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A general inverse DEA model for non-radial DEA☆

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ABSTRACT

Traditional inverse DEA models could be called inverse radial DEA because they are based on radial efficiency measures. Due to the neglect of slacks in evaluating the efficiency score, inverse radial DEA may mislead decision-making in some cases where slacks play important roles. In this paper, we proposed an integrated framework of inverse DEA called inverse non-radial DEA since it is based on non-radial DEA by multi-objective programming, which covers existing inverse DEA models. To further illustrate the inverse non-radial DEA, we construct the concrete mathematical formula of inverse SBM and some properties. In contrast to the radial approach, inverse non-radial DEA can overcome the error caused by ignoring slacks and provides more valuable information about inputs and outputs for decision-making by considering slacks. Although inverse non-radial DEA models are usually non-linear, we can convert it into a one-dimensional search problem about efficiency score, which can be solved by many existing efficient algorithms. A practical example is provided to demonstrate the advantages of inverse non-radial DEA models over inverse radial DEA models.

1. Introduction

Data envelopment analysis (DEA) is a linear programming based approach to measure the relative efficiencies of decision-making units (DMUs) with given input and output levels. The first DEA model CCR was proposed by Charnes et al. to deal with proportional changes of inputs or outputs (Charnes, Cooper, & Rhodes, 1978). Soon, this technique was widely used and developed by some scholars, see e.g., Banker, Charnes, and Cooper (1984), Charnes, Cooper, Wei, and Huang (1989), and Seiford and Thrall (1990). These models above are called radial model due to the neglect of slacks in evaluating the efficiency score. However, if these slacks play important roles in evaluating managerial efficiency, the radial approaches may result in decision error when we utilize the efficiency score as the only index for evaluating the performance of DMUs. Therefore, some researchers dealt directly with the input excesses and the output shortfalls of the DMU concerned by considering the effect of slacks, such as Russell Measure (RM) (Russell, 1985), Enhancement Russell Measure (ERM) (Pastor, 1999), Range Adjusted Measure (RAM) (Cooper, Park, & Pastor, 1999), Slacks-based Measure (SBM) (Tone, 2001), Bounded Adjusted Measure (BAM) (Cooper, Pastor, Borras, Aparicio, & Pastor, 2011).

The inverse DEA approach aims to estimate the feasible output levels under increasing inputs and preserving the efficiency score within the DEA framework. This inverse optimization problem can be served as a useful planning tool for management decisions by providing information such as how many resources should be invested to achieve a desired level of competitiveness. In contrast, conventional DEA concentrated mainly on evaluating organizational performance. The idea of inverse DEA was firstly introduced by Zhang and Cui (1999). Wei et al. subsequently formalize it by utilizing multiple-objective linear programming (MOLP) techniques (Wei, Zhang, & Zhang, 2000). After that, this problem has been studied in many theoretical and applied publications. Hadi-Vencheh et al. introduced another inverse DEA to estimate the inputs under given increasing outputs and preserving the efficiency score (Hadi-Vencheh, Foroughi, & Soleimani-damaneh, 2008). Zhang and Cui integrated inverse DEA models into 12 different scenarios (Zhang & Cui, 2016). Apart from these, a stream of literature have contributed to the methodological developments such as input-estimation for resource allocation (Xiaoya & Jinchuan, 2008; Yan, Wei, & Hao, 2002; Zhang, Wang, & Cui, 2018), inter-temporal application (Jahanshahloo, Soleimani-damaneh, & Ghobadi, 2015), fuzzy inverse DEA (Ghobadi & Jahangiri, 2015), inverse DEA with frontier changes (Lim, 2016), undesirable outputs or inputs (Eyni, Tohidi, & Mehrabeian, 2017), cost and revenue efficiency (Ghiyasi, 2017). In addition to illustrate the method provided in these literatures above, inverse DEA has been used widely to deal with various real-world

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problems including recent application on sustainability accession (Hassanzadeh, Yousefi, Saen, & Hosseininia, 2018), Enterprise merger (Amin, Emrouznejad, & Gattoufi, 2017; Gattoufi, Amin, & Emrouznejad, 2014), revenue target management (Lin, 2010), and pricing strategy (Frija, Wossink, Buysse, Speelman, & Van Huylenbroeck, 2011).

As far as we know, the above inverse DEA models are all based on radial DEA, which can be called inverse radial DEA. Since radial efficiency measure neglects the effect of slacks, the inverse radial DEA may mislead decision-making when we utilize it to estimate the feasible outputs (inputs) levels under increasing inputs (outputs) and preserving the efficiency score. The estimation based on inverse radial DEA may be unreasonable and impossible in some cases where slacks play an important role. Therefore, inverse DEA should consider slacks to estimate the output (inputs) levels.

To our knowledge, there is little literature considering inverse DEA based on non-radial measures except for Jahanshahloo, Hosseinzadeh Lotfi, Rostamy-Malkhalifeh, and Ghobadi (2014) and Ghobadi (2018). They proposed an inverse DEA model based on Enhanced Russell Model, assuming that the efficiency scores of each dimension remain unchanged. In this paper, we introduce a more general inverse DEA model called inverse non-radial DEA based on non-radial DEA, only assuming that the overall efficiency scores remain unchanged, not every dimension. What is more important, this approach provides an integrated framework of the inverse DEA problem and a corresponding solving algorithm, which covers all the non-radial and radial measures that are monotonous. In other words, we propose a basic form of all inverse DEA models, because monotonicity is one of the fundamental properties of DEA measures. To state it more clearly, we establish an inverse SBM model to illustrate inverse non-radial DEA models. From the perspective of applications, except for providing better estimations, inverse non-radial DEA can help decision-makers find out cost-effective indexes by considering the changes of slacks. In contrast, slacks are directly converted into outputs in inverse radial DEA. Individual work is therefore intended to enrich the literature not only on inverse DEA problems but also on various applications of DEA as a predictive decision support tool for management science.

