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Technical Note

On inverse DEA model: The case of variable returns to scale

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ABSTRACT

Lertworasirikul et al. (2011) proposed an inverse BCC model for the case of VRS. They used an MOLP model to assist them in the process of finding inputs. The proof of the main theorem contains considerable mistakes which invalidate the proof. Moreover, there are some ambiguities about the use of MOLP. The current article points out the drawbacks of Lertworasirikul et al. (2011) and then corrects the use of MOLP in a proposed inverse BCC model. This commentary continues with a revision of the aforementioned theorem with a valid proof. The proposed proof is entirely different from and simpler than Lertworasirikul et al's. (2011). A numerical example illustrates the proposed models.

1. Introduction

Data envelopment analysis (DEA) is a nonparametric technique based on an optimization method that assesses the relative efficiency of a group of decision making units (DMUs). Each DMU consumes some inputs and produces some outputs while the relative efficiency is measured based on inputs and outputs. Wei, Zhang, and Zhang (2000) were inspired by inverse optimization techniques and looked at DEA from a different perspective that yielded inverse DEA models. Inverse DEA models try to answer questions like: If a DMU perturbs its input (output), to what extent should its output (input) be changed to preserve the relative efficiency of the DMU?

Lertworasirikul, Charnsethikul, and Fang (2011) proposed an inverse DEA model assuming variable returns to scale (VRS). This model is named the inverse BBC model following the fact that the relative DEA model with VRS properties is known as the BCC model which was introduced by Banker, Charnes, and Cooper (1984). Lertworasirikul et al. (2011) first dealt with the inverse DEA model which is a non-linear programming and then they moved into a multiple-objective linear programming MOLP structure that has also been used in the literature regarding the difficulty associated with solving non-linear problems. However, this article contains some ambiguities about using MOLP in the inverse BCC model. They used the minimum solution of an MOLP (assuming minimization in the objective function) while it is well known that MOLP models do not have a unique solution. Regardless of the latter shortcoming, the proposed proof for the main theorem contains some considerable mistakes/flaws that influence and invalidate the proof.

The current article addresses the ambiguities in using MOLP in the inverse BCC model by Lertworasirikul et al. (2011). This shortcoming is eliminated by clarifying and using MOLP in a correct way. The theorem shows how usable MOLP is in finding the input levels that preserve the relative efficiency of DMUs for a given perturbation of output.

Section 2 briefly reviews basic DEA models. Section 3 deals with the inverse BCC model and points out the drawbacks in the existing literature. Section 4 revises the aforementioned drawbacks by proposing the correct use of MOLP in the inverse BCC model. This section is concluded with an illustrative example. Section 5 presents the conclusion of the article.

2. The DEA model

Suppose that there are n DMUs that consume m positive inputs and produce s positive outputs. Let's denote $x_{ij} > 0, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$ as the i-th input of the j-th DMU and $y_{rj} > 0, 1 \leqslant r \leqslant s, 1 \leqslant j \leqslant n$ as the r-th output of the j-th DMU. The following input-oriented DEA model –known as the BBC model-was proposed by Banker et al. (1984) for estimating the relative efficiency of DMU_0 $0 \in \{1, 2, \ldots, n\}$:

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0360-8352/

$$
\begin{aligned}
&Min\ \theta_0\\
&s.t \sum_{j=1}^{n}\lambda_j x_{ij} \leqslant \theta_0 x_{i0} \quad i=1,2,\ldots m\\
&\sum_{j=1}^{n}\lambda_j y_{rj} \geqslant y_{r0} \quad r=1,2,\ldots s\\
&\sum_{j=1}^{n}\lambda_j = 1\\
&\lambda_j \geqslant 0, \qquad j=1,2,\ldots,n
\end{aligned} \tag{1}
$$

which is based on the following production possibility set (PPS):

$$T=\left\{(x,y)\middle| x \geqslant \sum_{j=1}^{n}\lambda_j x_j, y \leqslant \sum_{j=1}^{n}\lambda_j y_j, \sum_{j=1}^{n}\lambda_j = 1, \lambda_j \geqslant 0, j=1,2,\ldots,n\right\}$$

where $x_j = \begin{pmatrix} x_{1j}\\ x_{2j}\\ \vdots\\ x_{mj}\end{pmatrix}$ and $y_j = \begin{pmatrix} y_{1j}\\ y_{2j}\\ \vdots\\ y_{sj}\end{pmatrix}$ are the input vector and the output vector of the j-the DMU respectively.

