SURA 2023

Data Envelopment Analysis for the Performance of Various Geographical Regions in Fighting COVID-19

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INTRODUCTION

- With the November 2019 outbreak of COVID-19, there was a considerable surge in hospital admissions worldwide. Due to disparities in political power, legislative framework, and health system capacity, different regions responded differently to this circumstance (both demographic and economic).
- Owing to the lack of recent similar experiences and the short reaction time, the bulk of responses were disorganised and ineffectual, leaving us with a wealth of valuable lessons.
- The pandemic resulted in several research advancements. Some of them were based on the mathematical study of the pandemic's dynamics and the optimization of resource use during the COVID era. Our study will be based on a similar approach.

PROBLEM STATEMENT

- To calculate the efficiencies of health sectors of states of India and rank them on this basis using DEA. (This will help identify efficiently performing states and analyse how the other states that are lower in ranking can do better).

- To predict the changes in the required provided input and extracted output to achieve target efficiencies under similar conditions at a future point in time using InvDEA.

OBJECTIVES

Modifications into existing models

Consider additional input and output parameters alongside the existing ones and rebuild a model with required changes.

Remark: Models are majorly built in the banking and finance sector, we have to analyse if it can be used in the case of our concern.

Data Collection, Analysis - Ranking of Indian states

Analyse the collected data and rank the various Indian states on the basis of their efficiency during COVID-19.

Application of InvDEA in this domain

Faced with a similar situation in future, predict the optimal output levels to reach a targeted efficiency.

Remark: There is a possibility to look into uncertainty and errors by making use of stochastic methods on our data.

BROAD MATHEMATICAL TOOLS

DATA ENVELOPMENT ANALYSIS (DEA)

It is a mathematical method that is used to examine and optimise the relative efficiency of similar Decision-Making Units(DMUs) utilising a collection of inputs and outputs categorised as performance indicators.

INVERSE DATA ENVELOPMENT ANALYSIS (InvDEA)

Whereas DEA calculates the (inputs and relative) technical efficiency of each project when the outcomes are known, InvDEA estimates the inputs required for similar projects to maintain DEA efficiency when only the (intended) results are known.

BASIC TERMINOLOGY

1. Decision Making Units
2. Input Parameters
3. Output Parameters
4. Benchmarking
5. Efficiency
6. Production Possibility Set

MATHEMATICAL MODELS

BCC Model - Inverse BCC Model

2.4 BCC Model(Output-Oriented)

$$\max \ \eta_B$$

subject to:

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \eta_B y_{ro}, \qquad r = 1, 2, \ldots, s$$

$$\lambda_j \geq 0, \qquad j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

2.5 Inverse-BCC Model(Output-Oriented)

$$\max \sum_{j=1}^{s} w_j * \beta_{jo}$$

subject to:

$$\sum_{j=1}^{n} \lambda_j x_{ij} \leq (x_{io} + \alpha_{io}), \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq \eta_B (y_{ro} + \beta_{ro}), \qquad r = 1, 2, \ldots, s$$

$$\lambda_j \geq 0, \qquad j = 1, 2, \ldots, n$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

Image source: SURA Report that has already been submitted.

MATHEMATICAL MODELS

SBM Model - Inverse SBM Model

$$\min \theta_0 = t - \frac{\sum_{i=1}^{m} \frac{S_i^-}{x_{io}}}{m}$$

subject to:

$$t + \frac{\sum_{j=1}^{s} \frac{S_r^+}{y_{ro}}}{s} = 1$$

$$\sum_{j=1}^{n} \Lambda_j x_{ij} = t x_{io} - S_i^-, \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \Lambda_j y_{rj} = t y_{ro} + S_r^+, \qquad r = 1, 2, \ldots, s$$

$$\Lambda_j \geq 0 \qquad j = 1, 2, \ldots, n,$$

$$t \geq 0,$$

$$S_i^- \geq 0, \qquad i = 1, 2, \ldots, m$$

$$S_r^+ \geq 0, \qquad r = 1, 2, \ldots, s$$

$$\max \beta_o = (\beta_{1o}, \beta_{2o}, \ldots, \beta_{so})$$

subject to:

