EE 7390

P R M L

HW I

a)

Sol^n: $P_X(k) = \begin{cases} \frac{1}{4} & \text{for } k=1 \\ \frac{1}{8} & \text{for } k=2 \\ \frac{1}{8} & \text{for } k=3 \\ \frac{1}{2} & \text{for } k=4 \\ 0 & \text{for otherwise} \end{cases}$

$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k=1 \\ \frac{1}{6} & \text{for } k=2 \\ \frac{1}{3} & \text{for } k=3 \\ \frac{1}{3} & \text{for } k=4 \\ 0 & \text{otherwise} \end{cases}$

It is given that X and Y are independent.

a) $P(X \leq 2 \text{ and } Y \leq 2) = P(X \leq 2) \cdot P(Y \leq 2)$

$= \left(\frac{1}{4} + \frac{1}{8}\right)\left(\frac{1}{6} + \frac{1}{6}\right)$

$= \frac{\cancel{3}}{\cancel{8}_4} \times \frac{\cancel{2}}{\cancel{3}} = \frac{1}{4}$

b) $P(X > 2 \text{ and } Y > 2) = P(X>2) \cdot P(Y>2)$

$= \left(\frac{1}{8} + \frac{1}{2}\right)\left(\frac{1}{3} + \frac{1}{3}\right)$

$= \frac{5}{\cancel{8}_4} \times \frac{\cancel{2}}{3} = \frac{5}{12}$

c) $P(X > 2 \,|\, Y > 2) = P(X > 2) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$

d) $P(X<Y) = P(Y=1 \text{ and } X<Y) + P(Y=2 \text{ and } X<Y) + P(Y=3 \text{ and } X<Y) + P(Y=4 \text{ and } X<Y)$

$$= \frac{1}{6}\times 0 + \frac{1}{6}\times\frac{1}{4} + \frac{1}{3}\times\left(\frac{1}{8}+\frac{1}{4}\right) + \frac{1}{3}\times\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{4}\right)$$

$$= \frac{1}{24} + \frac{1}{\cancel{3}}\times\frac{\cancel{3}}{8} + \frac{1}{3}\times\frac{1}{2}$$

$$= \frac{1}{24} + \frac{1}{8} + \frac{1}{6} = \frac{1+3+4}{24} = \frac{8}{24} = \frac{1}{3}$$

Q2)

Sol^n

$$P_X(k) = \begin{cases} 0.5 & \text{for } k=1 \\ 0.3 & \text{for } k=2 \\ 0.1 & \text{for } k=3 \\ 0.1 & \text{for } k=4 \\ 0 & \text{otherwise} \end{cases}$$

a) $E[X] = 1\times 0.5 + 2\times 0.3 + 3\times 0.1 + 4\times 0.1$

$= 0.5 + 0.6 + 0.3 + 0.4$

$= 1.8$

b) $Var(X) = E[X^2] - E[X]^2$

$= 0.5\times 1^2 + 0.3\times 4 + 0.1\times 9 + 0.1\times 16 - (1.8)^2$

$= 0.96$

$SD(X) = \sqrt{0.96} = 0.979$

Q3)

Sol^n $P(X \geq m) \geq \frac{1}{2}$ and $P(X \leq m) \geq \frac{1}{2}$.

a) For $m=2$,

$$P(x \geq m) = 0.6 \geq \frac{1}{2}$$

& $P(x \leq m) = 0.3 + 0.4 = 0.4 \geq \frac{1}{2}$

b) $$P_x(k) = \begin{cases} \frac{1}{6} & \text{for } k=1 \\ \frac{1}{6} & \text{for } k=2 \\ \frac{1}{6} & \text{for } k=3 \\ \frac{1}{6} & \text{for } k=4 \\ \frac{1}{6} & \text{"} \quad k=5 \\ \frac{1}{6} & \text{"} \quad k=6 \end{cases}$$

$m = 3$ or 4

Q4)