Definition 1. The DMU_0 is called (at least weakly) efficient if the optimal value of model (1) is unity.

If a DMU is not efficient, we call it inefficient and the optimal value of the model (1) is between zero and unity for these DMUs. The efficient DMUs shape the efficient frontier while all the DMUs which are not on the efficient frontier are deemed inefficient.

3. The inverse BCC model

Inverse DEA models try to answer questions like: if DMU_0, for instance, changes its current output into$\beta_0 = y_0 + \Delta y_0, \Delta y_0 \in \mathbb{R}^s$, then how much input is required to preserve the relative efficiency of DMU_0. The following MOLP model is proposed in the literature for estimating the required input:

$$
\begin{aligned}
&Min \quad (\alpha_1, \alpha_2, \ldots \alpha_m) = (x_{10}+\Delta x_1, x_{20}+\Delta x_2, \ldots, x_{m0}+\Delta x_m)\\
&s.t \quad \sum_{j=1}^{n}\lambda_j x_{ij} \leqslant \theta_0^* \alpha_i \quad i=1,2,\ldots m\\
&\qquad \sum_{j=1}^{n}\lambda_j y_{rj} \geqslant \beta_{r0} \quad r=1,2,\ldots s\\
&\qquad \sum_{j=1}^{n}\lambda_j = 1\\
&\qquad \lambda_j \geqslant 0, \quad j=1,2,\ldots,n
\end{aligned} \tag{2}
$$

where θ_0^* is the optimal value of the model (1) and $\alpha = x_0 + \Delta x, \Delta x \in \mathbb{R}^m$ is the required input levels that guarantee unchanged relative efficiency for DMU_0. Lertworasirikul et al. (2011) also used the above model as the core model. As a matter of fact, they dealt with the inverse DEA Model as a non-linear program first and then moved into an MOLP model (2) due to the difficulty of solving a non-linear problem. Theorem 1 in Lertworasirikul et al. (2011) applied model (2) and stated that " the minimum Δx of the perturbed DMU_0, which does not make any changes to the relative efficiency values of all DMUs can be obtained by solving the MOLP model (2)" (see model (7) and Theorem 1 of Lertworasirikul et al. (2011)).

There are considerable flaws with the stated theorem and its proof which are described in what follows. First of all, Lertworasirikul et al. (2011) used the minimum of MOLP model (2) in Theorem 1 and it is not clear what minimum means in an MOLP problem. It is already known that MOLP does not have a unique solution and that is why the weak efficient solutions are defined.

Definition 2. Suppose $(\lambda, \alpha) = (\lambda_1, \ldots, \lambda_n, \alpha_1, \ldots, \alpha_m)$ is a feasible solution model (2). If there is no feasible solution $(\overline{\lambda}, \overline{\alpha})$ for this model such that $\overline{\alpha_i} < \alpha_i$ for all $i = 1, 2, \ldots, m$, then we can say the (λ, α) is a weak efficient solution for model (2).

Now, let us point to some critical faults/flaws in the Proof of Theorem 1. The model (9) of Lertworasirikul et al. (2011)(model $DBCC'_0$) is used to check the efficiency score of perturbed DMU_0, that is $(\alpha_0, \beta_0) = (x_0 + \Delta x_0, y_0 + \Delta y_0)$.

Following the Proof of Theorem 1 by Lertworasirikul et al. (2011), the set of constraints of model $DBCC'_0$ are rearranged by dividing the constraints of both sides into $1 - \lambda_{0'} > 0$ Considering $1 - \lambda_{0'} \neq 0$, Lertworasirikul et al. (2011) divided both sides of the above set of constraints into $1 - \lambda_{0'} > 0$. Lertworasirikul et al. (2011) wrongly assumed $\sum_{j=1}^{n} \bar{\lambda}_j + \frac{\lambda_{0'}}{1-\lambda_{0'}} = 1$ while $\sum_{j=1}^{n} \bar{\lambda}_j + \frac{\lambda_{0'}}{1-\lambda_{0'}} = \frac{1}{1-\lambda_{0'}}$ and this is the key relation for the rest of proof. Eventually, this indicates that the Proof of Theorem 1 by Lertworasirikul et al. (2011)is not valid. In the next section, we will propose a correct version of Theorem 1 by Lertworasirikul et al. (2011) and then we will prove this theorem by using the weak efficiency notion of MOLP.

