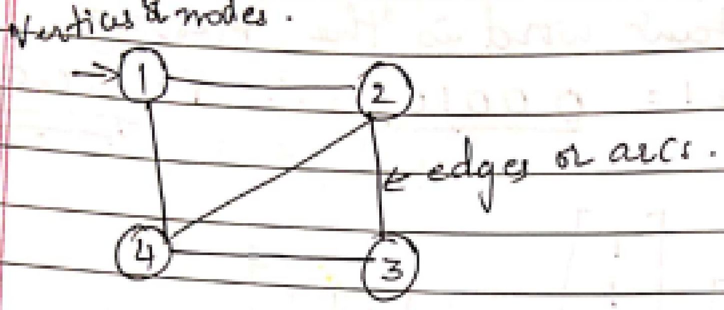
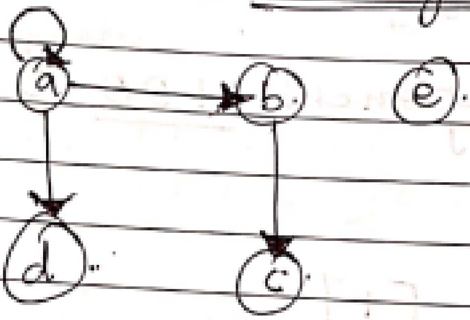
Graph Theory
(Introduced by Euler)

Graph . G = {V, E), Set of Vertices & Edges.

Graphs can be Undirected and directed

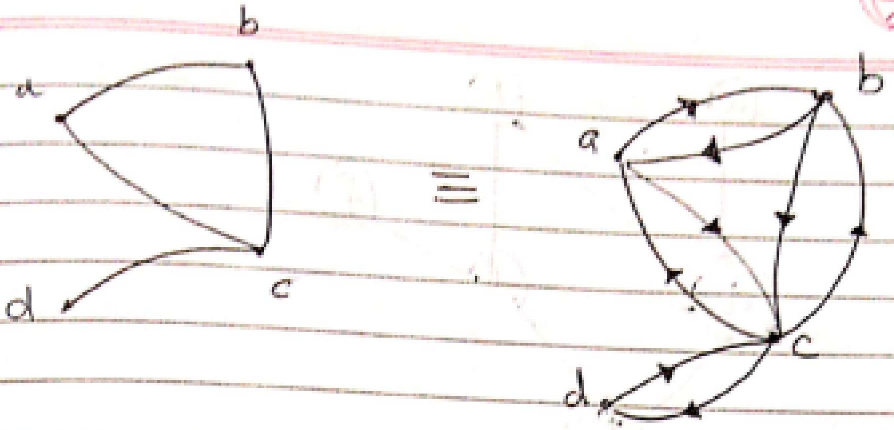


If the edges of Graph has direction then it is called directed graph or digraph.



V = {a, b, c, d, e} E = {(a, a) (a, b) (a, d) (b, C)}

- In the edge b/w (b, C), 'b' is said to be adjacent to 'C', whereas 'c' is adjacent from b
- 'b' is called the Origin or Source of the (b, c) and Vertex 'C' is called the terminating Vertex or terminating Vertex.
- The edge (a, a) is an example of a loop
- the Vertex e that has no incident edge is called an isolated Vertex



- A Graph that doesnot contain any loops is called loop-free Graph.

Definition :-

Walk :- (May repeat vertices & edges.

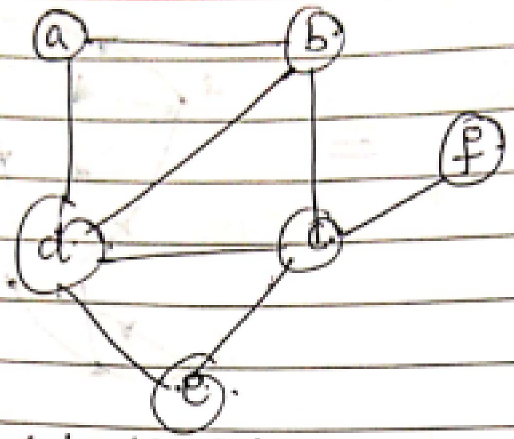
Let x, y be (not necessarily distinct) vertices in an Undirected Graph $G = (V, E)$,
An $x - y$ walk in G is a (loop-free) finite alternating sequence of vertices & edges from G.

$$x = x_0, e_1, x_1, e_2, x_2, e_3, \ldots, e_{n-1}, x_{n-1}, e_n, x_n$$

Starting from vertex x and ending at vertex y and involving the n edges $e_i = \{x_{i-1}, x_i\}$ where $1 \leq i \leq n$

- The length of this walk is 'n', the number of edges in the walk.
- Any $x - y$ walk where $x = y$ and $n > 1$ is called a **closed walk**. Otherwise the walk is called **open walk**.

