MTL712 - Assignment Report - 02

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1 Introduction

In the previous assignment, we looked at Lax-Friedreich and Lax-Wendroff Methods for solving differential equations. In this assignemnt, we explore more techniques through the lens of the five test cases that we were using earlier. Consider a PDE of the form:

$$u_t + f(u)_x = 0 \tag{1}$$

where:

- $u = u(x, t)$ is the unknown function of spatial variable x and time t.
- $f(u)$ is a given flux function that depends on u.
- u_t represents the partial derivative of u with respect to time t.
- $f(u)_x$ represents the partial derivative of the flux function $f(u)$ with respect to the spatial variable x.

The Godunov's and Roe's methods are iterative methods typically used to solve partial differential equations (PDEs) of the above form, given the initial and boundary conditions.

2 Theory

2.1 Godunov's Method

Godunov's first-order upwind method was discovered in 1959. In wave speed split form, Godunov's first-order upwind method is

$$u_i^{n+1} = u_i^n + C_{i+1/2}^+ \left(u_{i+1}^n - u_i^n\right) - C_{i-1/2}^- \left(u_i^n - u_{i-1}^n\right), \tag{2}$$

where

$$C_{i+1/2}^+ = -\lambda \min_{u \text{ between } u_i^n \text{ and } u_{i+1}^n} \left(\frac{f(u) - f(u_i^n)}{u_{i+1}^n - u_i^n}\right) \tag{3}$$

and

$$C_{i+1/2}^- = \lambda \max_{u \text{ between } u_i^n \text{ and } u_{i+1}^n} \left(\frac{f(u_i^n) - f(u)}{u_{i+1}^n - u_i^n}\right) \tag{4}$$

Of course, as always, there are infinitely many other wave speed split forms. However, this is the only wave speed split form with finite coefficients - in this sense, this is the unique natural wave speed split form.

2.2 Roe's Method

Our aim is to approximate the solutions of the Riemann problem. A common strategy is to linearize the nonlinear equations.

$$U_t + f(U)_x = 0$$

$$\Rightarrow \quad U_t + f'(U) \cdot U_x = 0$$

Here, f is a nonlinear flux function, and we will replace this $f'(U)$ locally by an approximation.

$$U_t + f'\left(\frac{U_j^n + U_{j+1}^n}{2}\right) U_x = 0.$$

$$\hat{A}_{j+\frac{1}{2}} = \begin{cases} \frac{f(U_{j+1}^n)-f(U_j^n)}{U_{j+1}^n-U_j^n}, & \text{if } U_{j+1}^n \neq U_j^n \\ f'(U_j^n), & \text{if } U_{j+1}^n = U_j^n. \end{cases}$$

where

$$\epsilon_{j+\frac{1}{2}}^n = \left|\hat{A}_{j+\frac{1}{2}}^n\right|$$

and

$$\epsilon_{j-\frac{1}{2}}^n = \left|\hat{A}_{j-\frac{1}{2}}^n\right|.$$

Finally, we write down the Roe solver as below:

$$U_j^{n+1} = U_j^n - \frac{\lambda}{2}\left[f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right)\right] + \frac{\lambda}{2}\left[\epsilon_{j+\frac{1}{2}}^n\left(U_{j+1}^n - U_j^n\right) - \epsilon_{j-\frac{1}{2}}^n\left(U_j^n - U_{j-1}^n\right)\right],$$

where

$$\lambda = \frac{\Delta t}{\Delta x}.$$

3 Testcase-wise Analysis of Methods

Please note that a detailed description of all the test cases and their exact solutions have been given in the previously submitted report (Assignment-01 report). We use the same parameters and conditions as used in the previous assignment for each test case.

