Example 198: Find the volume and surface of the solid of revolution of the curve $\mathbf{r = a(1-\cos\theta)}$ about the initial line.

Solution: Given Equation is $\mathbf{r = a(1-\cos\theta)}$--------------------------(i)

If we replace $-\theta$ for θ in equation (i) then it is unchanged hence the curve is symmetrical about the initial line.

Now Putting $\mathbf{r = 0}$ in (i)

$$\mathbf{r = a(1-\cos\theta)}$$
$$\Rightarrow \mathbf{0 = a(1-\cos\theta)}$$
$$\Rightarrow \mathbf{0 = a - a\cos\theta}$$
$$\Rightarrow \mathbf{-a = -a\cos\theta}$$
$$\Rightarrow \mathbf{1 = \cos\theta}$$
$$\Rightarrow \mathbf{\cos 0 = \cos\theta}$$
$$\Rightarrow \mathbf{0 = \theta}$$
$$\Rightarrow \mathbf{\theta = 0}$$

Now Putting $\mathbf{r = 2a}$ in (i)

$$\mathbf{r = a(1-\cos\theta)}$$
$$\Rightarrow \mathbf{2a = a(1-\cos\theta)}$$
$$\Rightarrow \mathbf{2a = a - a\cos\theta}$$
$$\Rightarrow \mathbf{2a - a = -a\cos\theta}$$
$$\Rightarrow \mathbf{a = -a\cos\theta}$$
$$\Rightarrow \mathbf{a = -a\cos\theta}$$
$$\Rightarrow \mathbf{1 = -\cos\theta}$$
$$\Rightarrow \mathbf{-1 = \cos\theta}$$
$$\Rightarrow \mathbf{\cos\pi = \cos\theta}$$
$$\Rightarrow \mathbf{\pi = \theta}$$
$$\Rightarrow \mathbf{\theta = \pi}$$

Therefore, the required volume is:

$$\mathbf{v = \frac{2}{3}\pi\int_0^\pi r^3 \sin\theta\, d\theta}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\pi\int_0^\pi \{a(1-\cos\theta)\}^3 \sin\theta\, d\theta} \qquad \mathbf{[\because r = a(1-\cos\theta)]}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\pi\int_0^\pi a^3(1-\cos\theta)^3 \sin\theta\, d\theta}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\pi a^3\int_0^\pi \left\{2\sin^2\frac{\theta}{2}\right\}^3 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta} \qquad \mathbf{[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2};\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}]}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\pi a^3\int_0^\pi 2^3\left\{\sin^2\frac{\theta}{2}\right\}^3 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\pi a^3\int_0^\pi 8\sin^6\frac{\theta}{2} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\times 8\pi a^3\int_0^\pi \sin^6\frac{\theta}{2} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\times 8\times 2\pi a^3\int_0^\pi \sin^6\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\times 8\times 2\pi a^3\int_0^\pi \sin^{6+1}\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}$$

$$\Rightarrow \mathbf{v = \frac{2}{3}\times 8\times 2\pi a^3\int_0^\pi \sin^7\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta}$$

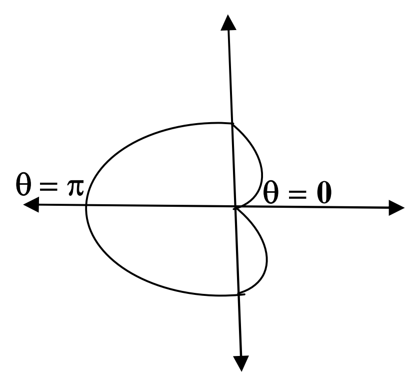

Figure # 100

$$\Rightarrow v=\frac{32}{3}\pi a^3\int_0^{\pi}\sin^7\frac{\theta}{2}\cos\frac{\theta}{2}d\theta \text{ ------------------------------(ii)}$$

Putting $\frac{\theta}{2}=t$

θ	0	π
t	$\frac{\theta}{2}=t$ $\frac{0}{2}=t$ $0=t$ $t=0$	$\frac{\theta}{2}=t$ $\frac{\pi}{2}=t$ $t=\frac{\pi}{2}$

$$\frac{d}{d\theta}\left(\frac{\theta}{2}\right)=\frac{d}{d\theta}(t)$$

$$\Rightarrow \frac{1}{2}=\frac{dt}{d\theta}$$

$$\Rightarrow 2dt=d\theta$$

$$\Rightarrow d\theta=2dt$$

From (ii),

$$\Rightarrow v=\frac{32}{3}\pi a^3\int_0^{\pi}\sin^7\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$\Rightarrow v=\frac{32}{3}\pi a^3\int_0^{\frac{\pi}{2}}\sin^7 t\cos t\times 2\,dt \qquad [\because \frac{\theta}{2}=t]$$

$$\Rightarrow v=\frac{64}{3}\pi a^3\int_0^{\frac{\pi}{2}}\sin^7 t\cos t\,dt$$

$$\Rightarrow v=\frac{64}{3}\pi a^3\int_0^{\frac{\pi}{2}}\sin^7 t\cos^1 t\,dt \text{ ------------------------------(iii)}$$

We have,

$$\beta(m,n)=2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta=\frac{\Gamma m\Gamma n}{\Gamma(m+n)} \qquad [\because \Gamma n=(n-1)!]$$

Here, from (iii)

$2m-1=7$ & $2n-1=1$

$\Rightarrow 2m=7+1$ $\qquad \Rightarrow 2n=1+1$

$\Rightarrow 2m=8$ $\qquad \Rightarrow 2n=2$

$\Rightarrow m=4$ $\qquad \Rightarrow n=1$

Hence From (iii)

$$v=\frac{64}{3}\pi a^3\int_0^{\frac{\pi}{2}}\sin^7 t\cos^1 t\,dt$$

$$\Rightarrow v=\frac{32}{3}\pi a^3\times 2\int_0^{\frac{\pi}{2}}\sin^7 t\cos^1 t\,dt$$

$$\Rightarrow v=\frac{32}{3}\pi a^3\times\beta(m,n) \qquad [\because \beta(m,n)=2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}d\theta]$$

$$\Rightarrow v = \frac{32}{3}\pi a^3 \times \beta(4,1)$$

$$\Rightarrow v = \frac{32}{3}\pi a^3 \times \frac{\overline{)4}\overline{)1}}{\overline{)4+1}} \qquad [\because \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}]$$

$$\Rightarrow v = \frac{32}{3}\pi a^3 \times \frac{(4-1)!\overline{)1}}{\overline{)5}} \qquad [\because \Gamma n = (n-1)!]$$

$$\Rightarrow v = \frac{32}{3}\pi a^3 \times \frac{(3)!\overline{)1}}{(5-1)!} = \frac{32}{3}\pi a^3 \times \frac{3.2.1.1}{(4)!} \qquad [\Gamma(1) = 1]$$

$$\Rightarrow v = \frac{32}{3}\pi a^3 \times \frac{3.2.1.1}{4.3.2.1} = \frac{32}{3}\pi a^3 \times \frac{1}{4} = \frac{8}{3}\pi a^3$$ Answer

And, the required surface of the curve is,

$$S = 2\pi\int_0^\pi y\, ds \text{ --------------------(ii)}$$

We know, $ds = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta$

Also, $\frac{dr}{d\theta} = a\sin\theta$ and $y = r\sin\theta$

From (ii), $S = 2\pi\int_0^\pi y\, ds$

$$\Rightarrow S = 2\pi\int_0^\pi y\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\pi\int_0^\pi r\sin\theta\{r^2 + a^2\sin^2\theta\}^{\frac{1}{2}} d\theta \qquad [\because \frac{dr}{d\theta} = a\sin\theta]$$

$$= 2\pi\int_0^\pi a(1-\cos\theta)\sin\theta\{a^2(1-\cos\theta)^2 + a^2\sin^2\theta\}^{\frac{1}{2}} d\theta \qquad [\because \text{Given}, r = a(1-\cos\theta)]$$

$$= 2\pi\int_0^\pi a(1-\cos\theta)\sin\theta \times [a^2\{(1-\cos\theta)^2 + \sin^2\theta\}]^{\frac{1}{2}} d\theta$$

$$= 2\pi\int_0^\pi a(1-\cos\theta)\sin\theta \times a[\{(1-\cos\theta)^2 + \sin^2\theta\}]^{\frac{1}{2}} d\theta$$

$$= 2\pi\int_0^\pi a^2(1-\cos\theta)\sin\theta[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\}]^{\frac{1}{2}} d\theta$$

$$= 2\pi\int_0^\pi a^2(1-\cos\theta)\sin\theta[1 - 2\cos\theta + 1\}]^{\frac{1}{2}} d\theta \qquad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$= 2\pi a^2 \int_0^{\pi} (1-\cos\theta)\sin\theta \{2 - 2\cos\theta\}^{\frac{1}{2}} d\theta$$

$$= 2\pi a^2 \int_0^{\pi} 2^{\frac{1}{2}} (1-\cos\theta)^{\frac{1}{2}+1} \sin\theta d\theta \ = 2\sqrt{2}\,\pi a^2 \int_0^{\pi} (1-\cos\theta)^{\frac{3}{2}} \sin\theta\, d\theta$$

$$= 2\sqrt{2}\,\pi a^2 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^{\frac{3}{2}} \sin\theta\, d\theta \qquad [\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}]$$

$$= 2\sqrt{2}\,\pi a^2 \int_0^{\pi} 2^{\frac{3}{2}} \sin^3\frac{\theta}{2} 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta \qquad [\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta]$$

$$= 2 \times 2^{\frac{1}{2}} \times 2^{\frac{3}{2}} \times 2\,\pi a^2 \int_0^{\pi} \sin^3\frac{\theta}{2} \sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$$

$$= 2^{1+\frac{1}{2}+\frac{3}{2}+1}\,\pi a^2 \int_0^{\pi} \sin^3\frac{\theta}{2} \sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta \ = 2^4 \pi a^2 \int_0^{\pi} \sin^4\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$$

Putting $\frac{\theta}{2} = t$

$\Rightarrow \theta = 2t$

$\Rightarrow \frac{d}{dt}(\theta) = \frac{d}{dt}(2t)$

$\Rightarrow \frac{d\theta}{dt} = 2.1$

$\Rightarrow d\theta = 2dt$

θ	0	π
t	$\frac{\theta}{2} = t$ $\frac{0}{2} = t$ $0 = t$ $t = 0$	$\frac{\theta}{2} = t$ $\frac{\pi}{2} = t$ $t = \frac{\pi}{2}$

Putting $\frac{\theta}{2} = t$ where, $d\theta = 2dt$ and when $\theta = 0$ then $t = 0$ and also when

$\theta = \pi$ Then $t = \frac{\pi}{2}$ Therefore,

$$\Rightarrow S = 2^4 \pi a^2 \int_0^{\pi} \sin^4\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$$

$$\Rightarrow S = 2^4 \pi a^2 \int_0^{\frac{\pi}{2}} \sin^4 t \cos t\, 2dt$$

$$\Rightarrow S = 2^4 \pi a^2 \times 2\int_0^{\frac{\pi}{2}} \sin^4 t \cos^1 t dt \text{ --------------------------------(i)}$$

We have,

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} [\because \Gamma n = (n-1)!]$$

Here, from (i)

