MTL712 - Assignment Report - 03

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1 Introduction

The heat equation is a widely studied parabolic partial differential equation (PDE) that describes the distribution of heat (or variation in temperature) in a given region over time. The 1D heat equation can be written as:

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t),$$

where $u(x,t)$ represents the temperature at a point x and time t, and α^2 is the thermal diffusivity constant.

This report discusses three numerical methods to approximate the solution of the heat equation:

1. Forward-Time Central-Space (FTCS) method
2. Backward-Time Central-Space (BTCS) method
3. Crank-Nicolson method

1.1 Forward-Time Central-Space (FTCS) Method

The FTCS method is an explicit scheme used to solve the heat equation. It uses a forward difference approximation for the time derivative and a central difference approximation for the spatial second derivative.

The finite difference approximations are:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{k},$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2},$$

where k is the time step and h is the spatial step. Substituting these into the heat equation gives the recursion relation:

$$u_i^{n+1} = u_i^n + \frac{\alpha^2 k}{h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right).$$

This method is conditionally stable, requiring that the time step satisfies the stability criterion:

$$k \leq \frac{h^2}{2\alpha^2}.$$

1.2 Backward-Time Central-Space (BTCS) Method

The BTCS method is an implicit scheme that uses backward differences for time and central differences for space. It is unconditionally stable, but requires solving a system of linear equations at each time step.

The finite difference approximations are:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{k},$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2}.$$

Substituting these into the heat equation gives the implicit relation:

$$u_i^{n+1} - \frac{\alpha^2 k}{h^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) = u_i^n.$$

This can be written as a tridiagonal system $A\mathbf{u}^{n+1} = \mathbf{u}^n$, where A is the coefficient matrix that needs to be inverted at each time step.

1.3 Crank-Nicolson Method

The Crank-Nicolson method is a semi-implicit method that averages the FTCS and BTCS methods. It is unconditionally stable and second-order accurate in both time and space.

The Crank-Nicolson method uses the following approximations:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{k},$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{1}{2}\left(\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}\right).$$

Substituting these into the heat equation gives the following relation:

$$u_i^{n+1} - \frac{\alpha^2 k}{2h^2}\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right) = u_i^n + \frac{\alpha^2 k}{2h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right).$$

Like the BTCS method, this leads to solving a tridiagonal system at each time step.

2 Problem Solutions

This section presents the solutions to the problems in the assignment, using the FTCS, BTCS, and Crank-Nicolson methods. The graphs compare the numerical solutions with the exact solutions.

Question 1:

Use the Finite-Difference method to approximate the solution to the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < 1, \quad 0 \le t,$$

with boundary conditions:

$$u(0,t) = 0 = u(1,t), \quad 0 \le t,$$

and initial condition:

$$u(x,0) = \sin(\pi x), \quad 0 \le x \le 1,$$

and compare the results at $t = 0.5$ with the exact solution $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

Part (a): $h = 0.1, k = 0.0005$

Observation: As shown in Figure 1, the FTCS method closely approximates the exact solution at $t = 0.5$ with a small time step $k = 0.0005$.

Part (b): $h = 0.1, k = 0.01$

Observation: With a larger time step $k = 0.01$, the FTCS solution becomes unstable, as shown in Figure 2. We see that the prediction from part-(a) is much more accurate. This behavior is expected, as FTCS is conditionally

