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Q1 Analytical work. Given data

$$A + B = N.$$

$$[\dot{B}] = \beta \frac{[B][A]}{N} - \gamma [B]$$

$$[\dot{A}] = -\beta \frac{[B][A]}{N} + \gamma [B]$$

Substituting $N - B = A$

$$[\dot{B}] = \beta \frac{[B][N-B]}{N} - \gamma [B]$$

$$[\dot{B}] = \beta \frac{[B]\cancel{N}}{\cancel{N}} - \beta \frac{[B^2]}{N} - \gamma [B]$$

$$= \beta [B] - \beta \frac{[B^2]}{N} - \gamma [B]$$

$$\boxed{[\dot{B}]} = B(\beta - \gamma) - \beta \frac{[B]^2}{N}$$

This equation no longer involves A. And is called mean field equation.

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But $[\dot{B}] = \beta \frac{[N-B]}{N}[B] - \gamma[B]$

at $[\dot{B}] = 0$.

$$\gamma[B] = \beta \left[\frac{NB - B^2}{N}\right]$$

$$\therefore \frac{N[B]}{NB - B^2} = \frac{\beta}{\gamma}.$$

$$\boxed{\frac{N}{N-B} = \frac{\beta}{\gamma}} \quad \text{--- (1)}$$

Writing the above equation in terms of R_0.

$$R_0 = \frac{\beta}{\gamma}.$$

$$\boxed{\therefore R_0 = \frac{N}{N-B} = N\left(1 - \frac{1}{B}\right).}$$

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(2) Using the meanfield equation.

$$[\dot{B}] = [B](\beta-\gamma) - \beta\frac{[B]^2}{N}$$

At equilibrium $[\dot{B}] = 0$.

$$\therefore\ 0 = [B](\beta-\gamma) - \beta\frac{[B^2]}{N}$$

$$\beta\frac{[B^{\cancel{2}}]}{N} = \cancel{[B]}(\beta-\gamma)$$

$$[B]\frac{\beta}{N} = (\beta-\gamma)$$

$$\therefore\ [B] = \frac{N(\beta-\gamma)}{\beta}$$

$$[B] = N\left(1-\frac{\gamma}{\beta}\right)$$

$$\boxed{[B]} = 0,\ N\left(1-\frac{\gamma}{\beta}\right)$$

Fixed points at $B=0$.

$\therefore$ But $\frac{\beta}{\gamma} = R_0$.

$$\boxed{\therefore\ [B] = 0,\ N\left(1-\frac{1}{R_0}\right).}$$

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Sol Taking other eq^n -

$$[\dot{A}] = -\beta\frac{(N-A)A}{N} + \gamma[N-\cancel{A}]$$

To find fixed points $[\dot{A}] = 0$.

$$\therefore \beta\frac{(N-[A])A}{N} = \gamma[N-A]$$

$$\beta\frac{A}{N} = \gamma.$$

$$A = \frac{\gamma}{\beta}N.$$

But $\beta/\gamma = R_0$

$$\therefore A = \frac{N}{R_0}.$$

$\therefore$ The fixed points of the system are

$$(N,0),\ \frac{N}{R_0},\ N\left(1-\frac{1}{R_0}\right).$$

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hence, non zero B is called B^*

$$\boxed{\therefore B^* = N - \frac{N}{R_0}}$$

Next step is to find the Jacobian.

$$J = \begin{bmatrix} f_A(A,B) & f_B(A,B) \\ g_A(A,B) & g_B(A,B) \end{bmatrix}$$

$\therefore$

$$J = \begin{bmatrix} -\frac{\beta[B]}{N} & -\frac{\beta[A]}{N} + \gamma \\ \frac{\beta[B]}{N} & \frac{\beta[A]}{N} - \gamma \end{bmatrix}$$

Consider point at point $(N, 0)$

$$J = \begin{bmatrix} 0 & -\beta + \gamma \\ 0 & \beta - \gamma \end{bmatrix}$$

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This is at start $(N, 0)$.

