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Inverse data envelopment analysis model to preserve relative efficiency values: The case of variable returns to scale ☆

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ABSTRACT

This paper studies the inverse Data Envelopment Analysis (inverse DEA) for the case of variable returns to scale (inverse BCC). The developed inverse BCC model can preserve relative efficiency values of all decision making units (DMUs) in a new production possibility set composing of all current DMUs and a perturbed DMU with new input and output values. We consider the inverse BCC model for a resource allocation problem, where increases of some outputs and decreases of the other outputs of the considered DMU can be taken into account simultaneously. The inverse BCC problem is in the form of a multi-objective nonlinear programming model (MONLP), which is not easy to solve. We propose a linear programming model, which gives a Pareto-efficient solution to the inverse BCC problem. However, there exists at least an optimal solution to the proposed model if and only if the new output vector is in the set of current production possibility set. The proposed approach is illustrated via a case study of a motorcycle-part company.

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1. Introduction

Data Envelopment Analysis (DEA), as proposed by Charnes, Cooper, and Rhodes (1978) is an effective approach for measuring relative efficiency values of a set of comparable decision making units (DMUs) that have multiple inputs and multiple outputs. In DEA, the so-called 'efficient frontier' or 'production frontier' is built as the envelope of all DMUs. The set of feasible activities or DMUs is called a production possibility set. The representation of the efficient frontier in DEA is critically influenced by the values of inputs and outputs of observations in the dataset. The change in input and output values can cause the change in the structure of the efficient frontier and the relative efficiency values of DMUs. An interesting problem is how to preserve the relative efficiency value of a considered DMU if the internal technical structure of the considered DMU slightly changes in a short term.

In recent years, inverse optimization of DEA models has been studied. For DEA models, constraint parameters are input and output values of DMUs. Therefore, inverse problems of DEA models can be classified into two types depending on what parameters are changed and what parameters need to be varied to keep the optimal objective value unchanged. The first type of inverse DEA models is a resource allocation problem. The resource allocation problem of DEA is an inverse DEA problem of determining the best possible inputs for given outputs such that the current efficiency value of a considered DMU (DMU_0) with respect to other DMUs remains unchanged. Another type of inverse DEA models is an investment analysis problem. The investment analysis problem of DEA is an inverse DEA problem of determining the best possible outputs for given inputs such that the current efficiency value of a considered DMU_0 with respect to other DMUs remains unchanged. Normally, the internal technical structure of a DMU should not change dramatically in a short term (Yan, Wei, & Hao, 2002). Therefore, the inverse DEA models can be used to such resource allocation and investment analysis problems.

Wei, Zhang, and Zhang (2000) proposed, for the first time, an inverse DEA model for input and output estimation. In their work, an inverse DEA model was discussed to answer the following question: among a group of DMUs, if we increase certain inputs of a particular unit and assume that the DMU maintains its current efficiency value with respect to other units, how much more outputs could the unit produce? or, if the outputs need to be increased to a certain value and the efficiency of the unit remains unchanged, how much more inputs should be provided to the unit? In their developed inverse DEA model, the increases in input and output values were assumed to be nonnegative values, and the inverse DEA model was transformed into and solved as a multi-objective linear programming (MOLP) problem.

Yan et al. (2002) discussed an inverse DEA problem with preference cone constraints to represent decision makers' preferences,

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which was useful in resource planning. Jahanshahloo, Lotfi, Shoja, Tohidi, and Razavyan (2004a) extended the inversed DEA problem and the developed solution method by Yan et al. (2002) to the case of determining outputs of the considered DMU when some or all of inputs were increased and the efficiency value of the considered DMU with respect to other DMUs needed to be improved by specified percentage of its current efficiency value. Jahanshahloo, Lotfi, Shoja, Tohidi, and Razavyan (2004b) showed that the inversed DEA models could be used to estimate inputs for a DMU when some or all outputs and the efficiency value of the DMU were increased or preserved, and also identified extra inputs when outputs were estimated using the proposed models by Yan et al. (2002) and Jahanshahloo et al. (2004a). Jahanshahloo, Lotfi, Shoja, Tohidi, and Razavyan (2005) proposed a modified inverse DEA model for sensitivity analysis of efficiency classifications of efficient and inefficient DMUs in which important policies over inputs, outputs and DMUs were represented by preference cones.

Hadi-Vencheh and Foroughi (2006) discussed an extended inverse DEA model where an increase of some inputs (outputs) and a decrease due to some of the other inputs (outputs) are taken into account at the same time. A proposed solving method was based on DEA and MOLP. In their paper, they also showed that the solution proposed by Wei et al. (2000) did not guarantee the efficiency result for input estimating, i.e., it might fail in a special case. Actually, in the paper of Wei et al. (2000), only the increase of inputs (outputs) is considered whereas each DMU may concern the increase of some of inputs (outputs) and the decrease of the other inputs (outputs) simultaneously. Recently, Alinezhad, Makui, and Kiani Mavi (2007) proposed a methodology that uses an interactive MOLP for solving the inverse DEA problems.

Previous studies on inverse DEA problems mostly consider the efficiency value of the considered DMU without considering the effect of input/output changes on efficiency values of other DMUs. In this paper, we discuss the inverse DEA model for the case of variable returns to scale (inverse BCC) to preserve relative efficiency values of all DMUs when output values of a considered DMU are changed. The proposed inverse BCC model is a type of resource allocation problem. The considered DMU after its input and output values are changed will be called as a "perturbed DMU." The objective of the inverse BCC model is to determine the best possible input values (minimum input values) of the perturbed DMU. We consider the inverse BCC model where increases of some inputs and decreases of the other inputs can be taken into account simultaneously. A solution approach is developed and illustrated via a case study of a motorcycle-part company in Thailand. The advantage of the inverse DEA model is for decision makers to find out how to allocate limited resources to a perturbed DMU according to the expected slightly increases of its outputs such that the changes will not affect relative efficiency levels of all DMUs.

The rest of this paper is organized as follows. In Section 2, we provide a brief review on DEA models. Section 3 states the inverse BCC model and presents our proposed model to determine the best possible values of inputs for the perturbed DMU to preserve relative efficiency values of all DMUs. In Section 4, a solution approach to the inverse BCC is presented and then illustrated via a case study of a motorcycle-part company. Finally, conclusions are given in Section 5.

2. The DEA model

DEA is a non-parametric technique for evaluating the performance of comparable DMUs. DEA evaluates the relative efficiency of a set of homogeneous DMUs by using a ratio of the weighted sum of outputs to the weighted sum of inputs. Specifically, it determines a set of weights such that the efficiency of a considered DMU (DMU_0) relative to the other DMUs is maximized. Input and output data for DEA are assumed to be semi-positive, i.e., all data are assumed to be nonnegative but at least one component of every input and output vector is positive. DEA has rapidly expanded its use in many applications such as performance analysis of health care units, agricultural production, airline operations, textile companies, university libraries, and military logistics (Arnade, 1994; Barros & Peypoch, 2009; Chandra, Cooper, Li, & Rahman, 1998; Charnes, Cooper, Lewin, & Seiford, 1994; Chen, 1997).