The remainder of the paper is organized as follows. In the next section, we introduce some preliminaries for inverse radial DEA and its defects. Section 3 provides an integrated framework of the inverse non-radial DEA by multi-objective programming. Section 4 establishes the inverse SBM model as an illustrative case to further explain inverse non-radial DEA. Section 5 provides a first-order algorithm to solve inverse non-radial DEA. Section 6 provides an example to specify the advantages of inverse non-radial DEA compared to inverse radial DEA. Finally, Section 7 provides a summary of the results and possible future research directions.

2. Inverse radial DEA models and its defects

DEA is a non-parametric technique based on an optimization method that assesses the relative efficiency of a group of DMUs. Each DMU consumes some inputs and produces some outputs while the relative efficiency is measured based on inputs and outputs. Suppose there are n decision-making units with input and output matrices $\boldsymbol{x} = x_{ij} \in \mathbb{R}^{m\times n}, \boldsymbol{x} \geqslant \boldsymbol{0}$ and $\boldsymbol{y} = y_{rj} \in \mathbb{R}^{s\times n}, \boldsymbol{y} \geqslant \boldsymbol{0}$. The general production possible set (PPS) is defined as:

$$\mathcal{T} = \left\{ \left(\boldsymbol{x}, \boldsymbol{y} \right) \middle| \sum_{j=1}^{n} \lambda_j \boldsymbol{x_j} \leqslant \boldsymbol{x}, \sum_{j=1}^{n} \lambda_j \boldsymbol{y_j} \geqslant \boldsymbol{y}, \lambda \in \boldsymbol{\Gamma} \right\}, \tag{1}$$

where $\boldsymbol{\Gamma}$ indicates 4 different DEA models like follows:

(i) $\boldsymbol{\Gamma} = \{\lambda, \lambda \geqslant \boldsymbol{0}\}$ corresponds to constant return to scale;

(ii) $\boldsymbol{\Gamma} = \left\{\lambda, \sum_{j=1}^{n} \lambda_j = 1, \lambda \geqslant \boldsymbol{0}\right\}$ corresponds to various return to scale;

(iii) $\boldsymbol{\Gamma} = \left\{\lambda, \sum_{j=1}^{n} \lambda_j \leqslant 1, \lambda \geqslant \boldsymbol{0}\right\}$ corresponds to non-increase return to scale;

(iv) $\boldsymbol{\Gamma} = \left\{\lambda, \sum_{j=1}^{n} \lambda_j \geqslant 1, \lambda \geqslant \boldsymbol{0}\right\}$ corresponds to non-decrease return to scale.

For DMU_o, $o \in [N]$, a general DEA model for estimating the relative efficiency is that:

$$\theta_o = \min Eff(\boldsymbol{X_o}, \boldsymbol{Y_o}) \tag{2}$$

$$\text{s. t. } (\boldsymbol{X_o}^*, \boldsymbol{Y_o}^*) \in \mathcal{T}. \tag{3}$$

Let the efficiency score measure of DMU_o, $Eff(\boldsymbol{X_o}, \boldsymbol{Y_o})$ be θ_o, whether it is a radial model or a non-radial model. In the traditional radial DEA model, $(\boldsymbol{X_o}^*, \boldsymbol{Y_o}^*)$ indicating the projection point of $(\boldsymbol{X_o}, \boldsymbol{Y_o})$ on the production front is $(\theta_o \boldsymbol{X_o}, \boldsymbol{Y_o})$, whereas $(\boldsymbol{X_o}^*, \boldsymbol{Y_o}^*)$ has different formulas in a variety of non-radial DEA models. Given the constant efficiency score θ_o, we need to estimate the output vector $\boldsymbol{\beta}_o$ of DMU_o when the inputs of DMU_o are increased from $\boldsymbol{x_o}$ to $\boldsymbol{\alpha}_o = \boldsymbol{x_o} + \boldsymbol{\Delta x_o}$, where the vector $\boldsymbol{\Delta x_o} \geqslant \boldsymbol{0}$. Here . Although there are 12 cases of radial inverse DEA model, Zhang and Cui (2016) pointed out the solution of $Output_Y$ can be transformed into that of $Input_X$. Therefore, this paper mainly considers the following input-oriented model $InputX_+$ because the other cases are similar:

$$V - \max \boldsymbol{\beta}_o = (\beta_1, \ldots, \beta_s) \tag{4}$$

$$\text{s. t. } (\theta_o \boldsymbol{\alpha}_o, \boldsymbol{\beta}_o) \in \mathcal{T}, \tag{5}$$

where θ_o is the efficiency score of DMU_o under model (2). It is an input-oriented inverse DEA model with multi-objectives whose optimal solution is the Pareto solution. The general definition of Pareto solution is as follows and we usually ignore the slack in the radial model:

Definition 1. $(\overline{\boldsymbol{\beta}}_{\boldsymbol{o}}, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+) \in \mathcal{R}^s \times \mathcal{R}^n \times \mathcal{R}^m \times \mathcal{R}^s$ is a Pareto solution of multi-objectives inverse DEA model if there is no solution $(\boldsymbol{\beta}_o, \boldsymbol{\lambda}, \boldsymbol{S}^-, \boldsymbol{S}^+)$ such that $\boldsymbol{\beta}_o \gneqq \overline{\boldsymbol{\beta}}_o$ and at least one component of the strict inequality is established.