Finding Eigen values.

$$A - \lambda I = 0.$$

$$\begin{bmatrix} 0 & -\beta+\gamma \\ 0 & \beta-\gamma \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0.$$

$$\begin{bmatrix} -\lambda & -\beta+\gamma \\ 0 & \beta-\gamma-\lambda \end{bmatrix} = 0.$$

$$-\beta\lambda + \gamma\lambda + \lambda^2 = 0.$$

$$\therefore \lambda^2 + \lambda\gamma - \beta\lambda = 0.$$

$$\lambda(\lambda + \gamma - \beta) = 0.$$

$$\therefore \lambda = 0 \quad \text{or} \quad \lambda + \gamma - \beta = 0.$$

$$\boxed{\therefore \lambda_1 = 0} \quad \text{or} \quad \boxed{\lambda_2 = \beta - \gamma}$$

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Now consider point $\frac{N}{R_0}$ & $N\left(1-\frac{1}{R_0}\right)$

$$J = \begin{bmatrix} -\beta\frac{\times N}{N}\left(1-\frac{\gamma}{\beta}\right) & -\frac{\beta}{N}\left(\frac{N\gamma}{\beta}+\gamma\right) \\ \beta\frac{N}{N}\left(1-\frac{\gamma}{\beta}\right) & \beta\frac{N\gamma}{N\beta}-\gamma \end{bmatrix}$$

$$J \begin{bmatrix} -\beta\left(1-\frac{\gamma}{\beta}\right) & 0 \\ \beta\left(1-\frac{\gamma}{\beta}\right) & 0 \end{bmatrix}$$

Continue to find eigen value

$$A - \lambda I = 0$$

$$\therefore \begin{bmatrix} -\beta\left(1-\frac{\gamma}{\beta}\right)-\lambda & 0 \\ \beta\cdot\left(1-\frac{\gamma}{\beta}\right) & 0 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

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$$A\lambda - I = 0$$

$$\begin{bmatrix} -\beta\left(1-\frac{\gamma}{\beta}\right)-\lambda & 0 \\ \beta\left(1-\frac{\gamma}{\beta}\right) & -\lambda \end{bmatrix} = 0.$$

$$\therefore \left(-\beta\left(1-\frac{\gamma}{\beta}\right)-\lambda\right)(-\lambda) - 0 = 0.$$

$$\therefore \lambda^2 + \beta\left(1-\frac{\gamma}{\beta}\right)\lambda = 0. \quad \therefore \lambda = 0$$

or

$$\lambda + \beta\left(1-\frac{\gamma}{\beta}\right) = 0.$$

$$\therefore \lambda = -\beta\left(1-\frac{\gamma}{\beta}\right).$$

$$\lambda = -\beta + \frac{\gamma\cancel{\beta}}{\cancel{\beta}}.$$

$$\boxed{\therefore \lambda = \gamma - \beta}$$

$$\boxed{\lambda = 0, \quad \lambda_2 = \gamma - \beta}$$

Consider the stability of system. marginal stability is only achieved if $1 - \frac{\gamma}{\beta} > 0$

$\therefore \frac{\gamma}{\beta} < 1$ But $\frac{\beta}{\gamma} = R_0$

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So the system is only stable when $\frac{\gamma}{\beta} < 1 \rightarrow$ so $\frac{\beta}{\gamma} > 1$

