrow \frac{du}{dx} = \sin^{m-1}(x).\{n.\cos^{n-1}(x).\frac{d}{dx}(\cos x)\} + \cos^n(x)(m-1)\{\sin^{m-1-1}(x)\frac{d}{dx}(\sin x)\}$$

$$\Rightarrow \frac{du}{dx} = \sin^{m-1}(x).\{n.\cos^{n-1}(x).(-\sin x)\} + \cos^n(x)(m-1)\{\sin^{m-2}(x)(\cos x)\}$$

$$\therefore du = [\sin^{m-1}(x).\{n.\cos^{n-1}(x).(-\sin x)\} + \cos^n(x)(m-1)\{\sin^{m-2}(x)(\cos x)\}]dx$$

-------------------------------------(ii)

Again Let $dv = \sin x dx$

$\Rightarrow \int dv = \int \sin x dx$

$\Rightarrow v = -\cos x$ -----------------------------------------------------(iii)

We have, $\int u\,dv = uv - \int v\,du$ -------------------------------------(iv)

Here,

$u = \sin^{m-1}(x)\cos^n(x)$

$\Rightarrow du = [\sin^{m-1}(x).\{n.\cos^{n-1}(x).(-\sin x)\} + \cos^n(x)(m-1)\{\sin^{m-2}(x)(\cos x)\}]dx$

$\Rightarrow dv = \sin x dx$

and $\mathbf{v = -\cos x}$

From (i),

$$\int \sin^m(x)\cos^n(x)dx = \int \sin^{m-1}(x)\cos^n(x)\sin(x)dx$$

$$= \{\sin^{m-1}(x)\cos^n(x)\}.(-\cos x) - \int(-\cos x)[\sin^{m-1}(x).\{n.\cos^{n-1}(x).(-\sin x)\}$$

$$+ \cos^n(x)(m-1)\{\sin^{m-2}(x)(\cos x)\}]dx$$

$$[\int u\,dv = uv - \int v\,du]$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) - \int[n\sin^{m-1+1}(x)\cos^{n-1+1}(x) - \cos^{n+1+1}(x)(m-1)\sin^{m-2}(x)]dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) - \int[n\sin^m(x)\cos^n(x) - \cos^{n+2}(x)(m-1)\sin^{m-2}(x)]dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) - \int n\sin^m(x)\cos^n(x)dx + \int\cos^{n+2}(x)(m-1)\sin^{m-2}(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) - n\int\sin^m(x)\cos^n(x)dx + (m-1)\int\cos^{n+2}(x).\sin^{m-2}(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) - n\int\sin^m(x)\cos^n(x)dx + (m-1)\int\sin^{m-2}(x)\cos^{n+2}(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^{n+2}(x)dx - n\int\sin^m(x)\cos^n(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^n(x).\cos^2 x dx - n\int\sin^m(x)\cos^n(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^n(x)(1-\sin^2 x)dx - n\int\sin^m(x)\cos^n(x)dx$$

$$[\because \sin^2 x + \cos^2 x = 1]$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^n(x)(1)dx$$

$$-(m-1)\int\sin^{m-2}(x)\cos^n(x)(\sin^2 x)dx - n\int\sin^m(x)\cos^n(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^n(x)(1)dx - (m-1)$$

$$\int\sin^{m-2+2}(x)\cos^n(x)dx \quad - n\int\sin^m(x)\cos^n(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^n(x)dx - (m-1)$$

$$\int\sin^{m-2+2}(x)\cos^n(x)dx - n\int\sin^m(x)\cos^n(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^n(x)dx - (m-1)\int\sin^m(x)\cos^n(x)dx$$

$$- n\int\sin^m(x)\cos^n(x)dx$$

$$\therefore \int\sin^m(x)\cos^n(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^n(x)dx - (m-1)\int\sin^m(x)\cos^n(x)dx$$

$$- n\int\sin^m(x)\cos^n(x)dx$$

$$\Rightarrow \int\sin^m(x)\cos^n(x)dx + (m-1)\int\sin^m(x)\cos^n(x)dx + n\int\sin^m(x)\cos^n(x)dx$$

$$= -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int\sin^{m-2}(x)\cos^n(x)dx$$

$$\Rightarrow \{1 + (m-1) + n\}\int\sin^m(x)\cos^n(x)dx = -\sin^{m-1}(x)\cos^{n+1}(x)$$

$$+ (m-1)\int\sin^{m-2}(x)\cos^n(x)dx$$

$$\Rightarrow (m+n)\int \sin^m(x)\cos^n(x)dx = -\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int \sin^{m-2}(x)\cos^n(x)dx$$

$$\Rightarrow \int \sin^m(x)\cos^n(x)dx$$

$$= \frac{1}{(m+n)}[-\sin^{m-1}(x)\cos^{n+1}(x) + (m-1)\int \sin^{m-2}(x)\cos^n(x)dx]$$

$$\Rightarrow \int \sin^m(x)\cos^n(x)dx = \frac{-\sin^{m-1}(x)\cos^{n+1}(x)}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}(x)\cos^n(x)dx$$

$$\Rightarrow \int \sin^m(x)\cos^n(x)dx = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}(x)\cos^n(x)dx$$

***(Proved)***

**Example 88:**
**Prove that**

$$\int \sin^m(x)\cos^n(x)dx = -\frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n} + \frac{n-1}{m+n}\int \sin^m(x)\cos^{n-2}(x)dx$$

***Answer:***

$$\int \sin^m(x)\cos^n(x)dx = \int \sin^m(x)\cos^{n-1}(x)\cos(x)dx \text{ --------(i)}$$

Let $u = \sin^m(x)\cos^{n-1}(x)$

$$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\{\sin^m(x)\cos^{n-1}(x)\}$$

$$\Rightarrow \frac{du}{dx} = \sin^m(x)\frac{d}{dx}\{\cos^{n-1}(x)\} + \cos^{n-1}(x)\frac{d}{dx}\sin^m(x)$$

$$[\because \frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u]$$

$$\Rightarrow \frac{du}{dx} = \sin^m(x).\{(n-1).\cos^{n-1-1}(x).\frac{d}{dx}(\cos x)\} + \cos^{n-1}(x)(m)\{\sin^{m-1}(x)\frac{d}{dx}(\sin x)\}$$

$$\Rightarrow \frac{du}{dx} = \sin^m(x).\{(n-1).\cos^{n-2}(x).(-\sin x)\} + \cos^{n-1}(x)(m)\{\sin^{m-1}(x)(\cos x)\}$$

$$\therefore du = [\sin^m(x).\{(n-1).\cos^{n-2}(x).(-\sin x)\} + \cos^{n-1}(x)(m)\{\sin^{m-1}(x)(\cos x)\}]dx$$

-----------------------------------(ii)

Again Let $dv = \cos x dx$

$$\Rightarrow \int dv = \int \cos x dx$$

$$\Rightarrow v = \sin x \text{ -------------------------------------------------------(iii)}$$

We have, $\int u\,dv = uv - \int v\,du$ -------------------------------------(iv)

Here, $u = \sin^m(x)\cos^{n-1}(x)$

$$\Rightarrow du = [\sin^m(x).\{(n-1).\cos^{n-2}(x).(-\sin x)\} + \cos^{n-1}(x)(m)\{\sin^{m-1}(x)(\cos x)\}]dx$$

$\mathbf{dv = \cos x dx}$
and $\mathbf{v = \sin x}$

From (i),

$$\int \sin^m(x)\cos^n(x)dx = \int \sin^m(x)\cos^{n-1}(x)\cos(x)dx$$
$$= \{\sin^m(x)\cos^{n-1}(x)\}.(\sin x) - \int (\sin x)[\sin^m(x).\{(n-1).\cos^{n-2}(x).(-\sin x)\}$$
$$+\cos^{n-1}(x)(m)\{\sin^{m-1}(x)(\cos x)\}]dx$$
$$[\because \int u\,dv = uv - \int v\,du]$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int (\sin x)[\sin^m(x).\{(n-1).\cos^{n-2}(x).(-\sin x)\}$$
$$+\cos^{n-1+1}(x)(m)\sin^{m-1}(x)]dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int (\sin x)[\sin^m(x)(n-1).\cos^{n-2}(x).(-\sin x)$$
$$+\cos^n(x)(m)\sin^{m-1}(x)]dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int (\sin x)[-\sin^{m+1}(x)(n-1).\cos^{n-2}(x) + \cos^n(x)(m)\sin^{m-1}(x)]dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int (\sin x)[\cos^n(x)(m)\sin^{m-1}(x) - \sin^{m+1}(x)(n-1).\cos^{n-2}(x)]dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int (\sin x)[m\cos^n(x)\sin^{m-1}(x) - (n-1)\sin^{m+1}(x)\cos^{n-2}(x)]dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int (\sin x)[m\sin^{m-1}(x)\cos^n(x) - (n-1)\sin^{m+1}(x)\cos^{n-2}(x)]dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int m\sin^{m-1}(x)\cos^n(x)\sin x dx + \int (n-1)\sin^{m+1}(x)\cos^{n-2}(x)\sin x dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int m\sin^{m-1+1}(x)\cos^n(x)dx + \int (n-1)\sin^{m+1+1}(x)\cos^{n-2}(x)dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - \int m\sin^m(x)\cos^n(x)dx + \int (n-1)\sin^{m+2}(x)\cos^{n-2}(x)dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)dx + (n-1)\int \sin^{m+2}(x)\cos^{n-2}(x)dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)dx + (n-1)\int \sin^m(x)\sin^2 x\cos^{n-2}(x)dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)dx + (n-1)\int \sin^m(x)(1-\cos^2 x)\cos^{n-2}(x)dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)dx + (n-1)\int \sin^m(x)(1-\cos^2 x)\cos^{n-2}(x)dx$$
$$[\because \sin^2 x + \cos^2 x = 1]$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)dx + (n-1)\int \sin^m(x)\cos^{n-2}(x)dx$$
$$-(n-1)\int \sin^m(x)\cos^2 x\cos^{n-2}(x)dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)dx + (n-1)\int \sin^m(x)\cos^{n-2}(x)dx$$
$$-(n-1)\int \sin^m(x)\cos^{n-2+2}(x)dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)dx + (n-1)\int \sin^m(x)\cos^{n-2}(x)dx$$
$$-(n-1)\int \sin^m(x)\cos^n(x)dx$$
$$\int \sin^m(x)\cos^n(x)dx = \sin^{m+1}(x)\cos^{n-1}(x) - m\int \sin^m(x)\cos^n(x)dx$$
$$+(n-1)\int \sin^m(x)\cos^{n-2}(x)dx - (n-1)\int \sin^m(x)\cos^n(x)dx$$

$$\Rightarrow \int \sin^m(x)\cos^n(x)dx + m\int \sin^m(x)\cos^n(x)dx + (n-1)\int \sin^m(x)\cos^n(x)dx$$
$$= \sin^{m+1}(x)\cos^{n-1}(x) + (n-1)\int \sin^m(x)\cos^{n-2}(x)dx$$
$$\Rightarrow \{1+m+(n-1)\}\int \sin^m(x)\cos^n(x)dx = \sin^{m+1}(x)\cos^{n-1}(x)$$
$$+ (n-1)\int \sin^m(x)\cos^{n-2}(x)dx$$
$$\Rightarrow (m+n)\int \sin^m(x)\cos^n(x)dx = \sin^{m+1}(x)\cos^{n-1}(x) + (n-1)\int \sin^m(x)\cos^{n-2}(x)dx$$
$$\therefore \int \sin^m(x)\cos^n(x)dx = \frac{1}{(m+n)}\{\sin^{m+1}(x)\cos^{n-1}(x)\} + \frac{n-1}{m+n}\int \sin^m(x)\cos^{n-2}(x)dx$$
$$\therefore \int \sin^m(x)\cos^n(x)dx = \frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n} + \frac{n-1}{m+n}\int \sin^m(x)\cos^{n-2}(x)dx \; \textit{(Proved)}$$

**Example 89:** Prove that

i. $$\int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}(x)\cos^n(x)dx$$

ii. $$\int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \frac{n-1}{m+n}\int_0^{\pi/2} \sin^m(x)\cos^{n-2}(x)dx$$

**i. Answer**: We have,

$$\int \sin^m(x)\cos^n(x)dx = -\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}(x)\cos^n(x)dx$$

$$\therefore \int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \left[-\frac{\sin^{m-1}(x)\cos^{n+1}(x)}{m+n}\right]_0^{\pi/2} + \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}(x)\cos^n(x)dx$$

$$\therefore \int_0^{\pi/2} \sin^m(x)\cos^n(x)dx$$
$$= \left[-\frac{\sin^{m-1}(\frac{\pi}{2})\cos^{n+1}(\frac{\pi}{2})}{m+n} - [-\frac{\sin^{m-1}(0)\cos^{n+1}(0)}{m+n}]\right] + \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}(x)\cos^n(x)dx$$

$$\therefore \int_0^{\pi/2} \sin^m(x)\cos^n(x)dx = \left[-\frac{1.0}{m+n} + \frac{0.1}{m+n}\right] + \frac{m-1}{m+n}\int_0^{\pi/2} \sin^{m-2}(x)\cos^n(x)dx$$

$$[\because \sin\frac{\pi}{2}=1;\ \cos\frac{\pi}{2}=0; \cos 0=1;\ \sin 0=0]$$

$$\therefore \int_0^{\pi/2}\sin^m(x)\cos^n(x)dx = \left[-\frac{0}{m+n}+\frac{0}{m+n}\right]+\frac{m-1}{m+n}\int_0^{\pi/2}\sin^{m-2}(x)\cos^n(x)dx$$

$$\therefore \int_0^{\pi/2}\sin^m(x)\cos^n(x)dx = \frac{m-1}{m+n}\int_0^{\pi/2}\sin^{m-2}(x)\cos^n(x)dx \quad \textit{(Proved)}$$

**ii.** We have,

$$\int\sin^m(x)\cos^n(x)dx = \frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n}+\frac{n-1}{m+n}\int\sin^m(x)\cos^{n-2}(x)dx$$

$$\Rightarrow \int_0^{\pi/2}\sin^m(x)\cos^n(x)dx = \left[\frac{\sin^{m+1}(x)\cos^{n-1}(x)}{m+n}\right]_0^{\pi/2}+\frac{n-1}{m+n}\int_0^{\pi/2}\sin^m(x)\cos^{n-2}(x)dx$$

$$\Rightarrow \int_0^{\pi/2}\sin^m(x)\cos^n(x)dx =$$

$$\left[\frac{\sin^{m+1}(\frac{\pi}{2})\cos^{n-1}(\frac{\pi}{2})}{m+n}-\frac{\sin^{m+1}(0)\cos^{n-1}(0)}{m+n}\right]+\frac{n-1}{m+n}\int_0^{\pi/2}\sin^m(x)\cos^{n-2}(x)dx$$

$$\Rightarrow \int_0^{\pi/2}\sin^m(x)\cos^n(x)dx = \left[\frac{1.0}{m+n}-\frac{0.1}{m+n}\right]+\frac{n-1}{m+n}\int_0^{\pi/2}\sin^m(x)\cos^{n-2}(x)dx$$

$$\Rightarrow \int_0^{\pi/2}\sin^m(x)\cos^n(x)dx = \left[\frac{0}{m+n}-\frac{0}{m+n}\right]+\frac{n-1}{m+n}\int_0^{\pi/2}\sin^m(x)\cos^{n-2}(x)dx$$

$$\therefore \int_0^{\pi/2}\sin^m(x)\cos^n(x)dx = \frac{n-1}{m+n}\int_0^{\pi/2}\sin^m(x)\cos^{n-2}(x)dx \ \textit{(Proved)}$$

**Example 90:** If n is a positive integer, Prove that $u_{n+2}+u_n = 2u_{n+1}$ Where,

$$u_n = \int_0^{\pi}\frac{1-\cos nx}{1-\cos x}dx$$

**Solution**: $u_n = \int_0^{\pi}\frac{1-\cos nx}{1-\cos x}dx$

$$u_n = \int_0^\pi \frac{2\sin^2\frac{nx}{2}}{2\sin^2\frac{x}{2}}dx \qquad [2\sin^2\frac{x}{2} = 1-\cos x]$$

$$u_n = \int_0^\pi \frac{\sin^2\frac{nx}{2}}{\sin^2\frac{x}{2}}dx \text{ ---------------------(i)}$$

Putting $n = n-1$

$$u_{n-1} = \int_0^\pi \frac{\sin^2\frac{(n-1)x}{2}}{\sin^2\frac{x}{2}}dx \text{ -------------(ii)}$$