Definition 2. $(\overline{\boldsymbol{\beta}}_{\boldsymbol{o}}, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+) \in \mathcal{R}^s \times \mathcal{R}^n \times \mathcal{R}^m \times \mathcal{R}^s$ is a weak-Pareto solution of multi-objectives inverse DEA model if there is no solution $(\boldsymbol{\beta}_o, \boldsymbol{\lambda}, \boldsymbol{S}^-, \boldsymbol{S}^+)$ such that $\boldsymbol{\beta}_o > \overline{\boldsymbol{\beta}}_o$.

For convenience, we introduce a new DMU, DMU_{n+1}, to represent $(\boldsymbol{\alpha}_o, \boldsymbol{\beta}_o)$, which is DMU_o after the changes of inputs and outputs. The efficiency evaluation model of DMU_{n+1} is like follows:

$$\min Eff(\boldsymbol{\alpha}_o, \boldsymbol{\beta}_o) \tag{6}$$

$$\text{s. t. } (\boldsymbol{\alpha}_o^*, \boldsymbol{\beta}_o^*) \in \mathcal{T}. \tag{7}$$

Theorem 2.1. *If* $(\overline{\boldsymbol{\beta}}_o, \overline{\lambda})$ *is a weak-Pareto solution of model (4), then the efficiency score of new* $\text{DMU}(\boldsymbol{\alpha}_o, \boldsymbol{\beta}_o)$ *is still* θ_o. *Conversely, let* $(\overline{\boldsymbol{\beta}}_o, \overline{\lambda})$ *be a feasible solution of problem (4). If the efficiency score of DMU* $(\boldsymbol{\alpha}_o, \boldsymbol{\beta}_o)$ *under model (6) is* θ_o, *then* $(\overline{\boldsymbol{\beta}}_o, \overline{\lambda})$ *must be a weak-Pareto solution of model (4).*

Proof. Firstly, we derive the proof of the first half of the theorem by contradiction. Let $(\overline{\boldsymbol{\beta}}_o, \overline{\lambda})$ be a weak-Pareto solution of problem (4). Since $(\theta_o, \overline{\lambda})$ is also a feasible solution of model (6) due to the same constraints $(\theta_o \boldsymbol{\alpha}_o, \boldsymbol{\beta}_o) = (\boldsymbol{\alpha}_o^*, \boldsymbol{\beta}_o^*) \in \mathcal{T}$, we can know $\hat{\theta} \leqslant \theta_o$, where $\hat{\theta}$ is the optimal value of the model (6). Now by contradiction assume that $\hat{\theta} < \theta_o$. We can obtain some new outputs $\hat{\boldsymbol{\beta}}_{\boldsymbol{o}} = \overline{\boldsymbol{\beta}}_o + \boldsymbol{\Delta}\overline{\beta}_o(\boldsymbol{\Delta}\overline{\beta}_o > \boldsymbol{0})$ such that $Eff(\boldsymbol{\alpha}_o, \hat{\boldsymbol{\beta}}_{\boldsymbol{o}}) = \theta_o$ in the model (6) because $\hat{\theta} < \theta_o$. Note that $(\boldsymbol{\Delta}\overline{\boldsymbol{\beta}}_{\boldsymbol{o}} > \boldsymbol{0})$ is necessary because there exists some outputs $\hat{\boldsymbol{\beta}}_{\boldsymbol{o}} \gneqq \overline{\boldsymbol{\beta}}_o$ but $Eff(\boldsymbol{\alpha}_o, \hat{\boldsymbol{\beta}}_{\boldsymbol{o}}) = Eff(\boldsymbol{\alpha}_o, \overline{\boldsymbol{\beta}}_o)$ due to slacks, which leads to some drawbacks of radial DEA. At the same time, we also obtain the parameters $\hat{\lambda}$ in the model (6). We claim that $(\hat{\boldsymbol{\beta}}_{\boldsymbol{o}}, \hat{\lambda})$ is a feasible solution of the model (4) to conclude the proof because all the constraints are identical. As $\hat{\boldsymbol{\beta}}_{\boldsymbol{o}} > \overline{\boldsymbol{\beta}}_o$, this contradicts the fact that $(\overline{\beta}_o, \overline{\lambda})$ is a weak-Pareto solution. Therefore, the assumption $\hat{\theta} < \theta_o$ is wrong, which means $\hat{\theta} = \theta_o$.

For the rest of the theorem, let $(\overline{\beta}_o, \overline{\lambda})$ be a feasible solution of problem (4) and $\overline{\theta}_o$ be the efficiency score of $\text{DMU}(\alpha_o, \overline{\beta}_o)$ under model (6). If $(\overline{\beta}_o, \overline{\lambda})$ is not a weak-Pareto solution of problem (4), we can claim $\overline{\theta}_o < \theta_o$, contradicting the given condition. Assuming that $(\overline{\beta}_o, \overline{\lambda})$ is not a weak-Pareto solution, there must exist another feasible solution $(\hat{\boldsymbol{\beta}}_o, \hat{\lambda})$ such that $\hat{\beta}_o > \overline{\beta}_o$. According to the first half of the theorem, we can obtain the efficiency score of $\text{DMU}(\alpha_o, \hat{\boldsymbol{\beta}}_o)$ is θ_o. It is clear that $\theta_o = Eff(\alpha_o, \hat{\beta}_o) > Eff(\alpha_o, \beta_o) = \overline{\theta}_o$ in the radial DEA model because $\hat{\beta}_o > \overline{\beta}_o$, which contradicts the given condition. Hence, the assumption is wrong, which means $(\overline{\beta}_o, \overline{\lambda})$ is a weak-Pareto solution. □

Note that this is slightly different from the models given in Wei et al. (2000) andZhang and Cui (2016) where they used the output-oriented model to evaluate the DMUs in prior. We construct a new approach to derive the results as extension of the inverse DEA method, which is similarly used to prove the situation under no-radial measures later.

