CHANGE OF AXES 4

Art. 31. Transformation of Co-ordinates.

The co-ordinates of a point or the equation of a curve are alway given with reference to a fixed origin and a set of axes of co-ordinates. The above co-ordinates of the equation of the curve changes when the origin is changed or the direction of axes changed or both. The process of changing the co-ordinate of a point or the equation of a curve is called **transformation of co-ordinates.** Now we have to investigate the mode of the change of the co-ordinates or the equation of the curve according to the transfer from one set to another set of co-ordinate axes.

Art. 32. Change of origin (Translation of axes)

To find the change in the co-ordinates of a point when the origin is shifted to another point $O'(\alpha, \beta)$ *where the direction of axes remains unalterad.*

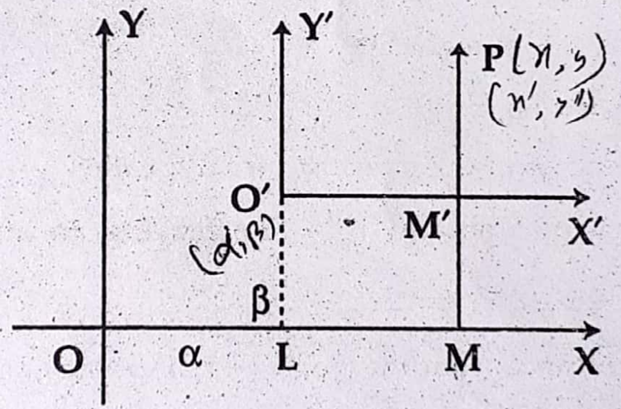


Fig. 10

Let us take a new pair of axes O′ X′ and O′Y′ parallel to the old pair OX and OY; O′ being the new origin whose co-ordinate are (α, β) referred to O′X′ and O′Y′.

let (x', y') be the co-ordinates referred to the axes O′X′ and O′Y′ or a point P, whose co-ordinate referred to the old axes are (x, y).

It is required to transform the co-ordinates (x, y) in terms or (x', y').

From O′ and P draw O′L and PM perpendiculars to OX. Let PM meet O′X′ in M′.

Then $OL = \alpha, LO' = \beta$, $OM = x$, $MP = y$

Also $O'M' = x'$ and $M'P = y'$

Therefore, $OM = OL + LM = OL + O'M'$

$\therefore x = \alpha + x'$ ... ... (i)

Similarly, $MP = MM' + M'P = LO' + M'P$

$\therefore\ y = \beta + y'$ ... ... (ii)

The transformed co-ordinates are

$$\left.\begin{aligned} x' &= x - \alpha \\ \text{and} \quad y' &= y - \beta \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

Rule : *In order to shift the origin to* (α, β) *the transformation is obtained by replacing* x *by* $x + \alpha$ *and* y *by* $y + \beta$. *If from the transformed equation we want to get the old equations then replace* x *by* $x - \alpha$ *and* y *by* $y - \beta$. *This is known as shifting the origin back.*

Art. 33. Rotation of axes (origin fixed)

To find the change in the co-ordinates of a point when the direction of axes is turned through an angle θ *where as the origin of co-ordinates remains the same.*

Let OX and OY be the old axes and OX′ and OY′ set the new axes. O is the common origin for the two sets of axes. Let the angle X′ OX through which the axes have rotated be represented by θ.

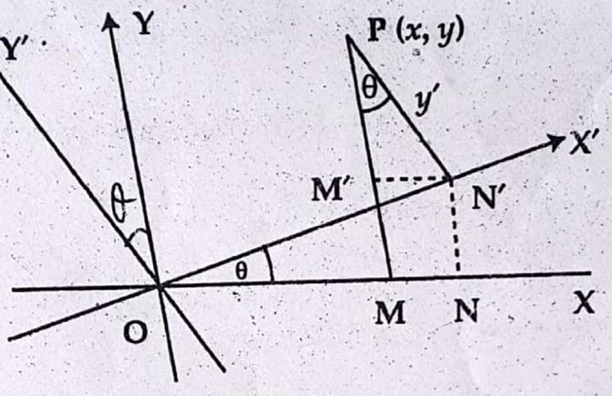


Fig. 11

Let P be any point in the plane and let its co-ordinates referred to the old axes be (x, y), and referred to the new axes be (x', y').

