# AN EIGENVALUE REPRESENTATION FOR RANDOM WALK HITTING TIMES AND ITS APPLICATION TO THE ROOK GRAPH

HOWARD LEVINSON

Abstract. Given an aperiodic random walk on a finite graph, an expression will be derived for the hitting times in terms of the eigenvalues of the transition matrix. The process of diagonalizing the transition matrix and its associated fundamental matrix will be discussed. These results will then be applied to a random walk on a rook graph. Lastly, a cover time bound depending on the hitting times will be proved. **Keywords.** Fundamental matrix, transition matrix eigenvalues, random walk, hitting times, cover times, rook graph.

## 1. The Fundamental Matrix

Consider an aperiodic random walk $X_k$ on a finite graph with $n$ nodes. By the convergence theorem for finite Markov chains [2], the associated transition matrix $P$ satisfies $\lim_{k\to\infty} P^k = P_\infty$, where $(P_\infty)_{ij} = \pi_j$, the $j$th component of the unique stationary distribution $\pi$. That is,

$$P_\infty = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_n \\ \vdots & \vdots & \vdots & \vdots \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix}.$$

Note that $P_\infty$ can be rewritten as $e\pi^T$, where $e$ is the column vector consisting of all ones. We now define the fundamental matrix $Z$ by

$$Z = (I - (P - P_\infty))^{-1}.$$

We will use the fundamental matrix to derive an expression for the expected hitting time from node $i$ to node $j$, $E_iT_j$. Let $H$ be the hitting time matrix for $P$, that is $h_{ij} = E_iT_j$. Also define the return time matrix $R$ by $r_{ij} = E_i\tau_j$, where we define $\tau_j = \min\{k \geq 1 : X_k = j\}$. Note that for $i = j$, $h_{ii} = 0$ and $r_{ii} = 1/\pi_i$, and for $i \neq j$, $h_{ij} = r_{ij}$. The following lemmas will prove some important properties about $P_\infty$ and $Z$ that will be used in further analysis.

**Lemma 1** (from [1])**. Commutativity of $P$ and $P_\infty$**
*One has the identity,*

$$PP_\infty = P_\infty P = P_\infty P_\infty = P_\infty.$$

*Proof.* As $P$ is stochastic, $Pe = e$, hence $PP_\infty = Pe\pi^T = e\pi^T = P_\infty$. Likewise, $P_\infty P = e\pi^T P = e\pi^T = P_\infty$. Lastly, we have

$$\begin{aligned}(P_\infty P_\infty)_{ij} =& \pi_1\pi_j + \cdots + \pi_n\pi_j \\ =& \pi_j(\pi_1 + \cdots + \pi_n) = \pi_j = (P_\infty)_{ij}.\end{aligned}$$

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□

**Lemma 2** (from [1]). *$Z$ is well defined and,*

$$Z = I + \sum_{n=1}^{\infty}(P^n - P_\infty) = \sum_{n=0}^{\infty}(P - P_\infty)^n.$$

*Proof.* First, equality between the second two expressions is proved. For $n \geq 1$,

$$\begin{aligned}(P - P_\infty)^n =& \sum_{k=0}^{n}\binom{n}{k}(-1)^k P^{n-k} P_\infty^k \\ =& P^n + \sum_{k=1}^{n}\binom{n}{k}(-1)^k P_\infty \\ =& P^n + P_\infty\left((1-1)^n - 1\right) = P^n - P_\infty.\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty}(P - P_\infty)^n = I + \sum_{n=1}^{\infty}(P - P_\infty)^n = I + \sum_{n=1}^{\infty}(P^n - P_\infty).$$

Now let $A = P - P_\infty$. Then,

$$(I - A)(I + A + \cdots + A^{n-1}) = I - A^n = I + P^n - P_\infty.$$

Letting $n \to \infty$ on each side, we have

$$(I - A)\left(\sum_{n=0}^{\infty} A^n\right) = I,$$

which shows that $Z$ is well defined as $I - (P - P_\infty)$ is invertible, and its inverse is precisely as desired. □

We are now ready to prove our main theorems relating the fundamental matrix to hitting times. In the following we will let $J = ee^T$ and $D$ be a diagonal matrix with $d_{ii} = \pi_i$. The following table summarizes our matrix notation up to this point.

