MTL712 - Assignment Report - 05

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1 Introduction

Boundary Value Problems (BVPs) are differential equations defined on a specified interval, where the solution must satisfy certain conditions (boundary conditions) at the endpoints of the interval. BVPs are common in fields such as physics, engineering, and applied mathematics, where they are used to model steady-state phenomena like heat distribution, beam deflection, and electrostatic potential.

In this report, we explore four numerical methods for solving BVPs: Linear Shooting, Nonlinear Shooting, Linear Finite Difference, and Nonlinear Finite Difference methods. Each method has its own approach, advantages, and limitations, which we discuss in detail.

1.1 Linear Shooting Method

The Linear Shooting Method transforms a BVP into an Initial Value Problem (IVP) by making an initial guess for the derivative at the starting point, integrating forward, and iteratively refining the guess until the solution satisfies the boundary conditions.

Recurrence Relation For a second-order BVP of the form:

$$y'' = f(x, y, y'), \quad y(a) = \alpha, \quad y(b) = \beta$$

we make an initial guess s for $y'(a)$, then use a numerical method (e.g., Runge-Kutta) to integrate forward:

$$y_{n+1} = y_n + hf(x_n, y_n, y'_n)$$

Advantages

- Suitable for both linear and nonlinear problems.
- Simple to implement, leveraging IVP solvers.

Drawbacks

- May fail for sensitive or stiff problems.
- Initial guess adjustments can be time-consuming, especially for nonlinear BVPs.

1.2 Nonlinear Shooting Method

The Nonlinear Shooting Method is similar to the Linear Shooting Method but specifically adapted for nonlinear differential equations. This method often uses Newton's method to iteratively adjust the initial slope guess to meet the boundary conditions.

Recurrence Relation For a BVP with nonlinear terms, we again make an initial guess s for $y'(a)$ and integrate:

$$y_{n+1} = y_n + hf(x_n, y_n, y'_n)$$

Newton's method then adjusts the guess iteratively by updating:

$$s_{\text{new}} = s - \frac{y(b) - \beta}{\frac{d}{ds}y(b; s)}$$

Advantages

- Handles nonlinear problems effectively.
- More robust for nonlinearities when using Newton's method.

Drawbacks

- Computationally intensive due to iterative Newton's method.
- Can be challenging to find convergence for highly nonlinear problems.

1.3 Linear Finite Difference Method

The Linear Finite Difference Method discretizes the interval into a grid and approximates derivatives using finite difference formulas. This converts the differential equation into a system of linear equations, which can be solved to approximate the solution at discrete points.

Recurrence Relation For a second-order BVP:

$$y'' = p(x)y' + q(x)y + r(x)$$

we approximate y''_i and y'_i at grid points as:

$$y''_i \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \quad y'_i \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

Substituting these into the BVP gives a system of equations for the values $y_1, y_2, \ldots, y_N$.

Advantages

- Simple and efficient for linear BVPs.
- Provides good approximations for moderate step sizes.

Drawbacks

- Limited to linear problems; nonlinear problems require modifications.
- Accuracy depends on the grid size; small step sizes increase computational cost.

1.4 Nonlinear Finite Difference Method

The Nonlinear Finite Difference Method is similar to the linear finite difference method but is adapted for nonlinear BVPs. It iteratively solves a system of nonlinear equations, often using Newton's method, to ensure that the solution satisfies the BVP conditions.

Recurrence Relation For a BVP with a nonlinear term, we set up a nonlinear system of equations by discretizing:

$$y_i'' \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \quad y_i' \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

and solving iteratively for $y_1, y_2, \ldots, y_N$.

Advantages

- Handles both linear and nonlinear BVPs.
- Effective for nonlinear problems where finite difference approximations are feasible.

Drawbacks

- Computationally demanding due to iterative solving.
- Requires careful tuning of parameters to ensure convergence, especially for strongly nonlinear problems.

This introduction provides an overview of BVPs and the four methods used for their numerical solution, along with their advantages and limitations. The rest of the report applies these methods to specific BVPs and compares the numerical solutions with exact solutions to evaluate accuracy and performance.

Problems

Question 1

BVP:

$$y'' = -\frac{2}{x} + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x}, \quad 1 \le x \le 2, \quad y(1) = 1, \quad y(2) = 2$$

Exact solution:

$$y(x) = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x)$$

where:

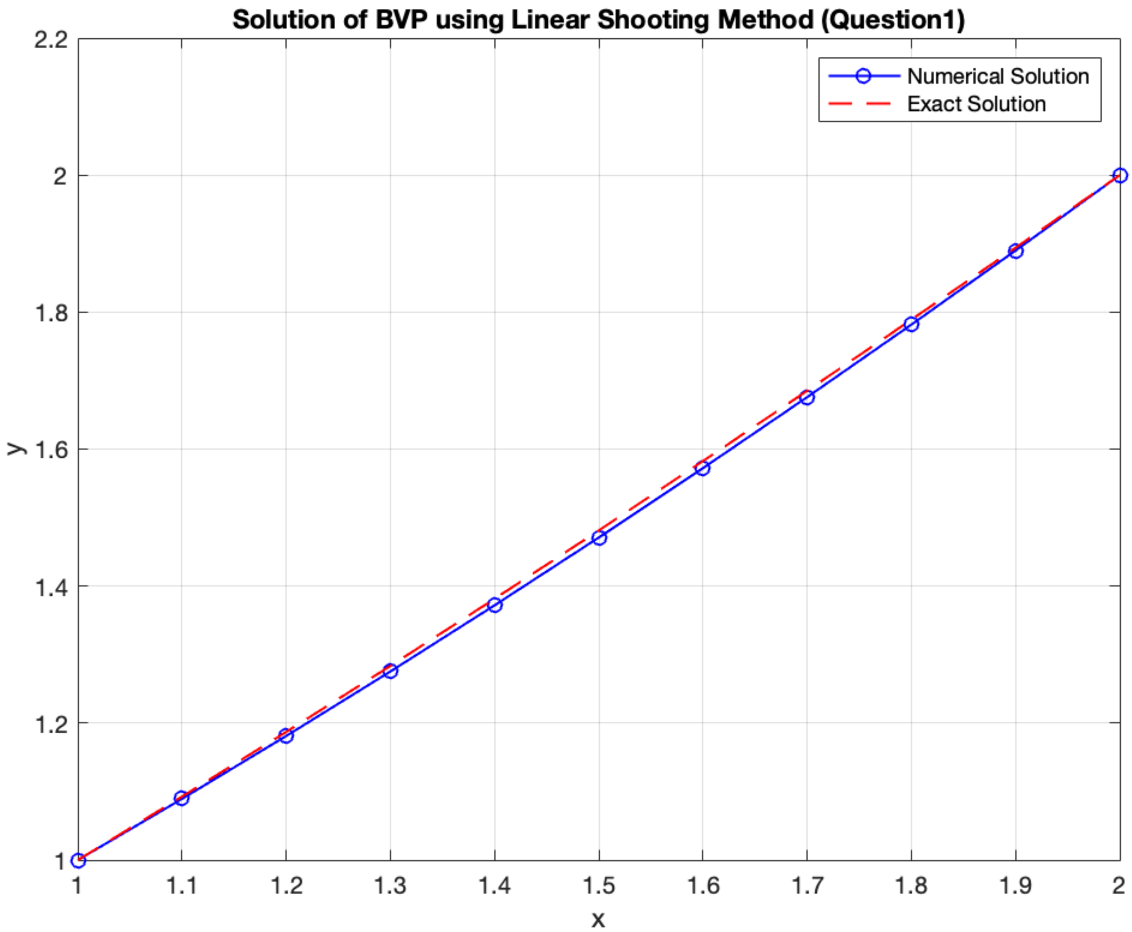
$$c_2 = \frac{1}{70}\left[8 - 12\sin(\ln 2) - 4\cos(\ln 2)\right] = -0.03920701320$$

