Dr. **A**mbedkar **I**nstitute of **T**echnology
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DEPARTMENT OF MATHEMATICS

Lecture Notes
on
Engineering Mathematics - 1
Subject Code: 22MAU101B
For
2023-24 Batch
By
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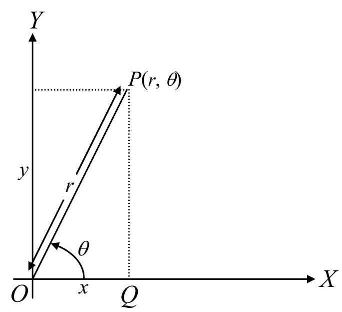
1 Differential Calculus

Syllabus: Introduction to polar coordinates and curvature relating to Computer Science and Engineering applications.
Polar coordinates, polar curves, angle between the radius vector and the tangent, angle between two curves. Pedal equations. Curvature and Radius of curvature-Cartesian, Parametric, Polar and Pedal forms (without proof).
Self-study: Center and circle of curvature, evolutes and involutes.
Applications: Tracing of polar curves.

1.1 Polar curves

Theory:- Consider a point $P(x, y)$ in xy-plane. In polar form we take positive x axis as initial line, origin O as a pole and OP as a radius vector. If r is the length of the radius vector OP and θ is the angle between the line OP and the initial line measured in positive sense (i.e. anti-clockwise direction), then (r, θ) is called polar co-ordinate of the point P. The length r is called radial distance and θ is called polar angle.



From the figure we see that $OP^2 = OQ^2 + PQ^2 \Rightarrow r^2 = x^2 + y^2$ and $\tan\theta = \frac{opp}{adj} = \frac{y}{x}$. Hence $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$., this relation is useful to obtain polar co-ordinate of the point whose Cartesian co-ordinate is known.

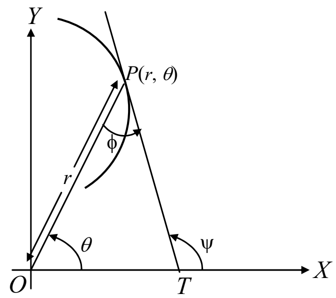
Further, $\sin\theta = \frac{opp}{hyp} = \frac{y}{r}$. and $\cos\theta = \frac{adj}{hyp} = \frac{x}{r} \Rightarrow x = r\cos\theta$ and $y = r\sin\theta$ this relation helps us to convert polar co-ordinate to cartesian form.

Now, consider the curve $y = f(x)$ in Cartesian form, changing this equation by using the above relation, yields $r = f(\theta)$ called polar equation of the curve. The curves represented in terms of r and θ is called *polar curves.*

1.2 Angle Between radius vector and a tangent.

Let $P(r, \theta)$ be any point on the polar curve $r = f(\theta)$. Let PT be the tangent drawn to the curve at P so as to meet the x-axis at T. Let ψ be the angle between the tangent and the positive x-axis measured in positive sense. Then the angle ϕ between radius vector OP and the tangent PT is given by

$$\tan\phi = r\frac{d\theta}{dr}$$



Proof. From the figure it is clear that

$$\psi = \phi + \theta \tag{1}$$

Taking tangent on both sides of equation (1), we get

$$\tan\psi = \tan(\phi+\theta) = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta} \tag{2}$$

If $y = f(x)$ is the Cartesian form of the polar curve, then we have by the transformation,

$$x = r\cos\theta, y = r\sin\theta$$

that is

$$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta \quad (\because r = f(\theta))$$

the slope of the tangent PT at $P(x, y)$ is

$$\tan\psi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + \cos\theta f(\theta)}{f'(\theta)\cos\theta - \sin\theta f(\theta)}$$

Dividing numerator and denominator of right hand side of this equation by $f'(\theta)\cos\theta$, we get.

$$\tan\psi = \frac{\frac{f'(\theta)\sin\theta+\cos\theta f(\theta)}{f'(\theta)\cos\theta}}{\frac{f'(\theta)\cos\theta-\sin\theta f(\theta)}{f'(\theta)\cos\theta}} = \frac{\frac{f'(\theta)\sin\theta}{f'(\theta)\cos\theta} + \frac{\cos\theta f(\theta)}{f'(\theta)\cos\theta}}{\frac{f'(\theta)\cos\theta}{f'(\theta)\cos\theta} - \frac{\sin\theta f(\theta)}{f'(\theta)\cos\theta}} = \frac{\frac{\sin\theta}{\cos\theta} + \frac{f(\theta)}{f'(\theta)}}{1 - \frac{\sin\theta f(\theta)}{f'(\theta)\cos\theta}} = \frac{\tan\theta + \frac{f(\theta)}{f'(\theta)}}{1 - \tan\theta\frac{f(\theta)}{f'(\theta)}} \tag{3}$$

