Application to Differential Equations

Q-103: Solve the following Initial Value Problem (IVP) by Laplace Transform: $Y'' + Y = t$, $Y(0) = 1$, $Y'(0) = -2$

Solution

Let $Y = f(t)$

$Y' = f'(t)$

$Y'' = f''(t)$

Given,

$Y'' + Y = t$

That is $\frac{d^2Y}{dt^2} + Y = t$

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$Y'' + Y = t$

$L\{Y''\} + L\{Y\} = L\{t\}$

We have,

$L(f''(t)) = s^2 L\{f(t)\} - s\,f(0) - f'(0)$

$L\{Y''\} + L\{Y\} = L\{t\}$

$s^2L\{Y\} - s\,f(0) - f'(0) + L\{Y\} = L\{t\}$

$s^2y - s\,f(0) - f'(0) + y = L\{t\}$ $\quad$ $[\text{let}, L\{Y\} = y]$

$s^2y - s\,f(0) - f'(0) + y = \frac{1}{s^2}$ $\quad$ $[\text{let}, L\{t\} = \frac{1}{s^2}]$

$s^2y - s.1 - (-2) + y = \frac{1}{s^2}$ $\quad$ $[\text{Given}, Y(0) = f(0) = 1 \qquad Y'(0) = f'(0) = -2]$

$s^2y + y - s.1 + 2 = \frac{1}{s^2}$

$s^2y + y - s.1 + 2 - \frac{1}{s^2} = 0$

$y(s^2+1) - s + 2 - \frac{1}{s^2} = 0$

$y(s^2+1) = s - 2 + \frac{1}{s^2}$

$y = \frac{s-2}{s^2+1} + \frac{1}{s^2(s^2+1)}$

$y = \frac{s}{s^2+1} - \frac{2}{s^2+1} + \frac{1}{s^2} - \frac{1}{(s^2+1)}$

$y = \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{3}{s^2+1}$

$$\therefore L\{Y\} = y = \frac{s}{s^2+1} + \frac{1}{s^2} - \frac{3}{s^2+1}$$

$$\therefore Y = L^{-1}(y) = L^{-1}[\frac{s}{s^2+1} + \frac{1}{s^2} - \frac{3}{s^2+1}]$$

$$\therefore Y = L^{-1}(y) = L^{-1}[\frac{s}{s^2+1}] + L^{-1}[\frac{1}{s^2}] - 3L^{-1}[\frac{1}{s^2+1}]$$

$\therefore Y = L^{-1}(y) = \cos t + t - 3\sin t$ Answer

Proof:

$$\therefore Y = \cos t + t - 3\sin t$$

$$\therefore Y' = -\sin t + 1 - 3\cos t$$

$$\therefore Y'' = -\cos t + 0 + 3\sin t$$

$$\therefore Y'' + Y = -\cos t + 0 + 3\sin t + \cos t + t - 3\sin t$$

$$\therefore Y'' + Y = t$$

Again,

$$\therefore Y = \cos t + t - 3\sin t$$

$$\therefore Y(0) = \cos 0 + 0 - 3\sin 0$$

$$\therefore Y(0) = 1$$

Again

$$\therefore Y' = -\sin t + 1 - 3\cos t$$

$$\therefore Y'(0) = -\sin 0 + 1 - 3\cos 0$$

$$\therefore Y'(0) = 0 + 1 - 3.1$$

$$\therefore Y'(0) = -2$$

Q-104: Solve the following Initial Value Problem (IVP) by Laplace Transform $Y'' + 4Y = 12t$

$$Y(0) = 0 \qquad Y'(0) = 7$$

Solution

Given,

$$Y'' + 4Y = 12t$$

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$$Y'' + 4Y = 12t$$

$$L\{Y''\} + 4L\{Y\} = 12L\{t\}$$

We have,

$$L(f''(t)) = s^2 L\{f(t)\} - s\,f(0) - f'(0)$$

$$L\{Y''\} + 4L\{Y\} = 12L\{t\}$$

$$s^2 L\{Y\} - s f(0) - f'(0) + 4L\{Y\} = 12L\{t\}$$

$$s^2 y - s f(0) - f'(0) + 4y = 12L\{t\} \qquad [let, L\{Y\} = y]$$

$$s^2y - sf(0) - f'(0) + 4y = 12\frac{1}{s^2}$$ $[\text{let}, L\{t\} = \frac{1}{s^2}]$