and:

$$c_1 = \frac{11}{10} - c_2 = 1.1392070132$$

Method to be used: Linear Shooting Technique

Graph:



Analysis: We will apply the Linear Shooting technique with $N = 10$ to solve this BVP.

The approximate solution generated by Linear Shooting closely approximate the exact solution. The deviation is more in the central region as compared to the boundaries (as expected).

Question 2

BVP:

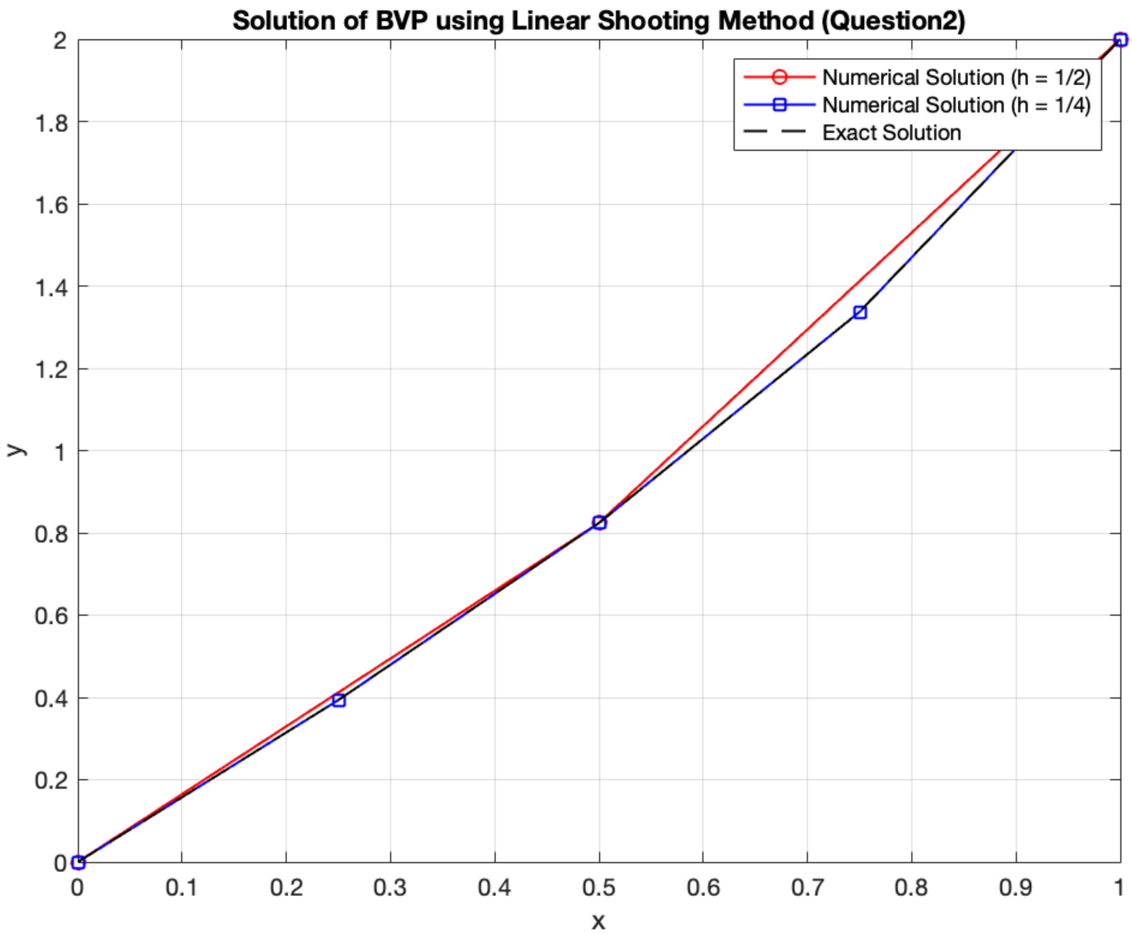
$$y'' = 4(y - x), \quad 0 \leq x \leq 1, \quad y(0) = 0, \quad y(1) = 2$$

Exact solution:

$$y(x) = x + \frac{e^2(e^{2x} - e^{-2x})}{e^4 - 1}$$

Method to be used: Linear Shooting Technique

Graph:



Analysis: The Linear Shooting technique will be applied with two different step sizes: $h = 1/2$ and $h = 1/4$.

The given plot shows the two approximate solutions and the exact solution on the same figure. We observe that the approximate solution generated by $h = 1/4$ is closely overlapping with the exact solution. In general, we observe that increasing the value of h results in better approximations.