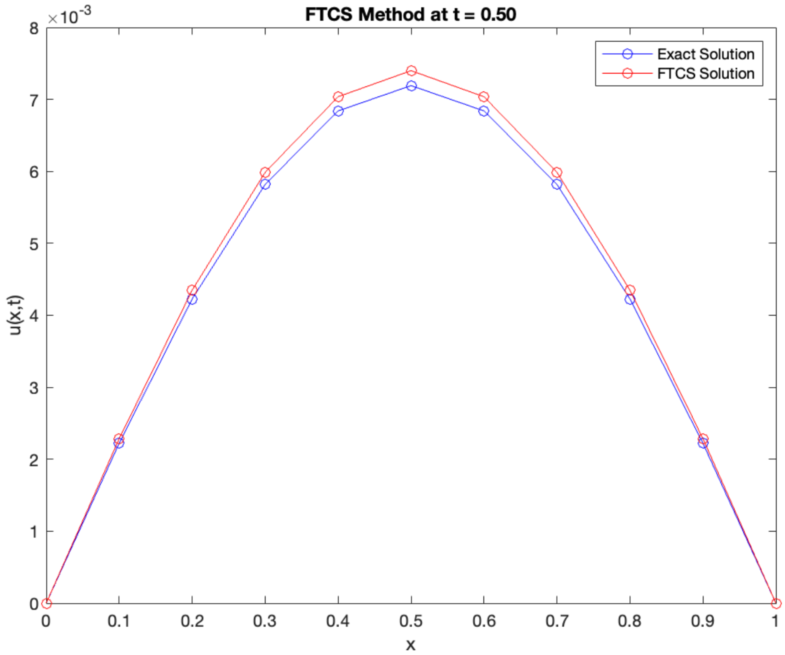

Figure 1: FTCS method solution with $h = 0.1$, $k = 0.0005$.

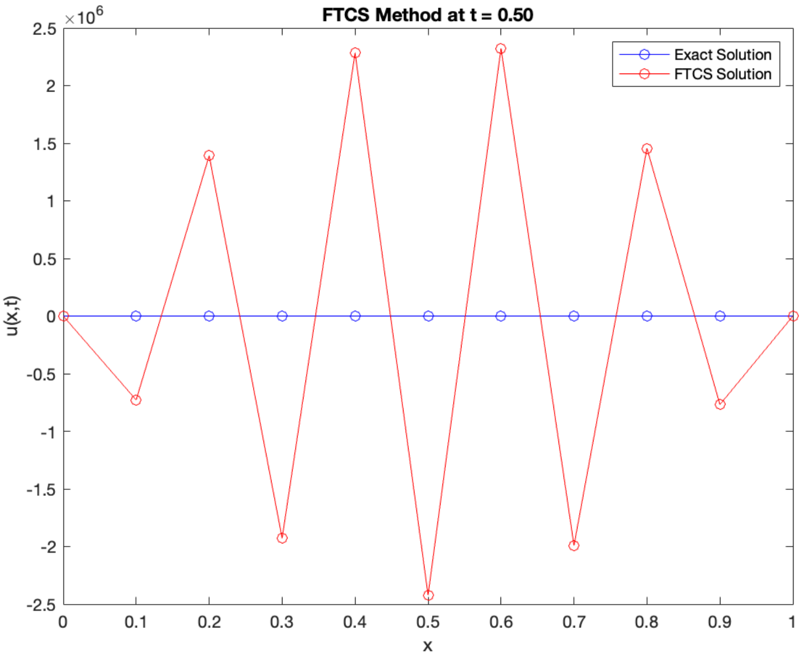

Figure 2: FTCS method solution with $h = 0.1$, $k = 0.01$.

