MTL712 - Assignment Report - 04

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1 Introduction

This report discusses several numerical methods for solving ordinary differential equations (ODEs). The methods covered include Euler's Method, Modified Euler Method, Taylor Series Method, and the Runge-Kutta Method. Each method has its strengths and is suitable for different types of problems.

1.1 Euler's Method

Euler's method is one of the simplest numerical methods for solving first-order ordinary differential equations. Given an initial value problem of the form:

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0, \tag{1}$$

Euler's method approximates the solution by iterating over discrete time steps. The formula for updating the solution is given by:

$$y_{n+1} = y_n + hf(t_n, y_n), \tag{2}$$

where h is the step size, t_n is the current time, and y_n is the current value of the function. The main drawback of Euler's method is its low accuracy, especially for larger step sizes.

1.2 Modified Euler Method

The Modified Euler method, also known as Heun's method, improves upon Euler's method by averaging the slopes at the beginning and end of the interval. The update formula is:

$$k_1 = f(t_n, y_n), \tag{3}$$

$$k_2 = f(t_n + h, y_n + hk_1), \tag{4}$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2). \tag{5}$$

This method offers better accuracy than the standard Euler method and is still relatively simple to implement.

1.3 Taylor Series Method

The Taylor series method for solving ODEs expands the solution in a Taylor series about the initial point. For a function $y(t)$ that is sufficiently differentiable, we can express it as:

$$y(t) = y(t_0) + (t - t_0)y'(t_0) + \frac{(t - t_0)^2}{2!}y''(t_0) + \dots \tag{6}$$

In practice, the Taylor series method involves truncating the series after a few terms. The more terms included, the more accurate the approximation, but at the cost of increased computational effort.

1.4 Runge-Kutta Methods

Runge-Kutta methods are a family of iterative methods for approximating the solutions of ODEs. The most commonly used is the fourth-order Runge-Kutta method (RK4), which provides a good balance between accuracy and computational efficiency. The update formula for RK4 is:

$$k_1 = f(t_n, y_n), \tag{7}$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \tag{8}$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \tag{9}$$

$$k_4 = f(t_n + h, y_n + hk_3), \tag{10}$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{11}$$

RK4 is widely used due to its accuracy and relatively simple implementation.