$\therefore$ But $\frac{\beta}{\gamma} = R_0$

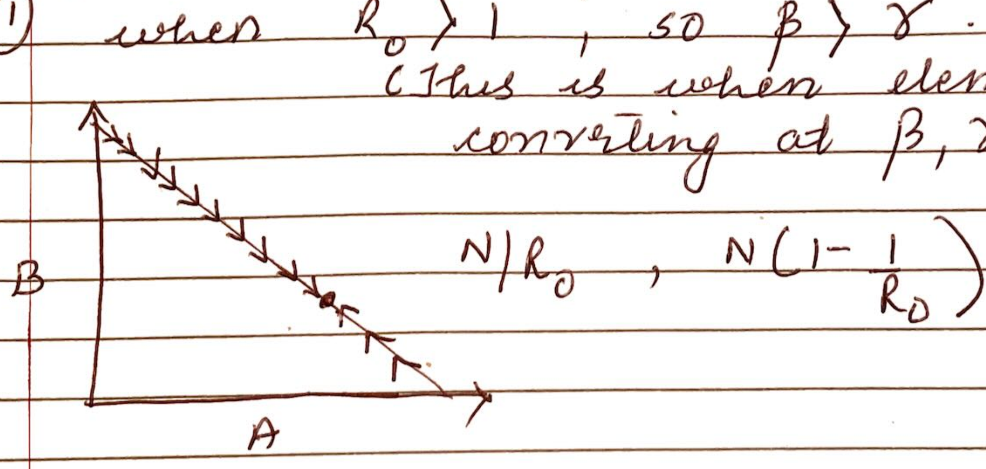
$$\boxed{\therefore \; R_0 > 1.}$$

Q2: Phase ~~pe~~ potrait.

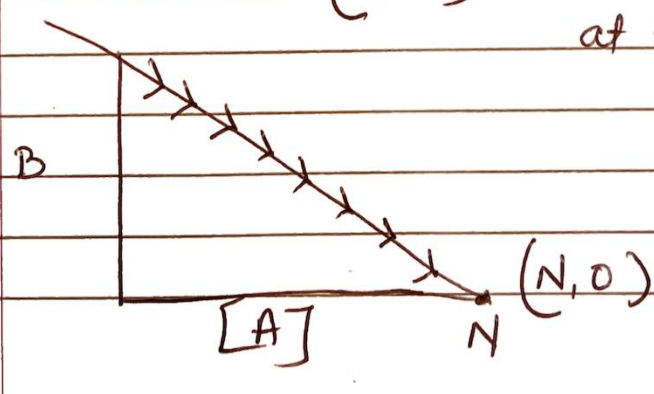
As per the points $(N, 0)$ $\left(\frac{N}{R_0}, N\left(1-\frac{1}{R_0}\right)\right)$

Consider two possibilities.

1. when $R_0 > 1$, so $\beta > \gamma$.
(This is when elements are converting at β, γ rate)



2. when the elements are at the start $(N, 0)$ so $\gamma > \beta$, $R_0 \leq 1$
at t_0.



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Q4] Integrate $[\dot{B}]$ analytically to obtain $[B](t)$.

Consider the starting with $[\dot{B}]$

$$[\dot{B}] = [B]\frac{[B][A]}{N} - \gamma[B]$$

we end up with mean field eq^n.

$$[\dot{B}] = B(\beta-\gamma) - \beta\frac{(B)^2}{N}.$$

$$\frac{[\dot{B}]}{B^2} = \frac{\cancel{B}(\beta-\gamma)}{B^{\cancel{2}}} - \beta\frac{\cancel{(B)^2}}{N\cancel{B^2}}.$$

$$\boxed{\therefore \frac{[\dot{B}]}{B^2} = \frac{\beta-\gamma}{B} - \frac{\beta}{N}.}$$

Considering $\cancel{\frac{1}{y}=}$ $\cancel{B} = y = \frac{1}{[B]}$

$$\therefore [\dot{B}]y^2 = (\beta-\gamma)y - \frac{\beta}{N}$$

~~yn~~ But $y = \frac{1}{B}$

$$\boxed{\therefore \dot{y} = \frac{-1}{B^2}\cdot[\dot{B}]}$$

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Now substituting the above results

$$\frac{\mathring{B}}{B} = (\beta-\gamma)\frac{1}{B} - \frac{\beta}{N}.$$

$$-\mathring{y} = (\beta-\gamma)y - \frac{\beta}{N}$$

$$\mathring{y} = \frac{\beta}{N} \cancel{*} (\beta-\gamma)y.$$

But $B^* = N\left(1-\frac{1}{R_0}\right)$ where $R_0 = \frac{\beta}{\gamma}$.