Question 3

BVP:

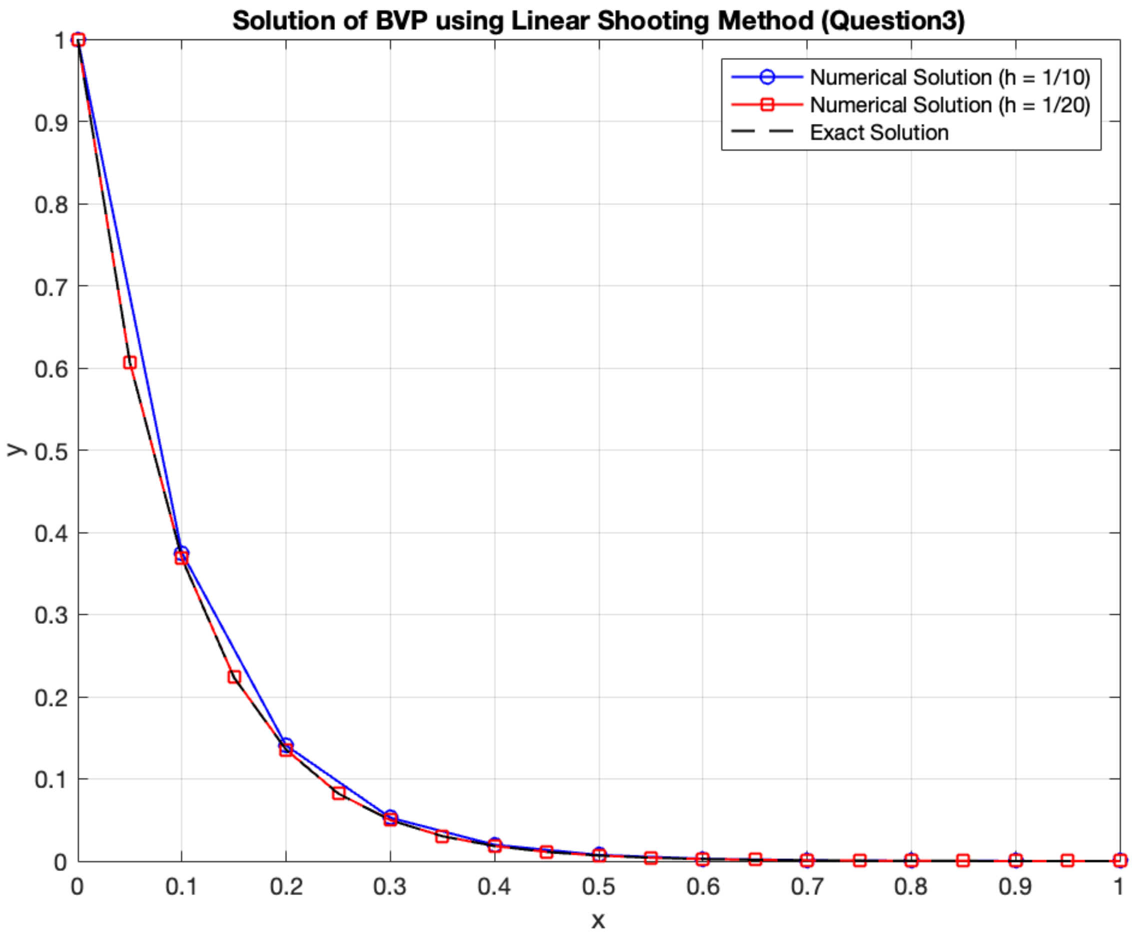
$$y'' = 100y, \quad 0 \le x \le 1, \quad y(0) = 1, \quad y(1) = e^{-10}$$

Exact solution:

$$y(x) = e^{-10x}$$

Method to be used: Linear Shooting Technique

Graph:



Analysis: For this BVP, we will apply the Linear Shooting technique with $h = 1/10$ and $h = 1/20$.

The given plot shows the two approximate solutions and the exact solution on the same figure. We observe that the approximate solution generated by $h = 1/20$ is closely overlapping with the exact solution. In general, we observe that increasing the value of h results in better approximations.

Question 4

BVP:

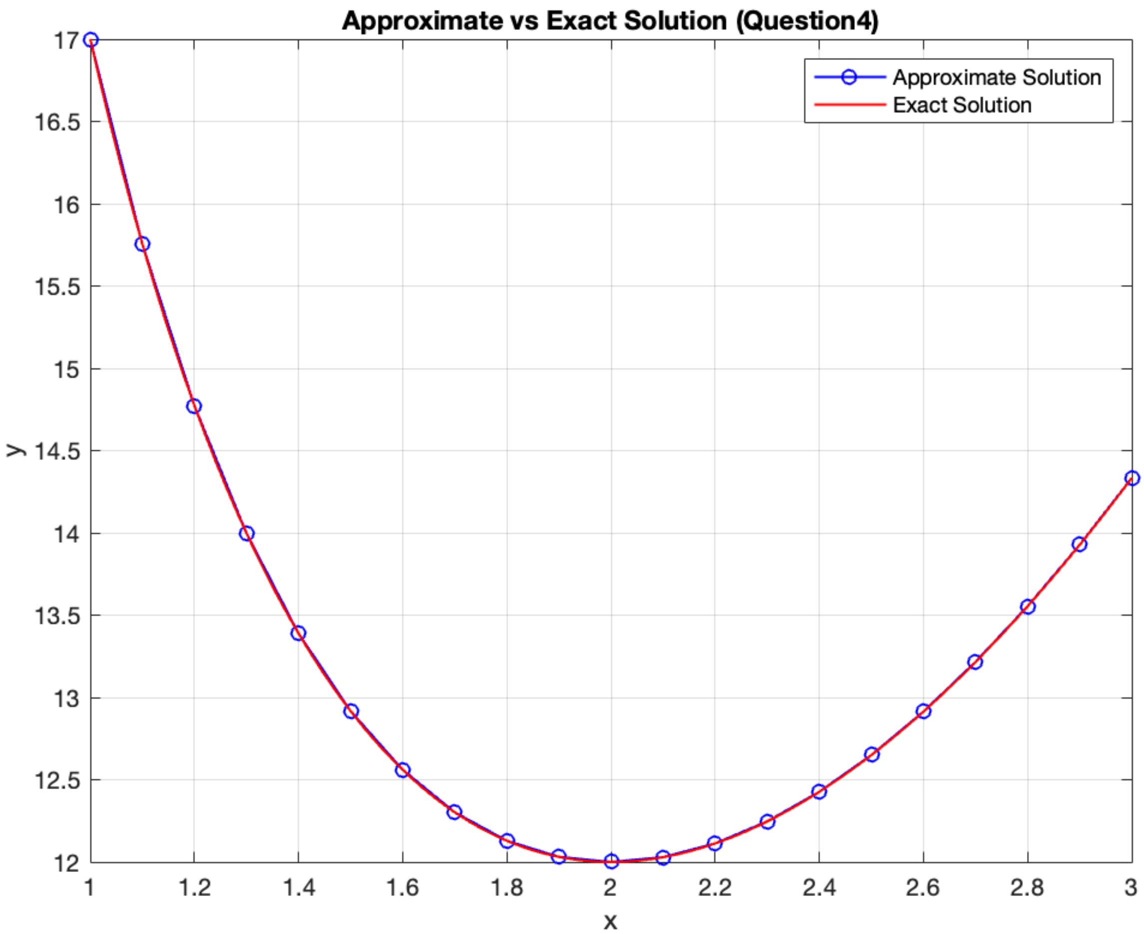
$$y'' = \frac{1}{8}\left[32 + 2x^3 - yy'\right], \quad 1 \le x \le 3, \quad y(1) = 17, \quad y(3) = \frac{43}{3}$$

Exact solution:

$$y(x) = x^2 + \frac{16}{x}$$

Method to be used: Nonlinear Shooting Method with Newton's Method

Graph:



Analysis: We will apply the Nonlinear Shooting Method with Newton's Method for the given BVP with $\text{TOL} = 10^{-5}$.