The frequently used DEA models are the CCR model, named after Charnes, Cooper, and Rhodes (Charnes et al., 1978), and the BCC model, named after Banker, Charnes, and Cooper (Banker, Charnes, & Cooper, 1984). Other models are extensions of the CCR model obtained by either modifying the production possibility set of the CCR model or adding slack variables in the objective function (Banker et al., 1984; Cooper, Seiford, & Tone, 2000). In this paper, we are interested in the inverse DEA of the BCC model.

Assume that input and output data are semi-positive, i.e.,

$$\mathbf{x}_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{mi} \end{bmatrix} \geqslant 0,\ \mathbf{x}_i \neq 0,\ \mathbf{y}_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{ri} \end{bmatrix} \geqslant 0,\ \mathbf{y}_i \neq 0,$$

$$\mathbf{x}_0 = \begin{bmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{m0} \end{bmatrix} \geqslant 0,\ \mathbf{x}_0 \neq 0,\ \mathbf{y}_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \vdots \\ y_{r0} \end{bmatrix} \geqslant 0,\ \mathbf{y}_0 \neq 0.$$

The primal and dual BCC models are in the following form.

$$\begin{aligned} (\mathrm{PBCC}_0)\quad & \text{maximize} \quad \sum_{k=1}^{r} u_k y_{k0} - u_0 \\ & \text{s.t.} \quad \sum_{j=1}^{m} v_j x_{j0} = 1 \\ & \qquad -\sum_{j=1}^{m} v_j x_{ji} + \sum_{k=1}^{r} u_k y_{ki} - u_0 \leqslant 0 \quad \text{for } i = 1,\ldots,n \\ & \qquad u_0 \ \text{ is free},\ \ v_j, u_k \geqslant 0,\ \ j = 1,2,\ldots,m,\ k = 1,2,\ldots,r \end{aligned} \tag{1}$$

$$\begin{aligned} (\mathrm{DBCC}_0)\quad & \text{minimize} \quad \theta_0 \\ & \text{s.t.} \quad \sum_{i=1}^{n} \lambda_i x_{ji} \leqslant \theta_0 x_{j0} \quad \text{for } j = 1,\ldots,m \\ & \qquad \sum_{i=1}^{n} \lambda_i y_{ki} \geqslant y_{k0} \quad \text{for } k = 1,\ldots,r \\ & \qquad \sum_{i=1}^{n} \lambda_i = 1 \\ & \qquad \lambda_i \geqslant 0, \quad i = 1,2,\ldots,n \end{aligned} \tag{2}$$

where $i = 1,\ldots,n$, $j = 1,\ldots,m$, $k = 1,\ldots,r$, x_{j0} is the input j of the considered DMU (DMU_0), x_{ji} is the input j of DMU_i, y_{k0} is the output k of DMU_0, y_{ki} is the output k of DMU_i, u_k is the weight of output k, v_j is the weight of input j, u_0 is scalar, λ_i is the convex combination of DMU_i, θ_0 is the objective function or the technical efficiency value of DMU_0. The indices and notation will be used throughout this paper.

From the BCC models, DMU_0 will be technically efficient if the maximal efficiency, θ_0^*, is equal to 1. If $\theta_0^* < 1$, it is possible to produce the given outputs using smaller input values, which may be obtained as a convex combination of inputs of other DMUs. The λ_i, $i = 1,\ldots,n$ obtained from the $DBCC_0$ model provide a reference set for inefficient DMUs. The convex combination of the reference set is the projected point on the production frontier of the inefficient DMUs. The set of feasible activities or all DMUs is called production possibility set.

The BCC model has its production frontier spanned by the convex hull of the existing DMUs. The production frontier is piecewise linear and concave, which leads to variable returns to scale as shown in Fig. 1 for the case of one input and one output. From economic theory, there are three types of "returns to scale": increasing returns to scale (IRS), decreasing returns to scale (DRS) and constant returns to scale (CRS). From Fig. 1, IRS occurs on the first line segment of the production frontier followed by DRS on the second line segment. Finally, CRS occurs at the point of transition from IRS to DRS. The production frontier of the BCC model is composed of IRS and DRS. IRS and DRS are referred to as "variable returns to scale" because the outputs produced increase more or less than proportionally to the increase in the inputs (Thanassoulis, 2001).

From the production frontier of the BCC model, a DMU is inefficient if it is possible to reduce any input without increasing any other inputs and achieve the same value of outputs or it is possible to increase any output without reducing any other outputs and use the same values of inputs (Lertworasirikul, Fang, Nuttle, & Joines, 2003).

Because the production frontier of the BCC model is not spanned by the linear combination of the existing DMUs, any changes in input or output values of efficient DMUs, which lie on the production frontier, will shift the production frontier and then efficiency values of DMUs in the production possibility set might be changed. If a considered DMU changes its output (or input) values, to preserve relative efficiency values of all DMUs, input (or output) values of the considered DMU have to be changed such that the production frontier is still the same. Therefore, in this paper we propose an inverse BCC model to preserve efficiency values of all DMUs relative to other DMUs based on the current production frontier. With the same production frontier, relative efficiency values of all DMUs can be maintained. To preserve the production frontier, the production possibility set is composed of all current DMUs and the considered DMU with the changes in its input and output values.

3. The inverse BCC model

Denote the considered DMU with current input and output values by DMU_0 and the considered DMU_0 with its input and output changes (perturbed DMU_0) by $DMU_{0'}$. The developed inverse BCC model for a resource allocation problem is introduced to answer the following question.

For a group of current DMUs with their relative efficiency values of $\theta_1^*, \theta_2^*, \ldots, \theta_n^*$, suppose that the output values of DMU_0 are changed from $\mathbf{y}_0$ to $\mathbf{y}_0 + \Delta\mathbf{y}_0 \geqslant \mathbf{0}$, $\Delta\mathbf{y}_0 \neq \mathbf{0}$, we want to find the minimum $\mathbf{x}_0 + \Delta\mathbf{x}_0$ where $\mathbf{x}_0 + \Delta\mathbf{x}_0$ is a semi-positive vector such that $DMU_{0'}$ with new input and output values ($\mathbf{x}_0 + \Delta\mathbf{x}_0$, $\mathbf{y}_0 + \Delta\mathbf{y}_0$) still has its relative efficiency value of θ_0^*, and all other DMUs still have their relative efficiency values of $\theta_1^*, \theta_2^*, \ldots, \theta_n^*$.

