INTEGRAL CALCULUS

By

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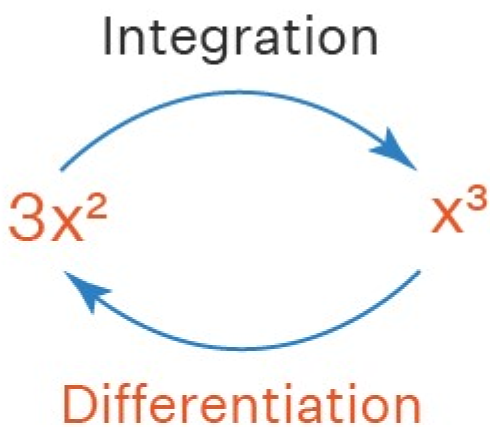
Chapter One

Integration an Inverse Process of Differentiation



01. Applications of the Indefinite Integral

1.1 Displacement from Velocity, and Velocity from Acceleration

A very useful application of calculus is displacement, velocity and acceleration.
Recall

$$\mathbf{v} = \frac{\mathbf{ds}}{\mathbf{dt}} \text{---(i)}$$

Similarly, we can find the expression for the **acceleration** by differentiating the expression for velocity, and this is equivalent to finding the second derivative of the displacement:

$$\mathbf{a} = \frac{\mathbf{dv}}{\mathbf{dt}} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\mathbf{ds}}{\mathbf{dt}}\right) \quad [\because \mathbf{v} = \frac{\mathbf{ds}}{\mathbf{dt}}]$$

$$\mathbf{a} = \frac{\mathbf{dv}}{\mathbf{dt}} = \frac{\mathbf{d}}{\mathbf{dt}}\left(\frac{\mathbf{ds}}{\mathbf{dt}}\right) = \frac{\mathbf{d^2s}}{\mathbf{dt^2}} \text{-----------------(ii)}$$

It follows (since integration is the opposite process to differentiation) that to obtain the **displacement**, *s* of an object at time *t* (given the expression for velocity, *v*) we would use:
From (i),

$$\mathbf{v} = \frac{\mathbf{ds}}{\mathbf{dt}}$$

$$\Rightarrow \mathbf{ds} = \mathbf{vdt}$$

$$\Rightarrow \int \mathbf{ds} = \int \mathbf{vdt}$$

$$\Rightarrow \mathbf{s} = \int \mathbf{vdt} \text{---------------------------------(iii)}$$

Similarly, the **velocity** of an object at time *t*, given the acceleration *a*, is given by:

From (ii),

$$\mathbf{a = \frac{dv}{dt}}$$

$$\Rightarrow \mathbf{dv = adt}$$
$$\Rightarrow \mathbf{\int dv = \int adt}$$
$$\Rightarrow \mathbf{v = \int adt} \text{ ---------------------------------(iv)}$$

Example 1: A proton moves in an electric field such that its acceleration (in cms^{-2}) is $\mathbf{a = -\frac{20}{(1+2t)^2}}$; where t is in seconds. Find the velocity as a function of time if $v = 30$ cms^{-1} when $t = 0$.

Solution: We have from (iv),

$$\mathbf{v = \int adt}$$

So $\Rightarrow \mathbf{v = \int \frac{-20dt}{(1+2t)^2}}$ -------------------------------(i)

Put

$$\mathbf{u = 1 + 2t}$$
$$\Rightarrow \mathbf{\frac{du}{dt} = \frac{d}{dt}(1+2t)}$$
$$\Rightarrow \mathbf{\frac{du}{dt} = 0 + 2.1}$$
$$\Rightarrow \mathbf{\frac{du}{dt} = 2}$$
$$\Rightarrow \mathbf{dt = \frac{du}{2}}$$

From (i), $\mathbf{v = \int \frac{-20dt}{(1+2t)^2} = \int \frac{-20}{u^2}.\frac{du}{2} = \int \frac{-10}{u^2}du = -10\int \frac{1}{u^2}du = -10\int u^{-2}du}$

$$\Rightarrow \mathbf{v = -10 \times \frac{u^{-2+1}}{-2+1} = -10 \times \frac{u^{-1}}{-1} = 10 \times \frac{1}{u} = \frac{10}{u}}$$
$$\Rightarrow \mathbf{v = \frac{10}{1+2t} + c \;[\because u = 1+2t]} \text{--------------------(ii)}$$

When $t = 0$, $v = 30$
Putting these values in (ii),

$$\mathbf{v = \frac{10}{1+2t} + c}$$
$$\Rightarrow \mathbf{30 = \frac{10}{1+2\times 0} + c}$$
$$\Rightarrow \mathbf{30 = \frac{10}{1} + c}$$

$$\Rightarrow 30 = 10 + c$$
$$\Rightarrow 30 - 10 = c$$
$$\Rightarrow 20 = c$$
$$\Rightarrow c = 20$$

From (ii), So the expression for velocity as a function of time is:

$$v = \frac{10}{1+2t} + 20 \text{ Cm/sec}$$

Example 2: A flare is ejected vertically upwards from the ground at 15 m/s. Find the height of the flare after2.5 second.

Solution: [The object has acting on it the force due to gravity, so its acceleration is $-9.8\,m/sec^2$ [in MKS system]

We have, $v = \int a dt$

$$\Rightarrow v = \int -9.8 dt$$
$$\Rightarrow v = -9.8t + c \text{ --------------------------------(i) } [\because \int dt = t]$$

Now at $t = 0$, the velocity, $v = 15 m/sec$

Putting these values in (i)

$$v = -9.8t + c$$
$$\Rightarrow 15 = -9.8 \times 0 + c$$
$$\Rightarrow 15 = 0 + c$$
$$\Rightarrow 15 = c$$
$$\Rightarrow c = 15$$

From (i), $v = -9.8t + c$

$$\Rightarrow v = -9.8t + 15 \ [\because c = 15]$$

So the expression for velocity becomes: $v = -9.8t + 15$

Now, we need to find the displacement, so

We have: $s = \int vdt$

$$\Rightarrow s = \int vdt$$
$$\Rightarrow s = \int (-9.8t + 15)dt \ \ [\because v = -9.8t + 15]$$
$$\Rightarrow s = \int -9.8t\,dt + \int 15dt = -9.8\int t^1\,dt + 15\int dt$$
$$\Rightarrow s = -9.8 \times \frac{t^{1+1}}{1+1} + 15t \ [\because \int x^n dx = \frac{x^{n+1}}{n+1} + c\ ]$$
$$\Rightarrow s = -9.8 \times \frac{t^2}{2} + 15t = -4.9 \times t^2 + 15t$$
$$\Rightarrow s = -4.9t^2 + 15t + c \text{ ------------------------------(ii)}$$

Now, we know from the question that when $t = 0$, $s = 0$

Putting these values in (ii),

$$s = -4.9t^2 + 15t + c$$
$$\Rightarrow 0 = -4.9.0^2 + 15.0 + c$$

$$\Rightarrow 0 = 0 + 0 + c$$
$$\Rightarrow 0 = c$$
$$\Rightarrow c = 0$$

From (ii), $s = -4.9t^2 + 15t + c$

$$\Rightarrow s = -4.9t^2 + 15t + 0$$
$$\Rightarrow s = -4.9t^2 + 15t \text{ ------------------------------------(iii)}$$

At time t = 2.5

Putting this value in (iii),

$$s = -4.9t^2 + 15t$$
$$\Rightarrow s = -4.9 \times (2.5)^2 + 15 \times 2.5 = -4.9 \times 2.5 \times 2.5 + 15 \times 2.5 = -30.625 + 37.5 = 68.125$$

1. 2. Displacement and Velocity Formulas

Using integration, we can obtain the well-known expressions for displacement and velocity, given a constant acceleration *a*, initial displacement zero, and an initial velocity v_0.

We have, $v = \int a dt$

$$\Rightarrow v = at + c \text{ -----------------------------(i)}$$

Since the velocity at $t = 0$ is v_0. That is $v = v_0$

Putting these values in (i),

$$v = at + c$$
$$\Rightarrow v_0 = a \times 0 + c$$
$$\Rightarrow v_0 = c$$
$$\Rightarrow c = v_0$$

From (i), $v = at + c$

$$\Rightarrow v = at + v_0 \; [\because c = v_0] \text{ ----------------------(ii)}$$

Similarly, we have,

$$s = \int v dt$$
$$\Rightarrow s = \int (at + v_0) dt \; [\because \text{from (ii)}; v = at + v_0]$$
$$\Rightarrow s = \int at dt + \int v_0 dt$$
$$\Rightarrow s = a \int t dt + v_0 \int dt$$
$$\Rightarrow s = a \int t^1 dt + v_0 \int dt$$
$$\Rightarrow s = a \times \frac{t^{1+1}}{1+1} + v_0 \times t + c \; [\because \int dt = t]$$
$$\Rightarrow s = a \times \frac{t^2}{2} + v_0 t + c$$
$$\Rightarrow s = a\frac{t^2}{2} + v_0 t + c \text{ ------------------------------(iii)}$$

Since the displacement at $t = 0$ is $s = 0$

Putting these values in (iii),

$$s = a\frac{t^2}{2} + v_0 t + c$$

$$\Rightarrow 0 = a\frac{0^2}{2} + v_0 \times 0 + c$$

$$\Rightarrow 0 = c$$

$$\Rightarrow c = 0$$

From (iii), $s = a\frac{t^2}{2} + v_0 t + c$

$$\Rightarrow s = a\frac{t^2}{2} + v_0 t + 0 \; [\because c = 0]$$

$$\Rightarrow s = a\frac{t^2}{2} + v_0 t \; = v_0 t + \frac{1}{2} a t^2$$

1.3 Voltage across a Capacitor

Definition: The current, *i* (amperes), in an electric circuit equals the time rate of change of the **charge** *q*, (in coulombs) that passes a given point in the circuit. We can write this (with *t* in seconds) as: $\mathbf{i = \frac{dq}{dt}}$

By writing $\mathbf{i\,dt = dq}$ and **integrating**, we have:

$$\int i\,dt = \int dq$$

$$\Rightarrow \int i\,dt = q$$

$$\Rightarrow q = \int i\,dt \text{ ------------------------------------(i)}$$

The voltage, V_C (in volts) across a capacitor with capacitance *C* (in farads) is given by $V_C = \frac{q}{C}$

It follows that

$$V_C = \frac{q}{C}$$

$$\Rightarrow V_C = \frac{1}{C} q$$

$$\Rightarrow V_C = \frac{1}{C}\int i\,dt \;\; [\because q = \int i\,dt]$$

$$\Rightarrow V_C = \frac{1}{C}\int i\,dt$$

Example 3: The electric current (in mA) in a computer circuit as a function of time is $\mathbf{i = 0.3 - 0.2\,t}$. What total charge passes a point in the circuit in $\mathbf{0.050s}$?

