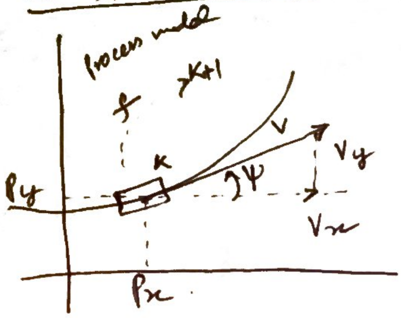
UKF Module

CTRV model:



State vector, $x = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \end{bmatrix}$

Process model

$$x_{k+1} = f(x_k, \nu_k)$$

noise vector

Change rate of state

$$\dot{x} = \begin{bmatrix} \dot{p_x} \\ \dot{p_y} \\ \dot{v} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix}$$

gives

Differential equation:

$$\dot{x} = g(x)$$

$$\dot{x} = \begin{bmatrix} v_x \\ v_y \\ 0 \\ \dot{\psi} \\ 0 \end{bmatrix} = \begin{bmatrix} v\cos\psi \\ v\sin\psi \\ 0 \\ \dot{\psi} \\ 0 \end{bmatrix}$$

Time difference

$$\Delta t = t_{k+1} - t_k$$

$$x_{k+1} = x_k + \begin{bmatrix} \frac{v_k}{\dot{\psi}_k}\left(\sin(\psi_k + \dot{\psi}_k \Delta t) - \sin\psi_k\right) \\ \frac{v_k}{\dot{\psi}_k}\left(-\cos(\psi_k + \dot{\psi}_k \Delta t) + \cos\psi_k\right) \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix}$$

What happens when $\dot{\psi}_k = 0$??

- vehicle drives in a straight line
- divide by zero !!

when $\dot{\psi}_k = 0$, we have:

$$\therefore x_{k+1} = x_k + \begin{bmatrix} v\cos\psi \, \Delta t \\ v\sin\psi \, \Delta t \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Integral

$$x_{k+1} = x_k + \int_{t_k}^{t_{k+1}} \begin{bmatrix} \dot{p_x}(t) \\ \dot{p_y}(t) \\ \dot{v}(t) \\ \dot{\psi}(t) \\ \ddot{\psi}(t) \end{bmatrix} dt$$

$$= x_k + \int_{t_k}^{t_{k+1}} \begin{bmatrix} v\cos(\psi_k + \dot{\psi}_k(t - t_k)) \\ v\sin(\psi_k + \dot{\psi}_k(t - t_k)) \\ 0 \\ \dot{\psi}_k \\ 0 \end{bmatrix} dt$$

$$= x_k + \begin{bmatrix} \frac{v_k}{\dot{\psi}_k}\sin(\psi_k + \dot{\psi}_k(t - t_k))\Big|_{t_k}^{t_{k+1}} \\ \frac{-v_k}{\dot{\psi}_k}\cos(\psi_k + \dot{\psi}_k(t - t_k))\Big|_{t_k}^{t_{k+1}} \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix}$$

Process model:

$$x_{k+1} = f(x_k, \nu_k)$$

Noise vector:

$$\nu_k = \begin{bmatrix} \nu_{a,k} \\ \nu_{\ddot{\psi},k} \end{bmatrix}$$

Assumptions about noise vector:

1) Longitudinal acceleration noise has below statistical properties:

$$\nu_{a,k} \sim N(0, \sigma_a^2) \quad \begin{bmatrix} \text{White noise with} \\ \mu = 0, \ \sigma^2 = \sigma_a^2 \end{bmatrix}$$

2). Yaw acceleration noise:

$$\nu_{\ddot{\psi},k} \sim N(0, \sigma_{\ddot{\psi}}^2)$$

3) Noise vector is constant between time steps k and k+1

4) Yaw rate, $\dot{\psi}$ is small so that car is driving nearly straight at angle $\hat{\psi}_k$.

5) Effect of $\nu_{\ddot{\psi},k}$ on p_x, p_y is small.