$$t + \frac{\sum_{j=1}^{s} \frac{S_r^+}{\beta_{ro}}}{s} = 1$$

$$\sum_{j=1}^{n} \Lambda_j \alpha_{ij} = t \alpha_{io} - S_i^-, \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \Lambda_j \beta_{rj} = t \beta_{ro} + S_r^+, \qquad r = 1, 2, \ldots, s$$

$$\theta_0 = t - \frac{\sum_{i=1}^{m} \frac{S_i^-}{\alpha_{io}}}{m}$$

$$\Lambda_j \geq 0 \qquad j = 1, 2, \ldots, n,$$

$$t \geq 0,$$

$$S_i^-, S_r^+ \geq 0, \qquad i = 1, 2, \ldots, m, r = 1, 2, \ldots, s$$

Image source: SURA Report that has already been submitted.

MATHEMATICAL MODELS

In our case, there is only one output parameter.We have proved that the SBM inverse model is equivalent to the following in our case:

$$\max \beta_{1o}$$

subject to:

$$\sum_{j=1}^{n} \Lambda_j \alpha_{ij} = t\alpha_{io} - S_i^-, \qquad i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \Lambda_j \beta_{1j} = t\beta_{1o} + S_r^+,$$

$$\theta_0 = 1 - \frac{\sum_{i=1}^{m} \frac{S_i^-}{\alpha_{io}}}{m}$$

$$\Lambda_j \geq 0 \qquad j = 1, 2, \ldots, n,$$

$$S_i^- \geq 0, \qquad i = 1, 2, \ldots, m$$

We have hence used this model for running our codes.

Note: The only plausible way to un the general model is through iterative procedures.

MATHEMATICAL MODELS (Further Scope)

Stochastic Model

$$\min \theta^*(\alpha) = \theta$$

subject to:

$$\sum_{j=1}^{n} \lambda_j x_{ij} - \Phi^{-1}(\alpha)\bar{\sigma}(p_i^+ + p_i^-) \leq \theta x_{io}, i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} + \Phi^{-1}(\alpha)\bar{\sigma}(q_r^+ + q_r^-) \geq y_{ro}, r = 1, 2, \ldots, s$$

$$\sum_{j=1}^{n} \lambda_j a_{ij} - \theta a_{io} = p_i^+ - p_i^-, i = 1, 2, \ldots, m$$

$$\sum_{j=1}^{n} \lambda_j b_{rj} - \theta b_{ro} = q_r^+ - q_r^-, r = 1, 2, \ldots, s$$

$$\lambda_j, p_i^+, p_i^-, q_r^+, q_r^- \geq 0, j = 1, 2, \ldots, n, i = 1, 2, \ldots, m, r = 1, 2, \ldots, s$$

Image source: SURA Report that has already been submitted.

