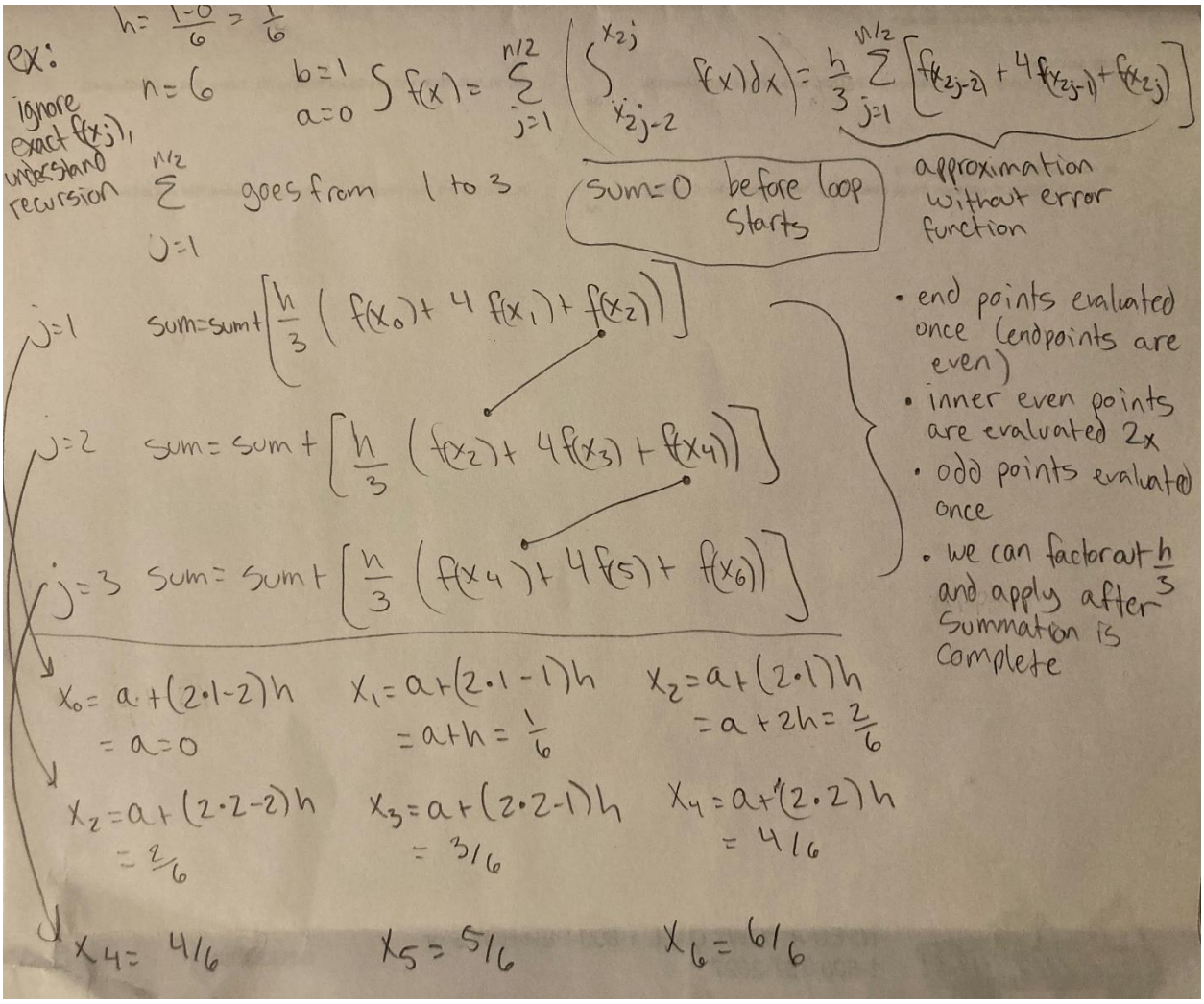
Project 3, Raktim Biswas

1. Composite Simpson's Rule

I've included this scratchwork to demonstrate how I programmed my second function for approximating integrals since this is still Composite Simpson's Rule, but written differently than the formula given to us in the project pdf as well as the sources I used.