$$s^2y - s.0 - 7 + 4y = 12\frac{1}{s^2}$$ $[\text{Given}, Y(0) = 0 \quad Y'(0) = 7]$

$$s^2y - 7 + 4y = \frac{12}{s^2}$$

$$s^2y + 4y - 7 - \frac{12}{s^2} = 0$$

$$y(s^2+4) - 7 - \frac{12}{s^2} = 0$$

$$y(s^2+4) = 7 + \frac{12}{s^2}$$

$$y = \frac{7}{(s^2+4)} + \frac{12}{s^2(s^2+4)}$$

$$y = \frac{7}{(s^2+4)} + 3\times\frac{4}{s^2(s^2+4)}$$

$$y = \frac{7}{(s^2+4)} + 3\times[\frac{1}{s^2} - \frac{1}{(s^2+4)}]$$

$$y = \frac{7}{(s^2+4)} + [\frac{3}{s^2} - \frac{3}{(s^2+4)}]$$

$$\therefore L\{Y\} = y = \frac{7}{(s^2+4)} + [\frac{3}{s^2} - \frac{3}{(s^2+4)}]$$

$$\therefore Y = L^{-1}(y) = L^{-1}[\frac{7}{(s^2+4)} + [\frac{3}{s^2} - \frac{3}{(s^2+4)}]]$$

$$\therefore Y = L^{-1}(y) = L^{-1}[\frac{7}{(s^2+4)}] + L^{-1}[\frac{3}{s^2} - \frac{3}{(s^2+4)}]$$

$$\therefore Y = L^{-1}(y) = 7L^{-1}[\frac{1}{(s^2+4)}] + 3L^{-1}[\frac{1}{s^2} - \frac{1}{(s^2+4)}]$$

$$\therefore Y = L^{-1}(y) = 7L^{-1}[\frac{1}{(s^2+4)}] + 3L^{-1}[\frac{1}{s^2}] - 3L^{-1}[\frac{1}{(s^2+4)}]$$

$$\therefore Y = L^{-1}(y) = 4L^{-1}[\frac{1}{(s^2+4)}] + 3L^{-1}[\frac{1}{s^2}]$$

$$\therefore Y = L^{-1}(y) = 4\frac{1}{2}L^{-1}[\frac{2}{(s^2+4)}] + 3L^{-1}[\frac{1}{s^2}]$$

$$\therefore Y = L^{-1}(y) = 4\frac{1}{2}\sin 2t + 3t$$

$$\therefore Y = L^{-1}(y) = 2\sin 2t + 3t$$

Q-105: Solve the following Initial Value Problem (IVP) by Laplace Transform $Y'' - 3Y' + 2Y = 4e^{2t}$

$$Y(0) = -3 \qquad Y'(0) = 5$$

Solution

$$Y = f(t)$$

Given,

$$Y'' - 3Y' + 2Y = 4e^{2t}$$

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$$Y'' - 3Y' + 2Y = 4e^{2t}$$

$$L\{Y''\} - 3L\{Y'\} + 2L\{Y\} = 4L\{e^{2t}\}$$

We have,

$$L(f''(t)) = s^2 L\{f(t)\} - s\,f(0) - f'(0)$$

And

$$\therefore L(f'(t)) = sL\{f(t)\} - f(0)$$

$$\therefore L\{Y''\} - 3L\{Y'\} + 2L\{Y\} = 4L\{e^{2t}\}$$

$$s^2L\{f(t)\} - s f(0) - f'(0) - 3[sL\{f(t)\} - f(0)] + 2y = 4\frac{1}{s-2} \quad [\text{let}, L\{Y\} = y \;\&\; L(e^{at}) = \frac{1}{s-a}]$$

$$s^2L\{Y\} - s f(0) - f'(0) - 3[sL\{Y\} - f(0)] + 2y = 4\frac{1}{s-2} \quad [Y = f(t)]$$

$$s^2y - s f(0) - f'(0) - 3[sy - f(0)] + 2y = 4\frac{1}{s-2} \quad [\text{let}, L\{Y\} = y]$$