* No. of edges in a walk is called the **length of the walk.**



1) (a,b) (b,d) (d,c) (c,e) (e,d) (d,b)
is a an open walk from a-b of length 6 in which vertices d & b are repeated & edges (b,d) is repeated.

2) $b \to c \to d \to e \to c - f$ is a open walk from b-f of length 5 and vertex c is repeated but no edges repeated.

3) $b \to c \to d \to b$ is a closed walk from b-b of length 3.

Circuit :-
If no edges in the x-y walk is repeated then the walk is called an x-y trail. a close x-x trial is called Circuit

path :-
If no vertex of the x-y walk it is repeated called a path, a closed path is a Cycle.

For directed graphs, we use directed walk, directed path, directed cycles.

Summarization

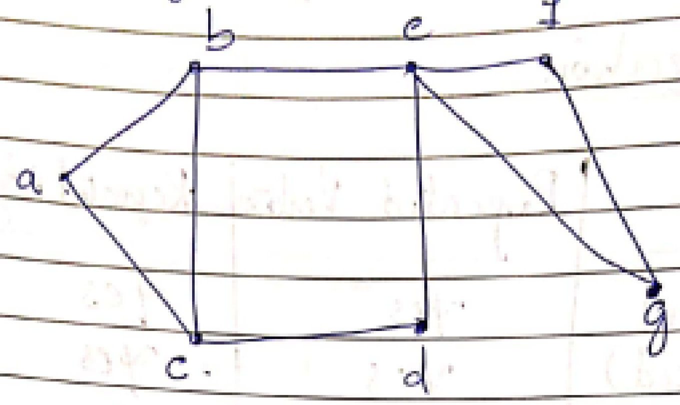
Name.	Repeated Vertex	Repeated Edge	open	close
walk (open)	yes	yes	yes	—
walk (closed)	yes	yes	—	yes
Trial.	yes	NO.	yes	—
Circuit.	yes	NO.		yes
path	NO	NO.	yes	
Cycle.	NO	NO	—	yes

problem:-

For the Graph in Fig determine
(a) a walk from n to p that is not a trail.
b) a n-p trail that is not a path
c) a path from n to p.
d) a closed walk from n to n that is not a Circuit;
e) a Circuit from n to n that is not a Cycle.
f) a Cycle from n to n.

1) for the Graph below determine.
a) walk from b to d that is not a trial.
b) a b-d trail that is not path.
c) a path from b to d.
d) a closed walk from b to d that is not a Circuit
e) a Circuit from b to b that is not a Cycle.

f) a cycle from b to b.



a) {b, e}, {e, f) (f, g) (g, e) (e, b) (b, c) (c, d)
- (b, e) is repeated so it's a walk but not trial.

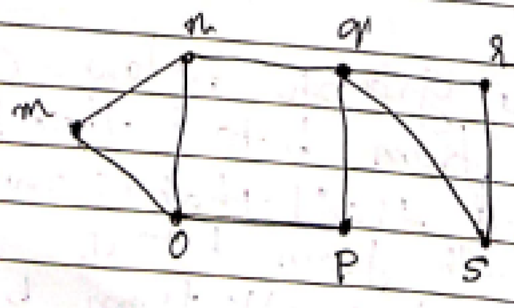
b) (b, e) (e, f) (f, g) (g, e) (e, d)
- it's a trail as no edges are repeated
- it's not a path as e vertex is reped.

c) (b, e) (e d)

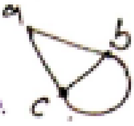
d) (b, e) (e, f) (f, g) (g, e) (e, b)

e) (b, e) (e, f), (f, g) (g, e), (e, d) (d, c), (c, b)

f) (b, a) (a, c) (c, b)