RESULTS AND DISCUSSION

Values obtained are too close by, not much analysis and distinction possible

State	BCC_Analysis	SBM_Analysis						
	Efficiency	Efficiency	s_doctors	s_nurses	s_hospitals	s_hospitalbeds	s_infections	s_recoveries
Andhra Pradesh	1	0.50367657	307.78252	618.4926	1.4252465	66.130055	0	0
Arunachal Pradesh	1.0013763	0.32010132	72.140239	411.95654	10.47169	109.91685	5.6847498	0
Assam	1.0070248	0.40499367	46.10558	52.43436	2.4896841	51.187632	0	0
Bihar	1.0093871	0.3683125	32.734459	19.599611	1.7712156	17.71733	0	0
Chhattisgarh	1	1	0	0	0	0	0	0
Delhi	1	1	0	0	0	0	0	0
Goa	1	1	0	0	0	0	0	0
Gujarat	1.0053789	0.37130857	198.58321	377.03072	5.5892921	67.881379	0	0
Haryana	1.0059035	0.6441038	14.846457	28.979164	0.46998369	9.3390017	0	0
Himachal Pradesh	1.0099724	0.3345673	39.259151	164.02563	5.1562543	102.43203	26.691412	0
Jammu & Kashmir	1	1	0	0	0	0	0	0
Jharkhand	1.0038815	0.52375273	4.8522864	6.9717143	10.774545	25.766997	0	0
Karnataka	1	0.33686464	405.20302	469.63405	73.264214	353.51337	85.394547	0
Madhya Pradesh	1.0041589	1	0	0	0	0	0	0
Maharashtra	1	0.46104854	149.95005	399.62027	0.67820245	95.671891	0	0
Mizoram	1	1	0	0	0	0	0	0
Nagaland	1	1	0	0	0	0	0	0
Odisha	1.0031392	0.38015496	0.99129389	5.4844971	1.3099466	18.304353	5.6527075	0
Punjab	1.020851	0.47166443	96.477268	216.66316	3.8881131	28.800462	18.617415	0
Rajasthan	1.0017793	0.35668901	60.827497	112.43187	1.1042915	28.025432	9.1740004	0
Sikkim	1.0052873	0.31635511	236.84136	914.68835	11.235001	238.78012	27.409235	0
Tamil Nadu	1.005112	0.43212718	11.929118	16.792793	0.2256894	20.794166	0	0
Kerala	1	0.47670161	220.63454	446.48116	2.6290854	42.507109	0	0
Uttar Pradesh	1.0076965	0.42441605	46.872815	117.96066	1.6967255	56.858928	4.6578406	0
Uttarakhand	1.0138096	0.37851669	7.4596289	15.261052	0.26489792	11.677459	1.7911458	0
West Bengal	1.0068185	0.37401057	81.627385	109.43699	5.0893865	62.901211	0	0
Tripura	1.0002132	0.47062714	52.646877	42.097452	2.0713659	27.952263	33.489076	0
Telengana	1.0014287	0.44700917	56.401132	52.034408	0.84187901	72.341102	0	0

Is this a coincidence?

Result: 0-slack is not a coincidence!

Lemma 4.1. *If an SBM model is applied to a set of Decision Making Units (DMUs) using multiple input parameters but only one output parameter, there is always an optimal solution where the slack associated with output is zero.*

Proof. Let P1 be the following problem(single output SBM DEA model):

$$\min \phi = t - \frac{\sum_{i=1}^{m} \frac{S_i^-}{x_{io}}}{m}$$

subject to:

$$\begin{aligned}
& t + \frac{s^+}{y_o} = 1 \\
& \sum_{j=1}^{n} \Lambda_j x_{ij} = t x_{io} - S_i^-, \qquad i = 1, 2, \ldots, m \\
& \Lambda Y = t y_o + s^+ \\
& \Lambda_j \geq 0 \qquad j = 1, 2, \ldots, n, \\
& t \geq 0, \\
& S_i^- \geq 0, \qquad i = 1, 2, \ldots, m \\
& s^+ \geq 0,
\end{aligned} \tag{13}$$

We now have to show that: If ($\phi, S^-, s^+, \Lambda, t$) be an optimal solution to P1, then there exist $\hat{S^-}$ and $\hat{\Lambda}$ such that $(\phi, \hat{S^-}, 0, \hat{\Lambda}, 1)$ is a solution of P1.

PROOF

Claim: $\hat{S^-} = S^- + s^+ \frac{x_o}{y_o}$ and $\hat{\Lambda} = \Lambda$

Check for feasibility:We know that $x_0 t = X\Lambda + S^-$, $y_0 t = Y\Lambda - s^+$, $t = 1 - \frac{s^+}{y_0}$

$$\hat{S^-} = S^- + s^+ \frac{x_o}{y_o} = x_0 t - X\Lambda + (Y\Lambda - y_0 t)\frac{x_0}{y_0}$$
$$\implies \hat{S^-} = Y\Lambda \frac{x_0}{y_0} - X\Lambda \ldots (1)$$

$$y_0 t = Y\Lambda - s^+$$
$$\implies y_0(1 - \frac{s^+}{y_0}) = Y\Lambda - s^+$$
$$\implies y_0 = Y\Lambda \ldots (2)$$

From (1), (2) we see $(\phi, \hat{S^-}, 0, \hat{\Lambda}, 1)$ is a feasible solution of P1 using $\hat{S^-} = S^- + s^+ \frac{x_o}{y_o}$ and $\hat{\Lambda} = \Lambda$.

