Non-Cooperative Bundling Games

[Extended Abstract]

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ABSTRACT

Bundling is a basic problem in commerce. We define and study the *Bundling Game*: each of n players with a unique product to sell can choose to sell their products in bundles with other players' products so as to maximize their payoff; the players all have knowledge of the valuations of m (potential) buyers for each of the products. We define two natural classes of payoff functions - Fair Profit Sharing Functions (*FPSFs*) and Incentive Functions (*IFs*). For the former we prove the surprising impossibility result that there exists no FPSF that can always guarantee stability (a pure Nash equilibrium). We counterbalance this result by showing that pure Nash equilibria always exist for *IFs* and obtain tight bounds on the Price of Anarchy. Motivated by the bundling game, we present hardness results and approximation algorithms for the optimization problems of finding revenue maximizing bundlings and bundles, and associated variants.

Keywords

Bundling, Game Theory, Market Economics, Profit Sharing

1. INTRODUCTION

Go to Amazon.com and look for any book. On the webpage for the book of interest you will also find an offer to sell another book along with the first as a bundle. Bundling is a well known strategy for increasing revenue [14]. A simple example will serve to illustrate how bundles can extract more revenue than offering goods individually: suppose there are two people, Alice and Bob each wanting an apple and an orange; Alice values an apple at \$1 and an orange at \$3 while Bob values an apple at \$3 and and orange at \$1. If apples and oranges are sold individually then the maximum revenue that can be extracted from Alice and Bob is \$6 (price apples and oranges each at \$3), however if they are sold in bundles consisting of an apple and an orange, the seller can make \$8 (price the bundle at \$4; both Alice and Bob will buy the bundle). Bundling is applicable not just to commerce, in fact, it is pervasive throughout life.

Consider the aggregation of individuals into firms. Individuals produce a unique good - their labor - that they choose to bundle with others, forming firms, in order to maximize their payoff. Motivated by such a scenario we define the Bundling Game: there are n players each selling one good, and m buyers who give their valuation for each of the products. Players are strategic agents looking to maximize their individual payoffs by forming bundles with others. Since the interests of the different players are not aligned, the bundling game is characterized by non-cooperative interactions.

1.1 Results

A natural question that arises is what are the Nash equilibria [10] or situations of stability in such games? Before we can characterize Nash equilibria we first need to decide what the payoff to an individual player is, when he has chosen to participate in a particular bundle.

– We define *Fair Profit Sharing Functions* (*FPSFs*) which are characterized by the natural principle of fairness, i.e. if, standing alone, player A makes as much revenue as B, then A makes as much as B when both A and B are in a bundle (that may contain other players' products) too. We obtain the surprising impossibility result that for all $n > 3$ there exists no such payoff function which always guarantees a pure Nash equilibrium.

Theorem: *For any $n > 3, m > 1$, there exist game instances with n players each selling a unique product, and m buyers, for which no pure Nash equilibrium exists under any* FPSF *payoff. The numbers* 3 *and* 1 *are tight, as in these cases, there exist **FPSFs** which always guarentee pure Nash equilibria.*

One implication of our result is that unfairness may be inherent in profit sharing schemes at firms; it is interesting to speculate whether the high pay of CEOs is an unavoidable artifact of bundling. Given that Fair Profit Sharing Functions do not guarantee stability it is natural to ask whether there are fair functions (for some suitable notion of fair) which guarantee stability even though they may not be necessarily profit sharing.

– We define *Incentive Functions* (*IFs*) where each player gets an incentive proportional to the value she adds to the bundle. Incentive Functions model the setting of a corporation where employees can strategically choose to join different departments (bundles) and their compensation is tied to their value addition to that department. We show that *pure Nash equilibria always exist for Incentive Functions.* In fact, any revenue maximizing bundling can be shown to be a pure Nash equilibrium. The significance of this result in the corporate setting is that, since the revenue maximizing bundling is also a pure Nash equilibrium, it is always possible for the firm to assign persons to departments such that the company maximizes its total revenue while simultaneously ensuring that all its employees are satisfied.[1] Since bundling games with Incentive Function payoffs can possess multiple pure Nash equilibria one would like to understand how the quality of these different stable operating points can vary in terms of total revenue.

– We obtain a tight bound on the *Price of Anarchy* [11] in bundling games with *IF* payoffs.

Theorem: *The Price of Anarchy of bundling games with payoffs as* IFs *is* $\Theta(\frac{1}{\log m})$*, when all valuations are non-negative.*

This implies that the revenue of a pure Nash equilibrium can be as bad as any arbitrary bundling which is also shown to be $\Omega(\frac{1}{\log m})$ of the revenue maximizing bundling. One can wonder whether this result goes some way towards explaining the phenomenon where all the employees of a firm seem individually satisfied even though the firm as a whole is rapidly going south. In other words local stability may be indistinguishable from complete anarchy in terms of collective efficiency.

– Motivated by the bundling game we study the computational complexity of the non game-theoretic optimization problems of finding revenue maximizing bundlings and bundles and present tight upper and lower bounds.

Theorem: *It is strongly NP-Hard to decide whether there is a bundling of n goods (a partition of the goods), that fetches a revenue of at least R.*

We also show that finding a revenue-maximizing bundling, given a set of acceptable bundles, is hard to approximate to within $O(n^{\frac{1}{2}-\epsilon})$. These results differ from existing work on coalitional games and single-minded bidder auctions as is explained in the following section.

1.2 Related Work

The idea of bundling goods in order to obtain greater revenue was introduced long ago. In [14], Stigler performed an early study on how bundling goods can benefit the seller when the buyer valuations are negatively correlated (like the apples and oranges example given in the introduction). In [1], stochastic variants of the problem were considered. The authors observed that when the buyer valuations for products are independently and randomly chosen from distributions, the law of large numbers ensures that the bundle comprising of all goods is the revenue maximizing one. In [5], Guruswami et al. consider bundling in the context of envy-free pricing - consumers have different valuations for the bundles they are interested in, and the seller is to price each product so as to maximize his total revenue.

The notion of bundling games, though related to, differs from the concept of combinatorial auctions. The key difference lies in the fact that *CA*s generally assume specific valuations (not implicitly computable in any fashion) for different bundles. We assume that a buyer's valuation of any bundle is just the sum of his valuations of individual products in that bundle. Also, in the general framework, the notion of single-mindedness does not arise in our setting as a buyer would buy any bundle provided the price the seller sets for the bundle is at most his valuation of the bundle. Therefore, our results on bundling hardness, differ in spirit, to the ones on hardness of coalition proved in [12] and [8]. Further, we are interested in the sellers welfare (they are the players) where as in common auctions literature, the buyers' welfare is the objective.

Another comparison that needs to be made is with the concept of Shapley value [13]. Here too, there are differences. It is not hard to see that the bundle revenue function (which we explain in the following section) is not super-additive in general, where as Shapley value assumes that the coalition values (analogous to bundle revenues) are super-additive. While FPSFs (for which we obtain the impossibility result) maintain budget-balancing and a notion of fairness of revenue distribution, IFs (which seem to resemble the key idea in the Shapley value and VCG pricing) fail to be budget-balanced. However, they are guaranteed to have pure Nash equilibria. Also, the fairness we consider, is not addressed by the Shapley values for a coalition.

2. PRELIMINARIES

We first describe our model.

2.1 Model

There are m buyers, and n players $A = \{a_1, a_2, ..., a_n\}$; each player or producer a_j has an unlimited supply of good g_j; G is the set of all goods g_j, $1 \leq j \leq n$. entry c_{ij} in the valuation matrix C specifies the valuation or reservation price of the i^{th} buyer for g_j (the maximum price he is willing to pay for g_j). Note that c_{ij} can take negative values.[2] Given a bundle $B = \{g_{j_1}, g_{j_2}, ..., g_{j_b}\} \subseteq G$ priced at $p(B)$, the i^{th} buyer buys B, iff $\sum_{g_j \in B} c_{ij} \geq p(B)$. The revenue $R(B)$ generated by a bundle B is the product of its price $p(B)$ and the number of people who buy it.