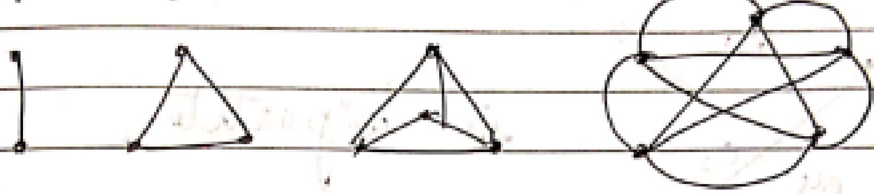


Multiple Graph: Graph that Contain Multiple edges ~~but no loops~~ b/w a pair of vertex



Complete Graph :-

A Simple Graph (Graph that doesnot contain selfloop or Multiple edges) and which has an edge b/w every pair of Vertices is called a Complete Graph. (full Graph)



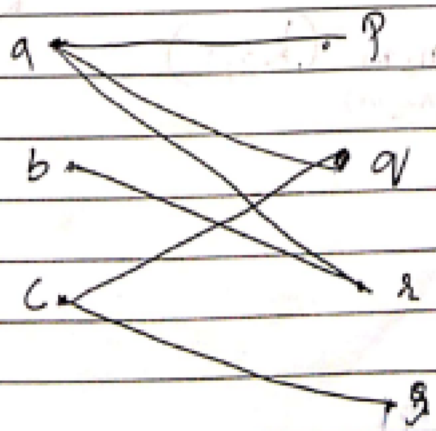
Bipartite Graph :-

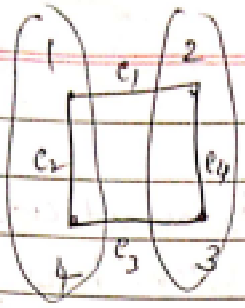
A Simple Graph G such that its Vertex set V is the union of two Mutually disjoint non-empty set V_1 & V_2 which are such that each edge in G joins a Vertex in V_1 & a Vertex in V_2. then Graph is Called as Bipartite Graph.

and denoted as $G = (V_1, V_2; E)$.

and Vertex set V_1 & V_2 are called bipartite of Vertex set V.

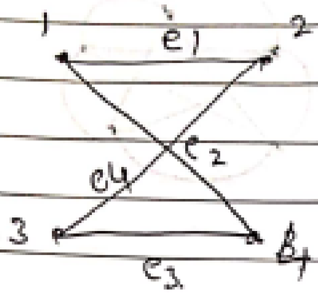
Eg: $V_1 = \{a, b, c\}$ $V_2 = \{p, q, r, s\}$





is not bipartite, because if we consider e_2 or e_4 the edges are adjacent edges of vertex of same set.

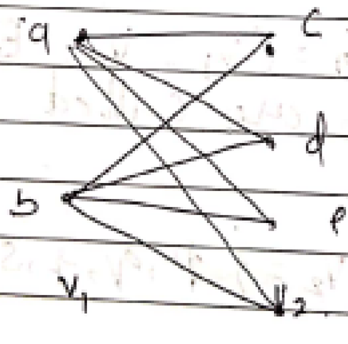
but

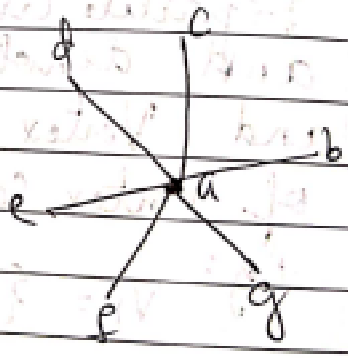


is bipartite.

Complete Bipartite Graph :

Is is bipartite Graph $G = (V_1 V_2\ E)$ that has an edge b/w every vertex of V_1 to every vertex in V_2.





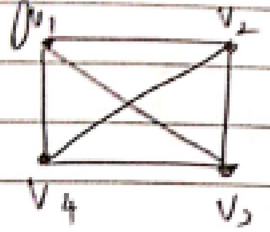
denoted as $k_{n,m}$ $(k_{2,3})$
$(n \leq m)$