Solution: The charge, q, is given by: $\mathbf{q = \int i\,dt}$

$$q = \int (0.3 - 0.2\,t)\,dt$$

$$\Rightarrow q = \int 0.3\,dt - \int 0.2\,t\,dt = 0.3\int dt - 0.2\int t\,dt = 0.3 \times t - 0.2 \times \frac{t^{1+1}}{1+1} + c$$

$$\Rightarrow \mathbf{q = 0.3 \times t - 0.2 \times \frac{t^2}{2} + c = 0.3 \times t - 0.1 \times t^2 + c}$$

$$\Rightarrow \mathbf{q = 0.3\,t - 0.1\,t^2 + c} \text{ -----------------------(i)}$$

At $t = 0$, $q = 0$

Putting these values in (i)

$$\mathbf{q = 0.3\,t - 0.1\,t^2 + c}$$

$$\Rightarrow \mathbf{0 = 0.3 \times 0 - 0.1 \times 0^2 + c}$$

$$\Rightarrow \mathbf{0 = 0. + c}$$

$$\Rightarrow \mathbf{0 = c}$$

$$\Rightarrow \mathbf{c = 0}$$

From (i), $\mathbf{q = 0.3\,t - 0.1\,t^2 + c}$

$$\Rightarrow \mathbf{q = 0.3\,t - 0.1\,t^2 + 0 [\because c = 0]}$$

$$\Rightarrow \mathbf{q = 0.3\,t - 0.1\,t^2} \text{ -----------------------------(ii)}$$

At time $\mathbf{t = 0.050}$

$$\mathbf{q = 0.3\,t - 0.1\,t^2}$$

$$\Rightarrow \mathbf{q = 0.3 \times (0.050) - 0.1 \times (0.050)^2 = 0.015 - 0.00025 = 0.01475}$$

Example 4: The voltage across a $\mathbf{8.50\,nF}$ capacitor in an FM receiver circuit is zero. Find the voltage after $2.00\ \mu s$ if a current $\mathbf{i = 0.042t}$ (in mA) charges the capacitor.

Solution:

We have $\mathbf{V_C = \frac{q}{C}}$

$$\Rightarrow \mathbf{V_C = \frac{1}{C}\int i\,dt} \text{ --------------------------------(i)}$$

$$\mathbf{1nF = 10^{-9}\,F} \text{ and } \mathbf{1\mu s = 10^{-6}\,s}$$

$$\Rightarrow \mathbf{0.042\mathit{t}\ \mathit{mA} = 0.042 \times 10^{-3}\ \mathit{t}\ \mathit{A}}$$

From (i), $\mathbf{V_C = \frac{1}{C}\int i\,dt}$

$$V_C = \frac{0.042 \times 10^{-3}}{8.5 \times 10^{-9}} \int t\,dt$$

$$= 4.94 \times 10^3 \frac{t^2}{2} + K$$

$$= 2.47 \times 10^3 t^2 + K \text{ ----------------(ii)}$$

Now, we are told that when $t = 0$, $V_C = 0$.

Putting these values in (ii)

$K = 0$.

Thus

$V_C = 2.47 \times 10^3 t^2$

So when $t = 2.00\ \mu s$, we have:

$$
\begin{aligned}
V_C &= 2.47\times10^3(2\times10^{-6})^2\\
&= 9.882\times10^{-9}\\
&= 9.88\text{ nV}
\end{aligned}
$$

02. Geometrical Interpretation

https://www.youtube.com/watch?v=XIdM2oxPttQ

Integration can be used to find areas, volumes, central points and many useful things. It is often used to find the **area underneath the graph of a function and the x-axis**

To find the area bounded by the curve $\mathbf{y = f(x)}$, the x-axis and the ordinates at $\mathbf{x = a}$ and $\mathbf{x = b}$ that is prove that $\int_a^b \mathbf{f(x)\,dx} = \lim_{h\to0} \sum_{r=0}^{n-1} \mathbf{h\,f(a+rh)}$

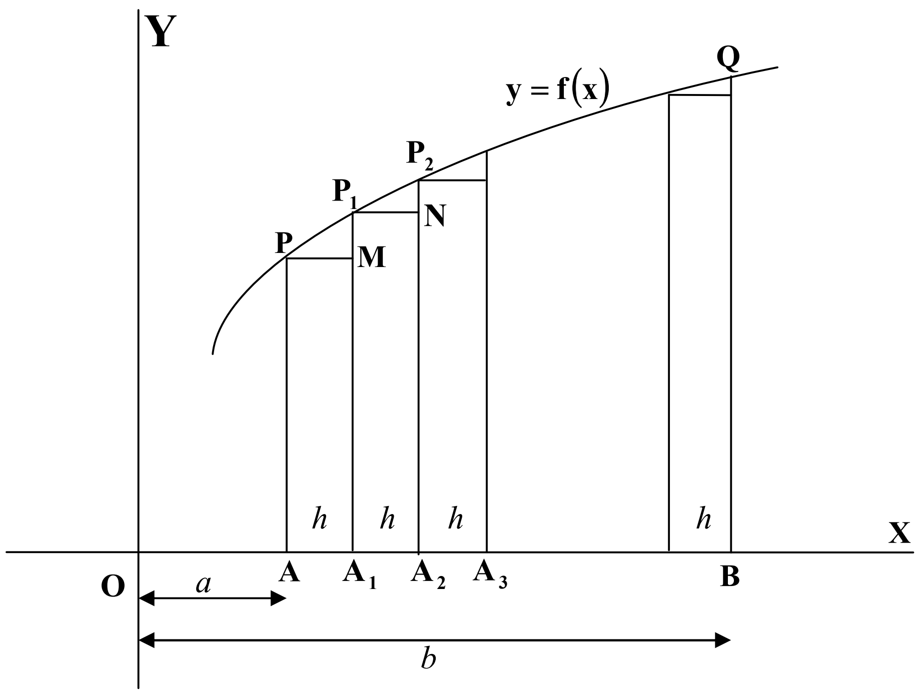


Figure 01

Let, $\mathbf{I} = \int_a^b \mathbf{f(x)dx}$

Suppose that the curve of $\mathbf{y = f(x)}$ shown above the figure.
Let, P & Q be the two points on the curve, Such that

$\mathbf{OA = a}$

$\mathbf{OB = b}$

$\mathbf{AA_1 = h,\ AA_2 = 2h}$

Then $\mathbf{AB = OB - OA = b - a}$ ---(i)

Let us, divide the interval $[a,b]$ into n equal subintervals of which each length is h. and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve $y = f(x)$ that is above the subinterval.

$$AA_1 = h$$
$$AA_2 = 2h$$
$$AA_3 = 3h$$

$$AB = nh$$

That is, $AB = OB - OA = b - a = nh$ -----------------------------------(ii)

Let $P(x, y)$ be a point on the curve $y = f(x)$

We have, $y = f(x)$ ---(iii)

Putting the values of x in (iii),

When $x = a$ then $y = f(a)$

When $x = a + h$ then $y = f(a + h)$

When $x = a + 2h$ then $y = f(a + 2h)$

When $x = a + 3h$ then $y = f(a + 3h)$

When $x = a + 4h$ then $y = f(a + 4h)$

$\therefore$ The coordinates of $P(x, y) = P(a, f(a))$

$\therefore$ The coordinates of $P_1(x, y) = P_1(a + h, f(a + h))$

$\therefore$ The coordinates of $P_2(x, y) = P_2(a + 2h, f(a + 2h))$

Now we see the area of all inner rectangles are

The area of 1st rectangle: $AA_1MP = AA_1 \times PA = h \times y = h \times f(a)$

The area of 2nd rectangle: $A_1A_2NP_1 = A_1A_2 \times P_1A_1 = h \times y = h \times f(a + h)$

The area of 3rd rectangle is $= h \times y = h \times f(a + 2h)$

The area of 4th rectangle is $= h \times y = h \times f(a + 3h)$

The area of nth rectangle is $= h \times y = h \times f(a + (n - 1)h)$

The total area is:

$$h \times f(a) + h \times f(a + h) + h \times f(a + 2h) + - - - - - - - - - - - - - + h \times f(a + (n - 1)h)$$

$$= h \sum_{r=0}^{n-1} f(a + rh)$$

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$$

Total area under the curve $\mathbf{y = f(x)}$ over the interval $\mathbf{[a,b]}$

$$= \int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh) \text{ ---------------(iv)}$$