$$s^2y - s f(0) - f'(0) - 3sy + 3f(0) + 2y = 4\frac{1}{s-2}$$

$$s^2y - s\{-3\} - 5 - 3sy + 3(-3) + 2y = 4\frac{1}{s-2}$$

$$s^2y + 3s - 5 - 3sy - 9 + 2y = 4\frac{1}{s-2}$$

$$s^2y + 3s - 3sy - 14 + 2y = 4\frac{1}{s-2}$$

$$s^2y - 3sy + 2y + 3s - 14 = 4\frac{1}{s-2}$$

$$s^2y - 3sy + 2y = -3s + 14 + 4\frac{1}{s-2}$$

$$y(s^2 - 3s + 2) = -3s + 14 + 4\frac{1}{s-2}$$

$$y(s^2 - 2s - s + 2) = -3s + 14 + 4\frac{1}{s-2}$$

$$y\{s(s-2) - 1(s-2) = -3s + 14 + 4\frac{1}{s-2}$$

$$y(s-1)(s-2)=-3s+14+4\frac{1}{s-2}$$

$$y=-3\frac{s}{(s-1)(s-2)}+14\frac{1}{(s-1)(s-2)}+4\frac{1}{(s-2)}\frac{1}{(s-1)(s-2)}$$

$$y=\frac{-3s}{(s-1)(s-2)}+\frac{14}{(s-1)(s-2)}+\frac{4}{(s-1)(s-2)^2}$$

$$y=\frac{-3s+14}{(s-1)(s-2)}+\frac{4}{(s-1)(s-2)^2}$$

$$y=\frac{(-3s+14)(s-2)+4}{(s-1)(s-2)^2}$$

$$y=\frac{-3s^2+6s+14s-28+4}{(s-1)(s-2)^2}$$

$$y=\frac{-3s^2+6s+14s-24}{(s-1)(s-2)^2}$$

Applying partial fraction

Let,

$$y=\frac{-3s^2+6s+14s-24}{(s-1)(s-2)^2}=\frac{A}{(s-1)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}$$

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$$y=\frac{-3s^2+6s+14s-24}{(s-1)(s-2)^2}=\frac{-7}{(s-1)}+\frac{4}{(s-2)}+\frac{4}{(s-2)^2}$$

$$\therefore L\{Y\}=y=\frac{-3s^2+6s+14s-24}{(s-1)(s-2)^2}=\frac{-7}{(s-1)}+\frac{4}{(s-2)}+\frac{4}{(s-2)^2}$$

$$\therefore Y=L^{-1}(y)=-7\frac{1}{2}L^{-1}[\frac{1}{(s-1)}]+4L^{-1}[\frac{1}{(s-2)}]+4L^{-1}[\frac{4}{(s-2)^2}]$$

$$\therefore Y=L^{-1}(y)=-7e^{t}+4e^{2t}+4te^{2t}$$

Q-106: Solve the following Initial Value Problem (IVP) by Laplace Transform $Y''+9Y=\cos 2t$

$$Y(0)=1 \qquad Y'(\frac{\pi}{2})=-1$$

Solution

$Y=f(t)$

Given,

$Y''+9Y=\cos 2t$

Taking the Laplace transform of both sides of the differential equation and using the given conditions, we have

$Y''+9Y=\cos 2t$

$$L\{Y''\}+L\{9Y\}=L\{\cos 2t\}$$

We have,

$$L(f''(t))=s^2L\{f(t)\}-s\,f(0)-f'(0)$$

$$\therefore L\{Y''\}+L\{9Y\}=L\{\cos 2t\}$$

$$\therefore L\{Y''\}+9L\{Y\}=L\{\cos 2t\}$$

$$s^2L\{f(t)\}-s\,f(0)-f'(0)+9y=\frac{s}{s^2+4} \qquad [\text{let}, \text{L}\{\text{Y}\}=\text{y}]$$

$$s^2L\{Y\}-s\,f(0)-f'(0)+9y=\frac{s}{s^2+4}$$

$$s^2y-s\,f(0)-f'(0)+9y=\frac{s}{s^2+4} \qquad [\text{let}, \text{L}\{\text{Y}\}=\text{y}]$$