Subtracting (i) & (ii),

$$u_n - u_{n-1} = \int_0^\pi \frac{\sin^2\frac{nx}{2}}{\sin^2\frac{x}{2}}dx - \int_0^\pi \frac{\sin^2\frac{(n-1)x}{2}}{\sin^2\frac{x}{2}}dx$$

$$u_n - u_{n-1} = \int_0^\pi \left[\frac{\sin^2\frac{nx}{2}}{\sin^2\frac{x}{2}} - \frac{\sin^2\frac{(n-1)x}{2}}{\sin^2\frac{x}{2}}\right]dx$$

$$u_n - u_{n-1} = \int_0^\pi \left[\frac{\sin^2\frac{nx}{2} - \sin^2\frac{(n-1)x}{2}}{\sin^2\frac{x}{2}}\right]dx \quad [\sin^2 A - \sin^2 B = \mathrm{Sin}(A+B)\mathrm{Sin}(A-B)]$$

$$u_n - u_{n-1} = \int_0^\pi \left[\frac{\sin\left[\frac{nx}{2} + \frac{(n-1)x}{2}\right]\sin\left[\frac{nx}{2} - \frac{(n-1)x}{2}\right]}{\sin^2\frac{x}{2}}\right]dx$$

$$u_n - u_{n-1} = \int_0^\pi \left[\frac{\sin\left[\frac{nx + (n-1)x}{2}\right]\sin\left[\frac{nx - (n-1)x}{2}\right]}{\sin^2\frac{x}{2}}\right]dx$$

$$u_n - u_{n-1} = \int_0^\pi \left[\frac{\sin\left[\frac{nx + nx - x}{2}\right]\sin\left[\frac{nx - nx + x}{2}\right]}{\sin^2\frac{x}{2}}\right]dx$$

$$u_n - u_{n-1} = \int_0^{\pi}\left[\frac{\sin\left[\frac{2nx-x}{2}\right]\sin\left[\frac{x}{2}\right]}{\sin^2\frac{x}{2}}\right]dx = \int_0^{\pi}\left[\frac{\sin\left[\frac{(2n-1)x}{2}\right]\sin\left[\frac{x}{2}\right]}{\sin^2\frac{x}{2}}\right]dx$$

$$u_n - u_{n-1} = \int_0^{\pi}\left[\frac{\sin\left[\frac{(2n-1)x}{2}\right]\sin\frac{x}{2}}{\sin^2\frac{x}{2}}\right]dx = \int_0^{\pi}\left[\frac{\sin\left[\frac{(2n-1)x}{2}\right]}{\sin\frac{x}{2}}\right]dx$$

$$u_n - u_{n-1} = \int_0^{\pi}\frac{\sin\frac{(2n-1)x}{2}}{\sin\frac{x}{2}}dx$$

$$u_n - u_{n-1} = I_n \text{------------------(iii)} \qquad \left[\text{Let, } I_n = \int_0^{\pi}\frac{\sin\frac{(2n-1)x}{2}}{\sin\frac{x}{2}}dx\right]$$

We have,

$$I_n = \int_0^{\pi}\frac{\sin\frac{(2n-1)x}{2}}{\sin\frac{x}{2}}dx \text{--------------------------(iv)}$$

Putting $n = n-1$

$$I_{n-1} = \int_0^{\pi}\frac{\sin\frac{\{2(n-1)-1\}x}{2}}{\sin\frac{x}{2}}dx = \int_0^{\pi}\frac{\sin\frac{(2n-2-1)x}{2}}{\sin\frac{x}{2}}dx$$

$$I_{n-1} = \int_0^{\pi}\frac{\sin\frac{(2n-3)x}{2}}{\sin\frac{x}{2}}dx \text{--------------------------(v)}$$

Subtracting (iv) & (v),

$$I_n - I_{n-1} = \int_0^{\pi}\frac{\sin\frac{(2n-1)x}{2}}{\sin\frac{x}{2}}dx - \int_0^{\pi}\frac{\sin\frac{(2n-3)x}{2}}{\sin\frac{x}{2}}dx$$

$$I_n - I_{n-1} = \int_0^{\pi}\left[\frac{\sin\frac{(2n-1)x}{2}}{\sin\frac{x}{2}} - \frac{\sin\frac{(2n-3)x}{2}}{\sin\frac{x}{2}}\right]dx = \int_0^{\pi}\left[\frac{\sin\frac{(2n-1)x}{2} - \sin\frac{(2n-3)x}{2}}{\sin\frac{x}{2}}\right]dx$$

$$I_n - I_{n-1} = \int_0^{\pi}\left[\frac{2\cos\dfrac{\frac{(2n-1)x}{2}+\frac{(2n-3)x}{2}}{2}\sin\dfrac{\frac{(2n-1)x}{2}-\frac{(2n-3)x}{2}}{2}}{\sin\frac{x}{2}}\right]dx$$

$$[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}]$$

$$I_n - I_{n-1} = \int_0^{\pi}\left[\frac{2\cos\dfrac{\frac{(2n-1)x+(2n-3)x}{2}}{2}\sin\dfrac{\frac{(2n-1)x-(2n-3)x}{2}}{2}}{\sin\frac{x}{2}}\right]dx$$

$$I_n - I_{n-1} = \int_0^{\pi}\left[\frac{2\cos\dfrac{\frac{2nx-x+2nx-3x}{2}}{2}\sin\dfrac{\frac{2nx-x-2nx+3x}{2}}{2}}{\sin\frac{x}{2}}\right]dx$$

$$I_n - I_{n-1} = \int_0^{\pi}\left[\frac{2\cos\dfrac{\frac{4nx-4x}{2}}{2}\sin\dfrac{\frac{2x}{2}}{2}}{\sin\frac{x}{2}}\right]dx = \int_0^{\pi}\left[\frac{2\cos\dfrac{\frac{4nx-4x}{2}}{2}\sin\frac{x}{2}}{\sin\frac{x}{2}}\right]dx$$

$$I_n - I_{n-1} = \int_0^{\pi}\left[2\cos\frac{\frac{4nx-4x}{2}}{2}\right]dx = \int_0^{\pi}\left[2\cos\frac{\frac{4(n-1)x}{2}}{2}\right]dx$$

$$I_n - I_{n-1} = \int_0^{\pi}\left[2\cos\frac{2(n-1)x}{2}\right]dx = \int_0^{\pi}\left[2\cos(n-1)x\right]dx = \frac{2}{n-1}\left[\sin(n-1)x\right]_0^{\pi}$$

$$I_n - I_{n-1} = \frac{2}{n-1}\left[\sin(n-1)\pi - \sin(n-1)0\right] = \frac{2}{n-1}\left[\sin(n-1)\pi - \sin 0\right]$$

$$I_n - I_{n-1} = \frac{2}{n-1}\left[0-0\right] \qquad [\sin(n-1)\pi = 0 \text{ for any values of n}]$$

$$I_n - I_{n-1} = 0$$

$I_n = I_{n-1}$ -----------------------------(vi)

From (vi),

$\mathbf{I_n = I_{n-1}}$

$\mathbf{I_{n-1} = I_{n-2}}$

$\mathbf{I_{n-2} = I_{n-3}}$

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$\mathbf{I_3 = I_2}$

$\mathbf{I_2 = I_1}$

So, $\mathbf{I_n = 1_1}$

From (iv)

$$\mathbf{I_n = \int_0^{\pi} \frac{\sin\frac{(2n-1)x}{2}}{\sin\frac{x}{2}} dx}$$

$$\mathbf{I_1 = \int_0^{\pi} \frac{\sin\frac{(2.1-1)x}{2}}{\sin\frac{x}{2}} dx \;\; I = \int_0^{\pi} \frac{\sin\frac{x}{2}}{\sin\frac{x}{2}} dx = \int_0^{\pi} dx = [x]_0^{\pi} = [\pi - 0]}$$

$\mathbf{I_1 = \pi}$ ---------------------------------(vii)

Since So, $\mathbf{I_n = 1_1}$

So, $\mathbf{I_n = 1_1 = \pi}$

From (iii),

$\mathbf{u_n - u_{n-1} = I_n}$

$\mathbf{u_n - u_{n-1} = \pi}$

$\mathbf{u_n = \pi + u_{n-1}}$ **-------------------------(viii)**

Putting $\mathbf{n = n+1, n+2\, in\, (viii)}$

$\mathbf{u_n = \pi + u_{n-1}}$

$\mathbf{u_{n+1} = \pi + u_{n+1-1}}$

$\mathbf{u_{n+1} = \pi + u_n}$ **---------------------------(ix)**

$\mathbf{u_{n+2} = \pi + u_{n+2-1}}$

$\mathbf{u_{n+2} = \pi + u_{n+1}}$ **--------------------------(x)**

Adding $\mathbf{u_n}$ both sides in (x)

$\mathbf{u_{n+2} = \pi + u_{n+1}}$

$\mathbf{u_{n+2} + u_n = \pi + u_{n+1} + u_n}$

$\mathbf{u_{n+2} + u_n = u_{n+1} + \pi + u_n}$

$$u_{n+2} + u_n = u_{n+1} + u_{n+1} \qquad \text{From (ix)} [\because u_{n+1} = \pi + u_n]$$
$$u_{n+2} + u_n = 2u_{n+1} \text{ (Proved)}$$

**Example 91:** If n is a positive integer, Prove that $I_n = \int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$

**Solution:** $I_n = \int_0^{\pi/2} \cos^n x \cos nx\, dx$ ----------(i)

**We have,** $\int uv dx = u\int v dx - \int \{\frac{d}{dx}(u)\int v dx\} dx$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \int \cos nx dx - \int \left\{\frac{d}{dx}(\cos^n x)\int \cos nx dx\right\} dx$$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} - \int \left\{(n \cos^{n-1} x)\frac{d}{dx}(\cos x)\frac{\sin nx}{n}\right\} dx$$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} - \int \left\{(n \cos^{n-1} x)(-\sin x)\frac{\sin nx}{n}\right\} dx$$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} + \frac{n}{n}\int \left\{\cos^{n-1} x \sin x \sin nx\right\} dx$$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} + \int \left\{\cos^{n-1} x \sin x \sin nx\right\} dx \text{ ---------(ii)}$$

**We have** $\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\therefore \cos(nx - x) = \cos nx \cos x + \sin nx \sin x$

$\Rightarrow \sin nx \sin x = \cos(nx - x) - \cos nx \cos x$ ----------------------------------(iii)

**From (i),**

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} + \int \left\{\cos^{n-1} x \sin x \sin nx\right\} dx$$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} + \int \left\{\cos^{n-1} x \left(\cos(nx - x) - \cos nx \cos x\right)\right\} dx$$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} + \int \left\{\cos^{n-1} x \cos(nx - x)\right\} dx - \int \left\{\cos^{n-1} x \cos nx \cos x\right\} dx$$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} + \int \left\{\cos^{n-1} x \cos(nx - x)\right\} dx - \int \left\{\cos^{n-1+1} x \cos nx\right\} dx$$

$$\therefore \int \cos^n x \cos nx\, dx = \cos^n x \frac{\sin nx}{n} + \int \left\{\cos^{n-1} x \cos(nx - x)\right\} dx - \int \left\{\cos^n x \cos nx)\right\} dx$$

**From (i),**

$$I_n = \int_0^{\pi/2} \cos^n x \cos nx\, dx$$

$$=\left[\cos^n x\frac{\sin nx}{n}\right]_0^{\pi/2}+\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(nx-x)\right\}dx-\int_0^{\pi/2}\left\{\cos^n x\cos nx\right\}dx$$

$$=\left[\cos^n(\frac{\pi}{2})\frac{\sin n\frac{\pi}{2}}{n}-\cos^n(0)\frac{\sin n.0}{n}\right]+\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(nx-x)\right\}dx-\int_0^{\pi/2}\left\{\cos^n x\cos nx\right\}dx$$

$$=[0-0]+\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(nx-x)\right\}dx-\int_0^{\pi/2}\left\{\cos^n x\cos nx\right\}dx$$

$$[\because \cos\frac{\pi}{2}=0,\sin 0=0,\sin\frac{\pi}{2}=1,\cos 0=1]$$

$$=\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(nx-x)\right\}dx-\int_0^{\pi/2}\left\{\cos^n x\cos nx\right\}dx$$

$$\therefore I_n=\int_0^{\pi/2}\cos^n x\cos nx\,dx=\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(nx-x)\right\}dx-\int_0^{\pi/2}\left\{\cos^n x\cos nx\right\}dx$$

$$\therefore I_n=\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(nx-x)\right\}dx-\int_0^{\pi/2}\left\{\cos^n x\cos nx\right\}dx$$

$$\therefore I_n=\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(nx-x)\right\}dx-I_n \qquad [\because I_n=\int_0^{\pi/2}\cos^n x\cos nx\,dx]$$

$$\therefore I_n+I_n=\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(nx-x)\right\}dx$$

$$\therefore I_n+I_n=\int_0^{\pi/2}\left\{\cos^{n-1}x\cos(n-1)x\right\}dx$$

$$\therefore I_n+I_n=I_{n-1} \qquad [\because I_n=\int_0^{\pi/2}\cos^n x\cos nx\,dx \;\therefore I_{n-1}=\int_0^{\pi/2}\cos^{n-1}x\cos(n-1)x\,dx]$$

$$\therefore 2I_n=I_{n-1}$$

$$\therefore I_n=\frac{I_{n-1}}{2}$$ ------------------------------(iv)

Putting $n=n-1,n-2,n-3,\dots\dots\dots\dots2,1$ in (iv)

$$\therefore I_{n-1}=\frac{I_{n-1-1}}{2}=\frac{I_{n-2}}{2}$$

$$\therefore I_{n-2}=\frac{I_{n-2-1}}{2}=\frac{I_{n-3}}{2}$$

$$\therefore I_{n-3}=\frac{I_{n-3-1}}{2}=\frac{I_{n-4}}{2}$$

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$$I_2 = \frac{I_{2-1}}{2} = \frac{I_1}{2}$$

$$I_1 = \frac{I_{1-1}}{2} = \frac{I_0}{2}$$

From (iv),

$$I_n = \frac{I_{n-1}}{2}$$

$$I_n = \frac{1}{2}\,\frac{I_{n-2}}{2} \qquad [\because I_{n-1} = \frac{I_{n-2}}{2}]$$

$$I_n = \frac{1}{2}\,\frac{1}{2}\,\frac{I_{n-3}}{2} \qquad [\because I_{n-2} = \frac{I_{n-3}}{2}]$$

$$I_n = \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\frac{I_{n-4}}{2} \qquad [\because I_{n-3} = \frac{I_{n-4}}{2}]$$

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$$I_n = \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\ldots\ldots\ldots\ldots\,\frac{I_0}{2} \text{-------------------------(v)}$$

From (i)

$$I_n = \int_0^{\pi/2} \cos^n x \cos nx\, dx$$

$$I_0 = \int_0^{\pi/2} \cos^0 x \cos o.x\, dx$$

$$I_0 = \int_0^{\pi/2} 1\, dx = [x]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

**From (v),**

$$I_n = \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\ldots\ldots\ldots\ldots\,\frac{I_0}{2}$$

$$I_n = \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2}\,\ldots\ldots\ldots\ldots\,\frac{1}{2}.\frac{\pi}{2}$$

$$I_n = \frac{1}{2^n}\,\frac{\pi}{2}$$

$$I_n = \frac{1}{2^{n+1}}\,\pi$$

$$I_n = \frac{\pi}{2^{n+1}}$$ ***Answer***

**Example 92:** Show that $\int_0^\infty t^n e^{-st} dt = \frac{n!}{s^{n+1}}$

Let, $I_n = \int_0^\infty t^n e^{-st} dt$ ---------------------------------- $(i)$

Now, $\int t^n e^{-st} dt$

$$= t^n \int e^{-st} dt - \int \left\{ \frac{d}{dt}(t^n) \int e^{-st} dt \right\} dt \quad [\because \int uv dx = u \int v dx - \int \{ \frac{d}{dx}(u) \int v dx \} dx]$$

$$= t^n \frac{e^{-st}}{-s} - \int n t^{n-1} \frac{e^{-st}}{-s} dt$$

$$= \frac{-t^n}{s} e^{-st} + \frac{n}{s} \int t^{n-1} e^{-st} dt \text{ ----------------------------- } (ii)$$

Since $I_n = \int_0^\infty t^n e^{-st} dt$

$\therefore I_{n-1} = \int_0^\infty t^{n-1} e^{-st} dt$ -------------------------------------------- $(iii)$

$\therefore$ From (i)

$$I_n = \int_0^\infty t^n e^{-st} dt$$

$$= \left[ \frac{-t^n}{s} e^{-st} \right]_0^\infty + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt \qquad \text{[From (ii)]}$$

$$= 0 + \frac{n}{s} I_{n-1} \qquad \text{[From (iii)]}$$

$$[\because e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0 \text{ and } 0^n = 0]$$

$\therefore I_n = \frac{n}{s} I_{n-1}$ ---------------------------------------------------- $(iv)$

