প্র For what values of λ, the equation $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ may represent a pair of straight lines.

Solution: Given equation is,

$$x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$$

or, $x^2 + 2\cdot\left(-\frac{\lambda}{2}\right)xy + 2y^2 + 2\cdot\frac{3}{2}x + 2\cdot\left(-\frac{5}{2}\right)y + 2 = 0 \longrightarrow (1)$

Equation (1) will represent a pair of straight lines if

$\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or, $1\cdot 2\cdot 2 + 2\cdot(-5/2)\cdot\frac{3}{2}\cdot(-\lambda/2) - 1\cdot(-5/2)^2 - 2\cdot(3/2)^2 - 2\cdot(-\lambda/2)^2 = 0$

or, $4 + \frac{15}{4}\lambda - \frac{25}{4} - \frac{9}{2} - \frac{\lambda^2}{2} = 0$

or, $16 + 15\lambda - 25 - 18 - 2\lambda^2 = 0$

or, $2\lambda^2 - 15\lambda + 27 = 0$

or, $2\lambda^2 - 6\lambda - 9\lambda + 27 = 0$

or, $2\lambda(\lambda - 3) - 9(\lambda - 3) = 0$

or, $(\lambda - 3)(2\lambda - 9) = 0$

$\therefore \lambda = 3, 9/2.$

Ans.

Q. Find angle between the straight lines $(x^2+y^2)(\cos^2\theta\sin^2\alpha + \sin^2\theta) = (x\tan\theta - y\sin\alpha)^2$.

Solution: Given that,

$(x^2+y^2)(\cos^2\theta\sin^2\alpha + \sin^2\theta) = (x\tan\theta - y\sin\alpha)^2$

or, $x^2\cos^2\theta\sin^2\alpha + x^2\sin^2\theta + y^2\cos^2\theta\sin^2\alpha + y^2\sin^2\theta - x^2\tan^2\theta$

$+ 2xy\tan\theta\sin\alpha - y^2\sin^2\alpha = 0$

or, $(\cos^2\theta\sin^2\alpha - \tan^2\theta + \sin^2\theta)x^2 + (\cos^2\theta\sin^2\alpha + \sin^2\theta - \sin^2\alpha)y^2$

$+ 2(\sin\alpha\cdot\tan\theta)xy = 0 \longrightarrow (1).$

Comparing (1) with $ax^2 + 2hxy + by^2 = 0$, we get

$a = \cos^2\theta\sin^2\alpha - \tan^2\theta + \sin^2\theta$

$= \cos^2\theta(\sin^2\alpha + \tan^2\theta) - \tan^2\theta$

$= k\cos^2\theta - \tan^2\theta.$

$h = \sin\alpha\tan\theta$

$b = \cos^2\theta\sin^2\alpha + \sin^2\theta - \sin^2\alpha$

$= \cos^2\theta(\sin^2\alpha + \tan^2\theta) - \sin^2\alpha$

$= k\cos^2\theta - \sin^2\alpha$

where, $k = \sin^2\alpha + \tan^2\theta$

let, φ be the angle between the straight lines, We have $\tan\phi = \dfrac{2\sqrt{h^2-ab}}{a+b}$ . . . . . (2)

Here, $a+b = k\cos^2\theta - \tan^2\theta + k\cos^2\theta - \sin^2\alpha$

$= 2k\cos^2\theta - (\sin^2\alpha + \tan^2\theta)$

$= 2k\cos^2\theta - k$

$= k(2\cos^2\theta - 1)$

$= k\cos 2\theta$.

& $h^2 - ab = (\sin\alpha \tan\theta)^2 - (k\cos^2\theta - \tan^2\theta)(k\cos^2\theta - \sin^2\alpha)$

$= \sin^2\alpha \tan^2\theta - k^2\cos^4\theta + k\cos^2\theta \sin^2\alpha + k\tan^2\theta\cos^2\theta$
$- \tan^2\theta \sin^2\alpha$

$= k\cos^2\theta(\tan^2\theta + \sin^2\alpha) - k^2\cos^4\theta$

$= k^2\cos^2\theta - k^2\cos^4\theta$

$= k^2\cos^2\theta(1-\cos^2\theta)$

$= k^2\cos^2\theta \sin^2\theta$.