4. Revised inverse BBC model

In this section, we modify the inverse BCC model of Lertworasirikul et al. (2011) and describe the relationship between the inverse BCC model and the relative MOLP. Weak efficient solutions for MOLP model (2) play a key role in this part. In fact, weak efficient solutions guarantee unchanged relative efficiencies for a given perturbation of outputs. The following theorem describes the role of weak efficient solutions in the inverse BCC model.

Theorem. *Assume that the relative efficiency of DMU_0 is θ_0^* and the output value of DMU_0 is perturbed into $\beta_0 = y_0 + \Delta y_0$. If (λ, α_0) is a weak efficient solution for MOLP model (2), where $\alpha_0 = x_0 + \Delta x_0$, then the relative efficiency of perturbed DMU – $DMU_{0'} = (\alpha_0, \beta_0) = (x_0 + \Delta x_0, y_0 + \Delta y_0)$ – is also θ_0^*. Moreover, the aforementioned perturbation also does not affect the efficiency score of other DMUs.*

Proof. Suppose that (λ, α_0) is a weak efficient solution for MOLP model (2); then, we will have:

$$
\begin{aligned}
&\sum_{j=1}^{n}\lambda_j x_{ij} \leqslant \theta_0^* \alpha_{i0} \quad i=1,2,\ldots,m\\
&\sum_{j=1}^{n}\lambda_j y_{rj} \geqslant \beta_{r0} \quad r=1,2,\ldots s\\
&\sum_{j=1}^{n}\lambda_j = 1\\
&\lambda_j \geqslant 0 \quad j=1,2,\ldots,n
\end{aligned} \tag{3}
$$

This means $(\alpha_0, \beta_0) = (x_0 + \Delta x_0, y_0 + \Delta y_0) \in T$. Model $DBCC'_0$ is used to check the relative efficiency of the perturbed DMU_0. Note that model $DBCC'_0$ is built based on the following extended PPS:

$$
\begin{aligned}
T' = \Bigg\{(x,y)\Bigg| x \geqslant \sum_{j=1}^{n}\lambda_j x_j + \lambda_{0'}\alpha_0, y \leqslant \sum_{j=1}^{n}\lambda_j y_j + \lambda_{0'}\beta_0,\\
\sum_{j=1}^{n}\lambda_j + \lambda_{0'} = 1, \lambda_j, \lambda_{0'} \geqslant 0; j=1,2,\ldots,n\Bigg\}
\end{aligned}
$$

The extended PPS T' consists of all current DMUs and a perturbed DMU with new input and output values, in contrast with the normal PPS of T. Now, we show that the shape of the efficient frontier made by extended PPS of T' is the same as the shape of the

Table 1
Input and output data of three DMUs.

DMUs	Input 1	Input 2	Output 1	Output 2
A	5	8	7	9
B	7	6	5	6
C	6	4	8	6

efficient frontier made by the normal PPS T. In order to achieve this aim, note that the only difference between T and T' is that T' is made by adding (α_0, β_0) as an observed DMU to T. On the other hand, $(\alpha_0, \beta_0) \in T$; thus, adding (α_0, β_0) to T does not change the T and consequently the shape of efficiency remains unchanged. Therefore, we use the following model based on PPS T instead of PPS T' – to check the relative efficiency of the perturbed DMU:

$$\begin{aligned}
&Min \quad \theta_{0'} \\
&s.t \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leqslant \theta_{0'} \alpha_{i0} \quad i = 1, 2, \ldots, m && (4)\\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geqslant \beta_{r0} \quad r = 1, 2, \ldots, s && (5)\\
&\qquad \sum_{j=1}^{n} \lambda_j = 1 \\
&\qquad \lambda_j \geqslant 0 \quad j = 1, 2, \ldots, n
\end{aligned}$$