Sol:- a) $\cancel{P(X=x) = \frac{1}{101}}$

$$f_x(x) = \frac{1}{100}$$

$$E[X] = \int_0^{100} x f_x(x)\, dx = \int_0^{100} \frac{x}{100}\, dx = \frac{[x^2]_0^{100}}{2 \times 100}$$

$$= \frac{100 \times \cancel{100}}{2 \times \cancel{100}}$$

$$= 50$$

$$E[X \mid X \geq 65] = \frac{\int_{65}^{100} x\, dx}{35} = \frac{\cancel{(100-65)}(100+65)}{2 \times \cancel{35}}$$

$$= \frac{165}{2} = 82.5$$

b) $$f_x(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \int_0^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \lambda \left[\int_0^{\infty} x e^{-\lambda x} dx \right]$$

$$= \lambda \left[\left[\frac{x e^{-\lambda x}}{\lambda} \right]_{\infty}^{0} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \right]$$

$$= \lambda \left[\frac{1}{\lambda} \int_0^{\infty} e^{-\lambda x} dx \right]$$

$$= \frac{\left[e^{-\lambda x} \right]_{\infty}^{0}}{\lambda} = \frac{1}{\lambda}$$

$$f_X(x|X>2) = \begin{cases} \lambda e^{-\lambda(x-2)} & x \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X|X \geq 2] = \lambda \int_2^{\infty} x e^{-\lambda(x-2)} dx$$

$$= \lambda \left[\left[\frac{x e^{-\lambda(x-2)}}{\lambda} \right]_{\infty}^{2} + \frac{1}{\lambda} \int_2^{\infty} e^{-\lambda(x-2)} dx \right]$$

$$= \lambda \left[\frac{2}{\lambda} + \frac{1}{\lambda} \frac{\left[e^{-\lambda(x-2)} \right]_{\infty}^{2}}{\lambda} \right]$$

$$= 2 + \frac{1}{\lambda}$$

Q5)

Sol:

$$f_X(x) = \begin{cases} c e^{-4x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a) $\int_{-\infty}^{\infty} f_X(x) dx = 1$

$\Rightarrow c \int_0^{\infty} e^{-4x} dx = 1 \quad \Rightarrow c = 4$

b) $F_x(x) = P(~~x<~~ \; X < x) = \int_0^x 4e^{-4x} dx$

$= \frac{4\left[e^{-4x}\right]_x^0}{4}$

$= 1 - e^{-4x}$

c) $P(2 < X < 5) = F_X(5) - F_X(2)$

$= 1 - e^{-20} - 1 + e^{-8}$

$= e^{-8} - e^{-20}$

d) $E[X] = \int_0^\infty 4x e^{-4x} dx = \frac{1}{4}$

Q6)

Sol:-

$Y = g(x)$

$$g(x) = \begin{cases} x & |x| \leq 2 \\ -2 & x < -2 \\ 2 & x > 2 \end{cases}$$

a) $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

For $y < -2$,

$F_Y(y) = Pr(Y < y) = Pr(Y < -2) = Pr(g(x) < -2)$
$= 0$

For $|y| < 2$,

$F_Y(y) = Pr(-2 \leq Y \leq 2)$

$= Pr(-2 \leq g(x) \leq 2) = Pr(-2 \leq X \leq 2)$
$= F_X(y)$

For $y > 2$,

$F_Y(y) = Pr(Y \leq 2) = Pr(g(x) \leq 2) = 1$

$$\therefore F_Y(y) = \begin{cases} 0 & y < -2 \\ F_X(y) & |y| \leq 2 \\ 1 & y > 2 \end{cases}$$

$$f_Y(y) = \begin{cases} f_X(y) & |y| \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