Inverse radial DEA has been widely used and achieved a series of contributions. However, since the effect of slacks is ignored, inverse radial DEA may cause unpredictable errors in decision-making. We now use a data set from Cook and Kress (1999) to explain this problem in more detail. Considering the data set in Table 1, we firstly use model (2) under the constant return to scale (CRS) assumption to evaluate it. The results are shown in Table 2.

Although we keep the inputs of DMU_3 unchanged, it is easy to check that the outputs of DMU_3 will increase to $\beta_3 = (75, 902.30)^\top$ by model (4). In other words, the outputs evaluated by model (4) will increase a lot that are from slacks, even if we do not increase the inputs, which is impossible in the real-world if the efficiency level is unchanged. From this general example, we can conclude that the inverse radial DEA models may lead to unreasonable results without considering slacks. Therefore, we need to pay attention to slacks to evaluate the outputs when inputs increases because slacks play a non-negligible role in the inverse DEA.

3. A general inverse DEA model for non-radial DEA

3.1. Inverse non-radial DEA

Since the non-radial DEA evaluates the efficiency score of DMU concerned by considering slacks, we can establish an inverse DEA based on non-radial DEA to overcome the defects involved in the last section. Before establishing a general form of inverse DEA model for non-radial DEA, we introduce a monotonic assumption about the non-radial DEA.

Definition 3 (*Tone (2001)*). A non-radial DEA model is monotonous if the measure is generally strictly monotone decreasing in each slack in inputs and outputs.

Now, we establish a general form of inverse DEA model that we call inverse non-radial DEA for monotonous non-radial DEA as follows:

$$V - \max \boldsymbol{\beta}_o = (\beta_1, \ldots, \beta_s) \tag{8}$$

$$\text{s. t. } (\alpha_o{}^*, \beta_o^*) \in \mathcal{T}, \tag{9}$$

$$Eff(\alpha_o, \boldsymbol{\beta}_o) = Eff(\boldsymbol{X_o}, \boldsymbol{Y_o}), \tag{10}$$

where $(\alpha_o{}^*, \beta_o^*)$ indicating the projection point of (α_o, β_o) on the production front is related to the slack variable $(\boldsymbol{S}^+, \boldsymbol{S}^-) \in \mathcal{R}^m \times \mathcal{R}^s$. For example, $\left(\alpha_o{}^*, \beta_o^*\right) = \left(t\alpha_o - \boldsymbol{S}^-, t\beta_o + \boldsymbol{S}^+\right), t = 1 + \sum_{r=1}^{s} \frac{s_r^+}{\beta_r}$ in SBM model. The constraints (9) indicate that this new inputs-outputs are still in production possible set. The second constraint (10) indicates that the efficiency score of DMU_o remains unchanged under this pair of new inputs-outputs. Because the monotonic assumption is an essential property of measure in the DEA models, such as SBM, RAM, and ERM, we can derive the following theorem by monotonicity, which is similar to radial DEA in Theorem 2.1.

Theorem 3.1. *Suppose that the efficiency score of* DMU_o, $Eff(\boldsymbol{X_o}, \boldsymbol{Y_o})$ *is* θ_o *under the monotonous measure in model (2), and the inputs of* DMU_o *are going to increase from* $\boldsymbol{x_o}$ *to* $\alpha_o = \boldsymbol{x_o} + \boldsymbol{\Delta x_o}$ $(\boldsymbol{\Delta x_o} \geqslant \boldsymbol{0})$*.*

(1) Let $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ be a Pareto solution of problem (8). Then, the efficiency score of DMU_{n+1} under model (6) is still θ_o. That is the efficiency score of DMU_o under this pair of new inputs-outputs remains unchanged.
(2) Conversely, Let $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ be a feasible solution of problem (8). If the efficiency score of $\text{DMU}(\alpha_o, \overline{\beta}_o)$ under model (6) is still θ_o, then $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ must be a Pareto solution of problem (8).

Proof. Firstly, we derive the proof of the first half of the theorem by contradiction. Let $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ be a Pareto solution of problem (8). Clearly, $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ is also a feasible solution of model (6) since the same constraints $(\alpha_o{}^*, \beta_o^*) \in \mathcal{T}$, then $\hat{\theta} \leqslant \theta_o$, where $\hat{\theta}$ is the optimal value of the model (6). Now by contradiction assume that $\hat{\theta} < \theta_o$. We can obtain some new outputs $\hat{\boldsymbol{\beta}}_o = \overline{\boldsymbol{\beta}}_o + \Delta\overline{\boldsymbol{\beta}}_o(\Delta\overline{\boldsymbol{\beta}}_o \gneqq \boldsymbol{0})$ such that $Eff(\alpha_o, \hat{\boldsymbol{\beta}}_o) = \theta_o$ in the model (6) since $\hat{\theta} < \theta_o$ and the measure is monotonous. At the same time, we get the parameters $(\hat{\lambda}, \hat{\boldsymbol{S}}^-, \hat{\boldsymbol{S}}^+)$ in the model (6). We claim that $(\hat{\boldsymbol{\beta}}_o, \hat{\lambda}, \hat{\boldsymbol{S}}^-, \hat{\boldsymbol{S}}^+)$ is a feasible solution of the model (8) to conclude the proof. Because $Eff(\alpha_o, \hat{\boldsymbol{\beta}}_o) = \theta_o = Eff(\boldsymbol{X_o}, \boldsymbol{Y_o})$, the constraint (10) is satisfied. And the remaining constraints are identical, $(\hat{\boldsymbol{\beta}}_o, \hat{\lambda}, \hat{\boldsymbol{S}}^-, \hat{\boldsymbol{S}}^+)$ is a feasible solution of the model (8). Because $\hat{\boldsymbol{\beta}}_o \gneqq \overline{\boldsymbol{\beta}}_o$, this contradicts the fact that $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ is a Pareto solution. Therefore, the assumption $\hat{\theta} < \theta_o$ is wrong, which means $\hat{\theta} = \theta_o$.