Put $n = n-1$ in $(iv)$

$\therefore I_{n-1} = \frac{n-1}{s} I_{n-2}$ -------------------------------------------- $(v)$

Again, Put $n = n-2$ in $(iv)$

$\therefore I_{n-2} = \frac{n-2}{s} I_{n-3}$ -------------------------------------------- $(vi)$

Again Put, $n = n-3$ in $(iv)$

$\therefore I_{n-3} = \frac{n-3}{s} I_{n-4}$ -------------------------------------------- $(vii)$

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Put $n = 2$ in $(iv)$

$$I_2 = \frac{2}{s} I_{2-1}$$

$$\therefore I_2 = \frac{2}{s} I_1 \text{------------------------------------------------------} (viii)$$

Put $n = 1$ in $(iv)$

$$\therefore I_1 = \frac{1}{s} I_{1-1}$$

$$= \frac{1}{s} I_0 \text{------------------------------------------------------} (ix)$$

Putting in values of $I_{n-1}, I_{n-2}, \ldots\ldots\ldots, I_2, I_1$ in $(iv)$

$$\therefore I_n = \frac{n}{s} I_{n-1}$$

$$= \frac{n}{s} . \frac{n-1}{s} I_{n-2} \qquad [form(v)]$$

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} I_{n-3} \qquad [form(vi)]$$

$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \frac{n-3}{s} I_{n-4} \quad [form(vii)]$$

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$$= \frac{n}{s} . \frac{n-1}{s} . \frac{n-2}{s} . \frac{n-3}{s} . \frac{n-4}{s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{2}{s} . \frac{1}{s} I_0 \quad [form(viii) \text{ and } (ix)]$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\ldots\ldots\ldots\ldots 2.1}{s^n} I_0$$

$$\therefore I_n = \frac{n!}{s^n} I_0 \text{------------------------------------------------------} (x)$$

We have, $I_n = \int_0^\infty t^n e^{-st} dt$

Put $n = 0$

$$I_0 = \int_0^\infty t^0 e^{-st} dt$$

$$I_0 = \int_0^\infty 1.e^{-st} dt$$

$$I_0 = \int_0^\infty e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = -\frac{1}{s}\left[e^{-\infty} - e^{-0}\right] = -\frac{1}{s}\left[\frac{1}{e^{\infty}} - \frac{1}{e^{0}}\right] = -\frac{1}{s}\left[\frac{1}{\infty} - \frac{1}{1}\right] = -\frac{1}{s}[0-1]$$

$$I_0 = \frac{1}{s} \text{ ----------------------------------------------------------- } (xi)$$

From $(x)$

$$I_n = \frac{n!}{s^n} I_0$$

$$\mathbf{I_n = \frac{n!}{s^n}.\frac{1}{s}} \qquad \text{[From } (xi)\text{]}$$

$$\mathbf{I_n = \frac{n!}{s^{n+1}}} \quad \textit{Answer}$$

**Example 93:** If $\mathbf{I_n = \int_0^{\pi/4} \tan^n\theta\, d\theta}$ and $\mathbf{n > 1}$, Prove that $\mathbf{n(I_{n-1} + I_{n+1}) = 1}$

**Solution:** Given, $\mathbf{I_n = \int_0^{\pi/4} \tan^n\theta\, d\theta}$ -------------------------(i)

Putting $\mathbf{n = n+1}$ in (i)

$$\mathbf{I_{n+1} = \int_0^{\pi/4} \tan^{n+1}\theta\, d\theta}$$

$$\mathbf{I_{n+1} = \int_0^{\pi/4} \tan^{n-1}\theta.\tan^2\theta\, d\theta}$$

$$\mathbf{I_{n+1} = \int_0^{\pi/4} \tan^{n-1}\theta.(\sec^2\theta - 1)\, d\theta} \qquad [\because \sec^2\theta - 1 = \tan^2\theta]$$

$$\mathbf{I_{n+1} = \int_0^{\pi/4} \tan^{n-1}\theta \sec^2\theta\, d\theta - \int_0^{\pi/4} \tan^{n-1}\theta\, d\theta} \text{ ------------------(ii)}$$

Now, $\mathbf{\int \tan^{n-1}\theta \sec^2\theta\, d\theta}$ ------------------------------------(iii)

From (iii)

$$\mathbf{\int \tan^{n-1}\theta \sec^2\theta\, d\theta}$$

$$= \mathbf{\int z^{n-1} dz}$$

$$= \mathbf{\frac{z^{n-1+1}}{n-1+1} = \frac{z^n}{n}}$$

$$= \mathbf{\frac{\tan^n\theta}{n}} \qquad [\mathbf{z = \tan\theta}]$$

| |
|---|
| Let $\mathbf{z = \tan\theta}$ |
| $\mathbf{\frac{dz}{d\theta} = \sec^2\theta}$ |
| $\mathbf{dz = \sec^2\theta d\theta}$ |

From (ii)

$$I_{n+1} = \int_0^{\pi/4} \tan^{n-1}\theta \sec^2\theta\, d\theta - \int_0^{\pi/4} \tan^{n-1}\theta\, d\theta$$

$$I_{n+1} = \left[\frac{\tan^n\theta}{n}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-1}\theta\, d\theta$$

$$I_{n+1} = \left[\frac{\tan^n\theta}{n}\right]_0^{\pi/4} - I_{n-1} \qquad [\because I_n = \int_0^{\pi/4} \tan^n\theta\, d\theta \therefore I_{n-1} = \int_0^{\pi/4} \tan^{n-1}\theta\, d\theta]$$

$$I_{n+1} + I_{n-1} = \left[\frac{\tan^n\theta}{n}\right]_0^{\pi/4}$$

$$I_{n+1} + I_{n-1} = \left[\frac{\tan^n\frac{\pi}{4}}{n} - \frac{\tan^n 0}{n}\right] = \left[\frac{(\tan\frac{\pi}{4})^n}{n} - \frac{(\tan 0)^n}{n}\right]$$

$$I_{n+1} + I_{n-1} = \left[\frac{1^n}{n} - \frac{0}{n}\right] = \left[\frac{1}{n}\right]$$

$$I_{n+1} + I_{n-1} = \frac{1}{n}$$

$n(I_{n+1} + I_{n-1}) = 1$ Proved

**Example 94:** If $I_n = \int_0^{\pi/4} \tan^n\theta\, d\theta$ and $n > 1$, Prove that $I_n + I_{n-2} = \frac{1}{n-1}$ and deduce the value of $I_5$

**Solution**: Given, $I_n = \int_0^{\pi/4} \tan^n\theta\, d\theta$ ------------------------(i)

$$I_n = \int_0^{\pi/4} \tan^{n-2}\theta . \tan^2\theta\, d\theta$$

$$I_n = \int_0^{\pi/4} \tan^{n-2}\theta . (\sec^2\theta - 1)\, d\theta \qquad [\because \sec^2\theta - 1 = \tan^2\theta]$$

$$I_n = \int_0^{\pi/4} \tan^{n-2}\theta \sec^2\theta\, d\theta - \int_0^{\pi/4} \tan^{n-2}\theta\, d\theta \text{ ------------------(ii)}$$

Now, $\int \tan^{n-2}\theta \sec^2\theta\, d\theta$ ----------------------------------(iii)

From (iii)

$\int \tan^{n-2}\theta \sec^2\theta\, d\theta$

Let $z = \tan\theta$

$\frac{dz}{d\theta} = \sec^2\theta$

$dz = \sec^2\theta\, d\theta$

$$= \int z^{n-2}\, dz$$

$$= \frac{z^{n-2+1}}{n-2+1} = \frac{z^{n-1}}{n-1}$$

$$= \frac{\tan^{n-1}\theta}{n-1} \qquad [z = \tan\theta]$$

From (ii)

$$I_n = \int_0^{\pi/4} \tan^{n-2}\theta \sec^2\theta\, d\theta - \int_0^{\pi/4} \tan^{n-2}\theta\, d\theta$$

$$I_n = \left[\frac{\tan^{n-1}\theta}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2}\theta\, d\theta$$

$$I_n = \left[\frac{\tan^{n-1}\theta}{n-1}\right]_0^{\pi/4} - I_{n-2} \qquad [\because I_n = \int_0^{\pi/4} \tan^{n}\theta\, d\theta \therefore I_{n-2} = \int_0^{\pi/4} \tan^{n-2}\theta\, d\theta]$$

$$I_n = \left[\frac{\tan^{n-1}(\pi/4)}{n-1} - \frac{\tan^{n-1} 0}{n-1}\right] - I_{n-2}$$

$$I_n = \left[\frac{1}{n-1} - \frac{0}{n-1}\right] - I_{n-2} = \left[\frac{1}{n-1} - 0\right] - I_{n-2}$$

$$I_n = \left[\frac{1}{n-1}\right] - I_{n-2} \text{-------------------(iv)}$$

$$I_n + I_{n-2} = \frac{1}{n-1} \quad \text{(Proved)}$$

From (iv)

$$I_n = \left[\frac{1}{n-1}\right] - I_{n-2}$$

$$I_n = \frac{1}{n-1} - I_{n-2} \text{----------------------(v)}$$

Putting $n = 5$ in (v)

$$I_5 = \frac{1}{5-1} - I_{5-2}$$

$$I_5 = \frac{1}{4} - I_3 \text{---------------------------(vi)}$$

Putting $n = 3$ in (v)

$$I_3 = \frac{1}{3-1} - I_{3-2}$$

$$I_3 = \frac{1}{2} - I_1 \text{---------------------------(vii)}$$

Putting the value of $\mathbf{I_3}$ in (vi)

$$\mathbf{I_5 = \frac{1}{4} - \frac{1}{2} + I_1} \text{ ----------------------(viii)}$$

From (i)

$$\mathbf{I_n = \int_0^{\pi/4} \tan^n \theta \, d\theta}$$

Putting $\mathbf{n = 1}$

$$\mathbf{I_1 = \int_0^{\pi/4} \tan^1 \theta \, d\theta = \int_0^{\pi/4} \tan \theta \, d\theta = [\log \sec \theta]_0^{\pi/4} = \left[\log \sec \pi/4 - \log \sec 0\right]}$$

$$\mathbf{I_1 = \left[\log \sqrt{2} - \log 1\right] = \left[\log \sqrt{2} - 0\right] = \log \sqrt{2}}$$

Putting the value of $\mathbf{I_1}$ in (viii)

$$\mathbf{I_5 = \frac{1}{4} - \frac{1}{2} + I_1}$$

$$\mathbf{I_5 = \frac{1}{4} - \frac{1}{2} + \log \sqrt{2} = \frac{1-2}{4} + \log \sqrt{2}}$$

$$\mathbf{I_5 = -\frac{1}{4} + \log \sqrt{2}} \text{ Answer}$$

**Example 95:** If $\mathbf{u_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx}$ and $\mathbf{t_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx}$

**Show that** i. $\mathbf{u_{n+1} = u_n = \frac{\pi}{2}}$ ii $\mathbf{t_{n+1} - t_n = u_{n+1}}$ iii. $\mathbf{t_n = \frac{n\pi}{2}}$

**Solution**: Given,

$$\mathbf{u_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx} \text{ -------------------------(i)}$$

Putting $\mathbf{n = n + 1}$ we get,

$$\mathbf{u_{n+1} = \int_0^{\pi/2} \frac{\sin\{2(n+1)-1)\}x}{\sin x} dx}$$

$$\mathbf{u_{n+1} = \int_0^{\pi/2} \frac{\sin(2n+2-1)x}{\sin x} dx}$$

$$\mathbf{u_{n+1} = \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx} \text{ ------------------------(ii)}$$

Again, Given, $\mathbf{t_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx}$ ---------------(iii)

Putting $\mathbf{n = n + 1}$ we get,

$$t_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$$

$$t_{n+1} = \int_0^{\pi/2} \frac{\sin^2 (n+1)x}{\sin^2 x} dx \text{ ------------------------(iv)}$$

From (i) & (ii), we get,

$$u_{n+1} - u_n = \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx - \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx$$

$$u_{n+1} - u_n = \int_0^{\pi/2} \left[ \frac{\sin(2n+1)x}{\sin x} - \frac{\sin(2n-1)x}{\sin x} \right] dx$$

$$u_{n+1} - u_n = \int_0^{\pi/2} \left[ \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} \right] dx$$

$$u_{n+1} - u_n = \int_0^{\pi/2} \left[ \frac{2\cos\frac{(2n+1)x + (2n-1)x}{2} \sin\frac{(2n+1)x - (2n-1)x}{2}}{\sin x} \right] dx$$

$$u_{n+1} - u_n = \int_0^{\pi/2} \left[ \frac{2\cos\frac{2nx + x + 2nx - x}{2} \sin\frac{2nx + x - 2nx + x}{2}}{\sin x} \right] dx$$

$$u_{n+1} - u_n = \int_0^{\pi/2} \left[ \frac{2\cos\frac{2nx + 2nx}{2} \sin\frac{x + x}{2}}{\sin x} \right] dx$$

$$u_{n+1} - u_n = \int_0^{\pi/2} \left[ \frac{2\cos\frac{4nx}{2} \sin\frac{2x}{2}}{\sin x} \right] dx = \int_0^{\pi/2} \left[ \frac{2\cos 2nx \sin x}{\sin x} \right] dx$$

$$u_{n+1} - u_n = \int_0^{\pi/2} 2\cos 2nx dx = 2\int_0^{\pi/2} \cos 2nx dx = 2\frac{1}{2n} [\sin 2nx]_0^{\pi/2}$$

$$u_{n+1} - u_n = \frac{1}{n} [\sin 2nx]_0^{\pi/2} = \frac{1}{n} \left[ \sin 2n\pi/2 - \sin 2n.0 \right] = \frac{1}{n} \left[ \sin 2n\pi/2 - 0 \right]$$

$$u_{n+1} - u_n = \frac{1}{n} [\sin n\pi]$$

$u_{n+1} - u_n = \frac{1}{n} \times 0$ [ $\sin n\pi = 0$ ; for any integer values of n]

$$u_{n+1} - u_n = 0$$

$$u_{n+1} = u_n \text{ ----------------------------(v)}$$

Putting $\mathbf{n = n-1, n-2, n-3, \ldots\ldots\ldots\ldots\ldots\ldots 2,1}$

$\mathbf{u_{n+1} = u_n}$

$\mathbf{u_{n-1+1} = u_{n-1}}$

$\mathbf{u_n = u_{n-1}}$ -------------------------------(vi)

$\mathbf{u_{n-2+1} = u_{n-2}}$

$\mathbf{u_{n-1} = u_{n-2}}$ ------------------------------(vii)

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$\mathbf{u_{2+1} = u_2}$

$\mathbf{u_3 = u_2}$ -----------------------------------(viii)

$\mathbf{u_{1+1} = u_1}$

$\mathbf{u_2 = u_1}$ ------------------------------------(ix)

From (v), (vi), (vii), (viii), (ix)

$\mathbf{u_{n+1} = u_n = u_{n-1} = u_{n-2}}$ ------------------- $\mathbf{u_3 = u_2 = u_1}$ -----(x)

We have, from (i)

$$\mathbf{u_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx}$$

$$\mathbf{\therefore u_1 = \int_0^{\pi/2} \frac{\sin(2.1-1)x}{\sin x} dx}$$

$$\mathbf{\therefore u_1 = \int_0^{\pi/2} \frac{\sin x}{\sin x} dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \left[\pi/2 - 0\right] = \pi/2}$$

Hence from (x), we can write

$\mathbf{u_{n+1} = u_n = u_{n-1} = u_{n-2}}$ ------------------- $\mathbf{u_3 = u_2 = u_1 = \pi/2}$ -----------------(xi)

$\mathbf{u_{n+1} = u_n = \pi/2}$ (Proved)

Again, from (iii) & (iv)

$$\mathbf{t_{n+1} - t_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x}{\sin^2 x} dx - \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx}$$

$$\mathbf{t_{n+1} - t_n = \int_0^{\pi/2} \left[\frac{\sin^2(n+1)x}{\sin^2 x} - \frac{\sin^2 nx}{\sin^2 x}\right] dx = \int_0^{\pi/2} \left[\frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x}\right] dx}$$

$$\mathbf{t_{n+1} - t_n = \int_0^{\pi/2} \left[\frac{2\{\sin^2(n+1)x - \sin^2 nx\}}{2\sin^2 x}\right] dx = \int_0^{\pi/2} \left[\frac{2\sin^2(n+1)x - 2\sin^2 nx}{2\sin^2 x}\right] dx}$$

$$\mathbf{t_{n+1} - t_n = \int_0^{\pi/2} \left[\frac{\{1-\cos 2(n+1)x\} - \{1-\cos 2nx\}}{2\sin^2 x}\right] dx}$$

$$t_{n+1} - t_n = \int_0^{\pi/2} \left[ \frac{1 - \cos 2(n+1)x - 1 + \cos 2nx}{2\sin^2 x} \right] dx$$

$$t_{n+1} - t_n = \int_0^{\pi/2} \left[ \frac{-\cos 2(n+1)x + \cos 2nx}{2\sin^2 x} \right] dx = \int_0^{\pi/2} \left[ \frac{\cos 2nx - \cos 2(n+1)x}{2\sin^2 x} \right] dx$$

$$t_{n+1} - t_n = \int_0^{\pi/2} \left[ \frac{2\sin \frac{2nx + 2(n+1)x}{2} \sin \frac{2(n+1)x - 2nx}{2}}{2\sin^2 x} \right] dx$$