Note that the current production possibility set before the changes of input and output values of DMU_0 is composed of n DMUs (DMU_i, $i = 1, \ldots, n$). However, after input and output values of DMU_0 are changed, we consider the new production possibility set composing of $n + 1$ DMUs (DMU_i, $i = 1, \ldots, n$, and $DMU_{0'}$) and try to preserve the production frontier as showed in Fig. 2. The mathematical models (primal and dual models) of the inverse BCC for a resource allocation problem are as follows.

3.1. Primal form of the inverse BCC model

$$
\begin{aligned}
(\mathrm{IBCC}_0)\quad & \text{minimize } \Delta\mathbf{x}_0 \\
\text{s.t.}\quad & \sum_{j=1}^{m} v_j(x_{j0} + \Delta x_{j0}) = 1 \\
& -\sum_{j=1}^{m} v_j x_{ji} + \sum_{k=1}^{r} u_k y_{ki} - u_0 \leqslant 0 \quad \text{for } i = 1, \ldots, n \\
& -\sum_{j=1}^{m} v_j(x_{j0} + \Delta x_{j0}) + \sum_{k=1}^{r} u_k^* y_{k0} - u_0 \leqslant 0 \\
& \mathbf{x}_0 + \Delta\mathbf{x}_0 \geqslant \mathbf{0} \\
& u_0 \ \text{ is free}, v_j, u_k \geqslant 0,\ \ j = 1, 2, \ldots, m,\ k = 1, 2, \ldots, r
\end{aligned} \tag{3}
$$

where $\mathbf{x}_0 + \Delta\mathbf{x}_0 \neq \mathbf{0}$ and $\sum_{k=1}^{r} u_k^* y_{k0} - u_0 = \sum_{k=1}^{r} u_k(y_{k0} + \Delta y_{k0}) - u_0 = \theta_0^*$ is the relative efficiency value of DMU_0 before the changes in its output values. Using $\Delta\mathbf{x}_0$ from the IBCC_0 model, the relative efficiency values of all DMU_l for $l = 1, 2, \ldots, n$ from solving the following IBCC_l model must be equal to θ_l^* where θ_l^* is the current relative efficiency value of DMU_l before DMU_0 changes its output values.

$$
\begin{aligned}
(\mathrm{IBCC}_l)\quad & \text{maximize } \sum_{k=1}^{r} u_k y_{kl} - u_0 \\
\text{s.t.}\quad & \sum_{j=1}^{m} v_j x_{jl} = 1 \\
& -\sum_{j=1}^{m} v_j x_{ji} + \sum_{k=1}^{r} u_k y_{ki} - u_0 \leqslant 0 \quad \text{for } i = 1, \ldots, n \\
& -\sum_{j=1}^{m} v_j(x_{j0} + \Delta x_{j0}) + \sum_{k=1}^{r} u_k(y_{k0} + \Delta y_{k0}) - u_0 \leqslant 0 \\
& u_0 \text{ is free},\ \ v_j, u_k \geqslant 0,\ \ j = 1, 2, \ldots, m,\ k = 1, 2, \ldots, r.
\end{aligned} \tag{4}
$$

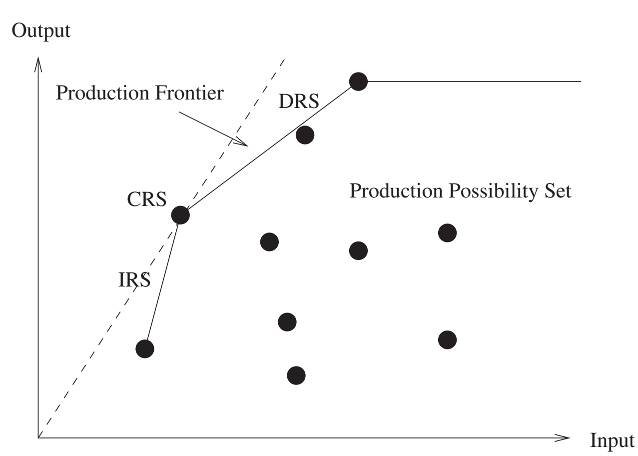

Fig. 1. Production frontier of the BCC model for the case of one input and one output.

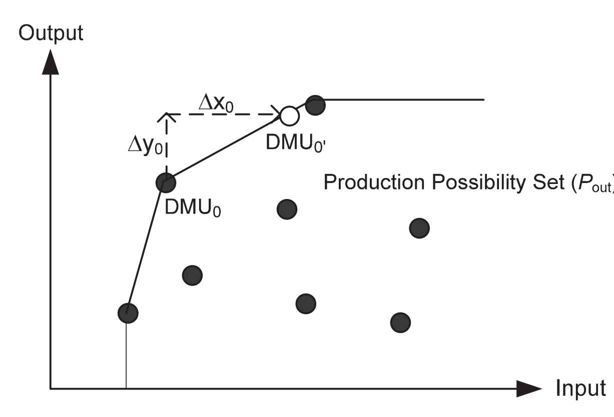

Fig. 2. Production possibility set and production frontier of the inverse BCC model for the case of one input and one output.

3.2. Dual form of the inverse BCC model

$$
\begin{aligned}
(\mathrm{DIBCC}_0)\ \ & \text{minimize } \Delta\mathbf{x}_0 \\
\text{s.t.}\ \ & \sum_{i=1}^{n}\lambda_i x_{ji} + \lambda_{0'}(x_{j0}+\Delta x_{j0}) \leqslant \theta_0^*(x_{j0}+\Delta x_{j0}) \quad \text{for } j=1,\ldots,m \\
& \sum_{i=1}^{n}\lambda_i y_{ki} + \lambda_{0'}(y_{k0}+\Delta y_{k0}) \geqslant y_{k0}+\Delta y_{k0} \quad \text{for } k=1,\ldots,r \\
& \sum_{i=1}^{n}\lambda_i + \lambda_{0'} = 1 \\
& \mathbf{x}_0 + \Delta\mathbf{x}_0 \geqslant \mathbf{0} \\
& \lambda_{0'},\ \ \lambda_i \geqslant 0,\ i=1,2,\ldots,n
\end{aligned} \tag{5}
$$

where $\mathbf{x}_0 + \Delta\mathbf{x}_0 \neq \mathbf{0}$ and θ_0^* is the relative efficiency value of DMU$_0$ before the changes in its output values. Using $\Delta\mathbf{x}_0$ from the DIBCC$_0$ model, the relative efficiency value of DMU$_l$ for $l = 1,2,\ldots,n$ from solving the DIBCC$_l$ model must be equal to θ_l^*.

$$
\begin{aligned}
(\mathrm{DIBCC}_l)\ \ & \text{minimize } \theta_l \\
\text{s.t.}\ \ & \sum_{i=1}^{n}\lambda_i x_{ji} + \lambda_{0'}(x_{j0}+\Delta x_{j0}) \leqslant \theta_l x_{jl} \quad \text{for } j=1,\ldots,m \\
& \sum_{i=1}^{n}\lambda_i y_{ki} + \lambda_{0'}(y_{k0}+\Delta y_{k0}) \geqslant y_{kl} \quad \text{for } k=1,\ldots,r \\
& \sum_{i=1}^{n}\lambda_i + \lambda_{0'} = 1 \\
& \lambda_{0'},\ \ \lambda_i \geqslant 0,\ i=1,2,\ldots,n.
\end{aligned} \tag{6}
$$

Note that the IBCC$_0$ and DIBCC$_0$ are in the form of multi-objective nonlinear programming (MONLP) form.

