OSCILLATION

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Simple Harmonic Motion:

Simple harmonic motion is a special type of periodic motion or oscillation motion where the restoring forceis directly proportional to the displacement and acts in the direction opposite to that of displacement.

In other words, the more you pull it one way, the more it wants to return to the middle. The classic example of this is a mass on a spring, because the more the mass stretches it, the more it feels a tug back towards the middle. A mass on a spring can be vertical, in which case gravity is involved, or horizontal on a smooth tabletop.

Differential Equation of simple harmonic motion:

Let a particle of mass m be executing simple harmonic motion . The acceleration of the particle at displacement x from a fixed point will be d^2x/dt^2.

Now, from the definition of simple harmonic oscillation, the restoring force, F will be proportional to displacement x,

So,

$$F = -kx$$

Where, k is the constant, which is called the force constant of the particle. Here the -ve sign tells that the direction of force acting on the particle is opposite to the direction of increase in displacement.

From Newton's Second law of motion, we get,

$$F = ma$$
$$= m\ (dv/dt)$$
$$= m\ (d^2x/dt^2)$$

Hence,

$$m\ (d^2x/dt^2) = -kx$$

or, $d^2x/dt^2 = -\ (k/m)x$

By considering $k/m = \omega^2$ we get,

$$d^2x/dt^2 = -\ \omega^2 x$$

or, $d^2x/dt^2 + \omega^2 x = 0$

This is the differential quation of simple harmonic oscillation.

Solution of simple harmonic motion:

Multiplying the above equation by dx/dt we get

$$(dx/dt)(d^2x/dt^2) + \omega^2 x\ (dx/dt) = 0$$

or, $(d/dt)(\ dx/dt)^2 + \omega^2\ dx^2/dt = 0$

By integrating above equation,

$(\ dx/dt)^2 + \omega^2\ x^2 = A$, where A is a integration constant

At the position of maximum displacement, i.e, at $x = a$,

velocity of particle $dx/dt = 0$

so

$A = \omega^2 a^2$

Then,

$$(dx/dt)^2 + \omega^2 x^2 = \omega^2 a^2$$

$$or,\ dx/dt = \omega\sqrt{(a^2 - x^2)}$$

this equation tells us the velocity of particle at position x

By integrating th above equation

$$x = a \sin (\omega t + \varphi)$$

here a is the amplitude of oscillations and φ is the initial phase of the motion of particle.

Total energy of simple harmonic equation:

The equation of simple harmonic oscillation of amplitude a, angular frequency ω, phase φ is,

$$x = a \sin (\omega t + \varphi)$$

Potential energy is the energy possesed by the particle when it is at rest.

Considering a particle of mass m performing simple harmonic oscillation, then restoring force

$$F = -kx$$

Now, the particle is given further infinite displacement dx against the restoring force F.

So, the workdone dw during the displacement is

$$dw = - F\,dx$$

$$= kxdx$$

Then the total workdone to displace the particle from 0 to x,

$$\int dw = \int kxdx$$

$$= k \int x\,dx$$

$$= (1/2)\,k\,x^2$$

This total workdone is stored in the form of potential energy,

$$U = (1/2)\,k\,x^2$$

$$= (1/2)\,ka^2 \sin^2 (\omega t + \varphi)$$

Kinetic energy is the energy possed by an object when it is in motion.