For the rest of the theorem, let $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ be a feasible solution of problem (8) and $\overline{\theta}_o$ be the efficiency score of $\text{DMU}(\alpha_o, \overline{\beta}_o)$ under model (6). If $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ is not a Pareto solution of problem (8), we can claim $\overline{\theta}_o < \theta_o$, contradicting the given condition. Assuming that $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ is not a Pareto solution, there must exist another feasible solution of (8), $(\hat{\boldsymbol{\beta}}_o, \hat{\lambda}, \hat{\boldsymbol{S}}^-, \hat{\boldsymbol{S}}^+)$, such that $\hat{\beta}_o \gneqq \overline{\beta}_o$ (at least one component $\hat{\beta}_r > \beta_r, r \in \{1, \ldots, s\}$). According to the first half of the theorem, we can obtain the efficiency score of $\text{DMU}(\alpha_o, \hat{\boldsymbol{\beta}}_o)$ is θ_o. Since $\hat{\beta}_o \gneqq \overline{\beta}_o$ and the efficiency measure is monotonous, we can know $\theta_o = Eff(\alpha_o, \hat{\beta}_o) > Eff(\alpha_o, \beta_o) = \overline{\theta}_o$, whichcontradicts the given condition. Hence, the assumption is wrong, which means $(\overline{\beta}_o, \overline{\lambda}, \overline{\boldsymbol{S}}^-, \overline{\boldsymbol{S}}^+)$ is a Pareto solution. □

Table 1
Data of 12 DMUs.

DMU_j	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8	DMU_9	DMU_{10}	DMU_{11}	DMU_{12}
Input 1	350	298	422	281	301	360	540	276	323	444	323	444
Input 2	39	26	31	16	16	29	18	33	25	64	25	64
Input 3	9	8	7	9	6	17	10	5	5	6	5	6
Output 1	67	73	75	70	75	83	72	78	75	74	25	104
Output 2	751	611	584	665	445	1070	457	590	1074	1072	350	1199

Table 2
Results of CCR.

DMU_j	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8	DMU_9	DMU_{10}	DMU_{11}	DMU_{12}
θ	0.76	0.92	0.75	1.00	1.00	0.96	0.86	1.00	1.00	0.83	0.33	1.00
s_1^-	0.00	0.00	0.00	0.00	0.00	0.00	175.65	0.00	0.00	46.91	0.00	0.00
s_2^-	4.41	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	28.28	0.00	0.00
s_3^-	2.42	0.56	0.00	0.00	0.00	10.34	2.44	0.00	0.00	0.00	0.00	0.00
s_1^+	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.86	0.00	0.00
s_2^+	0.00	0.00	318.30	0.00	0.00	0.00	0.00	0.00	0.00	0.00	8.00	0.00

3.2. The relationship between inverse radial DEA and inverse non-radial DEA

In this section, before demonstrating that the relationship between inverse radial DEA and non-radial DEA, we first introduce the following proposition:

Proposition 3.2. *Since the radial measure can be regarded as a special non-radial measure, the inverse non-radial DEA model (8) covers the inverse radial DEA model (4).*

Proof. Note that inverse radial DEA model (4) is identical to model (8), if we make the following transformation:

$$V-\max \boldsymbol{\beta}_o = (\beta_1, \ldots, \beta_s) \tag{11}$$

$$\text{s. t. } (\theta_o \boldsymbol{\alpha}_o, \boldsymbol{\beta}_o) = (Eff(\boldsymbol{\alpha}_o, \boldsymbol{\beta}_o)\alpha_o, \beta_o) = (\boldsymbol{\alpha}_o^*, \boldsymbol{\beta}_o^*) \in \mathcal{T} \tag{12}$$

$$Eff(\boldsymbol{\alpha}_o, \boldsymbol{\beta}_o) = Eff(\boldsymbol{X_o}, \boldsymbol{Y_o}) = \theta_o. \tag{13}$$

That is the same as inverse radial DEA (4), which completes the proof. □

Comparing with Theorems 2.1 and 3.1, we can derive the similarities and differences between these two models in the following ways.

