Changing Coordinates

01. A Cartesian coordinate system:

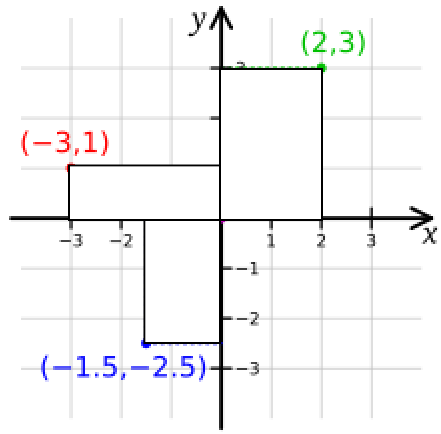


Figure # 143

Rectangular coordinates in 3-D:

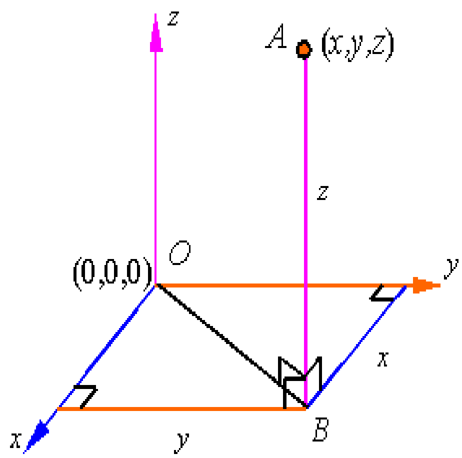


Figure # 144

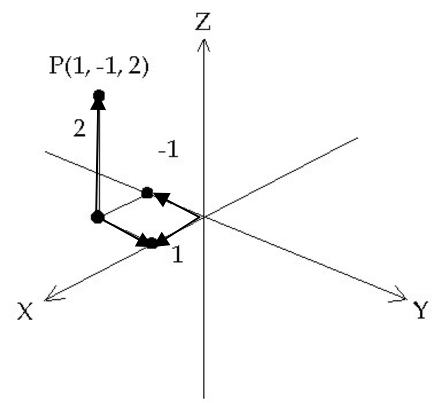


Figure # 145

The polar coordinate: the polar coordinate system in which each point on a plane is determined by a distance from a fixed point and an angle from a fixed direction.

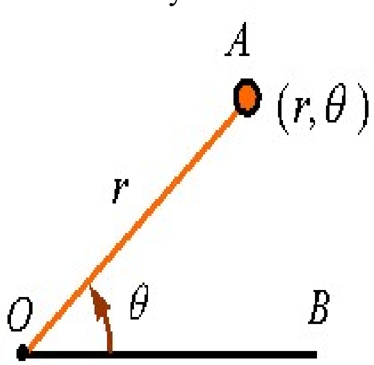


Figure # 146

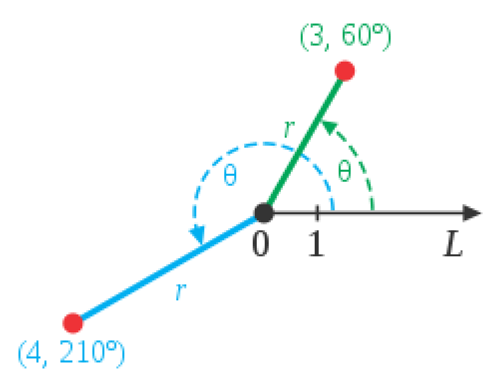


Figure # 147

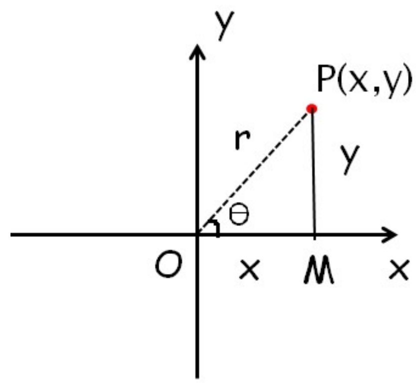
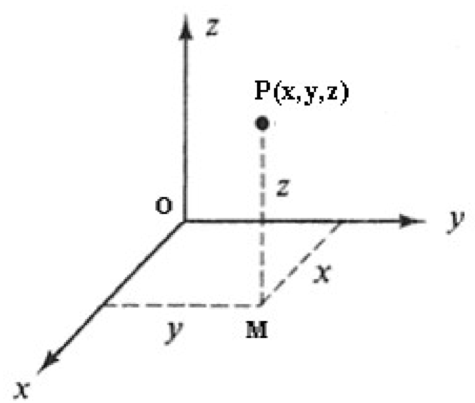


Figure # 148

$$\frac{\mathbf{x}}{\mathbf{r}} = \mathbf{cos}\,\theta \qquad \text{and} \qquad \frac{\mathbf{y}}{\mathbf{r}} = \mathbf{sin}\,\theta$$

$$\Rightarrow \mathbf{x} = \mathbf{r}\,\mathbf{cos}\,\theta \qquad \Rightarrow y = r\sin\theta$$