$$K = (1/2)\,mv^2$$

But velocity ,

$$v = dx/dt$$

$$= \omega a \cos (\omega t + \varphi)$$

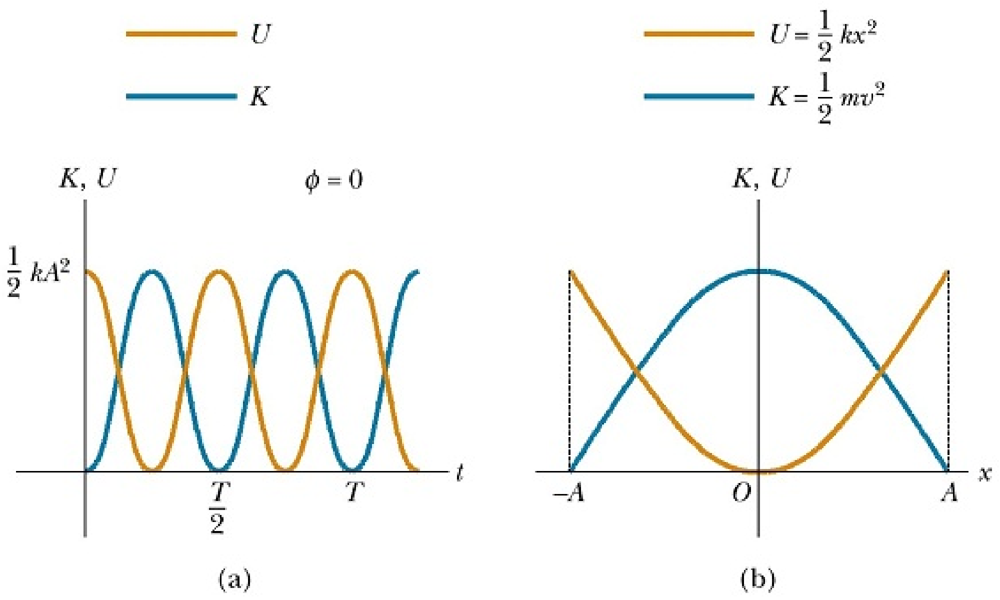
so,

$$K = (1/2)\,m\,\omega^2 a^2 \cos^2 (\omega t + \varphi)$$

$$= (1/2)\,k\,a^2 \cos^2 (\omega t + \varphi)$$

Therefore,

$$\textit{Total energy} = U+K$$
$$=(1/2)\, ka^2 \sin^2(\omega t + \varphi) + (1/2)\, k\, a^2 \cos^2(\omega t + \varphi)$$
$$=(1/2)\, k\, a^2$$



Average energy of simple harmonic oscillation:

Average potential energy,

$$U_{av} = (1/T)\int_0^T U\, dt$$
$$= (1/4)\, m\, \omega^2 a^2$$

Average kinetic energy,

$$K_{av} = (1/T)\int_0^T K\, dt$$
$$= (1/4)\,)\, m\, \omega^2 a^2$$

So,

$$U_{av} = K_{av} = (1/4)\,)\, m\, \omega^2 a^2 = (1/2)\, E$$

Compound harmonic oscillation:

Let us consider two simple harmonic oscillations of angular frequency ω, but have different amplitudes a_1 and a_2 .

So, the equation of displacement of these two waves can be of the form,

$$x_1 = a_1 \sin \omega t$$

And

$$x_2 = a_2 \sin (\omega t + \varphi)$$

Then the resultant equation becomes

$$x = x_1 + x_2$$

$$= a_1 \sin \omega t + a_2 \sin (\omega t + \varphi)$$

$$= a_1 \sin \omega t + a_2 (\sin\omega t \cos \varphi + \cos\omega t \sin \varphi)$$

$$= \sin \omega t (a_1 + a_2 \cos \varphi) + a_2 \sin\varphi \cos\omega t$$

We consider that,

$$A \cos\delta = (a_1 + a_2 \cos\varphi)$$

$$A \sin\delta = a_2 \sin\varphi$$

Hence the equation takes the form,

$$x = A \sin\omega t \cos\delta + A \cos\omega t \sin\delta$$

$$= A \sin(\omega t + \delta)$$

So the resultant wave will move with the amplitude A.