[Integration মানে কোন নির্দিষ্ট অঞ্চলের ক্ষেত্রফল (Area) বের করা। Figure # 01 এ আমরা PABQ অঞ্চলের ক্ষেত্রফল বের করলাম। উক্ত ক্ষেত্রফলের মান (iv) নং সমীকরণ হতে বের করা যায়]

Example 05: **Evaluate the Integral** $\mathbf{I = \int_0^1 x\,dx}$ by geometrically

Solution: Here, $\mathbf{f(x) = x}$

$$\therefore f(a+rh) = a+rh$$

$$\therefore f(0+rh) = 0+rh \quad [\text{Here } a=0,\ b=1]$$

$$\therefore f(rh) = rh \quad \text{-----------------------------(i)}$$

We have, $\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} hf(a+rh)$ [From equation iv]

$$\int_0^1 x\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} hf(a+rh)$$

$$\int_0^1 x\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} hf(0+rh) \quad [\text{Here } a=0,\ b=1]$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} hf(rh)$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\times(rh) \quad [\because f(rh)=rh]\ [\text{From (i)}]$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\times(rh) \quad \text{-----------------------------(ii)}$$

$$= \lim_{h\to 0}\sum_{r=1}^{n} h\times(rh) \quad \text{----------------------------(iii)}$$

$$= \lim_{h\to 0} h(h+2h+3h+-------------+nh)$$

$$= \lim_{h\to 0} h\times\{h(1+2+3+---------------+n)\}$$

$$= \lim_{h\to 0} h^2\,\frac{n(n+1)}{2} = \lim_{h\to 0}\frac{nh\,(nh+h)}{2} \quad [\because 1+2+3+---+n = \frac{n(n+1)}{2}]$$

$$= \lim_{h\to 0}\frac{1(1+h)}{2} \quad [nh = b-a = 1-0 = 1] \text{ From Page no 10, equation no (ii)]}$$

$$\therefore \int_0^1 x\,dx = \frac{1}{2}$$

Example 06 Evaluate the Integral $\mathbf{I = \int_0^1 x dx}$

Solution: $\mathbf{I = \int_0^1 x dx = \left[\frac{x^{1+1}}{1+1}\right]_0^1 = \left[\frac{x^2}{2}\right]_0^1 = \left[\frac{1^2}{2} - \frac{0^2}{2}\right] = \left[\frac{1}{2} - \frac{0}{2}\right] = \frac{1}{2}\ [\because \int x^n dx = \frac{x^{n+1}}{n+1}]}$

Example 07: $\mathbf{\int_0^1 e^x dx}$

Solution: Here, $\mathbf{a = 0,\ \ b = 1,\ \ nh = b - a = 1}$

We have, $\mathbf{f(x) = e^x}$

$$\therefore \mathbf{f(a + rh) = e^{a+rh}} \text{ --(i)}$$

Now, $\mathbf{\int_a^b f(x)\, dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h f(a + rh)}$ [Page no 10; From equation iv]

$$\mathbf{\int_0^1 e^x dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h f(a + rh)}$$

$$\therefore \mathbf{\int_0^1 e^x dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\, e^{a+rh}} \qquad [\therefore \mathbf{f(a + rh) = e^{a+rh}}\text{; From (i)}]$$

$$\mathbf{= h \lim_{h\to 0} \sum_{r=0}^{n-1} e^{0+rh}} \qquad \mathbf{[a = 0]}$$

$$\mathbf{= h \lim_{h\to 0}(e^0 + e^{0+h} + e^{0+2h} + e^{0+3h} + ------------ + e^{0+(n-1)h})}$$

$$\mathbf{= h \lim_{h\to 0}(1 + e^h + e^{2h} + e^{3h} + ------------ + e^{(n-1)h})\ [e^0 = 1]}$$

$$\mathbf{= \lim_{h\to 0} h(1 + e^h + e^{2h} + e^{3h} + ------------ + e^{(n-1)h})}$$

$$\mathbf{= \lim_{h\to 0} h(1 + e^h + (e^h)^2 + (e^h)^3 + --------- + (e^h)^{n-1})}$$

$$\mathbf{= \lim_{h\to 0} h \frac{(e^h)^n - 1}{e^h - 1}} \qquad [\because\ \mathbf{1 + x + x^2 + ---x^{n-1} = \frac{x^n - 1}{x - 1}\ \ ; x > 1]}$$

$$\mathbf{= \lim_{h\to 0} h \frac{e^{hn} - 1}{e^h - 1}}$$

$$\mathbf{= \lim_{h\to 0} h \frac{e - 1}{e^h - 1}\ [a = 0, b = 1, nh = b - a = 1 - 0 = 1]} \text{ [Page no 10, equation (ii)]}$$

$$\mathbf{= (e - 1) \lim_{h\to 0} \frac{h}{e^h - 1}}$$

$$\mathbf{= (e - 1) \lim_{h\to 0} \frac{h}{\left[1 + h + \frac{h^2}{2\,!} + \frac{h^3}{3\,!} + ---\right] - 1}} \qquad [\because \mathbf{e^h = 1 + h + \frac{h^2}{2\,!} + \frac{h^3}{3\,!} + ---]}$$

$$= (e-1)\lim_{h\to 0}\frac{h}{\left[h+\frac{h^2}{2!}+\frac{h^3}{3!}+---\right]} = (e-1)\lim_{h\to 0}\frac{h}{h\left(1+\frac{h}{2!}+\frac{h^2}{3!}+---\right)}$$

$$= (e-1)\lim_{h\to 0}\frac{1}{1+\frac{h}{2!}+\frac{h^2}{3!}+---} = (e-1)\frac{1}{1+\frac{0}{2!}+\frac{0}{3!}+---}$$

$$= (e-1)\frac{1}{1+0+0+---} = (e-1)\cdot\tfrac{1}{1} = (e-1)$$

$\int_0^1 e^x dx = (e-1)$ ***Answer***

Example 08: $\int_0^1 e^x dx$ Directly

Solution: $I = \int_0^1 e^x dx$

$$= \left[e^x\right]_0^1 = \left[e^1 - e^0\right] = \left[e-1\right] = e-1$$

Example 09: $\int_0^1 x^2 dx$

Solution: We have, $\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$

Given, $f(x) = x^2$

$f(a+rh) = (a+rh)^2$

Here $a = 0,\ b = 1$

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$$

$$\int_0^1 x^2 dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\,(a+rh)^2 = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,(0+rh)^2 = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,(rh)^2 = \lim_{h\to 0} h\sum_{r=0}^{n-1} r^2 h^2$$

$$= \lim_{h\to 0} h\times h^2\sum_{r=0}^{n-1} r^2 = \lim_{h\to 0} h^3\sum_{r=0}^{n-1} r^2 = \lim_{h\to 0} h^3\sum_{r=1}^{n} r^2$$

$$= \lim_{h\to 0} h^3(1^2+2^2+3^2+---------+n^2)$$

$$= \lim_{h\to 0} h^3\,\frac{n(n+1)(2n+1)}{6} \qquad \left[\because 1^2+2^2+3^2+---+n^2 = \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{h\to 0} \frac{nh(nh+h)(2nh+h)}{6} \quad [\because nh = 1]$$

$$= \lim_{h\to 0} \frac{1.(1+h)(2\times 1+h)}{6} = \lim_{h\to 0} \frac{(1+h)(2+h)}{6} = \frac{(1+0)(2+0)}{6} = \frac{2}{6} = \frac{1}{3} \textit{ Answer}$$

Example 10: $\int_0^1 x^3 dx$

Solution: We have, $\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$

Given, $f(x) = x^3$

$f(a+rh) = (a+rh)^3$

Here $a = 0,\ b = 1$

$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$$

$$\int_0^1 x^3 dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h\,(a+rh)^3 = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,(0+rh)^3 = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,(rh)^3 = \lim_{h\to 0} h \sum_{r=0}^{n-1} r^3 h^3$$

$$= \lim_{h\to 0} h\times h^3 \sum_{r=0}^{n-1} r^3 = \lim_{h\to 0} h^4 \sum_{r=0}^{n-1} r^3 = \lim_{h\to 0} h^4 \sum_{r=1}^{n} r^3$$

$$= \lim_{h\to 0} h^4 \left(1^3 + 2^3 + 3^3 + --------- + n^3\right)$$

$$= \lim_{h\to 0} h^4 \left\{\frac{n(n+1)}{2}\right\}^2 = \lim_{h\to 0} h^4 \frac{n(n+1)}{2}\frac{n(n+1)}{2} = \lim_{h\to 0} \frac{nh(nh+h)}{2}\frac{nh(nh+h)}{2}$$

$$= \lim_{h\to 0} \frac{1.(1+h)}{2}\frac{1.(1+h)}{2} \quad [\because nh = 1]$$

$$= \frac{1.(1+0)}{2}\frac{1.(1+0)}{2} = \frac{1.(1)}{2}\frac{1.(1)}{2} = \frac{1}{4} \textit{ Answer}$$