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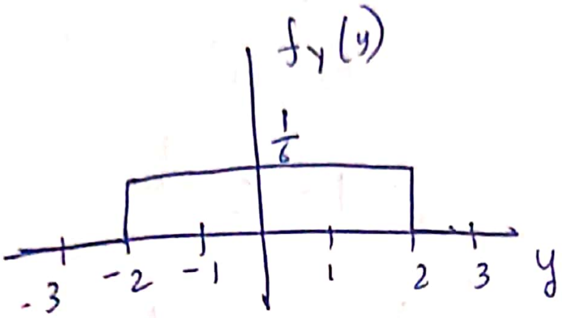
b) $X \sim U(-3, 3)$

$\therefore f_X(x) = \begin{cases} \frac{1}{6} & |x| < 3 \\ 0 & \text{otherwise} \end{cases}$

$\therefore f_Y(y) = \begin{cases} \frac{1}{6} & |y| \leq 2 \\ 0 & \text{otherwise} \end{cases}$



Q7)

$Y = X^2$

a) $f_X(x) = \begin{cases} \frac{x}{a} \exp\left(\frac{-x^2}{2a}\right) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{0}^{x} \frac{x}{a} \exp\left(\frac{-x^2}{2a}\right) dx$

let $\frac{x^2}{2a} = t$

$\Rightarrow \frac{2x}{2a} dx = dt \Rightarrow \frac{x}{a} dx = dt$

$\therefore F_X(x) = \int_{0}^{\frac{x^2}{2a}} \exp(-t)\,dt$

$= -\exp\left(\frac{-x^2}{2a}\right) + 1 = -e^{-\frac{x^2}{2a}} + 1$

$Y = X^2$

$\Rightarrow X = \sqrt{Y}$

~~For~~ $y < 0$,

$F_Y(y) \cancel{=} \cancel{Pr} (\cancel{Y < y}) = Pr(\cancel{X^2} < y < 0) = 0$

$F(x) \quad y > 0$

$F_Y(y) = Pr(Y < y) =$

$F_X(x) =$

$$F_Y(y) = Pr(Y < y) = Pr(X^2 < y)$$
$$= Pr(X < \sqrt{y})$$
$$= F_X(\sqrt{y})$$
$$= 1 - e^{-\frac{y}{2a}}$$

$$\therefore f_Y(y) = \frac{d\left(1 - e^{-\frac{y}{2a}}\right)}{dy} = \frac{e^{-\frac{y}{2a}}}{2a}$$

b) $$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$F_X(x) = \frac{1}{2}\left[1 + erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right]$$

$$\therefore F_Y(y) = \frac{1}{2}\left[1 + erf\left(\frac{\sqrt{y}}{\sigma\sqrt{2}}\right)\right]$$

$$\therefore f_Y(y) =$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{t^2}{2\sigma^2}}\, dt$$

$$F_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\sqrt{y}} e^{-\frac{t^2}{2\sigma^2}}\, dt$$

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{e^{-\frac{y}{2\sigma^2}}}{2\sqrt{y}}$$

Q) Sol. $f_X(x) = \begin{cases} x e^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$F_X(x) = \int_0^x t e^{-\frac{t^2}{2}} dt$

$\frac{t^2}{2} = m$

$t\,dt = dm$

$\therefore F_X(x) = \int_0^{\frac{x^2}{2}} e^{-m} dm$

$= [e^{-m}]_{\frac{x^2}{2}}^{0} = 1 - e^{-\frac{x^2}{2}}$

$\therefore F_X(x) = \begin{cases} 1 - e^{-\frac{x^2}{2}} ; & x \geq 0 \\ 0 & x < 0 \end{cases}$

a) $Z = X/Y$

$F_Z(z) = P_r(Z \leq z) = P_r\left(\frac{X}{Y} \leq z\right) = P_r\left(\frac{X}{y} \leq z\right)$