| Symbol | Name | Definition |
|---|---|---|
| $P$ | Transition Matrix | $p_{ij}$ |
| $P_\infty$ | Stationary Matrix | $e\pi^T$ |
| $Z$ | Fundamental Matrix | $(I - (P - P_\infty))^{-1}$ |
| $H$ | Hitting Time Matrix | $h_{ij} = E_i T_j$ |
| $R$ | Return Time Matrix | $r_{ij} = E_i \tau_j$ |
| $J$ | Ones Matrix | $ee^T$ |
| $D$ | Diagonal Matrix | $d_{ij} = \pi_i \delta_{ij}$ |

**Theorem 3** (from [7]). *The fundamental matrix can be written as*

$$Z = (J + P_\infty H - H)D.$$

*Proof.* From first step analysis, we obtain

$$r_{ij} = 1 + \sum_{k \neq j} p_{ik} r_{kj} = 1 + \sum_k p_{ik} h_{kj},$$

as we take one step and then must return to $j$ from there. Therefore, as $H$ agrees with $R$ except on the diagonal, we have

$$h_{ij} = r_{ij} - \delta_{ij} r_{ii} = 1 + \sum_k p_{ik} h_{kj} - \frac{\delta_{ij}}{\pi_i}.$$

Rewriting this in matrix notation we obtain, successively

$$\begin{aligned} H =& J + PH - D^{-1} \\ (I-P)H =& J - D^{-1} \\ Z(I-P)H =& Z(J - D^{-1}). \end{aligned} \tag{1}$$

Using Lemma 2 we have,

$$\begin{aligned} Z(I-P) =& I + \sum_{n=1}^{\infty} (P^n - P_\infty) - \left( P + \sum_{n=2}^{\infty} (P^n - P_\infty) \right) \\ =& I + P - P_\infty - P = I - P_\infty, \end{aligned}$$

and,

$$ZJ = Zee^T = \left( I + \sum_{n=1}^{\infty} (P^n - P_\infty) \right) ee^T = \left( e + \sum_{n=1}^{\infty} (e - e) \right) e^T = ee^T.$$

Simplifying (1) with these facts we obtain our result,

$$\begin{aligned} (I - P_\infty)H = J - ZD^{-1} \\ Z = (J + P_\infty H - H)D. \end{aligned}$$

□

In particular, this theorem gives us that

$$z_{ij} = \pi_j \left( 1 + \sum_{k=1}^{n} \pi_k h_{kj} - h_{ij} \right) = \pi_j (1 + E_\pi T_j - E_i T_j). \tag{2}$$

Using this, it is simple to derive a direct relationship between $Z$ and $H$ which will lay the foundation for the results in Section 3.

**Theorem 4** (from [7])**.** *The expected hitting time matrix $H$ has entries given by*

$$h_{ij} = E_i T_j = \frac{z_{jj} - z_{ij}}{\pi_j}.$$

*Proof.* By (2) we have

$$\begin{aligned} z_{ij} =& \pi_j (1 + E_\pi T_j - E_i T_j) \\ z_{jj} =& \pi_j (1 + E_\pi T_j). \end{aligned}$$

Subtracting the first line from the second proves the theorem. □

## 2. The Eigensystem of the Fundamental Matrix

We will now investigate the eigensystems of the matrices discussed in Section 1 to further expand on the hitting time formula given by Theorem 4 (as outlined in [1] and [4]). Given any aperiodic transition matrix $P$, conjugating $P$ by $D^{1/2}$ transforms $P$ into a symmetric matrix. This is because

$$(P^*)_{ij} = (D^{1/2} P D^{-1/2})_{ij} = \sqrt{\frac{d(i)}{d(j)}} p_{ij} = \sqrt{\frac{d(i)}{d(j)}} \frac{1}{d(i)} = \frac{1}{\sqrt{d(i)d(j)}}$$

is a symmetric function of $i$ and $j$. As this new matrix $P^*$ is symmetric, it is diagonalizable in the form

$$P^* = W \Lambda W^T,$$

where $\Lambda$ is the diagonal matrix consisting of the eigenvalues of $P^*$, $\lambda_1, \ldots, \lambda_n$, and $W$ is the associated matrix of orthogonal eigenvectors (columns $w_1, \ldots, w_n$). Substitution shows that $\lambda_1 = 1$ is an eigenvalue of $P^*$ with associated eigenvector $w_1 = (\sqrt{\pi(1)}, \ldots, \sqrt{\pi(n)})$. Now let,