Example 11: $\int_1^2 x^2 dx$

Solution: We have, $\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$

Given, $f(x) = x^2$

$f(a+rh) = (a+rh)^2$

Here $a = 1,\ b = 2$

We have, $b = a + nh$

$\Rightarrow 2 = 1 + nh$

$$\Rightarrow nh = 2 - 1 = 1$$
$$\Rightarrow nh = 1$$

$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$$

$$\int_1^2 x^2 dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h\,(a+rh)^2 = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,(1+rh)^2 = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,(1+2rh+r^2h^2)$$

$$= \lim_{h\to 0} h\sum_{r=0}^{n-1}(1) + \lim_{h\to 0} h\sum_{r=0}^{n-1} 2rh + \lim_{h\to 0} h\sum_{r=0}^{n-1}(r^2h^2)$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n}(1) + \lim_{h\to 0} h\sum_{r=1}^{n} 2rh + \lim_{h\to 0} h\sum_{r=1}^{n}(r^2h^2)$$

$$= \lim_{h\to 0} h(1+1+1+----+1) + \lim_{h\to 0} 2h^2\sum_{r=1}^{n} r + \lim_{h\to 0} h^3\sum_{r=1}^{n} r^2$$

$$= \lim_{h\to 0} h\times n + \lim_{h\to 0} 2h^2(1+2+3+----+n) + \lim_{h\to 0} h^3(1^2+2^2+3^3+----+n^2)$$

$$= \lim_{h\to 0}(nh) + \lim_{h\to 0} 2h^2\,\frac{n(n+1)}{2} + \lim_{h\to 0} h^3\left\{\frac{n(n+1)(2n+1)}{6}\right\}$$

$$= \lim_{h\to 0}(nh) + \lim_{h\to 0} 2\,\frac{nh(nh+h)}{2} + \lim_{h\to 0}\left\{\frac{nh(nh+h)(2nh+h)}{6}\right\}$$

$$= \lim_{h\to 0}(1) + \lim_{h\to 0} 2\,\frac{1.(1+h)}{2} + \lim_{h\to 0}\left\{\frac{1.(1+h)(2.1+h)}{6}\right\}$$

$$= (1) + 2\,\frac{1.(1+0)}{2} + \left\{\frac{1.(1+0)(2.1+0)}{6}\right\}$$

$$= (1) + 1 + \left\{\frac{(1)(2)}{6}\right\} = (1) + 1 + \frac{1}{3} = \frac{7}{3} \quad \textit{Answer}$$

Example 12: $\int_{-1}^{2} x^2 dx$

Solution: We have, $\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$

Given, $f(x) = x^2$

$f(a+rh) = (a+rh)^2$

Here $a = -1,\ b = 2$

We have, $b = a + nh$

$2 = -1 + nh$

$\Rightarrow nh = 2 + 1 = 3$

$\Rightarrow nh = 3$

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$$

$$\int_{-1}^{2} x^2 dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\,(a+rh)^2 = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,(-1+rh)^2$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\,(1-2rh+r^2h^2)$$

$$= \lim_{h\to 0} h\sum_{r=0}^{n-1}(1) - \lim_{h\to 0} h\sum_{r=0}^{n-1} 2rh + \lim_{h\to 0} h\sum_{r=0}^{n-1}(r^2h^2)$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n}(1) - \lim_{h\to 0} h\sum_{r=1}^{n} 2rh + \lim_{h\to 0} h\sum_{r=1}^{n}(r^2h^2)$$

$$= \lim_{h\to 0} h(1+1+1+-----+1) - \lim_{h\to 0} 2h^2\sum_{r=1}^{n} r + \lim_{h\to 0} h^3\sum_{r=1}^{n} r^2$$

$$= \lim_{h\to 0} h\times n - \lim_{h\to 0} 2h^2(1+2+3+-----+n) + \lim_{h\to 0} h^3(1^2+2^2+3^3+-----+n^2)$$

$$= \lim_{h\to 0}(nh) - \lim_{h\to 0} 2h^2\,\frac{n(n+1)}{2} + \lim_{h\to 0} h^3\left\{\frac{n(n+1)(2n+1)}{6}\right\}$$

$$= \lim_{h\to 0}(nh) - \lim_{h\to 0} 2\,\frac{nh(nh+h)}{2} + \lim_{h\to 0}\left\{\frac{nh(nh+h)(2nh+h)}{6}\right\}$$

$$= \lim_{h\to 0}(3) - \lim_{h\to 0} 2\,\frac{3.(3+h)}{2} + \lim_{h\to 0}\left\{\frac{3.(3+h)(2.3+h)}{6}\right\}$$

$$= (3) - 2\,\frac{3.(3+0)}{2} + \left\{\frac{3.(3+0)(6+0)}{6}\right\}$$

$$= (3) - 9 + \left\{\frac{(9)(6)}{6}\right\} = (3) - 9 + 9 = 3 \ \textit{Answer}$$

Example 13: $\int_a^b e^{-x}dx$

Solution: We have, $\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$

Given, $f(x) = e^{-x}$

We have,

$$f(x) = e^{-x}$$

$$\therefore f(a+rh) = e^{-(a+rh)}$$

Now, $\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$

$$\int_a^b e^{-x}dx = \lim_{h\to 0}\sum_{r=0}^{n-1} hf(a+rh)$$

$$\int_a^b e^{-x}dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\, e^{-(a+rh)}$$

$$\int_a^b e^{-x}dx = \lim_{h\to 0}\sum_{r=1}^{n} h\, e^{-(a+rh)}$$

$$= \lim_{h\to 0} h(e^{-(a+h)} + e^{-(a+2h)} + e^{-(a+3h)} + ------------ + e^{-(a+nh)})$$

$$= \lim_{h\to 0} h.(e^{-(a+h)} + e^{-(a+h+h)} + e^{-(a+h+2h)} + ------------ + e^{-(a+h+(n-1)h)})$$

$$= \lim_{h\to 0} h.(e^{-(a+h)} + e^{-(a+h)-h} + e^{-(a+h)-2h} + ------------ + e^{-(a+h)-(n-1)h)})$$

$$= \lim_{h\to 0} h.(e^{-(a+h)} + e^{-(a+h)}.e^{-h} + e^{-(a+h)}.e^{-2h} + ---------- + e^{-(a+h)}.e^{-(n-1)h)})$$

$$= \lim_{h\to 0} h.e^{-(a+h)}.[1 + e^{-h} + e^{-2h} + ---------- + e^{-(n-1)h}]$$

$$= \lim_{h\to 0} h.e^{-(a+h)}.[1 + (e^{-h})^1 + (e^{-h})^2 + ---------- + (e^{-h})^{n-1}]$$

$$`[\because 1 + x^1 + x^2 + ---------- + x^{n-1} = \frac{1-x^n}{1-x}]$$

$$= \lim_{h\to 0} h.e^{-(a+h)}.\frac{1-(e^{-h})^n}{1-e^{-h}} = \lim_{h\to 0} h.e^{-(a+h)}.\frac{1-e^{-nh}}{1-e^{-h}}$$

$$= \lim_{h\to 0} h.e^{-(a+h)}.\frac{1-e^{-(b-a)}}{1-e^{-h}} \qquad [\because nh = b-a]$$

$$= \lim_{h\to 0} h.\frac{1.e^{-(a+h)} - e^{-(b-a)}.e^{-(a+h)}}{1-e^{-h}} = \lim_{h\to 0} h.\frac{1.e^{-(a+h)} - e^{-b+a-a-h}}{1-e^{-h}}$$

$$= \lim_{h\to 0} h.\frac{1.e^{-(a+h)} - e^{-b-h}}{1-e^{-h}} = \lim_{h\to 0} h.\frac{1.e^{-a}.e^{-h} - e^{-b}.e^{-h}}{1-e^{-h}}$$

$$= \lim_{h\to 0} h.\frac{e^{-h}(e^{-a} - e^{-b})}{1-e^{-h}} = (e^{-a} - e^{-b})\lim_{h\to 0} h.\frac{e^{-h}}{1-e^{-h}}$$

$$= (e^{-a} - e^{-b})\lim_{h\to 0}\frac{h.e^{-h}}{1-e^{-h}} = (e^{-a} - e^{-b})\lim_{h\to 0}\frac{e-^{h}\ h}{e^{-h}(\frac{1}{e^{-h}} - 1)}$$

$$= (e^{-a} - e^{-b})\lim_{h\to 0}\frac{h}{(\frac{1}{e^{-h}} - 1)} = (e^{-a} - e^{-b})\lim_{h\to 0}\frac{h}{(e^{h} - 1)}$$

$$= (e^{-a} - e^{-b})\lim_{h\to 0}\frac{h}{(1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + --------- 1)}$$

$$= (e^{-a} - e^{-b})\lim_{h\to 0}\frac{h}{(\frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + -------)}$$

$$= (e^{-a} - e^{-b}) \lim_{h\to 0} \frac{h}{h\left(\frac{1}{1!} + \frac{h}{2!} + \frac{h^2}{3!} + --------\right)}$$

$$= (e^{-a} - e^{-b}) \lim_{h\to 0} \frac{1}{\left(\frac{1}{1!} + \frac{h}{2!} + \frac{h^2}{3!} + --------\right)}$$

$$= (e^{-a} - e^{-b}) \frac{1}{\left(\frac{1}{1!} + \frac{0}{2!} + \frac{0^2}{3!} + --------\right)}$$

$$= (e^{-a} - e^{-b}) \frac{1}{\left(\frac{1}{1!} + 0 + 0 + --------\right)}$$

$= (e^{-a} - e^{-b})$ Answer

Example 14: Evaluate $\int_0^{\pi/2} \sin x\, dx$

Solution: $\because nh = b - a$

$\because nh = \pi/2 - 0$

$\because nh = \pi/2$

Given, $f(x) = \sin x$

$\therefore f(a + rh) = \sin(a + rh)$

We have, $\int_a^b f(x)\, dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\, f(a + rh)$

$$\int_0^{\pi/2} \sin x\, dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\, f(a + rh)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h \sin(a + rh)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h \sin(0 + rh) \qquad [a = 0]$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h \sin rh = \lim_{h\to 0} \sum_{r=1}^{n} h \sin rh$$

$$\int_0^{\pi/2} \sin x\, dx = \lim_{h\to 0} h(\sin h + \sin 2h + \sin 3h + -------- + \sin nh) \text{------------------(i)}$$