Bundle Pricing: Given a bundle B, finding the price which maximizes the revenue generated can be done in time polynomial in $(m + n)$ as follows:

1. Compute $c_i(B) = (\sum_{g_j \in B} c_{ij})\ \forall i \in \{1, 2, ..., m\}$.
2. Sort $c_i(B)$ in descending order and let the sorted order be $p_1, p_2, ..., p_m$.
3. If $p_1 < 0$, set $p(B) = 0$, else set $p(B) = p_i$ where $ip_i = max(p_1, 2p_2, 3p_3, ..., mp_m) = R(B)$.

Intuitively, we add up the valuations for each buyer of all products in that bundle (that is precisely his valuation of

[1] In other words, the firm maximizes collective efficiency without sacrificing individual stability.

[2] Consumers have the freedom to devalue goods they would never want to buy.

the bundle in our model), and sort them. It is easy to see that pricing the bundle at a value not one of these would not benifit. Therefore, we compare the revenue by pricing it at each of the buyers valuations to determine the optimal price. Also, note that if all c_{ij}'s are non-negative, $R(B_1) >= R(B_2)$ when $B_2 \subseteq B_1$.

Revenue of a Bundling: A **Bundling** $\beta = \{B_1, B_2, ..., B_k\}$ where $B_1, B_2, ..., B_k$ are bundles is *feasible* iff $\forall i, j$ s.t $i \neq j$, $B_i \cap B_j = \phi$. We denote $R(\beta) = \sum_{j=1}^{k} R(B_j)$ to be revenue of the bundling. Essentially, the seller partitions the set of products into bundles, and collects revenue from each bundle. In our model, a product can not be sold in two different bundles.

2.2 Payoff Functions

We now introduce a class of functions, which shall be used as payoffs in the bundling games.

Fair Profit Sharing Functions: Let $B = \{g_{i_1}, g_{i_2}, ..., g_{i_l}\}$ be a bundle of products belonging to players $a_{i_1}, a_{i_2}, ..., a_{i_l}$ respectively. We define the profit of the bundle as $P(B) = R(B) - (\sum_{j=1}^{l} R(\{g_{i_j}\}))$. Let $f : A \times 2^G \rightarrow \mathbb{R}$ satisfy:
for every player a_i s.t $g_i \in B$, $f(a_i, B)$ represents the profit to a_i by forming the bundle B, and for $g_i \notin B$, $f(a_i, B) = 0$. Thus, the total revenue to a_i by being in B is $f(a_i, B) + R(\{g_i\})$. We say that f is a Fair Profit Sharing Function ($FPSF$), iff it satisfies the following properties:

Profit Conservation : $\forall B \in 2^G, \sum_{g_i \in B} f(a_i, B) = P(B)$
Non-Contribution : $\forall B$ and $\forall g_i \in B$ s.t $R(\{g_i\}) = 0$, $f(a_i, B) = 0$
One-For-All : $\forall B$, if $g_i \in B$ s.t $R(\{g_i\}) > 0, f(a_i, B) \geq 0 \Leftrightarrow P(B) \geq 0$ and $f(a_i, B) = 0 \Leftrightarrow P(B) = 0$
Fairness : $\forall g_{i_1}, g_{i_2} \in B$, if $R(\{g_{i_1}\}) \geq R(\{g_{i_2}\}), f(a_{i_1}, B) \geq f(a_{i_2}, B)$

The first requirement just states *revenue conservation* as one condition that an FPSF must satisfy. The second one says that players whose products dont get any revenue when sold alone, get no share of any bundle profit. Such players are said to "not contribute". The third one says that all contributing players enjoy a profit, if the bundle is profitable, and share the loss, otherwise. This means that if a bundle is profitable (fetches more revenue than sum of individual revenues), all contributing players must benifit by being part of it. The fourth one introduces a notion of "fairness" by stating that a player a_i whose product is more valuable that another players a_j (fetches more revenue when sold alone, i.e $R(\{g_i\}) >= R(\{g_j\})$) must not get a lower share of the bundle profit than a_j, when both the players are in some bundle. Note that there are *no* assumptions on the continuity and identity independence of the function f. Hence, the definition of an $FPSF$ is very general under realistic assumptions about profit distribution in day to day life. Intuitively, having any fewer restrictions would decrease the practical nature of the functions and having more constraints would make it less general.

Incentive Functions: We introduce another class of functions f, called Incentive Functions(IFs).
Let $B = \{g_{i_1}, g_{i_2}, ..., g_{i_l}\}$ be a bundle of products belonging to players $a_{i_1}, a_{i_2}, ..., a_{i_l}$ respectively. $f : A \times 2^G \rightarrow \mathbb{R}$ is an IF iff

$$\forall a_i, f(a_i, B) = k_1(R(B) - R(B \setminus \{g_i\})) + k_2 \qquad (1)$$

where constants $k_1 \in \mathbb{R}^+, k_2 \in \mathbb{R}$, are scaling and additive constants. This can be thought of as the value a player adds to a bundle, which is proportional to the difference in the revenue of the bundle, when he/she is present and when he/she is absent.

3. BUNDLING IN A NON-COOPERATIVE ENVIRONMENT

We now introduce the concept of non-cooperative bundling games. The model is as defined above. Each player a_i has a strategy set $S_i = \{1, 2, ..., n\}$. Once player a_i chooses a strategy[3] s_i, we obtain a strategy profile $\sigma = (s_1, s_2, ..., s_n)$. σ leads to a grouping of products defined by $B_i = \{g_j | s_j = i\}, 1 \leq i \leq n$. Also, let $\beta(\sigma) = \{B_i | 1 \leq i \leq n\ ,\ B_i \neq \phi\}$.[4] Given $\sigma = (s_1, s_2, ..., s_n)$, the payoff to a_i is $u_i(\sigma) = f(a_i, B)$ where $B = \{g_j | s_j = s_i\}$[5] and $f : A \times 2^G \rightarrow \mathbb{R}$ may be a profit sharing scheme, a revenue sharing scheme or any other payoff function. Once the players form the bundles, the configuration is a pure Nash equilibrium iff no player a_i has an incentive to leave its bundle and enter some other bundle. Formally, a strategy profile $\sigma = (s_1, s_2, ..., s_n)$ is a *pure Nash equilibrium* iff there exists no player a_i and strategy $s'_i \neq s_i$ such that $u_i((s_1, s_2, ..., s_i, ..., s_n)) < u_i((s_1, s_2, ..., s'_i, ..., s_n))$. We next investigate the equilibria of such games, when the payoffs are $FPSFs$.

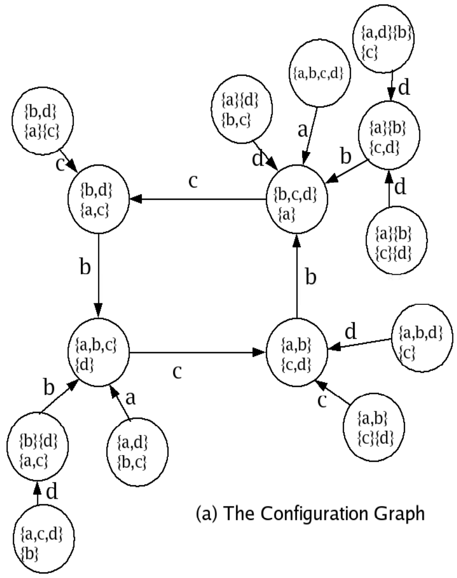
3.1 Bundling Games with FPSFs as Payoffs

For such games, mixed Nash equilibria are known to exist [10]. Though one would expect some $FPSF$ that always guarantees pure Nash equilibria to exist, we show negative results. The next theorem proves that *no Fair Profit Sharing scheme can always guarantee a stable configuration where all players are happy.*

THEOREM 1. *For any $n > 3, m > 1$, there exist game instances with n players, each selling a unique product and m buyers, for which no pure Nash equilibria exist under any* FPSF *payoff. The numbers* 3 *and* 1 *are tight, as in these cases, there exist* FPSFs *which always guarentee pure Nash equilibria.*

