Seminarthemen

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1 The geometry of Markov chains

In class, we've already seen relationships between P, Z and the hitting time matrix H and we've established that the commute times $C_{ij} = \frac{1}{2}(H_{ij} + H_{ji})$ are one way of forming a geometric embedding of the Markov chain.

topic 1a: potential theory and absorption probabilities The central objects of potential theory are harmonic functions. h is called harmonic on $\mathcal{X}_0 \subset \mathcal{X}$ if and only if $h = Ph$ holds on $\mathcal{X}_0$. Harmonic functions are the discrete analogon of solutions to the Laplace equation

$$\begin{aligned} \Delta h &= 0 \quad \text{in } \mathcal{X}_0 \\ h &= g \quad \text{on } \partial\mathcal{X} = \mathcal{X} \setminus \mathcal{X}_0. \end{aligned}$$

There is a close connection between these objects and absorption probabilities, which should be explored. Literature: [Woess Ch. 6].

topic 1b: Markov chains and electrical networks Reversible Markov chains have an electrical network analogy that allows thinking in terms of voltages, currents and resistors instead of probabilistic objects. This turns out to be a powerful conceptual and computational tool and is closely related to potential theory. One starts with Ohm's law

$$v(x) - v(y) = i(x,y)r(x,y)$$

that connects the voltage difference at x and y with the product of current $i(x,y)$ and resistance $r(x,y)$. Voltages are harmonic functions, currents are probability flows. One then has Kirchhoff' laws and network reduction techniques borrowed from electrical networks. Literature: [Lyons Ch. 2].

topic 1c: Commute time embedding of non reversible Markov chains For reversible Markov chains, we've seen in the class that there is an embedding function $\psi : \mathcal{X} \to \mathbb{R}^n$ of the Markov chain into some $\mathbb{R}^n$ such that

$$C_{ab} = \|\psi(a) - \psi(b)\|^2 \qquad \forall a, b \in \mathcal{X}.$$

But this required the spectral properties of symmetric matrices. It is possible to construct such an embedding explicitly also for non-reversible Markov chains. Literature: [Doyle, Steiner]

topic 1d: Wasserstein metric The state of a Markov chain is described by a probability distribution $\rho(x,t) = \mathbf{P}[X_t = x]$. It thus makes sense to consider distances between probability distributions in order to define a geometry. There are several notions of distance between probability distributions, one interesting one is the Wasserstein metric. The talk should look at convergence in wasserstein metric and investigate applications in image reconstruction. (this is a more challenging topic) Literature: [Gibbs 2007]

2 Probing geometry with Markov chains

Here we turn things around: Given sampled data $\{x_i\}_{i=1}^N$ from some geometrical object $\mathcal{M}$, we ask how much a Markov chain built on the data knows about $\mathcal{M}$. We will typically look for a map $\Psi : \mathcal{M} \to \mathbb{R}^n$ such that

$$\|\Psi(x_i) - \Psi(x_j)\|_2 = d(x_i, x_j)_{\mathcal{M}},$$

in other words the Euclidean distance in the embedding space provided by Ψ is equal to some meaningful intrinsic distance $d_{\mathcal{M}}$.

topic 2a: Laplacian Eigenmaps and Diffusion maps Laplacian eigenmaps and diffusion maps are built on the observation that there is a connection between the graph Laplacian $L = P - I$, where P is the transition matrix of the Markov chain that we built on the data, and the Laplace Beltrami operator $\Delta_{\mathcal{M}} = \text{div} \circ \text{grad}$. The 'diffusion map' Ψ is then defined via the spectral decomposition of L, and $\|\Psi(x_i) - \Psi(x_j)\|_2$ is equal to the so called diffusion distance. Literature: [Belkin 2009, Coifman 2006]. Applications: [Ferguson 2011, Nadler 2006].

topic 2b: Convergence of graph Laplacians in the infinite data limit This talk should investigate in which sense the graph Laplacian L, which is an $N \times N$ matrix, converges to the operator $\Delta_{\mathcal{M}}$ as the amount of data N goes to infinity. This is not trivial since the matrices in question grow as N grows. Literature: [Learning Theory p. 470 ff., Coifman 2006].

topic 2c: Topological and geometrical features encoded by the Laplacian This talk should investigate what the Laplace-Beltrami operator $\Delta_{\mathcal{M}}$, in particular its spectrum and eigenfunctions, actually knows about $\mathcal{M}$. Topological properties of $\mathcal{M}$ should be a focus, with applications in shape analysis. Literature: [McKeen '67, Reuter 2009].

topic 2d: Spectral clustering and applications This talk should talk about the theory behind spectral clustering [Literature: v. Luxburg 2007] and possible applications. One applications is image analysis [Shi and Malik 2000].