$[\because 2\sin \frac{C+D}{2} \sin \frac{D-C}{2} = \cos C - \cos D]$

$$t_{n+1} - t_n = \int_0^{\pi/2} \left[ \frac{2\sin \frac{2nx + 2nx + 2x}{2} \sin \frac{2nx + x - 2nx}{2}}{2\sin^2 x} \right] dx$$

$$t_{n+1} - t_n = \int_0^{\pi/2} \left[ \frac{2\sin \frac{2nx + 2nx + 2x}{2} \sin \frac{x}{2}}{2\sin^2 x} \right] dx = \int_0^{\pi/2} \left[ \frac{2\sin \frac{4nx + 2x}{2} \sin \frac{x}{2}}{2\sin^2 x} \right] dx$$

$$t_{n+1} - t_n = \int_0^{\pi/2} \left[ \frac{2\sin \frac{2(2n+1)x}{2} \sin \frac{x}{2}}{2\sin^2 x} \right] dx = \int_0^{\pi/2} \left[ \frac{\sin(2n+1)x \sin \frac{x}{2}}{\sin^2 x} \right] dx$$

$$t_{n+1} - t_n = \int_0^{\pi/2} \left[ \frac{\sin(2n+1)x}{\sin x} \right] dx$$

$t_{n+1} - t_n = u_{n+1}$ $\quad$ $[\because$ from (ii), $u_{n+1} = \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx]$

$t_{n+1} - t_n = u_{n+1}$ $\quad$ (Proved)

$t_{n+1} - t_n = u_{n+1} = \pi/2$ $\quad$ [From (xi)]------------------------(xii)

Putting $n = n-1, n-2, n-3, \ldots\ldots\ldots\ldots 3,2,1$ in (xii)

$t_n - \cancel{t_{n-1}} = \pi/2$

$\cancel{t_{n-1}} - \cancel{t_{n-2}} = \pi/2$

$\cancel{t_{n-2}} - \cancel{t_{n-3}} = \pi/2$

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$\cancel{t_4} - \cancel{t_3} = \pi/2$

$$\cancel{t_3} - \cancel{t_2} = \pi/2$$

$$\cancel{t_2} - t_1 = \pi/2$$

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$$t_n - t_1 = (n-1)\pi/2$$

$$t_n = t_1 + (n-1)\pi/2 \text{ --------------------------------(xiii)}$$

Given, $t_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$

$$\therefore t_1 = \int_0^{\pi/2} \frac{\sin^2 1.x}{\sin^2 x} dx$$

$$\therefore t_1 = \int_0^{\pi/2} \frac{\sin^2 x}{\sin^2 x} dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = [\pi/2 - 0] = \pi/2$$

From (xiii),

$$t_n = t_1 + (n-1)\pi/2$$

$$t_n = \frac{\pi}{2} + (n-1)\pi/2 = \frac{\pi}{2} + \frac{n\pi}{2} - \frac{\pi}{2} = \frac{n\pi}{2} \textit{ (Proved)}$$

**Example 96:** If $u_n = \int \cos n\theta \operatorname{cosec}\theta \, d\theta$, then show that $u_n - u_{n-2} = \frac{2\cos(n-1)\theta}{n-1}$, Hence find the value of $\int_0^{\pi/2} \frac{\sin 3\theta \sin 5\theta}{\sin\theta} d\theta$

**Solution**: Given, $u_n = \int \cos n\theta \operatorname{cosec}\theta \, d\theta$

$$u_n = \int \cos n\theta \frac{1}{\sin\theta} d\theta \qquad [\operatorname{cosec}\theta = \frac{1}{\sin\theta}]$$

$$u_n = \int \frac{\cos n\theta}{\sin\theta} d\theta \text{ ----------------------(i)}$$

Putting $n = n-2$ in (i)

$$u_n = \int \frac{\cos n\theta}{\sin\theta} d\theta$$

$$u_{n-2} = \int \frac{\cos(n-2)\theta}{\sin\theta} d\theta \text{ -----------------------------------(ii)}$$

From (i) & (ii)

$$u_n - u_{n-2} = \int \frac{\cos n\theta}{\sin\theta} d\theta - \int \frac{\cos(n-2)\theta}{\sin\theta} d\theta$$

$$u_n - u_{n-2} = \int \left[ \frac{\cos n\theta}{\sin\theta} - \frac{\cos(n-2)\theta}{\sin\theta} \right] d\theta$$

$$u_n - u_{n-2} = \int\left[\frac{\cos n\theta - \cos(n-2)\theta}{\sin\theta}\right]d\theta$$

$$[\because 2\sin\frac{C+D}{2}\sin\frac{D-C}{2} = \cos C - \cos D]$$

$$u_n - u_{n-2} = \int\left[\frac{2\sin\frac{n\theta+(n-2)\theta}{2}\sin\frac{(n-2)\theta-n\theta}{2}}{\sin\theta}\right]d\theta$$

$$u_n - u_{n-2} = \int\left[\frac{2\sin\frac{n\theta+n\theta-2\theta}{2}\sin\frac{n\theta-2\theta-n\theta}{2}}{\sin\theta}\right]d\theta$$

$$u_n - u_{n-2} = \int\left[\frac{2\sin\frac{2n\theta-2\theta}{2}\sin\frac{-2\theta}{2}}{\sin\theta}\right]d\theta = \int\left[\frac{2\sin\frac{2(n\theta-\theta)}{2}\sin(-\theta)}{\sin\theta}\right]d\theta$$

$$u_n - u_{n-2} = \int\left[\frac{2\sin(n\theta-\theta)\sin(-\theta)}{\sin\theta}\right]d\theta$$

$$u_n - u_{n-2} = -\int\left[\frac{2\sin(n\theta-\theta)\sin\theta}{\sin\theta}\right]d\theta \qquad [\sin(-\theta) - \sin\theta]$$

$$u_n - u_{n-2} = -\int\left[2\sin(n\theta-\theta)\right]d\theta = -\int 2\sin(n\theta-\theta)\,d\theta$$

$$u_n - u_{n-2} = -2\int\sin(n-1)\theta\,d\theta = \frac{-2}{n-1}\left[-\cos(n-1)\theta\right]$$

$$u_n - u_{n-2} = \frac{2}{n-1}\left[\cos(n-1)\theta\right]\text{------------------------(iii)} \qquad \textbf{\textit{(Proved)}}$$

***2nd part:***

$$\int_0^{\pi/2}\frac{\sin 3\theta\sin 5\theta}{\sin\theta}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\frac{2\sin 3\theta\sin 5\theta}{\sin\theta}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\frac{2\sin 5\theta\sin 3\theta}{\sin\theta}d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\frac{\cos(5\theta-3\theta)-\cos(5\theta+3\theta)}{\sin\theta}d\theta \qquad [2\sin A\sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}\int_0^{\pi/2}\frac{\cos 2\theta - \cos 8\theta}{\sin\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\frac{\cos 2\theta}{\sin\theta}d\theta-\frac{1}{2}\int_0^{\pi/2}\frac{\cos 8\theta}{\sin\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\pi/2}\cos 2\theta\ \operatorname{cosec}\theta d\theta-\frac{1}{2}\int_0^{\pi/2}\cos 8\theta \operatorname{cosec}\theta d\theta$$

$$=\left[\frac{1}{2}u_2-\frac{1}{2}u_8\right]_0^{\pi/2}$$ Since $u_n=\int\cos n\theta \operatorname{cosec}\theta\, d\theta$ (given)

$$\therefore \int_0^{\pi/2}\frac{\sin 3\theta\sin 5\theta}{\sin\theta}d\theta=\frac{1}{2}[u_2-u_8]_0^{\pi/2}\text{-----------------------------(iv)}$$

From (iii),

$$u_n-u_{n-2}=\frac{2}{n-1}[\cos(n-1)\theta]\text{------------(v)}$$

$$u_8-\cancel{u_6}=\frac{2}{7}[\cos 7\theta] \qquad [n=8 \text{ in (v)}]$$

$$\cancel{u_6}-\cancel{u_4}=\frac{2}{5}[\cos 5\theta] \qquad [n=6 \text{ in (v)}]$$

$$\cancel{u_4}-u_2=\frac{2}{3}[\cos 3\theta] \qquad [n=4 \text{ in (v)}]$$

---

Adding, $u_8-u_2=\frac{2}{7}[\cos 7\theta]+\frac{2}{5}[\cos 5\theta]+\frac{2}{3}[\cos 3\theta]$

$$\frac{1}{2}(u_8-u_2)=\frac{1}{2}\left[\frac{2}{7}[\cos 7\theta]+\frac{2}{5}[\cos 5\theta]+\frac{2}{3}[\cos 3\theta]\right]$$

$$\frac{1}{2}(u_8-u_2)=\frac{1}{7}[\cos 7\theta]+\frac{1}{5}[\cos 5\theta]+\frac{1}{3}[\cos 3\theta]$$

$$\frac{1}{2}(u_2-u_8)=-\left[\frac{1}{7}[\cos 7\theta]+\frac{1}{5}[\cos 5\theta]+\frac{1}{3}[\cos 3\theta]\right]$$

From (iv)

$$\int_0^{\pi/2}\frac{\sin 3\theta\sin 5\theta}{\sin\theta}d\theta=\frac{1}{2}[u_2-u_8]_0^{\pi/2}=-\left[\frac{1}{7}[\cos 7\theta]+\frac{1}{5}[\cos 5\theta]+\frac{1}{3}[\cos 3\theta]\right]_0^{\pi/2}$$

$$=-\left[\frac{1}{7}\left[\cos 7\frac{\pi}{2}\right]+\frac{1}{5}\left[\cos 5\frac{\pi}{2}\right]+\frac{1}{3}\left[\cos 3\frac{\pi}{2}\right]\right]+\left[\frac{1}{7}[\cos 7.0]+\frac{1}{5}[\cos 5.0]+\frac{1}{3}[\cos 3.0]\right]$$

$$=-\left[\frac{1}{7}.0+\frac{1}{5}.0+\frac{1}{3}.0\right]+\left[\frac{1}{7}.1+\frac{1}{5}.1+\frac{1}{3}.1\right]=\frac{1}{7}+\frac{1}{5}+\frac{1}{3}=\frac{71}{105}$$ Answer

## 02. Definite integral

**Method # 15:**

$$\int_a^b f(x)dx = \int_a^b f(z)dz$$

Let, $\int f(x)dx = \varphi(x) + c$

$$\therefore \int_b^a f(x)dx = [\varphi(x)]_a^b = \varphi(b) - \varphi(a) \text{-----------------(i)}$$

Again let, $\int f(z)dz = \varphi(z) + c$

$$\therefore \int_a^b f(z)dz = [\varphi(z)]_a^b$$

$$= \varphi(b) - \varphi(a) \text{------------------------------(ii)}$$

From (i) and (ii), we get,

$$\int_a^b f(x)dx = \int_a^b f(z)dz \quad \textit{(Proved)}$$

**Method # 16:**

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Let, $\int f(x)dx = \varphi(x) + c$

$$\therefore \int_b^a f(x)dx = [\varphi(x)]_a^b$$

$$= \varphi(b) - \varphi(a)$$

$$= -[\varphi(a) - \varphi(b)] = -[\varphi(x)]_b^a$$

$$= -\int_b^a f(x)dx$$

$$\therefore \int_a^b f(x)dx = -\int_b^a f(x)dx \quad \textit{(Proved)}$$

**Method # 17:**

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Answer: Let, $\int f(x)dx = \varphi(x) + c$

$$\therefore \int_a^b f(x)dx = [\varphi(x)]_a^b$$

$$= \varphi(b) - \varphi(a)$$

$$= \varphi(b) - \varphi(c) + \varphi(c) - \varphi(a)$$
$$= \varphi(c) - \varphi(a) + \varphi(b) - \varphi(c)$$
$$= [\varphi(x)]_a^c + [\varphi(x)]_c^b$$
$$= \int_a^c f(x)dx + \int_c^b f(x)dx$$
$$\therefore \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \textit{ (Proved)}$$

**<u>Method # 18:</u>**

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Example:
Let $f(x) = \sin x$
$\therefore f(a-x) = \sin(a-x)$

R.H.S. $\int_0^a f(a-x)dx$ ------------------------------------(i)

Let, $a - x = z$
$\Rightarrow z = a - x$
$\Rightarrow \dfrac{dz}{dx} = 0 - 1$
$\Rightarrow dz = -dx$

| x | 0 | a |
|---|---|---|
| $a - x = z$ | $a - x = z$ <br> $a - 0 = z$ <br> $z = a$ | $a - x = z$ <br> $a - a = z$ <br> $z = 0$ |

From (i), $\int_0^a f(a-x)dx$

$$= -\int_a^0 f(z)dz$$
$$= \int_0^a f(z)dz \qquad \text{[Method \# 16:]}$$
$$= \int_0^a f(x)dx \qquad \text{[Method \# 15]}$$
$$\therefore \int_0^a f(x)dx = \int_0^a f(a-x)dx \textit{ [Proved]}$$

## *Even and Odd function*

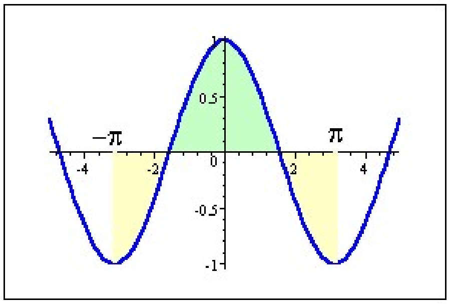


**Figure # 24** : An even signal

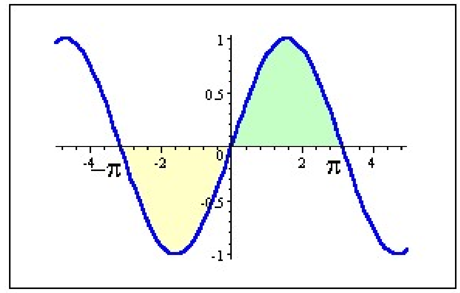


**Figure # 25** : An odd signal

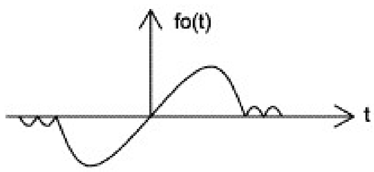


**Figure # 26**: An odd signal

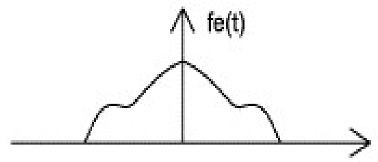


**Figure # 27**: An even signal

**Condition for odd function:**

$\mathbf{f(-x) = -f(x)}$

**Condition for even function:**

$\mathbf{f(-x) = f(x)}$

**Example 97:**

i. Let, $\mathbf{f(x) = x^2}$ --------------------------(i)

Put $\mathbf{x = -x}$ in (i)

$\Rightarrow \mathbf{f(x) = x^2}$

$\Rightarrow \mathbf{f(-x) = (-x)^2}$

$\Rightarrow \mathbf{f(-x) = x^2}$

$= \mathbf{f(x)}$ [From (i); $\mathbf{f(x) = x^2}$ ]

$\Rightarrow \mathbf{f(-x) = f(x)}$

$\therefore \mathbf{f(x) = x^2}$ is an even function

ii. Again, Let, $\mathbf{f(x) = \cos x}$ ---------------(ii)

Put $\mathbf{x = -x}$ in (ii)

$\Rightarrow \mathbf{f(x) = \cos x}$

$\Rightarrow \mathbf{f(-x) = \cos(-x)}$

$\Rightarrow \mathbf{f(-x) = \cos x [\because \cos(-x) = \cos x]}$

$= \mathbf{f(x)}$ [From (ii) $\mathbf{f(x) = \cos x}$ ]

$\Rightarrow \mathbf{f(-x) = f(x)}$

$\therefore \mathbf{f(x) = \cos x}$ is an even function

iii. Let, $f(x) = x^3$ ----------------------------(iii)

Put $x = -x$ in (iii)

$\Rightarrow f(x) = x^3$

$\Rightarrow f(-x) = (-x)^3$

$\Rightarrow f(-x) = -x^3$

$= -f(x)$ [From (iii); $f(x) = x^3$]

$\Rightarrow f(-x) = -f(x)$

$\therefore f(x) = x^3$ is an odd function

iv. Again, Let, $f(x) = \sin x$ ---------------(iv)

Put $x = -x$ in (iv)

$\Rightarrow f(x) = \sin x$

$\Rightarrow f(-x) = \sin(-x)$

$\Rightarrow f(-x) = -\sin x [\because \sin(-x) = -\sin x]$

$= -f(x)$ [From (iv) $f(x) = \sin x$]

$\Rightarrow f(-x) = -f(x)$

$\therefore f(x) = \sin x$ is an odd function

**Method # 19:**

**Example 98:**

Prove that i) $\int_{-a}^{+a} f(x)dx = \int_{0}^{a} \{f(x) + f(-x)\}dx = 0;$ When $f(x)$ is odd

ii) $\int_{-a}^{+a} f(x)dx = 2\int_{0}^{a} f(x)dx;$ When $f(x)$ is even.