$$s^2y-s.1-c+9y=\frac{s}{s^2+4} \qquad [let, f'(0)=c]$$

$$y(s^2+9)-s.1-c=\frac{s}{s^2+4}$$

$$y(s^2+9)=s.1+c+\frac{s}{s^2+4}$$

$$y=\frac{s}{(s^2+9)}+\frac{c}{(s^2+9)}+\frac{s}{(s^2+4)(s^2+9)}$$

$$y=\frac{s}{(s^2+4)(s^2+9)}+\frac{s}{(s^2+9)}+\frac{c}{(s^2+9)}$$

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Applying partial fraction

$$y=\frac{s}{5(s^2+4)}-\frac{s}{5(s^2+9)}+\frac{s}{(s^2+9)}+\frac{c}{(s^2+9)}$$

$$y=\frac{s}{5(s^2+4)}+\frac{s}{(s^2+9)}-\frac{s}{5(s^2+9)}+\frac{c}{(s^2+9)}$$

$$y=\frac{s}{5(s^2+4)}+\frac{5s-s}{(s^2+9)}+\frac{c}{(s^2+9)}$$

$$y=\frac{s}{5(s^2+4)}+\frac{4s}{5(s^2+9)}+\frac{c}{(s^2+9)}$$

$$\therefore L\{Y\}=y=\frac{s}{5(s^2+4)}+\frac{4s}{5(s^2+9)}+\frac{c}{(s^2+9)}$$

$$\therefore Y=L^{-1}(y)=L^{-1}\{\frac{s}{5(s^2+4)}\}+L^{-1}\{\frac{4s}{5(s^2+9)}\}+L^{-1}\{\frac{c}{(s^2+9)}\}$$

$$\therefore Y=L^{-1}(y)=L^{-1}\{\frac{s}{5(s^2+4)}\}+L^{-1}\{\frac{4s}{5(s^2+9)}\}+L^{-1}\{\frac{c}{(s^2+9)}\}$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}L^{-1}\{\frac{s}{(s^2+4)}\} + \frac{4}{5}L^{-1}\{\frac{s}{(s^2+9)}\} + \frac{c}{3}L^{-1}\{\frac{3}{(s^2+9)}\}$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}L^{-1}\{\frac{s}{(s^2+2^2)}\} + \frac{4}{5}L^{-1}\{\frac{s}{(s^2+3^2)}\} + \frac{c}{3}L^{-1}\{\frac{3}{(s^2+3^2)}\}$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t \text{ ....}(i)$$

Given,

$$\therefore Y(\frac{\pi}{2}) = -1$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t$$

$$\therefore Y(\frac{\pi}{2}) = \frac{1}{5}\cos 2(\frac{\pi}{2}) + \frac{4}{5}\cos 3(\frac{\pi}{2}) + \frac{c}{3}\sin 3(\frac{\pi}{2})$$

$$\therefore Y(\frac{\pi}{2}) = \frac{1}{5}\cos \pi + \frac{4}{5}\cos 3(\frac{\pi}{2}) + \frac{c}{3}\sin(\frac{3\pi}{2})$$

$$\therefore Y(\frac{\pi}{2}) = \frac{1}{5}(-1) + \frac{4}{5}\cos(\frac{3\pi}{2}) + \frac{c}{3}\sin(\frac{3\pi}{2})$$

$$\therefore Y(\frac{\pi}{2}) = \frac{1}{5}(-1) + \frac{4}{5}\times 0 + \frac{c}{3}(-1)$$

$$\therefore Y(\frac{\pi}{2}) = -\frac{1}{5} - \frac{c}{3}$$

$$-1 = \frac{1}{5}(-1) + \frac{c}{3}(-1) \qquad [\because Y(\frac{\pi}{2}) = -1]$$

$$-1 = -\frac{1}{5} - \frac{c}{3}$$

$$\frac{c}{3} = -\frac{1}{15} + 1$$

$$\frac{c}{3} = \frac{-1+15}{15}$$

$$\frac{c}{3} = \frac{14}{15}$$

$$c = \frac{14}{5}$$

Putting the value of c in (i)

$$\therefore Y = L^{-1}(y) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t \text{ ....}(i)$$

$$\therefore Y = L^{-1}(y) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{14}{15}\sin 3t$$

Answer