PROOF. We outline a sketch of the proof here. Given $n > 3, m > 1$, we consider the $m \times n$ valuation matrix given in Fig 1(b). Let a, b, c, d represent the products g_1, g_2, g_3, g_4, and A, B, C, D, the players a_1, a_2, a_3, a_4 respectively. We construct the Configuration Graph given in Fig 1(a), where each node (v) corresponds to a set of valid bundlings of the products. This graph partitions all the possible bundlings, into sets depending on the bundles into which a, b, c, d are distributed relative to each other. For instance, the node $v_{\{a,b\},\{c\},\{d\}}$ corresponds to all those bundlings where a and b fall into one bundle, c into another one and d into yet another. The other products may be distributed in any fashion. We will set b_1 and b_2 such that, in any bundling, the player whose product is marked on the outgoing edge from

[3]Note that all players have the same strategy set.
[4]Note that several strategy profiles may lead to the same bundling.
[5]Note that B $\in \beta(\sigma)$



(a) The Configuration Graph

$$\mathsf{C}(b_1, b_2) = \begin{pmatrix} 1 & b_1 & 12 & 2 & 0 \dots 0 \\ 11 & b_2 & 2 & 12 & 0 \dots 0 \\ 0 & 0 & 0 & 0 & 0 \dots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \ddots \\ 0 & 0 & 0 & 0 & 0 \dots 0 \end{pmatrix}$$

(b) Valuation matrix for FSPFs

$$\mathsf{C} = \begin{pmatrix} 2.5 & 2.5 & 2.5 & 2.5 & 2.5 & \dots & 2.5 \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{m-1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots & 1 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \cdots & \frac{1}{2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \\ \frac{1}{m-1} & 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{m-2} \end{pmatrix}$$

(c) Valuation matrix for tightness of Price of Anarchy

Figure 1:

the node this bundling corresponds to, has an incentive to deviate to some bundling corresponding to the neighbouring node. For this, we ensure that the following conditions are satisfied:

$P(\{a,b,c,d\}), P(\{b,d\}), P(\{b,c\}), P(\{a,d\}), P(\{a,b\}) \leq 0$
$P(\{a,b,d\}), P(\{a,c,d\}) < 0$
$P(\{a,c\}), P(\{b,c,d\}), P(\{c,d\}), P(\{a,b,c\}) > 0$
If $B_1 \cap \{a,b,c,d\} = \{c,d\}$ & $B_2 \cap \{a,b,c,d\} = \{a,b,c\}$, $f(C,B_1) > f(C,B_2)$
If $B_3 \cap \{a,b,c,d\} = \{a,c\}$ & $B_4 \cap \{a,b,c,d\} = \{b,c,d\}$, $f(C,B_3) > f(C,B_4)$

For all values of b_1, b_2 satisfying the following conditions (domain marked by the criss-crossed shaded region in the appendix A Fig.2.)

$$\begin{aligned} b_1 &> b_2 && (2)\\ b_2 &> 2b_1 - 8 && (3)\\ 2b_2 &> b_1 - 3 && (4)\\ \text{If } b_1 > 2b_2, b_1 &< 5 && \\ \text{else } b_1 - b_2 &< 2.5 && (5)\\ b_1 &> 2b_2 + 2.5\ . && (6) \end{aligned}$$

(say $b_1 = 2.7$ and $b_2 = 0.05$) we show in appendix A, that there can be no fair profit sharing payoff scheme such that a pure Nash equilibrium exists. Hence, in such games, there always exists a player with an incentive to deviate, for any bundling. Therefore, for such instances, no pure Nash equilibria can exist under any *FPSF*. The complete proof is in appendix A.

Tightness for $m > 1$ is simple, as when $m = 1$, any bundling is a pure Nash equilibria and all players always get no profit. To obtain tightness of $n > 4$, consider the *FPSF* $f(a_i, B) = \frac{P(B)}{z}$, if $g_i \in B$ and a_i is not a non-contributor, and 0 otherwise (z is the number of contributors in B). This models an equal sharing scheme. We show that for this function, any game instance with 3 contributing products always has a pure Nash equilibrium, by assuming the contrary. Let A, B, C be the 3 players, a, b, c their products. If $P(\{a,b,c\}) \geq 0$ then $\{\{a,b,c\}\}$ is a pure Nash equilibrium, as any unilateral deviation will not result in an increase of profit for any player. Therefore, $P(\{a,b,c\}) < 0$, meaning that, the profits for all 3 players is negative. Now, for $\{\{a\},\{b\},\{c\}\}$ not to be a pure Nash equilibrium, one of the profits $P(\{a,b\}), P(\{b,c\}), P(\{c,a\})$ must be > 0. Without loss of generality, let $P(\{a,b\}) > 0$. For this state $\{\{a,b\},\{c\}\}$ not to be a pure Nash equilibrium, one of A or B has to leave this bundle and join C(remember that C would do worse by joining {a,b}). Let us assume A gets an incentive to do so. This means that $P(\{a,c\})/2 > P(\{a,b\})/2$. For $\{\{a,c\},\{b\}\}$ not to be a pure Nash equilibrium, C must get an incentive by bundling with B. This means $P(\{c,b\})/2 > P(\{c,a\})/2$. Finally, for $\{\{c,b\},\{a\}\}$ not to be a pure Nash equilibrium, B must get an incentive by bundling with A. This means $P(\{b,a\})/2 > P(\{b,c\})/2$. This immediately leads to a contradiction. One of the states was a pure Nash equilibrium. Hence, when $n = 3$, there do exist *FPSFs* that actually converge to a stable state where all players are individually satisfied. Other cases, where 1,2, or 3 non-contributors exist are proved similarly. □

3.2 Bundling Games with IFs as Payoffs

We now focus on games where the payoffs are *IFs*. More formally, given $\sigma = (s_1, s_2, ..., s_n)$, the payoff to a_i is $u_i(\sigma) = f(a_i, B)$, where $B = \{g_j | s_j = s_i\}$, and f is any *IF*. This definition of *IF* is similar to the payoff in the *VCG* auction mechanisms. Incentive Functions closely model the setting of a corporation where employees can strategically choose to join different departments (bundles) and their compensation is tied to their value addition to that department.[6] Contrary to our result on the existence of pure Nash equilibria in

[6]Even though IFs are not profit conserving they are entirely natural in the corporate context where employees in a firm only get to share a *portion* of the profit set aside for them by the partners/owners.

bundling games with *FPSF* payoffs, we show that any *IF* always guarantees the existence of pure Nash equilibria, for any game instance.

THEOREM 2. *For bundling games with* IF *payoff, any game instance always has a pure Nash equilibrium. In fact, the revenue maximizing bundling is also a pure Nash equilibrium.*

PROOF. For the proof, we define a potential function [9] $\Phi(\sigma) = k_1 R(\beta(\sigma))$, with $\sigma, \beta(\sigma)$, and $R(\beta)$ being defined earlier.
Let $\sigma = (s_1, s_2, ..., s_i, ..., s_n)$ and $\sigma' = (s_1, s_2, ..., s_i', ..., s_n)$, $s_i' \neq s_i$, and let $\beta(\sigma) = \{B_1, B_2, ..., B_{s_i}, ..., B_{s_i'}, ..., B_l\}$.
Then,
$\beta(\sigma') = \{B_1, B_2, ..., B_{s_i} \setminus \{g_i\}, ..., B_{s_i'} \cup \{g_i\}, ..., B_l\}$.
Therefore,
$\Phi(\sigma) - \Phi(\sigma') = k_1(R(B_{s_i}) + R(B_{s_i'}) - (R(B_{s_i} \setminus \{g_i\}) + R(B_{s_i'} \cup \{g_i\}))) = u_i(\sigma) - u_i(\sigma')$.
Therefore, by [9], as Φ is an exact potential function, existence of a pure Nash equilibrium is guaranteed. Also by [9], the revenue maximizing bundling, which is the maximum possible value of Φ, is also a pure Nash equilibrium.[7] □

However, we can construct cases where other non-maximal bundlings too are pure Nash equilibria. This leads us to the problem of finding the *Price of Anarchy* [11], the worst case ratio between any pure Nash equilibria and the revenue maximizing bundling.