$= P_r(X \leq yz)$

$= F_X(yz)$

$\therefore F_Z(z|Y=y) = F_X(yz)$

$\therefore f_Z(z|Y=y) = y f_X(yz)$

$= y\left(yz\, e^{-\frac{y^2 z^2}{2}}\right)$

$= y^2 z\, e^{-\frac{y^2 z^2}{2}}$

b) $f_Z(z) = \int f(y)\, f(z|Y=y)\, dy$

$= \int_0^{\infty} y^3 z\, e^{-\frac{y^2 z^2}{2}} dy$

$= \frac{2z}{(z^2+1)^2}$

Q)

Sol:- $\underline{X} = [x_1 \quad x_2 \quad x_3]$

$C = \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \qquad \mu_X = [0 \quad 0 \quad 0]$

a) $\det(C) = 6(15-9) = 36$

$$f_{\underline{X}}(\underline{x}) = \frac{1}{(2\pi)^{3/2}\sqrt{36}} e^{-\frac{1}{2}(\underline{x}^T C^{-1} \underline{x})}$$

$$= \frac{1}{(2\pi)^{3/2} 6} e^{-\frac{1}{2}\underline{x}^T C^{-1} \underline{x}}$$

$$= 0.063\, e^{-\frac{1}{2}\underline{x}^T C^{-1} \underline{x}}$$

$$C^{-1} = \begin{bmatrix} 5/6 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$$

b) $Y = X_1 + 2X_2 - X_3$

$$= \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$E[Y] = 0$

$$\text{cov}[Y] = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 41$$

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi \times 41}} e^{-\frac{\|\underline{x}\|^2}{82}}$$

c) $$Z = \begin{bmatrix} 5 & -3 & -1 \\ -1 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix} X$$

$\mu[z]=0$

$$\text{cov}[z] = \begin{bmatrix} 5 & -3 & -1 \\ -1 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 5 & -1 & 1 \\ -3 & 3 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\sqrt{\det(\text{cov}(z))} = \sqrt{36 \times 36 \times 9} = 6 \times 6 \times 3 = 108$$

$$(\text{cov}(z))^{-1} = \begin{bmatrix} \frac{1}{36} & 0 & 0 \\ 0 & \frac{1}{36} & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix}$$

$$\therefore f_Z(z) = \frac{1}{(2\pi)^{3/2} \times 108} e^{-\frac{1}{2} \underline{z}^T \begin{bmatrix} \frac{1}{36} & 0 & 0 \\ 0 & \frac{1}{36} & 0 \\ 0 & 0 & \frac{1}{9} \end{bmatrix} \underline{z}}$$

2(ii)

Sol^n $F_X(x) = \begin{cases} 1 & , x \geq 1 \\ \frac{1}{2} + \frac{x}{2} & , 0 \leq x < 1 \\ 0 & , x < 0 \end{cases}$

a) ~~Mixed~~ Continuous

b) $f_X(x) = \begin{cases} \frac{1}{2} & ; 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$

c) $E[e^x] = \int_0^1 e^x f_X(x)\, dx = \frac{[e^x]_0^1}{2}$

$$= \frac{e - 1}{2}$$

d) $P(X=0 \mid X \leq 0.5)$

$$= \frac{P(X=0)}{P(X \leq 0.5)} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

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Q11)

Sol^n $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$ $\quad \because X \& Y$ are iid

$F_{Z,W}(z,w) = P_r[\max(X,Y) \le z, \min(X,Y) \le w]$

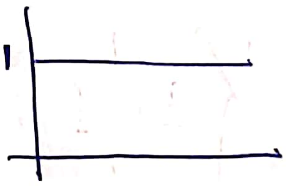
$= F_{X,Y}(w,\eta) + F_{X,Y}(z,w) - F_{X,Y}(w,w)\}$

$\therefore f_{Z,W}(z,w) = \begin{cases} f_{XY}(w,z) + f_{X,Y}(z,w) & , z \ge w \\ 0 & , z < w \end{cases}$