2. Error Analysis

We can set an appropriate upper bound for our error by noticing that the fourth derivative has 2 critical points on [0,1]; in other words, the fifth derivative has 2 zeros on our interval. After evaluating the zeros of the fifth derivative we find that the largest absolute value of the fourth derivative occurs at gamma equals 0. This lets us set a "lazy" error bound as we can see that our absolute error is less than our bound by about a whole power of 10 in each approximation. None the less, our error and bounds decrease as n increases. Python has its own erf(z) function as well to verify proper calculations of our absolute error.

Between both methods, we get very similar approximations, with variances only in the last few decimal places so we can assume their error bounds and absolute errors are almost identical. As we've discussed in class, we expect the error in Composite Simpson's Rule to be O(h^4). This means that as h is decreased (or increased) by a certain factor (the reciprocal of that factor is applied to n), the error will decrease (or increase) by that factor to the power of 4. This algorithm behaves exactly as we hope; each time h is reduced by ½, the error bound reduces exactly by (1/2)^4, although the absolute errors do no reduce by exactly 1/16.

To determine n such that our error is guaranteed to be less than 10^-12, we must set up an inequality with our error bound and 10^-12. With some algebra we can rearrange the inequality to give us n by first converting h^4 in our error bound to ((b-a)/n)^4. After some calculations we find that n must be greater than or equal to 524 to guarantee an error less than 10^-12.

Your input: find the local and global minima and maxima of $f = \dfrac{\left(32x^4 - 96x^2 + 24\right)e^{-x^2}}{\sqrt{\pi}}$

on the interval $[0, 1]$

CRITICAL POINTS

$$(x, f(x)) = \left(0, \frac{24}{\sqrt{\pi}}\right) \approx (0, 13.5405500051462)$$

$$(x, f(x)) = \left(\frac{\sqrt{10 - 2\sqrt{10}}}{2}, \frac{-216 + 2\left(10 - 2\sqrt{10}\right)^2 + 48\sqrt{10}}{\sqrt{\pi}e^{\frac{5}{2} - \frac{\sqrt{10}}{2}}}\right)$$

$$\approx (0.958572464613819, -8.37198799030836)$$

$f^4(x) = \frac{8}{\sqrt{\pi} \cdot e^{x^2}} \cdot (4x^4 - 12x^2 + 3)$

critical points on $[0;1]$

at $x = 0, \frac{\sqrt{10-2\sqrt{10}}}{2}$ (x_1, x_2)

$|f(x_1)| > |f(x_2)|$

$|E(f)| = \left|\frac{-h^4}{180}(b-a) f^4(\Theta_x)\right|$

$\leq \left|\frac{-h^4}{180}(b-a) f^4(0)\right| < 10^{-12}$ ← find n that guarantees this

bound on error

$\rightarrow \frac{1}{180} \cdot \left(\frac{b-a}{n}\right)^4 \cdot (b-a) f^4(0) < 10^{-12} \rightarrow \frac{(b-a)^5 f^4(0)}{180\, n^4} < 10^{-12}$