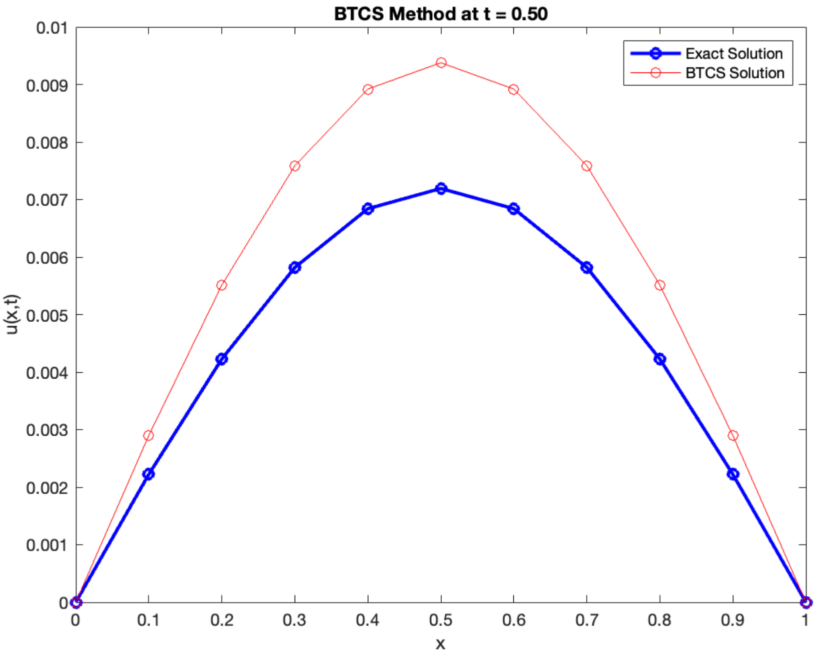

Figure 3: BTCS method solution with $h = 0.1$, $k = 0.01$.

stable and more sensitive to larger time steps.

Question 2:

Use the Backward-Difference method to approximate the solution to the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < 1, \quad 0 \leq t,$$

with the same boundary conditions and initial condition as in Question 1, but with $h = 0.1$ and $k = 0.01$. Compare the results at $t = 0.5$ with the exact solution.

Observation: The BTCS method produces a solution that matches the exact solution at $t = 0.5$ well, even with a larger time step $k = 0.01$, as seen in Figure 3. Since BTCS is unconditionally stable, it handles larger time steps without sacrificing accuracy, as seen in the close agreement between the numerical and exact solutions. We see that the BTCS prediction for this case is much more accurate as compared to the corresponding FTCS prediction (seen in Question 1).

Question 3:

Use the Backward-Difference method with $m = 4$, $T = 0.1$, and $N = 2$ to approximate the solution to the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < 2, \quad 0 \leq t,$$

with boundary conditions:

$$u(0,t) = 0 = u(2,t), \quad 0 \leq t,$$

and initial condition:

$$u(x,0) = \sin\left(\frac{\pi}{2}x\right), \quad 0 \leq x \leq 2,$$

and compare the results at $t = 0.5$ with the exact solution $u(x,t) = e^{-\frac{\pi^2}{4}t} \sin\left(\frac{\pi}{2}x\right)$.

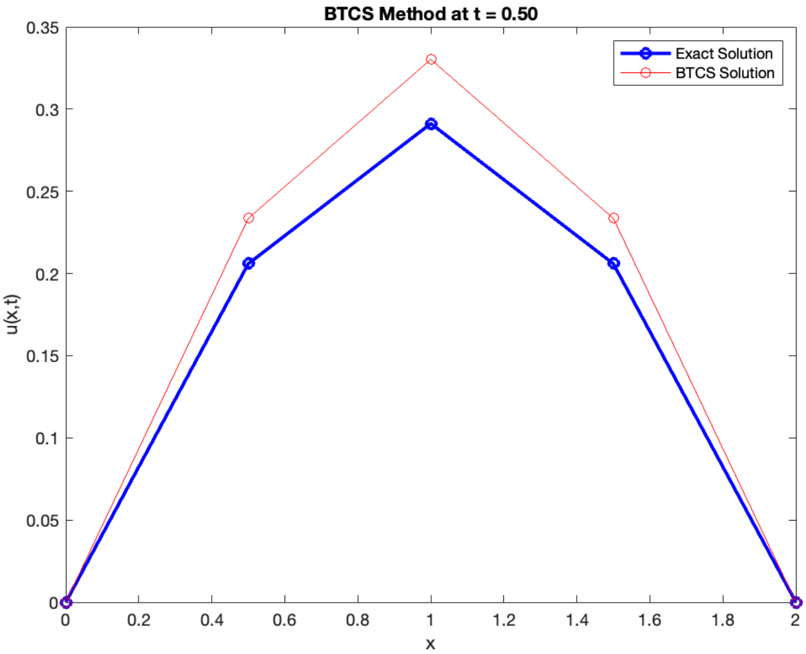

Figure 4: BTCS method solution with $m = 4$, $N = 2$, $T = 0.1$.