4. A solution approach to the inverse BCC model

To solve the inverse BCC model for the resource allocation problem, we need to find the value of $\Delta\mathbf{x}_0 = (\Delta x_{10}, \Delta x_{20}, \ldots, \Delta x_{m0})$, which keeps the relative efficiency values of all DMUs unchanged. This can be done by solving the IBCC$_0$ and IBCC$_l$ models or by solving DIBCC$_0$ and DIBCC$_l$ models. However, these models are in the form of MONLP, which is not easy to solve. In this section, we propose a multi-objective linear programming model (MLDIBCC$_0$), which gives an optimal solution for the inverse BCC model. Later we propose a linear programming model (LDIBCC$_0$) in Theorem 2, which gives a Pareto solution to the MLDIBCC$_0$ model. Therefore,

Theorem 1. *Assume that the relative efficiency value of DMU$_0$ with respect to other DMUs in a group of comparable DMUs ($i = 1,\ldots,n$) is θ_0^*. Given the changes in output values of DMU$_0$, $\Delta\mathbf{y}_0 \neq \mathbf{0}$, the minimum $\Delta\mathbf{x}_0$ of the perturbed DMU$_0$ (DMU$_{0'}$), which does not make any changes to the relative efficiency values of all DMUs ($l = 1,\ldots,n,0'$), can be obtained by solving the MLDIBCC$_0$ model.*

$$
\begin{aligned}
(\mathrm{MLDIBCC}_0)\ \ & \text{minimize } \Delta\mathbf{x}_0 = (\Delta x_{10},\ldots,\Delta x_{m0})^T \\
\text{s.t.}\ \ & \sum_{i=1}^{n}\lambda_i x_{ji} \leqslant \theta_0^*(x_{j0}+\Delta x_{j0}) \quad \text{for } j=1,\ldots,m \\
& \sum_{i=1}^{n}\lambda_i y_{ki} \geqslant y_{k0}+\Delta y_{k0} \quad \text{for } k=1,\ldots,r \\
& \sum_{i=1}^{n}\lambda_i = 1 \\
& \lambda_i \geqslant 0, \quad i=1,2,\ldots,n.
\end{aligned} \tag{7}
$$

Proof. The BCC model for DMU$_{0'}$ relative to other DMUs ($l = 1,\ldots,n$) is the DBCC$'_0$ model.

$$
\begin{aligned}
(\mathrm{DBCC}'_0)\ \ & \text{minimize } \theta_{0'} \\
\text{s.t.}\ \ & \sum_{i=1}^{n}\lambda_i x_{ji} + \lambda_{0'}(x_{j0}+\Delta x_{j0}) \leqslant \theta_{0'}(x_{j0}+\Delta x_{j0}) \quad \text{for } j=1,\ldots,m \\
& \sum_{i=1}^{n}\lambda_i y_{ki} + \lambda_{0'}(y_{k0}+\Delta y_{k0}) \geqslant y_{k0}+\Delta y_{k0} \quad \text{for } k=1,\ldots,r \\
& \sum_{i=1}^{n}\lambda_i + \lambda_{0'} = 1 \\
& \lambda_{0'},\ \ \lambda_i \geqslant 0,\ i=1,2,\ldots,n.
\end{aligned} \tag{8}
$$

The set of constraints in the DBCC$'_0$ model can be rearranged in the following form:

$$
\begin{aligned}
& \sum_{i=1}^{n}\lambda_i x_{ji} \leqslant (\theta_{0'}-\lambda_{0'})(x_{j0}+\Delta x_{j0}) \quad \text{for } j=1,\ldots,m \\
& \sum_{i=1}^{n}\lambda_i y_{ki} \geqslant (1-\lambda_{0'})(y_{k0}+\Delta y_{k0}) \quad \text{for } k=1,\ldots,r \\
& \sum_{i=1}^{n}\lambda_i + \lambda_{0'} = 1 \\
& \lambda_{0'}, \lambda_i \geqslant 0, \quad i=1,2,\ldots,n.
\end{aligned} \tag{9}
$$

Case 1: $\theta_0^* < 1$

From the set of constraints in (9), if $\lambda_{0'} = 1$, then $\lambda_i = 0$ for $i = 1,2,\ldots,n$ and $\theta_{0'} = 1$. The solution ($\lambda_i = 0$ for $i = 1,2,\ldots n$, $\lambda_{0'} = 1$, $\theta_{0'} = 1$) is not the optimal solution to the DBCC$'_0$ model when $\theta_0^* < 1$ because we can find a better solution at $\lambda_{0'} = 0$. When $\lambda_{0'} = 0$, the constraints of the DBCC$'_0$ model are in the same form as the constraints in the MLDIBCC$_0$ model. Thus, the objective value is θ_0^*, which is less than 1.

Since $(1-\lambda_{0'}) \neq 0$, we divide all constraints in the DBCC$'_0$ model by $(1-\lambda_{0'})$ and set $\bar{\theta}_{0'} = \frac{(\theta_{0'}-\lambda_{0'})}{(1-\lambda_{0'})}$ and $\bar{\lambda}_i = \frac{\lambda_i}{(1-\lambda_{0'})}$ for $i = 1,2,\ldots,n$. Then the DBCC$'_0$ model becomes

$$
\begin{aligned}
& \text{minimize} \quad \theta_{0'} \\
\text{s.t.}\ \ & \sum_{i=1}^{n}\bar{\lambda}_i x_{ji} \leqslant \bar{\theta}_{0'}(x_{j0}+\Delta x_{j0}) \quad \text{for } j=1,\ldots,m \\
& \sum_{i=1}^{n}\bar{\lambda}_i y_{ki} \geqslant y_{k0}+\Delta y_{k0} \quad \text{for } k=1,\ldots,r \\
& \sum_{i=1}^{n}\bar{\lambda}_i + \frac{\lambda_{0'}}{(1-\lambda_{0'})} = 1 \\
& \lambda_{0'}, \bar{\lambda}_i \geqslant 0, \quad i=1,2,\ldots,n.
\end{aligned}
$$