$2m - 1 = 4$ & $2n - 1 = 1$

$\Rightarrow 2m = 4 + 1$ $\qquad\qquad$ $\Rightarrow 2n = 1 + 1$

$\Rightarrow 2m = 5$ $\qquad\qquad$ $\Rightarrow 2n = 2$

$\Rightarrow m = \frac{5}{2}$ $\qquad\qquad$ $\Rightarrow n = 1$

From (i)

$$\Rightarrow S = 2^4 \pi a^2 \times 2\int_0^{\frac{\pi}{2}} \sin^4 t \cos^1 t dt$$

$$\Rightarrow S = 2^4 \pi a^2 \times 2\int_0^{\frac{\pi}{2}} \sin^4 t \cos^1 t dt$$

$$\Rightarrow S = 2^4 \pi a^2 \times \beta(m, n) \qquad [\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta]$$

$$= 16\pi a^2 \times \beta(\frac{5}{2}, 1) \qquad [\because m = \frac{5}{2} \text{ and } n = 1]$$

$$= 16\pi a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(1)}{\Gamma\left(\frac{5}{2}+1\right)} \qquad [\because \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}]$$

$$[\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} [\because \Gamma n = (n-1)!]$$

$$= 16\pi a^2 \frac{\Gamma\left(\frac{5}{2}\right).1}{\Gamma\left(\frac{5+2}{2}\right)} = 16\pi a^2 \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = 16\pi a^2 \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{5}{2}+1\right)} = 16\pi a^2 \frac{\Gamma\left(\frac{5}{2}\right)}{\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}$$

$[\because \Gamma\,(n+1) = n\,\Gamma\,(n)]$

$$= 16\pi a^2 \frac{1}{\frac{5}{2}} = 16\pi a^2 \frac{2}{5} = \frac{32}{5}\pi a^2$$

Therefore the required surface is: $\frac{32}{5}\pi a^2$ ***Answer***

Example 199: Find the volume and surface respectively of solid revolution of the curve $r = a(1 + \cos\theta)$ about the initial line

Solution: Given equation, $\quad r = a(1 + \cos\theta)$ -----------------------------(i)

If we replace $-\theta$ **for** θ in equation (i) then it is unchanged. Hence the curve is symmetrical about the initial line. Now when $r = 0$ then $\theta = \pi$, and also when $r = 2a$ then $\theta = 0$, Draw the curve,

If $r = 0$ **then,** $a(1 + \cos\theta) = 0$

$(1 + \cos\theta) = 0$

$\cos\theta = -1$

$\cos\theta = \cos\pi$

$\theta = \pi$

Again,

If $r = 2a$ then, $a(1+\cos\theta) = 2a$

$a(1+\cos\theta) = 2a$

$(1+\cos\theta) = 2$

$\cos\theta = 2-1$

$\cos\theta = 1$

$\cos\theta = \cos 0$

$\theta = 0$

Therefore the required volume,

$$v = \frac{2}{3}\pi\int_0^\pi r^3 \sin\theta d\theta$$

$$= \frac{2}{3}\pi a^3 \int_0^\pi (1+\cos\theta)^3 \sin\theta d\theta$$

$$= \frac{2}{3}\pi a^3 \int_0^\pi (2\cos^2\frac{\theta}{2})^3 \sin\theta d\theta \qquad [\because 2\cos^2\frac{\theta}{2} = 1+\cos\theta]$$

$$= \frac{2}{3}\pi a^3 \int_0^\pi 2^3 \cos^6\frac{\theta}{2}\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta \qquad [\because \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}]$$

$$= \frac{2^5}{3}\pi a^3 \int_0^\pi \cos^7\frac{\theta}{2}\sin\frac{\theta}{2} d\theta \text{ ------------------------------(ii)}$$

Putting $\frac{\theta}{2} = t$

$\Rightarrow \theta = 2t$

$\Rightarrow \frac{d}{dt}(\theta) = \frac{d}{dt}(2t)$

$\Rightarrow \frac{d\theta}{dt} = 2.1$

$\Rightarrow d\theta = 2dt$

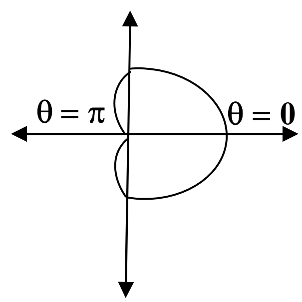

Figure # 101

Putting $\frac{\theta}{2} = t$ where, $d\theta = 2dt$ and when $\theta = 0$ then $t = 0$ and also when $\theta = \pi$ Then $t = \frac{\pi}{2}$ Therefore,

θ	0	π
t	$\frac{\theta}{2} = t$ $\frac{0}{2} = t$ $0 = t$ $t = 0$	$\frac{\theta}{2} = t$ $\frac{\pi}{2} = t$ $t = \frac{\pi}{2}$

From (ii)

$$= \frac{2^5}{3}\pi a^3 \int_0^\pi \cos^7\frac{\theta}{2}\sin\frac{\theta}{2} d\theta$$

$$\Rightarrow v = \frac{2^5}{3}\pi a^3 \int_0^{\pi/2} \cos^7 t \sin t \times 2dt$$

$$\Rightarrow v = \frac{2^6}{3}\pi a^3 \int_0^{\pi/2} \cos^7 t \sin t dt = \frac{2^5}{3}\pi a^3 \times 2\int_0^{\pi/2} \sin t \cos^7 t\, dt$$

$$\Rightarrow v = \frac{2^5}{3}\pi a^3 \times 2\int_0^{\pi/2} \sin^1 t \cos^7 t\, dt \text{ -----------------------------(iii)}$$

We have,

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1} d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} [\because \Gamma n = (n-1)!]$$

Here, from (iii)

$2m-1=1$ & $2n-1=7$
$\Rightarrow 2m=1+1$ $\Rightarrow 2n=7+1$
$\Rightarrow 2m=2$ $\Rightarrow 2n=8$
$\Rightarrow m=1$ $\Rightarrow n=4$

$$\therefore v=\frac{2^5}{3}\pi a^3\times 2\int_0^{\pi/2}\sin^1 t\cos^7 t\,dt$$

$$\therefore v=\frac{2^5}{3}\pi a^3\times\beta(m,n)=\frac{2^5}{3}\pi a^3\times\beta(1,4)$$

$$\therefore v=\frac{2^5}{3}\pi a^3\times\frac{\Gamma 1\Gamma 4}{\Gamma(1+4)} \qquad [\because \beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}]$$

$$\therefore v=\frac{2^5}{3}\pi a^3\times\frac{1.\Gamma 4}{\Gamma(4+1)} \qquad [\because \Gamma\, 1=1]$$

$$\therefore v=\frac{2^5}{3}\pi a^3\times\frac{1.\Gamma 4}{4\Gamma 4} \qquad [\because \Gamma\,(n+1)=n\,\Gamma\,(n)]$$

$$\therefore v=\frac{2^5}{3}\pi a^3\times\frac{1}{4}=\frac{2^5}{3}\pi a^3\times\frac{1}{2^2}=\frac{2^3}{3}\pi a^3=\frac{8}{3}\pi a^3$$

Therefore the required volume is $\frac{8}{3}\pi a^3$

Again, the required surface of the curve is,

$$S=2\pi\int_0^{\pi} y\,ds \text{ --------------------------(iv)}$$

We know that, $ds=\{r^2+(\frac{dr}{d\theta})^2\}^{1/2}d\theta$

And, Given $r=a(1+\cos\theta)$

$$\therefore \frac{dr}{d\theta}=-a\sin\theta$$

and $y=r\sin\theta$

Therefore, form (iv);

$$S=2\pi\int_0^{\pi} y\,ds$$

$$\Rightarrow S=2\pi\int_0^{\pi} r\sin\theta\{r^2+(\frac{dr}{d\theta})^2\}^{1/2}d\theta$$

$$\Rightarrow S=2\pi\int_0^{\pi}\{a(1+\cos\theta)\}\sin\theta[\{a(1+\cos\theta)\}^2+(-a\sin\theta)^2]^{1/2}d\theta$$

$$\Rightarrow S=2\pi\int_0^{\pi}\{a(1+\cos\theta)\}\sin\theta[\{a^2(1+\cos\theta)\}^2+a^2(\sin\theta)^2]^{1/2}d\theta$$

$$\Rightarrow S=2\pi\int_0^{\pi}a(1+\cos\theta)\sin\theta[a^2]^{1/2}[(1+\cos\theta)^2+\sin^2\theta]^{1/2}d\theta$$

$$\Rightarrow S=2\pi\int_0^{\pi}a(1+\cos\theta)\sin\theta\times a[(1+\cos\theta)^2+\sin^2\theta]^{1/2}d\theta$$

$$\Rightarrow S=2\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta[(1+\cos\theta)^2+\sin^2\theta]^{1/2}d\theta$$

$$\Rightarrow S = 2\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta[1+2\cos\theta+\cos^2\theta+\sin^2\theta]^{1/2}d\theta$$

$$\Rightarrow S = 2\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta[1+2\cos\theta+(\cos^2\theta+\sin^2\theta)]^{1/2}d\theta$$

$$\Rightarrow S = 2\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta[1+2\cos\theta+1]^{1/2}d\theta$$

$$\Rightarrow S = 2\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta[2+2\cos\theta]^{1/2}d\theta$$

$$\Rightarrow S = 2\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta\times 2^{1/2}\,[1+\cos\theta]^{1/2}d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\pi a^2\int_0^{\pi}(1+\cos\theta)^1\sin\theta\times[1+\cos\theta]^{1/2}d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\pi a^2\int_0^{\pi}(1+\cos\theta)^{1+\frac{1}{2}}\sin\theta d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\pi a^2\int_0^{\pi}(1+\cos\theta)^{\frac{3}{2}}\sin\theta d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\pi a^2\int_0^{\pi}(2\cos^2\frac{\theta}{2})^{\frac{3}{2}}\sin\theta d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\pi a^2\int_0^{\pi}(2\cos\frac{\theta}{2})^3\sin\theta d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\pi a^2\int_0^{\pi}(2\cos\frac{\theta}{2})^3\times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\pi a^2\int_0^{\pi}2^3\cos^3\frac{\theta}{2}\times 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\times 2^3\times 2\pi a^2\int_0^{\pi}\cos^3\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\times 2^3\times 2\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}d\theta$$

$$\Rightarrow S = 2\times 2^{1/2}\times 2^3\times \pi a^2\times 2\int_0^{\pi}\sin\frac{\theta}{2}\cos^4\frac{\theta}{2}d\theta$$

$$\Rightarrow S = 2^{1+\frac{1}{2}+3}\times \pi a^2\times 2\int_0^{\pi}\sin\frac{\theta}{2}\cos^4\frac{\theta}{2}d\theta \;= 2^{4+\frac{1}{2}}\times \pi a^2\times 2\int_0^{\pi}\sin\frac{\theta}{2}\cos^4\frac{\theta}{2}d\theta$$

$$\Rightarrow S = 2^{\frac{9}{2}}\times \pi a^2\times 2\int_0^{\pi}\sin^1\frac{\theta}{2}\cos^4\frac{\theta}{2}d\theta \text{-------------------------------(v)}$$

Putting $\frac{\theta}{2} = t$

$\Rightarrow \theta = 2t$

$\Rightarrow \frac{d}{dt}(\theta) = \frac{d}{dt}(2t)$

$\Rightarrow \frac{d\theta}{dt} = 2.1$

θ	0	π
t	$\frac{\theta}{2} = t$ $\frac{0}{2} = t$ $0 = t$ $t = 0$	$\frac{\theta}{2} = t$ $\frac{\pi}{2} = t$ $t = \frac{\pi}{2}$

$\Rightarrow d\theta = 2dt$

Putting $\frac{\theta}{2} = t$ where, $d\theta = 2dt$ and when $\theta = 0$ then $t = 0$ and also when

$\theta = \pi$ Then $t = \frac{\pi}{2}$ Therefore,

From (iv),

$$S = 2^{\frac{9}{2}} \times \pi a^2 \times 2\int_0^{\pi} \sin^1 \frac{\theta}{2} \cos^4 \frac{\theta}{2} d\theta$$

$$\Rightarrow S = 2^{\frac{9}{2}} \times \pi a^2 \times 2\int_0^{\pi/2} \sin^1 t \cos^4 t \times 2dt = 2^{\frac{9}{2}} \times 2 \times \pi a^2 \times 2\int_0^{\pi/2} \sin^1 t \cos^4 tdt$$

$$\Rightarrow S = 2^{\frac{9}{2}+1} \pi a^2 \times 2\int_0^{\pi/2} \sin^1 t \cos^4 tdt$$

$$\Rightarrow S = 2^{\frac{11}{2}} \pi a^2 \times 2\int_0^{\pi/2} \sin^1 t \cos^4 tdt \text{ -----------------------------(vi)}$$