THEOREM 3. *The Price of Anarchy of bundling games with payoffs as* IFs *is* $\Theta(\frac{1}{\log m})$*, when all valuations are non-negative.*

PROOF. We prove in Lemma 1 that, when all valuations are non-negative, any bundling is $\Omega(\frac{1}{\log m})$ of the revenue maximizing bundling, hence establishing $\frac{1}{\log m}$ as a lower bound on the price of anarchy. We now show that this is tight by constructing a game instance and a pure Nash equilibrium that is $O(\frac{1}{\log m})$ off the revenue maximizing bundling. Consider the valuation matrix given in Fig 1(c). In this game with $n = m - 1$ players and m buyers, each product individually fetches 2.5. We then show that the strategies that result in the bundling $\beta = \{\{1\}, \{2\}, \ldots, \{m-1\}\}$ (i.e each product sold individually) constitute a *pure Nash equilibrium* by showing in Lemma 2 that when any two products combine to form a bundle, it gives a payoff which is not more than the payoff obtained by selling individually ($= 2.5$). Hence the revenue of this pure Nash equilibrium is $2.5(m-1)$. Moreover, the max revenue that can be obtained $\geq m(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m-1})$ which is the revenue of the bundle($\{1, 2, 3, \ldots, m-1\}$) consisting of all products. Thus, the price of anarchy $\leq \frac{2.5(m-1)}{m(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m-1})}$ which is asymptotically $O(\frac{1}{\log m})$. □

LEMMA 1. *When all valuation entries are non-negative, any feasible bundling β fetches a revenue that is at most $\log m$ off the revenue of the revenue maximizing bundling, in the asymptotic limit.*

[7] The significance in the corporate setting that was introduced in Sect. 1 is that, since the revenue maximizing bundling is also a pure Nash equilibrium, it is always possible for the company to assign persons to departments such that the company maximizes its total revenue, also ensuring that all its employees are satisfied.

PROOF. Consider any feasible bundling of goods $\beta = \{B_1, B_2, \ldots, B_k\}$. Let $c_i(B) = \sum_{g_j \in B} c_{ij}$ be the valuation of bundle B by the i^{th} buyer. Let the revenue generated by β and the revenue maximizing bundling value be $R(\beta)$ and R_{max} respectively. Now, the sum of all entries of the reservation matrix, is clearly an upper bound on the maximum revenue that can be obtained. Also, since, the selling price $c_{l_j}(B_j)$ is such that $l_j c_{l_j}(B_j) \geq i c_i(B_j) \forall i$, we get

$$\sum_{g_t \in B_j} c_{it} = c_i(B_j) \leq \frac{1}{i} l_j c_{l_j}(B_j) \tag{7}$$

Therefore,

$$\sum_{t=1}^{n} \sum_{i=1}^{m} c_{ij} \leq \sum_{j=1}^{k} l_j c_{l_j}(B_j) \sum_{i=1}^{m} \frac{1}{i} \tag{8}$$

Further, $\sum_{i=1}^{m} \frac{1}{i}$ asymptotically goes to $\log m$ and $R = \sum_{j=1}^{n} l_j c_{l_j}(B_j)$.
Hence,

$$R_{max} \leq R(\beta) \sum_{i=1}^{m} \frac{1}{i} \tag{9}$$

Therefore, $R_{max} \leq (\log m) R(\beta)$, asymptotically for any β. □

LEMMA 2. *For the game defined in* Fig.1c*, with m buyers and $n = m - 1$ players, the strategies that result in the bundling $\beta = \{\{1\}, \{2\}, \ldots, \{m-1\}\}$ (i.e each product sold individually) constitute a* pure Nash equilibrium.

PROOF. Consider the valuation matrix C given in *Fig 1c*. Now, consider a strategy profile leading to the bundling $\beta = \{\{g_1\}, \{g_2\}, \ldots, \{g_{m-1}\}\}$. We now show that if any player unilaterally changes his strategy, his payoff does not increase (potential decreases, since it is a potential game). Let $n = m - 1$. By symmetry considerations, we just need to show that

$$R(\{g_1, g_r\}) \leq R(\{g_1\}) + R(\{g_r\}), 2 \leq r \leq \frac{n}{2}$$

The valuations of the bundle $B = \{g_1, g_r\}$ are $1 + \frac{1}{r}, \frac{1}{2} + \frac{1}{r+1}, \ldots, t_{i_1} = \frac{1}{i_1} + \frac{1}{i_1+r-1}, \ldots, \frac{1}{n-r+1} + \frac{1}{n}, \frac{1}{n-r+2} + 1, \ldots, t'_{i_2} = \frac{1}{n-r+i_2+1} + \frac{1}{i_2}, \ldots, \frac{1}{n} + \frac{1}{r-1}$.
We now prove that the Revenue of the bundle ≤ 5.
Note that $t_i \geq t'_i$ i.e $\frac{1}{i} + \frac{1}{i+r-1} \geq \frac{1}{i} + \frac{1}{i+n-r+1}$. Hence, if the bundle B is priced at $p(B) = \frac{1}{i_1} + \frac{1}{i_1+r-1}$,

$$\begin{aligned} R(B) \leq (2i_1+1)(\frac{1}{i_1} + \frac{1}{i_1+r-1}) &\leq 2 + \frac{1}{i_1} + \frac{2i_1+1}{i_1+r-1} \\ &\leq 5(\text{ since } r \geq 2) \end{aligned}$$

Again, $t'_{i_2} \leq t_{i_1} \Leftrightarrow \frac{1}{i_1} + \frac{1}{i+r-1} \geq t'_{i_2}$. Hence,

$$\begin{aligned} & i_1^2 + i_1(r - 1 - \frac{2}{t'_{i_2}}) - \frac{r-1}{t'_{i_2}} &\leq 0 \\ \Rightarrow \quad & i_1 \leq \frac{-(r-1-2/t'_{i_2}) + \sqrt{(r-1)^2 + \frac{r}{t'^2_{i_2}}}}{2} &\leq \frac{2}{t'_{i_2}} \end{aligned}$$

Hence if the bundle is priced at $t'_{i_2} = \frac{1}{n-r+1+i_2} + \frac{1}{i_2}$, the revenue

$$\begin{aligned} R(B) &\leq (i_2 + 1 + \frac{2}{t'_{i_2}})t'_{i_2} \\ &= 2 + \frac{i_2+1}{i_2} + \frac{i_2+1}{n-r+1+i_2} \\ &\leq 5(\text{ since } n \geq r) \end{aligned}$$

Hence $R(\{g1\}) + R(\{g_2\}) = 2.5 + 2.5 \geq R(\{g_1, g_2\})$. □

4. REVENUE MAXIMIZING BUNDLINGS AND BUNDLES

Motivated by the bundling game, we now focus on the computational aspects of the problems of finding the revenue maximizing bundlings and bundles in this section. We first examine the complexity of finding the bundling which maximizes the total revenue. However, in a more realistic scenario, not all bundles would be acceptable to the buyers. No buyer would like to purchase pizza and shoe polish in the same bundle. We therefore analyze the revenue maximizing bundling problem, when a set of allowable bundles is specified. Later in this section, we also study the complexity of finding single revenue maximizing bundles of a certain size. These are relevant in several practical scenarios, like a firm trying to decide all the features that it must include in a product to maximize consumer satisfaction.

4.1 Revenue Maximizing Bundling

We now look at the complexity of finding the revenue maximizing bundling.

