MTL712 - Assignment Report - 01

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1 Problem Statement

Approximate the solution of a given partial differential equation (PDE) using the Lax-Friedrichs and Lax-Wendroff numerical methods for a set of 5 predefined test cases.

2 Theory

Consider a PDE of the form:

$$u_t + f(u)_x = 0 \tag{1}$$

where:

- $u = u(x, t)$ is the unknown function of spatial variable x and time t.
- $f(u)$ is a given flux function that depends on u.
- u_t represents the partial derivative of u with respect to time t.
- $f(u)_x$ represents the partial derivative of the flux function $f(u)$ with respect to the spatial variable x.

The Lax-Friedrichs and Lax-Wendroff methods are iterative methods typically used to solve partial differential equations (PDEs) of the above form, given the initial and boundary conditions.

2.1 Lax - Friedrich Method

The iterative update equation for the Lax - Friedrich method is given by:

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\lambda}{2}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right) \tag{2}$$

2.2 Lax - Wendroff Method

The iterative update equation for the Lax - Wendroff method is given by:

$$u_i^{n+1} = u_i^n - \frac{\lambda}{2}\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right) + \frac{\lambda^2}{2}\left(f'(u_i^n)\left(f(u_{i+1}^n) - f(u_{i-1}^n)\right)\right) \tag{3}$$

3 Assignment Test Cases

3.1 Test Case 01

We need to find the approximate solution at $u(x, 30)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x,0) = -sin(\pi x)$$

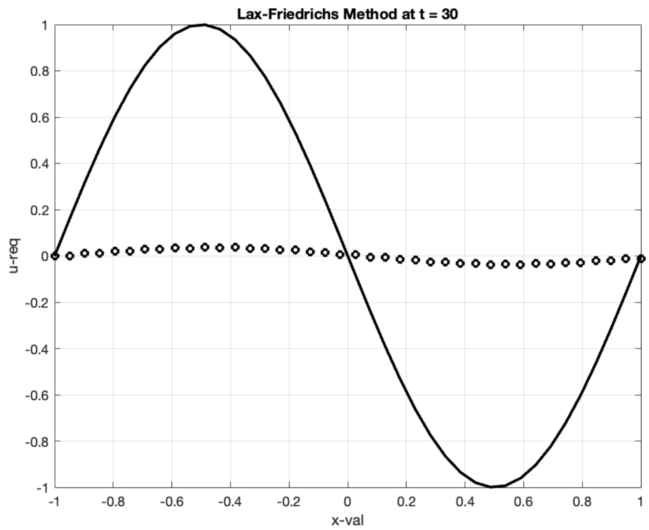
Solving the differential equation using method of characteristics, we find the exact solution to be the following:

$$u(x,30) = -sin(\pi x)$$

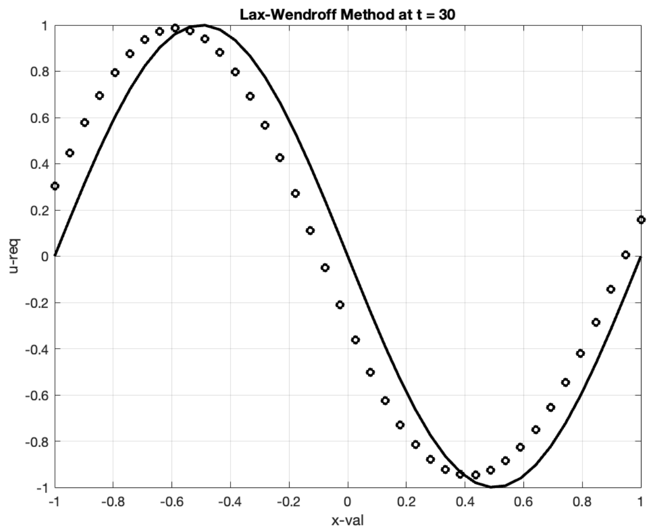
Parameters and Conditions:

- range of $x = [-1, 1]$
- $\lambda = 0.8$
- n = number of divisions in $x = 40$
- periodic boundary condition

3.1.1 Lax - Friedrich Method



3.1.2 Lax - Wendroff Method



3.1.3 Interpretation of Results

For test case 1, we see that Lax-Wendroff method gives a better approximation of the exact solution than Lax-Friedrich method. For LF, the sinusoid's shape is well preserved, and the phase error is relatively small, but the amplitude is much smaller than it should be. For LW, the sinusoid's shape and amplitude are well captured. The only visible error is a slight lagging phase error.