We have,

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} [\because \Gamma n = (n-1)!]$$

Here, from (vi)

$2m-1 = 1$ &	$2n-1 = 4$
$\Rightarrow 2m = 1+1$	$\Rightarrow 2n = 4+1$
$\Rightarrow 2m = 2$	$\Rightarrow 2n = 5$
$\Rightarrow m = 1$	$\Rightarrow n = \frac{5}{2}$

$$\Rightarrow S = 2^{\frac{11}{2}} \pi a^2 \times 2\int_0^{\pi/2} \sin^1 t \cos^4 tdt$$

$$\Rightarrow S = 2^{\frac{11}{2}} \pi a^2 \times \beta(m,n) \qquad [\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} d\theta]$$

$$\Rightarrow S = 2^{\frac{11}{2}} \pi a^2 \times \beta(1,\frac{5}{2}) \qquad [\because m = 1 \; \& \; n = \frac{5}{2}]$$

$$\Rightarrow S = 2^{\frac{11}{2}} \pi a^2 \times \frac{\Gamma 1 \, \Gamma\frac{5}{2}}{\Gamma\left(1+\frac{5}{2}\right)} \qquad [\because \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}]$$

$$\Rightarrow S = 2^{\frac{11}{2}} \pi a^2 \times \frac{1.\Gamma\frac{5}{2}}{\Gamma\frac{2+5}{2}} = 2^{\frac{11}{2}} \pi a^2 \times \frac{\Gamma\frac{5}{2}}{\Gamma\frac{7}{2}} \qquad = 2^{\frac{11}{2}} \pi a^2 \times \frac{\Gamma\frac{5}{2}}{\Gamma\left(\frac{5}{2}+1\right)}$$

$$\Rightarrow S = 2^{\frac{11}{2}}\pi a^2 \times \frac{\Gamma\left(\frac{5}{2}\right)}{\frac{5}{2}\Gamma\left(\frac{5}{2}\right)} \qquad [\because \Gamma(n+1) = n\,\Gamma(n)]$$

$$\Rightarrow S = 2^{\frac{11}{2}}\pi a^2 \times \frac{1}{\frac{5}{2}} = 2^{\frac{11}{2}}\pi a^2 \times \frac{2}{5}$$

Therefore the required surface is $2^{\frac{11}{2}}\pi a^2 \times \frac{2}{5}$

Example 200: Find the volume and (surface) respectively of the solid formed by the revolving one loop of the curve $r^2 = a^2 \cos 2\theta$

(i) about the initial line i,e $y = 0$

(ii) about the line $\theta = \frac{\pi}{2}$

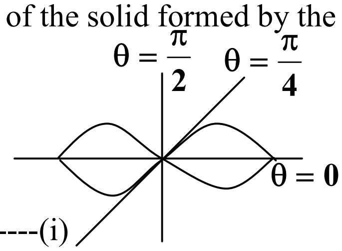

Figure # 102

Solution: Given equation $r^2 = a^2 \cos 2\theta$ -------------------------(i)

If we replace $-r$ for r and $-\theta$ for θ then equation (*i*) unchanged. So the curve is symmetrical about the pole also the curve is symmetrical about the initial line.

Putting the value of $r = 0$ in (i),

$$r^2 = a^2 \cos 2\theta$$
$$\Rightarrow 0 = a^2 \cos 2\theta$$
$$\Rightarrow a^2 \cos 2\theta = 0$$
$$\Rightarrow \cos 2\theta = 0$$
$$\Rightarrow \cos 2\theta = \cos\frac{\pi}{2}$$
$$\Rightarrow 2\theta = \frac{\pi}{2}$$
$$\Rightarrow \theta = \frac{\pi}{4}$$

Putting the value of $r = a$ in (i),

$$r^2 = a^2 \cos 2\theta$$
$$\Rightarrow a^2 = a^2 \cos 2\theta$$
$$\Rightarrow a^2 \cos 2\theta = a^2$$
$$\Rightarrow \cos 2\theta = 1$$
$$\Rightarrow \cos 2\theta = \cos 0$$
$$\Rightarrow 2\theta = 0$$
$$\Rightarrow \theta = 0$$

Given, $r^2 = a^2 \cos 2\theta$

$$\therefore r = a\cos^{\frac{1}{2}} 2\theta]$$

Therefore the volume of the curve about the initial line is,

$$v_1 = \frac{2}{3}\pi\int_0^{\frac{\pi}{4}} r^3 \sin\theta\, d\theta$$

$$\Rightarrow v_1 = \frac{2}{3}\pi\int_0^{\frac{\pi}{4}} (a\cos^{\frac{1}{2}} 2\theta)^3 \sin\theta\, d\theta \qquad [\because r = a\cos^{\frac{1}{2}} 2\theta]$$

$$\Rightarrow v_1 = \frac{2}{3}\pi\int_0^{\frac{\pi}{4}} a^3 (\cos^{\frac{1}{2}} 2\theta)^3 \sin\theta\, d\theta = \frac{2}{3}\pi\int_0^{\frac{\pi}{4}} a^3 \left(\cos 2\theta\right)^{\frac{3}{2}} \sin\theta\, d\theta$$

$$= \frac{2}{3}\pi a^3 \int_0^{\frac{\pi}{4}} \left(2\cos^2\theta - 1\right)^{\frac{3}{2}} \sin\theta\, d\theta \text{ -----------(ii) } [\because \cos 2\theta = 2\cos^2\theta - 1]$$

Putting, $\sqrt{2}\cos\theta = \sec\varphi$ in (ii)

$$\sqrt{2}\cos\theta = \sec\varphi$$

$$\Rightarrow \frac{d}{d\varphi}(\sqrt{2}\cos\theta) = \frac{d}{d\phi}(\sec\varphi)$$

$$\Rightarrow -\sqrt{2}\sin\theta.\frac{d\theta}{d\varphi} = \sec\varphi\tan\varphi$$

$$\Rightarrow -\sqrt{2}\sin\theta.d\theta = \sec\varphi\tan\varphi d\varphi$$

$$\Rightarrow \sqrt{2}\sin\theta.d\theta = -\sec\varphi\tan\varphi d\varphi$$

Putting $\theta = 0$ and $\theta = \frac{\pi}{4}$ in $\sqrt{2}\cos\theta = \sec\varphi$

Then we get,

$$\sqrt{2}\cos\theta = \sec\varphi$$

$$\Rightarrow \sqrt{2}\cos 0 = \sec\varphi \; [\theta = 0]$$

$$\Rightarrow \sqrt{2}.1 = \sec\varphi$$

$$\Rightarrow \sqrt{2} = \sec\varphi$$

$$\Rightarrow \sec\frac{\pi}{4} = \sec\varphi$$

$$\Rightarrow \frac{\pi}{4} = \varphi$$

$$\Rightarrow \varphi = \frac{\pi}{4}$$

Again,

$$\sqrt{2}\cos\theta = \sec\varphi$$

$$\sqrt{2}\cos\frac{\pi}{4} = \sec\varphi \; [\theta = \frac{\pi}{4}]$$

$$\sqrt{2}.\frac{1}{\sqrt{2}} = \sec\varphi$$

$$1 = \sec\varphi$$

$$\sec 0 = \sec\varphi$$

$$0 = \varphi$$

Therefore from (ii),

$$v_1 = \frac{2}{3}\pi a^3 \int_0^{\pi/4} \left(2\cos^2\theta - 1\right)^{\frac{3}{2}} \sin\theta\, d\theta$$

$$\Rightarrow v_1 = \frac{2}{3}\pi a^3 \int_0^{\pi/4} \left((\sqrt{2}\cos\theta)^2 - 1\right)^{\frac{3}{2}} \sin\theta\, d\theta$$

$$\Rightarrow v_1 = -\frac{2}{3}\pi a^3 \int_{\frac{\pi}{4}}^{0} \frac{1}{\sqrt{2}}\left(\sec^2\varphi - 1\right)^{\frac{3}{2}} \sec\varphi\tan\varphi\, d\varphi$$

$$[\because \sqrt{2}\cos\theta = \sec\varphi] \;\&\; [\because \sin\theta\, d\theta = -\frac{1}{\sqrt{2}}\sec\varphi\tan\varphi d\varphi]$$

$$\Rightarrow v_1 = \frac{2}{3} \times \frac{1}{\sqrt{2}}\pi a^3 \int_0^{\pi/4} \left(\sec^2\varphi - 1\right)^{\frac{3}{2}} \sec\varphi\tan\varphi\, d\varphi$$

$$\Rightarrow v_1 = \frac{2}{3}\times\frac{1}{\sqrt{2}}\pi a^3\int_0^{\pi/4}\left(\tan^2\varphi\right)^{\frac{3}{2}}\sec\varphi\tan\varphi\,d\varphi \qquad [\because \tan^2\varphi = \sec^2\varphi - 1]$$

$$\Rightarrow v_1 = \frac{2}{3}\times\frac{1}{\sqrt{2}}\pi a^3\int_0^{\pi/4}\tan^3\varphi\sec\varphi\tan\varphi\,d\varphi = \frac{\sqrt{2}}{3}\pi a^3\int_0^{\pi/4}\tan^4\varphi\sec\varphi\,d\varphi$$

$$\Rightarrow v_1 = \frac{\sqrt{2}}{3}\pi a^3\int_0^{\pi/4}(\tan^2\varphi)^2\sec\varphi\,d\varphi = \frac{\sqrt{2}}{3}\pi a^3\int_0^{\pi/4}(\sec^2\varphi - 1)^2\sec\varphi\,d\varphi$$

$$\Rightarrow v_1 = \frac{\sqrt{2}}{3}\pi a^3\int_0^{\pi/4}\left(\sec^4\varphi - 2\sec^2\varphi + 1\right)\sec\varphi\,d\varphi$$

$$v_1 = \frac{\sqrt{2}}{3}\pi a^3\int_0^{\pi/4}\left(\sec^5\varphi - 2\sec^3\varphi + \sec\varphi\right)d\varphi = \frac{\sqrt{2}}{3}\pi a^3\left[\frac{1}{4}\left\{\frac{3}{2}\log\left(1+\sqrt{2}\right) - \frac{\sqrt{2}}{2}\right\}\right]$$

$$\Rightarrow v_1 = \frac{1}{12}\pi a^3\left\{\frac{3}{\sqrt{2}}\log\left(1+\sqrt{2}\right) - 1\right\}$$; this is required volume

Therefore the volume of the curve about the line $\theta = \frac{\pi}{2}$ is

$$\Rightarrow v_2 = 2.\frac{2}{3}\pi\int_0^{\pi/4} r^3\cos\theta\,d\theta$$

$$\Rightarrow v_2 = 2.\frac{2}{3}\pi\int_0^{\pi/4}(a\cos^{\frac{1}{2}}2\theta)^3\cos\theta\,d\theta \; [\because r^2 = a^2\cos 2\theta;\ \therefore r = a\cos^{\frac{1}{2}}2\theta]$$

$$\Rightarrow v_2 = 2.\frac{2}{3}\pi a^3\int_0^{\pi/4}(\cos^{\frac{1}{2}}2\theta)^3\cos\theta\,d\theta = 2.\frac{2}{3}\pi a^3\int_0^{\pi/4}(\cos 2\theta)^{\frac{3}{2}}\cos\theta\,d\theta$$

$$\Rightarrow v_2 = 2.\frac{2}{3}\pi a^3\int_0^{\pi/4}\left(1-2\sin^2\theta\right)^{\frac{3}{2}}\cos\theta\,d\theta \text{ ---------------------(iii)}$$

Putting, $\sqrt{2}\sin\theta = \sin\varphi$ in (iii)