Now, the magnitude of amplitude will be,

$$A = \sqrt{(a_1^2 + a_2^2 + 2a_1a_2 \cos \varphi)}$$

We get,

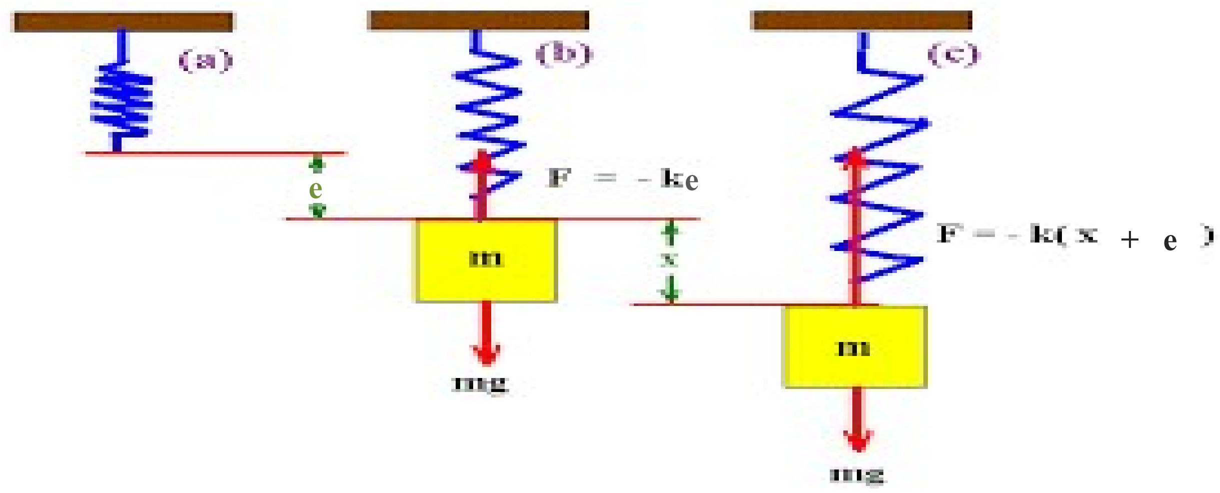
$$A_{max} = a_1 + a_2$$

$$A_{min} = a_1 - a_2$$

And

$$\tan\delta = (a_2 \sin \varphi) / (a_1 + a_2 \cos \varphi)$$

Spring mass system:



As shown in figure ,
A spring is suspended from point A. when a body of mass ***m*** is added, the stretch of the spring be ***e***.

At the equilibrium position the upward tension will be equal to the weight of the body. So,

$$mg = ke \text{ ……………………………….. (1)}$$

Now, if the spring is stretched further x by pulling the weight and then release it, then the spring will oscillate with the acceleration a for time t.

Then, the upward force,

$$T_1 = k(e+x)$$

According to Newton's Second law of motion,

$$F = ma$$

$$\text{or, } mg - T_1 = ma$$

$$\text{or, } ma = -kx$$

$$\text{or, } a = -(k/m)x$$

$$\text{or, } a = -\omega^2 x$$

Where, angular frequency, $\omega = \sqrt{(k/m)}$

Hence, the time period ,

$$T = (2\pi/\omega)$$

$$= 2\pi\sqrt{(m/k)}$$

$$= 2\pi\sqrt{(e/g)}$$

Effective mass:

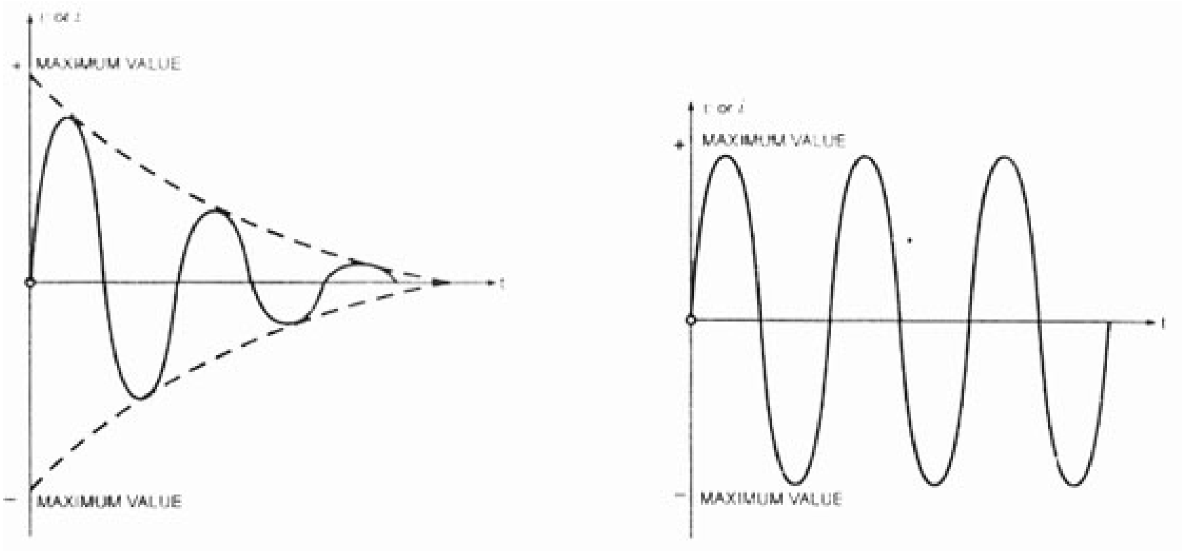
Effective mass is defined as the mass that needs to be added to suspended mass to correctly predict the behaviour of the system.

For uniform spring,
m* = (1/3)m

Damped Oscillation:

If a frictional force proportional to the velocity is present with the restoring force, the harmonic oscillator is described as a damped oscillator.

In real oscillators, friction, or damping, slows the motion of the system. Due to the frictional force, the velocity decreases in proportional to the acting frictional force.



(a) Damped Oscillations (b) Undamped or Sustained Oscillations

While simple harmonic motion oscillates with only the restoring force acting on the system, damped harmonic motion experiences friction. In many vibrating systems, the frictional force F_f can be modeled as being proportional to the velocity v of the object:

$F_f = -cv$, where c is called the viscous damping coefficient.

Balance of forces (Newton's second law) for damped harmonic oscillators is then

$$F_{ext} - kx - c\,dx/dt = m\, d^2x/dt^2$$

When no external forces are present (i.e. when $F_{ext} = 0$), this can be rewritten into the form,

$$m\, d^2x/dt^2 + c\,dx/dt + kx = 0$$

$$\text{or, } d^2x/dt^2 + (c/m)\, dx/dt + \omega^2 x = 0$$

where,

ω is called the 'undamped angular frequency of the oscillator.

Forced Oscillation:

If an oscillator is displaced and then released, it will begin to vibrate. If no more external forces are applied to the system it is a free oscillator. If a force is continually or repeatedly applied to keep the oscillation going, it is a forced oscillation.

In this case the equation of motion of the mass is given by,

$$m\frac{d^2x}{dt^2} = F_R - F_f + F$$

One common situation occurs when the driving force itself oscillates, in which case we may write

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos\omega_D t$$

where ω_D is the (angular) frequency of the driving force.

This equation has solutions of the form

$$x = B\sin(\omega_D t + \phi)$$

where the amplitude of these oscillations, B, depends on the parameters of the motion, $\omega,\ \omega_D, b, m, F_0$