Comparing equation (2) and equation (3), we get

$$\tan\theta = \frac{f(\theta)}{f'(\theta)} \Rightarrow \tan\theta = \frac{f(\theta)}{\frac{df(\theta)}{d\theta}} = \frac{r}{\frac{dr}{d\theta}} \Rightarrow \tan\theta = r\frac{d\theta}{dr}$$

□

Problem 1.2.1. *Find the angle between radius vector and tangent to the cardioid* $r = a(1+\cos\theta)$.

Problem 1.2.2. *Find the slope of the tangent at any point* (r,θ) *on the curve* $r = a(1+\cos\theta)$. *Also show that the tangent at the point* $\theta = \pi/3$ *is parallel to the initial line.*

Problem 1.2.3. *Find the angle between tangent and the radius vector for the following curves.*

1. $r^n = a^n \cos n\theta$.
2. $r^2 = a^2 \sin 2\theta$.
3. $r = ae^{b\theta}$.
4. $r = a(1-\cos\theta)$.
5. $r(a+\cos\theta) = a$.
6. $r = a\csc^2(\frac{\theta}{2})$.
7. $r = \sin\theta + \cos\theta$.
8. $r^n = a^n \sec(n\theta + \alpha)$.
9. $\frac{l}{r} = 1 + e\cos\theta$.

Problem 1.2.4. *Show that in the equiangular spiral* $r = ae^{\theta\cot\alpha}$, *the tangent is inclined at a constant angle to the radius vector.*

Problem 1.2.5. *Find the angle between initial line and tangent to the curve* $\frac{2a}{r} = 1 - \cos\theta$ *at any point* (r,θ).

Problem 1.2.6. *Show that the angle between the tangent at any point* P *and the line joining* P *to the origin is same at all points of the curve* $\log(x^2+y^2) = k\tan^{-1}\left(\frac{y}{x}\right)$.

Problem 1.2.7. *Find the angle between tangent and radius vector in the curve* $r^m = a^m(\cos m\theta + \sin m\theta)$.

Problem 1.2.8. *Find the angle between tangent and radius vector in the curve*

$$a\theta = \sqrt{r^2 - a^2} - a\cos^{-1}\left(\frac{a}{r}\right).$$

1.3 Angle between two polar curves

Let $P(r,\theta)$ be the point of intersection of two polar curves whose equations are $r = f(\theta)$ and $r = g(\theta)$. Let PT_1 and PT_2 be the tangents drawn to these curves at the point P, which make angles ϕ_1 and ϕ_2 with the radius vector

OP respectively. Then the angle $|\phi_1 - \phi_2|$ is the angle between the curves at the point of intersection P. Further,

$$\tan|\phi_1 - \phi_2| = \left|\frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}\right|$$

Hence if the two curves are orthogonal (perpendicular), then as $|\phi_1 - \phi_2| = \frac{\pi}{2}$, we get

$$\left|\frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}\right| = \infty \Rightarrow 1 + \tan\phi_1 \tan\phi_2 = 0 \Rightarrow \tan\phi_1 \tan\phi_2 = -1$$

this is the condition for orthogonality.

Problem 1.3.1. *Find the angle of intersection of the curves* $r = \sin\theta + \cos\theta$, $r = 2\sin\theta$. *Are they orthogonal?*

Problem 1.3.2. *Find the angle between the curves* $r = \sec^2(\frac{\theta}{2})$ *and* $r = \csc^2(\frac{\theta}{2})$. *Are the curves orthogonal?*

Problem 1.3.3. *Show that the curves* $r = a(1 + \cos\theta)$ *and* $r^2 = a^2\cos 2\theta$ *intersect at an angle* $3\sin^{-1}(\frac{3}{4})^{1/4}$.