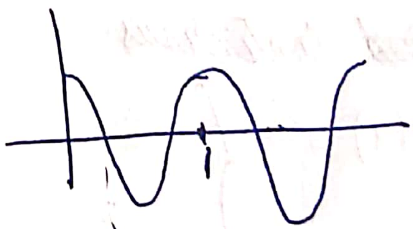
$\therefore f_{Z,W}(z,w) = \begin{cases} 2\lambda^2 e^{-\lambda(w+z)} & , z \ge 0 \\ 0 & , \text{otherwise} \end{cases}$

Q12)

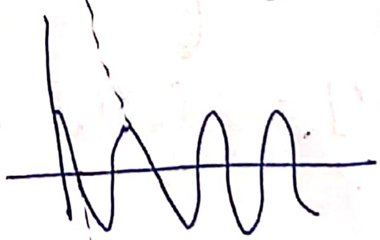
a) For $s=0$,



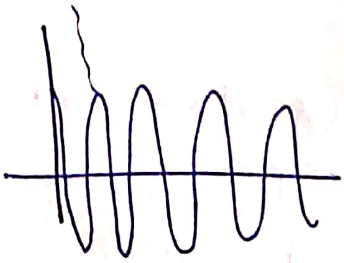
For $s=1$



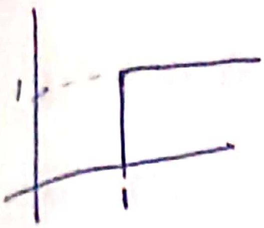
For $s=2$



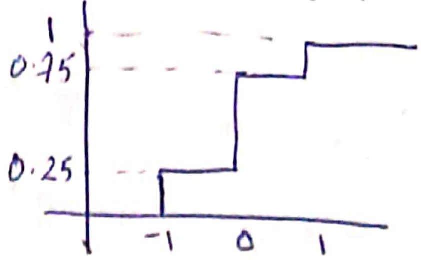
For $s=3$



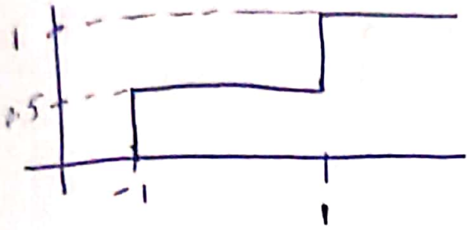
b) $x_0 = \cos 0 = 1$



$X_{\frac{1}{4}} = \cos\left(\frac{2\pi s}{4}\right) = \cos\left(\frac{\pi}{2} s\right)$



$X_{\frac{1}{2}} = \cos\left(2\pi \frac{1}{2}\right) = \cos(\pi s)$



c) $X_{0.25} = \cos\left(\frac{\pi}{2} s \cdot \frac{1}{4}\right) = \cos\left(\frac{\pi s}{2}\right)$

Given $X_{0.5} = -1$

$\Rightarrow \cos(\pi s) = -1$

$\therefore s = 1$ or 3

For $s = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$

For $s = 3$, $\cos\left(\frac{3\pi}{2}\right) = 0$

$\therefore P_H(X_{0.25} = 0 \mid X_{0.5} = -1) = \begin{cases} 1 & ; X_{0.25} = 0 \\ 0 & \text{otherwise} \end{cases}$

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d) $X_{0.5} = 1$

$\Rightarrow \cos\left(2\pi s \frac{1}{2}\right) = 1$

$\Rightarrow \cos(\pi s) = 1$

$\Rightarrow s = 0$ or 2

For $s = 0$

$X_{0.25} = \cos\left(2\pi \times 0 \times \frac{1}{4}\right) = 0 = 1$

For $s = 2$,

$X_{0.25} = \cos\left(2\pi \times 2 \times \frac{1}{4}\right) = -1$

$\therefore P_H(X_{0.25} \mid X_{0.5} = -1) = \begin{cases} \frac{1}{2} & \text{for } X_{0.25} = 1 \\ \frac{1}{2} & \text{for } X_{0.25} = -1 \end{cases}$

Q13)