Observation: The BTCS method with $m = 4$, $N = 2$, and $T = 0.1$ provides a good approximation for the solution at $t = 0.5$, as shown in Figure 4. The exact solution matches well with the numerical solution, demonstrating the robustness of BTCS for this configuration.

Question 4:

Use the Forward-Difference, Backward-Difference, and Crank-Nicolson methods with to approximate the solution to the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < 2, \quad 0 \le t,$$

with boundary conditions:

$$u(0,t) = 0 = u(2,t), \quad 0 \le t,$$

and initial condition:

$$u(x,0) = \sin(2\pi x), \quad 0 \le x \le 2,$$

and compare the results at $t = 0.5$ with the exact solution $u(x,t) = e^{-4\pi^2 t}\sin(2\pi x)$.

Part-(a): $h = 0.4$**,** $k = 0.1$

Observation: Figure 5 shows the FTCS solution with $h = 0.4$ and $k = 0.1$. The solution is unstable, as FTCS requires the time step to satisfy the stability condition $k \le \frac{h^2}{2\alpha^2}$, which is violated here. This results in significant oscillations and inaccuracies in the solution at $t = 0.5$.

Figure 6 demonstrates the BTCS method's performance with the same parameters. Even though BTCS is unconditionally stable, the numerical solution is unstable in this case. However, it has a lower magnitude of deviation as compared to FTCS.

Figure 7 shows the Crank-Nicolson method's solution. Similar to BTCS, it remains unstable. However, the approximation here has deviations of lesser magnitude as compared to the previous cases.

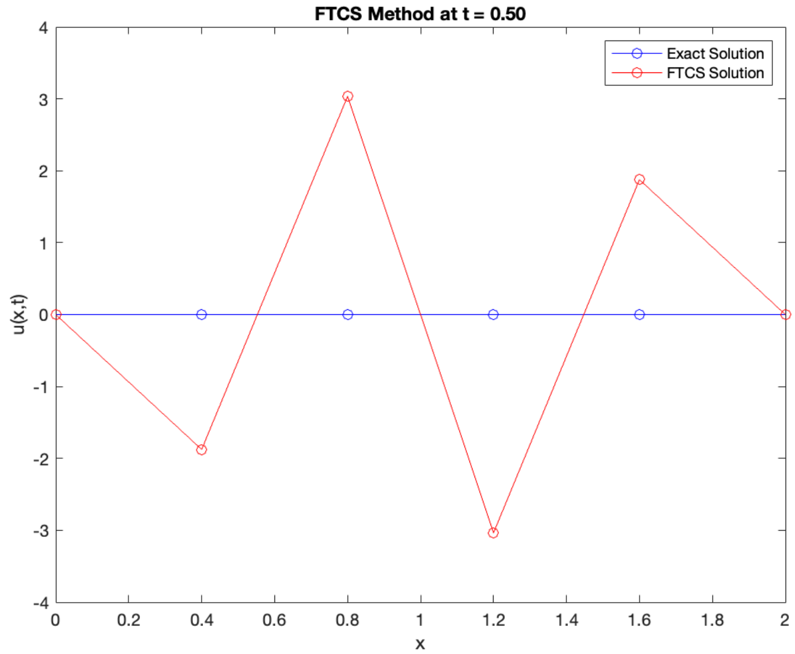

Figure 5: FTCS method solution with $h = 0.4$, $k = 0.1$.