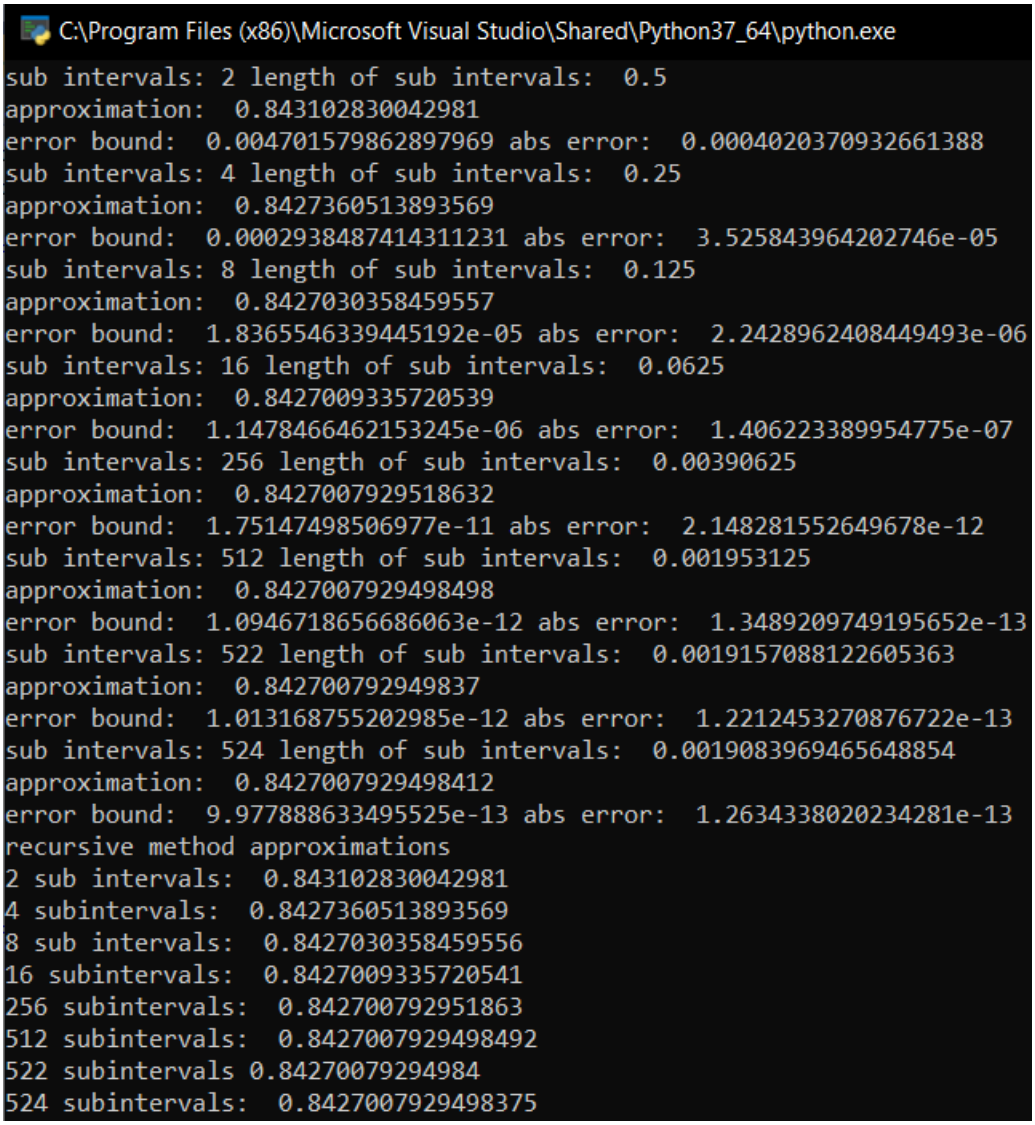
$\rightarrow \frac{f^4(0)}{n^4} < 1.8 \times 10^{-10} \rightarrow n^4 > \frac{f^4(0)}{1.8 \times 10^{-10}} \simeq \frac{13.54055}{1.8 \times 10^{-10}}$

$= \frac{24}{\sqrt{\pi} \cdot 1.8 \times 10^{-10}}$

$\rightarrow n > \sqrt[4]{\frac{24}{\sqrt{\pi} \cdot 1.8 \times 10^{-10}}} \simeq 523.710101$

n must by 524

- even
- guarantees error less than 10^{-12}



3. Resources

https://www.youtube.com/watch?v=3MNGU5w1mH8&list=PLQ9rYUh73b8SiBfAZJ88HGHiYSSMocQ8K&index=11

https://personal.math.ubc.ca/~pwalls/math-python/integration/simpsons-rule/