Sol^n: a) $X_t = A\cos(2\pi f t + \theta)$

$$E[X_{t+\tau} X_t] = E[A^2 \cos(2\pi f t + \theta) \cos(2\pi f(t+\tau) + \theta)]$$

$$= A^2 E\left[\frac{\cos(2\pi f \tau) + \cos(2\pi f(2t+\tau) + 2\theta)}{2}\right]$$

$$= \frac{A^2}{2} \cos(2\pi f \tau)$$

Q14)

Sol^n a) $E[X_n] = \rho E[X_{n-1}] + E[W_n]$

$\Rightarrow E[X_n] = \rho E[X_n]$

$\Rightarrow (1-\rho) E[X_n] = 0$

$\because (1-\rho) \neq 0 \quad \therefore E[X_n] = 0$

similarily,

$Var[X_n] = 1 + \rho^2 + \ldots + \rho^{2n-2}$

b) $E[X_n X_{n+k}] = E[X_n(\rho X_{n+k-1} + W_{n+k})]$

$= \rho E[X_n X_{n+k-1}]$

$= \rho^2 E[X_n X_{n+k-2}]$

$\vdots$

$= \rho^k E[X_n^2]$

$= \rho^k (1 + \rho^2 + \ldots + \rho^{2n-2})$

c) Correlation of X_n depends on n

$\therefore X_n$ is not W.S.S.

17)

$$x(t) = T + (1-t) \qquad T \sim U(0,1)$$

a) $$F_x(x) = \begin{cases} 0 & , x < 1-t \\ 1 & , x > 2-t \\ x+t-1 & , 1-t < x < 2-t \end{cases}$$

b) $$f_x(x) = \frac{d F_x(x)}{dx} = \begin{cases} 1 & , 1-t < x < 2-t \\ 0 & , \text{otherwise} \end{cases}$$

$$m_x(t) = \int_{1-t}^{2-t} x f_x(x)\, dx$$

$$= \frac{(2-t)^2 - (1-t)^2}{2}$$

$$= \frac{3-2t}{2}$$

$$R_x(t_1, t_2) = E\left[(T + (1-t_1))(T + (1-t_2))\right]$$

$$= E\left[T^2 + (2-t_1-t_2)T + (1-t_1)(1-t_2)\right]$$

$$= E[T^2] + (2-t_1-t_2)E[T] + (1-t_1)(1-t_2)$$

$$= \frac{1}{3} + \frac{(2-t_1-t_2)}{2} + 1 - t_1 - t_2 + t_1 t_2$$

$$= \frac{7}{3} - \frac{3}{2}t_1 - \frac{3}{2}t_2 + t_1 t_2$$

Q18)
Ans-

$$P_{Y|X}(y|x) = x(1-x)^{y-1}$$

Now, given $Y = 3$

$$P_{Y|X}(y|x) = P_{Y|X}(3|x) = x(1-x)^2$$

$$P_{(Y|X)}(y|x) f_X = x(1-x)^2 \cdot 2x = 2x^2(1-x)^2$$

For the map,

$\hat{x}_{MAP}$.

$$\frac{d}{dx}\left(x^2(1-x)^2\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(x^4 - 2x^3 + x^2\right) = 0$$

$$\Rightarrow 4x^3 - 6x^2 + 2x = 0$$

$$\Rightarrow x(4x^2 - 6x + 2) = 0$$

$$\Rightarrow x(2x^2 - 3x + 1) = 0$$

$$\therefore x = \frac{1}{2}$$

$$\therefore \hat{x}_{MAP} = \frac{1}{2}$$

Q19)

Sol^n - Similarly as the previous question,

$$P_{Y|X}(y|x)\, f_X(x) = 3x^3(1-x)^4$$

$$Q\ \frac{d}{dx}\left(x^3(1-x)^4\right) = 0$$

$$\Rightarrow 3x^2(1-x)^4 - 4(1-x)^3 x^3 = 0$$

$$\Rightarrow 3(1-x) - 4x = 0$$

$$\Rightarrow x = \frac{3}{7}$$

$$\therefore \hat{x}_{MAP} = \frac{3}{7}$$

Q16)