02. Changing Coordinate Systems: Rectangular and Cylindrical



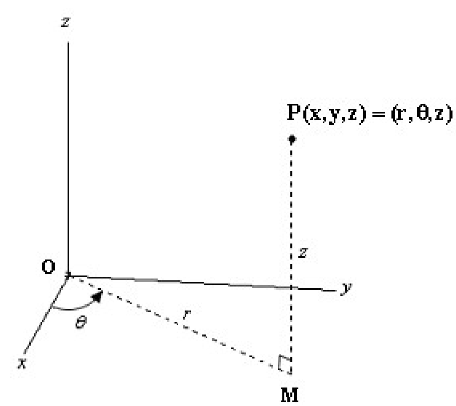
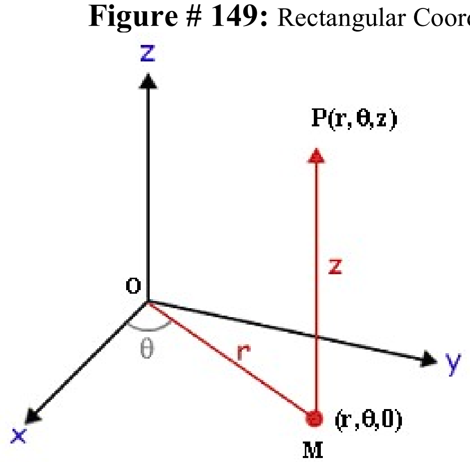


Figure # 149: Rectangular Coordinates (Cartesian Coordinates) **Figure # 150**



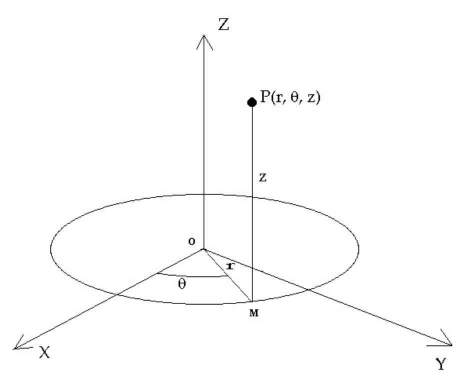


Figure # 151 **Figure # 152**

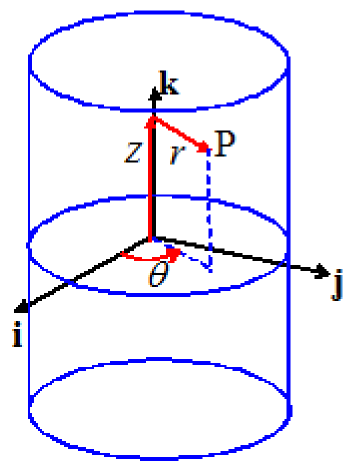


Figure # 153: Cylindrical Polar Coordinates

r is the radial distance of P from the axis of the cylinder
θ is the angle between the **i** direction and the projection of OP onto the xy- plane
z is the length of the projection of OP on the axis of the cylinder.

If the relationships between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (r, φ, z) are easily obtained from Figure below:

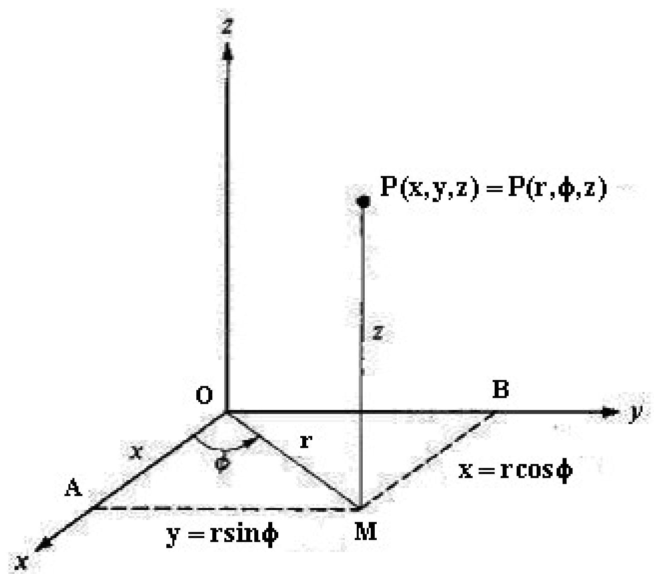


Figure # 154: Cylindrical Polar Coordinates

$$\frac{\mathbf{OA}}{\mathbf{OM}} = \cos\phi$$

$$\frac{\mathbf{x}}{\mathbf{r}} = \cos\phi$$

$$\mathbf{x} = \mathbf{r}\cos\phi \text{------------------------}\mathbf{(i)}$$

$$\frac{AM}{OM} = \sin\phi$$

$$\frac{y}{r} = \sin\phi$$

$y = r\sin\phi$ -----------------------(ii)

Finally we get, the Cylindrical Polar Coordinates are

$x = r\cos\phi$ -----------------------(iii)

$y = r\sin\phi$ -----------------------(iv)

$z = z$ ------------------------------(v)