3.1 Test Case 1

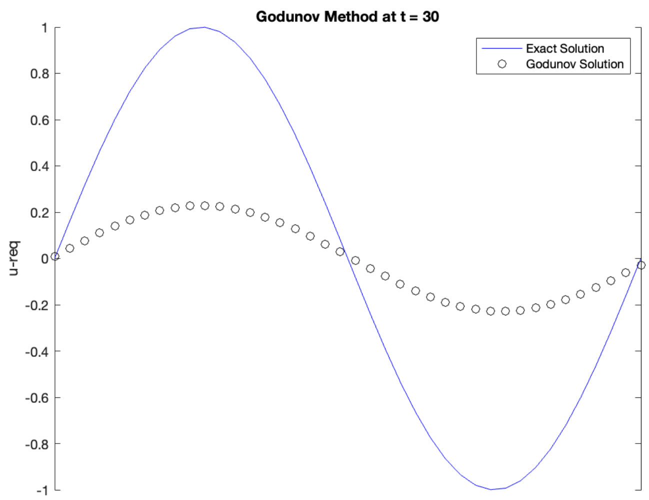
Here we need to find the approximate solution at $u(x, 30)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

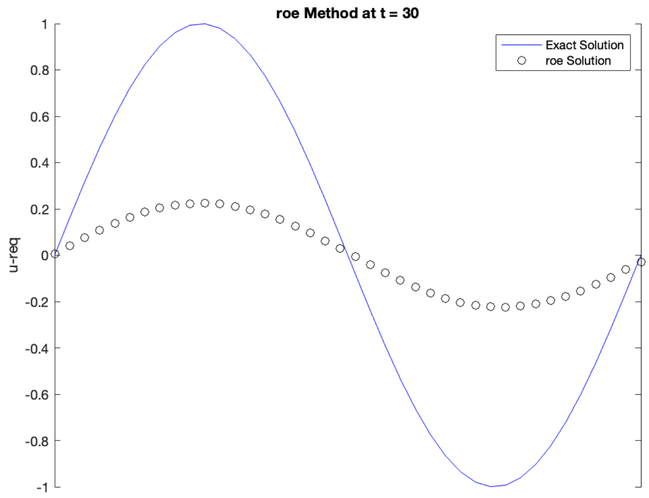
with initial condition:

$$u(x, 0) = -sin(\pi x)$$

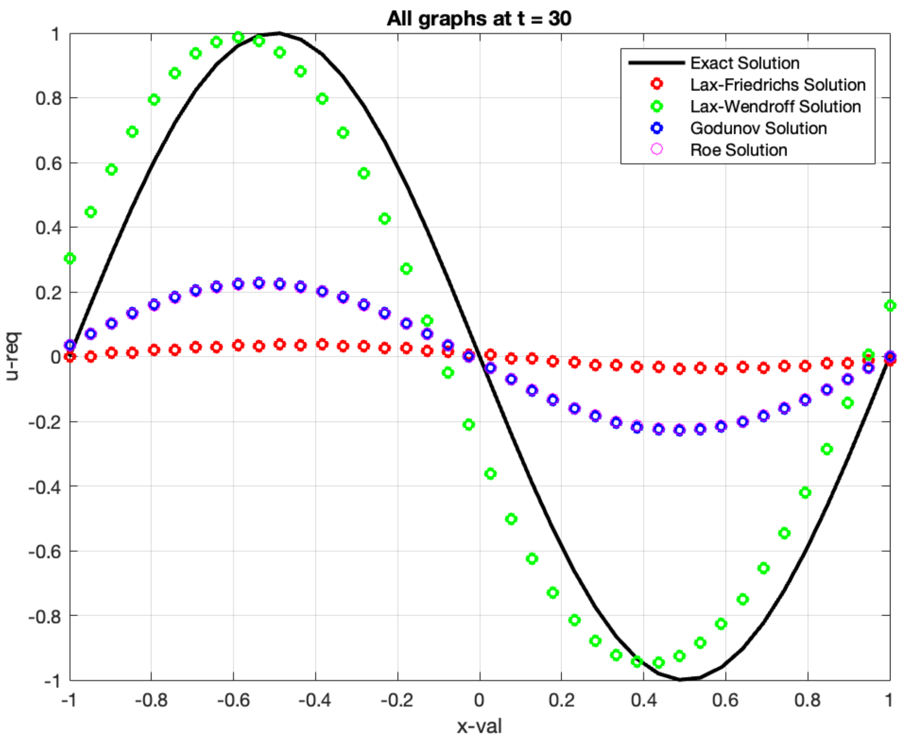
3.1.1 Godunov's Methods



3.1.2 Roe's Method



3.1.3 Comparison of All Four Methods



3.1.4 Summary of results

- For both Godunov's and Roe's method, the sinusoid's shape is well preserved, and its phase is approximately correct, but its amplitude has been reduced by a significant amount.
- We see that among all four methods, Lax-Friedrich performs best and Lax-Wendroff the worst. The other two give the exact same solution, and are intermediate in accuracy.

