Thursday, February 15, 2018 2:33 PM

PRIMAL DUAL ALGORITHMS

Combinatorial Algorithms where LP comes only in the proofs

PRIMAL

$$\min \; 2x_1 + 3x_2$$
$$4x_1 + 8x_2 \geq 12 \quad (y_1)$$
$$2x_1 + x_2 \geq 3 \quad (y_2)$$
$$3x_1 + 4x_2 \geq 4 \quad (y_3)$$
$$x_1, x_2 \geq 0$$

$$y_1(4x_1 + 8x_2) + y_2(2x_1 + x_2) + y_3(3x_1 + 4x_2) \geq 12y_1 + 3y_2 + 4y_3$$

(holds for any feasible (x_1, x_2) for LP & $y_1, y_2, y_3 \geq 0$)

by rearranging variables, this is equal to

$$x_1(4y_1 + 2y_2 + 3y_3) + x_2(8y_1 + y_2 + 4y_3) \geq 12y_1 + 3y_2 + 4y_3$$

Suppose I enforce that

$$2 \geq 4y_1 + 2y_2 + 3y_3 \quad \& \quad y_1\; y_2\; y_3 \geq 0$$
$$3 \geq 8y_1 + y_2 + 4y_3$$

then we get

$$2x_1 + 3x_2 \geq 12y_1 + 3y_2 + 4y_3$$

Create the following dual LP

DUAL

$$\text{Maximize} \quad 12y_1 + 3y_2 + 4y_3$$
$$\text{s.t.} \quad 4y_1 + 2y_2 + 3y_3 \leq 2$$
$$8y_1 + y_2 + 4y_3 \leq 3$$
$$y_1\; y_2\; y_3 \geq 0$$

If $y_1\, y_2\, y_3$ is <u>any</u> feasible soln to dual LP of value say V, then

$$\boxed{\text{OPT (Primal LP)} \geq V}$$

Weak Duality Theorem

Theorem If $\bar{x}$ is <u>any</u> feasible sol^n to primal & $\bar{y}$ is any feasible sol^n to dual LP,

$$\text{Obj}(\bar{x}) \geq \text{Obj}(\bar{y})$$

If $\bar{x}$ is any ...
any feasible sol^n to dual LP,
then $Obj_p(\bar{x}) \geq Obj_D(\bar{y})$

Proof: $Obj_p(\bar{x}) = 2\bar{x}_1 + 3\bar{x}_2$

$\geq (4\bar{y}_1 + 2\bar{y}_2 + 3\bar{y}_3)\bar{x}_1 + (8\bar{y}_1 + \bar{y}_2 + 2\bar{y}_3)\bar{x}_2$ ($\bar{y}$ is dual feasible)

$= \bar{y}_1(4\bar{x}_1 + 8\bar{x}_2) + \bar{y}_2(2\bar{x}_1 + \bar{x}_2) + \bar{y}_3(3\bar{x}_1 + 2\bar{x}_2)$

$\geq 12\bar{y}_1 + 3\bar{y}_2 + 4\bar{y}_3$ ($\bar{x}$ is primal feasible)

$\geq Obj_D(\bar{y})$

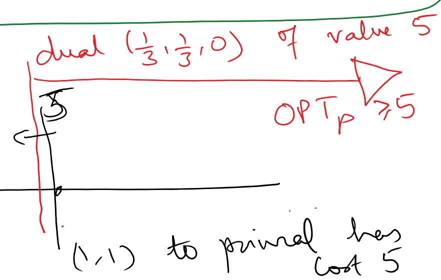
Consider $\bar{y}_1 = \frac{1}{3}$ & $\bar{y}_2 = \frac{1}{3}$, $\bar{y}_3 = 0$

This is feasible for dual LP

Summary

For above problem, to show that $OPT_p \leq 5$,
→ enough to produce 1 feasible sol^n to primal (in this example (1,1))

→ to show that $OPT_p \geq 5$, enough to produce 1 feasible sol^n to dual (in this example $(\frac{1}{3}, \frac{1}{3}, 0)$)



$\Rightarrow OPT_p \leq 5$

Theorem 2: If x^* y^* are feasible to primal & dual respectively &

$OBJ_P(x^*) = OBJ_D(y^*)$ then
x^* is optimal sol^n for primal
& y^* is optimal sol^n for dual.