$$u_i = D^{1/2} w_i \quad \text{and} \quad v_i = D^{-1/2} w_i. \tag{3}$$

As $P$ and $P^*$ are similar, they have the same eigenvalues, and the calculation

$$P v_i = P D^{-1/2} w_i = \lambda_i D^{-1/2} w_i = \lambda_i v_i$$

shows that $v_i$ is a right eigenvector of $P$ corresponding to $\lambda_i$. A similar calculation shows that the $u_i$'s are the left eigenvectors of $P$. Therefore, $P = V \Lambda U^T$, or in particular, as (3) implies that $u = Dv$,

$$P = V \Lambda V^T D. \tag{4}$$

A nice check shows that substituting $w_1 = (\sqrt{\pi(1)}, \ldots, \sqrt{\pi(n)})$ into (3) gives us corresponding to $\lambda_1 = 1$ the right eigenvector $v_1 = e$ (which is true as $P$ is stochastic) and the left eigenvector $u_1 = \pi$ (the definition of stationary measure). Moreover, as $P$ is aperiodic, we can conclude by the Perron-Frobenius Theorem that $|\lambda_i| < 1$ for all $i \neq 1$ (see [1]). Therefore, the diagonalization of $P$ expressed in (4) can be rewritten as

$$\begin{aligned} P &= e\pi^T + \sum_{j=2}^{n} \lambda_j v_j v_j^T D \\ &= P_\infty + \sum_{j=2}^{n} \lambda_j v_j v_j^T D. \end{aligned}$$

Therefore as $P^k = V \Lambda^k V^T D$,

$$P^k = P_\infty + \sum_{j=2}^{n} \lambda_j^k v_j v_j^T D, \tag{5}$$

and using the fact that $|\lambda_i| < 1$ we obtain the Markov chain convergence theorem for aperiodic random walks on graphs,

$$\lim_{k \to \infty} P^k = P_\infty.$$

*Remark* 5. The representation (5) gives an intuitive approach to the mixing rate, or the rate of convergence of $P^k$ to the stationary distribution. Let $\lambda^* = \max_{2\le i\le n} |\lambda_i|$. Then the rate of convergence of the second term in (5) to 0 only depends on the magnitude of $\lambda^*$.

Now we are ready to relate our diagonalization of $P$ to the diagonalization of the fundamental matrix.

**Theorem 6** (from [7]). *Let $P$ be the transition matrix for an aperiodic random walk, expressed as in (4). Then the fundamental matrix $Z$ can be diagonalized in the following form,*

$$Z = V\tilde{\Lambda}V^T D$$

*where $\tilde{\lambda}_1 = 1$, and for $i > 1$, $\tilde{\lambda}_i = (1-\lambda_i)^{-1}$.*

*Proof.* As $W^TW = I$,

$$I = W^TW = U^TD^{-1/2}D^{1/2}V = U^TV = V^TDV.$$

Using this and $Z^{-1} = I - (P - P_\infty)$, we obtain

$$\begin{aligned} V^TDZ^{-1}V =& V^TDV - V^TDPV + V^TDP_\infty V \\ =& I - \Lambda + V^TDP_\infty V, \end{aligned}$$

where

$$\begin{aligned} V^TDP_\infty V =& (\pi, \dots, v_n^TD)^T(\pi, \dots, \pi)^T(e, \dots, v_n) \\ =& (\pi, \dots, v_n^TD)^T(e, 0, \dots, 0) = \operatorname{diag}(1, 0, \dots, 0), \end{aligned}$$

where $\operatorname{diag}(1, 0, \dots, 0)$ is the diagonal matrix with the given entries on the diagonal. Therefore we have

$$\begin{aligned} V^TDZ^{-1}V =& I - \Lambda + \operatorname{diag}(1, 0, \dots, 0) \\ =& \operatorname{diag}(1, 1-\lambda_2, \dots, 1-\lambda_n), \end{aligned}$$

or equivalently,

$$\begin{aligned} Z =& \left((V^TD)^{-1}\operatorname{diag}(1, 1-\lambda_2, \dots, 1-\lambda_n)V^{-1}\right)^{-1} \\ =& V\operatorname{diag}(1, (1-\lambda_2)^{-1}, \dots, (1-\lambda_n)^{-1})V^TD = V\tilde{\Lambda}V^TD. \end{aligned}$$

□

Using this diagonalization of the fundamental matrix and Theorem 4, we can obtain an eigenvalue formula for the expected hitting times.