03. Changing Coordinate Systems: Rectangular and Spherical

A SPHERE

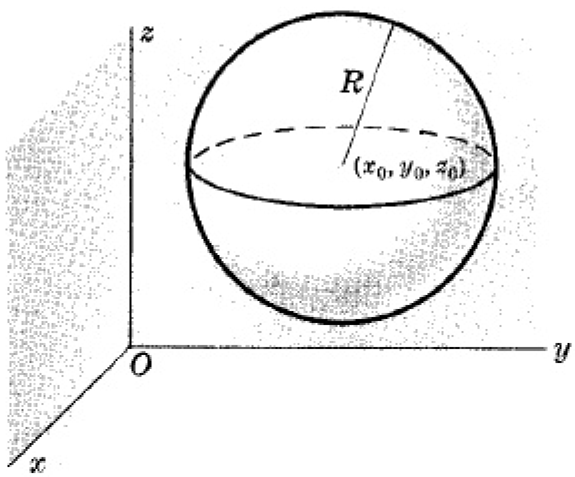


Figure # 155

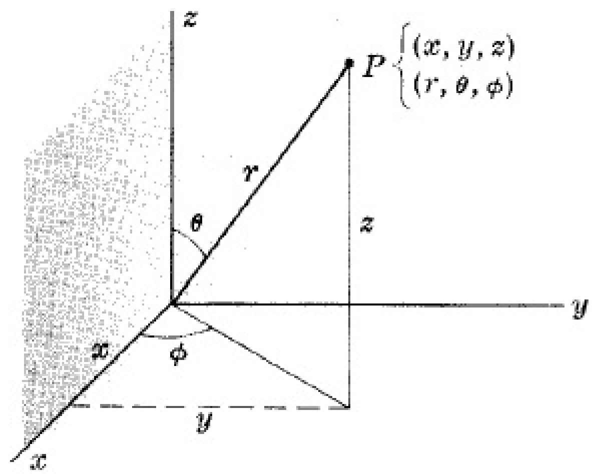


Figure # 156

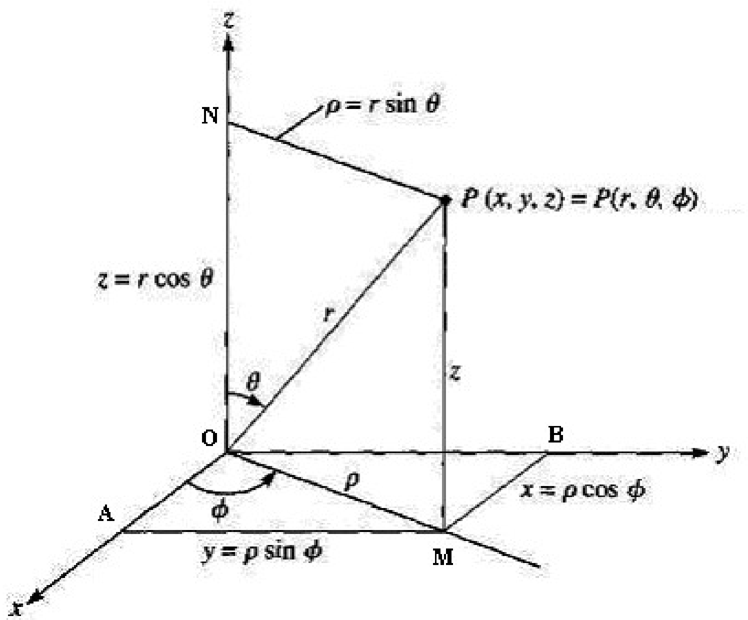


Figure # 157: Spherical Polar Coordinates

$$\frac{OA}{OM} = \cos\phi$$

$$\frac{x}{\rho} = \cos\phi$$

$$x = \rho\cos\phi \text{ -----------------------(i)}$$

&

$$\frac{AM}{OM} = \sin\phi$$

$$\frac{y}{\rho} = \sin\phi$$
$y = \rho\sin\phi$ ------------------------(ii)

Again,

$$\frac{ON}{OP} = \cos\theta$$
$$\frac{z}{r} = \cos\theta$$
$z = r\cos\theta$ ------------------------(iii)

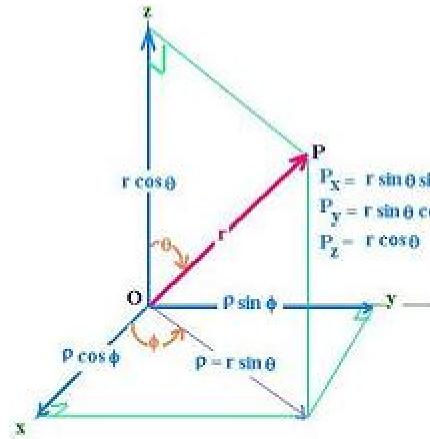


Figure # 158

&

$$\frac{PN}{OP} = \sin\theta$$
$$\frac{PN}{r} = \sin\theta$$
$\Rightarrow PN = r\sin\theta$
$\Rightarrow PN = OM = r\sin\theta$ $\quad [\because PN = OM]$
$\Rightarrow PN = \rho = r\sin\theta$ $\quad [\because OM = \rho]]$
$\therefore PN = \rho = r\sin\theta$ ----------------(iv)

Again,
ΔOAM,

$OM^2 = OA^2 + AM^2$
$\rho^2 = x^2 + y^2$ -----------------------(v)

Again,
ΔOPN,

$OP^2 = ON^2 + PN^2$
$r^2 = z^2 + \rho^2$ $\quad [\because PN = \rho]$
$r^2 = z^2 + \rho^2$
$r^2 = z^2 + x^2 + y^2$ $\quad [\because \rho^2 = x^2 + y^2]$

Again, putting the value of ρ in (i) and (ii),

$x = \rho\cos\phi$
$x = r\sin\theta\cos\phi$ $\quad [\because PN = \rho = r\sin\theta]$
&
$y = \rho\sin\phi$

$y = r\sin\theta\sin\phi$ [$\because PN = \rho = r\sin\theta$]

Finally we get, the Spherical Polar Coordinates are:

$\therefore\ x = r\sin\theta\cos\phi$ ------------------------(vi)

$y = r\sin\theta\sin\phi$ ------------------------(vii)

$z = r\cos\theta$ ------------------------------(viii)

R is the distance of P from the origin

θ is the angle between the **k** direction and OP

ϕ is the angle between the **i** direction and the projection of OP onto a plane through O normal to **k**

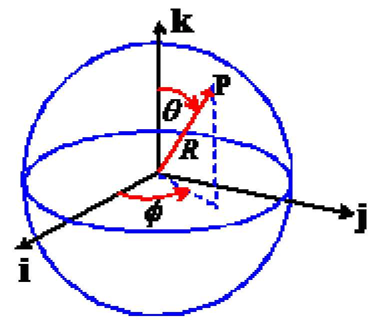


Figure # 159

04. **Jacobian Determinant**: In order to change variables in a double integral or triple integral we will need the **Jacobian** of the transformation. The **Jacobian** of the transformation $x = x(u, v)$ and $y = y(u, v)$ is denoted by:

$$|J(u,v)| = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Q # 105 : To see how this works we can start with one dimension. If we have an integral in rectangular coordinates such as $\int_{x_1}^{x_2} f(x)\,dx$

We can change coordinate systems if we define $x = x(u)$. Then we have $dx = \frac{dx}{du}du$.