3.2 Test Case 2

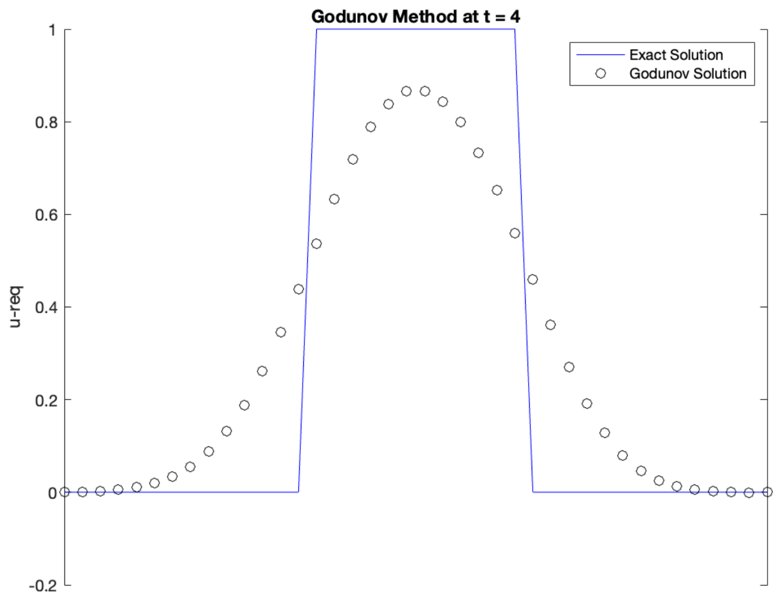
Here we need to find the approximate solution at $u(x, 4)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

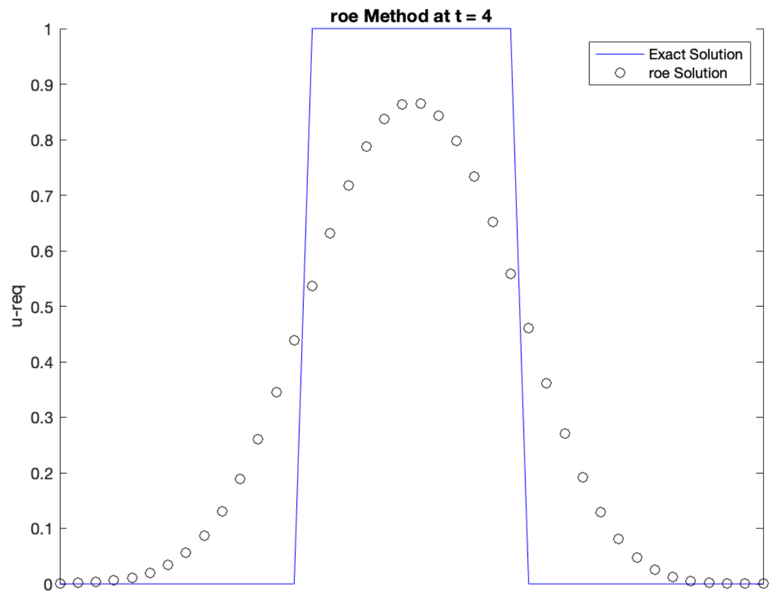
with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