Let us try to determine x and y in terms of x', y' and θ, Draw PM perpendicular to OX, PN′ perpendicular to OX′, and N′N perpendicular to OX, and N′M′ parallel to OX.

$$\begin{aligned} \text{Then} \quad x &= OM = ON - MN = ON - M'N' \\ &= ON' \cos\theta - PN' \sin\theta = x' \cos\theta - y' \sin\theta \\ y = MP &= MM' + M'P = NN' + M'P \\ &= ON' \sin\theta + PN' \cos\theta = x' \sin\theta + y' \cos\theta \end{aligned}$$

Hence the formula for the rotation of the axes through an angle θ are.

$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ \text{and} \quad y &= x' \sin\theta + y' \cos\theta \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3)$$

Art. 34. Change of origin with the change of the direction of axes.

It is really the combination of Art. 32 and Art. 33. The best method is to apply Art. 32 first and then Art. 33. Of course the two transformations may also be made simultaneously.

Let us suppose that the system of axes be rectangular. The origin is shifted to the point (α, β) and then the axes are rotated through an angle θ. If the co-ordinates of any point be (x, y) in the old system, and (x', y') in the new system, from Art. 32, and Art. 33.

Change of axes

~~Ex-1~~. Determine the equation of the parabola

$$x^2 - 2xy + y^2 + 2x - 4y + 3 = 0 \quad \cdots (1)$$ after rotating of axes through $45°$.

Sol^n. When the axes have been rotated through an angle $45°$, then

$$x = x'\cos 45° - y'\sin 45° = (x' - y')/\sqrt{2}$$

$$y = x'\sin 45° + y'\cos 45° = (x' + y')/\sqrt{2}$$

Put them in (1), then

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + \left(\frac{x'+y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x'-y'}{\sqrt{2}}\right) - 4\left(\frac{x'+y'}{\sqrt{2}}\right) + 3 = 0$$

or, $$\frac{x'^2 - 2x'y' + y'^2}{2} - (x'^2 - y'^2) + \frac{x'^2 + 2x'y' + y'^2}{2} + \frac{2}{\sqrt{2}}(x' - y') - \frac{4}{\sqrt{2}}(x' + y') + 3 = 0$$

or, $$x'^2 - 2x'y' + y'^2 - 2x'^2 + 2y'^2 + x'^2 + 2x'y' + y'^2 + 2\sqrt{2}(x' - y') - 4\sqrt{2}(x' + y') + 6 = 0$$

or, $4y'^2 \cancel{+ \sqrt{2}x' - \sqrt{2}y' - 2\sqrt{2}x' - 2\sqrt{2}y' + 3 = 0}$

or, $$4y'^2 + 2\sqrt{2}(x' - y') - 4\sqrt{2}(x' + y') + 6 = 0$$

or, $$2y'^2 + \sqrt{2}x' - \sqrt{2}y' - 2\sqrt{2}x' - 2\sqrt{2}y' + 3 = 0$$

or, $$2y'^2 - \sqrt{2}x' - 3\sqrt{2}y' + 3 = 0$$

Now dropping the suffixes, the equation is

$$2y^2 - \sqrt{2}x - 3\sqrt{2}y + 3 = 0$$ Ans

Ex-2 Determine the eq^n of the curve $2x^2+3y^2-8x+6y-7=0$ when origin is transferred to the point $(2,-1)$

Ex-3 Remove the first degree terms in $3x^2+4y^2-12x+4y+13=0$

Sol^n. Given eq^n $3x^2+4y^2-12x+4y+13=0$

or, $3(x^2-4x+4)+4(y^2+y+1/4)=0$

or, $3(x-2)^2+4(y+\frac{1}{2})^2=0$ ——— (ii)

Put, $x-2=x'$ and $y+\frac{1}{2}=y'$ in (ii)

$\therefore\ 3x'^2+4y'^2=0$

This is satisfied only by $x'=0,\ y'=0$, which is the new origin.

By shifting origin at $(2,-\frac{1}{2})$, the 1st degree terms can be removed.

Ex-04 By transforming to parallel axes through a proper chosen point (h,k), prove that the equation

$12x^2-10xy+2y^2+11x-5y+2=0$ can be reduced to on containing only the terms of the 2nd degree.