Question 5

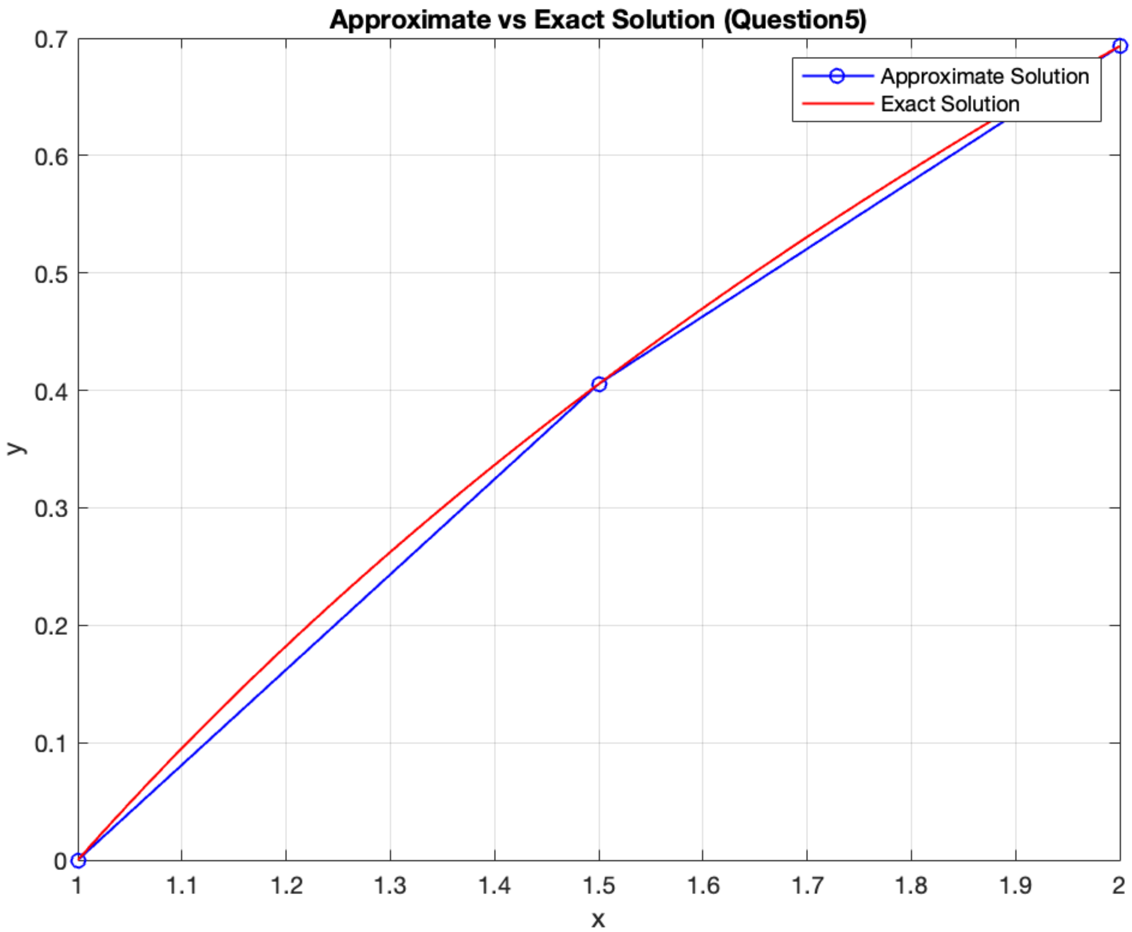
BVP:

$$y'' = -(y')^2 - y + \ln x, \quad 1 \le x \le 2, \quad y(1) = 0, \quad y(2) = \ln 2$$

Exact solution:

$$y(x) = \ln x$$

Method to be used: Nonlinear Shooting Method **Graph:**



Analysis: For this BVP, we will apply the Nonlinear Shooting Method with a step size $h = 0.5$. The numerical solution will be compared with the exact solution $y(x) = \ln x$, and the error will be analyzed.

Question 6

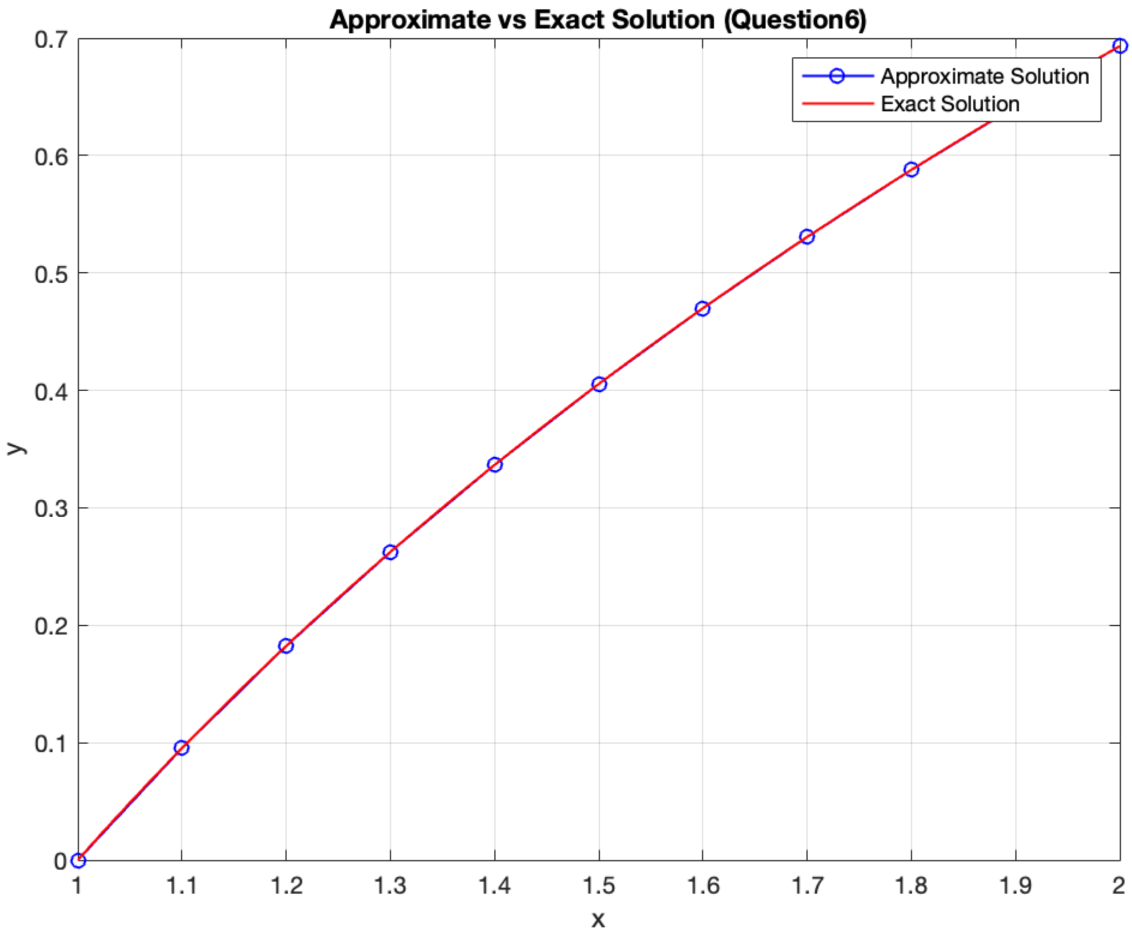
BVP:

$$y'' = -e^{-2y}, \quad 1 \le x \le 2, \quad y(1) = 0, \quad y(2) = \ln 2$$

Exact solution:

$$y(x) = \ln x$$

Method to be used: Nonlinear Shooting Method **Graph:**



Analysis: For this BVP, we will apply the Nonlinear Shooting Method with the specified tolerance TOL $= 10^{-4}$. The numerical solution will be compared to the exact solution $y(x) = \ln x$, and an error analysis will be performed to assess the method's accuracy.

Question 7

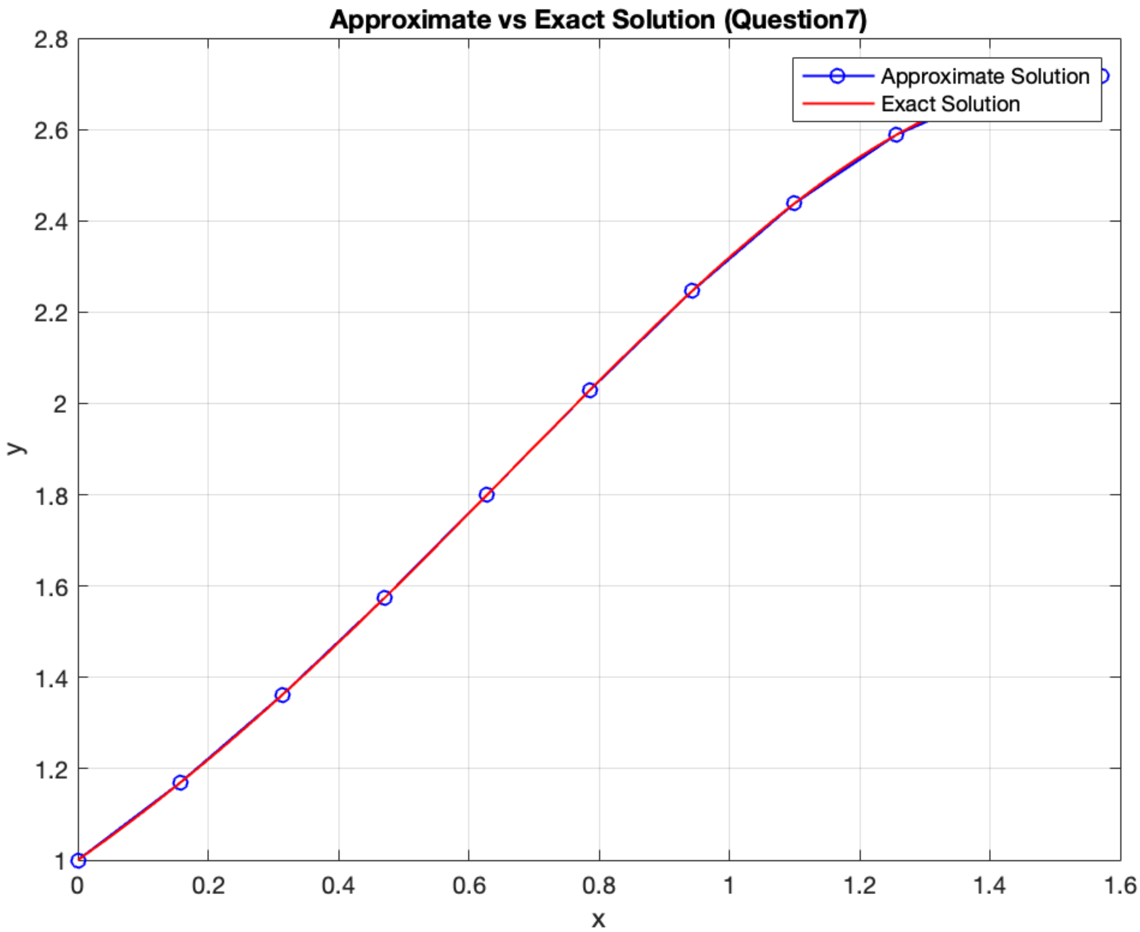
BVP:

$$y'' = y' \cos x - y \ln y, \quad 0 \leq x \leq \frac{\pi}{2}, \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = e$$

Exact solution:

$$y(x) = e^{\sin x}$$

Method to be used: Nonlinear Shooting Method **Graph:**



Analysis: We will apply the Nonlinear Shooting Method to solve this BVP. The solution will be compared with the exact solution $y(x) = e^{\sin x}$, and the accuracy of the method will be assessed.