Let $S = (\sin h + \sin 2h + \sin 3h + -------- + \sin nh)$ ----------------------------------(ii)

Multiplying by $2\sin\frac{h}{2}$

$$\Rightarrow 2\sin\frac{h}{2} S = (\sin h + \sin 2h + \sin 3h + -------- + \sin nh) 2\sin\frac{h}{2}$$

$\Rightarrow$

$$S\times 2\sin\frac{h}{2}=\sin h\times 2\sin\frac{h}{2}+\sin 2h\times 2\sin\frac{h}{2}+\sin 3h\times 2\sin\frac{h}{2}+----+\sin nh\times 2\sin\frac{h}{2}$$

$$\Rightarrow\; S\times 2\sin\frac{h}{2}=2\sin h\sin\frac{h}{2}+2\sin 2h\sin\frac{h}{2}+2\sin 3h\sin\frac{h}{2}+---------+2\sin nh\sin\frac{h}{2}$$

$\Rightarrow$

$$S\times 2\sin\frac{h}{2}=\left\{\cos(h-\frac{h}{2})-\cos(h+\frac{h}{2})\right\}+\left\{\cos(2h-\frac{h}{2})-\cos(2h+\frac{h}{2})\right\}+$$

$$\left\{\cos(3h-\frac{h}{2})-\cos(3h+\frac{h}{2})\right\}+---------------+\left\{\cos(nh-\frac{h}{2})-\cos(nh+\frac{h}{2})\right\}$$

$[\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)]$

$\Rightarrow$

$$S\times 2\sin\frac{h}{2}=\left\{\cos(\frac{h}{2})-\cos(\frac{3h}{2})\right\}+\left\{\cos(\frac{3h}{2})-\cos(\frac{5h}{2})\right\}+\left\{\cos(\frac{5h}{2})-\cos(\frac{7h}{2})\right\}+----$$

$$-----------------+\left\{\cos(nh-\frac{h}{2})-\cos(nh+\frac{h}{2})\right\}$$

$$\Rightarrow\; S\times 2\sin\frac{h}{2}=\cos(\frac{h}{2})-\cos(\frac{3h}{2})+\cos(\frac{3h}{2})-\cos(\frac{5h}{2})+\cos(\frac{5h}{2})-\cos(\frac{7h}{2})+------$$

$$----------------------+\left\{\cos(nh-\frac{h}{2})-\cos(nh+\frac{h}{2})\right\}$$

$$\Rightarrow\; S\times 2\sin\frac{h}{2}=\cos(\frac{h}{2})+\left\{-\cos(nh+\frac{h}{2})\right\}$$

$$\Rightarrow\; S\times 2\sin\frac{h}{2}=\cos(\frac{h}{2})-\cos(nh+\frac{h}{2})$$

$$\Rightarrow\; S=\frac{\cos(\frac{h}{2})-\cos(nh+\frac{h}{2})}{2\sin\frac{h}{2}}\text{---------------------------------(iii)}$$

From (1),

$$\Rightarrow\int_0^{\pi/2}\sin x\,dx=\lim_{h\to 0}h(\sin h+\sin 2h+\sin 3h+--------+\sin nh)$$

$$\Rightarrow\int_0^{\pi/2}\sin x\,dx=\lim_{h\to 0}h\times\frac{\cos(\frac{h}{2})-\cos(nh+\frac{h}{2})}{2\sin\frac{h}{2}}\qquad\text{[From (iii)]}$$

$$\Rightarrow\int_0^{\pi/2}\sin x\,dx=\lim_{h\to 0}\frac{\frac{h}{2}\left\{\cos(\frac{h}{2})-\cos(nh+\frac{h}{2})\right\}}{\sin\frac{h}{2}}$$

$$\Rightarrow \int_0^{\pi/2} \sin x\, dx = \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} . \left[\lim_{h\to 0}\left\{\cos(\frac{h}{2}) - \cos(nh+\frac{h}{2})\right\}\right]$$

$$\Rightarrow \int_0^{\pi/2} \sin x\, dx = 1. \lim_{h\to 0}\left\{\cos(\frac{h}{2}) - \cos(nh+\frac{h}{2})\right\} \qquad [\because \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1]$$

$$\Rightarrow \int_0^{\pi/2} \sin x\, dx = 1. \left\{\cos(\frac{0}{2}) - \cos(\frac{\pi}{2}+\frac{0}{2})\right\} \qquad [\because nh = \pi/2]$$

$$\Rightarrow \int_0^{\pi/2} \sin x\, dx = 1. \left\{\cos 0 - \cos(\frac{\pi}{2}+0)\right\}$$

$$\Rightarrow \int_0^{\pi/2} \sin x\, dx = 1. \left\{\cos 0 - \cos(\frac{\pi}{2})\right\}$$

$$\Rightarrow \int_0^{\pi/2} \sin x\, dx = 1. (1-0) = 1 \textit{ Answer}$$

Directly: $\int_0^{\pi/2} \sin x\, dx = [-\cos x]_0^{\pi/2} = -\left[\cos\frac{\pi}{2} - \cos 0\right] = -[0-1] = 1$

Example 15: Evaluate $\int_a^b \sin x\, dx$

Solution: $\because nh = b - a$

$\because nh = b - a$

Given, $f(x) = \sin x$

$\therefore f(a+rh) = \sin(a+rh)$

We have, $\int_a^b f(x)\, dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\, f(a+rh)$

$$\int_a^b \sin x\, dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\, f(a+rh)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h \sin(a+rh)$$

$$= \lim_{h\to 0} \sum_{r=1}^{n} h \sin(a+rh)$$

$$\int_a^b \sin x\, dx = \lim_{h\to 0} h[\sin(a+h) + \sin(a+2h) + \sin(a+3h) + --- + \sin(a+nh)] \text{---------(i)}$$

Let, $S = \sin(a+h) + \sin(a+2h) + \sin(a+3h) + --------- + \sin(a+nh)$ ------------(ii)

Multiplying by $2\sin\frac{h}{2}$

$$\Rightarrow 2\sin\frac{h}{2}S = \left[\sin(a+h)+\sin(a+2h)+\sin(a+3h)+---------+\sin(a+nh)\right]\sin\frac{h}{2}$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = \sin(a+h)\times 2\sin\frac{h}{2}+\sin(a+2h)\times 2\sin\frac{h}{2}+\sin(a+3h)\times 2\sin\frac{h}{2}+---$$

$$---------------+\sin(a+nh)\times 2\sin\frac{h}{2}$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = 2\sin(a+h)\sin\frac{h}{2}+2\sin(a+2h)\sin\frac{h}{2}+2\sin(a+3h)\sin\frac{h}{2}+------$$

$$---------------+2\sin(a+nh)\sin\frac{h}{2}$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = \left\{\cos(a+h-\frac{h}{2})-\cos(a+h+\frac{h}{2})\right\}+\left\{\cos(a+2h-\frac{h}{2})-\cos(a+2h+\frac{h}{2})\right\}+$$

$$\left\{\cos(a+3h-\frac{h}{2})-\cos(a+3h+\frac{h}{2})\right\}+---------+\left\{\cos(a+nh-\frac{h}{2})-\cos(a+nh+\frac{h}{2})\right\}$$

$$[\because 2\sin A\sin B = \cos(A-B)-\cos(A+B)]$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = \left\{\cos(a+\frac{h}{2})-\cos(a+\frac{3h}{2})\right\}+\left\{\cos(a+\frac{3h}{2})-\cos(a+\frac{5h}{2})\right\}+$$

$$\left\{\cos(a+\frac{5h}{2})-\cos(a+\frac{7h}{2})\right\}+-----------+\left\{\cos(a+nh-\frac{h}{2})-\cos(a+nh+\frac{h}{2})\right\}$$

$$\Rightarrow \begin{array}{l} S\times 2\sin\frac{h}{2} = \cos(a+\frac{h}{2})-\cos(a+\frac{3h}{2})+\cos(a+\frac{3h}{2})-\cos(a+\frac{5h}{2})+\cos(a+\frac{5h}{2}) \\ -\cos(a+\frac{7h}{2})+-----------------+\left\{\cos(a+nh-\frac{h}{2})-\cos(a+nh+\frac{h}{2})\right\} \end{array}$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = \cos(a+\frac{h}{2})+\left\{-\cos(a+nh+\frac{h}{2})\right\}$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = \cos(a+\frac{h}{2})-\cos(a+nh+\frac{h}{2})$$

$$\Rightarrow S = \frac{\cos(a+\frac{h}{2})-\cos(a+nh+\frac{h}{2})}{2\sin\frac{h}{2}} \text{-------------------------------------(iii)}$$

From (i),

$$\Rightarrow \int_a^b \sin x\,dx = \lim_{h\to 0} h\left[\sin(a+h)+\sin(a+2h)+\sin(a+3h)+---+\sin(a+nh)\right]$$

$$\Rightarrow \int_a^b \sin x\,dx = \lim_{h\to 0} h\ \frac{\cos(a+\frac{h}{2})-\cos(a+nh+\frac{h}{2})}{2\sin\frac{h}{2}} \text{ [From (iii)]}$$

$$\Rightarrow \int_a^b \sin x\,dx = \lim_{h\to 0} \frac{\frac{h}{2}\left\{\cos(a+\frac{h}{2}) - \cos(a+nh+\frac{h}{2})\right\}}{\sin\frac{h}{2}}$$

$$\Rightarrow \int_a^b \sin x\,dx = \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}}.\left[\lim_{h\to 0}\left\{\cos(a+\frac{h}{2}) - \cos(a+nh+\frac{h}{2})\right\}\right]$$

$$\Rightarrow \int_a^b \sin x\,dx = 1.\ \lim_{h\to 0}\left\{\cos(a+\frac{h}{2}) - \cos(a+nh+\frac{h}{2})\right\} \qquad [\because \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1]$$

$$\Rightarrow \int_a^b \sin x\,dx = 1.\ \left\{\cos(a+\frac{0}{2}) - \cos(a+b-a+\frac{0}{2})\right\} \ [\because nh = b-a]$$

$$\Rightarrow \int_a^b \sin x\,dx = 1.\ \{\cos a - \cos(b)\} = (\cos a - \cos b)$$

$$\Rightarrow \int_a^b \sin x\,dx = \cos a - \cos b \ \textit{Answer}$$

Directly: $\int_a^b \sin x\,dx = [-\cos x]_a^b = -[\cos b - \cos a] = \cos a - \cos b$ ***Answer***

Example 16: $\int_2^3 x^3 dx$

Solution: We have, $\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$

Given, $f(x) = x^3$

$f(a+rh) = (a+rh)^3$

Here $a = 2,\ b = 3$

$\because nh = b - a$

$\therefore nh = 3 - 2 = 1$

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$$

$$\int_2^3 x^3 dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\,(a+rh)^3$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\,(2+rh)^3 \qquad [\because a = 2,\ b = 3]$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} (2+rh)^3$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} \left\{2^3 + 3\times 2^2 \times rh + 3\times 2\times (rh)^2 + (rh)^3\right\}$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} \left(8 + 12rh + 6r^2h^2 + r^3h^3\right)$$

$$= \lim_{h\to 0} h\sum_{r=1}^{n} 8 + \lim_{h\to 0} h\sum_{r=1}^{n} 12rh + \lim_{h\to 0} h\sum_{r=1}^{n} 6r^2h^2 + \lim_{h\to 0} h\sum_{r=1}^{n} r^3h^3$$

$$= \lim_{h\to 0} h8\sum_{r=1}^{n} + \lim_{h\to 0} 12h^2\sum_{r=1}^{n} r + \lim_{h\to 0} 6h^3\sum_{r=1}^{n} r^2 + \lim_{h\to 0} h^4\sum_{r=1}^{n} r^3$$

$$= \lim_{h\to 0} h8(1+1+1+---+1) + \lim_{h\to 0} 12h^2(1+2+3+---+n) +$$

$$\lim_{h\to 0} 6h^3(1^2+2^2+3^2+-----+n^2) + \lim_{h\to 0} h^4(1^3+2^3+3^3+-----+n^3)$$

$$= \lim_{h\to 0} 8nh + \lim_{h\to 0} 12h^2 \times \frac{n(n+1)}{2} + \lim_{h\to 0} 6h^3 \times \frac{n(n+1)(2n+1)}{6} + \lim_{h\to 0} h^4 \times \left\{\frac{n(n+1)}{2}\right\}^2$$

$$= \lim_{h\to 0} 8nh + \lim_{h\to 0} 12\times \frac{nh(nh+h)}{2} + \lim_{h\to 0} 6\times \frac{nh(nh+h)(2nh+h)}{6} + \lim_{h\to 0}\left\{\frac{nh(nh+h)}{2}\right\}^2$$

$$= \lim_{h\to 0} 8\times 1 + \lim_{h\to 0} 12\times \frac{1\times(1+h)}{2} + \lim_{h\to 0}\times \frac{1\times(1+h)(2\times 1+h)}{6} + \lim_{h\to 0}\left\{\frac{1\times(1+h)}{2}\right\}^2$$

$$= 8\times 1 + 12\times \frac{1\times(1+0)}{2} + \frac{1\times(1+0)(2\times 1+0)}{6} + \left\{\frac{1\times(1+0)}{2}\right\}^2$$

$$= 8\times 1 + 12\times \frac{1}{2} + \frac{2}{6} + \left\{\frac{1}{2}\right\}^2 = 8+6+2+\frac{1}{4} = \frac{32+24+8+1}{4} = \frac{65}{4} \; \textit{Answer}$$

Example 17: Evaluate $\int_0^{\pi/2} \cos x\,dx$

Solution: $\because nh = b - a$

$\because nh = \pi/2 - 0$

$\because nh = \pi/2$

Given, $f(x) = \cos x$

$\therefore f(a+rh) = \cos(a+rh)$

We have, $\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$

$$\int_0^{\pi/2} \cos x\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$$

$$= \lim_{h\to 0} \sum_{r=0}^{n-1} h\cos(a+rh)$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\cos(0+rh) \qquad\qquad [a=0]$$

$$= \lim_{h\to 0}\sum_{r=0}^{n-1} h\cos rh \;\; = \lim_{h\to 0}\sum_{r=1}^{n} h\cos rh$$

$$\int_0^{\pi/2}\cos x\,dx \;\; = \lim_{h\to 0} h(\cos h+\cos 2h+\cos 3h+--------+\cos nh) \text{-------------------(i)}$$

Let $S = (\cos h+\cos 2h+\cos 3h+--------+\cos nh)$ -----------------------------------(ii)

Multiplying by $2\sin\frac{h}{2}$

$$2\sin\frac{h}{2}S = (\cos h+\cos 2h+\cos 3h+--------+\cos nh)2\sin\frac{h}{2}$$

$\Rightarrow$

$$S\times 2\sin\frac{h}{2} = \cos h\times 2\sin\frac{h}{2}+\cos 2h\times 2\sin\frac{h}{2}+\cos 3h\times 2\sin\frac{h}{2}+-----+\cos nh\times 2\sin\frac{h}{2}$$

$$\Rightarrow\;\; S\times 2\sin\frac{h}{2} = 2\cos h\sin\frac{h}{2}+2\cos 2h\sin\frac{h}{2}+2\cos 3h\sin\frac{h}{2}+---------+2\cos nh\sin\frac{h}{2}$$

$$\Rightarrow\;\; S\times 2\sin\frac{h}{2} = \left\{\sin(h+\frac{h}{2})-\sin(h-\frac{h}{2})\right\}+\left\{\sin(2h+\frac{h}{2})-\sin(2h-\frac{h}{2})\right\}+$$
$$\left\{\sin(3h+\frac{h}{2})-\sin(3h-\frac{h}{2})\right\}+---------------+\left\{\sin(nh+\frac{h}{2})-\sin(nh-\frac{h}{2})\right\}$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = \left\{\sin(\frac{3h}{2})-\sin(\frac{h}{2})\right\}+\left\{\sin(\frac{5h}{2})-\sin(\frac{3h}{2})\right\}+\left\{\sin(\frac{7h}{2})-\sin(\frac{5h}{2})\right\}+--$$
$$-------------+\left\{\sin(nh+\frac{h}{2})-\sin(nh-\frac{h}{2})\right\}$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = \sin(\frac{3h}{2})-\sin(\frac{h}{2})+\sin(\frac{5h}{2})-\sin(\frac{3h}{2})+\sin(\frac{7h}{2})-\sin(\frac{5h}{2})+--$$
$$---------+\sin(nh+\frac{h}{2})-\sin(nh-\frac{h}{2})$$

$$\boxed{\begin{aligned}\Rightarrow S\times 2\sin\frac{h}{2} &= \sin(\cancel{\tfrac{3h}{2}})-\sin(\frac{h}{2})\\ &\quad+\sin(\cancel{\tfrac{5h}{2}})-\sin(\cancel{\tfrac{3h}{2}})\\ &\quad+\sin(\cancel{\tfrac{7h}{2}})-\sin(\cancel{\tfrac{5h}{2}})+\\ &\quad\text{---------------------------}\\ &\quad\text{---------------------------}\\ &\quad+\sin(nh+\frac{h}{2})-\sin(\cancel{nh-\tfrac{h}{2}})\end{aligned}}$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = -\sin(\frac{h}{2})+\sin(nh+\frac{h}{2})$$

$$\Rightarrow S\times 2\sin\frac{h}{2} = \sin(nh+\frac{h}{2})-\sin(\frac{h}{2})$$

$$\Rightarrow\;\; S = \frac{\sin(nh+\frac{h}{2})-\sin(\frac{h}{2})}{2\sin\frac{h}{2}} \text{-------------(iii)}$$