**Proof:**

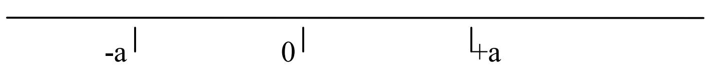


**Figure No # 28**

i) We can write,

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx \text{ -----------------(i)}$$

[Method#17: $\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$ ]

Let, $x = -z$ in first integral of (i),

$\Rightarrow z = -x$

$\Rightarrow \frac{dz}{dx} = -1$

$\Rightarrow dz = -dx$

| $x$ | $-a$ | $0$ |
|---|---|---|
| $z = -x$ | $z = -x$<br>$z = -(-a)$<br>$z = a$ | $z = -x$<br>$z = 0$ |

From (i),

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{a}^{0} f(-z)dz + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} f(-z)dz + \int_{0}^{a} f(x)dx \qquad \text{[Method-16: } \int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx \text{]}$$

$$= \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx \text{ ----------(ii)} \qquad \textbf{[Method \# 15]}$$

$$= -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx \qquad \text{[Condition for odd function: } f(-x) = -f(x) \text{]}$$

$$= 0 \textit{ (Proved)}$$

**ii) Again, from (ii)**

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx \qquad \textbf{[Condition for even function: } f(-x) = f(x) \textbf{]}$$

$$= 2\int_{0}^{a} f(x)dx \textit{ (Proved)}$$

**Example 99: Show that** $\int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$

**Answer:**

Let $I = \int_{0}^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ ------------------------------------(i)

$$= \int_{0}^{\pi/2} \frac{\sin(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)} dx \qquad \textbf{[Method \# 18: } \int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx \textbf{]}$$

$$= \int_{0}^{\pi/2} \frac{\sin(1.\frac{\pi}{2} - x)}{\sin(1.\frac{\pi}{2} - x) + \cos(1.\frac{\pi}{2} - x)} dx$$

$$I = \int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\cos x + \sin x} dx \text{ ------------------------------------(ii)}$$

$$[\because \sin(1.\frac{\pi}{2} - x) = \sin(1.90 - x) = \cos x; \quad \cos(1.\frac{\pi}{2} - x) = \cos(1.90 - x) = \sin x]$$

*From (i) + (ii)*

$$\therefore 2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx + \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx$$

$$= \int_0^{\pi/2} \frac{(\sin x + \cos x)}{(\sin x + \cos x)} dx$$

$$= \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right] = \left[\frac{\pi}{2}\right] = \frac{\pi}{2}$$

$$\therefore 2I = \frac{\pi}{2}$$

$$\therefore I = \frac{\pi}{4} \text{ *Answer*}$$

**Example 100:** Let, $I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$ ------------(i)

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} \quad \text{[Method \# 18: } \int_0^a f(x)dx = \int_0^a f(a - x)dx \text{ ]}$$

$[\because \sin(\pi - x) = \sin(2.90 - x) = \sin x; \quad \cos(\pi - x) = \cos(2.90 - x) = -\cos x]$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + (-\cos x)^2} dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx \text{ --------------------------(ii)}$$

*From (i) + (ii),*

$$2I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx$$

$$= \int_0^{\pi} \frac{x \sin x + (\pi - x)\sin x}{1 + \cos^2 x} dx$$

$$= \int_0^{\pi} \frac{x \sin x + \pi \sin x - x \sin x}{1 + \cos^2 x} dx$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx \text{ ---------------------------(iii)}$$

Let, $\cos x = z$
$\Rightarrow z = \cos x$
$\Rightarrow \frac{dz}{dx} = -\sin x$
$\Rightarrow dz = -\sin x dx$
$\Rightarrow \sin x dx = -dz$

| x | $\pi$ | 0 |
|---|---|---|
| $z = \cos x$ | $z = \cos x$<br>$z = \cos \pi$<br>$z = -1$ | $z = \cos x$<br>$z = \cos 0$<br>$z = 1$ |

From (iii), we get

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx$$

$$= -\pi \int_1^{-1} \frac{dz}{1 + z^2}$$

$$= \pi \int_{-1}^{1} \frac{dz}{1+z^2} \qquad [\text{Method-16} \int_a^b f(x)dx = -\int_b^a f(x)dx]$$

$$= \pi \left[\tan^{-1} z\right]_{-1}^{+1}$$

$$= \pi[\tan^{-1} 1 - \tan^{-1}(-1)]$$

$$= \pi[\tan^{-1} 1 + \tan^{-1}(1)] \qquad [\because \tan^{-1}(-1) = -\tan^{-1} 1]$$

$$= \pi[\tan^{-1} \tan\frac{\pi}{4} + \tan^{-1} \tan\frac{\pi}{4}] \; [\because \tan\frac{\pi}{4} = 1]$$

$$= \pi[\frac{\pi}{4} + \frac{\pi}{4}] = \pi.\frac{\pi}{2}$$

$$2I = \frac{\pi^2}{2}$$

$$\therefore I = \frac{\pi^2}{4}$$

**Example 101: Show that** $\int_0^{\pi} \frac{x}{1+\sin x} dx = \pi$

**Solution**: Let $I = \int_0^{\pi} \frac{x}{1+\sin x} dx$

$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)} dx \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi - x}{1+\sin x} dx \qquad [\because \sin(\pi - x) = \sin x]$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx - \int_0^{\pi} \frac{x}{1+\sin x} dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx - I$$

$$\Rightarrow 2I = \int_0^{\pi} \frac{\pi}{1+\sin x} dx = \pi \int_0^{\pi} \frac{1}{1+\sin x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{(1+\sin x)((1-\sin x))} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{\cos^2 x} dx - \pi \int_0^{\pi} \frac{\sin x}{\cos^2 x} dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \sec^2 dx - \pi \int_0^{\pi} \frac{\sin x}{\cos x} \frac{1}{\cos x} dx$$

$$\Rightarrow 2I = \pi\int_0^{\pi} \sec^2 dx - \pi\int_0^{\pi} \tan x \sec x dx$$
$$\Rightarrow 2I = \pi\int_0^{\pi} \sec^2 dx - \pi\int_0^{\pi} \sec x \tan x dx$$
$$\Rightarrow 2I = \pi[\tan x]_0^{\pi} - \pi[\sec x]_0^{\pi}$$
$$\Rightarrow 2I = \pi[\tan\pi - \tan 0] - \pi[\sec\pi - \sec 0]$$
$$\Rightarrow 2I = \pi[0 - 0] - \pi[-1 - 1]$$
$$\Rightarrow 2I = 2\pi$$
$\Rightarrow I = \pi$ *Answer*

**Example 102: Show that** $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \frac{\pi}{4}$

**Solution**: Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ --------------------------(i)

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}} dx \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$ --------------------------------------(ii)

$$[\because \sin(\frac{\pi}{2} - x) = \sin(1.\frac{\pi}{2} - x) = \sin(1.90^0 - x) = \cos x$$
$$\& \cos(\frac{\pi}{2} - x) = \cos(1.\frac{\pi}{2} - x) = \cos(1.90^0 - x) = \sin x]$$

(i)+(ii)

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$$
$$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$$
$$\Rightarrow 2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right]$$
$\Rightarrow I = \frac{\pi}{4}$ ***Answer***

**Example 103: Show that** $\int_0^{\pi/4} \log(1 + \tan\theta)d\theta = \frac{1}{8}\pi\log 2$

**Solution**:

Let, $\mathbf{I} = \int_0^{\pi/4} \log(1+\tan\theta)d\theta$ ------------------------(i)

$\Rightarrow \mathbf{I} = \int_0^{\pi/4} \log(1+\tan(\frac{\pi}{4}-\theta))d\theta \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$

$\Rightarrow \mathbf{I} = \int_0^{\pi/4} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}d\theta \qquad [\because \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}]$

$\Rightarrow \mathbf{I} = \int_0^{\pi/4} \log\left\{1+\frac{1-\tan\theta}{1+1.\tan\theta}\right\}d\theta \qquad [\because \tan\frac{\pi}{4} = 1]$

$\Rightarrow \mathbf{I} = \int_0^{\pi/4} \log\left\{\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right\}d\theta$

$\Rightarrow \mathbf{I} = \int_0^{\pi/4} \log\left\{\frac{2}{1+\tan\theta}\right\}d\theta$ -------------------------(ii)

(i)+(ii),

$\mathbf{I}+\mathbf{I} = \int_0^{\pi/4} \log(1+\tan\theta)d\theta + \int_0^{\pi/4} \log\left\{\frac{2}{1+\tan\theta}\right\}d\theta$

$\Rightarrow 2\mathbf{I} = \int_0^{\pi/4} \log(1+\tan\theta)d\theta + \int_0^{\pi/4} \log\left\{\frac{2}{1+\tan\theta}\right\}d\theta$

$\Rightarrow 2\mathbf{I} = \int_0^{\pi/4} \left\{\log(1+\tan\theta)+\log\left(\frac{2}{1+\tan\theta}\right)\right\}d\theta$

$\Rightarrow 2\mathbf{I} = \int_0^{\pi/4} \log\left\{(1+\tan\theta)\left(\frac{2}{1+\tan\theta}\right)\right\}d\theta \qquad [\because \log a + \log b = \log ab]$

$\Rightarrow 2\mathbf{I} = \int_0^{\pi/4} \log\left\{(1+\tan\theta)\frac{2}{(1+\tan\theta)}\right\}d\theta$

$\Rightarrow 2\mathbf{I} = \int_0^{\pi/4} \log 2 d\theta = \log 2\int_0^{\pi/4} d\theta = \log 2[\theta]_0^{\pi/4} = \log 2\left[\frac{\pi}{4}-0\right] = \frac{\pi}{4}\log 2$

$\Rightarrow \mathbf{I} = \frac{\pi}{8}\log 2$ ***Answer***

**Example 104: Show that** $\int_0^{\pi/2} \frac{dx}{1+\cos^2 x} = \frac{\pi}{2\sqrt{2}}$

Let, $\mathbf{I} = \int_0^{\pi/2} \frac{dx}{1+\cos^2 x}$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\frac{1}{\cos^2 x}}{\frac{1+\cos^2 x}{\cos^2 x}} dx \qquad [\text{dividing by } \cos^2 x]$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sec^2 x}{\frac{1}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x}} dx = \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 1} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sec^2 x}{1+\tan^2 x + 1} dx \qquad [\because \sec^2 x = 1 + \tan^2 x]$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x} dx$$

Let,

$z = \tan x$

$\Rightarrow \frac{dz}{dx} = \sec^2 x$

$\Rightarrow dz = \sec^2 x dx$

| x | 0 | $\frac{\pi}{2}$ |
|---|---|---|
| $z = \tan x$ | $z = \tan x$<br>$z = \tan 0 = 0$ | $z = \tan x$<br>$z = \tan \frac{\pi}{2} = \infty$ |

$$I = \int_0^{\pi/2} \frac{\sec^2 x}{2+\tan^2 x} dx$$

$$\Rightarrow I = \int_0^{\infty} \frac{1}{2+z^2} dz = \int_0^{\infty} \frac{1}{(\sqrt{2})^2 + z^2} dz$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \left[ \tan^{-1} \frac{z}{\sqrt{2}} \right]_0^{\infty} = \frac{1}{\sqrt{2}} \left[ \tan^{-1} \frac{\infty}{\sqrt{2}} - \tan^{-1} 0 \right] \qquad [\because \int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}]$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \left[ \tan^{-1} \infty - \tan^{-1} 0 \right]$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \left[ \tan^{-1} \tan \frac{\pi}{2} - \tan^{-1} \tan 0 \right]$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \left[ \frac{\pi}{2} - 0 \right] = \frac{\pi}{2\sqrt{2}} \textit{ Answer}$$

**Example 105: Show that** $\int_0^{\pi/2} \sin 2x \log \tan x dx = 0$

**Solution:**

$$I = \int_0^{\pi/2} \sin 2x \log \tan x dx$$

$$\Rightarrow I = \int_0^{\pi/2} \sin 2(\frac{\pi}{2} - x)\log\tan(\frac{\pi}{2} - x)dx \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

$$\Rightarrow I = \int_0^{\pi/2} \sin(\pi - 2x)\log\tan(\frac{\pi}{2} - x)dx$$

$$\Rightarrow I = \int_0^{\pi/2} \sin 2x \log\cot x dx$$

$$[\because \sin(\pi - 2x) = \sin(2.90^0 - 2x) = \sin 2x$$

$$\& \tan(\frac{\pi}{2} - x) = \tan(1.\frac{\pi}{2} - x) = \tan(1.90^0 - x) = \cot x]$$

$$\Rightarrow I = \int_0^{\pi/2} \sin 2x \log\frac{1}{\tan x}dx = \int_0^{\pi/2} \sin 2x \log(\tan x)^{-1} dx$$

$$\Rightarrow I = -\int_0^{\pi/2} \sin 2x \log\tan x dx \qquad [\log x^a = a\log x]]$$

$$\Rightarrow I = -I \qquad [\because I = \int_0^{\pi/2} \sin 2x \log\tan x dx]$$

$$\Rightarrow I + I = 0$$

$\Rightarrow I = 0$ ***Answer***

**Example 106: Show that $\int_0^{\pi} \cos^{15} x dx = 0$**

Solution:

$$I = \int_0^{\pi} \cos^{15} x dx$$

$$\Rightarrow I = \int_0^{\pi} \cos^{15}(\pi - x)dx \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

$$\Rightarrow I = -\int_0^{\pi} \cos^{15} x dx \qquad [\because \cos(\pi - x) = \cos(2.90^0 - x) = -\cos x]$$

$$\Rightarrow I = -I \qquad [\because I = \int_0^{\pi} \cos^{15} x dx]$$

$$\Rightarrow I + I = 0$$

$\Rightarrow I = 0$ ***Answer***

**Example 107: Show that $\int_0^{\pi} \frac{x}{1 + \cos^2 x}dx = \frac{\pi^2}{2\sqrt{2}}$**

Let $I = \int_0^{\pi} \frac{x}{1 + \cos^2 x}dx$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi-x}{1+\cos^2(\pi-x)}dx \qquad \left[\int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi dx}{1+\cos^2(\pi-x)}-\int_0^{\pi}\frac{xdx}{1+\cos^2(\pi-x)}$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi dx}{1+\{\cos(\pi-x)\}^2}-\int_0^{\pi}\frac{xdx}{1+\{\cos(\pi-x)\}^2}$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi dx}{1+(-\cos x)^2}-\int_0^{\pi}\frac{xdx}{1+(-\cos x)^2} \qquad [\because \cos(\pi-x)=\cos(2.90^0-x)=-\cos x]$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi}{1+\cos^2 x}dx-\int_0^{\pi}\frac{x}{1+\cos^2 x}dx$$

$$\Rightarrow I=\int_0^{\pi}\frac{\pi}{1+\cos^2 x}dx-I \qquad \left[\because I=\int_0^{\pi}\frac{x}{1+\cos^2 x}dx\right]$$

$$\Rightarrow I+I=\int_0^{\pi}\frac{\pi}{1+\cos^2 x}dx$$

$$\Rightarrow 2I=\int_0^{\pi}\frac{\pi}{1+\cos^2 x}dx=\pi\int_0^{\pi}\frac{1}{1+\cos^2 x}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\frac{1}{\cos^2 x}}{\frac{1+\cos^2 x}{\cos^2 x}}dx \qquad [\text{dividing by} \cos^2 x]$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sec^2 x}{\frac{1}{\cos^2 x}+\frac{\cos^2 x}{\cos^2 x}}dx=\pi\int_0^{\pi}\frac{\sec^2 x}{\sec^2 x+1}dx$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sec^2 x}{1+\tan^2 x+1}dx \qquad [\because \sec^2 x=1+\tan^2 x]$$