**Corollary 7.**

$$h_{ij} = E_iT_j = \sum_{k=2}^{n} \tilde{\lambda}_k(v_{jk}^2 - v_{ik}v_{jk}) \tag{6}$$

*Proof.* From theorem 4, we know that $h_{ij} = (z_{jj} - z_{ij})/\pi_j$. Using the previous result, we can calculate that

$$\begin{aligned} z_{jj} &= \pi_j\left(\sum_{k=1}^{n} \tilde{\lambda}_k v_{jk}^2\right), \\ z_{ij} &= \pi_j\left(\sum_{k=1}^{n} \tilde{\lambda}_k v_{ik}v_{jk}\right). \end{aligned}$$

The desired result is then obtained by noticing that as $v_1 = e$ and $v_{k1} = 1$ for all $k$,

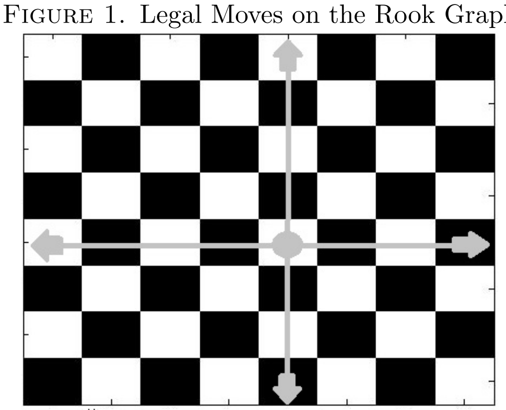
$$\tilde{\lambda}_1 \left(v_{j1}^2 - v_{i1}v_{j1}\right) = 1(1-1) = 0,$$

so this first term does not need to be added to the sum in (6). □

## 3. Application to the Rook Graph

We will now investigate the hitting times of a random walk on a rook graph using only the methods discussed in the previous sections. The rook graph represents all of the legal moves of a rook on a chessboard, which is any distance in a horizontal or vertical direction (see Figure 1). More specifically, consider an $n \times n$ chessboard graph consisting of the $n^2$ nodes, $\{(i,j)|1 \le i,j \le n\}$. Then edges connect any two nodes that agree in one coordinate, and hence the graph is regular of degree $2n-2$.

Figure 1. Legal Moves on the Rook Graph



The transition matrix for the random walk on the rook graph is given by

$$P_{(i,j)(k,l)} = \begin{cases} \frac{1}{2n-2} & \text{if } i=k \text{ or } j=l \\ 0 & \text{else }. \end{cases}$$

While this notation is useful conceptually, we will give the nodes a natural ordering for our calculations. $(1,1)$ will be the first node, and we will continue ordering to the right until reaching the edge $(1,n)$. Then $(1,2)$ will be node $(n+1)$, and then continue in this fashion. Now the transition matrix $R_n$ can be written as the $n^2 \times n^2$ block matrix

$$R_n = \frac{1}{2n-2}\begin{pmatrix} A_n & I_n & \dots & I_n \\ I_n & A_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & I_n \\ I_n & \dots & I_n & A_n \end{pmatrix},$$

where $I_n$ is the $n \times n$ identity matrix and $A_n$ is the $n \times n$ matrix given by

$$A_n = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}.$$

We will now find the eigenvalues of $R_n$. First, note that $R_n$ can be written as

$$R_n = \frac{1}{2n-2}[(A_n \otimes I_n) + (I_n \otimes A_n)], \tag{7}$$

where $\otimes$ is the Kronecker product of two matrices. We will use this form to easily find the eigenvalues and eigenvectors of $R_n$.

**Lemma 8** (from [3])**.** *Let $A, B \in \mathbb{R}^{n\times n}$ have full sets of eigenvalues $\lambda_i, \mu_i$, respectively. Then $(I_n \otimes A) + (B \otimes I_n)$ has $n^2$ eigenvalues given by*

$$\lambda_1 + \mu_1, \dots, \lambda_1 + \mu_n, \lambda_2 + \mu_1, \dots, \lambda_2 + \mu_n, \dots, \lambda_n + \mu_n.$$

*Additionally, if $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are linearly independent sets of eigenvectors for the $\lambda_i, \mu_i$ respectively, then $y_j \otimes x_i$ are linearly independent eigenvectors for $\lambda_i + \mu_j$.*

*Proof.* By the bilinearity, associativity, and the mixed product property of the Kronecker product,

$$\begin{aligned} [(I_n \otimes A) + (B \otimes I_n)](y_j \otimes x_i) =& (y_j \otimes Ax_i) + (By_j \otimes x_i) \\ =& (y_j \otimes \lambda_i x_i) + (\mu_j y_j \otimes x_i) \\ =& (\lambda_i + \mu_j)(y_j \otimes x_i). \end{aligned}$$

□

Therefore using (7) and Lemma 8, we can obtain the eigenvalues of $R_n$ directly from the eigenvalues of $A_n$.