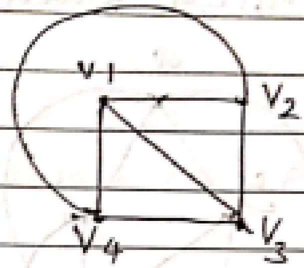
$K_{1,6}$

planar Graph :-

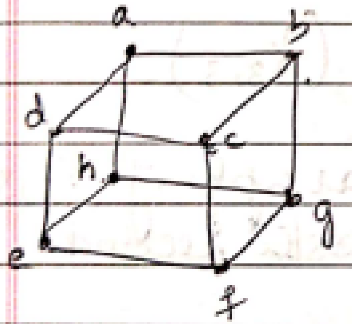
A Graph is called planar if it can be drawn in the plane without any edges crossing each other. And a Graph that cannot be drawn on a plane without a crossover b/w its edges is called Non-planar

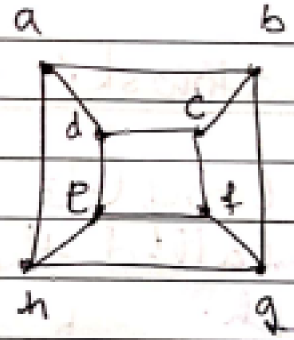
Eg:-





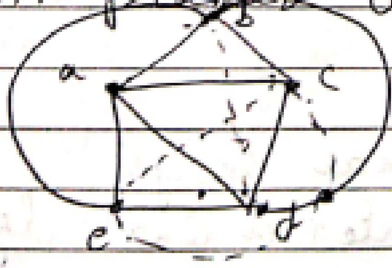
Non-planar. planar.





Non-planar planar.

** Note :- A Complete Graph of 5 Vertices is Non-planar Graph.



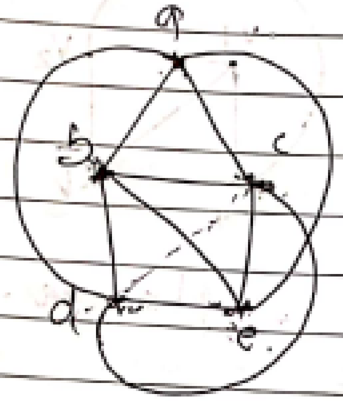
b - d.
e - c.

Kuratowski Graph :-

Kuratowski Graphs are Non-planar Graph.

Kuratowski Graph - I (K_5)

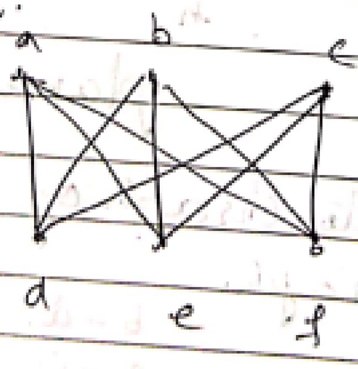
A Complete graph with 5 Vertices (K_5) is Called Kuratowski First Graph.



K_5 -

Kuratowski Graph - II ($K_{3,3}$)

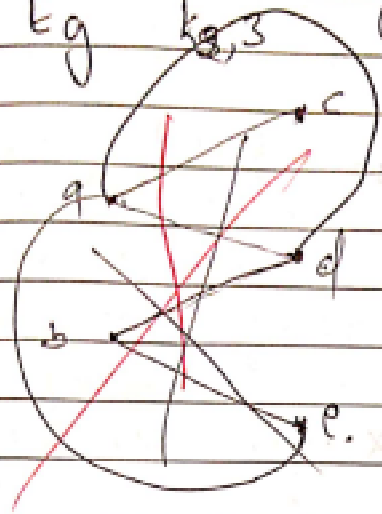
A Complete Bipartite Graph of ($K_{3,3}$) is Called the Kuratowski Second Graph.

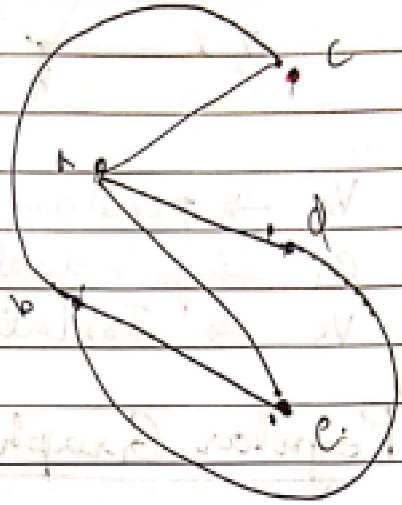


Note :-

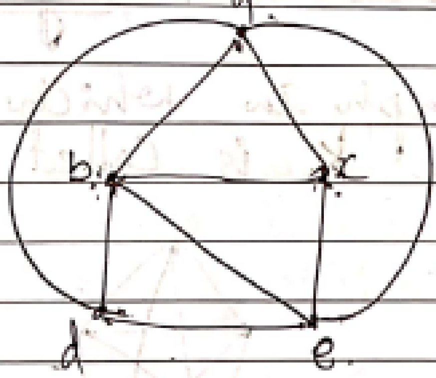
- Both are Non-planar Graph
- Removal of One Vertex or One edge makes the Graph planar

Eg $K_{2,3}$ (after removing an Vertex.