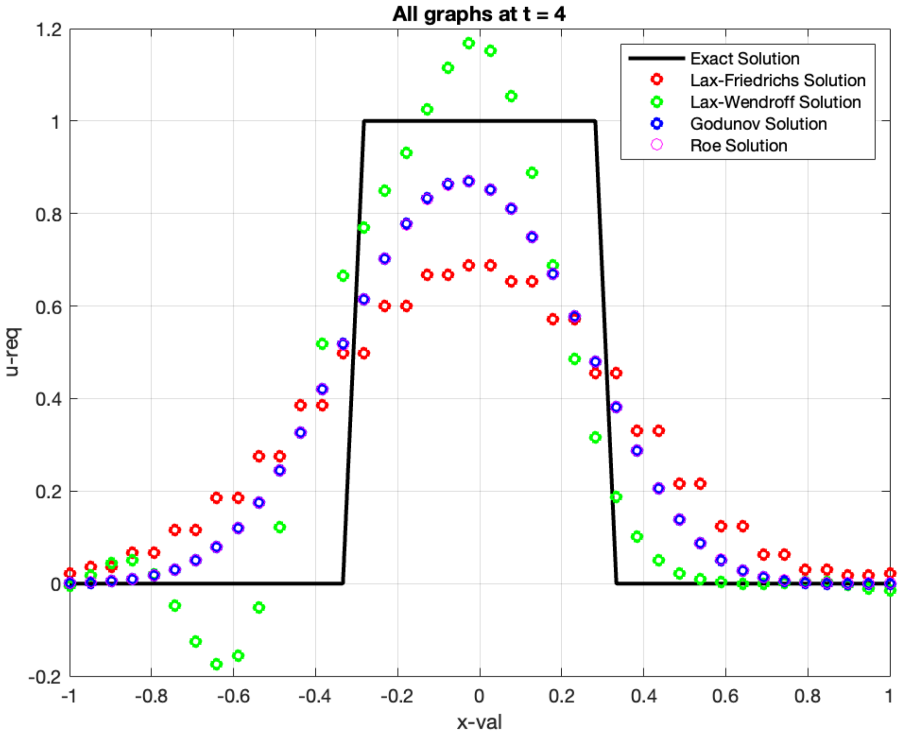
3.2.1 Godunov's Methods



3.2.2 Roe's Method



3.2.3 Comparison of All Four Methods



3.2.4 Summary of results

- For both Godunov's and Roe's method, the contacts are extremely smeared and the square wave's peak has reduced. But, the solution is symmetric, properly located, and free of spurious overshoots or oscillations.
- For LF, the contacts are extremely smeared and the peak of the square wave has been reduced. However, the solution seems symmetric and properly located. For LW, the solution overshoots and undershoots the exact solution. Also, the Lax-Wendroff method smears the contacts far less than the Lax-Friedrichs method. For Roe and Godunov, the solutions are exactly the same. They have thinner tails and are symmentric.

3.3 Test Case 3

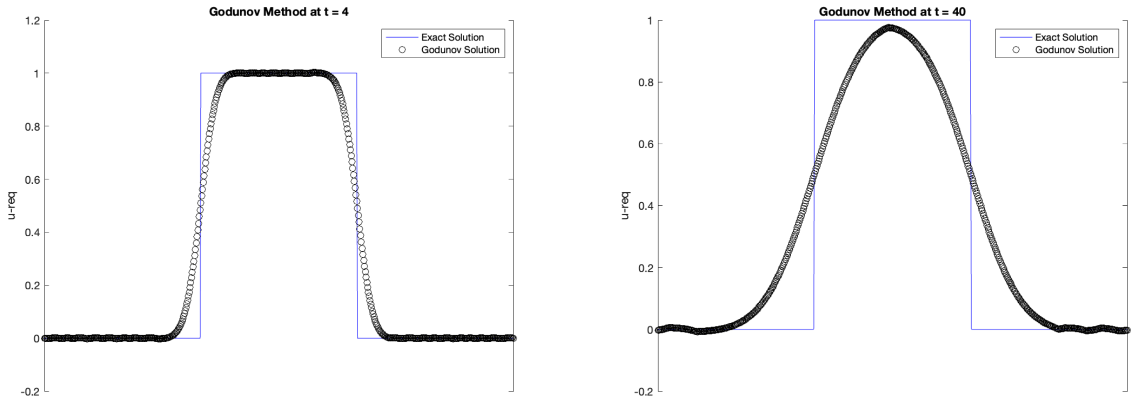
Here we need to find the approximate solution at $u(x, 4)$ and $u(x, 40)$ of the differential equation :