**Lemma 9.** *$A_n$ has eigenvalues $\lambda_1 = n - 1$ and for $2 \le i \le n$, $\lambda_i = -1$. The corresponding eigenvector for $\lambda_1$ is $x_1 = e$, and for $2 \le i \le n$, we have the linearly independent set of eigenvectors*

$$X = \{x_2, \dots, x_n\} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

*Proof.* $A_n = ee^T - I_n$. Therefore,

$$A_n e = ee^T e - I_n e = ne - e = (n-1)e.$$

For each $x \in X$, $e^T x = 0$. Therefore $A_n x = -x$, completing the proof. □

As a direct application of the two previous lemmas, we obtain the following result.

**Theorem 10.** *$R_n$ has the following eigenvalues:*

*(1)* $\lambda_1 = 1$ *with eigenvector* $x_1 \otimes x_1 = e_{n^2}$ *(multiplicity 1)*

*(2)* $\lambda_2 = (n-2)/(2n-2)$ *with eigenvectors* $\{x_1 \otimes x_i,\, x_i \otimes x_1 | 2 \le i \le n\}$ *(multiplicity* $2(n-1)$*)*

*(3)* $\lambda_3 = -1/(n-1)$ *with eigenvectors* $\{x_i \otimes x_j | 2 \le i, j \le n\}$ *(multiplicity* $(n-1)^2$*)*

Note that to ensure that $|\lambda_i| < 1$ for $i > 1$, we will assume that $n > 2$.

We have almost diagonalized $R_n$ into the form $R_n = V\Lambda V^{-1}$, where $V$ is the eigenvector matrix with columns given by

$$V = (x_1 \otimes x_1, x_1 \otimes x_2, \ldots, x_1 \otimes x_n, x_2 \otimes x_1, \ldots, x_n \otimes x_n),$$

and $\Lambda$ is the diagonal matrix of eigenvalues. Thus we are only left with calculating $V^{-1}$. Let $y_1 = (1/n, \ldots, 1/n)$ and for $2 \le j \le n$,

$$y_j = \Bigg( \underbrace{\frac{n-j+1}{n}, \ldots \frac{n-j+1}{n}}_{j-1 \text{ terms}}, \underbrace{\frac{1-j}{n}, \ldots, \frac{1-j}{n}}_{n-j+1 \text{ terms}} \Bigg).$$

Then a direct calculation shows that

$$V^{-1} = (y_1 \otimes y_1, y_1 \otimes y_2, \ldots, y_1 \otimes y_n, y_2 \otimes y_1, \ldots, y_n \otimes y_n)^T.$$

Our desired result is to derive a hitting time formula for the rook random walk using (6). However, while our choice of eigenvectors is natural, we can not use (6) directly as our eigenvectors are not orthogonal, hence $V^{-1} \neq DV^T$. Tweaking the equation, we still obtain for the rook graph

$$h_{ij} = E_i T_j = \frac{1}{\pi_j} \sum_{k=2}^{n^2} \tilde{\lambda}_k v_{kj}^{-1} (v_{jk} - v_{ik}), \tag{8}$$

where

$$\tilde{\lambda}_k = 1,\ \frac{2(n-1)}{n},\ \text{or}\ \frac{(n-1)}{n},\ \text{and}\ \pi_j = \frac{1}{n^2}.$$

By the symmetries of the rook graph, we only need to consider two cases. These are if the distance from node $i$ to node $j$ is 1, and if the distance from node $i$ to node $j$ is 2. Hence, we will only calculate $h_{1,2}$ and $h_{1,(n+2)}$.