3.2 Test Case 02

Here we need to find the approximate solution at $u(x, 4)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

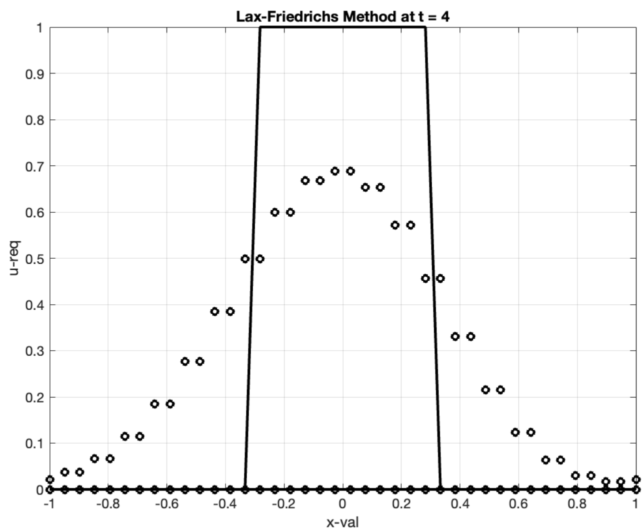
Solving the differential equation using method of characteristics, we find the exact solution to be the following:

$$u(x,4) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

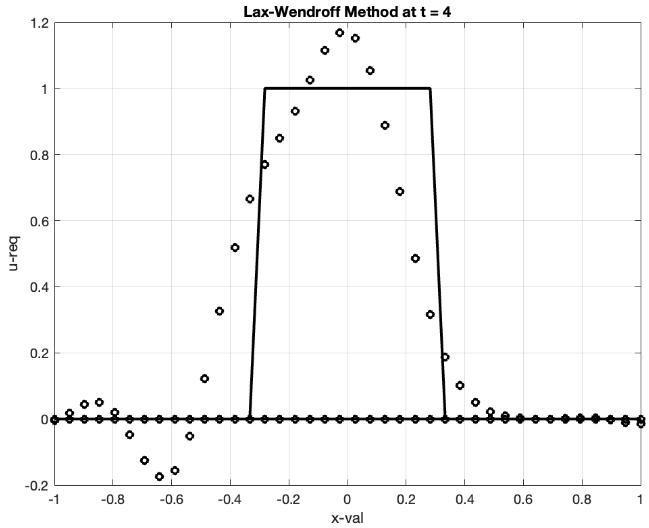
Parameters and Conditions:

- range of $x = [-1, 1]$
- $\lambda = 0.8$
- $n =$ number of divisions in $x = 40$
- periodic boundary condition

3.2.1 Lax - Friedrich Method



3.2.2 Lax - Wendroff Method



3.2.3 Interpretation of Results

For LF, the contacts are extremely smeared and the peak of the square wave has been reduced. However, the solution seems symmetric and properly located. For LW, the solution overshoots and undershoots the exact solution. Also, the Lax-Wendroff method smears the contacts far less than the Lax-Friedrichs method.

3.3 Test Case 3

Here we need to find the approximate solution for $u(x, 4)$, $u(x, 40)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition:

$$u(x, 0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

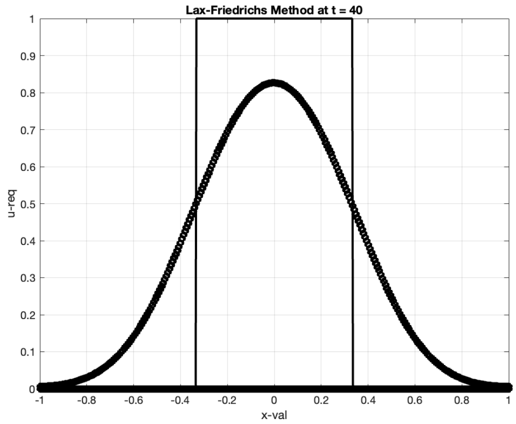
Solving the differential equation using method of characteristics, we find the exact solutions to be the following:

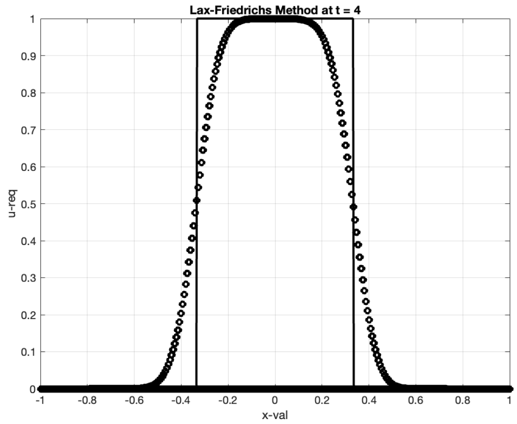
$$u(x, 4) = u(x, 40) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