Sol^n. Transforming to parallel axes through (h,k) we ha

$12(x'+h)^2-10(x'+h)(y'+k)+2(y'+k)^2+11(x'+h)-5(y'+k)+2=0$

or, $12x'^2-10x'y'+2y'^2+(24h-10k+11)x'+(-10h+4k-5)y'$
$+(12h^2-10hk+2k^2+11h-5k+2)=0$ - (i)

Now, equate the co-efficients of x', y' to zero then —

$24h - 10k + 11 = 0$ and $-10h + 4k - 5 = 0$

and $12h^2 - 10hk + 2k^2 + 11h - 5k + 2 = 0$ —(ii)

Solve the 1st two equations $h = -3/2$, $k = -5/2$ and it clear that these values of h, k satisfy the eq^n (ii)

Hence the equation (i) becomes

$$12x'^2 - 10x'y' + 2y'^2 = 0$$

Ex-5 Transform to parallel axes through the new origin $(3, 1)$ of the equation. $x^2 + 2y^2 - 6x + 7 = 0$

Sol^n. If we transfer the origin to the point $(3, 1)$ then the co-ordinates

$$x = x' + 3 \quad , \quad y = y' + 1$$

The transformed eq^n will be —

$$(x'+3)^2 + 2(y'+1)^2 - 6(x'+3) + 7 = 0$$

or, $x'^2 + 2y'^2 + 4y' = 0$

Hence the required transformed eq^n is $x^2 + 2y^2 + 4y = 0$

Ex-6 Transform to axes inclined at $45°$ to the original axes the equation $17x^2 - 16xy + 17y^2 = 225$.

Sol^n. If we rotate the axes by an angle $45°$ without changing the origin then ~~the co-ordinates of a variable~~

$$x = x' \cos 45° - y' \sin 45°$$

$$= \frac{x'}{\sqrt{2}} - \frac{y'}{\sqrt{2}} = \frac{1}{\sqrt{2}}(x' - y')$$

$y = x'\sin\theta + y'\cos\theta = x'\sin 45^\circ + y'\cos 45^\circ = x'/\sqrt{2} + y'/\sqrt{2}$

$= \frac{1}{\sqrt{2}}(x'+y')$

The transformed eq^n is

$17\left(\frac{x'-y'}{\sqrt{2}}\right)^2 - 16\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + 17\left(\frac{x'+y'}{\sqrt{2}}\right)^2 = 225$

or, $17(x'-y')^2 - 16(x'-y')(x'+y') + 17(x'+y')^2 = 450$

or, $18x'^2 + 50y'^2 = 450$

or, $9x'^2 + 25y'^2 = 225$

Hence, the required eq^n is $9x^2 + 25y^2 = 225$.

Ex-7 Transform the eq^n $14x^2 - 4xy + 11y^2 - 36x + 48y + 41 = 0$ to rectangular axes through the point $(1,-2)$ inclined at an angle $\tan^{-1}(-\frac{1}{2})$ to the original axes.

Sol^n Let $P(x,y)$ be a moving point on the curve $14x^2 - \dots\dots + 41 = 0$. If we transfer the origin at $(1,-2)$, then co-ordinate will be (x',y')

$\therefore\ x = x'+1$ and $y = y'-2$

The transformed eq^n will be

$14(x'+1)^2 - 4(x'+1)(y'-2) + 11(y'-2)^2 - 36(x'+1) + 48(y'-2) + 41 = 0$

or, $14x'^2 - 4x'y' + 11y'^2 = 25$

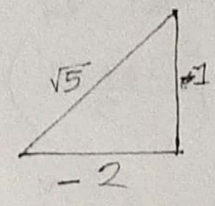
Hence the locus of (x',y') is

$14x^2 - 4xy + 11y^2 = 25$

If the axes turn through an angle $\theta = \tan^{-1}(-\frac{1}{2})$

or, $\tan\theta = -\frac{1}{2}$

$\therefore$ Sin $\theta = 1/\sqrt{5}$ and $\cos\theta = -2/\sqrt{5}$



$\therefore$ $x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{5}}(2x'+y')$

$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{5}}(2y'-x')$

The eqn. becomes

$$14\left(\frac{2x'+y'}{\sqrt{5}}\right)^2 - 4\cdot\frac{1}{\sqrt{5}}(2x'+y')\cdot\frac{1}{\sqrt{5}}(2y'-x') + 11\cdot\cancel{\frac{1}{\sqrt{5}}}\left(\frac{2y'-x'}{\sqrt{5}}\right)^2 = 25$$

or, $14(4x'^2+4x'y'+y'^2) - 4(-2x'^2+3x'y'+2y'^2) + 11(x'^2-4x'y'+4y'^2) - 125 = 0$

or, $3x'^2+2y'^2-5=0$

Then the required transformed eqn. $3x^2+2y^2-5=0$

Ex-8 Transform the eqn. $11x^2+24xy+4y^2-20x-40y-5=0$ to rectangular axes through the point $(2,-1)$ inclined at an angle $\tan^{-1}(-4/3)$ to the original axes.