$$P = \min\ c^T x \qquad\qquad D = \max\ b^T y$$
$$Ax \geq b \qquad\qquad A^T y \leq c$$
$$x \geq 0 \qquad\qquad y \geq 0$$

If $\bar{x}$ is primal feasible & $\bar{y}$ is dual feasible then
$c^T \bar{x} \geq b^T \bar{y}$

$$c^T \bar{x} \geq y^T A \bar{x} \geq y^T b = b^T y.$$

($\bar{y}$ is dual feasible) ($\bar{x}$ is primal feasible)

More generally

$$P = \min\ c^T x$$
$$a_i^T x \geq b_i \quad i \in I_1$$
$$a_i^T x = b_i \quad i \in I_2$$
$$x_j \geq 0 \quad j \in J_1$$
$$x_j \in \mathbb{R} \quad j \in J_2$$

$$D = \max\ b^T y$$
$$y_i \geq 0 \quad \forall i \in I_1$$
$$y_i \in \mathbb{R} \quad \forall i \in I_2$$
$$A^j \cdot y \geq c_j \quad \forall j \in J_1$$
$$\boxed{A^j y = c_j} \quad \forall j \in J_2$$

Nice Fact

$$\boxed{\text{Dual}(\text{Dual}) = \text{Primal}}$$

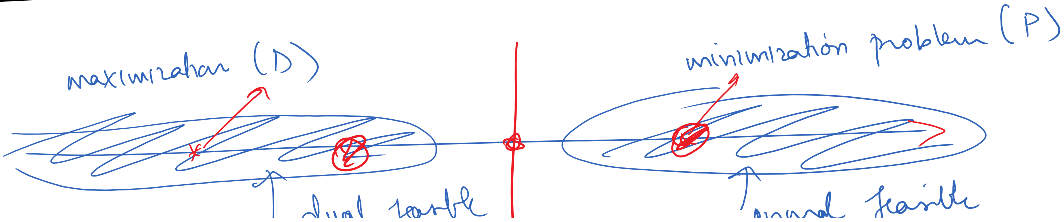
$P \rightleftarrows D$

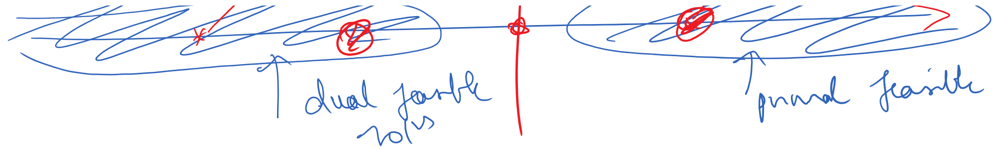
(primal, dual) pair

$P = \min$; $D = \max$

Every constraint (P) $\rightleftarrows$ dual variable
($\geq b$ constraint) — (≥ 0 variable)
($= b$ constraint) — ($\in \mathbb{R}$ variable)

Every variable (P) $\rightleftarrows$ dual constraint
(≥ 0 variable) — ($\leq c$ constraint)
($\in \mathbb{R}$ variable) — ($= c$ constraint)





Strong duality Theorem

For any primal dual pair, the following situations can arise

A) Primal is infeasible, dual is infeasible
B) Primal is Unbounded, dual is infeasible
C) Dual is unbounded, primal is infeasible
D) both are feasible & $OPT(P) = OPT(D) = V^*$ for some V^*

Examples:

$$\begin{array}{ll} \text{Min} & 2x_1 + 3x_2 \\ & x_1 \le 2 \\ & x_2 \le 3 \end{array} \quad = \quad \begin{array}{llll} \text{Min} & 2x_1 + 3x_2 & & \\ & -x_1 \geq -2 & & (y_1) \\ & -x_2 \geq -3 & & (y_2) \end{array}$$