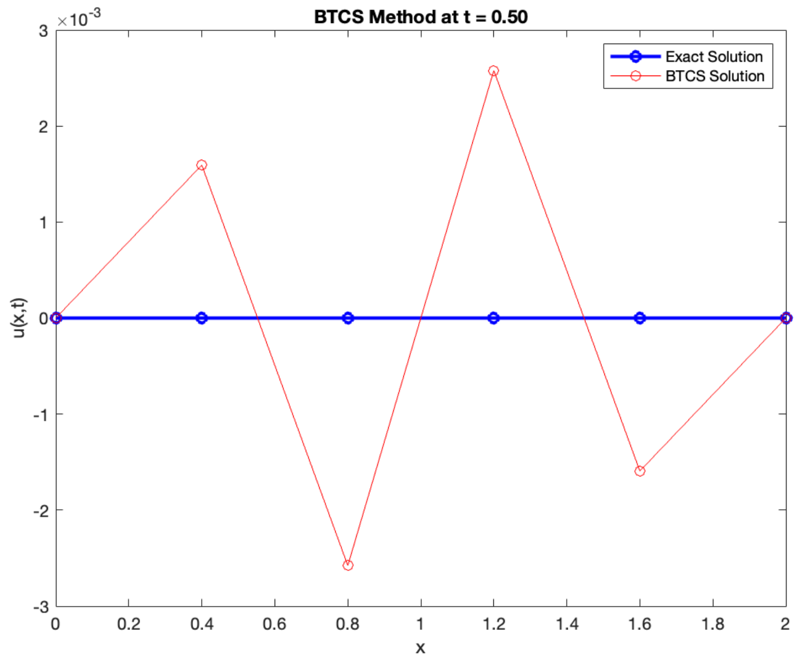

Figure 6: BTCS method solution with $h = 0.4$, $k = 0.1$.

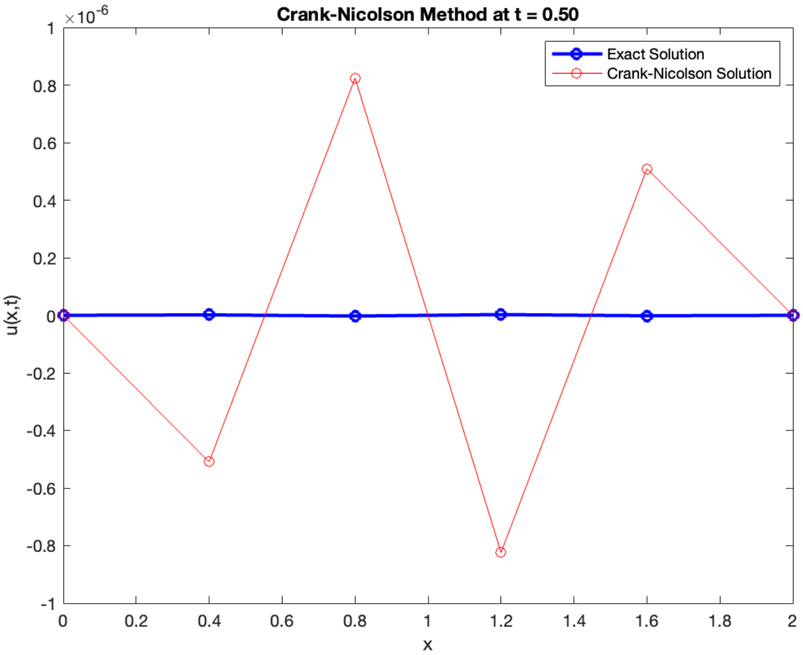

Figure 7: Crank-Nicolson method solution with $h = 0.4$, $k = 0.1$.