$\therefore 2\sqrt{h^2-ab} = 2\sqrt{k^2\cos^2\theta\sin^2\theta}$

$= 2k\cos\theta\sin\theta$

$= k\sin 2\theta$

Then equation (2) becomes, $\tan\phi = \dfrac{k\sin 2\theta}{k\cos 2\theta}$

or, $\tan\phi = \tan 2\theta$

$\therefore \phi = 2\theta$.

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✓ The equation that the pair of lines $ax^2+2hxy+by^2=0$ and $a'x^2+2h'xy+b'y^2=0$ should have one line in common is $4(ah'-a'h)(hb'-h'b) = (ab'-a'b)^2$

<u>Solution:</u> given lines are,

$$ax^2+2hxy+by^2=0 \quad \text{---} (i)$$

and

$$a'x^2+2h'xy+b'y^2=0 \quad \text{---} (ii)$$

Let, $y=mx$ is common to both the lines (i) & (ii).

$$\therefore ax^2+2hmx^2+bm^2x^2=0$$

or, $$bm^2+2hm+a=0 \quad \text{---} (iii)$$

and.

$$a'x^2+2h'mx^2+b'm^2x^2=0$$

or, $$b'm^2+2h'm+a'=0 \quad \text{---} (iv).$$

From (iii) & (iv) by cross-multiplication.....

$$\frac{m^2}{2(a'h-ah')} = \frac{m}{ab'-a'b} = \frac{1}{2(bh'-b'h)}$$

or, $$\frac{m^2}{2(ah'-a'h)} = \frac{m}{-(ab'-a'b)} = \frac{1}{2(hb'-h'b)}$$

or, $$m = \frac{2(ah'-a'h)}{-(ab'-a'b)} = \frac{-(ab'-a'b)}{2(hb'-h'b)}$$

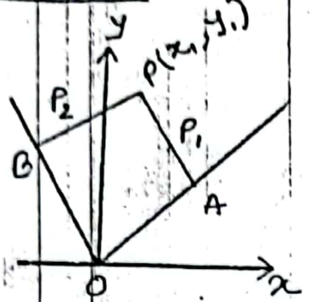
$$\therefore 4(ah'-a'h)(hb'-h'b) = (ab'-a'b)^2$$

[Proved]

Prove that the product of perpendicular from the po (x_1, y_1) at the lines $ax^2+2hxy+by^2=0$ is

$$\frac{ax^2+2hxy+by^2}{\sqrt{(a-b)^2+4h^2}}.$$

Solution:



The given lines are $ax^2+2hxy+by^2=0$ —— (1)

Let,

the lines given by (1) are

$y-m_1x=0$ —— (2)

and, $y-m_2x=0$ —— (3)

where, $m_1+m_2=-\frac{2h}{b}$ & $m_1m_2=\frac{a}{b}$ —— (4)

Now, product of the perpendicular from (x_1, y_1) to the lines (2) & (3) is

$$\frac{y_1-m_1x_1}{\sqrt{1+m_1^2}}\cdot\frac{y_1-m_2x_1}{\sqrt{1+m_2^2}}=\frac{y_1^2-(m_1+m_2)x_1y_1+(m_1m_2)x_1^2}{\sqrt{1+m_1^2+m_2^2+m_1^2m_2^2}}$$

$$=\frac{y_1^2+\frac{2h}{b}\cdot x_1y_1+\frac{a}{b}x_1^2}{\sqrt{1+(m_1+m_2)^2-2m_1m_2+m_1^2m_2^2}}$$

$$=\frac{ax_1^2+2hx_1y_1+by_1^2}{b\sqrt{1+\frac{4h^2}{b^2}-\frac{2a}{b}+\frac{a^2}{b^2}}}$$

$$=\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{a^2-2ab+b^2+4h^2}}$$

$$=\frac{ax_1^2+2hx_1y_1+by_1^2}{\sqrt{(a-b)^2+4h^2}}.$$

[Proved]

