Successive Differentiation

If $y=f(x)$ be any function then
It's 1st order differentiate can be express as,
$Dy, \frac{dy}{dx}, y_1, y', f'(x)$

2nd order derivative can be express as,
$D^2y, \frac{d^2y}{dx^2}, y_2, y'', f''(x)$

3rd order derivative can be express as,
$D^3y, \frac{d^3y}{dx^3}, y_3, y''', f'''(x)$

Similarly the nth order derivative can be express as,
$D^ny, \frac{d^ny}{dx^n}, y_n, y^n, f^n(x)$

Q. Find the nth derivative of the function, $y=x^n$.

Solution: let,
$$y=f(x)=x^n.$$

Now differentiating successevely w.r.to x
We have,
$$y_1 = nx^{n-1}$$
$$y_2 = \cancel{nx^n}\, n(n-1)x^{n-2}$$
$$y_3 = n(n-1)(n-2)x^{n-3}$$
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Similarly the nth derivatives is given by,

$$y_n = n(n-1)(n-2)\cdots 3.2.1.\, x^{n-n}$$

$$= n(n-1)(n-2)\cdots 3.2.1.\, x^0$$

$$= n(n-1)(n-2)\cdots 3.2.1$$

$$= \lfloor n \,.$$ Answer.

Home Task:

1. $y = \sin(ax+b)$.
2. $y = \cos(ax+b)$.
3. ~~$y = n^r[1+(-1)^r \sin 2nx]^{1/2}$.~~
4. $y = \ln x$.
5. $y = \cos x$.
3. If $y = \sin nx + \cos nx$ then show that,

$$y_r = n^r\left[1+(-1)^r \sin 2nx\right]^{1/2}.$$

Leibnitz Theorem

<u>Statement:</u> If u and v are the two functions of x then the nth. derivative of the product of u and v is given by,

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + {}^nC_3 u_{n-3} v_3 + \cdots$$
$$\cdots + {}^nC_r u_{n-r} v_r + \cdots + u v_n$$

Where the suffixes of u and v are indicates the order of differentiation with respect to x. Which is called the Leibnitz theorem.

Q. If $x = \sin\left(\frac{1}{m}\ln y\right)$ then show that,

$$(1+x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$$

<u>Solution:</u> Given that,

$$x = \sin\left(\frac{1}{m}\ln y\right)$$

or, $\sin^{-1}x = \frac{1}{m}\ln y$

or, $m\sin^{-1}x = \ln y$

or, $e^{m\sin^{-1}x} = e^{\ln y}$

or, $e^{m\sin^{-1}x} = y \quad [\because e^{\ln x} = x]$

$$\therefore \; y = e^{m\sin^{-1}x} \longrightarrow (1)$$

Now diff. w.r.to x, We get,

$$y_1 = e^{m\sin^{-1}x} \cdot m \cdot \frac{1}{\sqrt{1-x^2}}.$$

$$= \frac{my}{\sqrt{1-x^2}} \quad [\text{by } (1)]$$

$$\text{or, } y_1^2 = \frac{m^2y^2}{1-x^2} \quad [\text{by squring}]$$

$$\text{or, } (1-x^2)\,y_1^2 = m^2y^2$$

Again, diff. w.r.to x We get,

$$(1-x^2)\cdot 2y_1\cdot y_2 + y_1^2(0-2x) = m^2\cdot 2y\cdot y_1$$

$$\text{or, } (1-x^2)\,2y_1y_2 - 2xy_1^2 = m^2\,2yy_1$$

$$\text{or, } (1-x^2)\,y_2 - xy_1 = m^2y \quad [\text{dividing by } 2y_1]$$

$$\text{or, } (1-x^2)\,y_2 - xy_1 - m^2y = 0 \longrightarrow (2)$$

~~Now diff. n times~~

Now, differentiating n times by Leibnitz theorem

We have, $(1-x^2)\,y_{n+2} + {}^nC_1(-2x)\cdot y_{n+1} + {}^nC_2(-2)\cdot y_n$

$$- [xy_{n+1} + {}^nC_1\cdot 1\cdot y_n] - m^2y_n = 0$$

or, $(1-x^2)y_{n+2} + n(-2x)\cdot y_{n+1} + \frac{n(n-1)}{2}(-2)\cdot y_n$

$- xy_{n+1} - ny_n - m^2 y_n = 0$

or, $(1-x^2)y_{n+2} - 2nxy_{n+1} \cancel{-(n^2-n)y_n}$

$- n^2 y_n + ny_n - xy_{n+1} - ny_n - m^2 y_n = 0$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

(Shown).

Home Task:

1. $\longrightarrow$ 3(a) $\longrightarrow$ P313.

2. $\longrightarrow$ 5(b) $\longrightarrow$ P319.

3. $\longrightarrow$ 6(b) $\longrightarrow$ P320.

4. $\longrightarrow$ 8(a) $\longrightarrow$ P322.