From $\sum_{i=1}^{n}\bar{\lambda}_i + \frac{\lambda_{0'}}{(1-\lambda_{0'})} = 1$, we find that $\lambda_{0'} = \frac{1-\sum_{i=1}^{n}\bar{\lambda}_i}{2-\sum_{i=1}^{n}\bar{\lambda}_i}$. By substituting $\lambda_{0'} = \frac{\left(1-\sum_{i=1}^{n}\bar{\lambda}_i\right)}{\left(2-\sum_{i=1}^{n}\bar{\lambda}_i\right)}$ into $\bar{\theta}_{0'} = \frac{(\theta_{0'}-\lambda_{0'})}{(1-\lambda_{0'})}$, we can find that the objective function of the DBCC$'_0$ model is to minimize $\theta_{0'} = \frac{\left(\bar{\theta}_{0'}+1-\sum_{i=1}^{n}\bar{\lambda}_i\right)}{\left(2-\sum_{i=1}^{n}\bar{\lambda}_i\right)}$. Thus, the DBCC$'_0$ model becomes:

$$
\begin{aligned}
& \text{minimize } \frac{\left(\bar{\theta}_{0'}+1-\sum_{i=1}^{n}\bar{\lambda}_i\right)}{\left(2-\sum_{i=1}^{n}\bar{\lambda}_i\right)} \\
\text{s.t.}\ \ & \sum_{i=1}^{n}\bar{\lambda}_i x_{ji} \leqslant \bar{\theta}_{0'}(x_{j0}+\Delta x_{j0}) \quad \text{for } j=1,\ldots,m \\
& \sum_{i=1}^{n}\bar{\lambda}_i y_{ki} \geqslant y_{k0}+\Delta y_{k0} \quad \text{for } k=1,\ldots,r \\
& \bar{\lambda}_i \geqslant 0, \quad i=1,2,\ldots,n.
\end{aligned} \tag{10}
$$

Note that a fractional number is invariant under multiplication of both numerator and denominator by the same nonzero number. We set the denominator of the model (10) equal to 1, move it to a constraint, and minimize the numerator. This results in the following model.

$$\begin{aligned}
\text{minimize} \quad & \bar{\theta}_{0'} + 1 - \sum_{i=1}^{n} \bar{\lambda}_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} \bar{\lambda}_i x_{ji} \leqslant \bar{\theta}_{0'}(x_{j0} + \Delta x_{j0}) \quad \text{for } j = 1, \ldots, m \\
& \sum_{i=1}^{n} \bar{\lambda}_i y_{ki} \geqslant y_{k0} + \Delta y_{k0} \quad \text{for } k = 1, \ldots, r \\
& 2 - \sum_{i=1}^{n} \bar{\lambda}_i = 1 \\
& \bar{\lambda}_i \geqslant 0, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

Since $\sum_{i=1}^{n} \bar{\lambda}_i = 1$, the above model becomes the model (11).

$$\begin{aligned}
\text{minimize} \quad & \bar{\theta}_{0'} \\
\text{s.t.} \quad & \sum_{i=1}^{n} \bar{\lambda}_i x_{ji} \leqslant \bar{\theta}_{0'}(x_{j0} + \Delta x_{j0}) \quad \text{for } j = 1, \ldots, m \\
& \sum_{i=1}^{n} \bar{\lambda}_i y_{ki} \geqslant y_{k0} + \Delta y_{k0} \quad \text{for } k = 1, \ldots, r \\
& \sum_{i=1}^{n} \bar{\lambda}_i = 1 \\
& \bar{\lambda}_i \geqslant 0, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{11}$$

The optimal solution of the model (11) is also optimal for the model (10) since the above transformation is reversible. The models (10) and (11) therefore have the same optimal objective value. Note that the constraints in the model (11) are in the same form of the constraints in the MLDIBCC$_0$ model. Using the minimum $\Delta\mathbf{x_0}$ obtained from solving the MLDIBCC$_0$ model, all constraints in the model (11) are satisfied and the objective function is minimized at $\bar{\theta}_{0'} = \theta_0^*$, otherwise $\Delta\mathbf{x_0}$ is not optimal for the MLDIBCC$_0$ model. Therefore, the relative efficiency value of DMU$_{0'}$ with respect to the set of DMU$_l$ ($l = 1,\ldots,n,0'$) will remain equal to θ_0^*.

Case 2: $\theta_0^* = 1$

If $\lambda_{0'} \neq 1$ in the DBCC$'_0$ model, we can prove that the optimal objective value of the DBCC$'_0$ model ($\theta_{0'}$) is equal to $\theta_0^* = 1$ by using the same way for the proof of case 1. And if $\lambda_{0'} = 1$, then $\lambda_i = 0$ for $i = 1,2,\ldots,n$ and $\theta_{0'} = \theta_0^* = 1$.

Now let us consider the inverse BCC (DIBCC$_0$) model. If $\lambda_{0'} = 1$, then $\lambda_i = 0$ for $i = 1, 2, \ldots, n, \theta_{0'} = 1$, and $\mathbf{x_0} + \Delta\mathbf{x_0} = \mathbf{0}$. However, we assume at the beginning that $\mathbf{x}_0 + \Delta\mathbf{x}_0$ must be a semi-positive vector. Therefore, the solution $\lambda_{0'} = 1$, $\lambda_i = 0$ for $i = 1, 2, \ldots, n$, $\theta_{0'} = 1$, and $\mathbf{x_0} + \Delta\mathbf{x_0} = \mathbf{0}$ is not an optimal solution for the inverse BCC model.

Consequently, given the changes in output values of DMU$_0$, $\Delta\mathbf{y}_0 \neq \mathbf{0}$, the minimum $\Delta\mathbf{x}_0$ of DMU$_{0'}$, which does not make any change to the relative efficiency value of DMU$_{0'}$, can be obtained by solving the MLDIBCC$_0$ model.