Eg:- (after removing an edge in K_5)



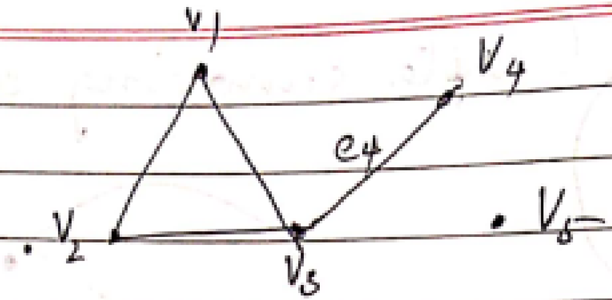
(c,d) removed

Note:- Number of Vertices in a Graph is called order of the Graph.

Number of edges in called the Size of the Graph.

A Vertex in a graph which is not an end Vertex of any edge of the Graph is called an Isolated Vertex. (Degree 0)

A Vertex of degree 1 is called pendent vertex. A edge which is incident on a pendent Vertex is called pendent edge

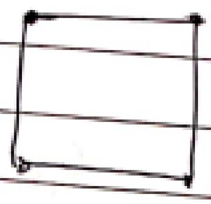


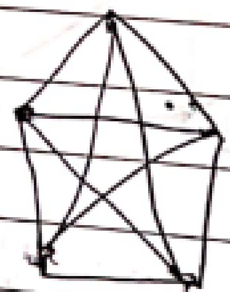
V_4 → pendant Vertex
e_4 → pendant edge
V_5 → Isolated Vertex.

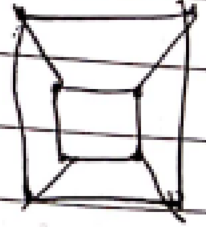
Regular Graph :-

A Graph in which all the Vertices are of same degree is called Regular graph.

A Regular graph in which all Vertices are of degree 'K' is called K-regular Graph.







Isomorphic Graphs :-

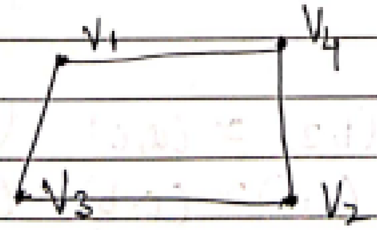
The Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called Isomorphic if there is One-to-one and onto fun^n from V_1 to V_2 with the property that 'a' and 'b' are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2 for all 'a' and 'b' in V_1

Suppose that an edge 'e' is incident on V_1 and V_2 in G_1 then Correspondig edge in G_2 Must be incident on Vertices V_1 & V_2 that Corresponds to V_1 & V_2 in G_1.

Properties :-

1. Same no. of Vertices
2. Same no of Edges.
3. Equal no. of Vertices with ~~given~~ Same degree.

Eg:- Show that G and H are Isomorphic



Mapping: b/w Vertice

$f(u_1) = V_1$

$f(u_3) : V_3$

$f(u_2) : V_4$

$f(u_4) = V_2$

Mapping b/w edge.

$(u_1, u_2) \cong (V_1 V_4)$

$(u_1, u_3 \cong (V_1 V_3)$

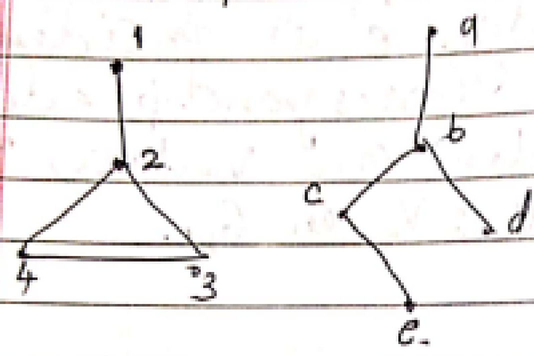
$(u_3, u_4) \cong (v_3, v_2)$

$(u_4, u_2) \cong (v_2, v_4)$

$\therefore$ G and H are Isomorphic.