Using (8) to calculate $h_{1,2}$, any eigenvectors of $R_n$ such that their first two entries agree will have no contribution. For any $n$, there are only four eigenvectors that do not have this property. This is because for $j \ge 3$, $x_j$ has a 0 as its first entry, hence for any $i$, at least the first $n$ entries of $x_j \otimes x_i$ are 0. Likewise, for $j \ge 4$, $x_j$ has a 0 as its first two entries, thus for any $i$, $x_i \otimes x_j$ has 0 for its first two entries. Lastly, as $x_1 = e$, for any $i$, $x_i \otimes x_1$ agrees in its first two entries. Therefore, we are left with only four eigenvectors, $x_1 \otimes x_2$, $x_1 \otimes x_3$, $x_2 \otimes x_2$, and $x_2 \otimes x_3$. The same reasoning applied to $h_{1,(n+2)}$ shows that we only need to consider the seven eigenvectors $x_1 \otimes x_2$, $x_1 \otimes x_3$, $x_2 \otimes x_1$, $x_2 \otimes x_3$, $x_3 \otimes x_1$, $x_3 \otimes x_2$, and $x_3 \otimes x_3$. These symmetries and the chosen diagonalization of $R_n$ allow us to reduce a calculation involving $n^2$ terms to a manageable calculation independent of $n$ to find a hitting time formula. We now arrive at our hitting time formulas for the rook graph.

**Theorem 11.** *Let $i, j$ be two nodes on the rook graph $R_n$, with $n > 2$.*
*(a) If $d(i,j) = 1$, then $h_{ij} = n^2 - 1$.*

*(b) If $d(i,j) = 2$, then $h_{ij} = n^2 + n - 2$.*

*Proof.* The proof of (a) will be given, and (b) is an identical calculation. By the discussion above, we only need to calculate $h_{1,2}$ and can reduce (8) to a sum of only four terms. Therefore,

$$\begin{aligned}
h_{1,2} =& n^2 \Bigg( \frac{2(n-1)}{n} (y_1 \otimes y_2)(2) \Big[ (x_1 \otimes x_2)(2) - (x_1 \otimes x_2)(1) \Big] \\
&+ \cdots + \frac{n-1}{n} (y_2 \otimes y_3)(2) \Big[ (x_2 \otimes x_3)(2) - (x_2 \otimes x_3)(1) \Big] \Bigg) \\
=& n^2 \left( \frac{2(n-1)}{n} \left( \frac{-1}{n^2} \right) (-2) + \frac{2(n-1)}{n} \left( \frac{n-2}{n^2} \right) (1) \right. \\
&\left. + \frac{n-1}{n} \left( \frac{1-n}{n^2} \right) (-2) + \frac{n-1}{n} \left( \frac{(n-1)(n-2)}{n^2} \right) (1) \right) \\
=& n^2 - 1
\end{aligned}$$

□

3.1. **Generalized Biased Random Walk on Rook Graph.** We will now generalize the random walk on the rook graph to allow a bias towards visiting either horizontal or vertical neighbors. We introduce a parameter $\alpha > 0$, such that the random walk is $\alpha$ times more likely to visit a horizontal neighbor than a vertical one. The transition matrix for this biased random walk is given by

$$\hat{P}_{(i,j)(k,l)} = \begin{cases} \frac{\alpha}{(n-1)(\alpha+1)} & \text{if } i = k \\ \frac{1}{(n-1)(\alpha+1)} & \text{if } j = l \\ 0 & \text{else.} \end{cases}$$

Therefore, using the same notation as earlier, we have

$$\hat{R}_n = \frac{1}{(n-1)(\alpha+1)} [(A_n \otimes I_n) + (I_n \otimes \alpha A_n)].$$

Applying Lemmas 8 and 9 to our expression for $\hat{R}_n$, we see that $\hat{R}_n$ has the following eigendecomposition.

**Theorem 12.** *$\hat{R_n}$ has the same eigenvectors as $R_n$ with the following eigenvalues:*

*(1) $\hat{\lambda}_1 = 1$ with eigenvector $x_1 \otimes x_1 = e_{n^2}$ (multiplicity 1)*

*(2) $\hat{\lambda}_2 = (n-1-\alpha)/[(n-1)(\alpha+1)]$ with eigenvectors $\{x_1 \otimes x_i | 2 \leq i \leq n\}$ (multiplicity $(n-1)$)*

*(3) $\hat{\lambda}_3 = (\alpha(n-1)-1)/[(n-1)(\alpha+1)]$ with eigenvectors $\{x_i \otimes x_1 | 2 \leq i \leq n\}$ (multiplicity $(n-1)$)*

*(4) $\hat{\lambda}_4 = -1/(n-1)$ with eigenvectors $\{x_i \otimes x_j | 2 \leq i, j \leq n\}$ (multiplicity $(n-1)^2$)*