THEOREM 4. *The problem ($MaxRevBund$) of deciding whether there is a bundling of n goods (a partition of the goods), that fetches a revenue of at least R is strongly NP-Hard.*

We first outline a sketch of the proof. It is a reduction from the *general MAXAGREE* problem [2], where it was shown that, in a graph where each edge is labeled '+' or '-', the problem of deciding whether there is a partition of the vertices into clusters such that the sum of the $-$ edges across clusters and $+$ edges inside the clusters (henceforth referred to as *agreement*) is at least K, is strongly NP-Hard.
In the reduction, for every Gr, we first construct another instance $H(V', E')$ of the $MAXAGREE$ problem, where for each vertex in Gr, we add a gadget of $|V|$ extra vertices with a $+$ edge between each extra vertex and its parent (the corresponding vertex in $V(Gr)$). We then construct an instance I of $MaxRevBund$ from H by having a product for every vertex $v \in V'$ and a consumer for every edge $e \in E'$. We define the valuation matrix C by having non-zero valuations only for those vertices (products) that a given edge (consumer) is incident on. For each $+$ edge, the valuations are 1 for both vertices, and for each $-$ edge, the valuations are 2 for one of the vertices and -2 for the other vertex, which that particular edge is incident with. Finally, we show that I has revenue $K + n^2$ iff Gr has an agreement of K. Formally,

PROOF. In [2], the following result is proved.
*Given a graph $Gr = (V, E)$, of n vertices and m edges each labelled $+$ or $-$, the problem of deciding if there is a partition of the vertices into clusters such that the sum of the total number of $-$ edges across clusters and $+$ edges inside the clusters (*agreement*) is at least K, is strongly NP-Hard.*

We now reduce the above problem to the revenue maximizing bundling Problem ($MaxRevBund$). Given a graph $Gr = (V, E)$ with $V = \{v_1, v_2, ..., v_n\}$, we construct a graph $H = (V', E')$ such that, for each vertex v_i, there are n extra child vertices $v_i^1, v_i^2, ..., v_i^n$ and edges $e_i^j = (v_i, v_i^j)$, $1 \leq j \leq n$ with $+$ label. Let $V_i = \{v_i^1, v_i^2, ..., v_i^n\}$ and $E_i = \{e_i^j, 1 \leq j \leq n\}$.

Now, given a partition of the vertices of Gr of agreement at least K, the same partition of V, along with all the extra vertices of H added to the clusters containing their parents, is a partition of the vertices of H of agreement at least $K+n^2$ (all the n^2 extra edges, fall into clusters).
Further, given a partition of the vertices of H of agreement at least $K+n^2$, by removing the extra vertices v_i^j, the agreement can decrease by at most n^2 (if all the extra vertices are in the same clusters containing their parents), thus forming a partition of V of agreement at least K.
Hence, Gr has a partition of agreement at least K iff H has a partition of agreement at least $K + n^2$.

Note 1. We also note that if P_1 is some cluster of Gr, i.e $P_1 \subseteq V$, the total number of $+$ edges coming out of P_1 is $\leq (n - |P_1|)|P_1|$. The same upper bound is true for the corresponding cluster P'_1 (the set of vertices in P_1 and all the extra vertices whose parent vertex is in P_1) of H, as each extra vertex is connected only to its parent.

We create an instance I of the revenue maximizing bundling problem($MaxRevBund$) for $H(V', E')$ as follows. For every vertex $v \in V'$, we have a product(player) and for every edge $e \in E'$ we have a consumer in I. We define the valuation matrix C as follows. Let $c_{e,v}$ refer to the valuation of product e by player v.

$$c_{e,v} = \begin{cases} 1 & \text{if } l(e = (u, v)) = + \text{ or } l(e = (v, u)) = + \\ 2 & \text{if } l(e = (v = v_i, v_j)) = - \\ & \text{ where } v_i, v_j \in V, \text{ and } i < j \\ -2 & \text{ if } l(e = (v = v_i, v_j)) = - \\ & \text{ where } v_i, v_j \in V, \text{ and } i > j \\ 0 & \text{otherwise} \end{cases}$$

We now show that H has a clustering with an *agreement* of at least K iff I has a bundling of revenue at least $2K$.
Now, let $(P_1, P_2, \ldots, P_l)$ be a clustering of H. If $v_i \in P_j$, we can assume the correponding set of extra vertices $V_i \subseteq P_j$. If not, the labels of the corresponding edges $l(e_i^t) = +$ and they do not contribute to *agreement*. Further the extra vertices are of degree 1. Hence the *agreement* only increases by moving v_i^t to P_i. Now, consider the bundling $\beta = \{B_i | v \in B_i \text{ iff } v \in P_i\}$. By pricing B_i at 2 we get $R(\beta) \geq 2 \times$ *agreement*.
Conversely let bundling $\beta' = \{B'_1, B'_2, \ldots, B'_l\}$ have revenue $R(\beta') \geq 2K$. We now construct the bundling $\beta = \{B_i | 1 \leq i \leq l\}$ from β' s.t. $V_j \subseteq B_i$ if $v_j \in B_i$, and $V_j \cap B_i = \phi$ if $v_j \notin B_i$, by including the extra vertices V_i into the bundle containing v_i. By Lemma 4, $R(\beta) \geq R(\beta')$. Now, consider the partitions $P_i = \{v | v \in B_i\}$. By Lemma 3, each of the bundles B_i have been priced at 2. Note that the valuation of a buyer (edge) for a bundle is 2 iff it is a $+$ edge with both vertices in the bundle or it is a $-$ edge with only the smaller index vertex in the bundle.

$R(B_i) = 2$ ($|\{e = (u, v) \mid l(e) = +$ and $v_j, v_k \in B_i\}| +$ $|\{e = (v_j, v_k) | j < k, l(e) = -$ and $v_j \in B_i$ & $v_k \notin B_i\}|)$ Hence, $R(\beta) = \sum_{i=1}^{l} R(B_i) = 2 \times$ agreement. Therefore *agreement* $\geq K$. $\square$

Lemma 3. *If a bundle $B = \{v_{i_1}, v_{i_2}, ..., v_{i_s}\}$ is such that, $v_i \in B \Rightarrow V_i \subseteq B$ and $v_i \notin B \Rightarrow B \cap V_i = \phi$, then pricing B at 2 fetches maximal revenue for B.*

Proof. Such a B is basically a set of vertices of Gr along with all the extra nodes associated with them. Hence, if the number of vertices of Gr in B is x, the nx edges between the vertices of G that are in B and the extra notes associated with them, value B at 2. Further, each $-$ edge (v_i, v_j) such that $v_i \in B$ and $v_j \notin B$ and $i < j$ values the bundle at 2. Let there be y_1 such edges. Other $-$ edges value B at 0. Each $+$ edge (v_i, v_j) such that $v_i \in B$ and $v_j \notin B$ value B at 1. Let there be y_2 such edges. $+$ edges (v_i, v_j) where $v_i \in B$ and $v_j \in B$ value B at 2. Let there be y_3 such edges. All other edges value B at 0. When priced at 1, the revenue is $nx+y_1+y_2+y_3$. When priced at 2, the revenue is $2(nx + y_1 + y_3)$. Further, by 1, $y_2 \leq x(n - x) \leq nx$. Hence, pricing it at 2 fetches at least as much revenue obtained as when priced at 1. $\square$

Lemma 4. *For any bundling $\beta' = \{B'_1, B'_2, \ldots, B'_l\}$, the bundling*

$$beta = \{B_i | 1 \leq i \leq l \text{ and } B_i = (B'_i \cap V) \cup U_i\}$$

where

$$U_i = \bigcup_{v_i \in B'_i \cap V} V_i$$

fetches more revenue. i.e. $R(\beta') \geq R(\beta)$. Note that V is the vertex set of Gr defined earlier (and not of H).