$$\Rightarrow 2I=\pi\int_0^{\pi}\frac{\sec^2 x}{2+\tan^2 x}dx=\pi\times 2\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{2+\tan^2 x}dx$$

$$\Rightarrow 2I=2\pi\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{2+\tan^2 x}dx$$

$$\Rightarrow 2I=2\pi\int_0^{\pi/2}\frac{\sec^2 x}{2+\tan^2 x}dx$$

$$\Rightarrow 2I=2\pi\int_0^{\infty}\frac{1}{2+z^2}dz$$

Let,
$z=\tan x$
$\frac{dz}{dx}=\sec^2 x$
$dz=\sec^2 xdx$

$$\Rightarrow 2I = 2\pi\int_0^{\infty}\frac{1}{(\sqrt{2})^2+z^2}dz$$

| x | 0 | $\frac{\pi}{2}$ |
|---|---|---|
| $z=\tan x$ | $z=\tan x$<br>$z=\tan 0=0$ | $z=\tan x$<br>$z=\tan\frac{\pi}{2}=\infty$ |

$$\Rightarrow 2I = \frac{2\pi}{\sqrt{2}}\left[\tan^{-1}\frac{z}{\sqrt{2}}\right]_0^{\infty}$$

$$\Rightarrow 2I = \frac{2\pi}{\sqrt{2}}\left[\tan^{-1}\frac{\infty}{\sqrt{2}}-\tan^{-1}0\right]$$

$$\Rightarrow 2I = \frac{2\pi}{\sqrt{2}}\left[\tan^{-1}\infty-\tan^{-1}0\right]$$

$$\Rightarrow 2I = \frac{2\pi}{\sqrt{2}}\left[\tan^{-1}\tan\frac{\pi}{2}-\tan^{-1}\tan 0\right]$$

$$\Rightarrow 2I = \frac{2\pi}{\sqrt{2}}\left[\frac{\pi}{2}-0\right]=\frac{\pi^2}{\sqrt{2}}$$

$$\Rightarrow I = \frac{\pi^2}{2\sqrt{2}}$$ ***Answer***

**Example 108:** Show that $\int_0^{\pi/2}\frac{d\theta}{1+\tan\theta}=\frac{\pi}{4}$

**Solution:**

Let $I = \int_0^{\pi/2}\frac{d\theta}{1+\tan\theta}$

$$\Rightarrow I = \int_0^{\pi/2}\frac{d\theta}{1+\frac{\sin\theta}{\cos\theta}}$$

$$\Rightarrow I = \int_0^{\pi/2}\frac{d\theta}{\frac{\cos\theta+\sin\theta}{\cos\theta}} = \int_0^{\pi/2}\frac{\cos\theta}{\cos\theta+\sin\theta}d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{2\cos\theta}{\cos\theta+\sin\theta}d\theta$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2}\frac{\cos\theta+\cos\theta}{\cos\theta+\sin\theta}d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{\cos\theta+\cos\theta+\sin\theta-\sin\theta}{\cos\theta+\sin\theta}d\theta$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2}\frac{\cos\theta+\sin\theta+\cos\theta-\sin\theta}{\cos\theta+\sin\theta}d\theta$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2}\frac{(\cos\theta+\sin\theta)+(\cos\theta-\sin\theta)}{\cos\theta+\sin\theta}d\theta$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2}\frac{(\cos\theta+\sin\theta)}{(\cos\theta+\sin\theta)}d\theta+\frac{1}{2}\int_0^{\pi/2}\frac{(\cos\theta-\sin\theta)}{(\cos\theta+\sin\theta)}d\theta$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \frac{(\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta)} d\theta$$

$$\Rightarrow I = \frac{1}{2}[\theta]_0^{\pi/2} + \frac{1}{2}[\log(\cos\theta + \sin\theta)]_0^{\pi/2}$$

$[\because \frac{d}{d\theta}(\cos\theta + \sin\theta) = -\sin\theta + \cos\theta]$ & $[\because \frac{d}{d\theta}(\cos\theta + \sin\theta) = \cos\theta - \sin\theta]$

[নিচের ফাংশনকে ডিফারেন্সিয়েট করলে যদি উপরের ফাংশন পাওয়া যায় তাহলে তার ইন্টিগ্রেশন হল লগ অফ নিচের ফাংশন]

$$\Rightarrow I = \frac{1}{2}\left[\frac{\pi}{2} - 0\right] + \frac{1}{2}\left[\log\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \log(\cos 0 + \sin 0)\right]_0^{\pi/2}$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{\pi}{2}\right] + \frac{1}{2}[\log(0+1) - \log(1+0)]$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{\pi}{2}\right] + \frac{1}{2}[\log 1 - \log 1]$$

$$\Rightarrow I = \frac{1}{2}\left[\frac{\pi}{2}\right] = \frac{\pi}{4} \text{ \textit{Answer}}$$

**Example 109: Show that** $\int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx = \frac{\pi}{2}\log\frac{1}{2}$

**Solution: Let** $I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$

Put $x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta\, d\theta$

$$I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\log\sin\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\, d\theta$$

| x | 0 | 1 |
|---|---|---|
| $x = \sin\theta$<br>$\therefore \theta = \sin^{-1} x$ | $\theta = \sin^{-1} 0 = 0$ | $\theta = \sin^{-1} 1$<br>$= \sin^{-1}\sin\frac{\pi}{2}$<br>$= \frac{\pi}{2}$ |

$$\Rightarrow I = \int_0^{\pi/2} \frac{\log\sin\theta}{\sqrt{\cos^2\theta}} \cos\theta\, d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\log\sin\theta}{\cos\theta} \cos\theta\, d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \log\sin\theta\, d\theta \text{ -----------------------(i)}$$

$$\Rightarrow I = \int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - \theta\right) d\theta \qquad \left[\int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\Rightarrow I = \int_0^{\pi/2} \log\cos\theta\, d\theta \text{ -----------------------(ii)}$$

$$[\because \sin(\frac{\pi}{2} - x) = \sin(1.\frac{\pi}{2} - x) = \sin(1.90^0 - x) = \cos x$$

(i)+(ii)

$$I + I = \int_0^{\pi/2} \log\sin\theta\, d\theta + \int_0^{\pi/2} \log\cos\theta\, d\theta$$

$$\Rightarrow 2I = \int_0^{\pi/2} \log\sin\theta\, d\theta + \int_0^{\pi/2} \log\cos\theta\, d\theta$$

$$\Rightarrow 2I = \int_0^{\pi/2} [\log\sin\theta + \log\cos\theta]\, d\theta$$

$$\Rightarrow 2I = \int_0^{\pi/2} [\log\sin\theta\cos\theta]\, d\theta \qquad [\because \log a + \log b = \log ab]$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left[\log\frac{1}{2} \times 2\sin\theta\cos\theta\right] d\theta$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left[\log\frac{1}{2} \times \sin 2\theta\right] d\theta \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left[\log\left(\frac{1}{2} \times \sin 2\theta\right)\right] d\theta$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left[\log\left(\frac{1}{2}\right) + \log(\sin 2\theta)\right] d\theta \qquad [\because \log a + \log b = \log ab]$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left[\log\left(\frac{1}{2}\right)\right] d\theta + \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta$$

$$\Rightarrow 2I = \log\left(\frac{1}{2}\right)\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta = \log\left(\frac{1}{2}\right)[\theta]_0^{\pi/2} + \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta$$

$$\Rightarrow 2I = \log\left(\frac{1}{2}\right)\left[\frac{\pi}{2} - 0\right] + \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta = \log\left(\frac{1}{2}\right)\left[\frac{\pi}{2}\right] + \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta$$

$$\Rightarrow 2I = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta$$

**Let,** $z = 2\theta$

| $\theta$ | $0$ | $\frac{\pi}{2}$ |
|---|---|---|
| $z = 2\theta$ | $z = 2\theta$<br>$z = 2.0 = 0$ | $z = 2\theta$<br>$z = 2 \times \frac{\pi}{2} = \pi$ |

$$\Rightarrow \frac{dz}{d\theta} = 2$$

$$\Rightarrow d\theta = \frac{dz}{2}$$

$$2I = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + \int_0^{\pi/2} \log(\sin 2\theta)\, d\theta$$

$$\Rightarrow 2I = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + \int_0^{\pi} \log(\sin z)\frac{dz}{2}$$

$$\Rightarrow 2I = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + \frac{1}{2}\int_0^{\pi} \log(\sin z)dz = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + \int_0^{\pi/2} \log(\sin z)dz$$

$$\Rightarrow 2I = \frac{\pi}{2}\log\left(\frac{1}{2}\right) + I$$

$$\Rightarrow 2I - I = \frac{\pi}{2}\log\left(\frac{1}{2}\right)$$

$$\Rightarrow I = \frac{\pi}{2}\log\left(\frac{1}{2}\right) \textit{ Answer}$$

**Example 110:**

Show that $\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2$

**Solution:** Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}dx$

| |
|---|
| Put $x = \tan\theta$ <br> $\frac{dx}{d\theta} = \sec^2\theta$ <br> $dx = \sec^2\theta\, d\theta$ |

| x | 0 | 1 |
|---|---|---|
| $x = \tan\theta$ <br> $\therefore \theta = \tan^{-1} x$ | $\theta = \tan^{-1} 0$ <br> $= \tan^{-1}\tan 0 = 0$ | $\theta = \tan^{-1} 1$ <br> $= \tan^{-1}\tan\frac{\pi}{4}$ <br> $= \frac{\pi}{4}$ |

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}dx$$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\, d\theta$$

$$I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta \qquad [\sec^2\theta = 1+\tan^2\theta]$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \text{ --------------------------(i)}$$

$$I = \int_0^{\pi/4} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\}d\theta \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

$$I = \int_0^{\pi/4} \log\left\{1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}d\theta \qquad [\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}]$$

$$I = \int_0^{\pi/4} \log\left\{\frac{1+\tan\frac{\pi}{4}\tan\theta+\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}d\theta$$

$$I = \int_0^{\pi/4} \log\left\{\frac{1+1.\tan\theta+1-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}d\theta = \int_0^{\pi/4} \log\left\{\frac{1+\tan\theta+1-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right\}d\theta$$

$$I = \int_0^{\pi/4} \log\left\{\frac{2}{1+\tan\frac{\pi}{4}\tan\theta}\right\}d\theta = \int_0^{\pi/4} \log\left\{\frac{2}{1+1.\tan\theta}\right\}d\theta = \int_0^{\pi/4} \log\left\{\frac{2}{1+\tan\theta}\right\}d\theta$$

$$I = \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta \qquad [\log\frac{a}{b} = \log a - \log b]$$

$$I = \int_0^{\pi/4} \log 2\, d\theta - I \qquad \text{[From (i)]}$$

$$I + I = \int_0^{\pi/4} \log 2\, d\theta$$

$$2I = \int_0^{\pi/4} \log 2\, d\theta = \log 2\int_0^{\pi/4} d\theta = \log 2[\theta]_0^{\pi/4} = \log 2\left[\frac{\pi}{4}-0\right]$$

$$I = \frac{1}{2}\log 2\left[\frac{\pi}{4}\right] = \frac{\pi}{4}\frac{1}{2}\log 2 = \frac{\pi}{8}\log 2 \quad \textit{(Proved)}$$

**Example 111: Show that** $\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2}+(\cos x)^{3/2}}dx = \frac{1}{4}\pi$

**Solution: Let** $I = \int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2}+(\cos x)^{3/2}}dx$ **-----------------------(i)**

$$I = \int_0^{\pi/2} \frac{\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}^{3/2}}{\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}^{3/2} + \left\{\cos\left(\frac{\pi}{2} - x\right)\right\}^{3/2}} dx \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx\,]$$

$$I = \int_0^{\pi/2} \frac{\left\{\sin\left(1.\frac{\pi}{2} - x\right)\right\}^{3/2}}{\left\{\sin\left(1.\frac{\pi}{2} - x\right)\right\}^{3/2} + \left\{\cos\left(1.\frac{\pi}{2} - x\right)\right\}^{3/2}} dx$$

$$I = \int_0^{\pi/2} \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} dx \text{ ----------------------------(ii)}$$

(i)+(ii)

$$I + I = \int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} dx + \int_0^{\pi/2} \frac{(\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} dx$$

$$2I = \int_0^{\pi/2} \frac{(\sin x)^{3/2} + (\cos x)^{3/2}}{(\cos x)^{3/2} + (\sin x)^{3/2}} dx = \int_0^{\pi/2} \frac{\left[(\sin x)^{3/2} + (\cos x)^{3/2}\right]}{\left[(\sin x)^{3/2} + (\cos x)^{3/2}\right]} dx$$

$$2I = \int_0^{\pi/2} dx = \left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2}$$

$$I = \frac{\pi}{4} \text{ Proved}$$

**Example 112: Show that** $\int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a} \frac{1}{\sqrt{a^2 - 1}}$

**Solution: Let,** $I = \int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x}$ -----------------------(i)

$$I = \int_0^{\pi} \frac{\pi - x}{a^2 - \cos^2(\pi - x)} dx \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx\,]$$

$$I = \int_0^{\pi} \frac{\pi - x}{a^2 - \{\cos(\pi - x)\}^2} dx$$

$$I = \int_0^{\pi} \frac{\pi - x}{a^2 - (-\cos x)^2} dx \qquad [[\because \cos(\pi - x) = \cos(2.90 - x) = -\cos x]$$

$$I = \int_0^{\pi} \frac{\pi - x}{a^2 - (\cos x)^2} dx = \int_0^{\pi} \frac{\pi - x}{a^2 - \cos^2 x} dx$$

$$I=\int_0^{\pi}\frac{\pi}{a^2-\cos^2 x}dx-\int_0^{\pi}\frac{x}{a^2-\cos^2 x}dx$$

$$I=\int_0^{\pi}\frac{\pi}{a^2-\cos^2 x}dx-I \qquad \text{[From (i)]}$$

$$I+I=\int_0^{\pi}\frac{\pi}{a^2-\cos^2 x}dx$$

$$2I=\int_0^{\pi}\frac{\pi}{a^2-\cos^2 x}dx$$

$$2I=\int_0^{\pi}\frac{\frac{\pi}{\cos^2 x}}{\frac{a^2-\cos^2 x}{\cos^2 x}}dx \qquad \text{[Dividing by } \cos^2 x\text{ ]}$$

$$2I=\int_0^{\pi}\frac{\pi\sec^2 x}{\frac{a^2}{\cos^2 x}-\frac{\cos^2 x}{\cos^2 x}}dx=\int_0^{\pi}\frac{\pi\sec^2 x}{a^2\sec^2 x-1}dx$$

$$2I=\int_0^{\pi}\frac{\pi\sec^2 x}{a^2(1+\tan^2 x)-1}dx \qquad [\because \sec^2 x=1+\tan^2 x]$$

$$2I=\int_0^{\pi}\frac{\pi\sec^2 x}{a^2+a^2\tan^2 x-1}dx=\int_0^{\pi}\frac{\pi\sec^2 x}{a^2-1+a^2\tan^2 x}dx$$

$$2I=2\int_0^{\pi/2}\frac{\pi\sec^2 x}{a^2-1+a^2\tan^2 x}dx \text{ ------------------------------(ii)}$$

| $x$ | $0$ | $\frac{\pi}{2}$ |
|---|---|---|
| $z=\tan x$ | $z=\tan x$<br>$z=\tan 0=0$ | $z=\tan x$<br>$z=\tan\frac{\pi}{2}=\infty$ |