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

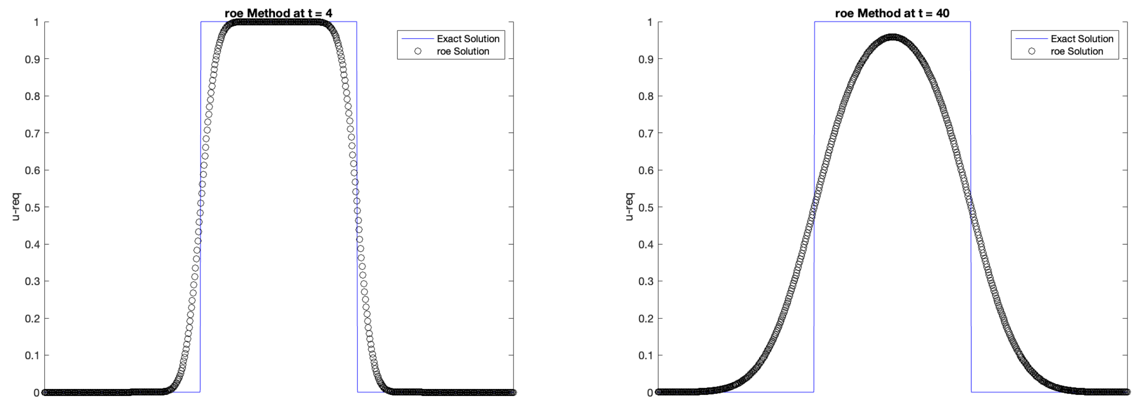
with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

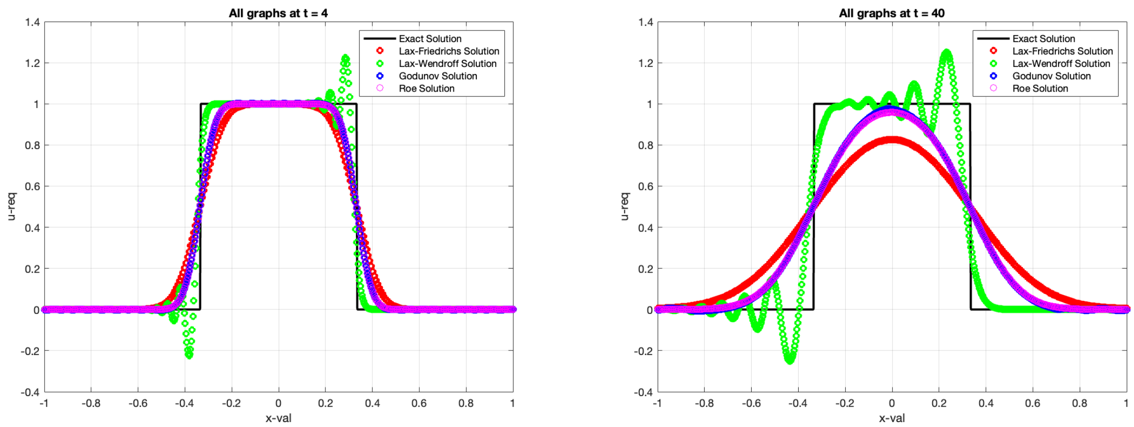
3.3.1 Godunov's Methods



3.3.2 Roe's Method



3.3.3 Comparison of All Four Methods



3.3.4 Summary of results

- For both Godunov's and Roe's method, increasing the number of grid points, and decreasing Δx and Δt, dramatically improve the approximation for u(x, 4),
- Here again, Godunov and Roe both show the same solution, which lies in between the solutions given by LF and LW methods.

3.4 Test Case 4

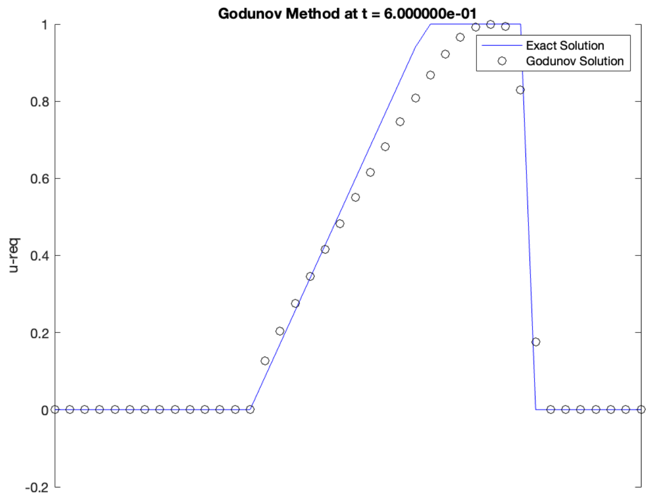
Here we need to find the approximate solution at $u(x, 0.6)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