$\sqrt{2}\sin\theta = \sin\varphi$

$\Rightarrow \frac{d}{d\phi}(\sqrt{2}\sin\theta) = \frac{d}{d\phi}(\sin\phi)$

$\Rightarrow \sqrt{2}\cos\theta.\frac{d\theta}{d\phi} = \cos\phi$

$\Rightarrow \sqrt{2}\cos\theta.d\theta = \cos\phi\,d\phi$

Putting $\theta = 0$ and $\theta = \frac{\pi}{4}$ in $\sqrt{2}\sin\theta = \sin\varphi$

Then we get,

$\sqrt{2}\sin\theta = \sin\varphi$

Again, $\sqrt{2}\sin\theta = \sin\varphi$

$\sqrt{2}\sin\frac{\pi}{4} = \sin\varphi \; [\theta = \frac{\pi}{4}]$

$\sqrt{2}.\frac{1}{\sqrt{2}} = \sin\varphi$

$1 = \sin\varphi$

$\sin\frac{\pi}{2} = \sin\varphi$

$\frac{\pi}{2} = \varphi$

$\varphi = \frac{\pi}{2}$

$$\Rightarrow \sqrt{2}\sin 0 = \sin\varphi \;[\theta = 0]$$
$$\Rightarrow \sqrt{2}.0 = \sin\varphi$$
$$\Rightarrow 0 = \sin\varphi$$
$$\Rightarrow \sin 0 = \sin\varphi$$
$$\Rightarrow 0 = \varphi$$
$$\Rightarrow \varphi = 0$$

Therefore $v_2 = 2\frac{2}{3}\pi a^3 \frac{1}{\sqrt{2}}\int_0^{\pi/2}(1-\sin^2\varphi)^{\frac{3}{2}}\cos\varphi d\varphi$

$$\Rightarrow v_2 = 2\frac{2}{3}\pi a^3 \frac{1}{\sqrt{2}}\int_0^{\pi/2}(\cos^2\varphi)^{\frac{3}{2}}\cos\varphi d\varphi$$

$$\Rightarrow v_2 = 2\frac{\sqrt{2}}{3}\pi a^3 \int_0^{\pi/2}\cos^3\varphi\cos\varphi\, d\varphi$$

$$\Rightarrow v_2 = 2\frac{\sqrt{2}}{3}\pi a^3 \int_0^{\pi/2}\cos^4\varphi\, d\varphi$$

$$= \frac{2\sqrt{2}}{3}\pi a^3 \times \frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma\left(\frac{4+1}{2}\right)}{\Gamma\left(\frac{4+2}{2}\right)} \qquad \left[\int_0^{\pi/2}\cos^m x dx = \frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}\right]$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \times\sqrt{\pi}\,\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{4+2}{2}\right)} = \frac{\sqrt{2}}{3}\pi a^3 \frac{\Gamma\left(\frac{5}{2}\right).\sqrt{\pi}}{\Gamma\left(\frac{6}{2}\right)} = \frac{\sqrt{2}}{3}\pi a^3 \frac{\Gamma\left(\frac{5}{2}\right).\sqrt{\pi}}{\Gamma(3)}$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \frac{\Gamma\left(\frac{3}{2}+1\right).\sqrt{\pi}}{\Gamma(3)} = \frac{\sqrt{2}}{3}\pi a^3 \frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right).\sqrt{\pi}}{\Gamma(3)} \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \frac{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right).\sqrt{\pi}}{\Gamma(3)} = \frac{\sqrt{2}}{3}\pi a^3 \frac{\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right).\sqrt{\pi}}{\Gamma(3)} \qquad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{\Gamma(3)} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{(3-1)!} \qquad [\because \Gamma n = (n-1)!]$$

$$=\frac{\sqrt{2}}{3}\pi a^3\frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{2!}\quad=\frac{\sqrt{2}}{3}\pi a^3\frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{2.1}$$

$$=\frac{\sqrt{2}}{3}\pi a^3\frac{\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}.\sqrt{\pi}}{2}\quad=\frac{\sqrt{2}}{3}\pi a^3\frac{3.\sqrt{\pi}.\sqrt{\pi}}{2.2.2}$$

$$=\frac{\sqrt{2}}{3}\pi a^3\frac{3\pi}{8}\quad=\frac{\sqrt{2}}{8}\pi^2 a^3$$; this is the required volume

Example 201: Find the surface of the solid generated by the revolution of the curve $\mathbf{r^2 = a^2\cos 2\theta}$ about the initial line

Solution: Given curve, $\mathbf{r^2 = a^2\cos 2\theta}$ -------------------------(i)

If we replace $-\theta$ for θ and $-\mathbf{r}$ for $\mathbf{r}$ then equation (i) unchanged. Hence the curve is symmetrical about the initial line and the line $\mathbf{x=0}$ .Now for $\mathbf{r=0}$ then $\theta=\frac{\pi}{4}$ and $\mathbf{r=\pm a}$ then $\theta=\mathbf{0}$. Draw the curve.

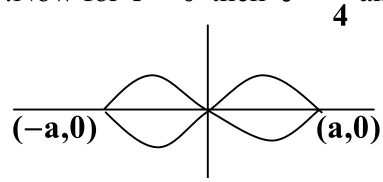

Figure # 103

Putting the value of $\mathbf{r=0}$ and $\mathbf{r=\pm a}$ in (i),

$\mathbf{r^2 = a^2\cos 2\theta}$

$\Rightarrow \mathbf{0^2 = a^2\cos 2\theta\ [r=0]}$

$\Rightarrow \mathbf{0 = a^2\cos 2\theta}$

$\Rightarrow \mathbf{0 = \cos 2\theta}$

$\Rightarrow \cos\frac{\pi}{2} = \cos 2\theta$

$\Rightarrow \frac{\pi}{2} = 2\theta$

$\Rightarrow \frac{\pi}{4} = \theta$

$\Rightarrow \theta = \frac{\pi}{4}$

Again, $\mathbf{r^2 = a^2\cos 2\theta}$

$\mathbf{(\pm a)^2 = a^2\cos 2\theta\ [r=\pm a]}$

$\mathbf{a^2 = a^2\cos 2\theta}$

$\mathbf{1 = \cos 2\theta}$

$\mathbf{\cos 0 = \cos 2\theta}$

$\mathbf{0 = 2\theta}$

$\mathbf{0 = \theta}$

$\mathbf{\theta = 0}$

Therefore the required surface is, $\mathbf{s = 4\pi\int_{\theta_1}^{\theta_2} y\,ds}$ -------------------------(ii)

Given $\mathbf{r^2 = a^2\cos 2\theta}$

$$\Rightarrow \frac{d}{d\theta}(r^2) = \frac{d}{d\theta}(a^2\cos 2\theta)$$

$$\Rightarrow 2r\frac{dr}{d\theta} = a^2\frac{d}{d\theta}(\cos 2\theta)$$

$$\Rightarrow 2r\frac{dr}{d\theta} = a^2(-\sin 2\theta)\frac{d}{d\theta}(2\theta)$$

$$\Rightarrow 2r\frac{dr}{d\theta} = -a^2(\sin 2\theta)\times 2$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{-a^2(\sin 2\theta)\times 2}{2r}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{-a^2(\sin 2\theta)}{r}$$

Now $y = r\sin\theta$ and also

$$ds = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} \text{ and } \frac{dr}{d\theta} = -\frac{a^2}{r}\sin 2\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} r\sin\theta\left\{r^2 + \frac{a^4}{r^2}\sin^2 2\theta\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} r\sin\theta\left\{\frac{r^4 + a^4\sin^2 2\theta}{r^2}\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} r\sin\theta \times \frac{(r^4 + a^4\sin^2 2\theta)^{1/2}}{(r^2)^{1/2}} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} r\sin\theta \times \frac{(r^4 + a^4\sin^2 2\theta)^{1/2}}{r} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times (r^4 + a^4\sin^2 2\theta)^{1/2} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times \{(r^2)^2 + a^4\sin^2 2\theta\}^{1/2} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times \{(a^2\cos 2\theta)^2 + a^4\sin^2 2\theta\}^{1/2} d\theta$$

$[\because r^2 = a^2\cos 2\theta; \therefore r^4 = a^4\cos^2 2\theta]$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times \{a^4\cos^2 2\theta + a^4\sin^2 2\theta\}^{1/2} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times \{a^4(\cos^2 2\theta + \sin^2 2\theta)\}^{1/2} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times (a^4)^{1/2}(\cos^2 2\theta + \sin^2 2\theta)^{1/2} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times a^2(\cos^2 2\theta + \sin^2 2\theta)^{1/2} d\theta$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times a^2(1)^{1/2} d\theta \; [\because \cos^2 2\theta + \sin^2 2\theta = 1]$$

$$\Rightarrow s = 4\pi\int_0^{\frac{\pi}{4}} \sin\theta \times a^2 d\theta = 4\pi a^2\int_0^{\frac{\pi}{4}} \sin\theta d\theta = 4\pi a^2\left[-\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow s = -4\pi a^2\left[\cos\theta\right]_0^{\frac{\pi}{4}} = -4\pi a^2\left[\cos\frac{\pi}{4} - \cos 0\right] = -4\pi a^2\left[\frac{1}{\sqrt{2}} - 1\right]$$

$$\Rightarrow s = 4\pi a^2\left[1 - \frac{1}{\sqrt{2}}\right]$$

Therefore the required surface is $4\pi a^2\left[1-\frac{1}{\sqrt{2}}\right]$

Example 202: For the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ show that the volume of the solid formed by the revolution of the curve about the axis is $\frac{32}{105}\pi a^3$ and the area of the surface so formed is $\frac{12}{5}\pi a^2$

Solution: Given equation,

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}} \text{---------------------------(i)}$$

$$y^{\frac{2}{3}}=a^{\frac{2}{3}}-x^{\frac{2}{3}} \text{--------------------------(ii)}$$

If we replace $-x$ for x and $-y$ for y in equation (i), then it is unchanged. Hence this curve is symmetrical about the both axis.

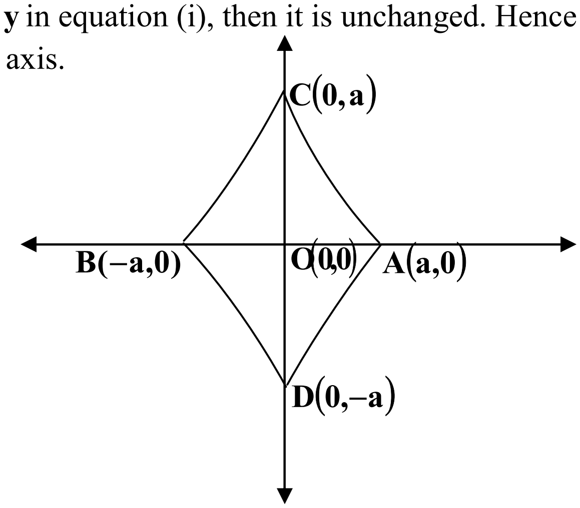

Figure # 104

Putting $x=0$ in (ii)

$y^{\frac{2}{3}}=a^{\frac{2}{3}}-x^{\frac{2}{3}}$

$\Rightarrow y^{\frac{2}{3}}=a^{\frac{2}{3}}-0^{\frac{2}{3}}$

$\Rightarrow y^{\frac{2}{3}}=a^{\frac{2}{3}}-0$

$\Rightarrow y^{\frac{2}{3}}=a^{\frac{2}{3}}$

$\Rightarrow y=a$

Putting $y=0$ in (ii)

$y^{\frac{2}{3}}=a^{\frac{2}{3}}-x^{\frac{2}{3}}$

$\Rightarrow 0^{\frac{2}{3}}=a^{\frac{2}{3}}-x^{\frac{2}{3}}$

$\Rightarrow 0=a^{\frac{2}{3}}-x^{\frac{2}{3}}$

$\Rightarrow a^{\frac{2}{3}}=x^{\frac{2}{3}}$

$\Rightarrow a=x$

$\Rightarrow x=a$

When $x=0$ then $y=\pm a$ when $y=0$ then $x=\pm a$. Hence the curve cut the x-axis at $A(a,0)$, $B(-a,0)$ and y-axis $C(0,a)$ and $(0,-a)$. Draw the curve.