(1) Although we can know inverse radial DEA and non-radial DEA have the same formula from Proposition 3.2, we should point out that the inverse radial DEA corresponds to the weak Pareto solution, while the inverse non radial DEA corresponds to the Pareto solution. The reason is that there exist DMU$(\boldsymbol{\alpha}_o, \overline{\boldsymbol{\beta}}_o)$ and DMU$(\boldsymbol{\alpha}_o, \widehat{\boldsymbol{\beta}}_o)$ such that $\widehat{\boldsymbol{\beta}}_o \gneqq \overline{\boldsymbol{\beta}}_o$ but $Eff(\boldsymbol{\alpha}_o, \widehat{\boldsymbol{\beta}}_o) = Eff(\boldsymbol{\alpha}_o, \overline{\boldsymbol{\beta}}_o)$ due to slacks in radial DEA, which leads to some drawbacks of radial DEA. In other words, radial measure is not strictly monotonic. On the other hands, if there exist some non-radial DEA models whose measures are not strictly monotonic such as input-oriented SBM model, we can only obtain weak-Pareto results similar to radial DEA.
(2) The biggest difference between these two models is that model (8) considers non-radial measures, which means that the role of slacks can be taken into account, while model (4) ignores the effect of slacks;
(3) Although both are multi-objective programming models, the former (4) is a linear model and the latter (8) is a non-linear model because non-radial measures are often non-linear, which can be seen in inverse SBM model in the next section. Regarding inverse non-radial DEA is a non-linear model, we will present an iterative algorithm for large-scale problems in Section 5;
(4) With same increasing inputs, the level of outputs for DMU concerned based on inverse non-radial DEA is lower than the level of outputs based on inverse radial DEA. This is because slacks can be converted into outputs in inverse radial DEA and the efficiency score in non-radial DEA is lower than the efficiency score in radial DEA. We will further illustrate this in Section 6 by comparing the inverse SBM model with the inverse CCR model.

4. A inverse DEA model for SBM model and a special case

4.1. A inverse DEA model for SBM model

To further illustrate the inverse non-radial DEA model, we take the SBM model as an example in this section. The linear form of SBM model is like follows:

$$\theta_o = \min_{t, S^-, S^+, \Lambda} t - \frac{1}{m}\sum_{i=1}^{m} \frac{S_i^-}{x_{io}} \tag{14}$$

$$\text{s. t.} \quad 1 = t + \frac{1}{s}\sum_{r=1}^{s} \frac{S_r^+}{y_{ro}}, \tag{15}$$

$$t\boldsymbol{x_o} = \boldsymbol{X\Lambda} + \boldsymbol{S}^-, \tag{16}$$

$$t\boldsymbol{y_o} = \boldsymbol{Y\Lambda} - \boldsymbol{S}^+, \tag{17}$$

$$\boldsymbol{\Lambda} \geqslant 0, \boldsymbol{S}^- \geqslant \boldsymbol{0}, \boldsymbol{S}^+ \geqslant \boldsymbol{0}, t > 0. \tag{18}$$

Suppose that the inputs of DMU_o are increased from $\boldsymbol{x_o}$ to $\boldsymbol{\alpha}_o = \boldsymbol{x_o} + \boldsymbol{\Delta x_o}$ ($\boldsymbol{\Delta x_o} \geqslant \boldsymbol{0}$), then the inverse SBM model is as follows:

$$V-\max_{t, S^-, S^+, \Lambda, \beta_o} \boldsymbol{\beta}_o = \left(\beta_1, \ldots, \beta_s\right) \tag{19}$$

$$\text{s. t. } 1 = t + \frac{1}{s}\sum_{r=1}^{s} \frac{S_r^+}{\beta_{ro}}, \tag{20}$$

$$t\boldsymbol{\alpha}_o = \boldsymbol{X\Lambda} + \boldsymbol{S}^-, \tag{21}$$

$$t\boldsymbol{\beta}_o = \boldsymbol{Y\Lambda} - \boldsymbol{S}^+, \tag{22}$$

$$t - \frac{1}{m}\sum_{i=1}^{m} \frac{S_i^-}{\alpha_{io}} = \theta_o \tag{23}$$

$$\boldsymbol{\Lambda} \geqslant 0, \boldsymbol{S}^- \geqslant \boldsymbol{0}, \boldsymbol{S}^+ \geqslant \boldsymbol{0}, t > 0. \tag{24}$$

where θ_o is given as the optimal value of problem (14). The efficiency score evaluation model for new DMU, DMU_{n+1} is like follows:

$$\hat{\theta} = \min_{t, S^-, S^+, \Lambda} t - \frac{1}{m}\sum_{i=1}^{m} \frac{S_i^-}{\alpha_{io}} \tag{25}$$

$$\text{s. t. } 1 = t + \frac{1}{s}\sum_{r=1}^{s} \frac{S_r^+}{\beta_{ro}}, \tag{26}$$

$$t\boldsymbol{\alpha}_o = \boldsymbol{X\Lambda} + \boldsymbol{S}^-, \tag{27}$$

$$t\boldsymbol{\beta}_o = \boldsymbol{Y\Lambda} - \boldsymbol{S}^+, \tag{28}$$

$$\boldsymbol{\Lambda} \geqslant 0, \boldsymbol{S}^- \geqslant \boldsymbol{0}, \boldsymbol{S}^+ \geqslant \boldsymbol{0}, t > 0. \tag{29}$$

As SBM model is monotonous, we can derive the following corollary according to the Theorem 3.1.

Corollary 4.1. *Suppose that the efficiency score of* DMU_o $Eff(\boldsymbol{X_o}, \boldsymbol{Y_o})$ *is* θ_o, *and the inputs of* DMU_o *are going to increase from* $\boldsymbol{x_o}$ *to* $\boldsymbol{\alpha}_o = \boldsymbol{x_o} + \boldsymbol{\Delta x_o}$ *(*$\boldsymbol{\Delta x_o} \geqslant \boldsymbol{0}$*).*

(1) Let $(\overline{\beta}_o, \overline{\lambda}, \overline{S}^-, \overline{S}^+)$ be a Pareto solution of problem (19). Then, the efficiency score of DMU_{n+1} under model (25) is still θ_o. That is the efficiency score of DMU_o under this pair of new inputs-outputs remains unchanged.
(2) Conversely, let $(\overline{\beta}_o, \overline{\lambda}, \overline{S}^-, \overline{S}^+)$ be a feasible solution of problem (19). If the efficiency score of $\text{DMU}(\alpha_o, \overline{\beta}_o)$ under model (25) is still θ_o, then $(\overline{\beta}_o, \overline{\lambda}, \overline{S}^-, \overline{S}^+)$ must be a Pareto solution of problem (19).