For other DMUs, the IBCC$_l$ model in (4) can be written in the vector-metric form as follows:

$$\begin{aligned}
\text{maximize} \quad & \mathbf{u}^{\mathbf{T}}\mathbf{y}_l - u_0 \\
\text{s.t.} \quad & \mathbf{v}^{\mathbf{T}}\mathbf{x}_l = 1 \\
& -\mathbf{v}^{\mathbf{T}}\mathbf{x}_i + \mathbf{u}^{\mathbf{T}}\mathbf{y}_i - u_0 \leqslant 0 \quad \text{for } i = 1, \ldots, n \\
& -\mathbf{v}^{\mathbf{T}}(\mathbf{x_0} + \Delta\mathbf{x_0}) + \mathbf{u}^{\mathbf{T}}(\mathbf{y_0} + \Delta\mathbf{y_0}) - u_0 \leqslant 0 \\
& u_0 \quad \text{is free}, \ \mathbf{u}, \mathbf{v} \geqslant \mathbf{0}
\end{aligned}$$

where

$$\mathbf{u}^{\mathbf{T}} = [u_1,\ u_2,\ \ldots,\ u_r], \quad \mathbf{v}^{\mathbf{T}} = [v_1,\ v_2,\ \ldots,\ v_m],$$

$$\mathbf{x}_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{mi} \end{bmatrix}, \quad \mathbf{y}_i = \begin{bmatrix} y_{1i} \\ y_{2i} \\ \vdots \\ y_{ri} \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{m0} \end{bmatrix}, \quad \Delta\mathbf{x}_0 = \begin{bmatrix} \Delta x_{10} \\ \Delta x_{20} \\ \vdots \\ \Delta x_{m0} \end{bmatrix}.$$

From the constraints in the MLDIBCC$_0$ model, $(\mathbf{x_0} + \Delta\mathbf{x}_0, \mathbf{y_0} + \Delta\mathbf{y}_0) \in P$ where P is a production possibility set of all DMU$_i$, $i = 1,\ldots,n$ and $P = \{(\mathbf{x},\mathbf{y}) | \mathbf{x} \geqslant \mathbf{X}\lambda, \mathbf{y} \leqslant \mathbf{Y}\lambda, \mathbf{e}^{\mathbf{T}}\lambda = 1,\ \lambda \geqslant \mathbf{0}\}$, $\mathbf{X} = [x_{ji}]_{mxn}$, $\mathbf{Y} = [y_{ki}]_{rxn}$, and $\lambda = (\lambda_i)_{n\times 1}$, $\lambda \in R^n$.

If $(\mathbf{x_0} + \Delta\mathbf{x}_0, \mathbf{y_0} + \Delta\mathbf{y}_0) \in P$, then we have

$$\begin{aligned}
& -\mathbf{v}^{\mathbf{T}}(\mathbf{x_0} + \Delta\mathbf{x_0}) + \mathbf{u}^{\mathbf{T}}(\mathbf{y_0} + \Delta\mathbf{y_0}) - u_0 \\
& \leqslant -\mathbf{v}^{\mathbf{T}}(\mathbf{X}\lambda) + \mathbf{u}^{\mathbf{T}}(\mathbf{Y}\lambda) - u_0 \leqslant -\sum_{i=1}^{n} \mathbf{v}^{\mathbf{T}}\mathbf{x}_i\lambda_i + \sum_{i=1}^{n} \mathbf{u}^{\mathbf{T}}\mathbf{y}_i\lambda_i - u_0 \\
& \leqslant \sum_{i=1}^{n} (-\mathbf{v}^{\mathbf{T}}\mathbf{x}_i + \mathbf{u}^{\mathbf{T}}\mathbf{y}_i)\lambda_i - u_0.
\end{aligned}$$

From the IBCC$_l$ model, $-\mathbf{v}^{\mathbf{T}}\mathbf{x}_i + \mathbf{u}^{\mathbf{T}}\mathbf{y}_i \leqslant u_0$ for $i = 1,\ldots,n$. Therefore,

$$-\mathbf{v}^{\mathbf{T}}(\mathbf{x_0} + \Delta\mathbf{x_0}) + \mathbf{u}^{\mathbf{T}}(\mathbf{y_0} + \Delta\mathbf{y_0}) - u_0 \leqslant \sum_{i=1}^{n} u_0\lambda_i - u_0 \leqslant 0.$$

This shows that $-\mathbf{v}^{\mathbf{T}}(\mathbf{x_0} + \Delta\mathbf{x_0}) + \mathbf{u}^{\mathbf{T}}(\mathbf{y_0} + \Delta\mathbf{y_0}) - u_0 \leqslant \mathbf{0}$ in the IBCC$_l$ model is redundant and can be dropped out from the model without changing the solution set and the optimal objective value. In other words, the IBCC$_l$ model is equivalent to the BCC model for DMU$_l$ before DMU$_0$ changes its output values. This implies that the relative efficiency values of all DMU$_l$, ($l = 1,\ldots,n$) remains unchanged. □

Lemma 1. *Assume that the relative efficiency value of DMU$_0$ with respect to other DMUs in a group of comparable DMUs ($i = 1,\ldots,n$) is θ_0^*. Also, assume the output values of DMU$_0$ are changed from $\mathbf{y}_0$ to $\mathbf{y}_0 + \Delta\mathbf{y}_0 \geqslant \mathbf{0}$, $\Delta\mathbf{y}_0 \neq \mathbf{0}$. There exists at least an optimal solution to the MLDIBCC$_0$ model, if and only if $\mathbf{y}_0 + \Delta\mathbf{y}_0 \in P_{out}$, where $P_{out} = \{\mathbf{y} | \mathbf{y} \leqslant \mathbf{Y}\lambda, \mathbf{e}^{\mathbf{T}}\lambda = 1, \lambda \geqslant \mathbf{0}\}$, $\mathbf{Y} = [y_{ki}]_{rxn}$, $\lambda = (\lambda_i)_{n\times 1}$, $\lambda \in R^n$.*

Proof. If $\mathbf{y}_0 + \Delta\mathbf{y}_0 \in P_{out}$, then the constraints $\sum_{i=1}^{n} \lambda_i y_{ki} \geqslant y_{k0} + \Delta y_{k0}$ for $k = 1, \ldots, r$ and $\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \geqslant 0,\ i = 1, 2, \ldots, n$ in the MLDIBCC$_0$ model are satisfied. The constraints $\sum_{i=1}^{n} \lambda_i x_{ji} \leqslant \theta_0^*(x_{j0} + \Delta x_{j0})$ for $j = 1, \ldots, m$ can be satisfied by finding the appropriate value of Δx_{j0}. Also, from the constraints $\sum_{i=1}^{n} \lambda_i x_{ji} \leqslant \theta_0^*(x_{j0} + \Delta x_{j0})$ for $j = 1, \ldots, m$, we know that $x_{j0} + \Delta x_{j0} \geqslant 0$ for $j = 1,\ldots,m$. The objective of the MLDIBCC$_0$ model is to minimize $\Delta\mathbf{x}_0$, therefore, there exists at least an optimal solution to the MLDIBCC$_0$ model. This proves that there exits at least an optimal solution to the MLDIBCC$_0$ model if $\mathbf{y}_0 + \Delta\mathbf{y}_0 \in P_{out}$.