Problem 1.3.4. *Find the angle between the following pair of curves:*

1. $r = a\cos\theta$; $2r = a$.
2. $r = a(1 - \cos\theta)$; $r = 2a\cos\theta$.
3. $r^m = a^m \cos m\theta$; $r^m = a^m \sin m\theta$.
4. $r^2 = a^2\cos 2\theta + b^2$; $r = b$.
5. $r = a\log\theta$; $r = \frac{a}{\log\theta}$.
6. $r = \frac{a\theta}{1+\theta}$; $r = \frac{a}{1+\theta^2}$.
7. $r^2\sin 2\theta = 4$; $16\cos 2\theta = r^2$.
8. $r = a\cos 2\theta$; $r = a\sin 2\theta$.
9. $r = 6\cos\theta$; $r = 2(1 + \cos\theta$.
10. $r^2\sin 2\theta = a^2$; $2r^2\cos 2\theta = b^2$.
11. $r = a\theta$; $r\theta = a$.
12. $r^2\sin 2\theta = 4$; $r^2 = 16\sin 2\theta$.
13. $r = \frac{a}{1+\cos\theta}$; $r = \frac{b}{1-\cos\theta}$.
14. $r = a(1 + \sin\theta)$; $r = a(1 - \sin\theta)$.
15. $(\frac{r}{a})^n = \sec(n\theta + \alpha)$; $(\frac{r}{b})^n = \sec(n\theta + \beta)$
16. $r = a(1 + \cos 3\theta)$; $r = a(1 - \cos 3\theta)$

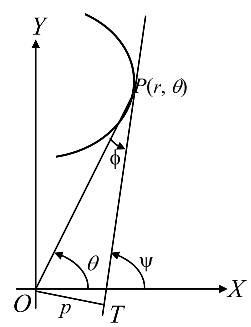
Problem 1.3.5. *Show that the following pair of curves intersect orthogonally*

1. $r^n = a^n \cos n\theta$; $r^n = a^n \sin n\theta$.
2. $r = a\sec^2(\frac{\theta}{2})$; $r = b\csc^2(\frac{\theta}{2})$.
3. $r = ae^\theta$; $re^\theta = b$.
4. $r = a(1+\sin\theta)$; $r = a(1-\sin\theta)$.
5. $r = ae^\theta$; $r = be^{-\theta}$.
6. $r = a(1+\cos\theta)$; $r = a(1-\cos\theta)$.
7. $r^n = a(1+\cos n\theta)$; $r^n = a(1-\cos n\theta)$.
8. $r^m = a^m \cos m\theta$; $r^m = b^m \sin m\theta$.
9. $b^2 = r^2\cos 2\theta$; $a^2 = r^2 \sin 2\theta$.
10. $r(1+\cos\theta) = a$; $r(1-\cos\theta) = b$.
11. $r = a\theta$; $r\theta = a$.
12. $r = 2a\cos\theta$; $r = 2a\sin\theta$.

Problem 1.3.6. *For what values of n the curves $r = a(1-\cos n\theta)$ and $r = a(1+\cos n\theta)$ are orthogonal.*

1.4 Pedal Equations

Theory: Let PT be the tangent drawn to the polar curve at P and OT be a normal to PT. Let length of OT be p. Then, as P moves on the curve, the value of r as well as the value of p changes. The relation between p and r is called $p-r$ equation or *pedal equation* to the curve. To obtain the relation we observe from the figure, in the right angled triangle, that



$$\sin\phi = \frac{opp}{hyp} = \frac{OT}{OP} = \frac{p}{r} \Rightarrow p = r\sin\phi \tag{4}$$

Further,

$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{1}{r^2}csc^2\phi = \frac{1}{r^2}(1+\cot^2\phi) = \frac{1}{r^2}\left(1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right) \tag{5}$$

Now the pedal equation for the curve $r = f(\theta)$ can be obtained by eliminating θ using its equation with equation (4) or with equation (5) whichever is convenient.

Finally, if $u = \frac{1}{r}$, then $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, substituting this in above equation (5), we get

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

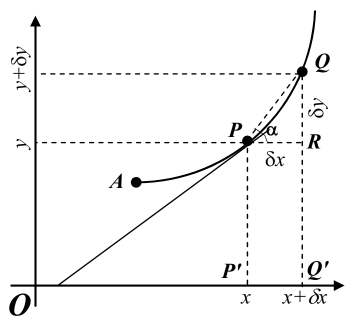
.