Proof. Since the SBM model is a non-radial model, we can derive the results from Theorem 3.1 immediately. □

4.2. Special cases for inverse SBM model

Although model (19) is a non-linear multi-objective programming problem, we can convert it into a linear multi-objective programming to get a (weak) Pareto solution for some special cases. In the next Proposition, we derive a Pareto solution by a multi-objective linear model considering a special case with only one output.

Proposition 4.2. *Suppose that there is the only one output $\beta_o \in \mathcal{R}$. We can derive a Pareto solution of the inverse SBM model (19) by the following linear model:*

$$V - \max_{t, S^-, \Lambda, \beta_o} \beta_o \tag{30}$$

$$\text{s. t. } \alpha_o = X\Lambda + S^-, \tag{31}$$

$$\beta_o = Y\Lambda, \tag{32}$$

$$1 - \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{\alpha_{io}} = \theta_o \tag{33}$$

$$\Lambda \geqslant 0,\ S^- \geqslant \mathbf{0}. \tag{34}$$

Proof. Given that $(\overline{\beta}_o, \overline{\lambda}, \overline{S}^+, \overline{S}^-, \overline{t})$ be a Pareto solution of problem (19), we can claim that $(\hat{\beta}_o, \hat{\lambda}, \hat{S}^+, \hat{S}^-, \hat{t})$ is a Pareto solution of problem (19), where

$$\hat{\beta}_o = \overline{\beta}_o,\ \hat{\lambda} = \overline{\lambda},\ \hat{S}^+ = 0,\ \hat{S}^- = \overline{S}^- + \overline{S}^+\frac{\alpha_o}{\overline{\beta}_o},\ \hat{t} = 1.$$

Now verify their feasibility one by one. Clearly, $\hat{\Lambda} \geqslant 0, \hat{S}^+ \geqslant \mathbf{0}, \hat{S}^- \geqslant \mathbf{0}, \hat{t} = 1 > 0.$

$$X\overline{\Lambda} = \overline{t}\alpha_o - \overline{S}^- = \alpha_o - \hat{S}^-, \tag{35}$$

$$Y\overline{\Lambda} = \overline{t}\overline{\beta} + \overline{S}^+ = \hat{\beta}, \tag{36}$$

$$\theta_o = \overline{t} - \frac{1}{m}\sum_{i=1}^{m}\frac{\overline{S}_i^-}{\alpha_{io}} = 1 - \frac{1}{m}\sum_{i=1}^{m}\frac{\hat{S}_i^-}{\alpha_{io}} \tag{37}$$

Therefore, $(\hat{\beta}_o, \hat{\lambda}, \hat{S}^+, \hat{S}^-, \hat{t})$ is a feasible solution of problem (19), which implies that $(\hat{\beta}_o, \hat{\lambda}, \hat{S}^+, \hat{S}^-, \hat{t})$ is a Pareto solution because $\hat{\beta}_o = \overline{\beta}_o$. Substituting $(\hat{\beta}_o, \hat{\lambda}, \hat{S}^+, \hat{S}^-, \hat{t})$ into problem (19), we can derive the model (30). □

Inspired by idea that S^+ is replaced by S^- in the Proposition 4.2, we can derive the following Proposition for input-oriented SBM models.

Proposition 4.3. *The input-oriented inverse SBM model is:*

$$V - \max_{S^-, \Lambda, \beta_o} \beta_o \tag{38}$$

$$\text{s. t. } X\Lambda + S^- = \alpha_o, \tag{39}$$

$$Y\Lambda \geqslant \beta_o, \tag{40}$$

$$1 - \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{\alpha_{io}} = \theta_o \tag{41}$$

$$\Lambda \geqslant 0,\ S^- \geqslant \mathbf{0}. \tag{42}$$

Proof. The proof process is similar to Theorem 3.1, but it should be pointed out that we can only get a weak-Pareto solution because there may exist some slacks in the second constraints in the model (38). □

5. An algorithm for the inverse non-radial DEA

We propose an iterative algorithm to solve non-linear multi-objective programming model (8) in this Section. Inspired by the idea that we can improve the efficiency score of DMUs by increasing the outputs due to the monotonicity of non-radial measures, we design the following algorithm:

Algorithm 1. An iterative algorithm for the inverse non-radial DEA

```
Input:
   The data set for DMU(X_j, Y_j), j = 1, ..., n;
   New inputs α_o for DMU_o ;
   Precision control coefficient ϵ;
Output:
   New outputs β_o for DMU_o ;
1: Calculate θ_o, the efficiency score of DMU_o by model (2), and obtain the slacks of
     outputs S+
2: for k = 1, 2, ... do
3:    β_o(k) = y_o + k * ϵ * S+;
4:    Calculate θ(k), the efficiency score of DMU (α_o, β_o(k)) by model (2);
5:    if |θ_o − θ(k)| < ϵ then
6:       β_o = β_o(k), break;
7:    end if
8: end for
9: return β_o;
```

Because $\beta_o(k)$, $\theta(k)$ increase monotonously in the k th iteration of the Algorithm 1 and the upper bound of $\theta(k)$ is θ_o, we can get the convergence of algorithm immediately. Besides, it is a first-order algorithm due to $\lim_{k\to+\infty}\frac{\|\beta_o(k+1)-\beta_o^*\|}{\|\beta_o(k)-\beta_o^*\|} = 1$. Note that S^+ is just one of the possible iterative directions, decision-makers can choose different iterative directions according to their own preferences. It should be pointed out that the inverse non-radial DEA model can be regarded as a one-dimensional search problem about efficiency score θ_o when iterative direction is given. Therefore, lots of searching algorithms including second order methods can be used to solve this problem.

