

## CHAPTER SIX

### RANDOM VARIABLE AND MATHEMATICAL EXPECTATION

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#### Random variable with example.

A random variable is a real valued function whose values are determined with the outcomes of a random experiment. It is usually denoted by  $X, Y, Z$  etc and the value of the random variable denoted by  $x, y, z$ .

Let us consider the experiment of tossing two fair coins. The sample space of the experiment is  $S = \{HH, HT, TH, TT\}$

Let  $X$  denotes the number of heads. So,

Sample point	Number of Head ( $X$ )
HH	2
HT	1
TH	1
TT	0

Here,  $X$  can take the values 0, 1 and 2. Therefore  $X$  is a random variable.

#### Types of random variable.

There are two types of random variable:

1. Discrete random variable.
2. Continuous random variable.

**Discrete random variable:** A random variable is called discrete random variable if it can take only isolated values.

Example: Family members, mobile number etc.

**Continuous random variable:** A random variable is called continuous random variable if it can take any values between certain limits.

Example: Age, weight, height etc.

### Probability function and probability density function.

**Probability function:** A function  $f(x)$  of a discrete random variable  $X$  is called a probability function if it satisfies the following two conditions:

- (i)  $f(x) \geq 0$
- (ii)  $\sum f(x) = 1$

**Probability density function:** A function  $f(x)$  of a continuous random variable  $X$  is called a probability function if it satisfies the following two conditions:

- (i)  $f(x) \geq 0$
- (ii)  $\int_{-\infty}^{\infty} f(x) dx = 1$

**Cumulative Distribution Function:** The **cumulative distribution function (cdf)**  $F(x)$  of a discrete random variable  $X$  with probability distribution  $f(x)$  is  $F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$

**Cumulative Distribution Function:** The cumulative distribution function  $F(x)$  of a continuous random variable  $X$  with density function  $f(x)$  is  $F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt$  for  $-\infty < x < \infty$

The consequence of above definition

$$P(a < x < b) = \int_{-\infty}^b f(x) dx - \int_{-\infty}^a f(x) dx = F(b) - F(a) \quad \text{and} \quad f(x) = \frac{dF(x)}{dx}$$

**Application problem:** A discrete random variable  $x$  has the following probability function:

x	0	1	2	3	4
f(x)	k	2k	3k	3k	k

For what value of  $k$  the function will be a probability function?

Ans: Since,  $\sum f(x) = 1$

$$\text{Or, } (k + 2k + 3k + 3k + k) = 1$$

$$\text{Or, } 10k = 1$$

$$\text{Or, } k = 0.10$$

Therefore,

x	0	1	2	3	4
f (x)	0.10	0.20	0.30	0.30	0.10

**Application problem:** A discrete random variable X has the following probability function:

Values of X:x	0	1	2	3	4
f (x)	0.12	0.18	k	0.30	0.16

(i) Find the value of k, Compute (ii)  $P[X > 3]$  ; (iii)  $P[1 < X < 4]$  ; and (iv)  $P[X < 1]$ .

(i) Since,  $\sum f(x) = 1$

$$\text{Or, } (0.76 + k) = 1$$

$$\text{Or, } k = 1 - 0.76$$

$$\text{Or, } k = 0.24$$

(ii)  $P[X > 3] = P[X = 4] = 0.16$

(iii)  $P[1 < X < 4] = P[X = 2] + P[X = 3]$

$$= k + 0.30$$

$$= 0.24 + 0.30 = 0.54$$

(iv)  $P[X < 1] = P[X = 0]$   
 $= 0.12$

**Assignment problem:** A discrete random variable X has the following probability function:

Values of X:x	0	1	2	3	4
f (x)	0.10	0.30	0.20	0.25	0.15

Find the value of (i)  $P[X = 1]$ ; (ii)  $P[X > 3]$  ; (iii)  $P[1 < X < 4]$  ; and (iv)  $P[X < 1]$ .

**Assignment problem:** Suppose x is discrete random variable with probability function.

Values of X:x	-2	-1	0	1	2
f (x)	0.3	0.2	0.1	0.25	0.15

Find the value of (i)  $P[X = 1]$ ; (ii)  $P[-1 < X < 2]$ ; (iii)  $P[X > 0]$  and  $P[X < -1]$

**Application problem:** A continuous random variable X has the following probability density function:

$$f(x) = kx^2 ; 0 \leq x \leq 1$$

- (i) Find the value of K
- (ii) probability that X lies between 0.2 and 0.50
- (iii) probability that X less than 0.30 and
- (iv) probability that X greater than 0.75

Ans:(i) Since,  $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\text{Or, } \int_0^1 kx^2 dx = 1$$

$$\text{Or, } k \left[ \frac{x^3}{3} \right]_0^1 = 1$$

$$\text{Or, } \frac{k}{3} = 1$$

$$\text{Or, } k = 3$$

- (ii) probability that X lies between 0.2 and 0.50

$$P[0.20 < X < 0.50] = \int_{0.20}^{0.50} kx^2 dx$$

$$= k \left[ \frac{x^3}{3} \right]_{0.20}^{0.50}$$

$$= 3 \left[ \frac{1}{3} (0.50^3 - 0.20^3) \right]$$

$$= 0.125 - 0.008 = 0.117$$

- (iii) probability that X less than 0.30

$$P[X < 0.3] = \int_0^{0.3} kx^2 dx$$

$$= k \left[ \frac{x^3}{3} \right]_0^{0.30} = 3 \left[ \frac{1}{3} (0.30^3 - 0^3) \right] = 0.027$$