Check for Optimality:

$$\text{Objective} = t - \frac{\sum_{i=1}^{m} \frac{S_i^-}{x_{io}}}{m} = 1 - \frac{\sum_{i=1}^{m} \frac{S_i^-}{x_{io}}}{m}$$
$$= 1 - \frac{\sum_{i=1}^{m} \frac{S_i^- + s^+ \frac{x_0}{y_0}}{x_{io}}}{m}$$
$$= 1 - \frac{\sum_{i=1}^{m} (\frac{S_i^-}{x_{io}} + \frac{s^+}{y_0})}{m}$$
$$= 1 - t + \phi - \frac{s^+}{y_0} = \phi$$

Hence, we also see that $(\phi, \hat{S^-}, 0, \hat{\Lambda}, 1)$ is an optimal solution to P1. □

Let P2 be the following problem:

$$\min \phi = 1 - \frac{\sum_{i=1}^{m} \frac{S_i^-}{x_{io}}}{m}$$

subject to:

$$\Lambda X = x_o - S^-, \qquad (14)$$
$$\Lambda Y = y_o,$$
$$\Lambda \geq 0,$$
$$S^- \geq 0,$$

Lemma 4.2. *If $\bar{\phi}$ is the optimal value of P2 then it is the optimal value of P1.*

Proof. Let $(\widetilde{S}_-, \widetilde{\Lambda})$ be the optimal value of P2.

So we have:

$$(\bar{\phi}, \widetilde{S}_-, \widetilde{\Lambda}) \rightarrow \text{optimal to P2.} \ldots (a)$$
$$\Rightarrow (\bar{\phi}, \widetilde{S}_-, 0, \widetilde{\Lambda}, 1) \rightarrow \text{feasible to P1} \ldots (b)$$
$$\exists(\widetilde{\phi}, \widetilde{S}_-, \widetilde{S}_+, \widetilde{\Lambda}, \widetilde{t}) \rightarrow \text{optimal to P1} \ldots (c)$$
$$\Rightarrow (\widetilde{\phi}, \widetilde{S}_- + \widetilde{S}_+ * \frac{x_o}{y_o}, \widetilde{\Lambda}) \rightarrow \text{feasible to P2} \ldots (d)$$

From (a) and (d):

$$\widetilde{\phi} \geq \bar{\phi}$$

From (b) and (c):

$$\widetilde{\phi} \leq \bar{\phi}$$

Hence:

$$\widetilde{\phi} = \bar{\phi}$$

□

From the previous two lemmas, we conclude that solving P1 and P2 is equivalent for the single output DEA case in SBM.

It is also worth noting that all efficient states have associated slack values of zero, which aligns with expectations, as achieving an efficiency score of 1 has already been attained and there is no need to do anything for achieving the same.

RESULTS AND DISCUSSION

Lemma 4.3. *There exists a linear optimization model that is equivalent to the InvDEA SBM model when the dataset consists of a single output parameter.*

The implication of this lemma is significant to our project To use the general inverse SBM odel (non-linear optimisation), the only known way is through iterative numerical procedures. We do not know about the run-time efficiency or termination of the procedure. Using this lemma, for a single-output case, the optimisation problem becomes linear.