Put $z=\tan x$

$\frac{dz}{dx}=\sec^2 x$

$dz=\sec^2 x\,dx$

From (ii)

$$2I=2\int_0^{\pi/2}\frac{\pi\sec^2 x}{a^2-1+a^2\tan^2 x}dx$$

$$2I=2\int_0^{\infty}\frac{\pi dz}{a^2-1+a^2z^2}$$

$$2I = 2\int_0^\infty \frac{\frac{\pi}{a^2}dz}{\frac{a^2-1+a^2z^2}{a^2}} \qquad \text{[Dividing by } a^2\text{]}$$

$$2I = 2\int_0^\infty \frac{\frac{\pi}{a^2}dz}{\frac{a^2-1}{a^2}+\frac{a^2z^2}{a^2}} = 2\int_0^\infty \frac{\frac{\pi}{a^2}dz}{\frac{a^2-1}{a^2}+z^2}$$

$$2I = 2\frac{\pi}{a^2}\int_0^\infty \frac{dz}{\frac{a^2-1}{a^2}+z^2} = 2\frac{\pi}{a^2}\int_0^\infty \frac{dz}{\left(\sqrt{\frac{a^2-1}{a^2}}\right)^2+z^2}$$

$$2I = 2\frac{\pi}{a^2}\times\frac{1}{\sqrt{\frac{a^2-1}{a^2}}}\left[\tan^{-1}\frac{z}{\sqrt{\frac{a^2-1}{a^2}}}\right]_0^\infty$$

$$2I = 2\frac{\pi}{a^2}\times\frac{1}{\sqrt{\frac{a^2-1}{a^2}}}\left[\tan^{-1}\frac{\infty}{\sqrt{\frac{a^2-1}{a^2}}}-\tan^{-1}\frac{0}{\sqrt{\frac{a^2-1}{a^2}}}\right]$$

$$2I = 2\frac{\pi}{a^2}\times\frac{1}{\sqrt{\frac{a^2-1}{a^2}}}\left[\tan^{-1}\infty-\tan^{-1}0\right]$$

$$2I = 2\frac{\pi}{a^2}\times\frac{1}{\sqrt{\frac{a^2-1}{a^2}}}\left[\tan^{-1}\tan\frac{\pi}{2}-\tan^{-1}\tan 0\right]$$

$$2I = 2\frac{\pi}{a^2}\times\frac{1}{\sqrt{\frac{a^2-1}{a^2}}}\left[\frac{\pi}{2}-0\right] = 2\frac{\pi}{a^2}\times\frac{1}{\sqrt{\frac{a^2-1}{a^2}}}\frac{\pi}{2}$$

$$I = \frac{\pi^2}{2a^2}\times\frac{a}{\sqrt{a^2-1}} = \frac{\pi^2}{2a}\frac{1}{\sqrt{a^2-1}}$$

**Example 113: Show that** $\int_0^\pi \log(1+\cos x)dx = \pi\log\frac{1}{2}$

Solution: Let $I = \int_0^{\pi} \log(1+\cos x)dx$ --------------------(i)

$I = \int_0^{\pi} \log(1+\cos(\pi - x)dx$ $\qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$

$I = \int_0^{\pi} \log(1-\cos x)dx$ $\qquad [\because \cos(\pi - x) = \cos(2.90 - x) = -\cos x]$

$I = \int_0^{\pi} \log(1-\cos x)dx$ ---------------------------------------(ii)

(i)+(ii)

$I + I = \int_0^{\pi} \log(1+\cos x)dx + \int_0^{\pi} \log(1-\cos x)dx$

$2I = \int_0^{\pi} \log(1+\cos x)dx + \int_0^{\pi} \log(1-\cos x)dx$

$2I = \int_0^{\pi} \log[(1+\cos x)(1-\cos x)]dx$ $\qquad [\because \log a + \log b = \log ab]$

$2I = \int_0^{\pi} \log(1-\cos^2 x)dx$

$2I = \int_0^{\pi} \log\sin^2 x dx$ $\qquad [1-\cos^2 x = \sin^2 x]$

$2I = \int_0^{\pi} \log(\sin x)^2 dx$

$2I = \int_0^{\pi} 2\log\sin x dx$ $\qquad [\log x^b = b\log x]$

$2I = 2\int_0^{\pi} \log\sin x dx$

$I = \int_0^{\pi} \log\sin x dx$

$I = 2\int_0^{\pi/2} \log\sin x dx$ -------------------------------(iii)

$I = 2\int_0^{\pi/2} \log\sin\left(\frac{\pi}{2} - x\right)dx$ $\qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$

$I = 2\int_0^{\pi/2} \log\cos x dx$ $\qquad [\because \sin(\frac{\pi}{2} - x) = \sin(1.90 - x) = \sin x]$

$I = 2\int_0^{\pi/2} \log\cos x dx$ -------------------------------(iv)

(iii)+(iv)

$$I + I = 2\int_0^{\pi/2} \log \sin x dx \;\; 2\int_0^{\pi/2} \log \cos x dx$$

$$2I = 2\int_0^{\pi/2} \log \sin x \cos x dx \qquad [\because \log a + \log b = \log ab]$$

$$2I = 2\int_0^{\pi/2} \log \frac{1}{2} 2 \sin x \cos x dx$$

$$2I = 2\int_0^{\pi/2} \log \frac{1}{2} \sin 2x dx \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$I = \int_0^{\pi/2} \log \frac{1}{2} \sin 2x dx = \int_0^{\pi/2} \log\left[\frac{1}{2} \sin 2x\right] dx$$

$$I = \int_0^{\pi/2} \log \frac{1}{2} dx + \int_0^{\pi/2} \log \sin 2x dx$$

$$I = \log \frac{1}{2} \int_0^{\pi/2} dx + \int_0^{\pi/2} \log \sin 2x dx$$

$$I = \log \frac{1}{2} [x]_0^{\pi/2} + \int_0^{\pi/2} \log \sin 2x dx = \log \frac{1}{2}\left[\frac{\pi}{2} - 0\right] + \int_0^{\pi/2} \log \sin 2x dx$$

$$I = \frac{\pi}{2} \log \frac{1}{2} + \int_0^{\pi/2} \log \sin 2x dx \text{ ---------------------(v)}$$

Now Let $I_1 = \int_0^{\pi/2} \log \sin 2x dx$

| x | 0 | $\frac{\pi}{2}$ |
|---|---|---|
| $z = 2x$ | $z = 2x$<br>$z = 2.0 = 0$ | $z = 2x$<br>$z = 2 \times \frac{\pi}{2} = \pi$ |

Let, $z = 2x$

$$\Rightarrow \frac{dz}{dx} = 2$$

$$\Rightarrow dx = \frac{dz}{2}$$

$$I_1 = \int_0^{\pi/2} \log \sin 2x dx$$

$$I_1 = \int_0^{\pi} \log \sin z \frac{dz}{2} = \frac{1}{2}\int_0^{\pi} \log \sin z dz$$

$$I_1 = \frac{1}{2} I \text{ ----------(vi)} \qquad [I = \int_0^{\pi} \log \sin z dz] \qquad [\int_a^b f(x) dx = \int_a^b f(z) dz]$$

Hence from (v)

$$I = \frac{\pi}{2}\log\frac{1}{2} + \int_0^{\pi/2} \log\sin 2x\,dx$$

$$I = \frac{\pi}{2}\log\frac{1}{2} + I_1$$

$$I = \frac{\pi}{2}\log\frac{1}{2} + \frac{1}{2}I \qquad\qquad [I_1 = \frac{1}{2}I]$$

$$I - \frac{1}{2}I = \frac{\pi}{2}\log\frac{1}{2}$$

$$\frac{1}{2}I = \frac{\pi}{2}\log\frac{1}{2}$$

$I = \pi\log\frac{1}{2}$ ***Answer***

**Example 114: Show that** $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx = \frac{1}{\sqrt{2}}\log\sqrt{2}$

**Solution**: $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx$ ------------------------------------(i)

$$= \int_0^{\pi/2} \frac{\sin^2(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)}dx \qquad [\text{Method \# 18}: \int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

$$= \int_0^{\pi/2} \frac{\left\{\sin\left(\frac{\pi}{2} - x\right)\right\}^2}{\sin(\frac{\pi}{2} - x) + \cos(\frac{\pi}{2} - x)}dx$$

$$= \int_0^{\pi/2} \frac{\left\{\sin\left(1.\frac{\pi}{2} - x\right)\right\}^2}{\sin(1.\frac{\pi}{2} - x) + \cos(1.\frac{\pi}{2} - x)}dx$$

$I = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\cos x + \sin x}dx$ -----------------------------------(ii)

$[\because \sin(1.\frac{\pi}{2} - x) = \sin(1.90 - x) = \cos x; \quad \cos(1.\frac{\pi}{2} - x) = \cos(1.90 - x) = \sin x]$

***From (i) + (ii)***

$$\therefore 2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}dx + \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}dx$$

$$= \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{(\sin x + \cos x)} dx = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{2} \times \frac{1}{\sqrt{2}} (\sin x + \cos x)} dx = \int_0^{\pi/2} \frac{1}{\sqrt{2}(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x)} dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{2}(\sin x \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cos x)} dx = \int_0^{\pi/2} \frac{1}{\sqrt{2}(\sin x \frac{1}{\sqrt{2}} + \cos x \frac{1}{\sqrt{2}})} dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{2}(\cos x \frac{1}{\sqrt{2}} + \sin x \frac{1}{\sqrt{2}})} dx$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{2}(\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4})} dx \qquad [\cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}]$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{2} \cos(x - \frac{\pi}{4})} dx \qquad [\cos(x - \frac{\pi}{4}) = \cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4}]$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{1}{\cos(x - \frac{\pi}{4})} dx$$

$$2I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sec(x - \frac{\pi}{4}) dx$$

$$I = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sec(x - \frac{\pi}{4}) dx = \frac{1}{2\sqrt{2}} \times 2 \int_0^{\pi/4} \sec(x - \frac{\pi}{4}) dx$$

$$I = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec(x - \frac{\pi}{4}) dx = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec\left\{\left(\frac{\pi}{4} - x\right) - \frac{\pi}{4}\right\} dx$$

$$I = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec\left(\frac{\pi}{4} - x - \frac{\pi}{4}\right) dx = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec(-x) dx$$

$$I = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \sec x dx \qquad [\sec(-x) = \sec x]$$

$$I = \frac{1}{\sqrt{2}} \log[\sec x + \tan x]_0^{\pi/4}$$

$$I = \frac{1}{\sqrt{2}} \log\left[\sec \frac{\pi}{4} + \tan \frac{\pi}{4} - \sec 0 - \tan 0\right]$$

$$I = \frac{1}{\sqrt{2}} \log\left[\sqrt{2} + 1 - 1 - 0\right] = \frac{1}{\sqrt{2}} \log \sqrt{2}$$

$$\int_0^{\pi/2}\frac{\sin^2 x}{\sin x+\cos x}dx=\frac{1}{\sqrt{2}}\log\sqrt{2}$$ Proved

**Example 115: Show that** $\int_0^{\pi/2}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx=\frac{\pi}{6}$

**Solution:** $I=\int_0^{\pi/2}\frac{\cos^2 x}{\cos^2 x+4\sin^2 x}dx$

$$I=\int_0^{\pi/2}\frac{\frac{\cos^2 x}{\cos^2 x}}{\frac{\cos^2 x+4\sin^2 x}{\cos^2 x}}dx \qquad \text{[Dividing by } \cos^2 x\text{]}$$

$$I=\int_0^{\pi/2}\frac{1}{\frac{\cos^2 x}{\cos^2 x}+\frac{4\sin^2 x}{\cos^2 x}}dx$$

$$I=\int_0^{\pi/2}\frac{1}{1+4\tan^2 x}dx \text{ ---------------------------(i)}$$

Put $z=\tan x$

$\frac{dz}{dx}=\sec^2 x$

$dz=\sec^2 x\,dx$

$dx=\frac{dz}{\sec^2 x}$

| $x$ | $0$ | $\frac{\pi}{2}$ |
|---|---|---|
| $z=\tan x$ | $z=\tan x$<br>$z=\tan 0=0$ | $z=\tan x$<br>$z=\tan\frac{\pi}{2}=\infty$ |

$$I=\int_0^{\pi/2}\frac{1}{1+4\tan^2 x}dx$$

$$I=\int_0^{\infty}\frac{1}{1+4z^2}\frac{dz}{\sec^2 x}=\int_0^{\infty}\frac{1}{1+4z^2}\frac{dz}{1+\tan^2 x}=\int_0^{\infty}\frac{1}{1+4z^2}\frac{dz}{1+z^2}$$

$$I=-\frac{1}{3}\int_0^{\infty}\left(\frac{1}{1+z^2}-\frac{4}{1+4z^2}\right)dz$$

$$I=-\frac{1}{3}\int_0^{\infty}\left(\frac{1}{1+z^2}\right)dz-\frac{1}{3}\int_0^{\infty}\left(-\frac{4}{1+4z^2}\right)dz$$

$$I=-\frac{1}{3}\int_0^{\infty}\left(\frac{1}{1+z^2}\right)dz+\frac{4}{3}\int_0^{\infty}\left(\frac{1}{1+4z^2}\right)dz$$

$$I=-\frac{1}{3}\int_0^{\infty}\frac{1}{1+z^2}dz+\frac{4}{3}\int_0^{\infty}\frac{1}{1+4z^2}dz$$

$$I=-\frac{1}{3}\int_0^\infty\frac{1}{1+z^2}dz+\frac{4}{3}\int_0^\infty\frac{1}{4\left(\frac{1}{4}+z^2\right)}dz$$

$$I=-\frac{1}{3}\int_0^\infty\frac{1}{1+z^2}dz+\frac{1}{3}\int_0^\infty\frac{1}{\left(\frac{1}{4}+z^2\right)}dz$$

$$I=-\frac{1}{3}\left[\tan^{-1}z\right]_0^\infty+\frac{1}{3}\times\int_0^\infty\frac{1}{\left(\left(\frac{1}{2}\right)^2+z^2\right)}dz$$

$$I=-\frac{1}{3}\left[\tan^{-1}z\right]_0^\infty+\frac{1}{3}\times\frac{1}{\frac{1}{2}}\left[\tan^{-1}\frac{z}{\frac{1}{2}}\right]_0^\infty$$

$$I=-\frac{1}{3}\left[\tan^{-1}z\right]_0^\infty+\frac{1}{3}\times\frac{1}{\frac{1}{2}}\left[\tan^{-1}\frac{z}{\frac{1}{2}}\right]_0^\infty$$

$$I=-\frac{1}{3}\left[\tan^{-1}\infty-\tan^{-1}0\right]+\frac{1}{3}\times2\left[\tan^{-1}2z\right]_0^\infty$$

$$I=-\frac{1}{3}\left[\tan^{-1}\infty-\tan^{-1}0\right]+\frac{2}{3}\left[\tan^{-1}\infty-\tan^{-1}0\right]$$

$$I=-\frac{1}{3}\left[\tan^{-1}\tan\frac{\pi}{2}-\tan^{-1}\tan0\right]+\frac{2}{3}\left[\tan^{-1}\tan\frac{\pi}{2}-\tan^{-1}\tan0\right]$$

$$I=-\frac{1}{3}\left[\frac{\pi}{2}-0\right]+\frac{2}{3}\left[\frac{\pi}{2}-0\right]=-\frac{1}{3}\left[\frac{\pi}{2}\right]+\frac{2}{3}\left[\frac{\pi}{2}\right]=-\frac{\pi}{6}+\frac{2\pi}{6}=\frac{\pi}{6}$$ ***Proved***

**Example 116: Show that** $\int_0^\infty\frac{x}{(1+x)(1+x^2)}dx=\frac{\pi}{4}$

Solution: Let $I=\int_0^\infty\frac{x}{(1+x)(1+x^2)}dx$ -------------------(i)

Put $x=\tan\theta$

$\frac{dx}{d\theta}=\sec^2\theta$

$dx=\sec^2\theta\, d\theta$

| $x$ | $0$ | $\infty$ |
|---|---|---|
| $x=\tan\theta$<br>$\therefore\theta=\tan^{-1}x$ | $\theta=\tan^{-1}0$<br>$=\tan^{-1}\tan0=0$ | $\theta=\tan^{-1}\infty$<br>$=\tan^{-1}\tan\frac{\pi}{2}$<br>$=\frac{\pi}{2}$ |

From (i)

$$I=\int_0^{\infty}\frac{x}{(1+x)(1+x^2)}dx$$

$$I=\int_0^{\pi/2}\frac{\tan\theta}{(1+\tan\theta)(1+\tan^2\theta)}\sec^2\theta\,d\theta$$

$$I=\int_0^{\pi/2}\frac{\tan\theta}{(1+\tan\theta)\sec^2\theta}\sec^2\theta\,d\theta=\int_0^{\pi/2}\frac{\tan\theta}{(1+\tan\theta)}d\theta$$

$$I=\int_0^{\pi/2}\frac{1-1+\tan\theta}{(1+\tan\theta)}d\theta=\int_0^{\pi/2}\frac{1+\tan\theta-1}{(1+\tan\theta)}d\theta$$

$$I=\int_0^{\pi/2}\frac{1+\tan\theta}{(1+\tan\theta)}d\theta-\int_0^{\pi/2}\frac{1}{(1+\tan\theta)}d\theta$$

$$I=\int_0^{\pi/2}d\theta-\int_0^{\pi/2}\frac{1}{(1+\tan\theta)}d\theta=[\theta]_0^{\pi/2}-\int_0^{\pi/2}\frac{1}{(1+\tan\theta)}d\theta$$

$$I=\left[\frac{\pi}{2}-0\right]-\frac{\pi}{4}$$ [Example 108]

$$I=\left[\frac{\pi}{2}\right]-\frac{\pi}{4}=\frac{\pi}{2}-\frac{\pi}{4}=\frac{2\pi-\pi}{4}=\frac{\pi}{4}$$ ***Proved***