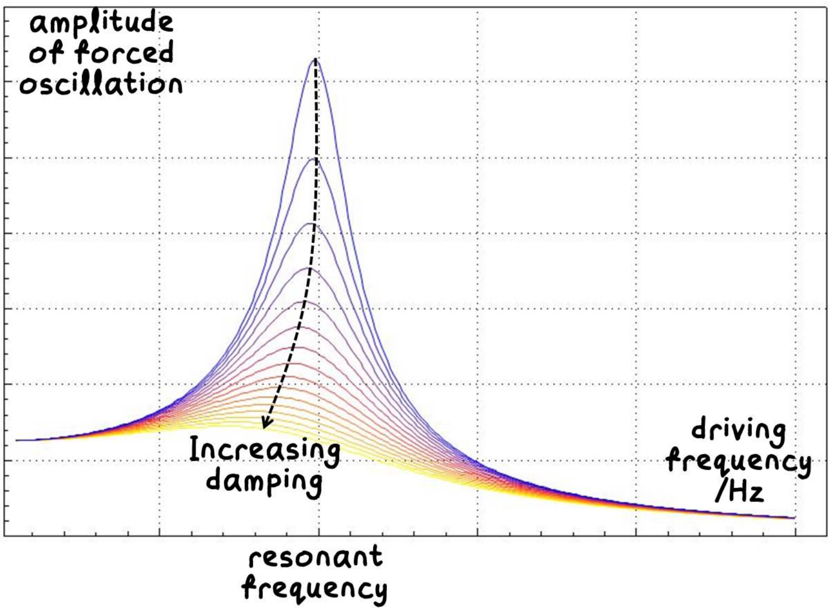


Figure: ***Forced oscillation and resonance***

Resonance:

Resonance is a phenomenon in which a vibrating system or external force drives another system to oscillate with greater amplitude at specific frequencies.

Frequencies at which the response amplitude is a relative maximum are known as the system's resonant frequencies or resonance frequencies.

There are various types of resonances. Such as-

- Mechanical and acoustic resonance
- Electrical resonance
- Optical resonance
- Orbital resonance
- Atomic, particle, and molecular resonance

Problem 1:

Determine the angular frequency and initial phase of the simple harmonic motion described by the equation,

$$y = 10 \sin(\omega t+\delta)$$

In which time period is 30s and displacement is 0.05 m initially.

Solution:

Here,

Time period, $T = 30s$

So,

$$\text{angular frequency, } \omega = 2\pi/T = 0.21 \text{ rad /s}$$

Again,

Initially,

Time, $t = 0s$

Displacement , $y = 0.05m$

So,

$$y = 10\sin(\omega t+\delta)$$

$$\text{or, } \delta = \sin^{-1}(y/10) - \omega t$$

$$\text{or, } \delta = 0.286 \text{ degree}$$

$$\text{or, } \delta = 0.005\text{rad}$$

Problem 2:

A body of mass 50g is attached to a spring. The motion of the spring will have the amplitude 12cm and time period 1.70s.

Find the frequency, spring constant, maximum velocity, maximum acceleration, velocity at time 5s, acceleration at distance 0.06m.

Solution:

We are given,

Mass of the body, m = 50g = 0.05kg

Amplitude, A = 12 cm =0.12m

Time period,T = 1.70s

Angular frequency, ω=?

Frequency, f =?

Spring constant, k=?

Maximum velocity, v_{max} =?

Maximum acceleration, a_{max} = ?

We know that,

$$\omega = 2\pi/T$$

$$= 3.69 \text{rad/s}$$

$$f = 1/T$$

$$= 0.59 \text{ hz}$$

$$k = \omega^2 m$$

$$=0.68 \text{ N/m}$$

$$v_{max} = \omega a$$

$$=0.44 \text{m/s}$$

$$a_{max} = \omega^2 A$$

$$=1.63 \text{ m/s}^2$$

Velocity at time 5s,

$$v = dx/dt$$

$$= A\omega \cos(\omega t)$$

$$a = -\omega^2 x$$

$$= -0.82 \text{ m/s}^2$$

<u>Problem 3:</u>

A spring is stretched 8cm when a body of mass 5g is attached to it. What is the time period ?

Solution:

Given that,

Extension of spring, $e = 8cm = 0.08\ m$

Acceleration due to gravity, $g = 9.8\ m/s^2$

Time period, $T = ?$

We know that,

$$T = 2\pi\sqrt{(e/g)}$$
$$= 0.57s$$