*Moreover, the related fundamental matrix eigenvalues are given by*

$$\tilde{\lambda}_1 = 1, \quad \tilde{\lambda}_2 = \frac{(n-1)(\alpha+1)}{n\alpha}, \quad \tilde{\lambda}_3 = \frac{(n-1)(\alpha+1)}{n}, \ \textit{and}\ \tilde{\lambda}_4 = \frac{n-1}{n}.$$

As $\hat{R}_n$ has the same eigenvectors as $R_n$,

$$\hat{R}_n = V\hat{\Lambda}V^{-1}$$

where $\hat{\Lambda}$ is the diagonal matrix of eigenvalues of $\hat{R}_n$. Moreover, as $\hat{R}_n$ is a doubly stochastic matrix, its stationary distribution is still given by $\pi = (1/n^2, \ldots, 1/n^2)$. Therefore we can apply (8) directly to calculate the hitting times for $\hat{R}_n$. For this biased random walk, we need to consider three cases: $d(i,j) = 1$ with $i$ and $j$ in the same row, $d(i,j) = 1$ with $i$ and $j$ in the same column, and $d(i,j) = 2$. Hence, we only need to calculate $h_{1,2}$, $h_{1,n+1}$, and $h_{1,n+2}$. Calculating $h_{1,2}$ and $h_{1,n+2}$ is identical to our calculation before, except substituting in the new eigenvalues $\tilde{\lambda}$ given in Theorem 12.

Similar reasoning shows that we can reduce the calculation of $h_{1,n+1}$ by considering only the four eigenvectors $x_2 \otimes x_1$, $x_2 \otimes x_2$, $x_3 \otimes x_1$, and $x_3 \otimes x_2$. These calculations yield the following result.

**Theorem 13.** *Let $i,j$ be two nodes on the rook graph $\hat{R}_n$, with $n > 2$.*
*(a) If $d(i,j) = 1$ with $i$ and $j$ in the same row, then $h_{ij} = \frac{1}{\alpha}[\alpha n^2 + (1-\alpha)n - 1]$.*

*(b) If $d(i,j) = 1$ with $i$ and $j$ in the same row, then $h_{ij} = n^2 + (1-\alpha)n - \alpha$.*

*(c) If $d(i,j) = 2$, then $h_{ij} = \frac{1}{\alpha}[\alpha n^2 + (\alpha^2 - \alpha + 1)n - \alpha^2 - 1]$.*

A nice check by setting $\alpha = 1$ shows that these results coincide with the hitting time results for $R_n$.

As mentioned in Remark 5, the mixing rate is the rate of convergence of powers of the transition matrix $P^k$ to the stationary matrix $P_\infty$. This rate of convergence only depends on the magnitude of $\lambda^* = \max_{2\le i\le n} |\hat{\lambda}_i|$. For large enough $n$, as $\hat{\lambda}_4 \sim O(1/n)$ and $\hat{\lambda}_2, \hat{\lambda}_3 \sim O(1)$, the mixing rate will be determined by $\max\{\hat{\lambda}_2, \hat{\lambda}_3\}$. To study the mixing rate of $\hat{R}_n$, we will fix $n$ and set the eigenvalues as functions of $\alpha$, that is

$$\hat{\lambda}_2(\alpha) = \frac{n-1-\alpha}{(n-1)(\alpha+1)} \qquad \text{and} \qquad \hat{\lambda}_3(\alpha) = \frac{\alpha(n-1)-1}{(n-1)(\alpha+1)}.$$

Calculating the derivatives of these functions, we see that

$$\begin{aligned} \hat{\lambda}_2'(\alpha) =& \frac{-(n-1)-(n-1)^2}{[(n-1)(\alpha+1)]^2} < 0 \\ \text{and } \hat{\lambda}_3'(\alpha) =& \frac{(n-1)+(n-1)^2}{[(n-1)(\alpha+1)]^2} > 0. \end{aligned}$$

Therefore, $\hat{\lambda}_2$ is a decreasing function of $\alpha$, and $\hat{\lambda}_3$ is an increasing function of $\alpha$, and as $\hat{\lambda}_2(1) = \hat{\lambda}_3(1)$,

$$\inf_\alpha \lambda^*(\alpha) = \lambda^*(1) = \hat{\lambda}_2(1) = \hat{\lambda}_3(1).$$

Thus, perhaps as intuitively expected, the rook random walk converges to the stationary distribution quickest when there is no bias present.