Algorithm 1. An iterative algorithm for the inverse non-radial DEA

Input:
The data set for DMU($\boldsymbol{X}_j$, $\boldsymbol{Y}_j$), $j = 1, \ldots, n$;
New inputs α_o for DMU_o ;
Precision control coefficient ϵ;
Output:
New outputs β_o for DMU_o ;
1: Calculate θ_o, the efficiency score of DMU_o by model (2), and obtain the slacks of outputs $\boldsymbol{S}^+$
2: **for** $k = 1, 2, \ldots$ **do**
3: $\beta_o(k) = \boldsymbol{y}_o + k * \epsilon * \boldsymbol{S}^+$;
4: Calculate $\theta(k)$, the efficiency score of DMU $(\alpha_o, \beta_o(k))$ by model (2);
5: **if** $|\theta_o - \theta(k)| < \epsilon$ **then**
6: $\beta_o = \beta_o(k)$, break;
7: **end if**
8: **end for**
9: **return** β_o;

Proposition 4.2. *Suppose that there is the only one output $\beta_o \in \mathcal{R}$. We can derive a Pareto solution of the inverse SBM model (19) by the following linear model:*

$$V - \max_{t, \boldsymbol{S}^-, \boldsymbol{\Lambda}, \beta_o} \beta_o \tag{30}$$

$$\text{s. t. } \alpha_o = \boldsymbol{X}\boldsymbol{\Lambda} + \boldsymbol{S}^-, \tag{31}$$

$$\beta_o = \boldsymbol{Y}\boldsymbol{\Lambda}, \tag{32}$$

$$1 - \frac{1}{m}\sum_{i=1}^{m}\frac{S_i^-}{\alpha_{io}} = \theta_o \tag{33}$$

$$\boldsymbol{\Lambda} \geqslant 0, \boldsymbol{S}^- \geqslant \boldsymbol{0}. \tag{34}$$

RESULTS AND DISCUSSION

	SBM_Analysis	BCC_Analysis
Andhra Pradesh	9.823525073	10.00000004
Arunachal Pradesh	10.00469709	9.999999715
Assam	10.00987331	9.999999935
Bihar	10.02397148	9.999999904
Chhattisgarh	3.771851349	10.00000006
Delhi	6.67304049	9.999999772
Goa	2.320150816	10.00000003
Gujarat	10.00290894	9.999999853
Haryana	10.00010712	10.00000036
Himachal Pradesh	10.00704992	9.999996937
Jammu & Kashmir	3.93027636	9.999999958
Jharkhand	10.03377316	9.999998436
Karnataka	9.803440641	10.00000145
Madhya Pradesh	10.00004515	10
Maharashtra	2.69008E-11	9.999999988
Mizoram	0.268023701	9.999999983
Nagaland	10.30991658	9.999998524
Odisha	9.928769299	9.999999656
Punjab	10.01287857	9.999999263
Rajasthan	9.918538913	9.99999924
Sikkim	10.0356175	9.999999887
Tamil Nadu	9.892639143	9.999999966
Kerala	3.470557848	9.999999997
Uttar Pradesh	10.00714704	10.00000001
Uttarakhand	10.00772573	10.00000007
West Bengal	10.00895139	10.00000023
Tripura	10.06014435	9.999999907
Telengana	10.00674691	9.999999838

This data table has been generated using the BCC and SBM InvDEA models. It offers a representative example by illustrating the necessary increase in the output parameter when all input parameters are augmented by 10%.

Note: It is not necessary to augment all quantities by the same amount, this is just an illustrative example.

FUTURE SCOPE

1. Stochastic DEA
2. "Inv-Stochastic" model
3. Feedback mechanisms based on the insights from stochastic models
4. Account for noise in the data sets.

In summary, further work can be done to make the analysis proceed dynamically. Use of Stochastic models can help. Noise correction also needs to be implemented.

OUR JOURNEY

1. (week-1,2) Study and modification of existing models.
2. Data Collection and Data Cleaning (week-3)
3. (week-4) Link the data sets to the modified models
4. Analysis of Data and ranking of geographical regions (week-5)
5. (week-6,7) Running the InvDEA process
6. Overview and wrapping up (week-8)

Please note: Sources of Data that we used: Kaggle, WHO data archives, and MOHFW(Govt of India) data archives. More sources can be incorporated as and when the requirement arises.

References

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