Aug 27, 2010

## 0.1 Basic Probability

Pick a random integer $x$ uniformly in $\{1, \ldots, 1000\}$.

1. What is $\mathbb{P}[x \text{ is divisible by } 5]$?
2. What is $\mathbb{P}[x \text{ is divisible by } 10]$?
3. What is $\mathbb{P}[x \text{ is divisible by } 11]$?
4. What is $\mathbb{P}[x \text{ is divisible by } 5 \text{ or } x \text{ is divisible by } 11]$?
5. What is $\mathbb{P}[x \text{ is divisible by } 5 \text{ or } x \text{ is divisible by } 10]$?
6. What is $\mathbb{P}[x \text{ is divisible by } 2 \text{ or } x \text{ is divisible by } 5 \text{ or } x \text{ is divisible by } 11]$?

## 0.2 Target Practice

Al and George go target shooting. Al hits the target with probability $p_1$ and George hits the target with probability $p_2$. Assuming they booth shoot simultaneously and someone hits the target,

1. What is the probability that both Al and George hit the target?
2. What is the probability that Al hit the target?
3. Suppose Al and George take turns shooting each other, with Al going first. If $p_2 = \frac{1}{2}$, how good does Al have to be so both Al and George are equally likely to die?

## 0.3 Pick a white ball and win

You have 50 White balls, 50 Black balls and two jars. You can put the balls into the jars however you please. You will then be blindfolded and the jars will be shaken. You then have to select a ball (still blindfolded), and you win if you pick a white ball. How can you maximize your chances of winning?

## 0.4 Random Permutation

Suppose you have access to a function `RandInt(i)` that returns an integer uniformly distributed in the set $\{1, \ldots, i\}$. Subsequent calls to the function produce independent results.

1. Starting with an array $A$ containing the elements $1, \ldots, n$ in sorted order, show how to compute a random permutation of $A$ in $O(n)$ time. Prove that your algorithm is correct.
2. What does your algorithm produce if the array $A$ is not initially in the sorted order, but contains the elements $1, \ldots, n$ in some arbitrary order?
3. What if you only have access to `Bernoulli(1/2)`?

## 0.5 Random graph on 4 vertices

Consider two ways of generating a random graph on 4 labeled vertices (V = {A,B,C,D}).
**Method 1:** Edge $AB \leftarrow$`Bernoulli(1/2)`. That is, edge $AB$ is added if the Bernoulli random variable is 1, and removed if it is 0. We do this for each edge independently.
**Method 2:** We put all possible the graphs on $V$ in a bag, and pick one graph uniformly at random.

1. Prove that these two methods of generating a random graph on $V$ are equivalent.
2. What is $\mathbb{P}[A \text{ and } B \text{ are connected}]$?
3. Method 1 is somewhat more powerful than Method 2. Can you think why?

## 0.6 Deferred Decision

Let $k$ be a positive integer, and consider the following experiment:
$X \leftarrow \texttt{RandInt}(2^k) - 1$

$\texttt{for } i \leftarrow 1 \dots 2^k - 1$

$\quad T_i \leftarrow \texttt{parity}(X\&i)$

Here & denotes bitwise-AND (when $X$ and $i$ are treated as $k$-bit strings) and `parity`() computes the bitwise-XOR of the bits in a $k$-bit string. E.g., with $k = 3$, we have

$$\texttt{parity}(5\&1) = \texttt{parity}(101\&001) = \texttt{parity}(001) = 0 \oplus 0 \oplus 1 = 1.$$

1. Thinking of $X$ as a $k$-bit string, prove that it is equivalent to generate $X$ by choosing each bit by a call to `Bernoulli`(1/2).
2. Let $A_i$ be the event that $T_i = 1$. Show that $\Pr[A_i] = 1/2$.
3. Assume $i \neq j$. Show that $\Pr[A_i \cap A_j] = 1/4$.