To transform the limits of the integral, we need to invert the definition to get $u = u(x)$.
Then the integral becomes

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{u(x_1)}^{u(x_2)} f(x(u))\,dx\,\frac{dx}{du}\,du = \int_{u(x_1)}^{u(x_2)} f(x(u))|J(u)|\,du \qquad [\text{Here } |J(u)| = \frac{dx}{du}]$$

That is $\int f(x)\,dx = \int f(u)|J(u)|\,du$ -----------------------------(i)

Q # 106:

Find $\int \frac{dx}{(3x+2)^5}$

Here, $f(x) = \frac{1}{(3x+2)^5}$

Solution: Let

$$\mathbf{u = 3x + 2.}$$

Now directly,

$$\mathbf{u = 3x + 2.}$$

$$\frac{du}{dx} = 3$$

$$\frac{dx}{du} = \frac{1}{3}$$

$$dx = \frac{1}{3} du \text{ ------------------------------------(i)}$$

$$\int \frac{\mathbf{dx}}{\mathbf{(3x+2)^5}} = \int \frac{1}{(u)^5} \frac{1}{3} du$$

$$\therefore \int \underbrace{\frac{1}{(3x+2)^5}}_{f(x)} dx = \int \underbrace{\frac{1}{(u)^5}}_{f(u)} \underbrace{\frac{1}{3}}_{\text{Jacobian Determinant}} du \text{ -------------------------------------(*)}$$

Another way, Using Jacobian **Determinant,**

Here Variable $x \to u$

Then $\int f(x)\, dx = \int f(u) |J(u)|\, du$ ------------------------(ii)

That is, $dx = |J(u)| du$ --(iii)

Where, $|J(u)| = \left|\frac{\partial x}{\partial u}\right|$ --(iv)

Now, we have to find out the value of $|J(u)|$

$$\mathbf{u = 3x + 2.}$$

$$\frac{\delta u}{\delta x} = 3$$

$$\frac{\delta x}{\delta u} = \frac{1}{3}$$

From (iv),

$$|J(u)| = \frac{\delta x}{\delta u} = \frac{1}{3} \text{ ----------------------------(v)}$$

Putting the value of $|J(u)|$ in (iii),

$$\therefore dx = |J(u)| du$$

$$\therefore dx = \frac{1}{3} du \text{ ---(vi)}$$

We see, (i) & (vi) are same.

That is,

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{u(x_1)}^{u(x_2)} f(x(u))\frac{dx}{du}\,du = \int_{u(x_1)}^{u(x_2)} f(x(u))\,|J(u)|\,du \text{ ------(vii)}$$

Here, $f(x) = \dfrac{1}{(3x+2)^5}$ and $f(x(u)) = \dfrac{1}{u^5}$

From (ii),

$$\int f(x)\,dx = \int f(u)\,|J(u)|\,du$$

$$\int \frac{1}{(3x+2)^5}dx = \int \frac{1}{u^5}\frac{1}{3}du = \int \frac{1}{u^5}|J(u)|du$$

$$= \int u^{-5}\frac{1}{3}du = \frac{1}{3}.\frac{u^{-5+1}}{-5+1} = \frac{1}{3}.\frac{u^{-4}}{-4} + c \qquad [\because |J(u)| = \frac{1}{3}]$$

$$= \frac{-1}{12}(3x+2)^{-4} + c$$

$$= \frac{-1}{12(3x+2)^4} + c$$

Q # 107: In two dimensions, $\iint f(x,y)\,dx\,dy$ ----------------------------(i)

Now if we want to switch to another coordinate system, we define $x = x(u,v)$ and $y = y(u,v)$

$$\iint_R f(x,y)dx\,dy = \iint_{R'} f(u,v)\,|J(u,v)|\,du\,dv$$

That is: $\iint_R f(x,y)dx\,dy = \iint_{R'} f[x(u,v), y(u,v)]\,|J(u,v)|\,du\,dv$ ---------(ii)

Where $|J(u,v)| = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Q # 108: Find area of a circle whose center $(0,0)$ and radius r that is find $\iint f(x,y)\,dx\,dy$

Answer: $f(x,y) = x^2 + y^2 = r^2$

Let $x = r\cos\theta \quad y = r\sin\theta$

Hence, $dxdy = |J(r,\theta)|drd\theta$ ---------------------(i)

When $x = r$, $y = 0$ and $r = r$ then

$x = r\cos\theta$

$\Rightarrow r = r\cos\theta$

$\Rightarrow 1 = \cos\theta$

$\Rightarrow \cos 0 = \cos\theta$

$\Rightarrow \theta = 0$

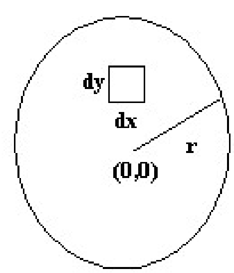


Figure # 160

Hence the limit of θ for a circle is $0 \le \theta \le 2\pi$

Again

When $x = r$ and $\theta = 0$ then

$$\mathbf{x = r\cos\theta}$$
$$\mathbf{r = r\cos 0}$$
$$\mathbf{\Rightarrow r = r \times 1}$$
$$\mathbf{\Rightarrow r = r}$$
$$\mathbf{\Rightarrow r = r}$$