$\downarrow$

Dual

$$\begin{array}{ll} \text{Max} & -2y_1 - 3y_2 \\ & -y_1 = 2 \\ & -y_2 = 3 \\ & y_1, y_2 \geq 0 \end{array}$$

$$\begin{array}{lll} & \text{Min} & 0 \\ y_1 & & x_1 \geq 2 \\ y_2 & & -x_1 \geq -1 \\ & & x_1 \in \mathbb{R} \end{array} \quad \Bigg| \quad \begin{array}{ll} \text{Max} & 2y_1 - y_2 \\ & y_1 - y_2 \le 0 \\ & y_1 \geq 0 \\ & y_2 \geq 0 \end{array}$$

dual unbounded !!
primal infeasible

Corollary

* if P / D is unbounded $\Rightarrow$ D / P is infeasible

* if P or D has bounded optimum value, then both have equal optimum value

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Complementary Slackness

Complementary Slackness

let (x^*, y^*) be an optimal primal dual pair

$$\boxed{P} \quad \min \ c^T x \quad Ax \geq b \quad x \geq 0 \qquad \Bigg| \qquad \boxed{D} \quad \max \ b^T y \quad A^T y \leq c \quad y \geq 0$$

STRONG DUALITY THM: $c^T x^* = b^T y^*$

if $x_j^* > 0$ for some variable, then corresponding dual constraint is "Tight"

$$\underline{A^j \cdot y^* = c_j}$$

if $y_i^* > 0$ for some dual variable, then corr. primal constraint is tight

(ie) $A_i \cdot x^* = b_i$

Proof:-

Dual $\underline{b^T y^*} \leq \underline{x^T A} y^*$ (uses primal feasibility of x^*)

Ⓐ: If $y_i^* > 0$ & $A_i x > b_i$ then we'll have

$b^T y^* < x^T A y^*$

Primal

$c^T x^* \geq y^T A x^* > b^T y^*$ (if Ⓐ holds)

contradicts strong duality because $c^T x^* = b^T y^*$.

Often times we want to solve the minimization primal

$$\boxed{P} \quad \text{Min} \sum w_v x_v \quad x_u + x_v \geq 1 \ \ \forall (u,v) \in E \quad x_u \geq 0 \ \ \forall u \qquad \Bigg| \qquad \boxed{D} \quad \text{Max} \sum y_e \quad \boxed{\sum_{e \in \delta(v)} y_e \leq w_v} \ \forall v \quad y_e \geq 0 \ \ \forall e$$

WANT (goal)

Algo which computes $\boxed{\text{Integer}}$ soln to primal & "good" feasible soln to dual

A) if $\underline{\text{primal variable} > 0}$ $\Rightarrow$ dual constraint is tight

B) if dual variable > 0 $\Rightarrow$ primal constraint is "approx" tight

Start with feasible dual ($y_e = 0$)

repeat until I find a vertex cover

Increase the y_e's in some manner until some constraint becomes tight ($\sum_{e \in \delta(v)} y_e = w_v$)

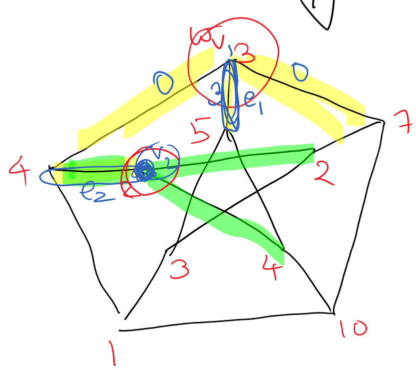
$\Rightarrow$ Intuitively, I can set $x_v = 1$ & "freeze" all the y_e Incident on vertex v

Start with all $\tilde{y}_e = 0$ (all e are unfrozen)

Increase UNFROZEN $\tilde{y}_e$'s in some manner slowly until $\sum_{e \in \delta(v)} \tilde{y}_e = w_v$ for some v

Set $X_v = 1$ for v & freeze all y_e for $e \in \delta(v)$

Repeat until we have vertex cover



set all $\tilde{y}_e = 0 \ \forall e$

~~when~~ Increase $\tilde{y}_{e_1} = 3$

Set $\hat{x}_{v_1} = 1$

Increase $\tilde{y}_{e_2} = 2$

& set $\hat{x}_{v_2} = 1$

Invariant: My $\{\tilde{y}\}$ is always dual feasible

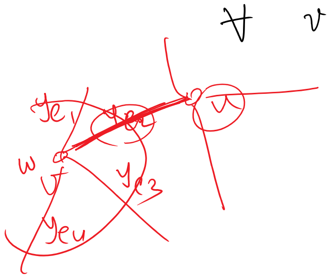
Weak duality

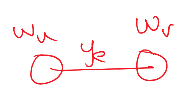
If x^* is any primal sol^n (feasible)