Considering the set of constraints (3) that satisfy the constraints of model (4), we get $\theta^*_{0'} \leqslant \theta^*_0$, where $\theta^*_{0'}$ is the optimal value of model (4). Now, we just need to show that $\theta^*_{0'} \not< \theta^*_0$ to prove $\theta^*_{0'} = \theta^*_0$. On the contrary, assume that $\theta^*_{0'} < \theta^*_0$; then, we have $\theta^*_{0'} = k\theta^*_0, 0 < k < 1$ that yields:

$$\begin{aligned}
&\sum_{j=1}^{n} \lambda_j x_{ij} \leqslant \theta^*_{0'} \alpha_{i0} = k\theta^*_0 \alpha_{i0} \quad i = 1, 2, \ldots, m \\
&\sum_{j=1}^{n} \lambda_j y_{rj} \geqslant \beta_{r0} \quad r = 1, 2, \ldots, s \\
&\sum_{j=1}^{n} \lambda_j = 1 \\
&\lambda_j \geqslant 0 \quad j = 1, 2, \ldots, n
\end{aligned}$$

This means that $(\lambda, k\alpha_0)$ is a feasible solution for MOLP model (2), but this contradicts the fact that (λ, α_0) is a weak efficient solution for MOLP model (2) since $k\alpha_0 < \alpha_0$ and this invalidates the proof. Note that as a consequence of unchanged normal PPS T, the efficiency score of other DMUs also remain unchanged. We already showed that the efficiency score of DMU_0 does not change after perturbation ($\theta^*_{0'} = \theta^*_0$). Therefore, it can be concluded that model (2) preserves the relative efficiency of all DMUs. □

Note if the output of the perturbed DMU is greater than the maximum output of all original DMUs, there is no feasible solution to model (2). This is predictable considering the variable returns to scale assumption. In fact, VRS properties do not allow a decision maker to increase outputs as much as s/he wants and model (2) accounts for this fact by infeasibility. Obviously, if we assume constant returns to scale or increasing returns to scale, then model (2) will always be feasible, no matter how much we perturb DMUs. Technically, when we assume constant returns to scale or increasing returns to scale, $\sum_{j=1}^{n} \lambda_j = 1$ is dropped or replaced by $\sum_{j=1}^{n} \lambda_j > 1$ respectively in model (2). This leads to an arbitrary perturbation of DMUs.

A numerical example; Consider Table 1 which shows the data of three DMUs. Each DMU uses two inputs and produces two outputs.

Consider DMU B, for instance; the relative efficiency of DMU B is 0.8235 using the BCC model (1). DMU B is the only inefficient DMU i.e., other DMUs are efficient. Now, assume that this DMU needs to increase its second output to 6.5. The MOLP model (2) shows to what extent we should change the input of DMU B to preserve the relative efficiency of this DMU. We use the weighted sum method (see Steuer, 1986) with unity weights for solving the MOLP model. $(\lambda_1, \lambda_2, \lambda_3, \Delta x_{B1}, \Delta x_{B2}) = (0.1666667, 0, 0.833333, -1.9402955, -0.3331309)$ is a weakly efficient solution for MOLP model. This solution suggests that DMU B could save $-\Delta x_{1B}$ and $-\Delta x_{2B}$ units from its first and second input respectively to preserve its relative efficiency. Now, we need to check the relative efficiency of perturbed DMU B, that is, $(\alpha_{1B}, \alpha_{2B}, \beta_{1B}, \beta_{2B}) = (5.059705, 5.666869, 5, 6.5)$. Model (4) fulfills this aim and checks the validity of the latter statement. Using model (4), the relative efficiency of perturbed DMU B is 0.8235 which is the same as the relative efficiency of DMU B before perturbation. This shows that if we use a weakly efficient solution for MOLP model (2) for input levels, then the relative efficiency of the DMU stays unchanged for the given perturbation of output levels. Moreover, efficient DMUs remain efficient after perturbation of DMU B. In fact, model (2) preserves the efficiency scores of all DMUs.

5. Conclusion

This article dealt with some problematic issues of a rather recent published paper by Lertworasirikul et al. (2011) on inverse DEA. The use of the MOLP by Lertworasirikul et al. (2011) is vague due to the fact that it does not determine what type of solution for MOLP is used in their paper. On the other hand, the proof of their proposed theorem contains some mistakes/flaws and is not valid. The current article clarifies the use of MOLP on input estimation by using the inverse DEA model. Moreover, it proposes the correct form of the theorem for input estimation and then provides a valid and simple proof for the aforementioned theorem. A numerical example is also provided to illustrate the proposed models.

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