Given, $y^{\frac{2}{3}}=a^{\frac{2}{3}}-x^{\frac{2}{3}}$

$\Rightarrow \left(y^{\frac{2}{3}}\right)^3=\left(a^{\frac{2}{3}}-x^{\frac{2}{3}}\right)^3$

$$\Rightarrow (y)^{\frac{2}{3}\times 3} = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3$$

$$\Rightarrow (y)^2 = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3$$

$$\Rightarrow y^2 = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 \text{-----------------------------------(iii)}$$

Therefore the required volume is,

$$v = 2\int_0^a \pi y^2 dx$$

$$= 2\times\pi\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx \ [\because y^2 = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3] \text{---------------------(iv)}$$

Putting, $x = a\sin^3\theta$ in (iv)

$$\Rightarrow \frac{dx}{d\theta} = \frac{d}{d\theta}(a\sin^3\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a\frac{d}{d\theta}(\sin^3\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a\times 3\sin^2\theta\frac{d}{d\theta}(\sin\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a\times 3\sin^2\theta\cos\theta$$

$$\Rightarrow dx = a\times 3\sin^2\theta\cos\theta\, d\theta$$

$$\Rightarrow dx = 3a\sin^2\theta\cos\theta\, d\theta$$

$x = a\sin^3\theta$	0	a
θ	$x = a\sin^3\theta$ $0 = a\sin^3\theta$ $0 = \sin^3\theta$ $0 = \sin\theta$ $\sin 0 = \sin\theta$ $0 = \theta$ $\theta = 0$	$x = a\sin^3\theta$ $a = a\sin^3\theta$ $1 = \sin^3\theta$ $1 = \sin\theta$ $\sin\frac{\pi}{2} = \sin\theta$ $\frac{\pi}{2} = \theta$ $\theta = \frac{\pi}{2}$

and θ varies 0 to $\frac{\pi}{2}$

From (iv),

$$v = 2\pi\int_0^a \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx$$

$$\Rightarrow v = 2\pi\int_0^a \left(a^{\frac{2}{3}} - (a\sin^3\theta)^{\frac{2}{3}}\right)^3 dx \ [\because x = a\sin^3\theta]$$

$$\Rightarrow v = 2\pi\int_0^{\frac{\pi}{2}} \left(a^{\frac{2}{3}} - a^{\frac{2}{3}}\sin^{3\times\frac{2}{3}}\theta\right)^3 3a\sin^2\theta\cos\theta\, d\theta \ [\because dx = 3a\sin^2\theta\cos\theta\, d\theta]$$

$$\Rightarrow v = 2\pi\int_0^{\frac{\pi}{2}} \left(a^{\frac{2}{3}} - a^{\frac{2}{3}}\sin^2\theta\right)^3 3a\sin^2\theta\cos\theta\, d\theta$$

$$\Rightarrow v = 2\pi\int_0^{\frac{\pi}{2}}\left(a^{\frac{2}{3}}(1-\sin^2\theta)\right)^3 3a\sin^2\theta\cos\theta\,d\theta$$

$$\Rightarrow v = 2\pi\int_0^{\frac{\pi}{2}}\left(a^{\frac{2}{3}}\cos^2\theta\right)^3 3a\sin^2\theta\cos\theta\,d\theta$$

$$\Rightarrow v = 2\pi\int_0^{\frac{\pi}{2}}\left(a^{\frac{2}{3}\times 3}\cos^{2\times 3}\theta\right)3a\sin^2\theta\cos\theta\,d\theta$$

$$\Rightarrow v = 2\pi\int_0^{\frac{\pi}{2}}\left(a^2\cos^6\theta\right)3a\sin^2\theta\cos\theta\,d\theta$$

$$\Rightarrow v = 2\pi\int_0^{\frac{\pi}{2}}a^2\cos^6\theta\,3a\sin^2\theta\cos\theta\,d\theta = \frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$\Rightarrow v = 3\pi a^3\times 2\int_0^{\frac{\pi}{2}}\cos^7\sin^2\theta\,d\theta = 3\pi a^3\times 2\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^7\theta\,d\theta \text{ -------------(v)}$$

We have,

$$\beta(m,n) = 2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m\Gamma n}{\Gamma(m+n)}[\because \Gamma n = (n-1)!]$$

Here, from (v)

$2m-1=2$ &	$2n-1=7$
$\Rightarrow 2m = 2+1$	$\Rightarrow 2n = 7+1$
$\Rightarrow 2m = 3$	$\Rightarrow 2n = 8$
$\Rightarrow m = \frac{3}{2}$	$\Rightarrow n = 4$

Hence From (v),

$$v = 3\pi a^3\times 2\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^7\theta\,d\theta$$

$$\Rightarrow v = 3\pi a^3\times\beta(m,n)$$

$$\Rightarrow v = 3\pi a^3\times\beta(\tfrac{3}{2},4) \qquad [\because m = \frac{3}{2} \text{ and } n = 4]$$

$$\Rightarrow v = 3\pi a^3\frac{\Gamma\frac{3}{2}\Gamma 4}{\Gamma\left(\frac{3}{2}+4\right)} \qquad [\because \beta(m,n) = \frac{\Gamma m\Gamma n}{\Gamma(m+n)}]$$

$$\beta(m,n) = 2\int_0^{\pi/2}(\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m\Gamma n}{\Gamma(m+n)}[\because \Gamma n = (n-1)!]$$

$$\Rightarrow v = 3\pi a^3 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(4)}{\Gamma\left(\frac{3+8}{2}\right)} = 3\pi a^3 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(4)}{\Gamma\left(\frac{11}{2}\right)} = 3\pi a^3 \frac{\Gamma\left(\frac{3}{2}\right)(4-1)!}{\Gamma\left(\frac{11}{2}\right)} \qquad [\because \Gamma n = (n-1)!]$$

$$\Rightarrow v = 3\pi a^3 \frac{\Gamma\left(\frac{3}{2}\right)3!}{\Gamma\left(\frac{11}{2}\right)} = 3\pi a^3 \frac{\Gamma\left(\frac{3}{2}\right)3.2.1}{\Gamma\left(\frac{11}{2}\right)} = 3\pi a^3 \frac{3.2.1.\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{9}{2}+1\right)}$$

$$\Rightarrow v = 3\pi a^3 \frac{3.2.1.\frac{1}{2}\Gamma(1)}{\frac{9}{2}\Gamma\left(\frac{9}{2}\right)} \qquad [\Gamma\ (n+1) = n\,\Gamma\ (n)]$$

$$\Rightarrow v = 3\pi a^3 \frac{3.2.1.\frac{1}{2}.1}{\frac{9}{2}\Gamma\left(\frac{9}{2}\right)} \qquad [\Gamma\ (1) = 1\ ]$$

$$\Rightarrow v = 3\pi a^3 \frac{3.2.1.\frac{1}{2}.1}{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}\frac{1}{2}\Gamma(1)} = 3\pi a^3 \frac{3.2.1.\frac{1}{2}.1}{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}\frac{1}{2}.1} = 3\pi a^3 \frac{3.2.1.}{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}}$$

$$\Rightarrow v = \pi a^3 \frac{2.1.}{\frac{3}{2}.\frac{7}{2}.\frac{5}{2}.\frac{1}{2}} = \pi a^3 \frac{2.2.2.2.2}{3.7.5.1} = \pi a^3 \frac{32}{105}$$

Therefore the volume of the curve is $\pi a^3 \frac{32}{105}$

The area of the surface is,

$$s = 2\int_0^a 2\pi y ds \text{ ----------------------(iii)}$$

$$ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx$$

Also, $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$

From (iii),

$$s = 2\int_0^a 2\pi y ds$$

$$\Rightarrow s = 4\pi\int_0^a y\,ds$$

$$\Rightarrow s = 4\pi\int_0^a y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = 4\pi\int_0^a y\left\{1+\left(-\left(\frac{y}{x}\right)^{\frac{1}{3}}\right)^2\right\}^{\frac{1}{2}} dx$$

$$\Rightarrow s = 4\pi\int_0^a y\left\{1+\left(\frac{y}{x}\right)^{\frac{2}{3}}\right\}^{\frac{1}{2}} dx = 4\pi\int_0^a y\left\{1+\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right\}^{\frac{1}{2}} dx = 4\pi\int_0^a y\left\{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right\}^{\frac{1}{2}} dx$$

$$\Rightarrow s = 4\pi\int_0^a y\left\{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right\}^{\frac{1}{2}} dx \qquad [\because x^{\frac{2}{3}}+y^{\frac{2}{3}} = a^{\frac{2}{3}}]$$

$$\Rightarrow s = 4\pi\int_0^a y\times\frac{\left(a^{\frac{2}{3}}\right)^{\frac{1}{2}}}{\left(x^{\frac{2}{3}}\right)^{\frac{1}{2}}} dx = 4\pi\int_0^a y\times\frac{a^{\frac{1}{3}}}{x^{\frac{1}{3}}} dx = 4\pi\times a^{\frac{1}{3}}\int_0^a y\times\frac{1}{x^{\frac{1}{3}}} dx$$

$$\Rightarrow s = 4\pi\times a^{\frac{1}{3}}\int_0^a \frac{y}{x^{\frac{1}{3}}} dx \text{ ---------------------------------(iv)}$$

Putting $y = a\sin^3\theta,\ x = a\cos^3\theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos^3\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta\frac{d}{d\theta}(\cos\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = -3a\cos^2\theta.\sin\theta$$

$$\Rightarrow dx = -3a\cos^2\theta.\sin\theta\,d\theta$$

$x = a\cos^3\theta$	0	a
θ	$x = a\cos^3\theta$ $0 = a\cos^3\theta$ $0 = \cos^3\theta$ $0 = \cos\theta$ $\cos\frac{\pi}{2} = \cos\theta$ $\frac{\pi}{2} = \theta$ $\theta = \frac{\pi}{2}$	$x = a\cos^3\theta$ $a = a\cos^3\theta$ $1 = \cos^3\theta$ $1 = \cos\theta$ $\cos 0 = \cos\theta$ $0 = \theta$ $\theta = 0$

Therefore, from (iv),

$$s = 4\pi\times a^{\frac{1}{3}}\int_0^a \frac{y}{x^{\frac{1}{3}}} dx$$

$$\Rightarrow s = 4\pi\times a^{\frac{1}{3}}\int_{\frac{\pi}{2}}^0 \frac{a\sin^3\theta}{(a\cos^3\theta)^{\frac{1}{3}}}(-3a\cos^2\theta.\sin\theta\,d\theta)$$

$$\Rightarrow s = -12\pi a^{\frac{1}{3}} a \int_{\pi/2}^{0} \frac{a\sin^3\theta}{a^{\frac{1}{3}}(\cos^{\frac{1}{3}}\theta)^{\frac{1}{3}}} \cos^2\theta \sin\theta \, d\theta$$

$$\Rightarrow s = -12\pi a \int_{\pi/2}^{0} \frac{a\sin^3\theta}{\cos\theta} \cos^2\theta \sin\theta \, d\theta$$

$$\Rightarrow s = -12\pi a^2 \int_{\pi/2}^{0} \sin^4\theta \cos\theta \, d\theta$$

$$\Rightarrow s = 12\pi a^2 \int_{0}^{\pi/2} \sin^4\theta \cos\theta \, d\theta \; [\because \int_a^b f(x)dx = -\int_b^a f(x)dx]$$

We have,

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \, d\theta = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} [\because \Gamma n = (n-1)!]$$

$$s = 12\pi a^2 \int_0^{\pi/2} \sin^4\theta \cos^1\theta \, d\theta$$

Here,

$2m - 1 = 4$	&	$2n - 1 = 1$
$\Rightarrow 2m = 4 + 1$		$\Rightarrow 2n = 1 + 1$
$\Rightarrow 2m = 5$		$\Rightarrow 2n = 2$
$\Rightarrow m = \frac{5}{2}$		$\Rightarrow n = 1$

$$\Rightarrow s = 12\pi a^2 \int_0^{\pi/2} \sin^4\theta \cos^1\theta \, d\theta = 6\pi a^2 \times 2\int_0^{\pi/2} \sin^4\theta \cos^1\theta \, d\theta$$

$$\Rightarrow s = 6\pi a^2 \times \beta(\frac{5}{2}, 1) = 6\pi a^2 \times \frac{\Gamma\frac{5}{2}\Gamma 1}{\Gamma(\frac{5}{2}+1)} \quad [\because \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}]$$

$$\Rightarrow s = 6\pi a^2 \times \frac{\Gamma\frac{5}{2}.1}{\Gamma(\frac{5}{2}+1)} \quad [\Gamma(1) = 1]$$

$$\Rightarrow s = 6\pi a^2 \times \frac{\Gamma\frac{5}{2}.1}{\frac{5}{2}\Gamma(\frac{5}{2})} \quad [\Gamma(n+1) = n\,\Gamma(n)]$$

$$\Rightarrow s = 6\pi a^2 \times \frac{1}{\frac{5}{2}} = 6\pi a^2 \times \frac{2}{5} = \frac{12}{5}\pi a^2$$

Therefore, the required surface is $\frac{12}{5}\pi a^2$. (Proved)

Example 203: For the curve $\frac{x^2}{a^2}+\frac{a^2}{b^2}=1$ show that the volume of the solid formed by the revolution of the curve about major axis is $\frac{4}{3}\pi ab^2$ and the area of surface so formed is.