Problems

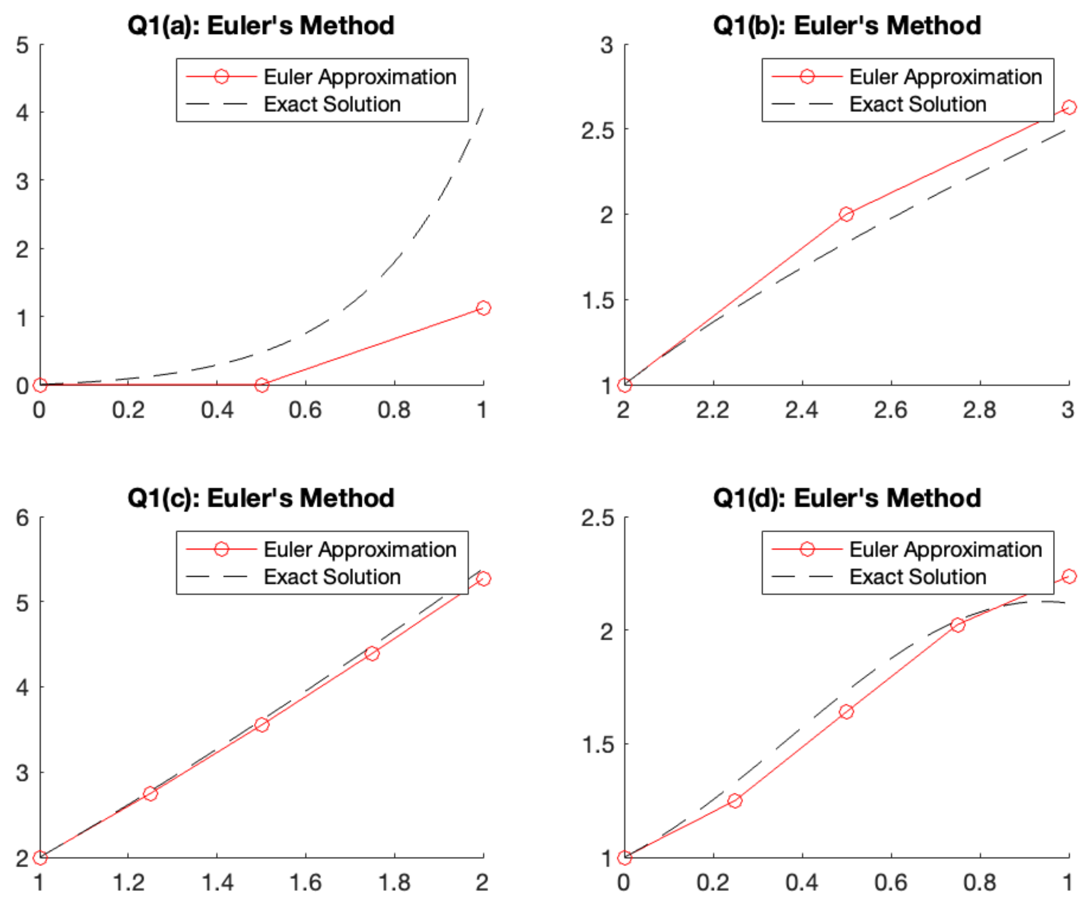
1. Euler's Method

a. $y' = te^{3t} - 2y; \quad 0 \le t \le 1; \quad y(0) = 0$

b. $y' = 1 + (t-y)^2; \quad 2 \le t \le 3; \quad y(2) = 1$

c. $y' = 1 + \frac{y}{t}; \quad 1 \le t \le 2; \quad y(1) = 2$

d. $y' = \cos(2t) + \sin(3t); \quad 0 \le t \le 1; \quad y(0) = 1$



Analysis: We observe that for the first part, the approximation is far from the exact solution. However, for the other parts, Euler's method gives good approximations. It is a general observation that as the value of h is increased, we get better results.