From (i),

$$\Rightarrow \int_0^{\pi/2}\cos x\,dx \;\; = \lim_{h\to 0} h(\cos h+\cos 2h+\cos 3h+--------+\cos nh)$$

$$\Rightarrow \int_0^{\pi/2} \cos x\,dx = \lim_{h\to 0} h \; \frac{\sin(nh + \frac{h}{2}) - \sin(\frac{h}{2})}{2\sin\frac{h}{2}} \qquad \text{[From (iii)]}$$

$$\Rightarrow \int_0^{\pi/2} \cos x\,dx = \lim_{h\to 0} \frac{\frac{h}{2}\left\{\sin(nh + \frac{h}{2}) - \sin(\frac{h}{2})\right\}}{\sin\frac{h}{2}}$$

$$\Rightarrow \int_0^{\pi/2} \cos x\,dx = \lim_{h\to 0} \frac{\frac{h}{2}}{\sin\frac{h}{2}} \cdot \left[\lim_{h\to 0} \left\{\sin(nh + \frac{h}{2}) - \sin(\frac{h}{2})\right\}\right]$$

$$\Rightarrow \int_0^{\pi/2} \cos x\,dx = 1.\ \lim_{h\to 0} \left\{\sin(nh + \frac{h}{2}) - \sin(\frac{h}{2})\right\} \qquad [\because \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1]$$

$$\Rightarrow \int_0^{\pi/2} \cos x\,dx = 1.\ \left\{\sin(\frac{\pi}{2} + \frac{0}{2}) - \sin(\frac{0}{2})\right\} \qquad [\because nh = \pi/2]$$

$$\Rightarrow \int_0^{\pi/2} \cos x\,dx = 1.\ \left\{\sin(\frac{\pi}{2}) - \sin 0\right\} = 1.\ (1-0) = 1 \qquad \textit{Answer}$$

Example 18: Evaluate $\lim_{n\to\infty} \sum_{r=0}^{n} \frac{n}{r^2 + n^2}$

Solution:: Let $S = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n}{r^2 + n^2}$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n}{n^2 + r^2}$$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n}{n^2(1 + \frac{r^2}{n^2})}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{n(1 + \frac{r^2}{n^2})} = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{n\left\{1 + \left(\frac{r}{n}\right)^2\right\}}$$

$$\Rightarrow S = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n} \frac{1}{\left\{1 + \left(\frac{r}{n}\right)^2\right\}}$$

Therefore, $S = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n}{r^2 + n^2}$

Putting $\frac{1}{n} = h$

$\Rightarrow nh = 1$

and if $n \to \infty$ then $\frac{1}{n} = h$

$\Rightarrow h = \frac{1}{n}$

$\Rightarrow h = \frac{1}{\infty}$

$\Rightarrow h = 0$

That is $n \to \infty$ then $h \to 0$

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\, f(a + rh)$$

Where, $b - a = nh$

$\therefore b = a + nh$

Where $x = rh$

$\Rightarrow x = 0 + rh$

$\Rightarrow x = a + rh$

That is $a = 0$, is the lower limit.

Also, $b = a + nh$

$b = 0 + 1$

$b = 1$; is the upper limit

$$S = \lim_{n\to\infty} \frac{1}{n}\sum_{r=0}^{n} \frac{1}{\left\{1+\left(\frac{r}{n}\right)^2\right\}}$$

$$= \lim_{h\to 0} h\sum_{r=0}^{n} \frac{1}{\left\{1+r^2\left(\frac{1}{n}\right)^2\right\}}$$

$$= \lim_{h\to 0} h\sum_{r=0}^{n} \frac{1}{\{1+r^2h^2\}}$$

$$S = \lim_{h\to 0} h\sum_{r=0}^{n} \frac{1}{\{1+r^2h^2\}}$$

$$= \int_0^1 \frac{dx}{1+x^2} \qquad [h = dx]$$

$$= \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \tan^{-1}\tan\frac{\pi}{4} - \tan^{-1}\tan 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\therefore \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n}{r^2+n^2} = \frac{\pi}{4} \quad \textit{Answer}$$

Example 19: Evaluate $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{r^4+n^4}$

Solution: Let, $S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{r^4+n^4}$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{n^4+r^4}$$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^3}{n^4\left\{1+\frac{r^4}{n^4}\right\}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\frac{r^3}{n^3}}{n\left\{1+\left(\frac{r}{n}\right)^4\right\}}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^3}{n\left\{1+\left(\frac{r}{n}\right)^4\right\}}$$

Putting $\frac{1}{n} = h$

$$\Rightarrow nh = 1$$

and if $n \to \infty$ then $\frac{1}{n} = h$

$$\Rightarrow h = \frac{1}{n}$$

$$\Rightarrow h = \frac{1}{\infty}$$

$$\Rightarrow h = 0$$

That is $n \to \infty$ then $h \to 0$

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$$

Where, $b - a = nh$

$\therefore b = a + nh$

Where $x = rh$

$$\Rightarrow x = 0 + rh$$

$$\Rightarrow x = a + rh$$

That is $a = 0$, is the lower limit.

Also, $b = a + nh$

$b = 0 + 1$

$b = 1$; is the upper limit

$$S = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^3}{\left\{1+\left(\frac{r}{n}\right)^4\right\}}$$

Therefore,

$$S = \lim_{n\to\infty} \frac{1}{n}\sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^3}{\left\{1+\left(\frac{r}{n}\right)^4\right\}}$$

$$S = \lim_{h\to 0} h\sum_{r=1}^{n} \frac{(rh)^3}{\{1+(rh)^4\}}$$

$$= \int_0^1 \frac{x^3dx}{1+x^4} = \frac{1}{4}\int_0^1 \frac{4x^3dx}{1+x^4} = \frac{1}{4}\left[\log\left(1+x^4\right)\right]_0^1 = \frac{1}{4}\left[\log(1+1^4)-\log(1+0)\right]$$

$$= \frac{1}{4}[\log 2 - \log 1] = \frac{1}{4}[\log 2 - 0]\ S = \frac{1}{4}\log 2 \textit{ Answer}$$

Example 20: Evaluate $\lim_{n\to\infty}\left[\frac{1}{1+n^3}+\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}+\ldots\ldots\ldots\ldots\ldots+\frac{n^2}{n^3+n^3}\right]$

Solution: Let $S = \lim_{n\to\infty}\left[\frac{1}{1+n^3}+\frac{2^2}{2^3+n^3}+\frac{3^2}{3^3+n^3}+\ldots\ldots\ldots\ldots\ldots+\frac{n^2}{n^3+n^3}\right]$

$$\Rightarrow S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3}$$

$$\Rightarrow S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3+r^3} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3\left\{\frac{n^3}{n^3}+\frac{r^3}{n^3}\right\}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3\left\{1+\frac{r^3}{n^3}\right\}}$$

$$\Rightarrow S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{n^3\left\{1+\left(\frac{r}{n}\right)^3\right\}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\frac{r^2}{n^2}}{n\left\{1+\left(\frac{r}{n}\right)^3\right\}}$$

$$S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2}{n\left\{1+\left(\frac{r}{n}\right)^3\right\}} \text{ ----------------------------(i)}$$

Putting $\frac{1}{n} = \mathbf{h}$

$\mathbf{nh = 1}$

and if $\mathbf{n \to \infty}$ then $\frac{1}{n} = \mathbf{h}$

$h = \frac{1}{n}$

$h = \frac{1}{\infty}$

$\mathbf{h = 0}$

That is $\mathbf{n \to \infty}$ then $\mathbf{h \to 0}$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^2}{n\left\{1+\left(\frac{r}{n}\right)^3\right\}}$$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\frac{1}{n}\left(r\times\frac{1}{n}\right)^2}{\left\{1+\left(r\times\frac{1}{n}\right)^3\right\}}$$

$$\Rightarrow S = \lim_{h\to 0} \sum_{r=1}^{n} \frac{h(rh)^2}{\{1+(rh)^3\}}$$

$$\Rightarrow S = \lim_{h\to 0} h \sum_{r=1}^{n} \frac{(rh)^2}{\{1+(rh)^3\}}$$

$$\Rightarrow S = \int_0^1 \frac{x^2}{1+x^3}\,dx$$

$$\Rightarrow \therefore S = \int_0^1 \frac{x^2}{1+x^3}\,dx$$

$$\Rightarrow S = \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3}\,dx$$

$$\Rightarrow S = \frac{1}{3}\left[\ln(1+x^3)\right]_0^1$$

$$\Rightarrow S = \frac{1}{3}\left[\ln(1+1^3)-\ln(1+0^3)\right]$$

$$\Rightarrow S = \frac{1}{3}\left[\ln 2-\ln 1\right]$$

$$\Rightarrow S = \frac{1}{3}\left[\ln\frac{2}{1}\right] \qquad [\because \log a-\log b=\log\frac{a}{b}]$$

$\Rightarrow S = \frac{1}{3}\ln 2$ *Answer*

We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\sum_{r=0}^{n-1} h\,f(a+rh)$$

Where, $b-a=nh$

$\therefore b=a+nh$

Where $x=rh$

$x=0+rh$

$x=a+rh$

That is $a=0$, is the lower limit.

