1. $f(t) = e^{-\alpha|t|}$, with $\alpha > 0$. To deal with the absolute value, we break the transform integral into two regions:

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{0} e^{\alpha t + i\omega t}\, dt + \sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} e^{-\alpha t + i\omega t}\, dt$$

$$= \sqrt{\frac{1}{2\pi}} \left[\frac{1}{\alpha + i\omega} + \frac{1}{\alpha - i\omega} \right] = \sqrt{\frac{1}{2\pi}}\, \frac{2\alpha}{\alpha^2 + \omega^2}. \qquad (20.13)$$

We note two features of this result: (1) It is real; from the form of the transform, we can see that if $f(t)$ is even, its transform will be real. (2) The more localized is $f(t)$, the less localized will be $g(\omega)$. The transform will have an appreciable value until $\omega \gg \alpha$; larger α corresponds to greater localization of $f(t)$.

it has the same value for all ω.

3. $f(t) = 2\alpha\sqrt{1/2\pi}\,/(\alpha^2 + t^2)$, with $\alpha > 0$. One way to evaluate this transform is by contour integration. It is convenient to start by writing initially

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\alpha\, e^{i\omega t}}{(t - i\alpha)(t + i\alpha)}\, dt.$$

The integrand has two poles: one at $t = i\alpha$ with residue $e^{-\alpha\omega}/i$ and one at $t = -i\alpha$ with residue $e^{+\alpha\omega}/(-i)$. If $\omega > 0$, our integrand will become negligible on a large semicircle in the upper half-plane, so an integral over the contour shown in Fig. 20.2(a) will be that needed for $g(\omega)$. This contour encloses only the pole at $t = i\alpha$, so we get

$$g(\omega) = \frac{1}{2\pi}\,(2\pi i)\frac{e^{-\alpha\omega}}{i} \quad (\omega > 0). \tag{20.15}$$

However, if $\omega < 0$, we must close the contour in the lower half-plane, as in Fig. 20.2(b), circling the pole at $t = -i\alpha$ in a clockwise sense (thereby generating a minus sign). This procedure yields

$$g(\omega) = \frac{1}{2\pi}\,(-2\pi i)\frac{e^{+\alpha\omega}}{-i} \quad (\omega < 0). \tag{20.16}$$

If $\omega = 0$, we cannot perform a contour integration on either of the paths shown in Fig. 20.2, but we then do not need this sophisticated an approach, as we have the

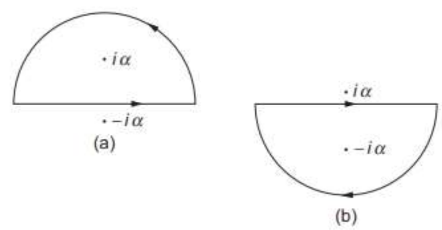

FIGURE 20.2 Contours for third transform in Example 20.2.1.

elementary integral

$$g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\alpha}{t^2 + \alpha^2}\, dt = 1. \tag{20.17}$$

Combining Eqs. (20.15)–(20.17) and simplifying, we have

$$g(\omega) = e^{-\alpha|\omega|}.$$

Here we Fourier transformed the transform from our first example, recovering the original untransformed function. This provides an interesting clue as to the form to be expected for the inverse Fourier transform. It is only a clue, because our example involved a transform that was real (i.e., not complex). ■

An important Fourier transform follows.

20.2.4 Find the Fourier transform of the triangular pulse (Fig. 20.6),

$$f(x) = \begin{cases} h(1 - a|x|), & |x| < 1/a, \\ 0, & |x| > 1/a. \end{cases}$$

20.2.4. $g(\omega) = \sqrt{\dfrac{2}{\pi}}\,\dfrac{ha}{\omega^2}\left[1 - \cos\left(\dfrac{\omega}{a}\right)\right].$

20.2.2 The function

$$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

is a symmetrical finite step function.

(a) Find $g_c(\omega)$, Fourier cosine transform of $f(x)$.

(b) Taking the inverse cosine transform, show that

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin\omega \, \cos\omega x}{\omega} \, d\omega.$$

(c) From part (b) show that

$$\int_0^\infty \frac{\sin\omega \cos\omega x}{\omega} d\omega = \begin{cases} 0, & |x| > 1, \\ \frac{\pi}{4}, & |x| = 1, \\ \frac{\pi}{2}, & |x| < 1. \end{cases}$$

20.2.2. (a) $g_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^1 \cos\omega x \, dx = \sqrt{\frac{2}{\pi}} \frac{\sin\omega}{\omega}$.