Parameters and Conditions:

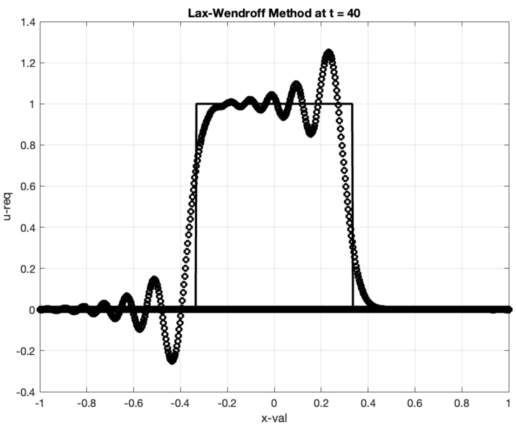
- range of $x = [-1, 1]$
- $\lambda = 0.8$
- $n =$ number of divisions in $x = 600$
- periodic boundary condition

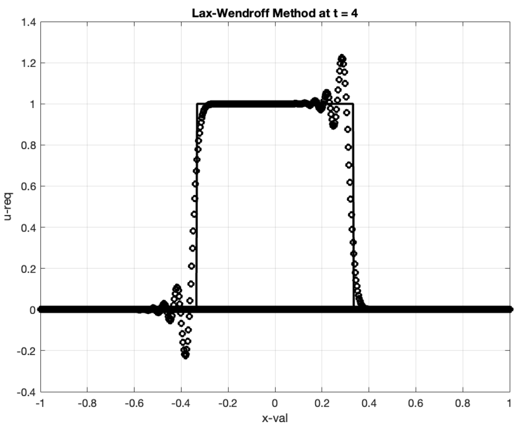
3.3.1 Lax - Friedrich Method





3.3.2 Lax - Wendroff Method





3.3.3 Interpretation of Results

The Lax Wendroff method seems to be giving a much closer approximation to the actual solution than Lax Friedrich method at many points. For LF, the approximations are symmetric and properly located. However, the peak and tails are smeared. For LW method, increasing the number of grid points creates large ringing oscillations to the left of the jump discontinuities. Increasing the final time also increases the ringing oscillations.

3.4 Test Case 4

Here we need to find the approximate solution at $u(x, 0.6)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

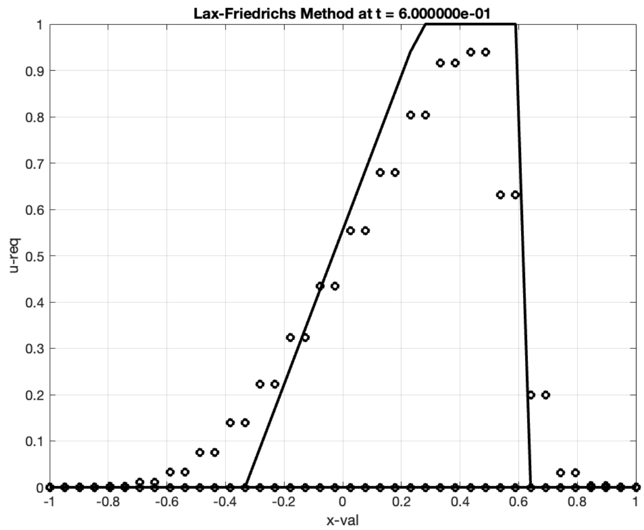
Solving the differential equation using method of characteristics, we find the exact solutions to be the following:

$$u(x,0.6) = \begin{cases} 0 & \text{if } |x| \leq -\frac{1}{3} \\ \frac{x+\frac{1}{3}}{0.6} & \text{if } -\frac{1}{3}+0.6 \geq |x| \geq -\frac{1}{3} \\ 1 & \text{if } \frac{1}{3}+0.6/2 \geq |x| \geq -\frac{1}{3}+0.6 \\ 0 & \text{if } |x| \geq \frac{1}{3}+0.6/2 \end{cases}$$

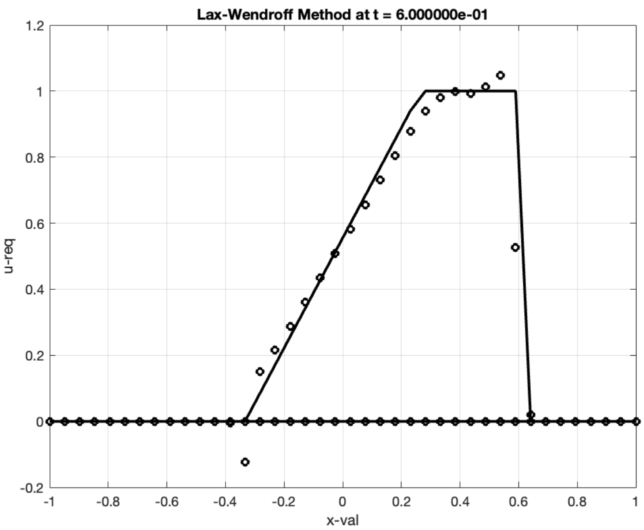
Parameters and Conditions:

- range of $x = [-1, 1]$
- $\lambda = 0.8$
- $n =$ number of divisions in $x = 40$
- periodic boundary condition

3.4.1 Lax - Friedrich Method



3.4.2 Lax - Wendroff Method



3.4.3 Interpretation of Results

For LF, we see that the solution undershoots near the shock and overshoots near the tail of the expansion. For LW, we see that the solution overshoots near the shock and undershoots and overshoots near the tail of the expansion.

3.5 Test Case 5

Here we need to find the approximate solution at $u(x, 0.3)$ of the differential equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{u^2}{2}) = 0$$

with initial condition:

$$u(x,0) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{3} \\ -1 & \text{if } \frac{1}{3} \leq |x| \leq 1 \end{cases}$$

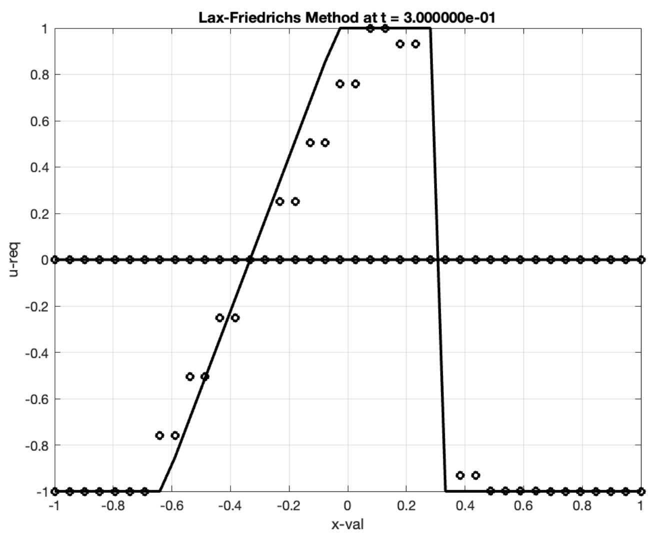
Solving the differential equation using method of characteristics, we find the exact solutions to be the following:

$$u(x,0.3) = \begin{cases} -1 & \text{if } |x| \leq -\frac{1}{3} - 0.3 \\ -1 + \frac{x+\frac{1}{3}+0.3}{0.3} & \text{if } -\frac{1}{3} + 0.3 \geq |x| \geq -\frac{1}{3} - 0.3 \\ 1 & \text{if } \frac{1}{3} \geq |x| \geq -\frac{1}{3} + 0.3 \\ -1 & \text{otherwise} \end{cases}$$

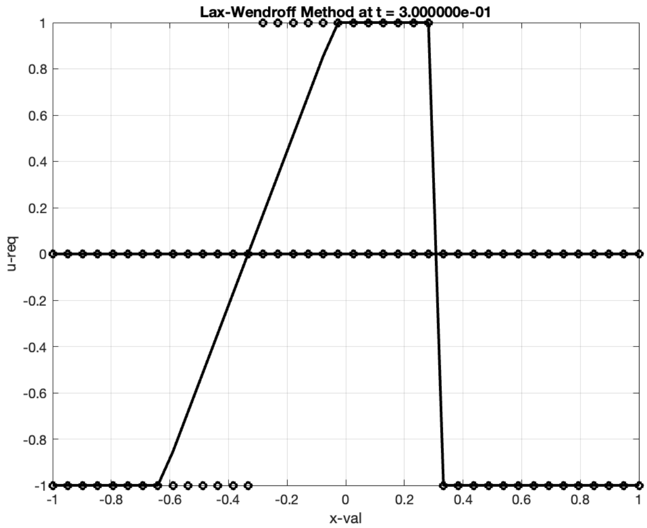
Parameters and Conditions:

- range of $x = [-1, 1]$
- $\lambda = 0.8$
- n = number of divisions in $x = 40$
- periodic boundary condition

3.5.1 Lax - Friedrich Method



3.5.2 Lax - Wendroff Method



3.5.3 Interpretation of Results

The Lax Friedrich method seems to be giving a more accurate approximation than Lax Wendroff method. For LF, the expansion fan is captured well. For LW, the expansion fan is captured as an expansion shock.