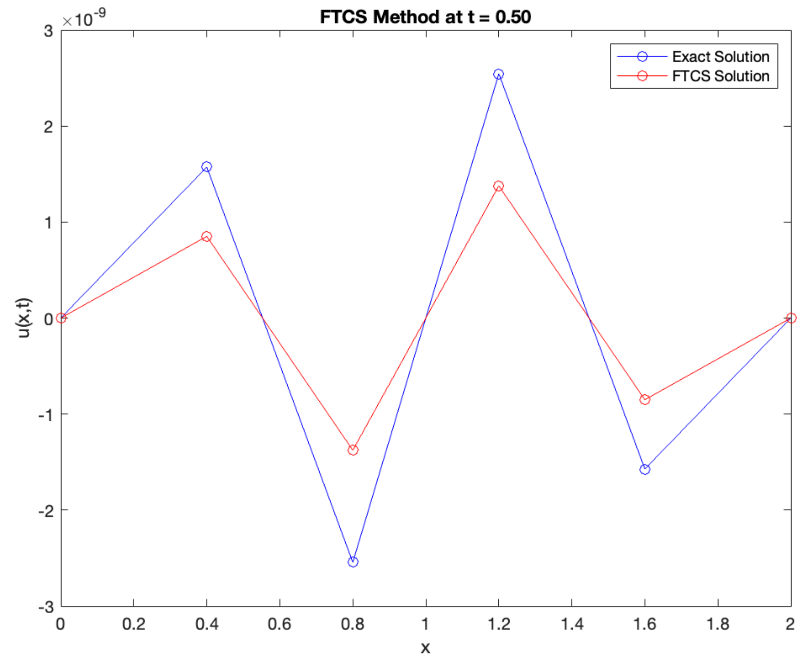

Figure 8: FTCS method solution with $h = 0.4$, $k = 0.05$.

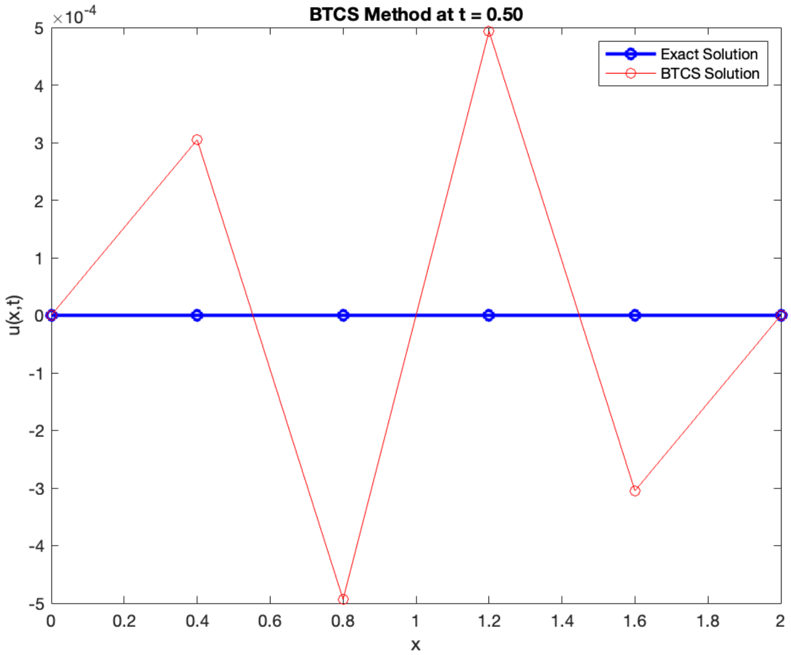

Figure 9: BTCS method solution with $h = 0.4$, $k = 0.05$.