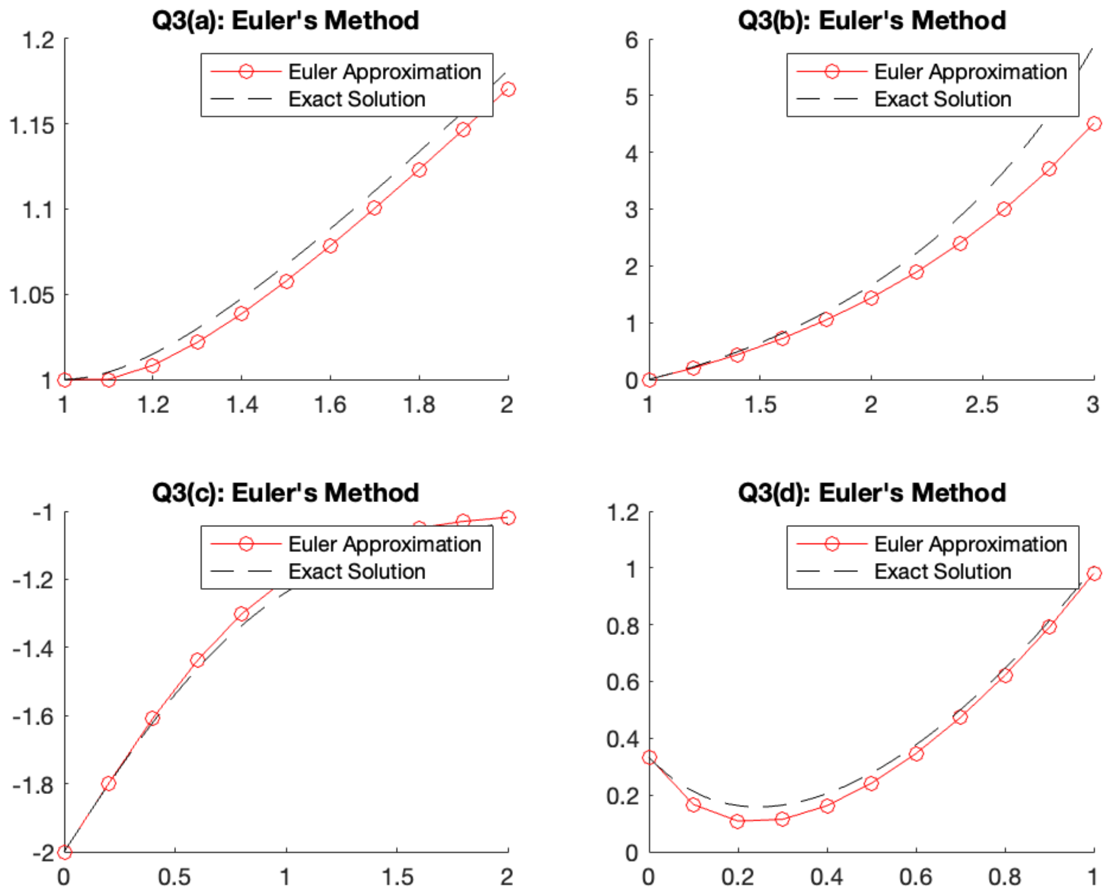
3. Euler's Method

a. $y' = \frac{y}{t} - \left(\frac{y}{t}\right)^2; \quad 1 \le t \le 2; \quad y(1) = 1$

b. $y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2; \quad 1 \le t \le 3; \quad y(1) = 0$

c. $y' = -(y+1)(y+3); \quad 0 \le t \le 2; \quad y(0) = -2$

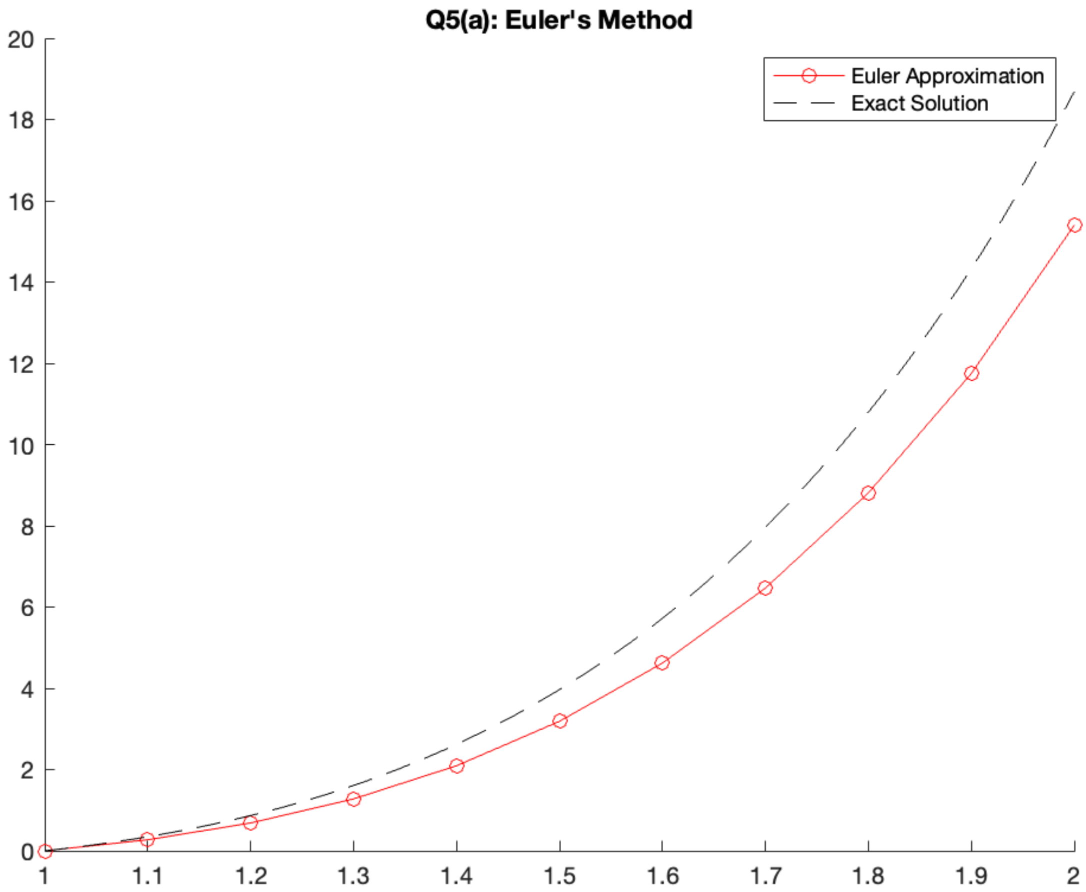
d. $y' = -5y + 5t^2 + 2t; \quad 0 \le t \le 1; \quad y(0) = \frac{1}{3}$