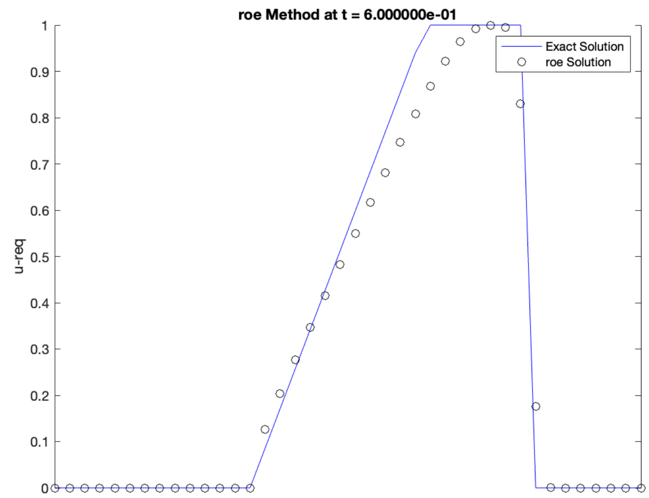
with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

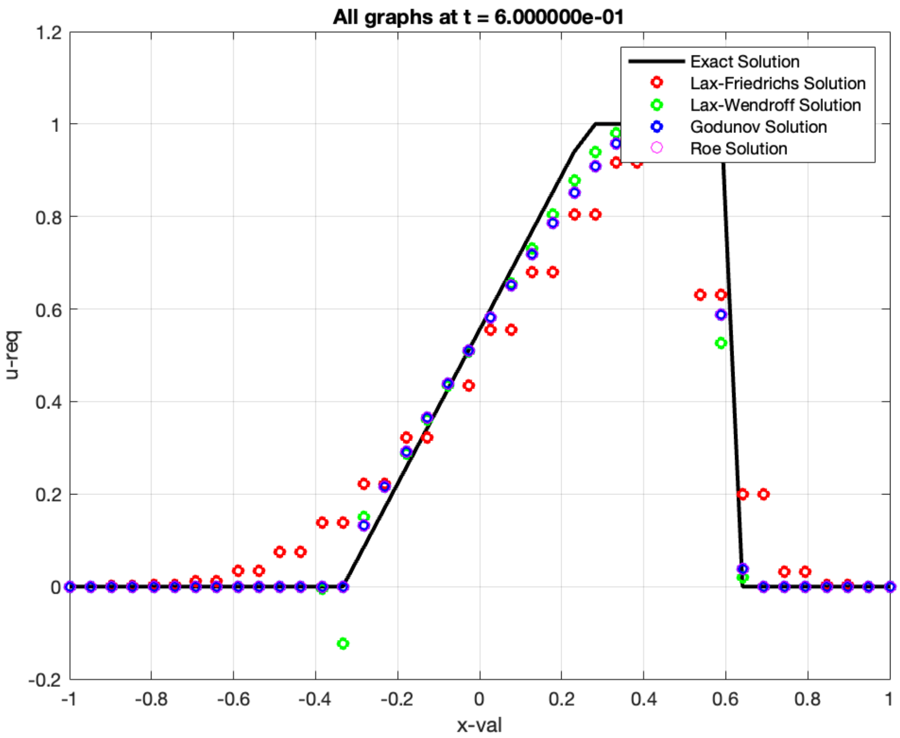
3.4.1 Godunov's Methods



3.4.2 Roe's Method



3.4.3 Comparison of All Four Methods



3.4.4 Summary of results

- For both Godunov's and Roe's method, the shock is captured across only two grid points and without any spurious overshoots or oscillations. The corner at the head of the expansion fan has been slightly rounded off.
- Godunov and Roe methods give the same solutions. Solutions given by all the four methods are very close to one another for this case.