Sol^n - a) $P_{X_i}(x_i, \theta) = {}^3C_{x_i}\theta^{x_i}(1-\theta)^{3-x_i}$

$$L(x_1 x_2 x_3 x_4) = P_{X_1 X_2 X_3 X_4}(x_1 x_2 x_3 x_4 ; \theta)$$

$$= {}^3C_{x_1}\theta^{x_1}(1-\theta)^{1-x_1}\, {}^3C_{x_2}\theta^{x_2}(1-\theta)^{1-x_2} \cdots$$

$$\cdots {}^3C_{x_4}\theta^{x_4}(1-\theta)^{1-x_4}$$

For $x_1 = 1$, $x_2 = 3$, $x_3 = 2$, $x_4 = 2$,

$$L(1,3,2,2;\theta) = 27\theta^8(1-\theta)^4$$

$$= 27\left[8\theta^7(1-\theta)^4 - 4\theta^8(1-\theta)^3\right] = 0$$

$$\frac{\partial L}{\partial \theta} = 27\left[8\theta^7(1-\theta)^4 - 4\theta^8(1-\theta)^3\right] = 0$$

$$\Rightarrow 2(1-\theta) = \theta$$

$$\Rightarrow \theta_{MLE} = \frac{2}{3}$$

b) $f_{X_i}(x_i;\theta) = \theta e^{-\theta x}$

$$L(x;\theta) = \theta^4 e^{-\theta(x_1+x_2+x_4+x_4)}$$

$$= \theta^4 e^{-\theta(8)} = \theta^4 e^{-8\theta}$$

$$\log L(x,\theta) = 4\log\theta - 8\theta$$

$$\frac{d \log L(x,\theta)}{d\theta} = 0$$

$$\Rightarrow \frac{4}{\theta} - 8 = 0$$

$$\Rightarrow \theta = \frac{1}{2} \qquad \therefore \theta_{MLE} = \frac{1}{2}$$

Q(7)

Ans - a) Binomial distribution -

$$f(x;p) = {}^nC_x\, p^x (1-p)^{n-x}$$

$$L(x;p) = \prod_{i=1}^{N} {}^nC_{x_i}\, p^{x_i}(1-p)^{n-x_i}$$

$$\log L = \sum_{i=1}^{N} \log\left[{}^nC_{x_i}\, p^{x_i}(1-p)^{n-x_i}\right]$$

$$\frac{\partial \log L}{\partial p} = \frac{d}{dp}\left(\sum_{i=1}^{n} \log {}^nC_{x_i} + x_i \log p + (n-x_i)\log(1-p)\right]$$

$$= \sum^{N}\left(\frac{x_i}{p}\left(\frac{n-x_i}{1-p}\right)\right) = 0$$

$$\Rightarrow \sum^{N}(x_i - np) = 0$$

$$\Rightarrow P_{MLE} = \frac{\sum^{N} x_i}{nN}$$

b) Poisson's distribution

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$L(x;\lambda) = \prod_{i=1}^{N} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$\log L = \sum_{i=1}^{N} (x_i \log\lambda - \lambda - \log(x_i!))$$

$$\frac{d\log L}{d\lambda} = \sum_{i}^{N} \left(\frac{x_i}{\lambda} - 1\right)$$

$$\Rightarrow \frac{\sum x_i}{\lambda} = N \qquad \Rightarrow \lambda_{MLE} = \frac{1}{N}\sum_{i}^{N} x_i$$