$$OPT(VC) \geq \sum w_v x^*_v \geq \sum_{e \in E} \tilde{y}_e$$

Our Cost

$\forall v \in$ Final Sol^n, $w_v = \sum_{e \in E, e \in \delta(v)} \tilde{y}_e$





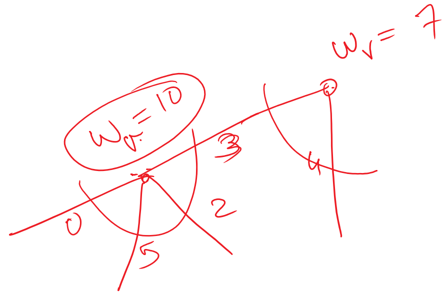
$$\sum_{v \in Sol^n} w_v = \sum_{v \in Sol^n} \sum_{e \in E, e \in \delta(v)} \tilde{y}_e$$

$$\leq 2 \sum_{e \in E} \tilde{y}_e \leq 2\, OPT(VC)$$

(... provided we find a vertex cover at the end)

Q: Why did we find a vertex cover at the end?

A: If not, some edge e would have $\tilde{y}_e$ unfrozen,
& algo would not have terminated.



$$\text{Min} \sum w_v x_v$$

$$x_u + x_v \geq 1 \quad \forall (u,v) \in E \qquad y_e$$

$$x_u \geq 0$$

P

$$\text{Max} \sum y_e$$

$$\sum_{e \in \delta(v)} y_e \leq w_v$$

$$y_e \geq 0$$

D

Weak Duality Theorem

all feasible primal soln x^*

y^*

$\forall$ Primal feasible x^*
$\forall$ dual feasible y^* $\quad \text{Obj}_D(y^*) \leq \text{Obj}_P(x^*)$

(i.e) $\boxed{\sum_e y_e^* \leq \sum_u w_u x_u^*}$

In other words, if we find a good dual feasible solution with large $\sum y_e^*$, then that gives us a lower bound on

$$OPT(VC) \geq \sum y_e^*$$

goal in Primal Dual approx algos
→ a) try to find a good dual soln &
b) using it construct a good primal soln.

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start with all $\tilde{y}_e = 0 \quad \forall e, \; S = \phi$

Start with all $\tilde{y}_e = 0 \ \forall e, S = \phi$

set all edges as unfrozen

Until S is a valid vertex cover

we pick arbitrary unfrozen edge 'e'

& increase $\tilde{y}_e$ till some endpoint becomes tight (ie) $\sum_{e \in \delta(u)} \tilde{y}_e = w_u$

Freeze 'e' & add u to S

Fact 1 : $OPT(VC) \geq \sum_{e \in E} \tilde{y}_e$ (weak Duality)

Fact 2 :

$$\text{Alg Cost} = \sum_{u \in S} w_u = \sum_{u \in S} \sum_{e \in \delta(u)} \tilde{y}_e$$

$$\leq 2 \sum_{e \in E} \tilde{y}_e \leq 2 OPT(VC)$$

Fact 3 Why is S feasible?

- Firstly we run the algo till S is feasible

If all edges are frozen, then S is a VC

- If some edge is unfrozen, algo can continue.

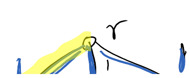
PRIMAL DUAL ALGORITHMS FOR GRAPH CONNECTIVITY PROBLEMS

Steiner Tree Problem

Given | Given a graph $G = (V, E)$ & there are ≥ 0 costs on edges $c_e \geq 0$ for $\forall e \in E$

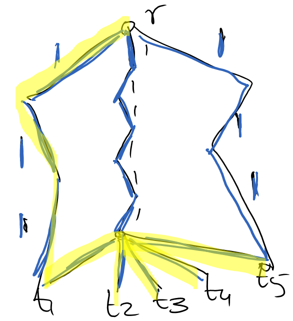
& there are collection of terminals $t_1, t_2, \ldots t_k \in V$ and one root $r \in V$

Goal | Find a minimum cost subgraph $F \subseteq E$ s.t for all t_i, there exists a $r - t_i$ path in F.