Solution: Given that,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{----------------------------(i)}$$

or, $y^2=\frac{b^2}{a^2}(a^2-x^2)$ ----------------(ii)

If we replace –x for x and – y for y in equation (i) then it unchanged hence the curve is symmetrical about the both axis. When x = 0 then $y=\pm b$ when y = 0 then $x=\pm a$ therefore the curve cut the axis of A (a, 0) B (-a, 0) and C (0, b), D (0,-b) draw the curve, Therefore the volume of the curve, about the major axis

$$v=2\int_0^a \pi y^2 dx$$

$$=2\pi\int_0^a \frac{b^2}{a^2}(a^2-x^2)dx \quad [\because y^2=\frac{b^2}{a^2}(a^2-x^2)]$$

$$=2\pi\frac{b^2}{a^2}[\int_0^a a^2dx-\int_0^a x^2dx]$$

$$=2\pi\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=2\pi\frac{b^2}{a^2}\left[a^2.a-\frac{a^3}{3}-0-0\right]$$

$$=2\pi\frac{b^2}{a^2}\left[a^3-\frac{a^3}{3}\right]=2\pi\frac{b^2}{a^2}\left[\frac{3a^3-a^3}{3}\right]$$

$$=2\pi\frac{b^2}{a^2}\frac{2a^3}{3}=\frac{4}{3}\pi ab^2$$

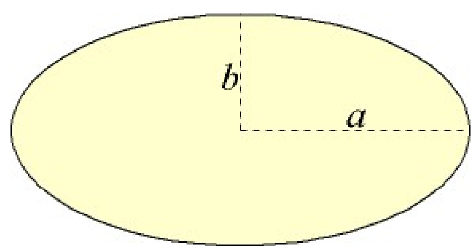

Figure # 105

Therefore, the required volume is $\frac{4}{3}\pi ab^2$.

The area of surface of the curve about the major axis is,

$$s=\int_{-a}^{a} 2\pi y ds$$

$$\Rightarrow s=2\times\int_0^a 2\pi y ds \quad [\because \int_{-a}^{+a} f(x)dx==2\int_0^a f(x)dx]$$

$$\Rightarrow s = 4\pi\int_0^a yds$$

$$\Rightarrow s = 4\pi\int_0^a \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}} ds \ [\because y^2 = \frac{b^2}{a^2}(a^2 - x^2);\ \therefore y = \frac{b}{a}(a^2 - x^2)^{1/2}]$$

Also, $ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx$ and

Given,

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\Rightarrow y^2 = \frac{b^2}{a^2}a^2 - \frac{b^2}{a^2}x^2$$

$$\Rightarrow y^2 = 1 - \frac{b^2}{a^2}x^2$$

$$\Rightarrow 2ydy = 0 - \frac{b^2}{a^2}2xdx$$

$$\Rightarrow 2ydy = -\frac{b^2}{a^2}2xdx$$

$$\Rightarrow ydy = -\frac{b^2}{a^2}xdx$$

$$\Rightarrow \therefore \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

Hence $s = 4\pi\int_0^a \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}} ds$

$$\Rightarrow s = 4\pi\int_0^a \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}.\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} \ [\because ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx]$$

$$\Rightarrow s = 4\pi\frac{b}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}}\left\{1 + \frac{b^4}{a^4}.\frac{x^2}{y^2}\right\}^{\frac{1}{2}} dx \ [\because \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}]$$

$$\Rightarrow s = 4\pi\frac{b}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}} \times \frac{(a^4y^2 + b^4x^2)^{\frac{1}{2}}}{(a^4y^2)^{\frac{1}{2}}} dx$$

$$\Rightarrow s = 4\pi.\frac{b}{a}\int_0^a (a^2 - x^2)^{\frac{1}{2}} \frac{1}{a^2y}\left\{a^4y^2 + b^4x^2\right\}^{\frac{1}{2}} dx$$

$$\Rightarrow s = 4\pi.\frac{b}{a}\times\frac{1}{a^2y}\int_0^a (a^2-x^2)^{\frac{1}{2}}\left\{a^4y^2+b^4x^2\right\}^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{1}{a^2y}\int_0^a \frac{b}{a}(a^2-x^2)^{\frac{1}{2}}\left\{a^4y^2+b^4x^2\right\}^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{1}{a^2y}\int_0^a y\left\{a^4y^2+b^4x^2\right\}^{\frac{1}{2}}dx \qquad [\because y=\frac{b}{a}(a^2-x^2)^{1/2}]$$

$$\Rightarrow s = 4\pi\times\frac{1}{a^2}\int_0^a \left\{a^4y^2+b^4x^2\right\}^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{1}{a^2}\int_0^a \left\{a^4\frac{b^2}{a^2}(a^2-x^2)+b^4x^2\right\}^{\frac{1}{2}}dx \quad [\because y^2=\frac{b^2}{a^2}(a^2-x^2)]$$

$$\Rightarrow s = 4\pi\times\frac{1}{a^2}\int_0^a \left\{a^2b^2(a^2-x^2)+b^4x^2\right\}^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{1}{a^2}\int_0^a \left[b^2\left\{a^2(a^2-x^2)+b^2x^2\right\}\right]^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{1}{a^2}\int_0^a b\left[\left\{a^2(a^2-x^2)+b^2x^2\right\}\right]^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{b}{a^2}\int_0^a \left[\left\{a^2(a^2-x^2)+b^2x^2\right\}\right]^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{b}{a^2}\int_0^a \left[\left\{a^4-a^2x^2)+b^2x^2\right\}\right]^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{b}{a^2}\int_0^a \left[\left\{a^4+b^2x^2-a^2x^2\right\}\right]^{\frac{1}{2}}dx \;= 4\pi\times\frac{b}{a^2}\int_0^a \left[\left\{a^4+(b^2-a^2)x^2\right\}\right]^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{b}{a^2}\int_0^a \left[(b^2-a^2)\times\frac{1}{(b^2-a^2)}\left\{a^4+(b^2-a^2)x^2\right\}\right]^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{b}{a^2}\int_0^a \left[(b^2-a^2)\left\{\frac{a^4}{(b^2-a^2)}+\frac{(b^2-a^2)x^2}{(b^2-a^2)}\right\}\right]^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\times\frac{b}{a^2}\int_0^a \left[(b^2-a^2)\left\{\frac{a^4}{(b^2-a^2)}+x^2\right\}\right]^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi \times \frac{b}{a^2}(b^2 - a^2)^{\frac{1}{2}} \int_0^a \left[\left\{ \frac{a^4}{(b^2 - a^2)} + x^2 \right\} \right]^{\frac{1}{2}} dx$$

$$\Rightarrow s = 4\pi \times \frac{b}{a^2}(b^2 - a^2)^{\frac{1}{2}} \int_0^a \left[\left\{ \frac{(a^2)^2}{\left[(b^2 - a^2)^{\frac{1}{2}} \right]^2} + x^2 \right\} \right]^{\frac{1}{2}} dx$$

$$\Rightarrow s = 4\pi \times \frac{b}{a^2}(b^2 - a^2)^{\frac{1}{2}} \int_0^a \left[\left\{ \left[\frac{a^2}{(b^2 - a^2)^{\frac{1}{2}}} \right]^2 + x^2 \right\} \right]^{\frac{1}{2}} dx$$

$$\Rightarrow s = \frac{4\pi b(b^2 - a^2)^{\frac{1}{2}}}{a^2} \int_0^a \left[\left\{ \left[\frac{a^2}{(b^2 - a^2)^{\frac{1}{2}}} \right]^2 + x^2 \right\} \right]^{\frac{1}{2}} dx$$

$$[\because \int \sqrt{x^2 + a^2}\, dx = \frac{x\sqrt{x^2 + a^2}}{2} \pm \frac{a^2}{2} \sin^{-1} \frac{x}{a}]$$

$$s = \frac{4\pi b(b^2 - a^2)^{\frac{1}{2}}}{a^2} \left[\frac{x\sqrt{x^2 + \left[\frac{a^2}{(b^2 - a^2)^{\frac{1}{2}}} \right]^2}}{2} \pm \frac{\left[\frac{a^2}{(b^2 - a^2)^{\frac{1}{2}}} \right]^2}{2} \sin^{-1} \frac{x}{\frac{a^2}{(b^2 - a^2)^{\frac{1}{2}}}} \right]_0^a$$

$$\Rightarrow s=\frac{4\pi b(b^2-a^2)^{\frac{1}{2}}}{a^2}\left[\begin{array}{l}\dfrac{a\sqrt{a^2+\left[\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}\right]^2}}{2}\pm\dfrac{\left[\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}\right]^2}{2}\sin^{-1}\dfrac{a}{\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}}-\\ \dfrac{0\sqrt{0^2+\left[\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}\right]^2}}{2}\pm\dfrac{\left[\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}\right]^2}{2}\sin^{-1}\dfrac{0}{\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}}\end{array}\right]$$

$$\Rightarrow s=\frac{4\pi b(b^2-a^2)^{\frac{1}{2}}}{a^2}\left[\dfrac{a\sqrt{a^2+\left[\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}\right]^2}}{2}\pm\dfrac{\left[\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}\right]^2}{2}\sin^{-1}\dfrac{a}{\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}}\pm 0\right]$$

$$\Rightarrow s=\frac{4\pi b(b^2-a^2)^{\frac{1}{2}}}{a^2}\left[\dfrac{a\sqrt{a^2+\left[\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}\right]^2}}{2}\pm\dfrac{\left[\dfrac{a^2}{(b^2-a^2)^{\frac{1}{2}}}\right]^2}{2}\sin^{-1}\dfrac{a(b^2-a^2)^{\frac{1}{2}}}{a^2}\right]$$