Hence the limit of $\mathbf{r}$ is $\mathbf{0 \le r \le r}$

$$\therefore |\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \begin{vmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{r}} & \frac{\partial \mathbf{x}}{\partial \theta} \\ \frac{\partial \mathbf{y}}{\partial \mathbf{r}} & \frac{\partial \mathbf{y}}{\partial \theta} \end{vmatrix}$$

$$|\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \begin{vmatrix} \cos\theta & -\mathbf{r}\sin\theta \\ \sin\theta & \mathbf{r}\cos\theta \end{vmatrix}$$

$$|\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \mathbf{r}\cos^2\theta - (-\mathbf{r}\sin^2\theta)$$

$$|\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \mathbf{r}\cos^2\theta + \mathbf{r}\sin^2\theta$$

$$|\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \mathbf{r}(\cos^2\theta + \sin^2\theta)$$

$$|\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \mathbf{r.1}$$

$$|\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \mathbf{r} \text{ ----------------------------------(ii)}$$

Putting the value of $|\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \mathbf{r}$ in (i), we get

$$dxdy = |J(r,\theta)|drd\theta$$
$$dxdy = rdrd\theta \text{ --(iii)}$$

Now, we can write

$$\iint \mathbf{f(x,y)\,dx\,dy} = \iint \mathbf{f}\left[\mathbf{x(u,v),y(u,v)}\right] |\mathbf{J(u,v)}|\,\mathbf{du\,dv}$$

$$\iint \mathbf{f(x,y)\,dx\,dy} = \iint \mathbf{f}\left[\mathbf{x(r,\theta),y(r,\theta)}\right] |\mathbf{J(r,\theta)}|\,\mathbf{dr\,d\theta}$$

$$\iint \mathbf{dx\,dy} = \int_{\theta=0}^{2\pi}\int_{\mathbf{r}=0}^{\mathbf{r}} |\mathbf{J(r,\theta)}|\,\mathbf{dr\,d\theta}$$

$$\iint \mathbf{dx\,dy} = \int_{\theta=0}^{2\pi}\int_{\mathbf{r}=0}^{\mathbf{r}} \mathbf{r\,dr\,d\theta} \qquad [\,|\mathbf{J(r,\theta)}| = \frac{\partial(\mathbf{x,y})}{\partial(\mathbf{r},\theta)} = \mathbf{r}\,]$$

$$\iint \mathbf{dx\,dy} = \int_{\theta=0}^{2\pi}\left[\frac{\mathbf{r}^2}{2}\right]_0^{\mathbf{r}} \mathbf{d\theta}$$

$$\iint dx\,dy = \int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}-\frac{0^2}{2}\right]d\theta$$

$$\iint dx\,dy = \int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}-0\right]d\theta$$

$$\iint dx\,dy = \frac{r^2}{2}\int_{\theta=0}^{2\pi} d\theta$$

$$\iint dx\,dy = \frac{r^2}{2}\left[\theta\right]_0^{2\pi}$$

$$\iint dx\,dy = \frac{r^2}{2}\left[2\pi-0\right]$$

$$\iint dx\,dy = \frac{r^2}{2}\left[2\pi\right]$$

$$\iint dx\,dy = \pi r^2$$

Q # 109: In three dimensions, $\iiint\limits_V f(x,y,z)\,dx\,dydz$ ----------------------------(i)

Now if we want to switch to another coordinate system, we define $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$

That is: $\iiint\limits_V f(x,y,z)\,dx\,dydz = \iiint\limits_{V'} f(u,v,w)\left|J(u,v,w)\right| du\,dvdw$ ---------------(ii)

Where $$\left|J(u,v,w)\right| = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Q# 110: Verify Green's theorem for the integral $\oint\limits_C \{(x^2+y^2)dx+(x+2y)dy\}$ taken round the boundary curve c defined by

$y = 0; 0 \le x \le 2$

$x^2 + y^2 = 4; 0 \le x \le 2$

$x = 0; 0 \le y \le 2$

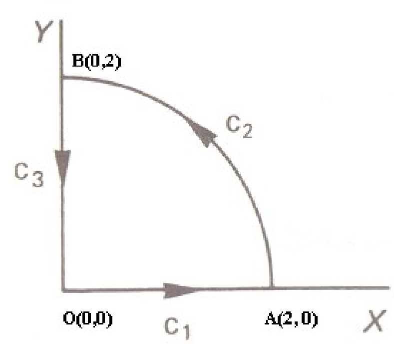


Figure # 161

Answer:

We have Green's theorem $\iint_S \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dxdy = \oint_c (Pdx + Qdy)$

LH.S.