Question 8

BVP:

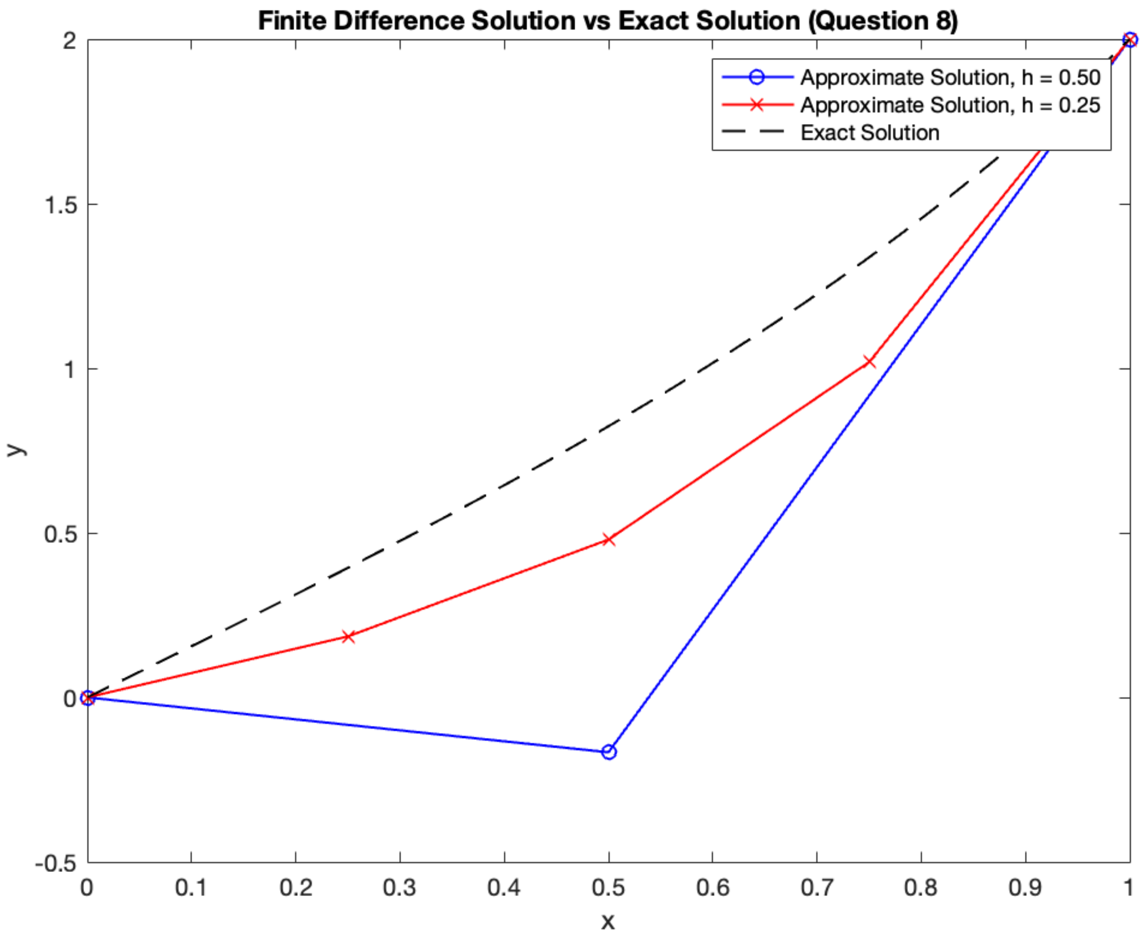
$$y'' = 4(y - x), \quad 0 \le x \le 1, \quad y(0) = 0, \quad y(1) = 2$$

Exact solution:

$$y(x) = x + \frac{e^2(e^{2x} - e^{-2x})}{e^4 - 1}$$

Method to be used: Linear Finite Difference Method

Graph:



Analysis: We will apply the Linear Finite Difference Method with two different step sizes: $h = 1/2$ and $h = 1/4$.

The given plot shows the two approximate solutions and the exact solution on the same figure. We observe that the approximate solution generated by $h = 1/4$ has lesser deviation than $h = 1/2$. In general, we observe that increasing the value of h results in better approximations. For this case, however, both approximations do not offer very accurate results.

Question 9

BVP:

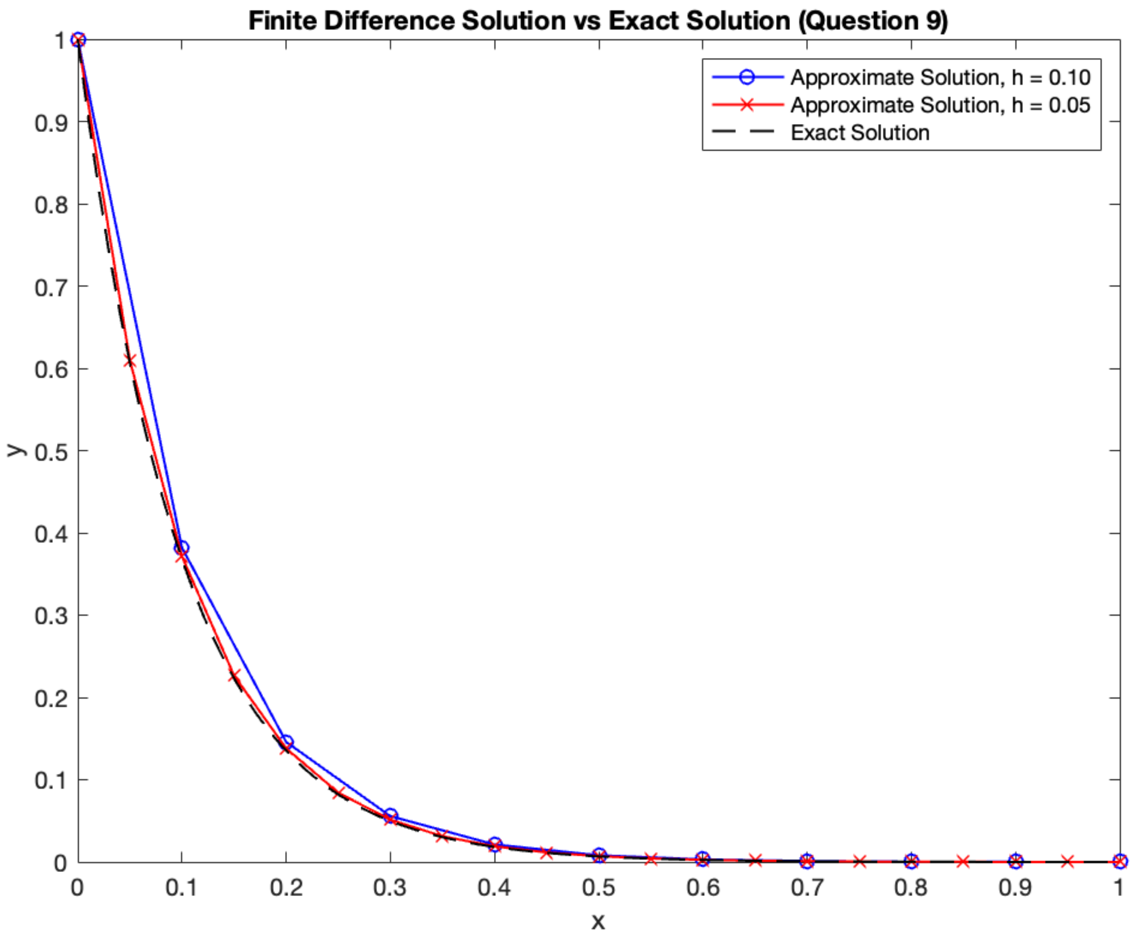
$$y'' = 100y, \quad 0 \le x \le 1, \quad y(0) = 1, \quad y(1) = e^{-10}$$