Problem 1.4.1. *Find the pedal equation for the following curves:*

1. $r^n = a^{2n} \sin n\theta$.
2. $r^m \cos m\theta = a^m$.
3. $r = ae^{\theta \cot \alpha}$.
4. $\frac{l}{r} = 1 + e \cos \theta$.
5. $r = ae^{\theta}$.
6. $r^m = a^m(\cos m\theta + \sin m\theta)$.
7. $r = a + b \cos \theta$.
8. $r(1 - \sin(\frac{\theta}{2}))^2 = a$.
9. $r = a(1 - \sin \theta)$.
10. $r\theta = a$.
11. $re^{-\theta} = b$.
12. $\frac{a(e^2-1)}{r} = 1 + e \cos \theta$.
13. $r = a\theta$.
14. $re^{\theta} = b$.
15. $r = asechn\theta$.
16. $r^m = a^m \sin m\theta$.
17. $r^m = a^m \cos m\theta$.
18. $r(1 - \cos \theta) = b$.
19. $r = a(1 - \cos \theta)$.
20. $r = a(1 + \cos \theta)$.
21. $r = 2a \sin m\theta$.
22. $r = 2a \cos \theta$.
23. $r = a \csc^2(\frac{\theta}{2})$.
24. $r = a \sec m\theta$.
25. $r^2 = a^2 \sec 2\theta$.
26. $\frac{2a}{r} = 1 - \cos \theta$.

Problem 1.4.2. *Show that the pedal equation of the polar curve* $r^n = a^n \sin n\theta + b^n \cos n\theta$ *is* $p^2 = \frac{r^{2(n+1)}}{a^{2n}+b^{2n}}$.

1.5 Radius of curvature



Let s be the distance from the point A to the point P, measured along the curve. Then s is called *arc length* or length of the arc from A to P and is denoted by arc $AP = s$. Now as the point P moves on the curve the value of x as well as s changes. The rate of change of s with respect to x is called the *derivative of arc length with respect to* x and is denoted by $\frac{ds}{dx}$. Similarly, other derivatives are defined.

Let P be any point on the curve. Let ψ be the inclination of the tangent drawn to the curve at P, measured with positive x-axis. If P moves on the curve to a position Q with arc $PQ = \delta s$, then the angle ψ also changes to $\psi + \delta\psi$, that is, the tangent turned through the angle $\delta\psi$. This is called **total bending** or *total curvature* of the arc PQ.

The average curvature of arc $PQ = \frac{\delta\psi}{\delta s}$. The limiting value of average curvature when Q approaches P ($\therefore \delta s \to 0$) is called the *curvature of the curve at* P and is denoted by κ (kappa). On the other hand,

Definition: The curvature at P is the rate at which the direction of the curve changes at P with respect to the arc length s. That is

$$\kappa = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$$

Remarks:

1. The unit of curvature is radian / unit length.

2. Similar to pedal form the curve can also be expressed in terms of r and ψ. Such an equation is known as *intrinsic form* of the curve.

Definition: The reciprocal of the curvature of a curve at any point P is called *radius of curvature* at P and is denoted by the symbol ρ, that is,

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi}$$

1.6 Expressions for radius of curvature

In Cartesian Form:

$$\rho = \frac{1}{\kappa} = \frac{ds}{d\psi} = \frac{(1+y_1^2)^{3/2}}{y_2} \tag{6}$$

Note: Since ρ is independent of coordinate axis, we can also write it as

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

In Parametric Form:

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}}$$

In Polar Form:

$$\rho = \frac{ds}{d\psi} = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

In Pedal Form: $\rho = \frac{ds}{d\psi} = r\frac{dr}{dp}$

1.7 Problems on radius of curvature in various forms

Problem 1.7.1. *Define curvature and radius of curvature of a polar curve at a given point on it.*

Problem 1.7.2. *Derive the expressions for radius of curvature in Cartesian and parametric form.*

Problem 1.7.3. *Find radius of curvature ρ at any point (s, ψ) for the tractrix whose equation in intrinsic form is $s = a \log \sec \psi$.*

Problem 1.7.4. *Find the radius of curvature ρ at any point on $y^2 = 4ax$.*

Problem 1.7.5. *If ρ_1 and ρ_2 be the radii of curvature at the extremities of a focal chord of parabola $y^2 = 4ax$, prove that*

$$\rho_1^{-2/3} + \rho_2^{-2/3} = (2a)^{-2/3}.$$

Problem 1.7.6. *Find the radius of curvature of the Catenary $y = c\cosh(x/c)$ at the point $(0, c)$.*