First we have to evaluate $\iint_S \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dxdy$ ----------------------(i)

Given, $\oint_C \{(x^2 + y^2)dx + (x + 2y)dy\}$ --------------------------------(ii)

Since, R.H.S $\oint_c (Pdx + Qdy)$ ---------------------------------(iii)

Comparing (ii) & (iii),

Here, $P = x^2 + y^2$

$$\therefore \frac{\delta P}{\delta y} = 0 + 2y = 2y$$

And $Q = x + 2y$

$$\therefore \frac{\delta Q}{\delta x} = 1 + 0 = 1$$

So, from (i), we get, *[Putting the values of* $\frac{\delta P}{\delta y}$ & $\frac{\delta Q}{\delta x}$ *in (i)]*

$$\iint_S \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dxdy = \iint_S (1 - 2y)\, dxdy \qquad \text{--------------------------(iv)}$$

It will be more convenient to work in polar coordinates, so we make the substitutions

Let $x = r\cos\theta \quad y = r\sin\theta$

Then $\iint_s f(x,y)dxdy = \iint_s f(r,\theta)|J(r,\theta)|\, dr\, d\theta$ --------------------(v)

Here $f(x,y) = 1 - 2y$

From **Jacobian Determinant,**

$$\therefore |J(r,\theta)| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \qquad \text{---------------------------------(vi)}$$

We have, $x = r\cos\theta$

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta)$$

$$\frac{\partial x}{\partial r} = \cos\theta \frac{\partial}{\partial r}(r)$$

$$\frac{\partial x}{\partial r} = \cos\theta.1$$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \text{------------------------------------(vii)}$$

Again

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta)$$

$$\frac{\partial x}{\partial \theta} = r\frac{\partial}{\partial \theta}(\cos\theta)$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \text{--------------------------------------(viii)}$$

Again, $\mathbf{y = r\sin\theta}$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta)$$

$$\frac{\partial y}{\partial r} = \sin\theta\frac{\partial}{\partial r}(r)$$

$$\frac{\partial y}{\partial r} = \sin\theta.1$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \text{-----------------------------------(ix)}$$

Again

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta)$$

$$\frac{\partial y}{\partial \theta} = r\frac{\partial}{\partial \theta}(\sin\theta)$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta \qquad \text{---------------------------------------(x)}$$

From (vi),

$$\therefore |J(r,\theta)| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$|J(r,\theta)| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$|J(r,\theta)| = \frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta - (-r\sin^2\theta)$$

$$|J(r,\theta)| = \frac{\partial(x,y)}{\partial(r,\theta)} = r\cos^2\theta + r\sin^2\theta$$

$$|J(r,\theta)| = \frac{\partial(x,y)}{\partial(r,\theta)} = r(\cos^2\theta + \sin^2\theta)$$

$$|J(r,\theta)| = \frac{\partial(x,y)}{\partial(r,\theta)} = r.1$$

$$\left|\mathbf{J}(\mathbf{r},\theta)\right| = \frac{\partial(\mathbf{x},\mathbf{y})}{\partial(\mathbf{r},\theta)} = \mathbf{r} \qquad \text{--(xi)}$$

From (v),

$$\iint_{s} f(x,y)dxdy = \iint_{s} f(r,\theta)\left|J(r,\theta)\right| dr\, d\theta \quad \text{---------------------------(*)}$$

Hence, From (iv),

$$\iint_{S} (1-2y)dxdy = \iint_{S} (1-2r\sin\theta)r\, dr \times d\theta \text{----------------------------(xii)}$$

[Comparing * & xii,

[Here, $f(x,y) = 1-2y,\ y = r\sin\theta,\ f(r,\theta) = 1-2r\sin\theta,\ \left|J(r,\theta)\right| = r$ ***]***

Now,

When $\mathbf{x = 2}$, $\mathbf{y = 0}$ ***and*** $\mathbf{r = 2}$ ***then***

$$\mathbf{x = r\cos\theta}$$
$$\Rightarrow \mathbf{2 = 2\cos\theta}$$
$$\Rightarrow \mathbf{1 = \cos\theta}$$
$$\Rightarrow \mathbf{\cos 0 = \cos\theta}$$
$$\Rightarrow \theta = \mathbf{0}$$

Again

When $\mathbf{x = 0}$, $\mathbf{y = 2}$ ***and*** $\mathbf{r = 2}$ ***then***

$$\mathbf{y = r\sin\theta}$$
$$\Rightarrow \mathbf{2 = 2\sin\theta}$$
$$\Rightarrow \mathbf{1 = \sin\theta}$$
$$\Rightarrow \mathbf{\sin\frac{\pi}{2} = \sin\theta}$$
$$\Rightarrow \theta = \frac{\pi}{\mathbf{2}}$$

Hence the limit of θ is $\mathbf{0} \le \theta \le \pi/\mathbf{2}$

Again

When $\mathbf{x = 2}$ ***and*** $\theta = \mathbf{0}$ ***then***

$$\mathbf{x = r\cos\theta}$$
$$\mathbf{2 = r\cos 0}$$
$$\Rightarrow \mathbf{2 = r \times 1}$$
$$\Rightarrow \mathbf{2 = r}$$
$$\Rightarrow \mathbf{r = 2}$$

When $\mathbf{y = 2}$ ***and*** $\theta = \pi/\mathbf{2}$ ***then***

$$\mathbf{y = r\sin\theta}$$
$$\mathbf{2 = r\sin\frac{\pi}{2}}$$
$$\Rightarrow \mathbf{2 = r \times 1}$$
$$\Rightarrow \mathbf{2 = r}$$
$$\Rightarrow \mathbf{r = 2}$$