Also, $b=a+nh$

$b=0+1$

$b=1$; is the upper limit

Example 21: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots\dots\dots\dots\dots+\frac{1}{8n}\right]$

Solution: Let $S=\lim_{n\to\infty}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\dots\dots\dots\dots\dots+\frac{1}{8n}\right]$

$$\Rightarrow S = \lim_{n\to\infty}\left[\frac{1}{(n+0)^3}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots\ldots\ldots\ldots\ldots+\frac{n^2}{(n+n)^3}\right]$$

$$\Rightarrow S = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{(n+r)^3} = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{n^3\left\{1+\frac{r}{n}\right\}^3}$$

$$\Rightarrow S = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{n^2}{n^3\left\{1+\frac{r}{n}\right\}^3}$$

$$\Rightarrow S = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n\left\{1+\frac{r}{n}\right\}^3} \text{---------------(i)}$$

Putting $\frac{1}{n} = h$

$nh = 1$

and if $n \to \infty$ then $\frac{1}{n} = h$

$h = \frac{1}{n}$

$h = \frac{1}{\infty}$

$h = 0$

That is $n \to \infty$ then $h \to 0$

$$\Rightarrow S = \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{n\left\{1+\frac{r}{n}\right\}^3}$$

$$\Rightarrow S = \lim_{h\to 0}\sum_{r=0}^{n}\frac{\frac{1}{n}}{\left\{1+r\times\frac{1}{n}\right\}^3}$$

$$\Rightarrow S = \lim_{h\to 0}\sum_{r=0}^{n}\frac{h}{\{1+rh\}^3}$$

$$\Rightarrow S = \lim_{h\to 0} h\sum_{r=0}^{n}\frac{1}{\left\{1+(rh)^3\right\}} \text{-----------------(ii)}$$

Where, $x = 1 + rh$

$x = a + rh$

$\therefore a = 1$ is the lower limit.

Also, $b = a + nh$

$b = 1 + nh$

$b = 1 + 1$

$b = 2$

$$\Rightarrow S = \lim_{h\to 0} h\sum_{r=0}^{n}\frac{1}{\left\{1+(rh)^3\right\}}$$

$$\Rightarrow S = \int_1^2\frac{dx}{x^3} = \int_1^2 x^{-3}dx = \left[\frac{x^{-3+1}}{-3+1}\right]_1^2$$

$$\Rightarrow S = \left[\frac{x^{-2}}{-2}\right]_1^2 = \left[\frac{2^{-2}}{-2}-\frac{1^{-2}}{-2}\right]$$

$$\Rightarrow S = \frac{1}{-2}\left[\frac{1}{2^2}-\frac{1}{1^2}\right] = \frac{1}{-2}\left[\frac{1}{4}-1\right]$$

$$\Rightarrow S = \frac{1}{-2}\left[\frac{1-4}{4}\right] = \frac{1}{-2}\left[\frac{-3}{4}\right] = \frac{3}{8} \textit{ Answer}$$

Example 22: Evaluate $\lim\limits_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\ldots\ldots\ldots\ldots..+\frac{1}{nb}\right]$

Solution: Let $S=\lim\limits_{n\to\infty}\left[\frac{1}{na}+\frac{1}{na+1}+\ldots\ldots\ldots\ldots..+\frac{1}{nb}\right]$

$$\Rightarrow S=\lim_{n\to\infty}\left[\frac{1}{na+0}+\frac{1}{na+1}+\frac{1}{na+2}+\ldots\ldots\ldots\ldots..+\frac{1}{nb+na-na}\right]$$

$$\Rightarrow S=\lim_{n\to\infty}\sum_{r=0}^{n(b-a)}\frac{1}{na+r}$$

$$\Rightarrow S=\lim_{n\to\infty}\sum_{r=0}^{n(b-a)}\frac{1}{n\left(a+\frac{r}{n}\right)}$$

$$\Rightarrow S=\lim_{n\to\infty}\sum_{r=0}^{n(b-a)}\frac{1/n}{\left(a+r\times\frac{1}{n}\right)}$$

$$\Rightarrow S=\lim_{h\to 0}\sum_{r=0}^{n(b-a)}\frac{h}{(a+rh)}$$

$$\Rightarrow S=\lim_{h\to 0}h\sum_{r=0}^{n(b-a)}\frac{1}{(a+rh)}$$

$$\Rightarrow S=\lim_{h\to 0}h\sum_{r=0}^{n(b-a)}\frac{1}{(a+rh)}$$

$$\Rightarrow S=\int_a^b\frac{dx}{x}$$

$\Rightarrow S=[\ln x]_a^b=[\ln b-\ln a]$ *Answer*

Putting $\frac{1}{n}=h$

$nh=1$

and if $n\to\infty$ then $\frac{1}{n}=h$

$h=\frac{1}{n}$

$h=\frac{1}{\infty}$

$h=0$

That is $n\to\infty$ then $h\to 0$

Where $x=a+rh$

$\therefore$ the lower limit is a

and the upper limit is

$a+nh(b-a)$

$=a+nh(b-a)$

$=a+(b-a)$

$=b$

Example 23: Evaluate $\lim\limits_{n\to\infty}\left[\frac{1^{10}+2^{10}+3^{10}+\ldots\ldots\ldots\ldots\ldots+n^{10}}{n^{11}}\right]$

Solution: Let $S=\lim\limits_{n\to\infty}\left[\frac{1^{10}+2^{10}+3^{10}+\ldots\ldots\ldots\ldots\ldots+n^{10}}{n^{11}}\right]$

$$\Rightarrow S=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1^{10}+2^{10}+3^{10}+\ldots\ldots\ldots\ldots\ldots+n^{10}}{n^{10}}\right]$$

$$\Rightarrow S=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1^{10}}{n^{10}}+\frac{2^{10}}{n^{10}}+\frac{3^{10}}{n^{10}}+\ldots\ldots\ldots\ldots\ldots+\frac{n^{10}}{n^{10}}\right]$$

$$\Rightarrow S=\lim_{n\to\infty}\frac{1}{n}\left[1^{10}(\frac{1}{n})^{10}+2^{10}(\frac{1}{n})^{10}+3^{10}(\frac{1}{n})^{10}+\ldots\ldots\ldots\ldots\ldots+n^{10}(\frac{1}{n})^{10}\right]$$

$$\Rightarrow S = \lim_{h\to 0} h\left[1^{10}(h)^{10} + 2^{10}(h)^{10} + 3^{10}(h)^{10} + \dots\dots + n^{10}(h)^{10}\right]$$

$$\Rightarrow S = \lim_{h\to 0} h \sum_{r=1}^{n} (rh)^{10}$$

$$\Rightarrow S = \lim_{h\to 0} h \sum_{r=1}^{n} (rh)^{10}$$

$$\Rightarrow S = \int_0^1 x^{10}dx$$

$$\Rightarrow S = \left[\frac{x^{10+1}}{10+1}\right]_0^1$$

$$\Rightarrow S = \left[\frac{x^{11}}{11}\right]_0^1$$

$$\Rightarrow S = \left[\frac{1^{11}}{11} - \frac{0^{11}}{11}\right] = \left[\frac{1^{11}}{11} - 0\right] = \frac{1^{11}}{11} = \frac{1}{11} \quad \textit{Answer}$$

Putting $\frac{1}{n} = h$

$nh = 1$

and if $n \to \infty$ then $\frac{1}{n} = h$

$h = \frac{1}{n}$

$h = \frac{1}{\infty}$

$h = 0$

That is $n \to \infty$ then $h \to 0$

We have,

$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$

Where, $b - a = nh$

$\therefore b = a + nh$

Where $x = rh$

$x = 0 + rh$

$x = a + rh$

That is $a = 0$, is the lower limit.

Also, $b = a + nh$

$b = 0 + 1$

$b = 1$; is the upper limit

Example 24: Evaluate $\lim_{n\to\infty}\left[\frac{1}{n+1m} + \frac{1}{n+2m} + \dots\dots + \frac{1}{n+nm}\right]$

Solution: Let $\lim_{n\to\infty}\left[\frac{1}{n+1m} + \frac{1}{n+2m} + \dots\dots + \frac{1}{n+nm}\right]$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n + r\,m}$$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n\left(1 + \frac{rm}{n}\right)}$$

$$\Rightarrow S = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1/n}{\left(1 + rm \times \frac{1}{n}\right)}$$

$$\Rightarrow S = \lim_{h\to 0} \sum_{r=1}^{n} \frac{h}{(1 + rm\,h)}$$

$$\Rightarrow S = \lim_{h\to 0} h \sum_{r=1}^{n} \frac{1}{(1 + rmh)}$$

$$\Rightarrow S = \lim_{h\to 0} h \sum_{r=1}^{n} \frac{1}{(1 + mrh)}$$

$$\Rightarrow S = \int_0^1 \frac{dx}{1+mx} = \frac{1}{m}\int_0^1 \frac{mdx}{1+mx}$$

Putting $\frac{1}{n} = h$

$nh = 1$

and if $n \to \infty$ then $\frac{1}{n} = h$

$h = \frac{1}{n}$

$h = \frac{1}{\infty}$

$h = 0$

That is $n \to \infty$ then $h \to 0$

We have,

$\int_a^b f(x)\,dx = \lim_{h\to 0} \sum_{r=0}^{n-1} h\,f(a+rh)$

Where, $b - a = nh$

$\therefore b = a + nh$

Where $x = rh$

$x = 0 + rh$

$x = a + rh$

That is $a = 0$, is the lower limit.

Also, $b = a + nh$

$b = 0 + 1$

$b = 1$; is the upper limit

$$\Rightarrow S = \frac{1}{m}\left[\ln(1+mx)\right]_0^1$$

$$\Rightarrow S = \frac{1}{m}\left[\ln(1+m.1) - \ln(1+m.0)\right]$$

$$\Rightarrow S = \frac{1}{m}\left[\ln(1+m) - \ln(1)\right]$$

$$\Rightarrow S = \frac{1}{m}\left[\ln(1+m) - 0\right]$$

$$\Rightarrow S = \frac{1}{m}\left[\ln(1+m)\right] \quad \textit{Answer}$$