$\therefore$,

$$x_{k+1} = x_k + \begin{bmatrix} \frac{v_k}{\dot{\psi}_k}\left[\sin(\psi_k + \dot{\psi}_k \Delta t) - \sin\psi_k\right] \\ \frac{v_k}{\dot{\psi}_k}\left[-\cos(\psi_k + \dot{\psi}_k \Delta t) + \cos\psi_k\right] \\ 0 \\ \dot{\psi}_k \Delta t \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \nu_{a,k} \Delta t^2 \cos(\psi_k) \\ \frac{1}{2} \nu_{a,k} \Delta t^2 \sin(\psi_k) \\ \Delta t \cdot \nu_{a,k} \\ \frac{1}{2} \nu_{\ddot{\psi},k} \Delta t^2 \\ \nu_{\ddot{\psi},k} \Delta t \end{bmatrix}$$

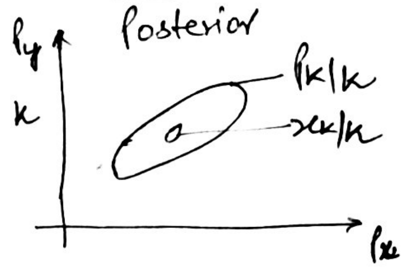
UKF process chain:

Prediction step (CTRV model) $\longrightarrow$ Update step (Radar or laser measurement model)

Note: Top level process chain of Extended KF is same as that of Unscented KF. The only difference is how Unscented KF processes non-linear process or measurement models using unscented transformation.

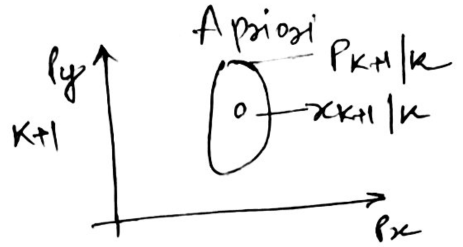
Problem with non-linear models:

Let's say we have,



K: est. time step | k: latest considered meas.

We want to predict,



k+1: est. time step | k: latest considered meas.

For linear case:

$$x_{k+1} = F x_k + \nu_k$$

covariance, $Q = E\{\nu_k \cdot \nu_k^T\}$

KF solution $\Rightarrow$

$$x_{k+1|k} = F x_{k|k}$$
$$P_{k+1|k} = F P_{k|k} F^T + Q$$

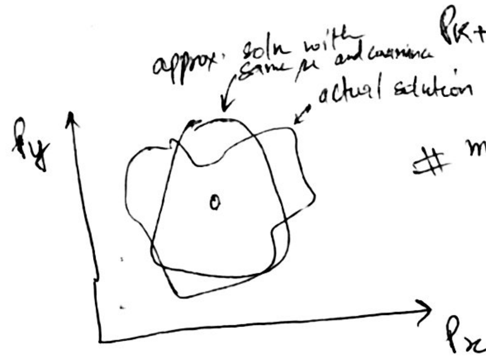
For non-linear case:

$$x_{k+1} = f(x_k + \nu_k)$$
$$Q = E\{\nu_k \cdot \nu_k^T\}$$

Solution:

$x_{k+1|k} =$

$P_{k+1|k} =$



\# may not be normally distributed and has to be solved numerically using some fancy algorithm

\# Unscented transformation provides us with an approx. normal distribution that has same mean and covariance as the (may not be normally distributed) using numerical solution (may not be normally distributed) using sigma points.