Analysis: Euler's method gives good approximations for all the parts in this question.

5. Euler's Method

a. $y' = \frac{2}{t}y + t^2 e^t; \quad 1 \leq t \leq 2; \quad y(1) = 0$



b. Results of interpolation: Linear interpolation between consecutive points generated by the Euler's method gives following reults:

i. y(1.04) = 0.10873 (exact value = 0.11999)

ii. y(1.55) = 3.90413 (exact value = 4.78864)

iii. y(1.97) = 14.30316 (exact value = 17.27930)

Analysis: Euler's method gives a good approximation near $t = 1$, however the deviation increases as we move towards $t = 2$, as can be seen from the interpolation results as well.

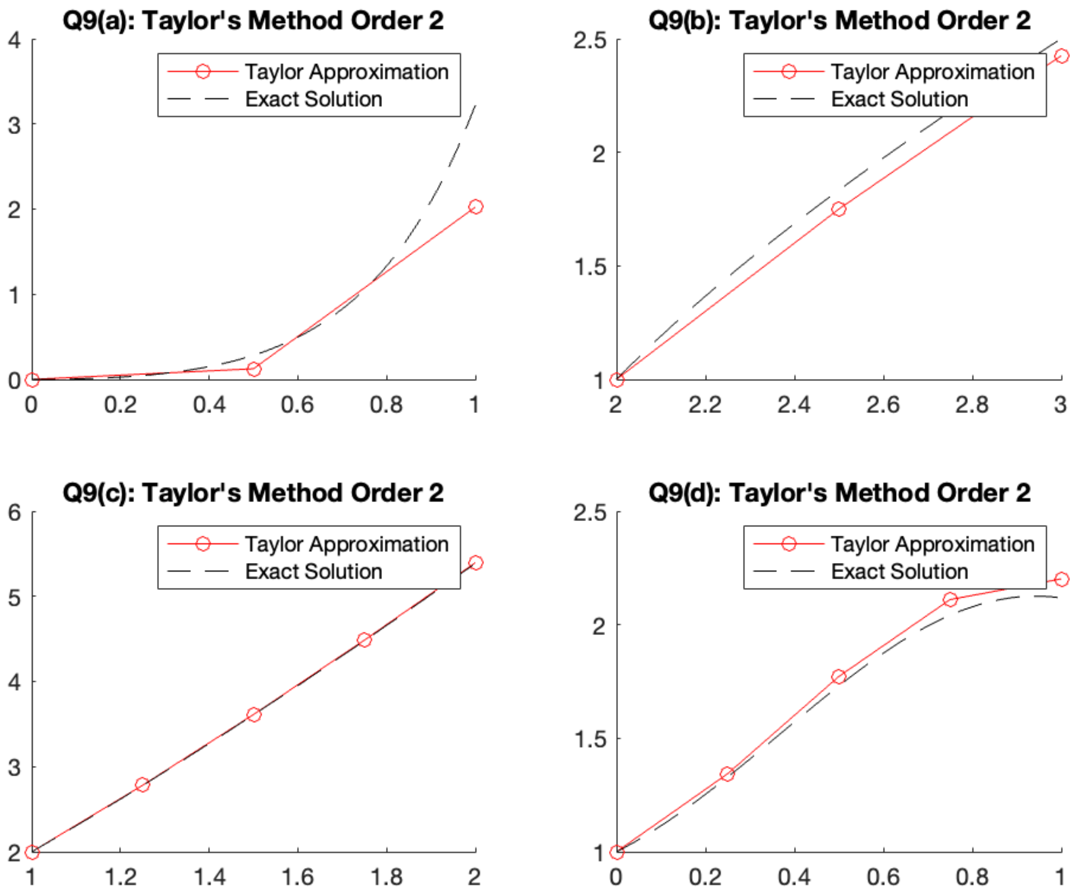
9. Taylor's Method (Order 2)