If there exists at least an optimal solution to the MLDIBCC$_0$ model, from the set of constraints $\sum_{i=1}^{n} \lambda_i y_{ki} \geqslant y_{k0} + \Delta y_{k0}$ for $k = 1, \ldots, r$ in the MLDIBCC$_0$ model, then $\mathbf{y}_0 + \Delta\mathbf{y}_0 \in P_{out}$. □

From Lemma 1, we can check whether $\mathbf{y}_0 + \Delta\mathbf{y}_0$ is in P_{out} or not by determining a set of non-dominated DMUs based on the output comparison. Then if all elements of $\mathbf{y}_0 + \Delta\mathbf{y}_0$ is less than or equal to all elements of the outputs of at least one DMU in the non-dominated set, then $\mathbf{y}_0 + \Delta\mathbf{y}_0$ is in P_{out}.

Theorem 2. $\Delta\mathbf{x}_0 = (\Delta x_{10}, \Delta x_{20}, \ldots, \Delta x_{m0})$ *obtained by solving the LDIBCC$_0$ model is a Pareto solution for the MLDIBCC$_0$ model.*

Table 1
Inputs and outputs for the efficiency analysis of die press machines in the press division of a motorcycle-part company.

Details	Type of data
Overtime hour of direct labor (h)	Input 1
Number of stopping times to change die and adjust die press machine in a month	Input 2
Number of testing press before a real press (stroke)	Input 3
Time of moving die from forklift (min)	Input 4
Time for preventive maintenance (h)	Input 5
Time for repair and adjustment (labor hour)	Input 6
Stopping time of die press machine for quality error detection (h)	Input 7
Number of total press resulted in good parts (stroke)	Output 1
Process capability ratio (C_p)	Output 2
Percent of on-schedule press (%)	Output 3

$$
\begin{aligned}
(\mathrm{LDIBCC}_0)\quad & \text{minimize } \mathbf{W}^{\mathbf{T}}\Delta\mathbf{x}_0 \\
\text{s.t.}\quad & \sum_{i=1}^{n}\lambda_i x_{ji} \leqslant \theta_0^*(x_{j0}+\Delta x_{j0}) \quad \text{for } j=1,\ldots,m \\
& \sum_{i=1}^{n}\lambda_i y_{ki} \geqslant y_{k0}+\Delta y_{k0} \quad \text{for } k=1,\ldots,r \\
& \sum_{i=1}^{n}\lambda_i = 1 \\
& \lambda_i \geqslant 0, \quad i=1,2,\ldots,n
\end{aligned}
\tag{12}
$$

where $\mathbf{W}^{\mathbf{T}} \in R^m$.

Proof. Assume that $\Delta\mathbf{x}_0^* \in R^m$ and $\boldsymbol{\lambda}^* = (\lambda_1^*,\ldots,\lambda_n^*)$ are the optimal solution from solving the LDIBCC$_0$ model but they were not Pareto solution to the MLDIBCC$_0$ model. There should be a possible $\overline{\Delta\mathbf{x}_0} \in R^m$ and $\bar{\boldsymbol{\lambda}} = (\overline{\lambda_1},\ldots,\overline{\lambda_n})$ from the MLDIBCC$_0$ model where $\overline{\Delta\mathbf{x}_0} \leqslant_p \Delta\mathbf{x}_0^*$, and thus $\mathbf{W}^{\mathbf{T}}\overline{\Delta\mathbf{x}_0} < \mathbf{W}^{\mathbf{T}}\Delta\mathbf{x}_0^*, \mathbf{W}^{\mathbf{T}} > \mathbf{0}$. Note that $\overline{\Delta\mathbf{x}_0} \leqslant_p \Delta\mathbf{x}_0^*$ represents a set of inequalities $\overline{\Delta x_{j0}} \leqslant \Delta x_{j0}^*, j = 1,2,\ldots,m$ with at least one strict inequality, $\overline{\Delta x_{j0}} < \Delta x_{j0}^*$.

Since the MLDIBCC$_0$ model and the LDIBCC$_0$ model have the same constraint sets, $\overline{\Delta\mathbf{x}_0} \in R^m$ and $\bar{\lambda}$ are also the solution to the LDIBCC$_0$ model. This leads to a contradiction; therefore, $\Delta\mathbf{x}_0^* \in R^m$ and λ^* from the LDIBCC$_0$ model would also be a Pareto solution to the MLDIBCC$_0$ model. □

From Theorem 2, if we find any positive vector, $\mathbf{W}^{\mathbf{T}} \in R^m$, we would be able to find a Pareto solution for the MLDIBCC$_0$ model from solving the LDIBCC$_0$ model, which is a linear programming model. Consequently, the input and output vector of DMU$_{0'}$ $(\mathbf{x_0}+\Delta\mathbf{x_0}, \mathbf{y}_0+\Delta\mathbf{y}_0)$ obtained from the LDIBCC$_0$ model will be a Pareto-efficient solution to the inverse BCC model.

4.1. Evidence base in a case study in the production plant of motorcycle parts

The data is taken from the die press division of a motorcycle-part company. The die press division is responsible for forming motorcycle parts with die press machines. There are 25 die press machines with 80-ton press in the die press division. The die press machines are considered as decision making units in this study. Types of inputs and outputs for the efficiency analysis of the die press machines are given in Table 1. There are 7 inputs and 3 outputs.

Input and output values of the die press machines in December 2009 are given in Table 2 for the efficiency analysis.

After solving the BCC models (PBCC$_0$ or DBCC$_0$) for all DMUs with the data from Table 2, the relative efficiency values (θ_i^*) of all DMUs are given in Table 3.

From Table 3, there are only six technically inefficient DMUs, which are DMU$_4$, DMU$_9$, DMU$_{15}$, DMU$_{20}$, DMU$_{23}$ and DMU$_{24}$. All other DMUs are technically efficient.

If we compare the performance of all DMUs based on outputs only, the set of non-dominated DMUs includes DMU$_6$, DMU$_{11}$, DMU$_{12}$, DMU$_{14}$, DMU$_{16}$, DMU$_{18}$, DMU$_{19}$ and DMU$_{22}$.

Let us consider an inefficient DMU$_4$, the optimal objective value is $\theta_4^* = 0.75$, which is less than 1. Suppose that the output vector of DMU$_4$ is changed from $(672, 1.27, 96)^T$ to $(650, 1.4, 97)^T$ and let $\mathbf{W}^{\mathbf{T}} = (1,1,1,1,1,1,1)$ for input weights. Solving the LDIBCC$_0$ model by using a solver in Microsoft Excel 2007 for DMU$_4$, we find that there is no feasible solution to this problem. The results follow

Table 2
Input and output values of die press machines in December 2009.