Steiner Forest Problem

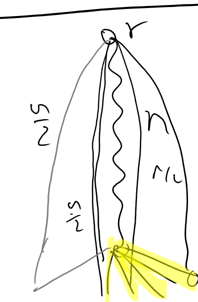
Same input
$G = (V, E)$
c_e cost ≥ 0



(s_1, t_1)
(s_2, t_2)
$\vdots$
(s_k, t_k)
} terminal pairs

Find min. cost $F \subseteq E$ st each s_i has a path to t_i

Thm: There is a 2-approximation using the PD framework



Union of shortest paths $m \cdot \frac{n}{2}$ cost
Opt cost : $\frac{n}{2} + m$

→ We know how to solve MST well (min. spanning tree)

→ For every pair of terminals (t_i, t_j) compute the shortest path between t_i & t_j and ~~create~~ add a fake edge of that length l_{ij}.

Do the same for (t_i, r) $\forall i$ & add these edges also

→ on a new graph on only $(k+1)$ vertices $r, t_1, t_2, \ldots, t_k$.

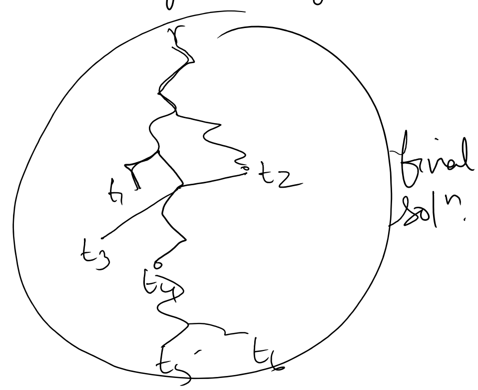
↓

Compute MST on new graph.

$\downarrow$ Compute MST on new graph.

$\downarrow$ Map it back to original edges

Suppose MST =



Lemma ① In fake graph G' suppose MST has cost C' then we can find a sol^n of cost $\le C'$ in the real graph.

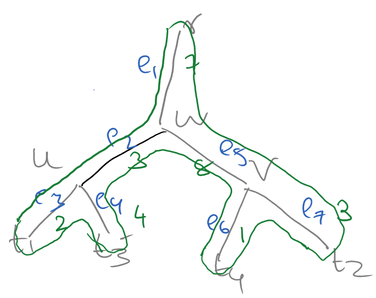
Need to show

$\exists$ Spanning Tree in G' of cost $\le 2 \cdot OPT$

Proof: Consider the Optimal Steiner Tree in G has cost C^*

Suppose OPT =

So $C^* = 28$



Goal: Show some spanning tree in fake graph of cost ≤ 56

Proof

Consider spanning path

$$\underbrace{r \underset{\le 12}{—} t_1 \underset{\le 6}{—} t_3 \underset{\le 16}{—} t_4 \underset{\le 4}{—} t_2 \underset{\le 18}{—} r}_{\le 56}$$

$= 56$

2-approximation for Steiner Forest

Primal Dual approach.

Special Case

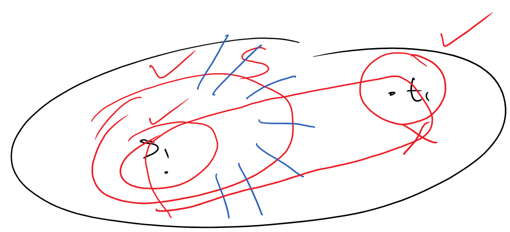
$G = (V, E)$ $c_e \geq 0$ on edges & (s_1, t_1)

What's a nice LP for this problem? x_e be a variable for whether 'e' is included or not.

$$\text{Min } \sum c_e x_e$$

$$(y_S) \quad \sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall S \subseteq V : |S \cap \{s_1, t_1\}| = 1$$

$$x_e \geq 0$$



Lemma

For any integral $\{x_e\}$ feasible for the above LP, s_1 & t_1 are connected.

$$\text{Min } \sum c_e x_e$$

$$y_S \quad \sum_{e \in \delta(S)} x_e \geq 1 \qquad \forall S \subseteq V : |S \cap \{s_1, t_1\}| = 1$$