a. $y' = te^{3t} - 2y; \quad 0 \leq t \leq 1; \quad y(0) = 0$

b. $y' = 1 + (t - y)^2; \quad 2 \leq t \leq 3; \quad y(2) = 1$

c. $y' = 1 + \frac{y}{t}; \quad 1 \leq t \leq 2; \quad y(1) = 2$

d. $y' = \cos(2t) + \sin(3t); \quad 0 \leq t \leq 1; \quad y(0) = 1$



Analysis: Taylor's 2nd order method gives a good approximation for parts (b), (c) and (d). However, for part (a), the approximate solution is not as good. As the value of h decreases, we see an improvement in approximation.

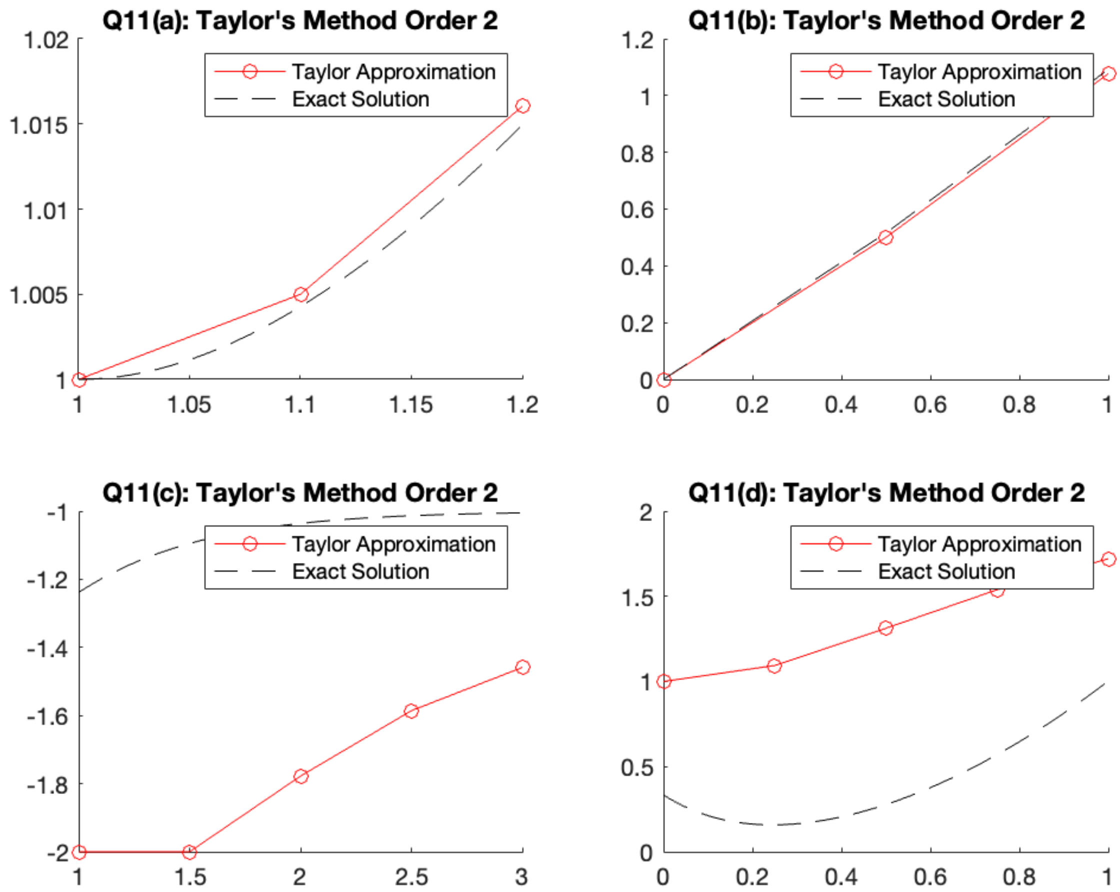
11. Taylor's Method (Order 2)