This is the required surface area

Example 204: Find the volume and surface of the solid formed by sphere of radius r.
Solution: Given sphere is: $x^2+y^2+z^2=r^2$ ------------------------(i)
The sphere is produced by the revolution of circle $x^2+y^2=r^2$ about its axis. The curve is symmetrical about the both axis. and it cut the axis at $\mathbf{A(r,o), B(-r,o)}$ and $\mathbf{C(o,r)}$ and $\mathbf{D(o,-r)}$. Therefore,
The volume of the curve about the x-axis is,

$$v=\int_{-r}^{r}\pi y^2dx$$

$$= 2\pi\int_0^r (r^2 - x^2)\,dx$$

$[\because x^2+y^2 = r^2 \;\&\; y^2 = r^2 - x^2]$ &

$[\because \int_{-a}^{+a} f(x)dx = 2\int_0^a f(x)dx]$

$$= 2\pi\left[r^2x - \frac{x^3}{3}\right]_0^r$$

$$= 2\pi\left\{r^3 - \frac{r^3}{3} - 0 + 0\right\} = 2\pi\left(\frac{3r^3 - r^3}{3}\right)$$

$$= 2\pi\left(\frac{2r^3}{3}\right) = \frac{4}{3}\pi r^3$$

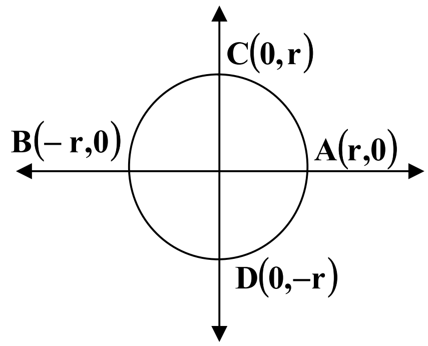

Figure # 106

Therefore the required volume is: $\frac{4}{3}\pi r^3$

Also, the area of surface about the x-axis is,

$$s = 2\int_{-r}^{r} \pi y ds$$

$$\Rightarrow s = 2\times 2\int_0^r \pi y ds \;[\because \int_{-a}^{+a} f(x)dx = 2\int_0^a f(x)dx]$$

$$\Rightarrow s = 4\pi\int_0^r y ds \text{ ------------------------------------(ii)}$$

Where, $ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx$

Now, $\frac{dy}{dx} = -\frac{x}{y}$

Therefore,

$$s = 4\pi\int_0^r y ds$$

$$\Rightarrow s = 4\pi\int_0^r y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = 4\pi\int_0^r y\left\{1 + \frac{x^2}{y^2}\right\}^{\frac{1}{2}} dx$$

$$\Rightarrow s = 4\pi\int_0^r y\left\{\frac{y^2 + x^2}{y^2}\right\}^{\frac{1}{2}} dx = 4\pi\int_0^r y \times \frac{(y^2 + x^2)^{\frac{1}{2}}}{(y^2)^{\frac{1}{2}}} dx$$

$$\Rightarrow s = 4\pi\int_0^r y \times \frac{(y^2+x^2)^{\frac{1}{2}}}{y}dx = 4\pi\int_0^r (y^2+x^2)^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\int_0^r \left(x^2+y^2\right)^{\frac{1}{2}}dx = 4\pi\int_0^r \left(r^2\right)^{\frac{1}{2}}dx = 4\pi\int_0^r r dx = 4\pi r\int_0^r dx$$

$$\Rightarrow s = 4\pi r[x]_0^r = 4\pi r[r-0] = 4\pi r^2$$

Therefore, the required surface area is $= \mathbf{4\pi r^2}$.

Example 205: Find the surface of the solid generated by revolving the area of $y^2 = 4ax$ bounded by its latus rectum about the x-axis.

Solution: Given parabola $y^2 = 4ax$ -------------------------(i)

If we replace – y for y then equation (i) is unchanged hence the curve is symmetrical about the x-axis also the equation of latus rectum $x = a$

i.e; $y^2 = 4ax$ $\qquad$ $y^2 = 4a^2$ [$\because x = a$]

i.e. $y = \pm 2a$

Hence the line $x = a$ cut the curve at $(a,2a)$ and $(a,-2a)$.

Therefore the required area of surface

$s = \int_0^a 2\pi y\, ds$ ------------------(ii)

Now, $ds = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx$

Also $\frac{dy}{dx} = \frac{2a}{y}$, Therefore,

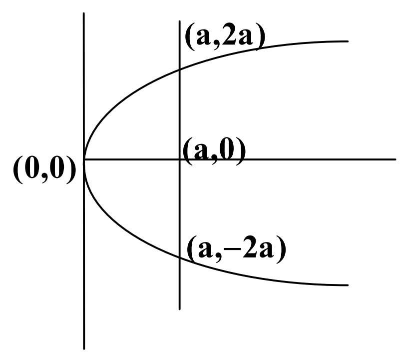

Figure # 107

$$\Rightarrow s = \int_0^a 2\pi y\, ds$$

$$\Rightarrow s = 2\pi\int_0^a y\, ds$$

$$\Rightarrow s = 2\pi\int_0^a y \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx \qquad \left[\because ds = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx\right]$$

$$\Rightarrow s = 2\pi\int_0^a y \left\{1+\left(\frac{2a}{y}\right)^2\right\}^{\frac{1}{2}} dx \qquad \left[\because \frac{dy}{dx} = \frac{2a}{y}\right]$$

$$\Rightarrow s = 2\pi\int_0^a y\left\{1+\frac{4a^2}{y^2}\right\}^{\frac{1}{2}} dx = 2\pi\int_0^a y\left\{\frac{y^2+4a^2}{y^2}\right\}^{\frac{1}{2}} dx$$

$$\Rightarrow s = 2\pi\int_0^a y \times \frac{\{y^2 + 4a^2\}^{\frac{1}{2}}}{y}\, dx = 2\pi\int_0^a \{y^2 + 4a^2\}^{\frac{1}{2}}\, dx$$

$$\Rightarrow s = 2\pi\int_0^a \{4ax + 4a^2\}^{\frac{1}{2}}\, dx \qquad [\because y^2 = 4ax]$$

$$\Rightarrow s = 2\pi\int_0^a \{4a(x+a)\}^{\frac{1}{2}}\, dx = 2\pi\int_0^a (4a)^{\frac{1}{2}}(x+a)^{\frac{1}{2}}dx$$

$$\Rightarrow s = 2\pi\int_0^a 2a^{\frac{1}{2}}(x+a)^{\frac{1}{2}}dx = 2\pi \times 2a^{\frac{1}{2}}\int_0^a (x+a)^{\frac{1}{2}}dx$$

$$\Rightarrow s = 4\pi\sqrt{a}\int_0^a (x+a)^{\frac{1}{2}}dx = 4\pi\sqrt{a}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^a$$

$$\Rightarrow s = 4\pi\sqrt{a}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a = 4\pi\sqrt{a}\left[\frac{2}{3}(x+a)^{\frac{3}{2}}\right]_0^a$$

$$\Rightarrow s = 4\pi\sqrt{a}\left[\frac{2}{3}(a+a)^{\frac{3}{2}} - \frac{2}{3}(0+a)^{\frac{3}{2}}\right] = 4\pi\sqrt{a}\left[\frac{2}{3}(2a)^{\frac{3}{2}} - \frac{2}{3}(a)^{\frac{3}{2}}\right]$$

$$\Rightarrow s = 4a^{\frac{1}{2}}\pi\left(\frac{2}{3}2^{\frac{3}{2}}a^{\frac{3}{2}} - \frac{2}{3}a^{\frac{3}{2}}\right) = 4a^{\frac{1}{2}}\pi \times \frac{2}{3}a^{\frac{3}{2}}\left(2^{\frac{3}{2}} - 1\right)$$

$$\Rightarrow s = 4\pi \times \frac{2}{3}a^{\frac{3}{2}+\frac{1}{2}}\left(2^1.2^{\frac{1}{2}} - 1\right) = 4\pi \times \frac{2}{3}a^{\frac{4}{2}}\left(2.2^{\frac{1}{2}} - 1\right)$$

$$\Rightarrow s = 4\pi \times \frac{2}{3}a^2\left(2.2^{\frac{1}{2}} - 1\right) = \frac{8}{3}a^2\pi\left(2\sqrt{2} - 1\right)$$

Which is the required surface area.

Example 206: Evaluate the surface area of the solid generated by revolving the cycloid $\mathbf{x = a(\theta - \sin\theta)}$, $\mathbf{y = a(1 - \cos\theta)}$; about the line $y = 0$ i.e. about the x-axis.

Solution: Given cycloid is,

$$\mathbf{x = a(\theta - \sin\theta)} \text{---------------------(i)}$$

$$\mathbf{y = a(1 - \cos\theta)} \text{---------------------(ii)}$$

When $y = 0$ then from (ii),

$$y = a(1 - \cos\theta)$$

$$0 = a(1 - \cos\theta)\ [\because y = 0]$$

$$0 = (1 - \cos\theta)$$

$$-1 = -\cos\theta$$

$$1 = \cos\theta$$

$$\cos 0 = \cos\theta$$

When $\mathbf{y = 2a}$ then from (ii),

$$\Rightarrow y = a(1 - \cos\theta)$$

$$\Rightarrow 2a = a(1 - \cos\theta)\ [\because y = 2a]$$

$$\Rightarrow 2 = (1 - \cos\theta)$$

$$\Rightarrow 2 - 1 = -\cos\theta$$

$$\Rightarrow 1 = -\cos\theta$$

$$\Rightarrow -1 = \cos\theta$$

$$\Rightarrow \cos\pi = \cos\theta$$

$$\Rightarrow \pi = \theta$$

$$\Rightarrow \theta = \pi$$

$$\mathbf{0 = \theta}$$
$$\mathbf{\theta = 0}$$

Draw the curve,
Therefore the required surface is,

$$S = \int_0^{2\pi} 2\pi y\, ds \text{ --------------------------(iii)}$$

$$\Rightarrow ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx$$

$$\Rightarrow ds = \left(dx^2 + dy^2\right)^{\frac{1}{2}}$$

$$\Rightarrow ds = \left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta$$

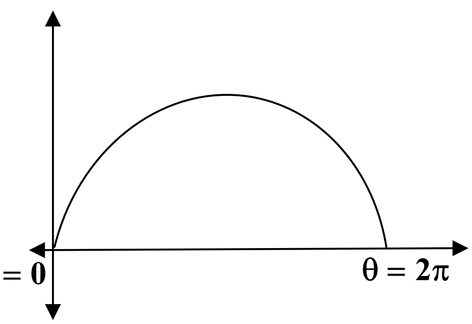

Figure # 108

Given cycloid is,

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

$$x = a(\theta - \sin\theta)$$

$$\therefore \frac{dx}{d\theta} = \frac{d}{d\theta}\{a(\theta - \sin\theta)\}$$

$$\therefore \frac{dx}{d\theta} = a(1 - \cos\theta)$$

and

$$y = a(1 - \cos\theta)$$

$$\therefore \frac{dy}{d\theta} = \frac{d}{d\theta}\{a(1 - \cos\theta)\}$$

$$\therefore \frac{dy}{d\theta} = a(0 + \sin\theta)$$

$$\therefore \frac{dy}{d\theta} = a\sin\theta$$

Therefore the required surface is,

$$S = \int_0^{2\pi} 2\pi y\, ds$$

$$\Rightarrow S = \int_0^{2\pi} 2\pi y\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\pi\int_0^{2\pi} a(1 - \cos\theta)\left\{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\pi\int_0^{2\pi} a(1 - \cos\theta)(a^2)^{1/2}\left\{(1 - \cos\theta)^2 + \sin^2\theta\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\pi\int_0^{2\pi} a(1 - \cos\theta)\, a\left\{(1 - \cos\theta)^2 + \sin^2\theta\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\pi a^2\int_0^{2\pi} (1 - \cos\theta)\left\{(1 - \cos\theta)^2 + \sin^2\theta\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\pi a^2 \int_0^{2\pi} (1-\cos\theta)\left\{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\pi a^2 \int_0^{2\pi} (1-\cos\theta)\left\{1 - 2\cos\theta + 1\right\}^{\frac{1}{2}} d\theta = 2\pi a^2 \int_0^{2\pi} (1-\cos\theta)\left\{2 - 2\cos\theta\right\}^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\pi a^2 \times 2^{1/2} \int_0^{2\pi} (1-\cos\theta)\left\{1 - \cos\theta\right\}^{\frac{1}{2}} d\theta = 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1-\cos\theta)(1-\cos\theta)^{\frac{1}{2}} d\theta$$

$$\Rightarrow S = 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1-\cos\theta)^{1+\frac{1}{2}} d\theta = 2\sqrt{2}\pi a^2 \int_0^{2\pi} (1-\cos\theta)^{\frac{3}{2}} d\theta$$

$$\Rightarrow S = 2\sqrt{2}\pi a^2 \int_0^{2\pi} \left(2\sin^2\frac{\theta}{2}\right)^{3/2} d\theta \qquad [\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}]$$

$$\Rightarrow S = 2\sqrt{2}\pi a^2 \int_0^{2\pi} 2^{3/2}\left(\sin^2\frac{\theta}{2}\right)^{3/2} d\theta = 2\sqrt{2}\pi a^2 \int_0^{2\pi} 2^{3/2}\sin^3\frac{\theta}{2} d\theta$$

$$\Rightarrow S = 2\sqrt{2} \times 2^{3/2}\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2} d\theta \Rightarrow = 2\times 2^{1/2} \times 2^{3/2}\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2} d\theta$$

$$\Rightarrow S = 2\times 2^{\frac{1}{2}+\frac{3}{2}}\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2} d\theta = 2\times 2^{\frac{4}{2}}\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2} d\theta$$

$$\Rightarrow S = 2\times 2^2\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2} d\theta \text{ --------------------------------------(iv)}$$