- (iv) probability that X greater than 0.75 is  $P[X > 0.75] = \int_{0.75}^1 kx^2 dx$

$$= k \left[ \frac{x^3}{3} \right]_{0.75}^1 = 3 \left[ \frac{1}{3} (1^3 - 0.75^3) \right] = 0.578$$

**Assignment problem:** The following is the probability density function of a random variable  $x$ :

$$f(x) = \frac{3}{4}(2x - x^2); 0 < x < 2$$

Find (i) the value of 'K' ; (ii)  $P(x > 1)$  and (iii)  $P(1.5 < x < 2.25)$

**Assignment problem:** A continuous random variable  $X$  has the following probability density function:

$$f(x) = K(x - 1); \quad 2 \leq x \leq 6$$

Compute (i) the value of 'K' ; (ii)  $P(X > 3)$  and (iii)  $P(3 < X < 4)$

**Assignment problem:** A continuous random variable  $X$  has the following probability density function:

$$f(x) = K(x + 1); \quad 2 \leq x \leq 5$$

Compute (i) the value of 'K' ; (ii)  $P(X > 3)$ ; (iii)  $P(X = 4)$  and (iv)  $P(3 < X < 4)$

**Assignment problem:** Let  $X$  be a continuous random variable with probability density function

$$f(x) = kx; 0 \leq x \leq 4 \quad \text{Find (i) the value of } k; \text{ (ii) } P(x \geq 1); \text{ (iii) } P(x \leq 2)$$

## MATHEMATICAL EXPECTATION

If  $X$  is a discrete or continuous random variable with probability function or probability density function  $f(x)$ , then the mathematical expectation of  $X$  is usually denoted by  $E[X]$  or  $\mu$  and defined by

$$\begin{aligned}\mu = E[X] &= \sum xf(x) ; \text{ If } X \text{ is a discrete random variable.} \\ &= \int_{-\infty}^{\infty} xf(x)dx ; \text{ If } X \text{ is a continuous random variable.}\end{aligned}$$

### Properties of mathematical expectation of a random variable:

- (i) If  $b$  is a constant then  $E[b] = b$
- (ii) If  $X$  is a random variable with expectation  $E[X]$ , then  $E[aX+b] = a E[X] + b$ , Where  $a$  and  $b$  constant.
- (iii) If  $X$  is a random variable with expectation  $E[X]$ , then  $E[X-E(X)] = 0$
- (iv) If  $X$  and  $Y$  are random variables then  $E[X+Y] = E[X] + E[Y]$
- (v) If  $X$  and  $Y$  are random variables then  $E[X-Y] = E[X] - E[Y]$

### Properties of variance of a random variable:

- (i) If  $b$  is a constant then  $V[b] = 0$
- (ii) If  $X$  is a random variable, then  $V[aX+b] = a^2 V[X]$ , Where  $a$  and  $b$  constant.
- (iii) If  $X$  is a random variable with expectation  $E[X]$ , then  
 $V(X) = E[X-E(X)]^2 = E(X^2) - [E(X)]^2$
- (iv) If  $X$  and  $Y$  are random variables then  $V[X+Y] = V[X] + V[Y]$
- (v) If  $X$  and  $Y$  are random variables then  $V[X-Y] = V[X] + V[Y]$

**Application problem:** Find mean variance and standard deviation of the following probability function:

Or, Find (i)  $E[X]$  (ii)  $V(X)$  and (iii) standard deviation

Values of $X:x$	0	1	2	3
$f(x)$	0.125	0.375	0.375	0.125

Solution: (i) **Mean:**

$$\begin{aligned}
 \text{We know, } E[X] &= \sum xf(x) \\
 &= (0)(0.125) + (1)(0.375) + (2)(0.375) + (3)(0.125) \\
 &= 1.5
 \end{aligned}$$

**(ii) Variance:**

$$\text{We know, } V(X) = E(X^2) - [E(X)]^2$$

$$\begin{aligned}
 \text{Here, } E(X^2) &= \sum x^2 f(x) \\
 &= (0)^2(0.125) + (1)^2(0.375) + (2)^2(0.375) + (3)^2(0.125) \\
 &= 3
 \end{aligned}$$

$$\begin{aligned}
 V(X) &= E(X^2) - [E(X)]^2 \\
 &= 3 - (1.5)^2 = 0.75
 \end{aligned}$$

**(iii) Standard deviation:**  $\sqrt{V(X)} = \sqrt{0.75} = 0.87$

**Assignment problem:** Suppose x is discrete random variable with probability function.

Values of X:x	-2	-1	0	1	2
f(x)	0.3	0.2	0.1	0.25	0.15

Compute (i)  $E[x]$  ; (ii)  $E[3x+3]$  ; (iii)  $V[x]$  and (iv)  $V[2x-3]$

**Assignment problem:** Let X be a continuous random variable with probability density function

$$f(x) = kx ; 0 \leq x \leq 4 \quad \text{Find (i) the value of } k; \text{ (ii) } P(x \geq 1); \text{ (iii) } P(x \leq 2)$$

(iv) Mean, variance and standard deviation of x