$$\therefore B^* = N\left(1-\frac{1}{\beta/\gamma}\right)$$

$$\therefore B^* = N\left(\frac{\beta-\gamma}{\beta}\right)$$

$$\boxed{\therefore \frac{\beta B^*}{N} = (\beta-\gamma)}$$ substitute in the eq.

$$\frac{\cancel{\mathring{B}}}{\cancel{B}}\mathring{y} = \frac{\beta}{N} \cancel{*} \frac{\beta B^*}{N}y.$$

$$\boxed{\therefore \mathring{y} = \frac{\beta}{N} - \frac{B^*\beta}{N}y.}$$

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Now substituting

$$\frac{\beta B^*}{N} = \lambda \ \& \ \frac{\beta}{N} = I.$$

$$\dot{y} = I - \lambda y.$$

Consider $y(t) = g(t) + h(t)$.

$$h(t) = ke^{-\lambda t}$$

$$y'(t) = g(t)h'(t) + h(t)g'(t)$$

$$= g(t) - \lambda h(t) + ke^{-\lambda t} g'(t)$$

$$= -\lambda g(t) \cancel{\&} h(t) + ke^{-\lambda t} g'(t).$$

$$y'(t) = -\lambda y + ke^{-\lambda t} g'(t)$$

which is in the form.

$$y'(t) = -\lambda y(t) + I$$

so $I = ke^{-\lambda t} g'(t)$

$$g'(t) = \frac{I}{k} e^{\lambda t}$$

Integrating with respect to t.

$$g(t) = \frac{I}{k\lambda} e^{\lambda t} + c \qquad c = \text{constant}.$$

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From above equations.

$$y(t) = g(t)h(t)$$

$$= \left(\frac{I}{k\lambda} e^{\lambda t} + c\right) k e^{-\lambda t}.$$

$$= \frac{I}{\lambda} + cke^{-\lambda t}.$$

$$\boxed{y(t) = \frac{I}{\lambda} + ke^{-\lambda t}.}$$

Resubstitution in our equation

$$y(t) = \overbrace{\frac{\beta}{N}}^{I} \times \overbrace{\frac{N}{\beta B^*}}^{\&\ \lambda} + ke^{-\frac{\beta B^* t}{N}}$$

$$y(t) = \frac{1}{B^*} + ke^{-\frac{\beta B^* t}{N}}$$

Now resubstitute $y = \frac{1}{B}$.

$$\therefore \frac{1}{B} = 1 + B^* k e^{-\frac{\beta B^*}{N} t}$$

$$\therefore B = \frac{B^*}{1 + B^* k e^{-\frac{\beta B^* t}{N}}}$$

$$B^* = N\left(\frac{\beta - \gamma}{\beta}\right)$$

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$$[B] = \frac{B^*}{1 + B^* k e^{\frac{-\cancel{\beta \times N}(\beta-\gamma)t}{\cancel{N+\beta}}}}$$

$$[B] = \frac{B^*}{1 + B^* k e^{-(+\beta-\gamma)t}}$$

~~B=~~

$$[B] = \frac{B^*}{1 + B^* k e^{-(\beta-\gamma)t}}$$

Now Consider $t=0$, $[B] = B_0$

$$B_0 = \frac{B^*}{1 + B^* k}$$

$$(1 + B^* k) = \frac{B^*}{B_0}$$

$$B^* k = \frac{B^*}{B_0} - 1$$

$$\boxed{k = \frac{1}{B^*}\left(\frac{B^*}{B_0} - 1\right)}$$

$$[B] = \frac{B^*}{1 + B^* \frac{1}{B^*}\left(\frac{B^*}{B_0} - 1\right) e^{-(\beta-\gamma)t}}$$

$$[B](t) = \frac{B^*}{1 + \left(\frac{B^*}{B_0} - 1\right) e^{-(\beta-\gamma)t}}$$