Put $\dfrac{\theta}{2} = \varphi$

$\therefore \dfrac{d}{d\varphi}\left(\dfrac{\theta}{2}\right) = \dfrac{d}{d\varphi}(\varphi)$

$\therefore \dfrac{1}{2}\dfrac{d}{d\varphi}(\theta) = 1$

$\therefore \dfrac{d\theta}{d\varphi} = 2$

$\Rightarrow d\theta = 2d\varphi$

θ	0	2π
$\dfrac{\theta}{2} = \varphi$ $\Rightarrow \varphi = \dfrac{\theta}{2}$	$\varphi = \dfrac{\theta}{2}$ $\Rightarrow \varphi = \dfrac{0}{2}$ $\Rightarrow \varphi = 0$	$\varphi = \dfrac{\theta}{2}$ $\varphi = \dfrac{2\pi}{2} = \pi$

$\therefore \varphi$ varies 0 to π

From (iv),

$$\Rightarrow S = 2\times 2^2\pi a^2 \int_0^{2\pi} \sin^3\frac{\theta}{2} d\theta$$

$$\Rightarrow S = 2\times 4\times\pi a^2 \int_0^{\pi} \sin^3\varphi \times 2d\varphi$$

$$\Rightarrow S = 2\times 4\times 2\times \pi a^2\int_0^{\pi}\sin^3\varphi\, d\varphi = 16\times\pi a^2\times 2\int_0^{\pi/2}\sin^3\varphi\, d\varphi \quad [\because \int_0^a f(x)dx = 2\int_0^{a/2} f(x)dx]$$

$$\Rightarrow S = 16\pi a^2\times 2\int_0^{\frac{\pi}{2}}\sin^3\varphi\, d\varphi = 32\pi a^2\int_0^{\frac{\pi}{2}}\sin^3\varphi\, d\varphi$$

$$\Rightarrow S = 32\pi a^2\times\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{3+1}{2}\right)}{\Gamma\left(\frac{3+2}{2}\right)} \qquad [\int_0^{\pi/2}\sin^m x\,dx = \frac{1}{2}\times\sqrt{\pi}\times\frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}]$$

$$S = 32\pi a^2\times\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{4}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = 32\pi a^2\times\frac{\sqrt{\pi}}{2}\frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)} = 16\pi a^2\,\frac{(2-1)!\sqrt{\pi}}{\Gamma\left(\frac{5}{2}\right)} \qquad [\because \Gamma n = (n-1)!]$$

$$\Rightarrow S = 16\pi a^2\,\frac{1!\sqrt{\pi}}{\Gamma\left(\frac{3}{2}+1\right)} = 16\pi a^2\,\frac{1!\sqrt{\pi}}{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)} \qquad [\because \Gamma\,(n+1) = n\,\Gamma\,(n)]$$

$$\Rightarrow S = 16\pi a^2\,\frac{1!\sqrt{\pi}}{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)} = 16\pi a^2\,\frac{1!\sqrt{\pi}}{\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \qquad [\because \Gamma\,(n+1) = n\,\Gamma\,(n)]$$

$$\Rightarrow S = 16\pi a^2\,\frac{1!\sqrt{\pi}}{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}} = 16\pi a^2\,\frac{1}{\frac{3}{2}\times\frac{1}{2}} = 16\pi a^2\,\frac{4}{3} = \frac{64}{3}\pi a^2$$

Which is the required surface.

Example 207: Find the volume of the solid generated by the revolution of an area between the curve $y^2 = 2x$ and the line $y = 3x$ about the x-axis.

Solution:

Given parabola

$y^2 = 2x$ -------------------------(i)

and $y = 3x$ ---------------------------(ii)

From (i) and (ii)

$y^2 = 2x$

$9x^2 = 2x$ [$\because y = 3x$]

$9x^2 - 2x = 0$

or $x(9x-2) = 0$

or $x = 0$ and $(9x-2) = 0$

or $x = 0$ and $9x = 2$

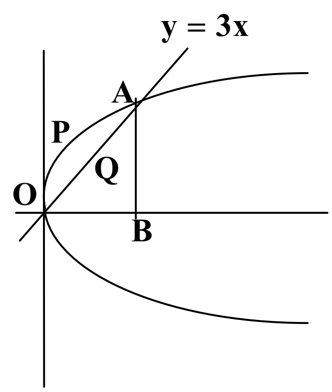

Figure # 109

or $\mathbf{x = 0}$ **and** $\mathbf{x = \frac{2}{9}}$

i,e $\mathbf{x = 0, \frac{2}{9}}$

Therefore the line cut the curve (i) at **O(0,0)** and A$\left(\frac{2}{9}, \frac{2}{3}\right)$

Therefore the required volume OPAQO is; $v =$ Volume of OPABO-volume of OQABO

$$\mathbf{v = \int_0^{2/9} \pi y_1^2 dx - \int_0^{2/9} \pi y_2^2 dx}$$

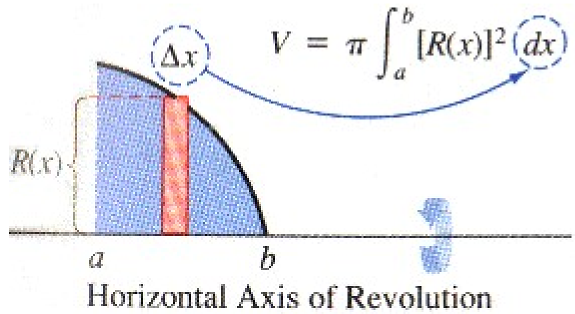

Figure # 110

$$\left[\text{Volume} = \int_a^b \pi(\textbf{radius})^2 dx = \int_a^b \pi(f(x))^2 dx = \int_a^b \pi y^2 dx\right]$$

$$\mathbf{v = \int_0^{2/9} \pi\, 2x dx - \int_0^{2/9} \pi(3x)^2 dx} \qquad [\because y_1^2 = y^2 = 2x \text{ and } y_2 = y = 3x]$$

$$\mathbf{v = \int_0^{2/9} \pi\, 2x dx - \int_0^{2/9} \pi 9x^2 dx = \pi \times \left[2\frac{x^2}{2}\right]_0^{\frac{2}{9}} - \pi\left[9\frac{x^3}{3}\right]_0^{\frac{2}{9}} = \pi \times \left[x^2\right]_0^{\frac{2}{9}} - \pi\left[3x^3\right]_0^{\frac{2}{9}}}$$

$$\mathbf{= \pi \times \left[(\frac{2}{9})^2 - 0\right] - \pi\left[3(\frac{2}{9})^3 - 0\right] = \pi \times \left[\frac{4}{81}\right] - \pi\left[\frac{3 \times 8}{729}\right] = \pi \times \left[\frac{4}{81}\right] - \pi\left[\frac{8}{243}\right]}$$

$$\mathbf{= \pi \times \left[\frac{4}{81} - \frac{8}{243}\right] = \pi \times \left[\frac{12-8}{243}\right] = \pi \times \left[\frac{4}{243}\right]}$$

Which is the required volume

Example 208: Find the area of the surface of a cone whose semi-vertical angle is α and base a circle of radius r.

Solution: The cone is generated by the revolution of the generator OB, which the axis of the cone which taken as x-axis. Since the semi-vertical angle of the curve is α. So the equation of OB is $\mathbf{y = x \tan \alpha}$ --------------------------(i)

Also let $\mathbf{PQ = ds}$ be an element of OB, where P(x,y) is any point on the line (i)

Also $\frac{dy}{dx} = \tan\alpha$ [From (i)]

Therefore,

$$\Rightarrow \frac{ds}{dx} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2}$$

$$\Rightarrow ds = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2} dx$$

$$\Rightarrow ds = \left\{1 + (\tan\alpha)^2\right\}^{1/2} dx$$

$$[\because \frac{dy}{dx} = \tan\alpha]$$

$$\Rightarrow ds = \left\{1 + \tan^2\alpha\right\}^{1/2} dx$$

$$\Rightarrow ds = \left\{\sec^2\alpha\right\}^{1/2} dx = \sec\alpha dx$$

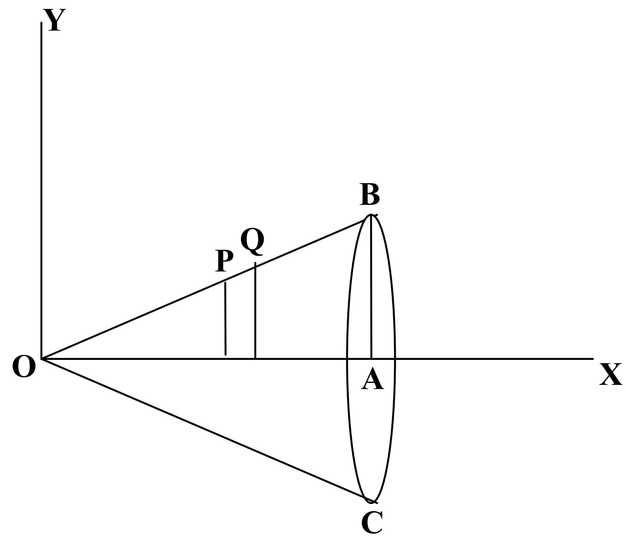

Figure # 111

From ΔOAB

$$\frac{OA}{AB} = \cot\alpha$$

$$OA = AB\cot\alpha = r\cot\alpha$$

Therefore, the required surface of the cone is:

$$S = \int_0^{r\cot\alpha} 2\pi y\, ds$$

$$\Rightarrow S = 2\pi \int_0^{r\cot\alpha} y\, ds \Rightarrow S = 2\pi \int_0^{r\cot\alpha} x\tan\alpha\, ds$$

$$\Rightarrow S = 2\pi \int_0^{r\cot\alpha} x\tan\alpha\sec\alpha dx \qquad [\because ds = \sec\alpha dx]$$

$$\Rightarrow S = 2\pi\tan\alpha\sec\alpha \int_0^{r\cot\alpha} x dx = 2\pi\tan\alpha\sec\alpha\left[\frac{x^2}{2}\right]_0^{r\cot\alpha}$$

$$\Rightarrow S = 2\pi\tan\alpha\sec\alpha\left[\frac{r^2\cot^2\alpha}{2} - 0\right] = 2\pi\tan\alpha\sec\alpha\left[\frac{r^2\cot^2\alpha}{2}\right]$$

$$\Rightarrow S = 2\pi\tan\alpha\sec\alpha\left[\frac{r^2}{2\tan^2\alpha}\right] = 2\pi\tan\alpha\sec\alpha \times \frac{r^2}{2\tan^2\alpha}$$

$$\Rightarrow S = \pi\sec\alpha \times \frac{r^2}{\tan\alpha} = \pi \times \frac{1}{\cos\alpha} \times \frac{r^2}{\frac{\sin\alpha}{\cos\alpha}} = \pi \times \frac{1}{\cos\alpha} \times \frac{r^2\cos\alpha}{\sin\alpha}$$

$$\Rightarrow S = \pi \times \frac{r^2}{\sin\alpha} = \pi r^2 \operatorname{cosec}\alpha$$

Which is the required surface.