c) Exponential distribution

$$f(x;\lambda) = \lambda e^{-\lambda x}$$

$$L(x;\lambda) = \prod_{i=1}^{N} \lambda e^{-\lambda x_i}$$

$$\log(L) = \sum_{i=1}^{N} (\log\lambda - \lambda \cancel{\log}\{x_i\})$$

$$\frac{d(\log L)}{d\lambda} = \sum_{i=1}^{N} \left(\frac{1}{\lambda} - \cancel{\log}\{x_i\}\right) = 0$$

$$\Rightarrow \frac{N}{\lambda} - \sum x_i = 0$$

$$\Rightarrow \lambda_{MLE} = \frac{N}{\sum x_i}$$

d) Gaussian Distribution

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$L = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

$$\log L = \sum_{i=1}^{N} \left(\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$\frac{\partial \log L}{\partial \mu} = 0$$

$$\Rightarrow \mu_{MLE} = \frac{1}{N}\sum^{N} x_i$$

$\frac{\partial \log L}{\partial \sigma} = 0$

$\Rightarrow \sigma_{MLE}^2 = \sum_{i=0}^{N} \frac{(x_i - \mu_{MLE})^2}{N}$

ii) Laplacian Distribution

$f(x; \mu, b) = \frac{1}{2b} e^{-\frac{|x-\mu|}{b}}$

$L = \prod_{i=1}^{N} \frac{1}{2b} e^{-\frac{|x_i - \mu|}{b}}$

$\log L = \sum_{i=1}^{N} \left(\log\left(\frac{1}{2b}\right) - \frac{|x_i - \mu|}{b} \right)$

$\frac{\partial \log L}{\partial b} = 0$

$\Rightarrow \sum_{i=1}^{N} \left(2b\left(\frac{-1}{2b^2}\right) + \frac{|x_i - \mu|}{b} \right) = 0$

$\Rightarrow b_{MLE} = \frac{\sum^{N} |x_i - \mu|}{N}, \quad \mu = \mu_{MLE}$

For μ_{MLE} :

$\frac{d \log L}{d\mu} = 0$

$\Rightarrow \sum_{i=1}^{N} \operatorname{sgn}(x_i - \mu) = 0$

$\therefore$ if N is odd,

~~μ_{MLE} = mid median~~

μ_{MLE} = median of $\{x_i\}^N$

else, if N is even,

μ_{MLE} = average of middle 2 value of $\{x_i\}^N$

Q20)

Sol^n: $f_{XY}(x,y) = \begin{cases} x + \frac{3}{2}y^2, & \text{if } 0 \le x,y < 1 \\ 0 & \text{otherwise} \end{cases}$

$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 \left(x + \frac{3}{2}y^2\right) dy$

$= \left[xy + \frac{y^3}{2}\right]_0^1$

$\therefore f_X(x) = x + \frac{1}{2}$

$\cancel{f_{X|Y}} = f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$

$= \frac{(x + \frac{3}{2}y^2)}{x + \frac{1}{2}}$

MAP estimate of X given $Y=y \rightarrow$

$f_{X|Y}(x|Y=y)$

$= f_X(x)\, f_{Y|X}(y|x)$

$P(x) = \frac{x + \frac{3}{2}y^2}{x + \frac{1}{2}}$

Max occurs at $x = 1$.

MLE estimate $\Rightarrow$

We need max. $f_{Y|X}(Y=y|x)$

$P(x) = \frac{x + \frac{3}{2}y^2}{x + \frac{1}{2}}$

$\frac{dP}{dx} = 0$

$\Rightarrow \frac{1}{x + \frac{1}{2}} - \frac{(x + \frac{3}{2}y^2)}{(x + \frac{1}{2})^2} = 0$

$\Rightarrow x + \frac{1}{2} = x + \frac{3}{2}y^2$

$\Rightarrow \frac{3}{2}y^2 = \frac{1}{2}$

$\because x$ gets cancelled, we check at the extremes.

For $x = 0$, $P(x) = 3y^2$

For $x = 1$, $P(x) = \frac{1 + \frac{3}{2}y^2}{\frac{3}{2}} = \frac{2}{3} + y^2$

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