2) Check whether G and H are Isomorphic are not

$|E_1| = 4$, $|E_2| = 4$; $|V_1| = 4$, $|V_2| = 5$.

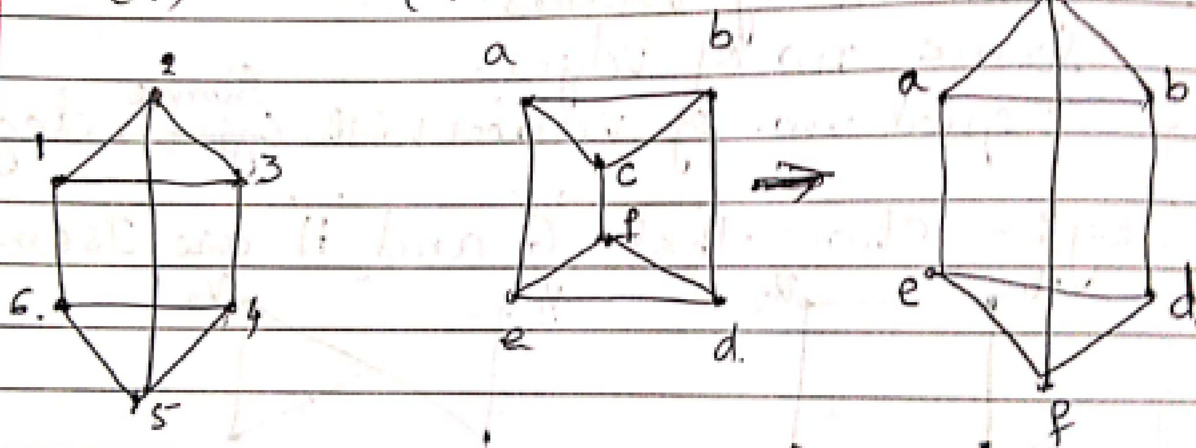


(G) (H)

As $|V_1|$ is not equal $|V_2|$

G and H not Isomorphic.

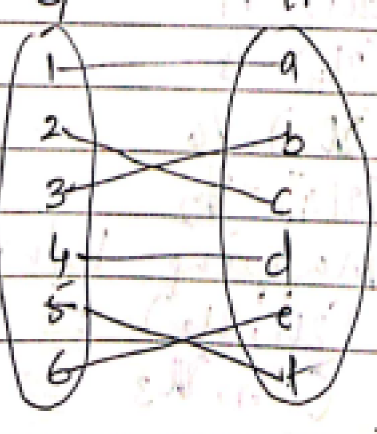
3)



$|V_1| = |V_2|$ and

$|E_1| = |E_2|$

G H



$(1,2) \cong (a,c)$ $(6,5) \cong (e,f)$

$(2,3) \cong (c,b)$ $(6,4) \cong (e,d)$

$(1,3) \cong (a,b)$ $(2,5) \cong (c,f)$

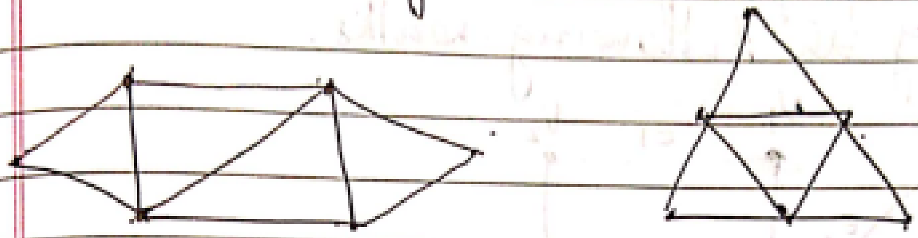
$(1,6) \cong (a,e)$

$(3,4) \cong (b,d)$

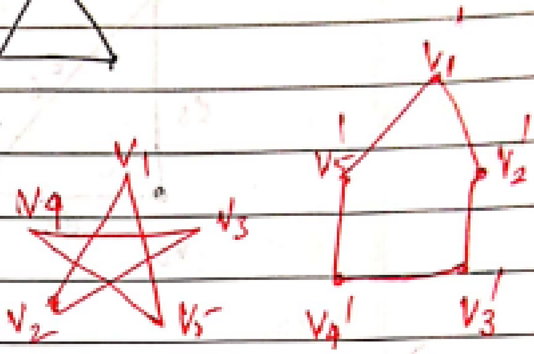
$(4,5) \cong (d,f)$

$\therefore$ G & H are Isomorphic.