$$\mathcal{S} = S : |S \cap \{s_1, t_1\}| = 1$$

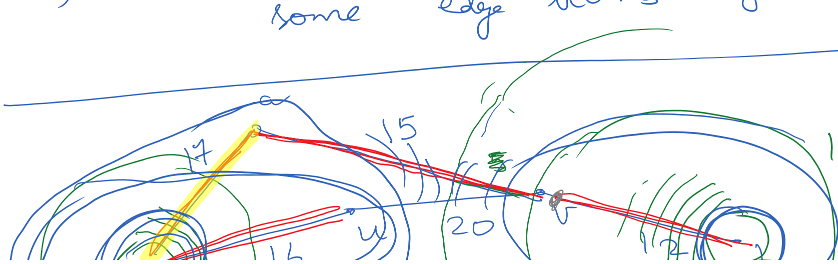
$$\text{Max } \sum_S y_S$$

$$\sum_{S : e \in \delta(S)} y_S \leq c_e$$

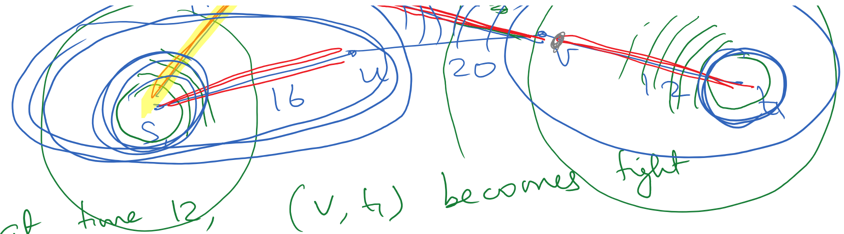
$$y_S \geq 0$$

Key Ideas

1) Start with $y_S = 0 \ \forall S$, $x_e = 0 \ \forall e$
2) Increase y_S for all minimally violated sets until some edge becomes tight,



Increase $y_{\{s_1\}}$ & $y_{\{t_1\}}$ at same rate.



$y_{\{s_1\}}$ & $y_{\{t_1\}}$ at same rate.

$\boxed{12+16+1+5+10 = 44}$

at time 12, (v, t_1) becomes tight

$\therefore \; y_{\{t_1\}} = 12$

$\Rightarrow$ set $x_e = 1$ for $e = (v, t_1)$

at time $12+\varepsilon$,

Increase $y_{\{s_1\}}$ & $y_{\{v, t_1\}}$ at same rate.

at time 16, (s_1, u) edge becomes tight.

$\Rightarrow$ set $x_e = 1$ for $e = (s_1, u)$

after that increase

$y_{\{s_1, u\}}$ & $y_{\{v, t_1\}}$ at same rate.

at time 17, (s_1, a) becomes tight

$\Rightarrow$ set $x_e = 1$ for $e = (s_1, a)$

after that, increase

$y_{\{s_1, u, a\}}$ & $y_{\{v, t_1\}}$ at same rate.

at time 22, edge (a, v) becomes tight

$\Rightarrow$ set $x_e = 1$ for $e = (a, v)$.

s_1 & t_1 are connected now! stop the dual growing process

& delete redundant edges.

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① ~~#~~ Algo will find a feasible sol^n. $\leftarrow$

② Our dual values are feasible for dual problem

$\Rightarrow$ OPT (Problem) $\geq \sum \tilde{y}_S$ (dual values we have maintained at the end)

$y_{\{t_1\}} = 12$
$y_{\{s_1\}} = 16$

Any feasible primal ≥ 44

$\Rightarrow$ optimal primal ≥ 44

$y_{\{t_1\}} = 12$
$y_{\{s_1\}} = 16$
$y_{\{s_1, u\}} = 1$
$y_{\{v, t_1\}} = 10$
$y_{\{s_1, u, a\}} = 5$

Any feasible ...

$\Rightarrow$ Optimal primal ≥ 44

$\boxed{\Rightarrow \text{ Optimal Integer } \cancel{\text{sol}^n \geq 44}}$

Cost of Alg :-

$\forall$ edge $e \in$ final sol^n, $c_e = \sum_{S: e \in \delta(S)} y_S$

let F^* be final sol^n after deletion

Final cost of algorithm

$$= \sum_{e \in F^*} c_e = \sum_{e \in F^*} \sum_{S: e \in \delta(S)} y_S = \sum_S \sum_{\substack{e \in F^*: \\ e \in \delta(S)}} y_S$$

$$= \sum_S y_S |F^* \cap \delta(S)|$$

$$= \sum_S y_S$$

Overall algorithm

Ⓐ Initialize $\hat{x}_e = 0 \ \forall e$, $\hat{y}_S = 0 \ \forall S$, $F^* = \phi$

Ⓐ Consider S_1 = connected component in F^* containing s_1

T_1 = connected component in F^* containing t_1.