a. $y' = \frac{y}{t} - \left(\frac{y}{t}\right)^2; \quad 1 \le t \le 1.2; \quad y(1) = 1$

b. $y' = \sin(t) + e^{-t}; \quad 0 \le t \le 1; \quad y(0) = 0$

c. $y' = \frac{y^2+y}{t}; \quad 1 \le t \le 3; \quad y(1) = -2$

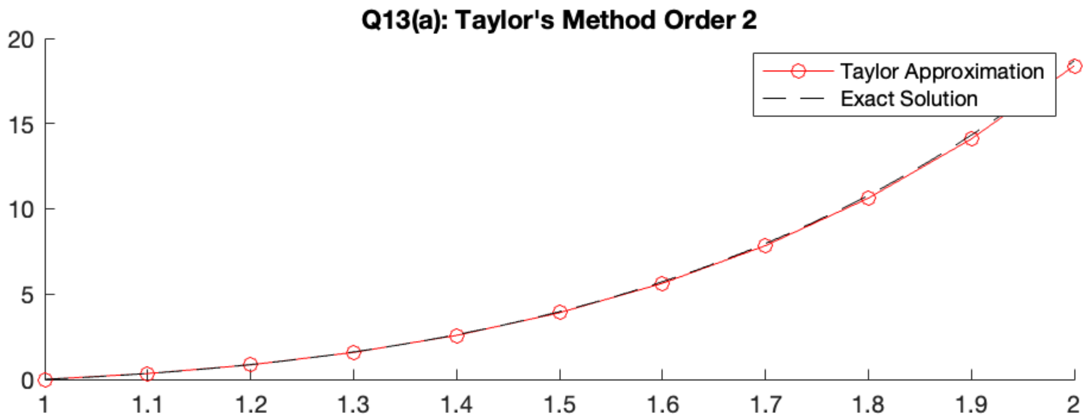
d. $y' = -ty + 4ty^{-1}; \quad 0 \le t \le 1; \quad y(0) = 1$

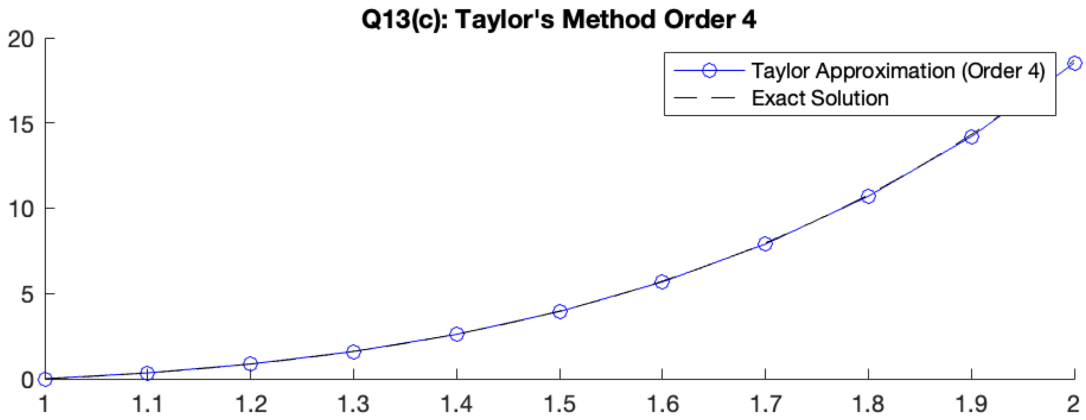


Analysis: The approximations for part (a) and (b) are close despite the high value of h. Parts (c) and (d) show a bad approximation result.

13. Taylor's Method

$y' = \frac{2}{t}y + t^2e^t; \quad 1 \le t \le 2; \quad y(1) = 0$





b. Results of linear interpolation using the 2nd Order method:

i. y(1.04) = 0.13591 (exact value = 0.11999)

ii. y(1.55) = 4.76784 (exact value = 4.78864)

iii. y(1.97) = 17.10101 (exact value = 17.27930)

d. Results of linear interpolation using the 4th Order method:

i. y(1.04) = 0.13750 (exact value = 0.11999)

ii. y(1.55) = 4.80799 (exact value = 4.78864)

iii. y(1.97) = 17.21570 (exact value = 17.27930)

Analysis: For this question, we get very good approximation results as seen from the graphs as well as interpolation results.