Proof. We first show that by removing all the q_i extra vertices $v_i^j \in V_i$ from bundles not containing v_i, the decrease in revenue is at most q_i. Consider a bundle B'_k. Let $v_i^j \in B'_k$ be an extra vertex of v_i s.t $v_i \notin B'_k$. Let x,y be the number of edges (buyers) who value B'_k at 2 and 1 respectively. If the price $p(B'_k) = 2$, e_i^j does not buy B'_k and hence by removing the degree 1 vertex v_i^j, the revenue $R(B'_k)$ is unaltered. If $p(B'_k) = 1$, $x + y \geq 2x$. After removing v_i^j, since pricing the modified bundle at 1 fetches revenue $x + y - 1$, the bundle revenue $\geq x + y - 1$ and hence the decrease in revenue is at most 1. We call the bundling resulting from the deletions β''. Hence $R(\beta'') \geq R(\beta') - q$ where $q = \sum_{i=1}^{n} q_i$. We can now construct β from β'' by adding the deleted (extra) vertices to the bundles containing their parents.
We now show that $R(\beta) \geq R(\beta'') + q$. Consider a bundle $B''_k \in \beta''$. Let $|B''_k \cap V| = z$. Let x_1 be the number of $-$ edges (v_i, v_j), s.t $i < j$, x_2 be the number of $+$ edges (v_i, v_j) s.t v_i and $v_j \in B''_k$, x_3 be the number of edges (v_i, v_i^j) s.t $v_i \in B''_k$ and $v_i^j \in B''_k$. Let y be the $+$ edges (v_i, v_j) s.t $v_i \in B''_k$ and $v_j \notin B''_k$. If $p(B''_k) = 2$, adding v_i^j to B''_k will increase the revenue by 2. If $p(B''_k) = 1$, after adding all $V_i \setminus B''_k$, $1 \leq i \leq n$, by Lemma 3, $p(B_k) = 2$. Hence, increase in revenue

$$\begin{aligned} R(B_k) - R(B''_k) &= 2x_1 + 2x_2 + 2nz - (x_1 + x_2 + x_3 + y_1) \\ &= x_1 + x_2 + (2nz - x_3) - y_1 \end{aligned}$$

But $nz + x_2 \geq y_1$ because $y_1 \leq (n - z)z$. Hence, $R(B_k) \geq R(B''_k) + (nz - x_3)$. Notice that the number of vertices added to B''_k is $nz - x_3$. Hence, the revenue increases by at least 1 per vertex added. Summing over all bundles, we get $R(\beta) \geq R(\beta'') + q$ $\square$

We now present an $O(\log m)$ approximation algorithm for this problem.

Algorithm 1. *ApproxBundling*

1. Sort each column in the valuation matrix $C = (c_{ij})$ in descending order of the entries. Let the sorted entries in the j^{th} column be $c'_{1j}, c'_{2j}, ..., c'_{mj}$.
2. If $c'_{1j} \geq 0$, form the bundle $B_j = \{p_j\}$ and set $p(B_j) = c'_{l_j j}$ such that $l_j c'_{l_j j} = max(c'_{1j}, 2c'_{2j}, ..., ic'_{ij}, ..., mc'_{mj})$.
3. Else, form the bundle $B_j = \{p_j\}$, and set $p(B_j) = 0$.
4. Return $\beta = \{B_1, B_2, ..., B_n\}$.

The above algorithm gives a $\log m$ approximation algorithm in the asymptotic limit.

Proof. The proof is exactly on the lines of Lemma 1, except that we consider only the non-negative values for each product (a bundle here) and the sum of non-negative valuations as an upper bound on the maximum possible revenue. $\square$

We now focus on the case where only some bundles are acceptable. We define $\mathscr{B} \subseteq 2^G$, to be the set of allowable bundles, i.e any bundle in a feasible bundling must belong to this set. We assume that $|\mathscr{B}|$ is within a polynomial in the number of products and that the values c_{ij} are non-negative, but all results hold even when negative values are allowed.

A **Bundling** $\beta = \{B_1, B_2, ...B_k\}$ is feasible iff $B_i \in \mathscr{B}$ and if $i \neq j$, $B_i \cap B_j = \phi$. We call the problem described above as *AllowBund*.

Theorem 5. *For every $\epsilon > 0$, there is no $O(b^{1-\epsilon})$ approximation algorithm for* AllowBund, *b being the number of allowed bundles, which runs in polynomial time unless $NP \subseteq ZPP$. Under the same assumptions, for every $\epsilon > 0$, there is no $O(n^{1/2-\epsilon})$ approximation algorithm, with n being the number of products, which runs in polynomial time.*

Proof. The proof involves a reduction from the Weighted Independent Set problem (WIS) whose hardness was shown in [6]. In the Weighted Independent Set Problem(WIS) [4], we are given a graph $G(V, E)$ with a weight $w : V \rightarrow \mathbb{R}^+$ associated with each vertex (say w_i associated with vertex v_i) and we are asked to find the set $S \subseteq V$ such that $\forall u, v \in S, (u, v) \notin E$ and $\sum_{v_i \in S} w_i$ is maximum.

Given a graph instance $Gr(V, E)$ of problem WIS with weights w_i for vertex v_i, we construct using *Algorithm 2* an instance I of *AllowBund* with one buyer and $|\mathscr{B}| = |V|$. Each vertex v corresponds to a bundle B in I and each edge of the graph $e \in E(Gr)$ corresponds to a product in I. When $(u, v) \in E$, in I bundles B_u and B_v have one product in common.

We now present the algorithm which performs the reduction in polynomial time in $|E|$. Let $c(g)$ be the valuation by the single player of the product g.

Algorithm 2. *ReduceWIStoBundling*

1. Initially, set working vertex set $V' = V$, $Gr'(V', E') = Gr(V, E)$ and weight function $w' = w$. Let $d'(v)$ denote the degree of v in Gr'. For every vertex $v \in V$, let B_v be the corresponding bundle. Set $B_v = \phi$ initially. Let $\beta = \{B_v | v \in V\}$.

2. If all vertices of $Gr'(V')$ are of weight 0 and degree 0 goto Step 7.

3. Choose a minimum weight vertex $v' \in V'$. Let its weight be $w'(v')$.

4. If $d'(v') \geq 1$, for every edge $e = (u, v') \in E$ where $u \in V'$, create a product g with the buyer's valuation $c(g) = \frac{w'(v')}{d'(v')}$. Set $B_u = B_u \cup \{g\}$ and $B_{v'} = B_{v'} \cup \{g\}$. Also set $w'(u) = w'(u) - c(g)$ and delete e from E'. Delete v' from V' as well. Since $w'(v') \leq w'(u)$, all the vertex weights are still ≥ 0.

5. If $d'(v') = 0$, set $B_{v'} = B_{v'} \cup \{g_{v'}\}$ with cost of the product $c(g_{v'}) = w'(v')$.

6. Goto Step 2.

7. Stop

The Algorithm 2, running in $O(m(m + 2|E|))$, clearly results in an instance I of *AllowBund*. Further, $p(B_v) = w(v)$ where $p(B_v)$ is the price of the bundle B_v. Notice that, if β is a bundling that gives a revenue k, $S = \{u | B_u \in \beta\}$ is an independent set of weight k because $\forall B_u, B_v \in \beta, B_u \cap B_v = \phi$ and $w(S) = \sum_{u \in S} w(u) = \sum_{B_v \in \beta} R(B_v) = k$.
We now prove that G has an independent set of weight k iff I has a bundling of revenue k, thereby showing that the optimal values of both problems are identical.
If there is an independent set of size(weight) k, then there is a set of vertices S s.t $\forall u, v \in S, (u, v) \notin E$. Therefore the bundling $\beta = \{B_u | u \in S\}$ is a feasible bundle with revenue $R(\beta) = \sum_{v \in S} R(B_v) = \sum_{v \in V} w(v) = k$.
Conversely, if there is a bundling β giving a revenue k, as shown above, $S = \{u | B_u \in \beta\}$ is an independent set of weight k. As the optimal values of both the problems are identical, the fact that any bundling of revenue k can be used to obtain an independent set of size k, along with the result in [6] that *for every $\epsilon > 0$, there is no $O(n^{1-\epsilon})$ approximation algorithm for the Weighted Independent Set problem (n being the number of vertices) which runs in time polynomial in n unless $NP \subseteq ZPP$* suffice to prove the theorem. □

The reduction holds in the reverse direction as well, when for each allowable bundle we create a node with weight equal to the revenue of that bundle and with edges between two nodes iff their respective bundles share an edge.

Note 2. Because of the reverse reduction, any approximation algorithm for the Weighted Independent Set problem or Maximal Independent Set problem will work for this problem too. Hence we have a simple $O(b)$ approximation algorithm for the revenue maximizing bundling problem.