Table 3
Results of SBM.

DMU_j	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6	DMU_7	DMU_8	DMU_9	DMU_{10}	DMU_{11}	DMU_{12}
θ	0.55	0.71	0.54	1.00	1.00	0.72	0.60	1.00	1.00	0.65	0.33	1.00
s_1^-	0.00	0.00	21.48	0.00	0.00	0.00	244.59	0.00	0.00	56.40	0.00	0.00
s_2^-	11.91	0.00	0.00	0.00	0.00	1.14	0.00	0.00	0.00	34.00	0.00	0.00
s_3^-	3.58	3.17	0.80	0.00	0.00	11.43	4.53	0.00	0.00	0.00	0.00	0.00
s_1^+	14.27	0.00	18.00	0.00	0.00	0.59	0.00	0.00	0.00	16.00	50.00	0.00
s_2^+	412.78	280.93	747.76	0.00	0.00	127.03	154.71	0.00	0.00	216.80	724.00	0.00

Table 4
Results of inverse SBM and CCR model for DMU_3.

method	slacks and outputs	$\Delta x_3 = (0, 0, 0)^\top$	$\Delta x_3 = (1, 0, 0)^\top$	$\Delta x_3 = (0, 1, 0)^\top$	$\Delta x_3 = (0, 0, 1)^\top$	$\Delta x_3 = (1, 1, 1)^\top$
SBM	s_1^-	82.52	83.50	81.66	81.10	80.98
	s_2^-	4.72	4.72	5.66	4.61	5.53
	s_3^-	1.74	1.74	1.73	2.72	2.71
	s_1^+	3.83	3.83	4.03	4.16	4.42
	s_2^+	544.81	544.14	537.56	512.04	503.47
	β_1	75.00	75.00	75.00	75.00	75.00
	β_2	584.00	584.71	594.11	621.47	633.77
CCR	s_1^-	0.00	0.00	0.00	0.00	0.00
	s_2^-	0.00	0.00	0.00	0.00	0.00
	s_3^-	0.00	0.00	0.00	0.00	0.13
	s_1^+	0.00	0.00	0.00	0.00	0.00
	s_2^+	0.00	0.00	0.00	0.00	0.00
	β_1	75.00	75.00	75.00	75.00	75.00
	β_2	902.30	912.49	934.55	946.84	981.35

6. Illustrative example

We now introduce an example to explain our inverse non-radial DEA model based on inverse SBM model (19). Consider the same data set on Table 1, we firstly estimate the efficiency score based on model (14), which is presented in the Table 3. Without loss of generality, we convert inverse DEA model into single objective programming through weighting for convenience. In our case, we set the weight to $\omega = (1, 1)$, then the object function of inverse DEA is $\omega^\top \beta$. To further explain the difference between inverse non-radial and radial DEA, we still take DMU_3 mentioned in Section 2 as an example. The results are shown in Table 4.

Comparing the performance of inverse SBM and CCR in Table 2, here are several findings that fully demonstrate the advantages of inverse non-radial DEA models over inverse radial DEA models. At first, even if $\Delta x_3 = (0, 0, 0)^\top$, the outputs of DMU_3 increase from $y_3 = (75, 584)^\top$ to $\beta_3 = (75, 902.30)^\top$ according to CCR model. As a contrast, the outputs of DMU_3 remain the same without additional inputs by the inverse SBM model. In other words, inverse non-radial DEA like inverse SBM (19) can overcome the error caused by neglecting the effect of slacks in inverse radial DEA. It is undeniable that the inverse non-radial DEA is the same as inverse radial DEA, such as DMU_5 when the slack is $\mathbf{0}$.

Besides, the inverse SBM model produces low-level outputs than the inverse CCR model with the same extra inputs, which may be related to the characteristics of the SBM model. The SBM model estimates the worst score for the objective DMU (lower than the CCR model shown in Table 3) because the SBM model evaluates the efficiency of DMUs referring to the furthest frontier point (Tone, 2001). From this point of view, The inverse non-radial DEA model may be a pessimistic prediction, while the inverse radial DEA model is an optimistic evaluation if we ignore the impact of slack.

Furthermore, inverse SBM model can help decision-makers find out cost-effective input indexes. Observing the three columns of data in the middle of Table 4, we know that increasing a unit of input 3 can bring more outputs, which means that input 3 may be a priority input for DMU_3. Hence, strengthening the management of input factor 3 can make the production system more profitable. In contrast, Decision-makers may make mistakes in the inverse CCR model, because they do not know if the output improvement is due to increasing inputs or slacks.

7. Conclusion

In this paper, we establish an integrated framework of inverse DEA, which covers all the inverse DEA models, including radial and non-radial models, as long as the measure is monotonous. To illustrate the inverse non-radial DEA model, we establish the inverse SBM model and design an iterative algorithm for large scale problems. Except for better estimation, inverse non-radial DEA can provide an approach to explore cost-effective input factors by considering slacks.

There are several follow-up research topics. One is to extend the application of the non-radial DEA model in practice. For example, the existing resource allocation models may overestimate the outputs based on the inverse radial DEA model where slacks are all converted into outputs. Another interesting topic for research is to study robust inverse DEA. This paper provides a complete framework of inverse DEA, but the data for DEA may be imprecise in production activities. Hence, imprecise inverse DEA should be considered. Finally, we can study some relationships between production factors under inverse non-radial DEA, such as substitution relationship and complementary relationship.

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