Soln.:- If change the origin to $(2,-1)$, the given eqn. is transformed to

$$11(x'+2)^2+24(x'+2)(y'-1)+4(y'-1)^2-20(x'+2)-40(y'-1)-5=0$$

or, $11x'^2+24x'y'+4y'^2=5$

The locus of (x',y') is $11x^2+24xy+4y^2=5$

If the axes turn through an angle $\theta = \tan^{-1}(-4/3)$

$\therefore$ $\cos\theta = 3/5$, $\sin\theta = -4/5$ $\qquad \tan\theta = -4/3$

$x = x'\cos\theta - y'\sin\theta$ $\qquad$, $y = x'\sin\theta + y'\cos\theta$

$= \frac{1}{5}(3x'+4y')$ $\qquad$ $= \frac{1}{5}(\cancel{0}\ 3y'-4x')$

So, the eq^n becomes :—

$$11\left(\frac{3x'+4y'}{5}\right)^2+24\left(\frac{3x'+4y'}{5}\right)\left(\frac{3y'-4x'}{5}\right)+4\left(\frac{3y'-4x'}{5}\right)^2=5$$

or, $x'^2-4y'^2+1=0$

$\therefore$ the required transformed eq^n $x^2-4y^2+1=0$

Ex-9 The equation $3x^2+2xy+3y^2-18x-22y+50=0$ is transformed to $4x^2+2y^2=1$ when referred to rectangular axes through the point $(2,3)$. Find the inclination of the latter axes to the former.

Sol^n: Given eq^n: $3x^2+ \quad - \quad - \quad - \quad +50=0$ —(i)

If we change the origin to $(2,3)$ then the given eq^n is transformed ~~to~~ to

$$3(x'+2)^2+2(x'+2)(y'+3)+3(y'+3)^2-18(x'+2)-22(y'+3)+50=0$$

or, $3x'^2+3y'^2+2x'y'=1$

$\therefore$ The locus of (x',y') is $3x^2+3y^2+2xy=1$ —(ii)

If we rotate eq^n (ii) to the axes by an angle θ then Co-ordinates

$$3(x'\cos\theta-y'\sin\theta)^2+3(x'\sin\theta+y'\cos\theta)^2+2(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)=1$$

or, $3(\cos^2\theta+\sin^2\theta)x'^2+3(\sin^2\theta+\cos^2\theta)y'^2+2(x'^2\sin\theta\cos\theta$
$+x'y'\cos^2\theta-x'y'\sin^2\theta-y'^2\sin\theta\cos\theta)=1$

or, $3(\cos^2\theta$ ~~$+\sin^2\theta$~~

equating co-efficient of $x'y'$, $\left[-6\cos\theta\sin\theta + 6\sin\theta\cos\theta + 2\cos^2\theta - 2\sin^2\theta\right] = 0$

$$2\left(\cos^2\theta - \sin^2\theta\right) = 0$$

or, $\cos 2\theta = 0 = \cos \pi/2$

or, $2\theta = \pi/2$

or, $\theta = \pi/4 = 45^\circ$

$$3\left(x'\cos 45^\circ - y'\sin 45^\circ\right)^2 + 3\left(x'\sin 45^\circ + y'\cos 45^\circ\right)^2 \cdots = 1$$

or, $\frac{3}{2}\left(x'-y'\right)^2 + \frac{3}{2}\left(x'+y'\right)^2 + 2(x+y)(x-y) = 1$

or, $3\left(x'-y'\right)^2 + 3\left(x'+y'\right)^2 + 2(x+y)(x-y) = 2$

or, $8x'^2 + 4y'^2 = 2$

or, $4x'^2 + 2y'^2 = 1$

$\therefore$ The locus of the point (x', y') is $4x^2 + 2y^2 = 1$.