3.5 Test Case 5

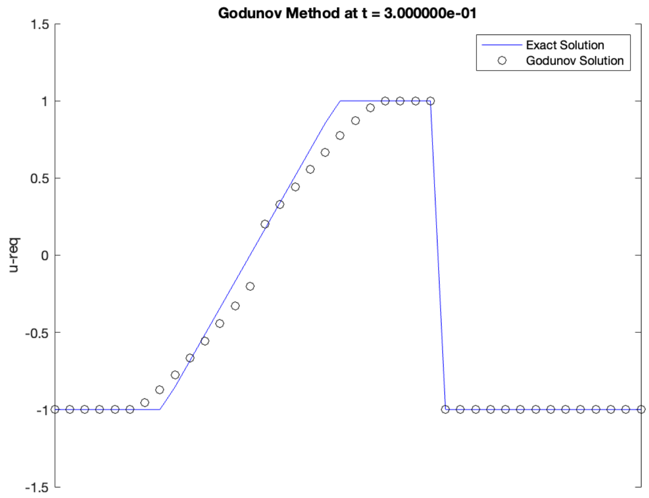
Here we need to find the approximate solution at $u(x, 0.3)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

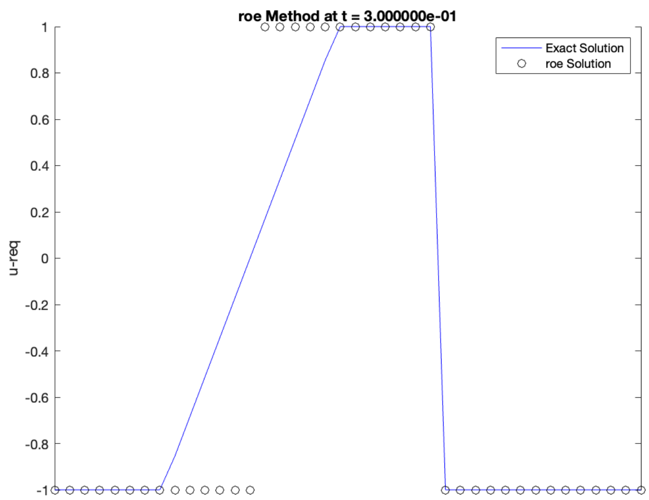
with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ -1 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

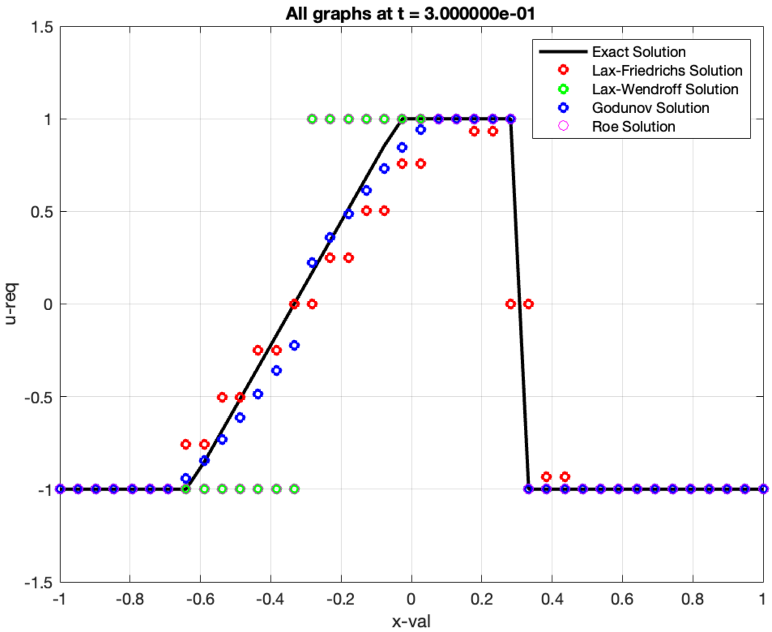
3.5.1 Godunov's Methods



3.5.2 Roe's Method



3.5.3 Comparison of All Four Methods



3.5.4 Summary of results

- Godunov's method captures the steady shock perfectly. It partially captures the expansion fan.
- In Roe's method, the steady shock is captured perfectly. Unfortunately, like the Lax-Wendroff method, Roe's method fails to alter the initial conditions in any way, which is a total disaster for the expansion. Roe's approximate Riemann solver cannot capture the finite spread of expansion fans, and this defect carries over to Roe's first-order upwind method.
- As another way to view the situation, Godunov's first-order method has somewhat inadequate artificial viscosity at expansive sonic points, whereas Roe's first-order upwind method has drastically inadequate artificial viscosity at expansive sonic points.
- This is the most interesting of the cases. Here the Godunov and Roe solutions are not the same. In fact, the Roe solution overlaps with the LW solution here. Both of them fail to capture the fan like behaviour of the exact solution and give a step function instead. LF and Godunov give much more accurate solutions.