Problem 1.7.7. *Find the point on the curve $y = c\cosh(x/c)$ at which the radius of curvature a minimum.*

Problem 1.7.8. *Find radius of curvature of the rectangular hyperbola $xy = c^2$,*

Problem 1.7.9. *For the curve $xy = a^2$, prove that $\rho = \frac{r^3}{2a^2}$, where r is the distance of the point from the origin.*

Problem 1.7.10. *Find the radius of curvature at the point $(3a/2, 3a/2)$ of the Folium $x^3 + y^3 = 3axy$.*

Problem 1.7.11. *Show that the radius of curvature at the point $(a/4, a/4)$ on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is $a\sqrt{2}$.*

Problem 1.7.12. *Show that the radius of curvature at $(a, 0)$ on the curve $y^2 = a^2(a - x)/x$ is $a/2$.*

Problem 1.7.13. *If ρ be the radius of curvature at the point P on the parabola $y^2 = 4ax$ and S be its focus, then show that ρ^2 2 varies with $(SP)^3$.*

Problem 1.7.14. *Find the radius of curvature at any point $(at^2, 2at)$ of the parabola $y^2 = 4ax$.*

Problem 1.7.15. *Prove for the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ that $\rho = \frac{a^2b^2}{p^3}$, where p is the $\perp$ from the centre to the tangent at (x, y).*

Problem 1.7.16. *Show that the radius of curvature at an end of the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is equal to the semi-latus rectum.*

Problem 1.7.17. *For the curve $y = \frac{ax}{a+x}$, show that $(\frac{2\rho}{a})^{2/3} = (\frac{x}{y})^2 + (\frac{y}{x})^2$.*

Problem 1.7.18. *Find radius of curvature for the curve $y = xe^{-x}$ at the point where y is maximum.*

Problem 1.7.19. *Find the radius of curvature at point (x, y) on the following curves:*

1. $x^2 - y^2 = a^2$
2. $y = \ln \sin x$
3. $x^2 + y^2 + 2gx + 2fy + c = 0.$
4. $x^m + y^m = 1.$
5. $a^2 y = x^3 - a^3.$
6. $x^2 = 4ay.$
7. $x^{2/3} + y^{2/3} = a^{2/3}.$
8. $y = c \ln \sec(x/c).$

Problem 1.7.20. *Find radius of curvature of the curve at any point t on the curve $x = a(\cos t + \ln \tan(t/2))$, $y = a \sin t$.*

Problem 1.7.21. *Prove that the radius of curvature at any point of the Astroid $x^{2/3} + y^{2/3} = a^{2/3}$, is three times the length of the perpendicular from the origin to the tangent at that point.*

Problem 1.7.22. *Find the radius of curvature at the points indicated on the following curves:*

1. $y = 2 \sin x(2 - \cos x)$ *at* $y = 0.$
2. $x^3 - y^2 + 8 = 0,$ *at* $x = 0.$
3. $xy = \log x$ *at* $x = 1.$
4. $x^3 + y^3 = 3axy,$ *at* $(\frac{3a}{2}, \frac{3a}{2}).$
5. $xy^2 = a^3 - x^3,$ *at* $(a, 0).$
6. $y^2 = x^3 + 8$ *at* $x = -8.$
7. xe^x, *at the point where y is Minimum.*
8. $(x^2 + y^2)^2 = a^2(y^2 - x^2),$ *at* $(0, a).$
9. $xy^3 = a^4,$ *at* $(a, a).$
10. $y^3 = x(x + 2y),$ *at* $(1, -1).$
11. $xy^2 = a^2(a - x),$ *at* $y = 0.$
12. $y = e^x$ *at* $x = 0.$
13. $x^3 = y(x - a)^2$, *at the point where the ordinate is maximum.*

14. $4ay^2 = 27(x-2a)^3$ *at* $y = a/2$

Problem 1.7.23. *Find* ρ *at any points for the following curves:*

1. $x = a\cos^3\theta, y = b\sin^3\theta$.
2. $x = at, y = \frac{a}{t}$.
3. $x = 6t^2 - 5t^4, y = 8t^3$.
4. $x = \log t, y = \frac{1}{2}(t + \frac{1}{t})$.
5. $x = a\log\sec t,\ y = a(\tan t - t)$.
6. $x = a\sin 2t(1+\cos 2t)$,
   $y = a\cos 2t(1-\cos 2t)$.
7. $x = 3a\cos t - a\cos 3t$,
   $y = 3a\sin t - a\sin 3t$.
8. $x = 2a\sin t + a\sin 2t$,
   $y = 2a\cos t + a\cos 2t$.