4) Show that the following graphs are not Isomorphic.



G H.



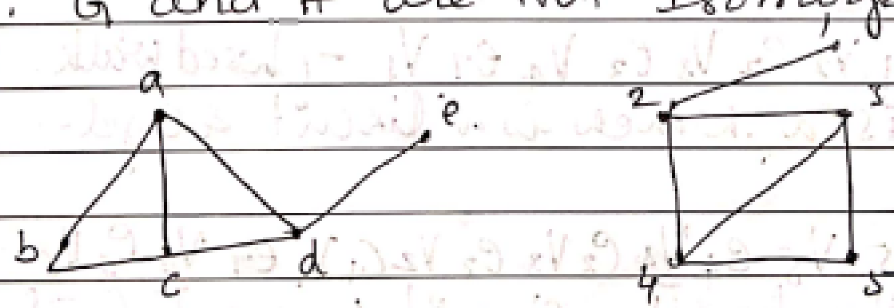
$|V_G| = 6 \qquad |E_G| = 9$

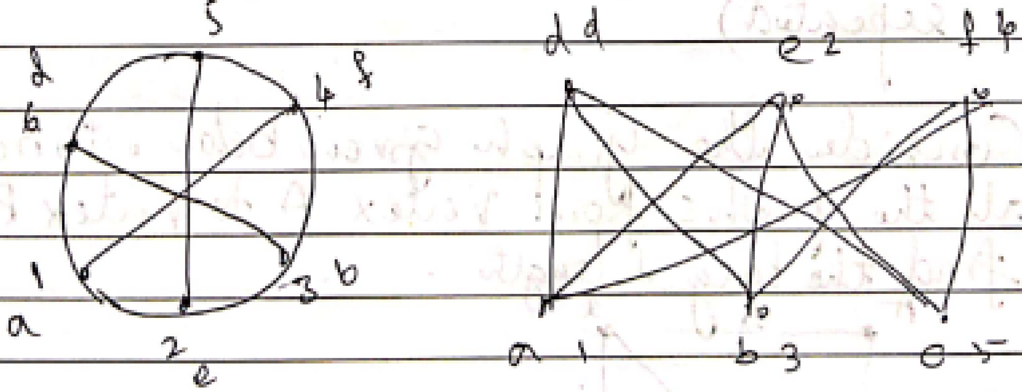
$|V_H| = 6 \qquad |E_H| = 9$

But G has 2 Vertices of degree 4 and H has 3 Vertices of degree 4.

$\therefore$ G and H Cannot have one-to-one Correspondence b/w Vertices & edges.

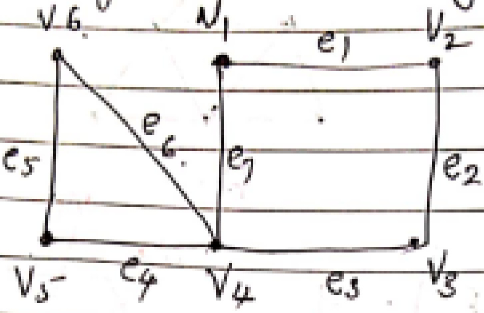
$\therefore$ G and H are Not Isomorphic.

5) 

(6) 

Problems on walk, trail, circuit, path & Cycle

1) For the Graph given below indicate the nature of the following walks.



a) $v_1 e_1 v_2 e_2 v_3 e_2 v_2$ — open walk which is not a trail.

b) $v_4 e_7 v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$ — Open walk which is a trail but not path.

c) $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$ — open walk which is a trail and also a path.

d) $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_7 v_1$ — Closed walk. which is a which is a circuit & cycle

e) $v_6 e_5 v_5 e_4 v_4 e_3 v_3 e_2 v_2 e_1 v_1 e_7 v_4 e_6 v_6$ — Closed walk, circuit but not cycle (v_4 is repeated)

2) Consider the Graph given below. Find all the paths from vertex A to vertex P. find the largest length.