Exact solution:

$$y(x) = e^{-10x}$$

Method to be used: Linear Finite Difference Method

Graph:



Analysis: For the given BVP, the Linear Finite Difference Method will be applied with two different step sizes, $h = 1/10$ and $h = 1/20$.

The given plot shows the two approximate solutions and the exact solution on the same figure. We observe that the approximate solution generated by $h = 1/20$ is closely overlapping with the exact solution. In general, we observe that increasing the value of h results in better approximations.

Question 10

BVP:

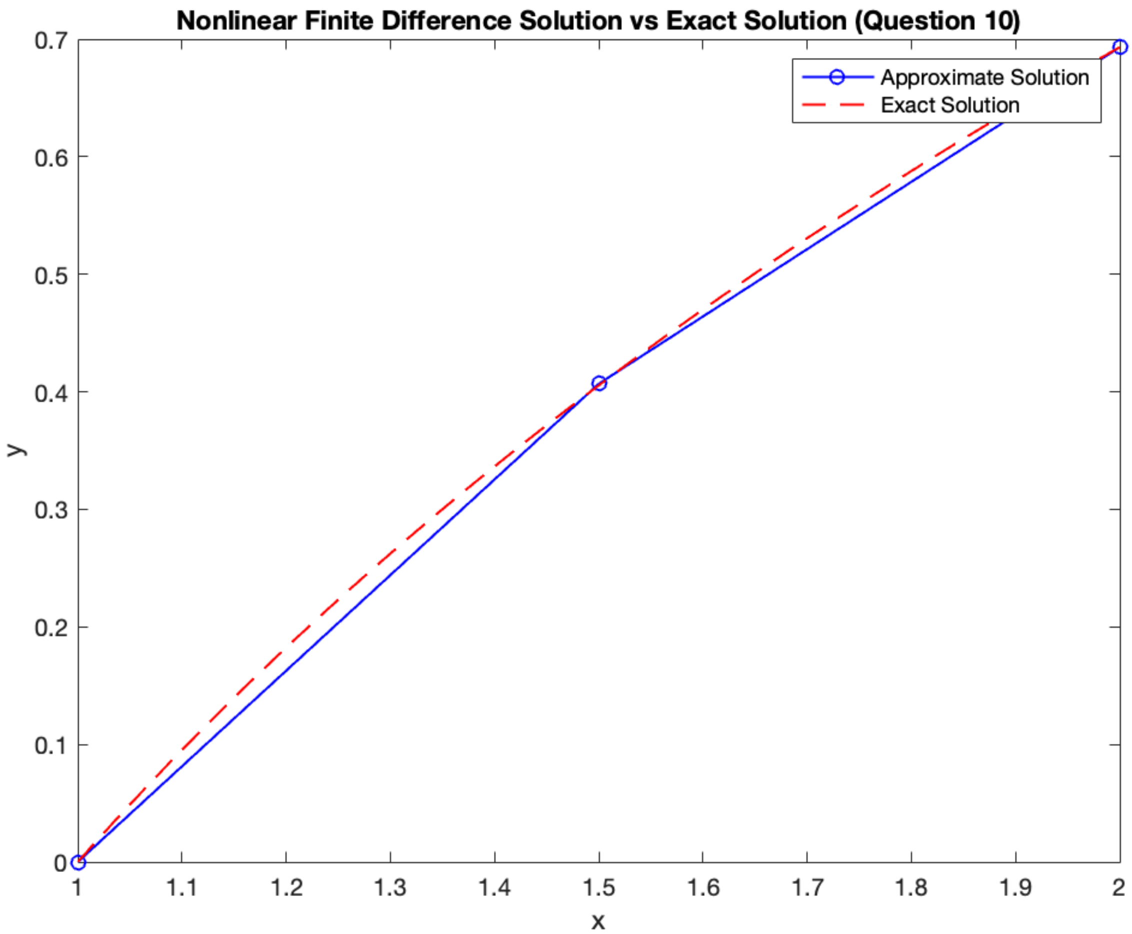
$$y'' = -(y')^2 - y + \ln x, \quad 1 \le x \le 2, \quad y(1) = 0, \quad y(2) = \ln 2$$

Exact solution:

$$y(x) = \ln x$$

Method to be used: Nonlinear Finite Difference Method

Graph:



Analysis: The Nonlinear Finite Difference Method will be applied with $h = 0.5$ for the given BVP.

We see that even with $h = 1/2$, the approximate solution gives a decently accurate approximation to the exact solution.

Question 11

BVP:

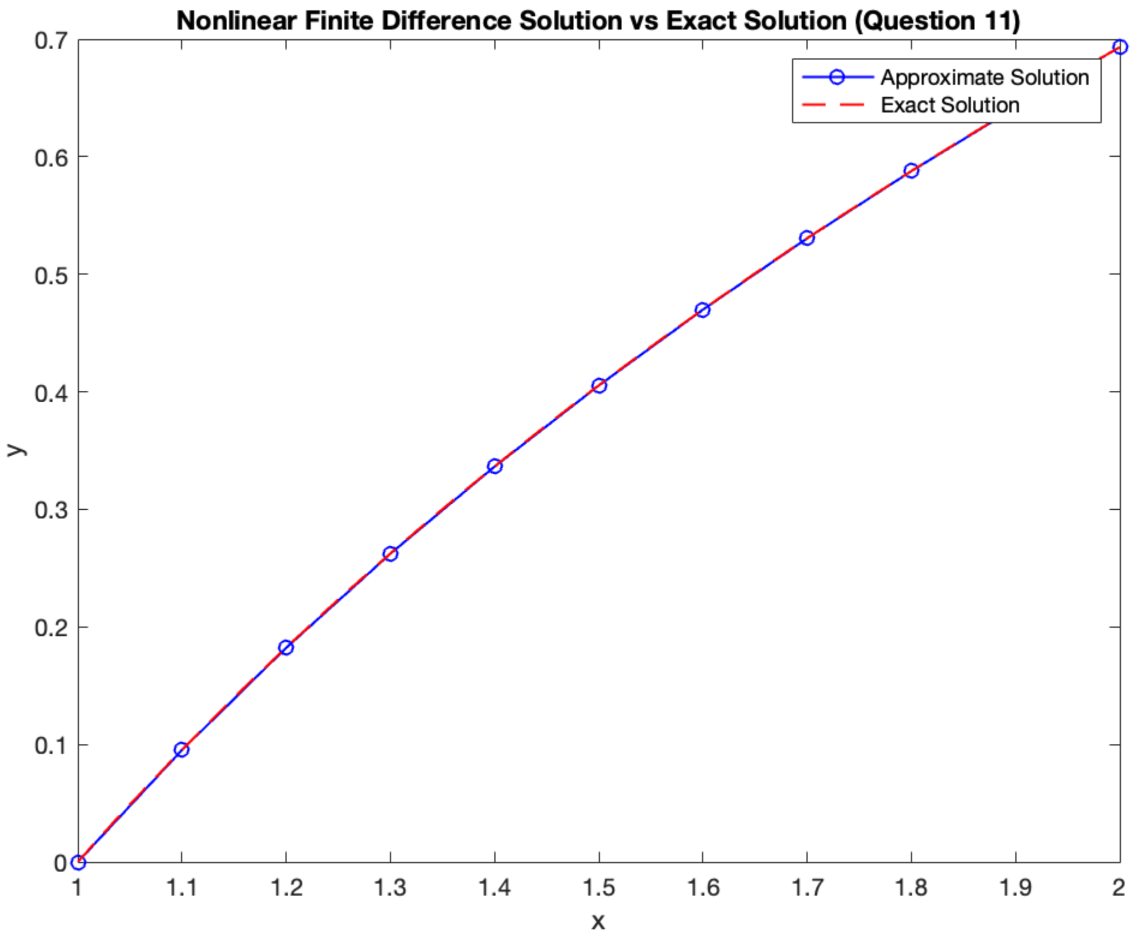
$$y'' = -e^{-2y}, \quad 1 \leq x \leq 2, \quad y(1) = 0, \quad y(2) = \ln 2$$

Exact solution:

$$y(x) = \ln x$$

Method to be used: Nonlinear Finite Difference Method

Graph:



Analysis: This question involves applying the Nonlinear Finite Difference Method with $N = 10$ and a tolerance of 10^{-4} for the given BVP.

We see that the approximation given by Nonlinear Finite Difference closely approximates the exact solution.

Question 12

BVP:

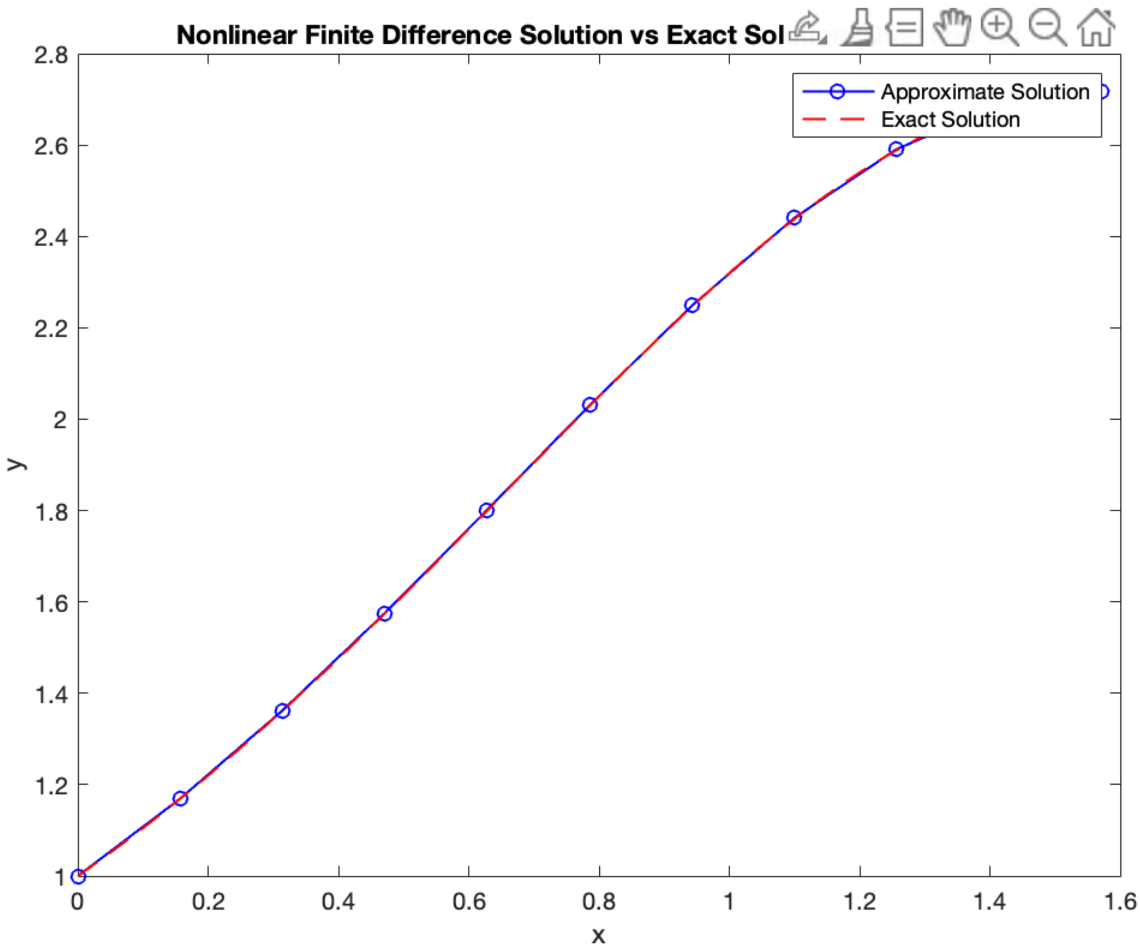
$$y'' = y' \cos x - y \ln y, \quad 0 \le x \le \frac{\pi}{2}, \quad y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = e$$

Exact solution:

$$y(x) = e^{\sin x}$$

Method to be used: Nonlinear Finite Difference Method

Graph:



Analysis: The Nonlinear Finite Difference Method will be used to solve the given BVP.

We see that the approximation given by Nonlinear Finite Difference closely approximates the exact solution.