Problem 1.7.24. *Derive the expressions for radius of curvature in Polar and Pedal form.*

Problem 1.7.25. *Show that the radius of curvature at any point of the cardioid* $r = a(1-\cos\theta)$ *varies as* $\sqrt{r}$.

Problem 1.7.26. *If* ρ_1 *and* ρ_2 *be the radii of curvature at the extremities of any chord of the cardioid* $r = a(1+\cos\theta)$ *which passes through the pole, show that* $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$.

Problem 1.7.27. *Find the radius of curvature at the points indicated on the following curves:*

1. $r^2 = a^2 - 3p^2$.
2. $\frac{1}{p^2} - \frac{1}{c^2} + \frac{3}{r^2} = 0$.
3. $r = a\sec^2(\theta/2)$.
4. $r^n = a^n\cos n\theta$
5. $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\frac{a}{r}$.
6. $pr = a^2$.
7. $r^3 = a^2p$.
8. $pa^n = r^{n+1}$.
9. $p = \frac{r^4}{r^2+a^2}$.
10. $r = a\theta$
11. $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$.
12. $r\theta = 1$.
13. $r = ae^{\theta\cot\alpha}$.
14. $2r = a(1-\cos\theta)$.
15. $r = a\sec 2\theta$.

Problem 1.7.28. *For the curve* $r = a(1+\cos\theta)$ *prove that* $\rho^2/2$ *is a constant.*

Problem 1.7.29. *For the curve* $r = a\cos\theta$, *prove that* $2\rho = 1$.

Problem 1.7.30. *For the curve* $r^2 = a^2\cos 2\theta$, *prove that* $\rho = \frac{a^2}{3r}$.

Problem 1.7.31. *For the curve* $\sqrt{r} = a\cos\frac{\theta}{2}$, *prove that* $3\rho = 2a\sqrt{r}$.

Problem 1.7.32. *For the curve* $a^2 = r^2\cos 2\theta$, *prove that* $a^2\rho = r^3$.

Problem 1.7.33. *For the curve* $r = a\sin^3\frac{\theta}{2}$, *show that* $\rho^3 \prec r^2$.

2 Unit - II: Differential Calculus-2

Syllabus: Introduction of series expansion and partial differentiation in Computer Science & Engineering.

Taylor's and McLaurin's series expansion of one variable (no proof). Partial differentiation, Euler's theorem, total derivative, differentiation of composite functions. Jacobian. Maxima and minima for a function of two variables.

Self-study: Extended Euler's theorem. Method of Lagrange's undetermined multipliers with single constraint.

Applications: Solution of first order ODE arises using Taylor's series method.

2.1 Taylor's theorem

Theorem 2.1.1 (Taylor's). *If* $f(x)$ *and its first* $n-1$ *derivatives be continues in* $[a,b]$ *and* $f^n(x)$ *exists for every value of* x *in* (a,b), *then there is a real* c *lies between* (a,b) *such that*

$$f(b) = f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\frac{(b-a)^3}{3!}f'''(a)+\cdots+\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(b-a)^n}{n!}f^n(c)$$

Note: The series in the above theorem is called Taylors series of $f(x)$ about a.

Problem 2.1.1. *Obtain Taylor's expansion of* $\sin x$ *about* $x = \pi/2$. *Hence find the value of* $\sin 91^o$ *correct to 4 decimal places.*

Problem 2.1.2. *Expand* $\sin ax$ *in powers of* $(x - \pi/2)$.

Problem 2.1.3. *Find the Taylor's series expansion for* $\log_e \cos x$ *about the point* $\pi/3$.

Problem 2.1.4. *Obtain Taylor's expansion for the function* $f(x) = e^x \sin x$ *up to the term containing* $(x - \pi/4)^5$.

Problem 2.1.5. *Obtain Taylor's series expansion of* $\tan^{-1} x$ *around* $x = 1$. *Hence approximate the value of* $\tan 0.9$.

Problem 2.1.6. *Expand* $\log_e x$ *in powers of* $(x - 1)$ *and hence evaluate* $\log_e 1.1$ *correct to 4 decimal places.*

Problem 2.1.7. *Expand* $\tan^{-1} x$ *in powers of* $(x - 1)$.