Hence the limit of $\mathbf{r}$ ***is*** $\mathbf{0 \le r \le 2}$

From (xii),

$$\iint_S (1-2y)\,dxdy = \iint_S (1-2r\sin\theta)\,r\,dr \times d\theta \text{ -}$$

$$= \int_0^{\pi/2}\int_0^2 (1-2r\sin\theta)rdrd\theta$$

$$= \int_0^{\pi/2}\int_0^2 (1-2r\sin\theta)rdrd\theta$$

$$= \int_0^{\pi/2}\int_0^2 (r-2r^2\sin\theta)drd\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2} - 2\frac{r^3}{3}\sin\theta\right]_0^2 d\theta$$

$$= \int_0^{\pi/2}\left[\frac{2^2}{2} - 2\frac{2^3}{3}\sin\theta - 0\right] d\theta$$

$$= \int_0^{\pi/2}\left[2 - \frac{16}{3}\sin\theta\right] d\theta$$

$$= \left[2\theta + \frac{16}{3}\cos\theta\right]_0^{\pi/2}$$

$$= \left[2\times\frac{\pi}{2} + \frac{16}{3}\cos\frac{\pi}{2} - 0 - \frac{16}{3}\cos 0\right]$$

$$= \left[\pi + \frac{16}{3}\times 0 - 0 - \frac{16}{3}\times 1\right]$$

$$= \left[\pi - \frac{16}{3}\right]$$

$$\therefore \iint_S \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right) dxdy = \iint_S (1-2y)\,dxdy = \left[\pi - \frac{16}{3}\right] \text{-----------------------------(xiii)}$$

Now we have to evaluate R.H.S of Green's Theorem $\oint_c (Pdx + Qdy)$

Given $\oint_C \{(x^2+y^2)dx + (x+2y)dy\}$

$\therefore \mathbf{P} = \mathbf{x}^2 + \mathbf{y}^2$ and $\mathbf{Q} = \mathbf{x} + 2\mathbf{y}$

We now take $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{c}_3$ in turn

i) $\mathbf{c}_1 : \mathbf{y} = \mathbf{0};\ \therefore \mathbf{dy} = \mathbf{0}$

$$\therefore \oint_{c_1}(Pdx+Qdy)=\int_0^2 (x^2+y^2)dx+(x+2y)dy=\int_0^2 (x^2+0^2)dx+(x+2.0).0$$

$$=\int_0^2 x^2 dx=\frac{8}{3}$$

ii) $c_2 : x^2+y^2=4 \therefore y^2=4-x^2 \therefore y=\sqrt{4-x^2}=(4-x^2)^{1/2}$

$$\therefore \frac{dy}{dx}=\frac{1}{2}(4-x^2)^{-1/2}.\frac{d}{dx}(4-x^2)$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}(4-x^2)^{-1/2}.(0-2x)$$

$$\therefore dy=-(4-x^2)^{-1/2}.xdx=\frac{-xdx}{\sqrt{4-x^2}}$$

$$\therefore \oint_{c_2}(Pdx+Qdy)=\oint_{c_2}(x^2+y^2)dx+(x+2y)dy$$

$$=\oint_{c_2}(x^2+4-x^2)dx+(x+2\sqrt{4-x^2}).\frac{-xdx}{\sqrt{4-x^2}}$$

$$=\oint_{c_2}4dx-(x+2\sqrt{4-x^2})\frac{xdx}{\sqrt{4-x^2}}$$

$$=\oint_{c_2}4dx-\frac{x^2dx}{\sqrt{4-x^2}}-2\sqrt{4-x^2}\,\frac{xdx}{\sqrt{4-x^2}}$$

$$=\oint_{c_2}4dx-\frac{x^2dx}{\sqrt{4-x^2}}-2xdx$$

$$=\oint_{c_2}(4-2x)dx-\frac{x^2dx}{\sqrt{4-x^2}}$$

$$=\oint_{c_2}(4-2x-\frac{x^2}{\sqrt{4-x^2}})dx$$

$$=\int_2^0 (4-2x-\frac{x^2}{\sqrt{4-x^2}})dx \text{ -------------------------------------(xiv)}$$

Let $x=2\sin\theta$

$\Rightarrow dx=2\cos\theta d\theta$

$\therefore \sqrt{4-x^2}=\sqrt{4-(2\sin\theta)^2}=\sqrt{4-4\sin^2\theta}=\sqrt{4(1-\sin^2\theta)}=\sqrt{4\cos^2\theta}=2\cos\theta$

Given, $x=2\sin\theta$

$\Rightarrow \theta=\sin^{-1}(x/2)$

x	0	2
θ	$\theta = \sin^{-1}(x/2)$ $\Rightarrow \theta = \sin^{-1}(0/2)$ $\Rightarrow \theta = \sin^{-1}(0)$ $\Rightarrow \theta = \sin^{-1}\sin 0$ $\Rightarrow \theta = 0$	$\theta = \sin^{-1}(x/2)$ $\Rightarrow \theta = \sin^{-1}(2/2)$ $\Rightarrow \theta = \sin^{-1}(1)$ $\Rightarrow \theta = \sin^{-1}(\sin\frac{\pi}{2})$ $\Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \oint_{c_2}(Pdx + Qdy) = \int_2^0 (4 - 2x - \frac{x^2}{\sqrt{4-x^2}})dx$$

$$= \int_{\pi/2}^0 \{(4 - 2\times 2\sin\theta - \frac{(2\sin\theta)^2}{2\cos\theta}\}2\cos\theta d\theta$$

$$= \int_{\pi/2}^0 \{(8\cos\theta - 4\times 2\sin\theta\cos\theta - 4\sin^2\theta)d\theta$$

$$\therefore \oint_{c_2}(Pdx + Qdy) = \int_{\pi/2}^0 8\cos\theta d\theta - \int_{\pi/2}^0 4\times 2\sin\theta\cos\theta d\theta - \int_{\pi/2}^0 4\sin^2\theta d\theta \text{ --------------(xv)}$$