(3)

Q. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$, represent a pair of straight lines, equidistant from the origin then show that,

(i) $f^4-g^4=c(bf^2-ag^2)$

(ii) $h(g^2-f^2)=fg(a-b)$

Solution: given equation is $ax^2+2hxy+by^2+2gx+2fy+c=0$ ——— (1)

Since, eq^n (1) represents a pair of st. lines so we must have

$$\Delta \equiv abc+2fgh-af^2-bg^2-ch^2=0 \text{ ——— (i)}$$

Let the st. lines represented by eq^n (1) are

$$l_1x+m_1y+n_1=0 \text{ ——— (2)}$$

and $$l_2x+m_2y+n_2=0 \text{ ——— (3)}$$

where, $l_1l_2=a$, $l_1m_2+l_2m_1=2h$, $m_1m_2=b$,
$l_1n_2+l_2n_1=2g$, $m_1n_2+m_2n_1=2f$, $n_1n_2=c$.

Since the st. lines (2) & (3) are equidistant from the origin.

$$\frac{l_1\cdot 0+m_1\cdot 0+n_1}{\sqrt{l_1^2+m_1^2}}=\frac{l_2\cdot 0+m_2\cdot 0+n_2}{\sqrt{l_2^2+m_2^2}}$$

or, $n_1^2(l_2^2+m_2^2)=n_2^2(l_1^2+m_1^2)$

or, $(l_1n_2)^2-(l_2n_1)^2=(m_2n_1)^2-(m_1n_2)^2$

or, $(l_1n_2+l_2n_1)(l_1n_2-l_2n_1)=(m_2n_1+m_1n_2)(m_2n_1-m_1n_2)$

or, $2g\sqrt{(l_1n_2+l_2n_1)^2-4l_1l_2n_1n_2}=2f\sqrt{(m_2n_1+m_1n_2)^2-4m_1m_2n_1n_2}$

or, $g^2\cdot 4(g^2-ac)=f^2\cdot 4(f^2-bc)$

$\therefore f^4-g^4=c(bf^2-ag^2)$ ———→ (ii) [Proved]

Eliminating c from (i) & (ii) and multiply —

$$(f^4-g^4)(ab-h^2) = (bf^2-ag^2)(af^2-bg^2-2fgh)$$

or, $abf^4 - abg^4 - h^2f^4 + h^2g^4 = abf^4 + b^2f^2g^2 - 2bfg^3h - a^2g^2f^2 - abg^4 + 2afg^3h$

or, $(hg^2)^2 - 2hg^2 \cdot afg + (afg)^2 = (hf^2)^2 - 2hf^2 \cdot bfg + (bfg)^2$

or, $(hg^2 - afg)^2 = (hf^2 - bfg)^2$

or, $hg^2 - afg = hf^2 - bfg$

or, $h(g^2 - f^2) = fg(a-b)$

[Proved].

☐ Prove that the equation

$$(a+2h+b)x^2 + 2(a-b)xy + (a-2h+b)y^2 = 0.$$

denotes a pair of straight lines each inclined at an angle 45° to one or other of the lines given by $ax^2 + 2hxy + by^2 = 0$.