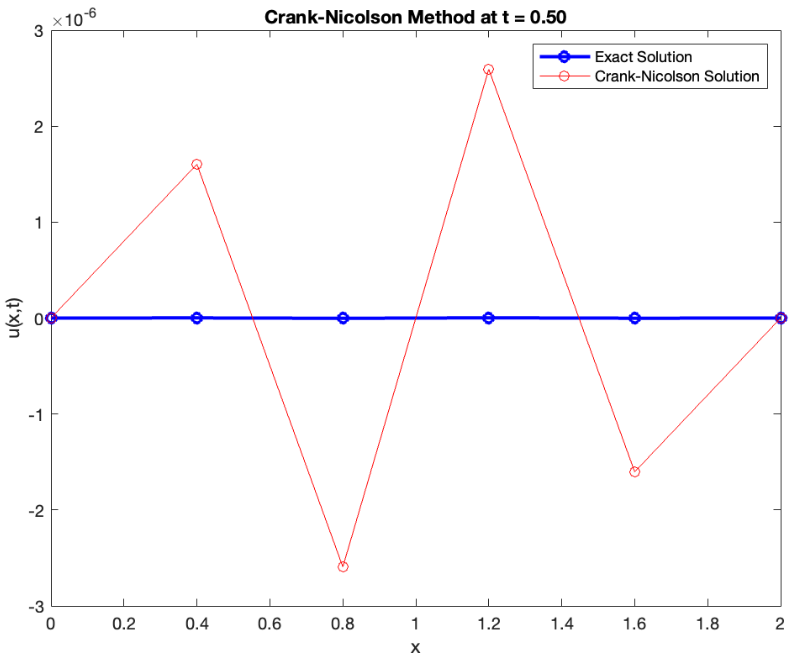

Figure 10: Crank-Nicolson method solution with $h = 0.4$, $k = 0.05$.

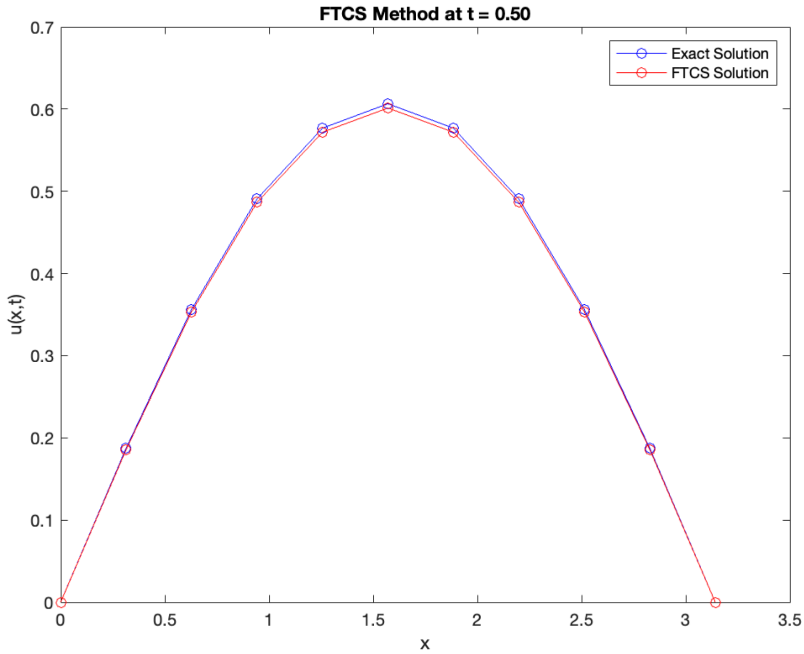

Figure 11: FTCS method solution with $h = \frac{\pi}{10}$, $k = 0.05$.

Part-(b): $h = 0.4$, $k = 0.05$

Observation: As shown in Figure 8, reducing the time step to $k = 0.05$ improves the FTCS method's stability. In fact, we get a more accurate solution using FTCS than we get for the other two cases.

Figure 9 shows that BTCS gives an unstable approximation.

In Figure 10, the Crank-Nicolson method with $k = 0.05$ continues to provide an inaccurate solution. However, the magnitude of deviation is lesser as compared to BTCS.

Question 5:

Use the Forward-Difference, Backward-Difference, and Crank-Nicolson methods with $h = \frac{\pi}{10}$ and $k = 0.05$ to approximate the solution to the heat equation:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t), \quad 0 < x < \pi, \quad 0 \leq t,$$

with boundary conditions:

$$u(0,t) = 0 = u(\pi,t), \quad 0 \leq t,$$

and initial condition:

$$u(x,0) = \sin(x), \quad 0 \leq x \leq \pi,$$

and compare the results at $t = 0.5$ with the exact solution $u(x,t) = e^{-t}\sin(x)$.

Observation: Figure 11 illustrates the FTCS solution with $h = \frac{\pi}{10}$ and $k = 0.05$. The FTCS method exhibits some stability and provides a close approximation to the exact solution.

Figure 12 shows the BTCS solution under the same conditions. The BTCS method remains stable and accurately matches the exact solution at $t = 0.5$. This demonstrates the robustness of the BTCS method, as it can handle larger time steps effectively.