DMUs	Inputs/outputs									
	Input 1	Input 2	Input 3	Input 4	Input 5	Input 6	Input 7	Output 1	Output 2	Output 3
DMU$_1$	18	8	93	140	5	110	2	861	1.27	98
DMU$_2$	18	7	87	120	5	80	0	947	1.27	96
DMU$_3$	18	7	83	110	5	70	0	906	1.29	96
DMU$_4$	24	14	163	260	15	210	0	672	1.27	96
DMU$_5$	21	5	62	120	5	90	3	974	1.27	98
DMU$_6$	21	7	79	150	20	110	0	867	1.28	100
DMU$_7$	18	9	102	210	20	150	2	806	1.27	98
DMU$_8$	18	8	95	110	20	80	2	869	1.29	97
DMU$_9$	24	14	136	320	15	240	0	696	1.28	97
DMU$_{10}$	24	9	98	220	20	170	0	817	1.28	100
DMU$_{11}$	24	8	94	110	15	120	1	913	1.23	100
DMU$_{12}$	24	6	82	170	15	120	0	989	1.27	99
DMU$_{13}$	18	8	97	130	15	170	0	823	1.29	95
DMU$_{14}$	18	7	89	90	15	130	0	862	1.32	97
DMU$_{15}$	21	7	79	180	15	140	0	888	1.31	98
DMU$_{16}$	21	7	87	110	10	130	1	847	1.34	97
DMU$_{17}$	21	9	113	140	10	80	0	793	1.27	98
DMU$_{18}$	21	6	74	130	10	130	0	957	1.29	99
DMU$_{19}$	21	7	79	110	10	110	0	842	1.32	99
DMU$_{20}$	21	8	95	200	10	170	2	839	1.28	97
DMU$_{21}$	18	7	74	170	15	110	3	887	1.26	98
DMU$_{22}$	21	7	76	90	5	120	0	898	1.31	98
DMU$_{23}$	24	16	187	290	15	280	1	614	1.29	98
DMU$_{24}$	24	13	175	210	15	260	0	735	1.28	95
DMU$_{25}$	18	8	91	130	20	120	0	885	1.31	96

Table 3
The relative efficient values (θ_i^*) of DMU_i, $i = 1,\ldots,25$.

DMU_i	θ_i^*	DMU_i	θ_i^*	DMU_i	θ_i^*	DMU_i	θ_i^*	DMU_i	θ_i^*
DMU_1	1	DMU_6	1	DMU_{11}	1	DMU_{16}	1	DMU_{21}	1
DMU_2	1	DMU_7	1	DMU_{12}	1	DMU_{17}	1	DMU_{22}	1
DMU_3	1	DMU_8	1	DMU_{13}	1	DMU_{18}	1	DMU_{23}	0.78
DMU_4	0.75	DMU_9	0.77	DMU_{14}	1	DMU_{19}	1	DMU_{24}	0.75
DMU_5	1	DMU_{10}	1	DMU_{15}	0.98	DMU_{20}	0.87	DMU_{25}	1

from Lemma 1, i.e., the new output values of DMU_4 is not in P_{out}, i.e., the second output element of DMU_4 is greater than the second output element of all non-dominated DMUs. Therefore, in this case we cannot find the new input vector, which can maintain the relative efficiency of DMU_4 when the output values of DMU_4 are changed to $(650, 1.4, 97)^T$.

Let us consider DMU_4 again but now let the output vector of DMU_4 be changed from $(672, 1.27, 96)^T$ to $(650, 1.30, 97)^T$ and let $\mathbf{W^T} = (1,1,1,1,1,1,1)$ for input weights. We can observe that the new output values are in P_{out}, i.e., all output elements of DMU_4 are less than or equal to those of DMU_{14}, DMU_{16}, DMU_{19} and DMU_{22}. By solving the $LDIBCC_0$ model for DMU_4, we find that the optimal solution is $\Delta x_{14} = 2$, $\Delta x_{24} = -4.67$, $\Delta x_{34} = -57$, $\Delta x_{44} = -126.67$, $\Delta x_{54} = -8.33$, $\Delta x_{64} = -83.33$, $\Delta x_{74} = 0$ and the optimal objective value is equal to -278. Therefore, the new input vector is $(26, 9.33, 106, 133.33, 6.67, 126.67, 0)^T$. Using the new input and output vectors for DMU_4, the relative efficiency values of all DMUs still remain the same.

Now we consider an efficient DMU. Let take DMU_5 into account. The optimal objective value of DMU_5 is $\theta_5^* = 1$ as given in Table 3. Suppose that the output vector of DMU_5 is changed from $(974, 1.27, 98)^T$ to $(980, 1.20, 100)^T$ and let $\mathbf{W^T} = (1,1,1,1,3,3,3)$ for input weights. By solving the $LDIBCC_0$ model for DMU_5, we cannot find an optimal solution because $(980, 1.20, 100)^T$ is not in P_{out}.

Now we consider DMU_5 again but the output vector of DMU_5 is changed from $(974, 1.27, 98)^T$ to $(980, 1.20, 99)^T$ and let $\mathbf{W^T} = (1,1,1,1,3,3,3)$ for input weights. We can find the optimal solution to the $LDIBCC_0$ model for DMU_5 because the new output values of DMU_5 are in P_{out}, i.e., all output elements of DMU_5 are less than or equal to those of DMU_{12}. The optimal solution is $\Delta x_{15} = 2.70$, $\Delta x_{25} = 1.10$, $\Delta x_{35} = 19.21$, $\Delta x_{45} = 39.12$, $\Delta x_{55} = 9.01$, $\Delta x_{65} = 27.03$, $\Delta x_{75} = -2.60$ and the optimal objective value is equal to 162.45. Therefore, the new input vector is $(23.70, 6.10, 81.21, 159.12, 14.01, 117.03, 0.4)^T$. Using the new input and output vectors for DMU_5, the relative efficiency values of all DMUs still remain the same.

5. Conclusions

The traditional inverse DEA model is used to determine the best possible values of inputs (outputs) for given values of outputs (inputs) of a considered DMU such that relative efficiency value of a considered DMU with respect to other DMUs remain unchanged. In this paper, we propose an inverse DEA for the case of variable returns to scale (inverse BCC model). The proposed inverse BCC model takes into account the relative efficiency values of all DMUs, which is different from the inverse DEA models by others (Hadi-Vencheh & Foroughi, 2006; Wei et al., 2000; Yan et al., 2002). The inverse BCC model is used to determine the best possible values of inputs for given values of outputs of a considered DMU such that relative efficiency values of all DMUs with respect to other DMUs in a new production possibility set remain unchanged. The new production possibility is a set of all current DMUs and a perturbed DMU with new input and output values. In this paper, we study the inverse BCC model for the resource allocation problem where increases of some outputs and decreases of the other outputs can be taken into account simultaneously. However, the proposed inverse BCC problem is in the form of a MONLP, which is not easy to solve. To find the optimal solution to the inverse BCC problem, we propose the $LDIBCC_0$ model, which is in the form of a linear programming model. However, there exists at least an optimal solution to the $LDIBCC_0$ model if and only if the new output vector is in the set of current production possibility set. We illustrated the proposed approach via a case study in the die press division of a motorcycle-part company.

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