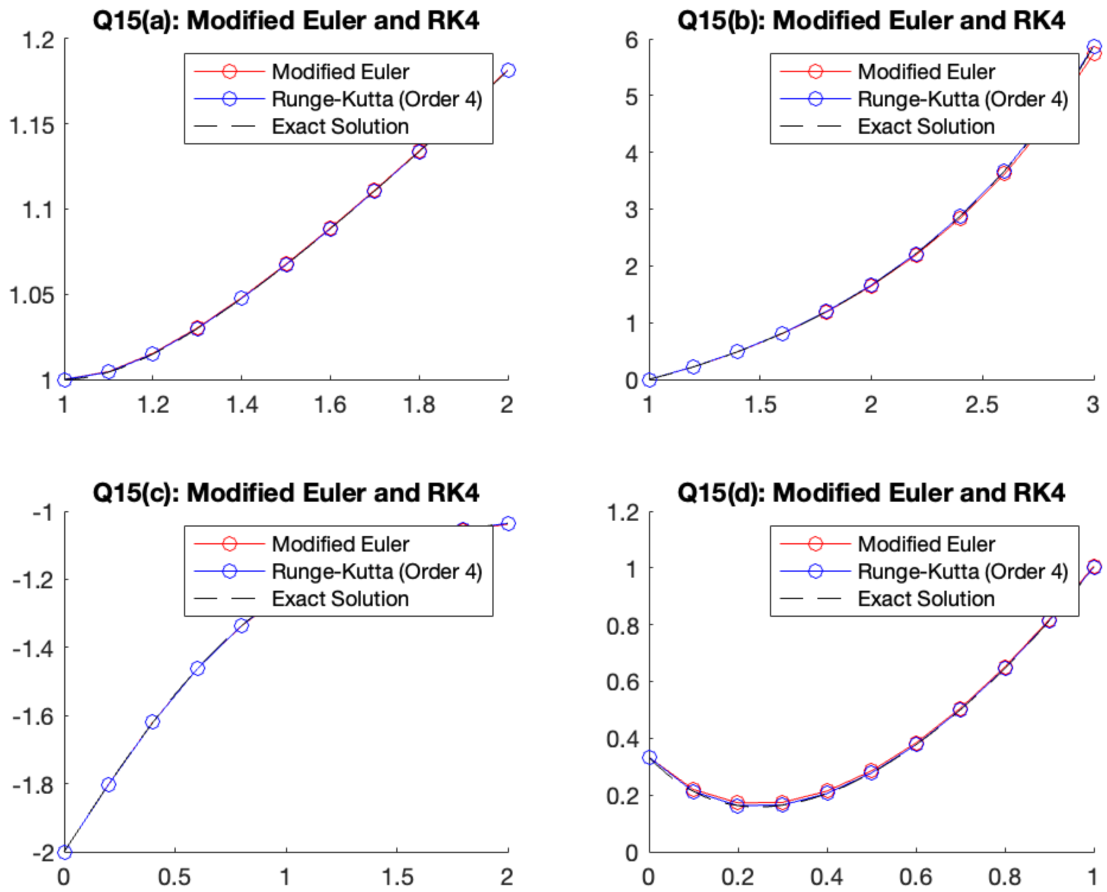
15. Runge-Kutta Method (Order 4)

a. $y' = \frac{y}{t} - \left(\frac{y}{t}\right)^2; \quad 1 \le t \le 2; \quad y(1) = 1$

b. $y' = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2; \quad 1 \le t \le 3; \quad y(1) = 0$

c. $y' = -(y+1)(y+3); \quad 0 \le t \le 2; \quad y(0) = -2$

d. $y' = -5y + 5t^2 + 2t; \quad 0 \le t \le 1; \quad y(0) = \frac{1}{3}$



Analysis: We get very good approximations using both Range-Kutta and Modified Euler's methods on all the four parts of the question.