4.2 Revenue Maximizing Bundle

There are certain scenarios where one would like to compute the single revenue maximizing bundle, rather than a bundling. There may also be constraints on the number of products that can be included in the bundle. For example, in a supermarket, a company may be alloted shelf space enough to sell only one bundle of its products. In this subsection, we present hardness results on the problems of finding the single revenue maximzing (optimal) bundle, when negative valuations are allowed, and that of finding the single largest revenue fetching bundle of size at most K, even when the valuations are non-negative.

THEOREM 6. *The problem of finding the single bundle yielding the largest revenue is hard to approximate within a factor of $2^{(\log n+m)^{\delta}}$ for some $\delta > 0$ where n is the number of products and m is the number of potential buyers under the plausible assumption that 3-SAT $\notin$ DTIME($2^{(m+n)^{3/4+\epsilon}}$) for some $\epsilon > 0$.*

PROOF. We outline the reduction from the Maximum Edge Biclique Problem [3]. Given a bipartite graph with n' nodes, we set one potential buyer for each vertex u_i in one partition, and one product for each vertex v_j in the other. For every edge (u_i, v_j), we set $c_{ij} = 1$. We set all other $c_{ij} = -n'^2$. Now, given any biclique of size $\geq R$, we can find k_1 buyers and k_2 products where k_1, k_2 are the sizes of the partitions in the biclique, such that the total revenue of the bundle comprising of the k_2 products, is at least $k_1 k_2 \geq R$ as each of the k_1 buyers value the bundle at k_2. If B is any bundle, say of k_2 products, that fetches a revenue R, by choosing the vertices corresponding to the maximal set of buyers buying the bundle (say, k_1 in number) in one partition and the products in another, we get a complete biclique of size R. This is because, for those k_1 buyers who buy B, each entry in the valuation matrix for all the k_2 products is 1 (else, there is at least one entry $-n'^2$, which would render it impossible for that buyer to buy the bundle). This argument also proves that if there is a bundle of k_2 products that fetches a revenue $\geq R$, there exists a biclique of size $\geq R$. Finding out the k_1 buyers who value the bundle at k_2 is simple. Thus, *there exists a biclique of size at least R iff there is a bundle of revenue at least R.* Further, the fact that any bundle which gives a revenue $\geq R$ can be used to obtain a biclique of size $\geq R$, and the fact that the optimal values for both the problems are equal, coupled with the result obtained in [3] that *the Maximum Edge Complete Bipartite Subgraph problem is hard to approximate within a factor of $2^{(logn)^{\delta}}$ for some $\delta > 0$, where n is the number of vertices, under the plausible assumption that 3-SAT $\notin$ DTIME($2^{n^{3/4+\epsilon}}$) for some $\epsilon > 0$* complete the proof. □

Note 3. When valuation prices can only be non-negative, the revenue maximizing bundle is efficiently computable. In fact, the best bundle is the biggest one, involving all products.

PROOF. By Contradiction, suppose the biggest bundle is not revenue maximizing and there exists a bundle $B \subset \{1, 2, ..., n\}$ which gives more revenue. Let the set M of buyers who buy B at the price which gives it maximal revenue be $M = \{i_1, i_2, ..., i_{m_1}\}$. The price of the bundle p is obviously the minimum value of the sum of the valuations for products in B, of each buyer in M. These buyers will therefore naturally purchase the bundle consisting of all the products, when it is priced at p (as the sum of their valuations for all products cannot decrease, as all values c_{ij} are non-negative). This means that the complete bundle generates a revenue at least the value generated by B, thereby completing the proof. □

However, since in most real-life scenarios, it is practically impossible to sell the entire set of products in one bundle, we analyze for the problem of finding a revenue maximizing bundle of size at most K.

THEOREM 7. *The problem of finding the single bundle B of size at most K products, which generates a revenue at least R, is strongly NP-Complete, given a valuation matrix of m buyers for n products even when all valuations are non-negative.*

We first outline the reduction from the *Maximum Clique Problem* [4].
Given a graph G, we add a gadget - a clique of $3n$ extra nodes, each of which is connected to all the nodes of G. Let the gadget appended graph be G'. We create an instance of the bundling problem, where we consider each vertex of G' as a product, and each edge of G' as a consumer who values the end vertices at 1 and all other products at 0. We show that there is a clique of size K in the original graph G iff there is a bundle containing $K+3n$ products that fetches a revenue of atleast $R=(K+3n)(K+3n-1)$. Formally,

PROOF. Given a graph $G=(V,E)$ where the vertex set $V=\{v_1,v_2,v_3,...,v_n\}$ and the edge set $E=\{e_1,e_2,e_3,...,e_m\}$, we construct an instance of the bundling problem as follows. There will be $N=4n$ products, $v_1,v_2,...,v_n,u_1,u_2,...,u_{3n}$ and $m+(3n(3n-1)/2+3n^2)$ buyers. The sets $C_1=\{e_1,e_2,...,e_m\}$ one for each edge in G, $C_2=\{f_{ij},2\leq i\leq 3n,1\leq j<i\}$, and $C_3=\{d_{ij},1\leq i\leq 3n,1\leq j\leq n\}$ all comprise of buyers. Each buyer $e_i=(v_{i_1},v_{i_2})$ will have a valuation of 1 for products v_{i_1} and v_{i_2} each and 0 for all the other products. Customer f_{ij} will have a valuation of 1 for products u_i and u_j and 0 for the others. Customer d_{ij} will have a valuation of 1 for products u_i and v_j and 0 for the others.Thus, our valuation matrix will be one filled with 0s and 1s, such that each row has exactly two 1s in it. And we will obtain a bundle of size at most $K+3n$ fetching a revenue at least $(K+3n)(K+3n-1)$ iff G contains a clique of size at least K
Now, given a clique of size at least K, we find a set of K vertices $V_K=v_{a_1},v_{a_2},...,v_{a_k}$ in the clique and form the bundle $B=V_K\cup\{u_1,u_2,...,u_{3n}\}$, and claim that B fetches a revenue of at least $(K+3n)(K+3n-1)$. This is because, by pricing B at 2, we know that all the buyers of the form f_{ij} will definitely buy it, as both u_i and u_j are present in B. Likewise, buyers of the form $d_{ia_j},1\leq i\leq 3n,1\leq j\leq K$ will buy the bundle, as the two products each values are in B. Similarly, the $K(K-1)/2$ buyers e_i where $e_i=(v_{a_{i_1}},v_{a_{i_2}})$ will buy the bundle. Hence the bundle revenue is at least $2(K(K-1)/2+3nK+3n(3n-1)/2)=(K+3n)(K+3n-1)$, as required.
Now, given that a bundle B of size at most $K+3n$ exists fetching a revenue of $(K+3n)(K+3n-1)$, we can expand this bundle to one of size exactly $K+3n$ products, as adding products will not hurt the revenue in any way. Further, we know that the maximum price of the bundle must have been 1 or 2, as the sum of each row of the valuation matrix adds up to 2. Had it been priced at 1, we would need at least $(K+3n)(K+3n-1)$ buyers buying the bundle, which is impossible for any $K\geq 0$, as the total number of buyers is $m+3n^2+3n(3n-1)/2\leq 4n(4n-1)/2<3n(3n-1)$. Hence, the bundle price must be 2. Let this bundle have x products from the set $\{u_i,1\leq i\leq 3n\}$ and hence, the number of buyers of the form f_{ij} and d_{ij} is bounded above by $x(x-1)/2+x(K+3n-x)$ as we are interested only in buyers who value B at 2. Further there can be at most $(K+3n-x)(K+3n-x-1)/2+x(x-1)/2+x(K+3n-x)$ buyers who will be interested in buying the bundle when priced at 2. This means that the number of buyers, of the form e_i who are interested in buying the bundle B is at least $(K+3n-x)(K+3n-x-1)/2$. All these buyers have interest only in products of the form v_j, of which there are only $K+3n-x$ in B. Hence, we have effectively found $(K+3n-x)(K+3n-x-1)/2$ buyers, each corresponding to an edge in G between 2 of the $K+3n-x$ vertices. The only possibility is the presence of a clique of size $K+3n-x$. As x is bounded above by $3n$, there is a clique of size at least K in G. Hence the result. $\square$

5. FUTURE WORK

Given that there exists no *FPSF* that always guarantees pure Nash equilibria, it would be interesting to come up with a polynomial time algorithm that finds out whether a game instance has pure Nash equilibria, under a given FPSF payoff. When *IF* payoffs are used, an open problem is to efficiently compute a pure Nash equilibrium (which is known to exist always). Obtaining better approximation algorithms to find the optimal bundling also pose an interesting challenge. It would also be good to solve the bundling problem not assuming prior knowledge of the buyer valuations. One possible approach would be to use *Price Discovery* [7] to find the buyers' valuations. Another direction of future research would be to analyze the bundling problems assuming limited stock for the products, which would be more valuable from a practical point of view.