## 4. Cover Times

An interesting question related to hitting times is the expected time to visit every node in a graph, known as the cover time. We will now discuss a theorem (from [5] and proof ideas from [6]) that gives upper and lower bounds for the time taken for a Markov chain to visit a fixed collection of states $A$, referred to as the cover time $C$. Given $A = \{a_0, a_1, \dots, a_n\}$, the bounds will be given in terms of the extremal hitting times $h_*(A), h^*(A)$ defined by:

$$h_*(A) = \min_{i,j,i \neq j} E_i(T_j) \qquad\qquad h^*(A) = \max_{i,j,i \neq j} E_i(T_j)$$

Given a permutation $\beta = (\beta(1), \dots, \beta(n)) \in S_n$, define $\tau_\beta(i)$ to be the first time when $\{a_{\beta(1)}, \dots, a_{\beta(i)}\}$ have all been visited. For simplicity, as $\beta$ will be fixed, $\tau_\beta(i)$ will be denoted more concisely as $\tau(i)$. This defines $\tau(i)$ to be a stopping time with respect to the filtration $\{\mathcal{F}_k = \sigma(X_0, \dots, X_k), k \geq 0\}$.

We now define $R_i = \tau(i) - \tau(i-1)$ to be the time to visit $a_{\beta(i)}$ from $X_{\tau(i-1)}$. If $X_k = a_{\beta(i)}$ for some $k < \tau(i-1)$, then $R_i = 0$. Thus the set $\{R_i \neq 0\} \in \mathcal{F}_{i-1}$, and we have the following lemma.

**Lemma 14** (from [5])**.** *$P(R_i \neq 0) = 1/i$.*

*Proof.* Let $(b_1, \dots, b_n)$ be the ordering in which the $a_i$'s are visited, $b_i \neq b_j$. That is, for all $i < j$ there exists $k$ such that $X_k = a_{b_i}$ and for all $m$ such that $X_m = a_{b_j}$, $k < m$. This ordering is only dependent on the Markov chain $X_n$, namely being independent of $\beta$. The probability that $R_i \neq 0$ can be rewritten as the probability that the coordinate where $\beta(i)$ appears in $(b_1, \dots, b_n)$ is further to the right then the coordinate where $\beta(i)$ appears in $(\beta(1), \dots, \beta(i-1))$. As $\beta$ is a random permutation, this probability is $1/i$. □

We are now ready to prove bounds on the cover time depending on the extreme hitting times.

**Theorem 15** (from [5])**.** *Let $(X_n, n \geq 0)$ be a Markov chain with finite state space. Let $C$ be the time taken for the Markov chain to visit a fixed collection of states $A = \{a_0, a_1, \dots, a_n\}$. Let $X_0 = a_0$. Then for $H_n$, the nth harmonic number,*

$$h_*(A) H_n \leq EC \leq h^*(A) H_n$$

*Proof.* Let $\beta = (\beta(1), \dots, \beta(n))$ be an element of the permutation group $S_n$. Then,

$$EC = E[\tau(n) - \tau(0)] = \sum_{i=1}^{n} ER_i = \sum_{i=1}^{n} E[E(R_i | \mathcal{F}_{i-1})],$$

where $E(R_i|\mathcal{F}_{i-1}) = \mathbb{1}(R_i \neq 0) E_{X_{\tau(i-1)}} T_{a_{\beta(i)}}$ by the strong Markov property.

But $h_*(A) \leq E_{X_{\tau(i-1)}} T_{a_{\beta(i)}} \leq h^*(A)$ by definition, giving us

$$\mathbb{1}(R_i \neq 0) h_*(A) \leq E(R_i|\mathcal{F}_{i-1}) \leq \mathbb{1}(R_i \neq 0) h^*(A)$$

Taking expectations and then summing from 1 to $n$ gives us our result,

$$h_*(A) H_n \leq EC \leq h^*(A) H_n.$$

□

As we know an exact formula for the hitting times of a random walk on a rook graph, we can neatly apply this theorem to our results from Section 3. Because we know both the minimum and maximum hitting time,

$$(n^2 - 1)H_n \leq EC \leq (n^2 + n - 2)H_n,$$

or generally that the cover time of $R_n$ is of order $n^2 \log n$. Similarly, the cover time of the biased rook graph $\hat{R}_n$ is also of order $n^2 \log n$.

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