Solution: The given lines are

$$ax^2 + 2hxy + by^2 = 0 \text{ ——— (1)}$$

and, $(a+2h+b)x^2 - 2(a-b)xy + (a-2h+b)y^2 = 0$ —— (2)

Let the lines given by (1) are

$$y = m_1x \;\&\; y = m_2x \text{ ——— (3)}$$

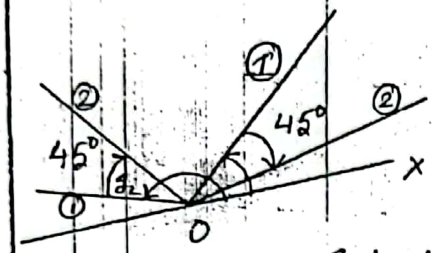
where, $m_1m_2 = \frac{a}{b}$ ——— (i)

$m_1 + m_2 = -\frac{2h}{b}$

(5)

Let m_1' and m_2' are the slopes of the lines making angle 45^0 with 1st and 2nd eq^n of (3).

$$\therefore \tan 45^\circ = \frac{m_1 - m_1'}{1 + m_1 m_1'}$$



$$\text{or, } 1 + m_1 m_1' = m_1 - m_1'$$

$$\therefore m_1' = \frac{m_1 - 1}{1 + m_1}$$

similarly, $m_2' = \dfrac{m_2 - 1}{1 + m_2}$.

$\therefore$ Eq^n of the pair of lines through the origin and having slope m_1' & m_2' is....

$$(y - m_1' x)(y - m_2' x) = 0$$

$$\text{or, } \left(y - \frac{m_1 - 1}{1 + m_1} \cdot x\right)\left(y - \frac{m_2 - 1}{1 + m_2} \cdot x\right) = 0$$

$$\text{or, } \{(m_1 - 1)x - (1 + m_1)y\}\{(m_2 - 1)x - (1 + m_2)y\} = 0$$

$$\text{or, } \{m_1 m_2 - (m_1 + m_2) + 1\}x^2 - \{(m_1 - 1)(1 + m_2) + (1 + m_1)(m_2 - 1)\}xy + (1 + m_1)(1 + m_2)y^2 = 0$$

$$\text{or, } \{m_1 m_2 - (m_1 + m_2) + 1\}x^2 - 2(m_1 m_2 - 1)xy + \{m_1 m_2 + (m_1 + m_2) + 1\} = $$

$$\text{or, } \left(\frac{a}{b} + \frac{2h}{b} + 1\right)x^2 - 2\left(\frac{a}{b} - 1\right)xy + \left(\frac{a}{b} - \frac{2h}{b} + 1\right)y^2 = $$

$$\text{or, } (a + 2h + b)x^2 - 2(a - b)xy + (a - 2h + b)y^2 = 0$$

[Proved]

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If $x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ represents a pair of straight lines at those straight lines makes angle α and β with x-axis. then prove that that —

$$\tan\alpha - \tan\beta = 2.$$

Solution:

The given eq^n is, $x^2(\tan^2\theta + \cos^2\theta) - 2xy\tan\theta + y^2\sin^2\theta = 0$ —— (1)

Let, the lines represented by (1) are,

$$y = m_1 x \quad \& \quad y = m_2 x$$

where, $m_1 m_2 = \dfrac{\tan^2\theta + \cos^2\theta}{\sin^2\theta} = \sec^2\theta + \cot^2\theta$

$$m_1 + m_2 = \frac{2\tan\theta}{\sin^2\theta} = \frac{2}{\sin\theta . \cos\theta}$$

Also given that, $m_1 = \tan\alpha$, & $m_2 = \tan\beta$

Now, $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$

$$= \frac{4}{\sin^2\theta . \cos^2\theta} - 4(\sec^2\theta + \cot^2\theta)$$

$$= 4.\sec^2\theta(\operatorname{cosec}^2\theta - 1) - 4.\cot^2\theta$$

$$= 4.\sec^2\theta.\cot^2\theta - 4\cot^2\theta$$

$$= 4.\cot^2\theta.(\sec^2\theta - 1)$$

$$= 4.\cot^2\theta.\tan^2\theta$$

$$= 4$$

$\therefore m_1 - m_2 = 2$

$\therefore \tan\alpha - \tan\beta = 2$

[Proved]