Figure 13 displays the Crank-Nicolson method's solution. As expected, Crank-Nicolson provides a solution that is very close to the exact solution, demonstrating both stability and high accuracy. Its semi-implicit nature allows it to maintain accuracy even with larger spatial steps.

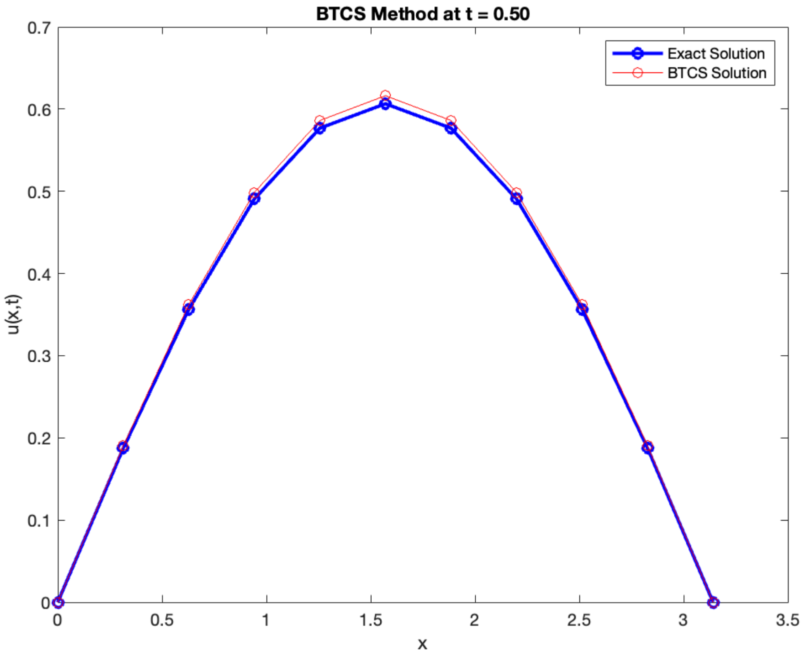

Figure 12: BTCS method solution with $h = \frac{\pi}{10}$, $k = 0.05$.

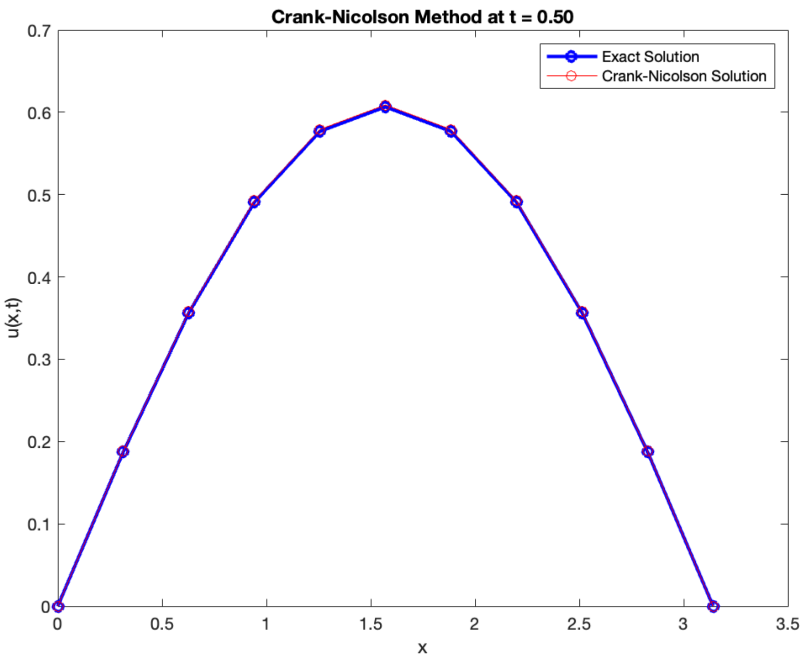

Figure 13: Crank-Nicolson method solution with $h = \frac{\pi}{10}$, $k = 0.05$.