UKF Road map

- Prediction
  - Generate sigma points
  - Predict sigma points
  - Predict mean and covariance
- Update
  - Predict Measurement
  - Update state

Generating sigma points:

State vector (CTRV model), $x = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \end{bmatrix}$

State dimension, $n_x = 5$

Number of sigma points, $n_\sigma = 2n_x + 1$
$= 11$

Matrix with sigma points (given posteriors $\underset{\text{mean state}}{x_{k|k}}$ and $\underset{\text{covariance}}{P_{k|k}}$)

$$\mathcal{X}_{k|k} = \left[x_{k|k} \quad x_{k|k} + \sqrt{(\lambda + n_x) P_{k|k}} \quad x_{k|k} - \sqrt{(\lambda + n_x) P_{k|k}} \right]$$

Design parameter: $\lambda = 3 - n_x$ (Spreading factor)

Square root of matrix: $A = \sqrt{P_{k|k}} \Leftarrow A^T A = P_{k|k}$

UKF Augmentation:

Process noise, $\nu_k = \begin{bmatrix} \nu_{a,k} \\ \nu_{\ddot{\psi},k} \end{bmatrix}$ ← Independent noise processes
- doesn't express effect on state vector
- independent of Δt

Stochastic properties,

$$\nu_{a,k} \sim N(0, \sigma_a^2)$$
$$\nu_{\ddot{\psi},k} \sim N(0, \sigma_{\ddot{\psi}}^2)$$

Process noise covariance matrix,

$$Q = E(\nu_k \cdot \nu_k^T) = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_{\ddot{\psi}}^2 \end{bmatrix}$$

Augmented state to include uncertainty caused by process noise:

$$x_{a,k} = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \\ \nu_a \\ \nu_{\ddot{\psi}} \end{bmatrix}$$

Augmented covariance matrix $P_{a,k|k} = \begin{bmatrix} P_{k|k} & 0 \\ 0 & Q \end{bmatrix}_{7\times7}$

Predict sigma points using process model:

$$x_{k+1|k} = f(\underset{5\times1}{x_k}, \underset{2\times1}{\nu_k}) = f(\underset{7\times1}{x_{a,k|k}}) = [\ \ ]_{5\times15}$$

Predicted Mean and Covariance:

Predicted Mean: $x_{k+1|k} = \sum_{i=1}^{n_\sigma} w_i x_{k+1|k,i}$

Predicted Covariance: $P_{k+1|k} = \sum_{i=0}^{2n_\sigma} w_i (x_{k+1|k,i} - x_{k+1|k})(x_{k+1|k,i} - x_{k+1|k})^T$

$$w_i = \frac{\lambda}{\lambda + n_a}, \quad i = [illegible]$$

$$w_i = \frac{1}{2(\lambda + n_a)}, \quad i = 2 \ldots n_a$$

Predict Measurement:

Process model: $x_{k+1} = f(x_k, \nu_k)$

Measurement model: $z_{k+1} = h(x_{k+1}) + w_{k+1}$ (Radar measurement)

↳ measurement noise is purely additive therefore, no need to augment state vector

Reuse the sigma points generated previously for ~~update~~ predict step to save computational effort.

Predicted sigma points: $\underset{5\times15}{\mathcal{X}_{k+1|k}}$, State vector, $x_{k+1|k} = \begin{bmatrix} p_x \\ p_y \\ v \\ \psi \\ \dot{\psi} \end{bmatrix}$

Measurement model: $z_{k+1} = h(x_{k+1}) + w_{k+1}$

Ignore here (will be added later)

Measurement sigma points: $Z_{k+1|k}$ & measurement vector $= z_{k+1|k} = \begin{bmatrix} \rho \\ \varphi \\ \dot{\rho} \end{bmatrix}$

Predicted measurement mean:

$$z_{k+1|k} = \sum_{i=1}^{n_\sigma} w_i Z_{k+1|k,i}$$

Radar Measurement model:

$$z_{k+1} = h(x_{k+1}) + w_{k+1}$$

$$\rho = \sqrt{p_x^2 + p_y^2} \qquad \varphi = \tan^{-1}\left(\frac{p_y}{p_x}\right)$$

$$\dot{\rho} = \frac{p_x\cos(\psi)v + p_y\sin(\psi)v}{\sqrt{p_x^2 + p_y^2}}$$

Predicted measurement covariance:

$$S_{k+1|k} = \sum_{i=0}^{2n_\sigma} w_i \left(Z_{k+1|k,i} - z_{k+1|k}\right)\left(Z_{k+1|k,i} - z_{k+1|k}\right)^T + R$$