Until some edge becomes tight (i.e. $c_e = \sum_{S: e \in \delta(S)} \hat{y}_S$)

increase $\hat{y}_{S_1}$ & $\hat{y}_{T_1}$ uniformly

If 'e' becomes tight, then include e in F^*

Until F^* has a path from $s_1 \rightsquigarrow t_1$

GROWTH

Consider edges added to F^* in reverse order

If deleting edge 'e' does not disconnect $s_1 \rightsquigarrow t_1$, delete 'e'.

Reverse Delete

Lemma

after the reverse delete process, if $y_S > 0$ for any set $S \subseteq V$, then

$$|F^* \cap \delta(S)| = 1$$

How to generalize to Steiner Forest?

$$\text{Min} \sum c_e x_e$$

$$(y_S) \quad \sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S : \exists i : |S \cap \{s_i, t_i\}| = 1$$

$$x_e \geq 0$$

$$\text{Max} \sum y_S$$

$$\sum_{S : e \in \delta(S)} y_S \leq c_e$$

$$y_S \geq 0$$

Initialize $\hat{x}_e = 0 \quad \forall e$

$\hat{y}_S = 0 \quad \forall S$

$F^* = \phi$

G R O W T H

Until F^* is feasible for connecting all $s_i \sim t_i$

let $A_1, A_2, \ldots, A_\ell$ denote the connected components in F^*

Call A_j to violated if $\exists i$ st $|A_j \cap \{s_i, t_i\}| = 1$

Increase all violated connected components at same rate until some edge becomes tight

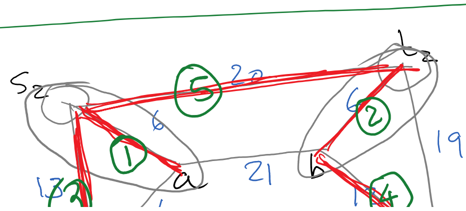
In that case, include this edge to F^*.

Reverse Delete

Consider edges added to F^* in reverse order

if any edge e can be deleted w/o disconnecting some $s_i - t_i$ pair, delete it.

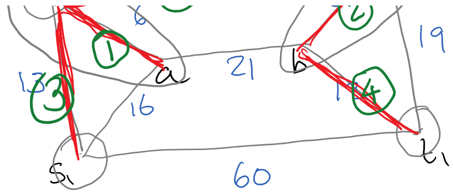
Output remaining F^*.



At time 6,

$y_{\{s_2\}} = 6 = c_{(s_2, a)}$

Include (s_2, a) in F^*



$\cup\{s_2\}$

Include (s_2, a) in F^*

& $y_{\{t_2\}} = 6 = c_{(b, t_2)}$

Include (b, t_2) in F^*

Violated Components

$\{s_1\}$, $\{t_1\}$, $\{s_2, a\}$, $\{t_2, b\}$

At time 6.5,

$$y_{\{s_2\}} + y_{\{s_2, a\}} + y_{\{s_1\}} = 6 + 0.5 + 6.5 = 13 = c_{(s_2, s_1)}$$

Include $(s_1, s_2) \in F^*$

~~So~~ Violated components:

$\{s_1, s_2, a\}$, $\{t_2, b\}$, $\{t_1\}$

At time 9

(b, t_1) becomes tight

Violated Components

$\{s_1, s_2, a\}$ & $\{b, t_1, t_2\}$

At time 10

(s_2, t_2) becomes tight

$\Rightarrow$ include in F^*

Reverse Delete only deletes (s_2, a).
All other edges remain

② $OPT(\text{Steiner Forest}) \geq \sum_S \hat{y}_S$ (total dual we've raised)

① $\{\hat{y}_S\}$ is dual feasible

③ If I show $\boxed{\sum_{e \in F^*} c_e \leq 2 \sum_S \hat{y}_S}$ we're done.

④ $\forall e \in F^*$, $c_e = \sum_{S: e \in \delta(S)} \hat{y}_S$

$\Rightarrow$ Remains to show

$$\sum_{e \in F^*} \sum_{S : e \in \delta(S)} \hat{y}_S \leq 2 \sum_S \hat{y}_S$$

Equivalently W.T.S

$$\boxed{\sum_S \hat{y}_S \, |F^* \cap \delta(S)| \leq 2 \sum_S \hat{y}_S}$$

What we'll show is that while all $|F^* \cap \delta(S)|$ may not be bounded by 2, it will be true on average.

lets do this over time (from $t=0$)