**Example 117: Show that** $\int_0^{\pi/2}\frac{\sin^2 x}{1+\sin x\cos x}dx=\frac{\pi}{3\sqrt{3}}$

**Solution: Let,** $I=\int_0^{\pi/2}\frac{\sin^2 x}{1+\sin x\cos x}dx$ ------------(i)

$$I=\int_0^{\pi/2}\frac{\sin^2(\frac{\pi}{2}-x)}{1+\sin(\frac{\pi}{2}-x)\cos(\frac{\pi}{2}-x)}dx$$ $[\int_0^a f(x)dx=\int_0^a f(a-x)dx]$

$$I=\int_0^{\pi/2}\frac{\left\{\sin(\frac{\pi}{2}-x)\right\}^2}{1+\sin(\frac{\pi}{2}-x)\cos(\frac{\pi}{2}-x)}dx=\int_0^{\pi/2}\frac{(\cos x)^2}{1+\sin(\frac{\pi}{2}-x)\cos(\frac{\pi}{2}-x)}dx$$

$$I=\int_0^{\pi/2}\frac{(\cos x)^2}{1+\cos x\sin x}dx$$

$$I=\int_0^{\pi/2}\frac{\cos^2 x}{1+\cos x\sin x}dx$$ --------------------------------(ii)

$$I + I = \int_0^{\pi/2} \frac{\sin^2 x}{1+\cos x \sin x} dx + \int_0^{\pi/2} \frac{\cos^2 x}{1+\cos x \sin x} dx$$

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{1+\cos x \sin x} dx$$

$$2I = \int_0^{\pi/2} \frac{1}{1+\cos x \sin x} dx \qquad [\sin^2 x + \cos^2 x = 1]$$

$$2I = \int_0^{\pi/2} \frac{1}{\sin^2 x + \cos^2 x + \cos x \sin x} dx \qquad [\sin^2 x + \cos^2 x = 1]$$

$$2I = \int_0^{\pi/2} \frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x + \cos^2 x + \cos x \sin x}{\cos^2 x}} dx \qquad \text{[Dividing by } \cos^2 x \text{]}$$

Put $z = \tan x$

$\frac{dz}{dx} = \sec^2 x$

$dz = \sec^2 x \, dx$

$dx = \frac{dz}{\sec^2 x}$

$$2I = \int_0^{\pi/2} \frac{\sec^2 x}{\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} + \frac{\cos x \sin x}{\cos^2 x}} dx$$

$$2I = \int_0^{\pi/2} \frac{\sec^2 x}{\tan^2 x + 1 + \frac{\cos x \sin x}{\cos x \cos x}} dx$$

| x | 0 | $\frac{\pi}{2}$ |
|---|---|---|
| $z = \tan x$ | $z = \tan x$<br>$z = \tan 0 = 0$ | $z = \tan x$<br>$z = \tan\frac{\pi}{2} = \infty$ |

$$2I = \int_0^{\pi/2} \frac{\sec^2 x}{\tan^2 x + 1 + \tan x} dx$$

$$2I = \int_0^{\pi/2} \frac{\sec^2 x}{\tan^2 x + 1 + \tan x} dx$$

$$2I = \int_0^{\infty} \frac{dz}{z^2 + 1 + z} = \int_0^{\infty} \frac{dz}{z^2 + z + 1}$$

$$2I = \int_0^{\infty} \frac{dz}{z^2 + 2.z.\frac{1}{2} + (\frac{1}{2})^2 - (\frac{1}{2})^2 + 1}$$

$$2I = \int_0^{\infty} \frac{dz}{z^2 + 2.z.\frac{1}{2} + (\frac{1}{2})^2 - \frac{1}{4} + 1}$$

$$2I = \int_0^{\infty} \frac{dz}{(z+\frac{1}{2})^2 + \frac{3}{4}} = \int_0^{\infty} \frac{dz}{(z+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$2I = \frac{1}{\frac{\sqrt{3}}{2}}\left[\tan^{-1}\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]_0^{\infty} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]_0^{\infty}$$

$$2I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{\infty+\frac{1}{2}}{\frac{\sqrt{3}}{2}} - \tan^{-1}\frac{0+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]$$

$$2I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\infty - \tan^{-1}\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] = \frac{2}{\sqrt{3}}\left[\tan^{-1}\infty - \tan^{-1}\frac{1}{\sqrt{3}}\right]$$

$$2I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\tan\frac{\pi}{2} - \tan^{-1}\tan\frac{\pi}{6}\right] = \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right]$$

$$I = \frac{1}{\sqrt{3}}\left[\frac{3\pi-\pi}{6}\right] = \frac{1}{\sqrt{3}}\left[\frac{2\pi}{6}\right] = \frac{1}{\sqrt{3}}\left[\frac{\pi}{3}\right] = \frac{1}{\sqrt{3}}\frac{\pi}{3}$$

$$\int_0^{\pi/2}\frac{\sin^2 x}{1+\sin x\cos x}dx = \frac{\pi}{3\sqrt{3}} \quad \textit{(Proved)}$$

**Example 118: Show that** $\int_0^1 \cot^{-1}(1-x+x^2)dx = \frac{\pi}{2} - \log 2$

**Solution:** Let, $I = \int_0^1 \cot^{-1}(1-x+x^2)dx$

$$I = \int_0^1 \tan^{-1}\frac{1}{1-x+x^2}dx$$

$$I = \int_0^1 \tan^{-1}\frac{x+1-x}{1-x+x^2}dx$$

$$I = \int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}dx$$

$$I = \int_0^1 \left[\tan^{-1}x + \tan^{-1}(1-x)\right]dx \qquad [\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}]$$

$$I = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx$$

$$I = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}x\,dx \qquad [\int_0^a \tan^{-1}(a-x)dx = \int_0^a \tan^{-1}x\,dx]$$

$$I = 2\int_0^1 \tan^{-1}x\,dx \text{ ------------------------------(i)} \qquad [\int_0^a f(x)dx = \int_0^a f(a-x)dx]$$

Now

$$\int 1.\tan^{-1} x dx = \tan^{-1} x \int 1 dx - \int \{\frac{d}{dx}(\tan^{-1} x)\int 1 dx\} dx$$

$$\int 1.\tan^{-1} x dx = \tan^{-1} x . x - \int \frac{1}{1+x^2} x dx$$

$$\int 1.\tan^{-1} x dx = x\tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$\int 1.\tan^{-1} x dx = x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2} dx$$

$$\int 1.\tan^{-1} x dx = x\tan^{-1} x - \frac{1}{2}\log(1+x^2)$$

From (i)

$$I = 2\int_0^1 \tan^{-1} x dx$$

$$I = 2\left[x\tan^{-1} x - \frac{1}{2}\log(1+x^2)\right]_0^1$$

$$I = 2\left[1.\tan^{-1} 1 - \frac{1}{2}\log(1+1^2) - \left\{0.\tan^{-1} 0 - \frac{1}{2}\log(1+0^2)\right\}\right]$$

$$I = 2\left[1.\tan^{-1} \tan\frac{\pi}{4} - \frac{1}{2}\log 2 - \left\{0 - \frac{1}{2}\log 1\right\}\right]$$

$$I = 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2 - \{0-0\}\right] = 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right] = \left[\frac{\pi}{2} - \log 2\right] = \frac{\pi}{2} - \log 2$$ Proved

**Example 119: Show that** $\int_0^{\pi/8} \frac{dx}{1+\tan 2x} = \frac{1}{16}\pi + \frac{1}{8}\log 2$

**Solution:** Let $I = \int_0^{\pi/8} \frac{dx}{1+\tan 2x}$

$$I = \int_0^{\pi/8} \frac{dx}{1+\frac{\sin 2x}{\cos 2x}} = \int_0^{\pi/8} \frac{dx}{\frac{\cos 2x + \sin 2x}{\cos 2x}} = \int_0^{\pi/8} \frac{\cos 2x\, dx}{\cos 2x + \sin 2x}$$

$$I = \frac{1}{2}\int_0^{\pi/8} \frac{(\cos 2x + \cos 2x) + \sin 2x - \sin 2x}{\cos 2x + \sin 2x} dx$$

$$I = \frac{1}{2}\int_0^{\pi/8} \frac{(\cos 2x + \sin 2x) + (\cos 2x - \sin 2x)}{\cos 2x + \sin 2x} dx$$

$$I=\frac{1}{2}\int_0^{\pi/8}\left[\frac{(\cos 2x+\sin 2x)}{\cos 2x+\sin 2x}+\frac{(\cos 2x-\sin 2x)}{\cos 2x+\sin 2x}\right]dx$$

$$I=\frac{1}{2}\int_0^{\pi/8}\left[1+\frac{(\cos 2x-\sin 2x)}{\cos 2x+\sin 2x}\right]dx=\frac{1}{2}\int_0^{\pi/8}1dx+\frac{1}{2}\int_0^{\pi/8}\frac{(\cos 2x-\sin 2x)}{\cos 2x+\sin 2x}dx$$

$$I=\frac{1}{2}\int_0^{\pi/8}1dx+I_1 \text{-------------(i)} \qquad \left[I_1=\frac{1}{2}\int_0^{\pi/8}\frac{(\cos 2x-\sin 2x)}{\cos 2x+\sin 2x}dx\right]$$

Let, $z=\cos 2x+\sin 2x$

$$\frac{dz}{dx}=\frac{d}{dx}(\cos 2x+\sin 2x)$$

$$\frac{dz}{dx}=-\sin 2x\frac{d}{dx}(2x)+\cos 2x\frac{d}{dx}(2x)$$

$$\frac{dz}{dx}=-2\sin 2x+2\cos 2x$$

$$\frac{dz}{dx}=2\cos 2x-2\sin 2x$$

$$\frac{dz}{dx}=2(\cos 2x-\sin 2x)$$

$$\frac{dz}{2}=(\cos 2x-\sin 2x)dx$$

| x | 0 | $\frac{\pi}{8}$ |
|---|---|---|
| z | $z=\cos 2x+\sin 2x$<br>$z=\cos 2.0+\sin 2.0$<br>$z=\cos 0+\sin 0$<br>$z=1+0$<br>$z=1$ | $z=\cos 2x+\sin 2x$<br>$z=\cos 2\times\frac{\pi}{8}+\sin 2\times\frac{\pi}{8}$<br>$z=\cos\frac{\pi}{4}+\sin\frac{\pi}{4}$<br>$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$<br>$z=\frac{2}{\sqrt{2}}=\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}}=\sqrt{2}$ |

$$I_1=\frac{1}{2}\int_0^{\pi/8}\frac{(\cos 2x-\sin 2x)}{\cos 2x+\sin 2x}dx$$

$$I_1=\frac{1}{2}\int_1^{\sqrt{2}}\frac{1}{z}\frac{dz}{2}$$

$$I_1=\frac{1}{4}\int_1^{\sqrt{2}}\frac{dz}{z}=\frac{1}{4}[\log z]_1^{\sqrt{2}}=\frac{1}{4}\left[\log\sqrt{2}-\log 1\right]=\frac{1}{4}\left[\log\sqrt{2}-0\right]$$

$$I_1=\frac{1}{4}\left[\log\sqrt{2}\right]$$

From (i)

$$I=\frac{1}{2}\int_0^{\pi/8}1dx+I_1$$

$$I=\frac{1}{2}\int_0^{\pi/8}1dx+\frac{1}{4}\left[\log\sqrt{2}\right] \qquad \left[I_1=\frac{1}{4}\left[\log\sqrt{2}\right]\right]$$

$$I=\frac{1}{2}[x]_0^{\pi/8}+\frac{1}{4}\left[\log\sqrt{2}\right]=\frac{1}{2}\left[\frac{\pi}{8}-0\right]+\frac{1}{4}\left[\log\sqrt{2}\right]$$

$$I=\frac{1}{2}\left[\frac{\pi}{8}\right]+\frac{1}{4}\left[\log\sqrt{2}\right]=\left[\frac{\pi}{16}\right]+\frac{1}{4}\left[\log\sqrt{2}\right]$$

$$\mathbf{I}=\left[\frac{\pi}{16}\right]+\frac{1}{4}\left[\log 2^{\frac{1}{2}}\right]=\left[\frac{\pi}{16}\right]+\frac{1}{4}\left[\frac{1}{2}\log 2\right]=\left[\frac{\pi}{16}\right]+\frac{1}{8}[\log 2]=\frac{\pi}{16}+\frac{1}{8}\log 2 \text{ (Proved)}$$

**Example 120: Show that** $\int_0^{\pi/4}\sin^4\theta\,d\theta=\frac{3\pi-8}{32}$

Solution: Let $\mathbf{I}=\int_0^{\pi/4}\sin^4\theta\,d\theta$

$$\mathbf{I}=\frac{1}{4}\int_0^{\pi/4}4\sin^4\theta\,d\theta=\frac{1}{4}\int_0^{\pi/4}(2\sin^2\theta)^2\,d\theta$$

$$=\frac{1}{4}\int_0^{\pi/4}(1-\cos 2\theta)^2\,d\theta \qquad [2\sin^2\theta=1-\cos 2\theta]$$

$$=\frac{1}{4}\int_0^{\pi/4}(1-2\cos 2\theta+\cos^2 2\theta)\,d\theta$$

$$=\frac{1}{4}\int_0^{\pi/4}1\,d\theta-\frac{1}{4}\int_0^{\pi/4}2\cos 2\theta\,d\theta+\frac{1}{4}\int_0^{\pi/4}\cos^2 2\theta\,d\theta$$

$$=\frac{1}{4}\int_0^{\pi/4}1\,d\theta-\frac{1}{4}\int_0^{\pi/4}2\cos 2\theta\,d\theta+\frac{1}{4}\frac{1}{2}\int_0^{\pi/4}2\cos^2 2\theta\,d\theta$$

$$=\frac{1}{4}\int_0^{\pi/4}1\,d\theta-\frac{1}{4}\int_0^{\pi/4}2\cos 2\theta\,d\theta+\frac{1}{4}\frac{1}{2}\int_0^{\pi/4}(1+\cos 2.2\theta)\,d\theta \qquad [2\cos^2\theta=1+\cos 2\theta]$$

$$=\frac{1}{4}\int_0^{\pi/4}1\,d\theta-\frac{1}{4}\int_0^{\pi/4}2\cos 2\theta\,d\theta+\frac{1}{4}\frac{1}{2}\int_0^{\pi/4}(1+\cos 4\theta)\,d\theta$$

$$=\frac{1}{4}[\theta]_0^{\pi/4}-\frac{1}{4}\left[2\frac{\sin 2\theta}{2}\right]_0^{\pi/4}+\frac{1}{8}\int_0^{\pi/4}(1+\cos 4\theta)\,d\theta$$

$$=\frac{1}{4}[\theta]_0^{\pi/4}-\frac{1}{4}\left[2\frac{\sin 2\theta}{2}\right]_0^{\pi/4}+\frac{1}{8}\int_0^{\pi/4}1\,d\theta+\frac{1}{8}\int_0^{\pi/4}\cos 4\theta\,d\theta$$

$$=\frac{1}{4}[\theta]_0^{\pi/4}-\frac{1}{4}\left[2\frac{\sin 2\theta}{2}\right]_0^{\pi/4}+\frac{1}{8}[\theta]_0^{\pi/4}+\frac{1}{8}\left[\frac{\sin 4\theta}{4}\right]_0^{\pi/4}$$

$$=\frac{1}{4}[\theta]_0^{\pi/4}-\frac{1}{4}[\sin 2\theta]_0^{\pi/4}+\frac{1}{8}[\theta]_0^{\pi/4}+\frac{1}{32}[\sin 4\theta]_0^{\pi/4}$$

$$=\frac{1}{4}\left[\frac{\pi}{4}-0\right]-\frac{1}{4}\left[\sin 2\frac{\pi}{4}-\sin 2.0\right]+\frac{1}{8}\left[\frac{\pi}{4}-0\right]+\frac{1}{32}\left[\sin 4.\frac{\pi}{4}-\sin 4.0\right]$$

$$=\frac{1}{4}\left[\frac{\pi}{4}\right]-\frac{1}{4}\left[\sin\frac{\pi}{2}-0\right]+\frac{1}{8}\left[\frac{\pi}{4}\right]+\frac{1}{32}[\sin\pi-0]$$

$$= \frac{1}{4}\frac{\pi}{4} - \frac{1}{4}[1] + \frac{1}{8}\frac{\pi}{4} + \frac{1}{32}[\sin\pi] \qquad [\sin\pi = 0; \quad \sin\frac{\pi}{2} = 1]$$

$$= \frac{\pi}{16} - \frac{1}{4} + \frac{\pi}{32} + \frac{1}{32}.0$$

$$= \frac{\pi}{16} - \frac{1}{4} + \frac{\pi}{32} = \frac{2\pi - 8 + \pi}{32} = \frac{3\pi - 8}{32}$$ ***(Proved)***