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APPENDIX

A. FPSF PURE NASH IMPOSSIBILITY

Theorem 1 *For any $n > 3, m > 1$, there exist game instances with n players, each selling a unique product and m buyers, for which no pure Nash equilibria exist under any fair profit sharing payoff. The numbers 3 and 1 are tight, as when there are 3 products or less, there exist fair profit sharing schemes which always have pure Nash equilibria and when there is only 1 buyer, any possible bundling is a pure Nash equilibrium.*

PROOF. Given $n > 3, m > 1$, we construct the $m \times n$ valuation matrix as given in Fig 1(b). We construct the Configuration Graph (directed) given in Fig 1(a) as follows. Each node (v) in the graph corresponds to a set of valid bundlings of all products. We shall use a, b, c, d to represent the products of the first four players (g_1, g_2, g_3, g_4), and A, B, C, D to represent the players a_1, a_2, a_3, a_4 respectively, for convenience. Once all strategies are chosen, a bundling of goods is formed. This graph partitions all the possible bundlings, into sets depending on the bundles into which a, b, c, d are distributed relative to each other. For instance, the node $v_{\{a,b\},\{c\},\{d\}}$ corresponds to all those bundlings where a and b fall into one bundle, c into another one and d into yet another. Note that for a particular node, any of the other products can be in any of the bundles that contain a, b, c and d. The other products can also occupy other bundles which do not contain a, b, c and d. We will set b_1 and b_2 such that the following conditions are satisfied:

$$P(\{a,b,c,d\}), P(\{b,d\}), P(\{b,c\}), P(\{a,d\}), P(\{a,b\}) \leq 0 \tag{10}$$

$$P(\{a,b,d\}), P(\{a,c,d\}) < 0 \tag{11}$$

$$P(\{a,c\}), P(\{b,c,d\}), P(\{c,d\}), P(\{a,b,c\}) > 0 \tag{12}$$

If $B_1 \cap \{a,b,c,d\} = \{c,d\}$ & $B_2 \cap \{a,b,c,d\} = \{a,b,c\}$,

$$f(C, B_1) > f(C, B_2) \tag{13}$$

If $B_3 \cap \{a,b,c,d\} = \{a,c\}$ & $B_4 \cap \{a,b,c,d\} = \{b,c,d\}$,

$$f(C, B_3) > f(C, B_4) \tag{14}$$

Considering that all the buyers' valuations for products g_j are zero (when $j \geq 5$), it can be easily shown that for any bundle B,

$$R(B) = R(B \cap \{a,b,c,d\}) \text{ and } P(B) = P(B \cap \{a,b,c,d\}) \tag{15}$$

An edge in the configuration graph indicates that a player (product marked on the edge) could gain by deviating unilaterally to some other bundle. This is true for all bundlings that correspond to that node because, for all edges (u, v) with player q whose product is marked, except the edge $e_1 = (v_{\{\{a\},\{b,c,d\}\}}, v_{\{\{b,d\},\{a,c\}\}})$ and the edge $e_2 = (v_{\{\{a,b,c\},\{d\}\}}, v_{\{\{a,b\},\{c,d\}\}})$, there is a sign change from $\mathrm{P}(B_u)$ to $\mathrm{P}(B_v)$, of either 0 to $+^{ve}$, $-^{ve}$ to 0, or $-^{ve}$ to $+^{ve}$, where B_u and B_v are bundles occupied by q in some bundling corresponding to the nodes u and v. Hence, by **One-For-All** property of f, $f(q, B_u) < f(q, B_v)$.
Further from (13) and (14), C increases his payoff by moving along e_1 and e_2 as well. Thus, each vertex in the configuration graph has at least one outgoing edge, and hence for every bundling, at least one player can increase his payoff by unilaterally deviating.
Hence if all the conditions (10) to (14) are satisfied, we are through. Therefore, we choose $b_1 = 2.7$ and $b_2 = 0.05$ to satisfy (10) to (14).
It is easy to verify (10), (11) and (12) by substituting the values. Let B_1, B_2, B_3 and B_4 be as defined in (13) and (14). Now, to verify (13), $P(B_1) = P(\{c,d\}) = 4$ amd $P(B_2) = P(\{a,b,c\}) = 0.4$.
Further, as $R(\{d\}) <= R(\{c\})$, $f(C, B_1) \geq f(C, B_2)$. Now, by **Profit Conservation** and **Non-Contribution** properties of f, $f(C, B_1) \geq 2 > f(C, B_2)$. Similarly, $f(C, B_3) \geq 1.5 > f(C, B_4)$, proving (14). Hence, for all payoff functions, there exists no pure Nash equilibrium.
In fact, it can also be shown that for all values of b_1, b_2 satisfying the following conditions (domain marked by the criss-crossed shaded region in the Fig.2.)

$$\begin{aligned}
b_1 &> b_2 && (16)\\
b_2 &> 2b_1 - 8 && (17)\\
2b_2 &> b_1 - 3 && (18)\\
\text{If } b_1 > 2b_2, b_1 &< 5 &&\\
\text{else } b_1 - b_2 &< 2.5 && (19)\\
b_1 &> 2b_2 + 2.5\ . && (20)
\end{aligned}$$

there can be no fair profit sharing payoff scheme such that a pure Nash equilibrium exists. $\square$

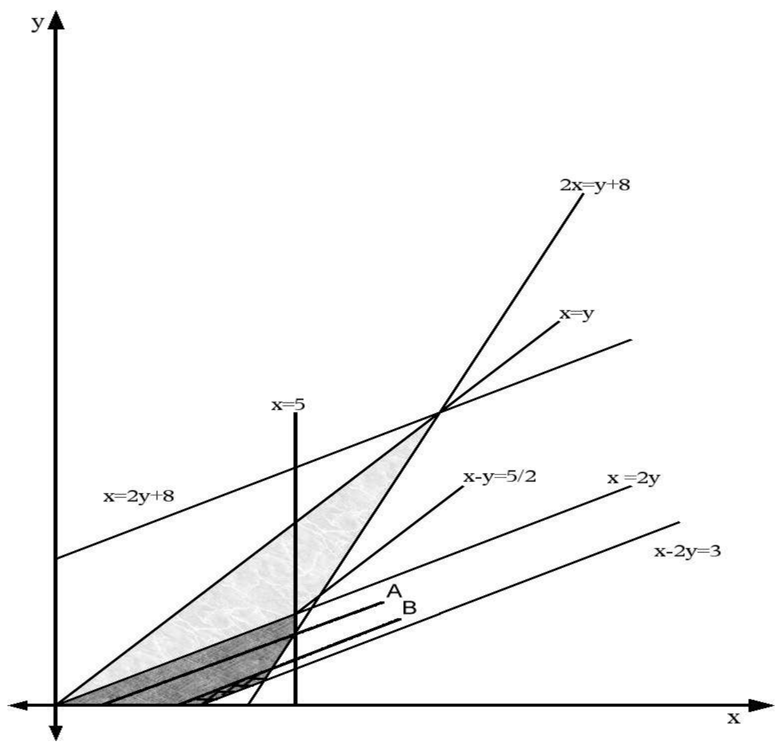

Figure 2: Graph showing feasible domain