Problem 2.1.8. *If* $f(x) = \ln(1 + x)$, $x > 0$, *show that for* $0 < \theta < 1$, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+\theta x)^3}$. *Hence deduce that* $\ln(1+x) < x < x - \frac{x^2}{2} + \frac{x^3}{3}$ *for* $0 < x < 1$.

Problem 2.1.9. *If* x *is positive, show that* $x > \ln(1 + x) > x - \frac{1}{2}x^2$.

Problem 2.1.10. *Prove that* $\ln \sin x = \ln \sin a + (x - a) \cot a - \frac{1}{2}(x - a)^2 \ cosec^2 a + \cdots$.

Problem 2.1.11. *By using Taylor's series compute, to four decimal places, the value of* $\cos 32^0$.

2.2 Power series

Theorem 2.2.1 (Maclaurin's)**.** *If* $f(x)$ *can be expanded as an infinite series, then*

$$f(x) = f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\cdots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+\frac{x^n}{n!}f^n(0)+\cdots$$

Note: The above expansion is also called Taylor's series expansion about the origin.

Problem 2.2.1. *Obtain Power series expansion of the following*

1. $\sin x$
2. $\cos x$
3. $\tan x$
4. e^x
5. $\frac{1}{1+x}$
6. $\tan^{-1} x$
7. $\sin^{-1} x$
8. $\cos^{-1} x$.

Problem 2.2.2. *Expand* $e^{\sin x}$ *by Maclaurin's series or otherwise up to the term containing* x^4.

Problem 2.2.3. *Expand* $\ln(1+x)$ *as a Maclaurin's series and hence obtain series expansion for* $\ln(1+\sin x)$ *in powers of* x.

Problem 2.2.4. *Expand* $\ln(1+x)$ *in powers of* x. *Hence deduce that* $\ln\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$.

Problem 2.2.5. *Expand* $\log_e(1+\sin x)$ *in powers of* x.

Problem 2.2.6. *Expand* $\log_e(1+\tan x)$ *in powers of* x.

Problem 2.2.7. *Obtain the power series expansion of* $\tan x$. *Hence deduce the power series of* $\log_e(\sec x)$

Problem 2.2.8. *Expand* $x^5 \tan x$ *in powers of* x.

Problem 2.2.9. *Obtain Power series expansion of* $\tan^{-1} x$ *and hence show that*

$$\sin^{-1}\frac{2x}{1+x^2} = 2\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right\}$$

Problem 2.2.10. *Prove that* $e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2x^4}{4!} \cdots$

Problem 2.2.11. *Prove that* $\log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots$. *Hence deduce the power series of* $\tan x$.

Problem 2.2.12. *Prove that* $e^{x\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$

Problem 2.2.13. *Prove that*

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = 2\left\{x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right\}$$

Problem 2.2.14. *Obtain power series expansion of* $\sin^{-1} x$. *Hence prove that*

$$\sin^{-1}(3x-4x^3) = 3\left\{x + \frac{x^3}{3} + \frac{3x^5}{40} + \cdots\right\}$$

Problem 2.2.15. *Expand* $e^{a\sin^{-1} x}$ *in ascending powers of* x.

Problem 2.2.16. *Expand* $\log(1+\sin^2 x)$ *in ascending powers of* x.

Problem 2.2.17. *Prove that* $\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \cdots$

Problem 2.2.18. *Prove that* $\sqrt{1+\sin x} = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{374} + \cdots$

Problem 2.2.19. *Given* $\log_{10} 4 = 0.6021$, *calculate approximately* $\log_{10} 404$ *using power series.*

2.3 Partial derivative

Consider a function $z = f(x, y)$. Let this function be continuous in the domain of definition. Supposing one of the independent variables x or y, say y is kept fixed and the other variable x is allowed to vary, then z becomes a function of a single variable x alone. Further, in this case, if the $\lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = l$ exists (i.e. finite and unique), then we denote this finite limit by $\frac{\partial z}{\partial x}$ (or $\frac{\partial f}{\partial x}$ or f_x) and is called partial derivative of the function z with respect to x. That is

$$\frac{\partial z}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the partial derivatives of $z = f(x, y)$ with respect to y is also defined and is denoted by $\frac{\partial z}{\partial y}$ (or $\frac{\partial f}{\partial y}$ or f_y). That is

$$\frac{\partial z}{\partial y} = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}, \quad \text{provided the limit exists}$$