Now,

a) $\int_{\pi/2}^0 8\cos\theta d\theta = [8\sin\theta]_{\pi/2}^0 = 8\sin 0 - 8\sin\frac{\pi}{2} = 0 - 8\times 1 = -8$

b) $\int_{\pi/2}^0 4\times 2\sin\theta\cos\theta d\theta$

Let $z = \sin\theta$

$\Rightarrow dz = \cos\theta d\theta$

Since $z = \sin\theta$

$\Rightarrow \theta = \sin^{-1}(z/2)$

θ	0	$\frac{\pi}{2}$
z	$z = \sin\theta$ $z = \sin 0$ $z = 0$	$z = \sin\theta$ $z = \sin\pi/2$ $z = 1$

$$\int_{\pi/2}^0 4\times 2\sin\theta\cos\theta d\theta = \int_1^0 4\times 2z dz = \left[8\times\frac{z^2}{2}\right]_1^0 = 4(0-1) = -4$$

c) $\int_{\pi/2}^{0} 4\sin^2\theta d\theta = 2\int_{\pi/2}^{0} 2\sin^2\theta d\theta = 2\int_{\pi/2}^{0}(1+\cos 2\theta)d\theta$

$$= 2\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/2}^{0} = 2(0 - \frac{1}{2}\sin 2.0 - \frac{\pi}{2} + \frac{1}{2}\sin 2.\frac{\pi}{2}) = 2(0 - 0 - \frac{\pi}{2} + \frac{1}{2}.\sin\pi) = 2(-\frac{\pi}{2} + \frac{1}{2}.0)$$

$$= -\pi$$

From (xv)

$$\therefore \oint_{c_2}(Pdx + Qdy) = \int_{\pi/2}^{0} 8\cos\theta d\theta - \int_{\pi/2}^{0} 4\times 2\sin\theta\cos\theta d\theta - \int_{\pi/2}^{0} 4\sin^2\theta d\theta$$

$$= -8 - (-4) - (-\pi)$$

$$= -8 + 4 + \pi$$

$$= \pi - 4$$

iii) $c_3 : x = 0;\ \therefore dx = 0$

$$\therefore \oint_{c_3}(Pdx + Qdy)$$

$$= \oint_{c_2}(x^2 + y^2)dx + (x + 2y)dy$$

$$= \oint_{c_3}(0^2 + y^2).0 + (0 + 2y)dy = \int_2^0 2ydy = \left[2\frac{y^2}{2}\right]_2^0 = -4$$

Collecting our three partial results

$\oint_c (Pdx + Qdy) = \frac{8}{3} + \pi - 4 - 4 = \pi - \frac{16}{3}$ --------------------(xvi)

From (xiii) and (xvi), we can write

L.H.S. = R.H.S.

$\therefore \iint_S \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)dxdy = \oint_c (Pdx + Qdy)$ ***(Proved)***

Q# 111: Use Green's theorem to evaluate $\int_C \mathbf{(x^2 + xy)dx + (x^2 + y^2)dy}$ where C is the square formed ABCD by the lines $\mathbf{y = \pm 1, x = \pm 1}$

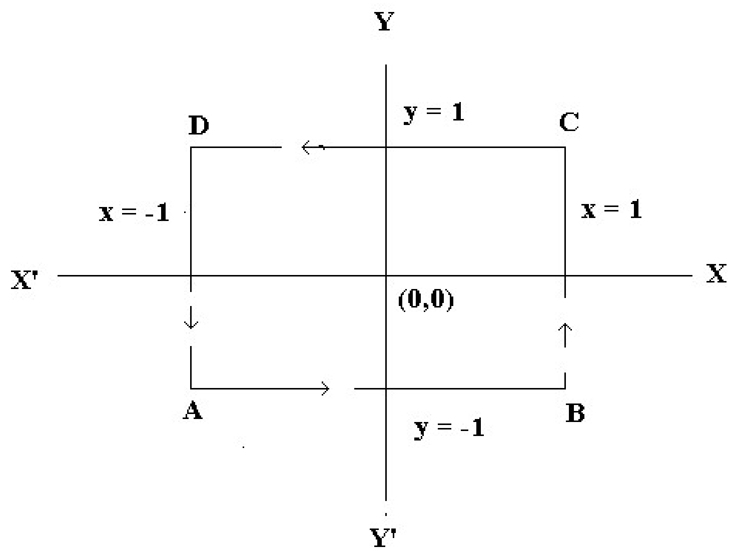


Figure # 162

Answer: From Green's theorem, we have,

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)dxdy$$

Here, $P = x^2 + xy$ and $Q = x^2 + y^2$

$$\int_C Pdx + Qdy = \iint_R \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)dxdy$$

L.H.S.

$\int_C \mathrm{Pdx} + \mathrm{Qdy} = \text{TRY YOURSELF}$

Here, R.H.S= $\int_{-1}^{1}\int_{-1}^{1} \{(\frac{\delta}{\delta x}(x^2 + y^2) - \frac{\delta}{\delta y}(x^2 + xy)\} dxdy$

$$= \int_{-1}^{1}\int_{-1}^{1} \mathbf{(2x - x)dxdy}$$

$$= \int_{-1}^{1}\int_{-1}^{1} \mathbf{xdxdy}$$

$$= \int_{-1}^{1} \left[\frac{\mathbf{x^2}}{\mathbf{2}}\right]_{-1}^{1} \mathbf{dy}$$

$$= \int_{-1}^{1} \left[\frac{1^2}{2} - \frac{(-1)^2}{2} \right] dy$$

$$= \int_{-1}^{1} \left[\frac{1^2}{2} - \frac{1^2}{2} \right] dy$$

$= \mathbf{0}$ Answer

Home Task:

State Green's theorem. Verify Green's theorem in the plane for $\int_C (2xy - x^2)dx + (x + y^2)dy$ where the close curve of the region bounded by $y = x, y^2 = x$