Measurement noise covariance: $R = E\{w_k * w_k^T\} = \begin{bmatrix} \sigma_\rho^2 & 0 & 0 \\ 0 & \sigma_\varphi^2 & 0 \\ 0 & 0 & \sigma_{\dot{\rho}}^2 \end{bmatrix}$

Update step for UKF:

Kalman Gain: $K_{k+1|k} = T_{k+1|k} S_{k+1|k}^{-1}$

measurement at timestep k+1

State Update: $x_{k+1|k+1} = x_{k+1|k} + K_{k+1|k}(z_{k+1} - z_{k+1|k})$

Covariance matrix update: $P_{k+1|k+1} = P_{k+1|k} - K_{k+1|k} S_{k+1|k} K_{k+1|k}^T$

Cross correlation b/w sigma points in state space and measurement space:

$$T_{k+1|k} = \sum_{i=0}^{2n_\sigma} w_i \left(\mathcal{X}_{k+1|k,i} - x_{k+1|k}\right)\left(Z_{k+1|k,i} - z_{k+1|k}\right)^T$$

Parameters and consistency:

Process model, $x_{k+1} = f(x_k, \nu_k)$

Process noise
$$\nu_k = \begin{bmatrix} \nu_{a,k} \\ \nu_{\ddot{\psi},k} \end{bmatrix}$$

Process noise covariance
$$\underbrace{Q = \begin{bmatrix} \sigma_a^2 & 0 \\ 0 & \sigma_{\ddot{\psi}}^2 \end{bmatrix}}_{\text{has to be selected by engineer based on application}}$$

Measurement model
$$z_{k+1} = h(x_{k+1}) + \omega_{k+1}$$

Radar measurement noise
$$\omega_k = \begin{bmatrix} \omega_{\rho,k} \\ \omega_{\varphi,k} \\ \omega_{\dot{\rho},k} \end{bmatrix}$$

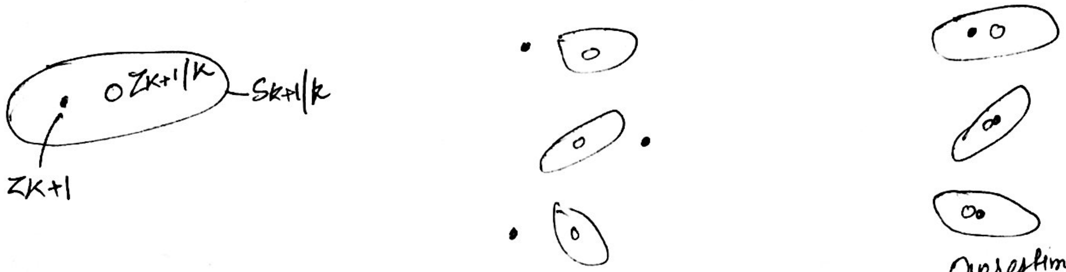
Measurement noise covariance:
$$\underbrace{R = \begin{bmatrix} \sigma_\rho^2 & 0 & 0 \\ 0 & \sigma_\varphi^2 & 0 \\ 0 & 0 & \sigma_{\dot{\rho}}^2 \end{bmatrix}}_{\text{come from manufacturer}}$$

Consistency check for process noise parameters:

Consistent

Inconsistent

Inconsistent



Underestimate of uncertainty

Overestimate of uncertainty

Consistency check through Normalized Innovation Squared (NIS)

$$\varepsilon = (z_{k+1} - z_{k+1|k})^T \cdot S_{k+1|k}^{-1} \cdot (z_{k+1} - z_{k+1|k})$$

follows χ^2 (Chi-squared) distribution based on degrees of freedom of sensor (n_z)

Advantages of UKF:

- → Input noisy sensor data and provide estimates of position & velocity without delay.
- → Provide estimate of orientation & yaw rate without having sensors with these.
- → For all these estimations we also get the information about uncertainty of our estimation from covariance
- → we can check whether the values are realistic if UKF perf passes consistency check.

* UKF Module Ends *