(b) The equation written here is just the inverse cosine transform of g_c and therefore has to yield $f(x)$.

(c) For all x such that $|x| \neq 1$ the integral of this part is $(\pi/2)f(x)$, in agreement with the answer in the text. For $x = 1$, the present integral can be evaluated as

$$\int_0^\infty \frac{\sin\omega \cos\omega}{\omega} \, d\omega = \frac{1}{2} \int_0^\infty \frac{\sin 2\omega}{\omega} \, d\omega = \frac{1}{2} \int \frac{\sin u}{u} \, du = \frac{1}{2} \frac{\pi}{2} = \frac{\pi}{4}.$$

The u integral is that in Eq. (11.107)

20.2.3 (a) Show that the Fourier sine and cosine transforms of e^{-at} are

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \frac{\omega}{\omega^2 + a^2}, \quad g_c(\omega) = \sqrt{\frac{2}{\pi}} \frac{a}{\omega^2 + a^2}.$$

Hint. Each of the transforms can be related to the other by integration by parts.

(b) Show that

$$\int_0^\infty \frac{\omega \sin\omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2} e^{-ax}, \quad x > 0,$$

$$\int_0^\infty \frac{\cos\omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2a} e^{-ax}, \quad x > 0.$$

These results can also be obtained by contour integration (Exercise 11.8.12).

20.2.3. (a) Integrating by parts twice we obtain

$$\int_0^\infty e^{-ax} \cos\omega x \, dx = -\frac{1}{a} e^{-ax} \cos\omega x \Big|_0^\infty - \frac{\omega}{a} \int_0^\infty e^{-ax} \sin\omega x \, dx$$

$$= \frac{1}{a} - \frac{\omega}{a} \left[-\frac{1}{a} e^{-ax} \sin\omega x \Big|_0^\infty + \frac{\omega}{a} \int_0^\infty e^{-ax} \cos\omega x \, dx \right].$$

Laplace problem

then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad \text{for} \quad s > 0. \tag{20.129}$$

Next, let

$$F(t) = e^{kt}, \quad t > 0.$$

The Laplace transform becomes

$$\mathcal{L}\left\{e^{kt}\right\} = \int_0^\infty e^{-st}\,e^{kt}\,dt = \frac{1}{s-k}, \quad \text{for} \quad s > k. \tag{20.130}$$

Using this relation, we obtain the Laplace transform of certain other functions. Since

$$\cosh kt = \tfrac{1}{2}(e^{kt} + e^{-kt}), \quad \sinh kt = \tfrac{1}{2}(e^{kt} - e^{-kt}), \tag{20.131}$$

we have

$$\mathcal{L}\{\cosh kt\} = \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) = \frac{s}{s^2-k^2}, \tag{20.132}$$

$$\mathcal{L}\{\sinh kt\} = \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) = \frac{k}{s^2-k^2}, \tag{20.133}$$

both valid for $s > k$.

From the relations

$$\cos kt = \cosh ikt, \quad \sin kt = -i\sinh ikt,$$

it is evident that we can obtain transforms of the sine and cosine if k is replaced by ik in Eqs. (20.132) and (20.133):

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \tag{20.134}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \tag{20.135}$$

both valid for $s > 0$. Another derivation of this last transform is given in Example 20.8.1. It is a curious fact that $\lim_{s\to 0}\mathcal{L}\{\sin kt\} = 1/k$ despite the fact that $\int_0^\infty \sin kt\,dt$ does not exist.

Finally, for $F(t) = t^n$, we have

$$\mathcal{L}\left\{t^n\right\} = \int_0^\infty e^{-st}t^n\,dt,$$

which is just a gamma function. Hence

$$\mathcal{L}\left\{t^n\right\} = \frac{\Gamma(n+1)}{s^{n+1}}, \quad s > 0,\ n > -1. \tag{20.136}$$

Note that in all these transforms we have the variable s in the denominator, so that it occurs as a negative power. From the definition of the transform, Eq. (20.126) and the existence condition, Eq. (20.127), it is clear that if $f(s)$ is a Laplace transform, then $\lim_{s\to\infty} f(s) = 0$. The significance of this point is that if $f(s)$ behaves asymptotically for large s as a positive power of s, then no inverse transform can exist.

In [ ]: