In the best case scenario, three main components are sufficient. The P correlated variables are then reduced to three uncorrelated variables that can easily be graphically represented.

2.8.3. *Interpretation of results*

The initial objective of extracting the most relevant information is generally reached. The number of variables is smaller (main components). They are uncorrelated and the individuals can easily be graphically represented without much distortion. There are two approaches: one is based on variables and the other on individuals.

2.8.3.1. *Method based on variables*

The correlation between the main components and the original variables is determined. If only the first r main components $Y_1, Y_2,\ldots,Y_r$ are kept, we obtain r.P correlation coefficients to be calculated: the correlation of Y_1 with $X_1, X_2,\ldots,X_p$, that of Y_2 with $X_1, X_2,\ldots,X_p$, ..., that of Y_p with $X_1, X_2,\ldots,X_p$. The main components are interpreted on the basis of the observed values of these coefficients.

2.8.3.2. *Method based on individuals*

The main components can be interpreted by means of the position of the individuals with respect to the main components. The individuals whose contributions related to the concerned axes are too small are considered poorly represented. The position of the individuals in the planes formed by the components can be interpreted.

2.9. Applications

2.9.1. *Rod mesh*

This example takes into account a rod structure (rods are identical to the rod dealt with in the previous example) such as the plane mesh presented in Figure 2.13. The chords are articulated to nodes and contribute to the overall stiffness only by their extension. Structural stiffness and mass matrices can be easily built using the rod finite element [RIT 08].

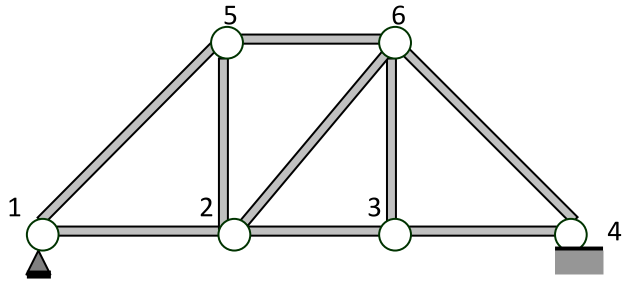

Figure 2.13. *Mesh structure*

The mass and stiffness matrices of the structure are according to Géradain [GÉR 96]:

$$[K]\frac{L}{ES}=\begin{bmatrix}
2+\frac{1}{2\sqrt{2}} & & & & & & & & \\
\frac{1}{2\sqrt{2}} & 1+\frac{1}{2\sqrt{2}} & & & & & & & \\
-1 & 0 & 2 & & & & & & \\
0 & 0 & 0 & 1 & & & sym. & & \\
0 & 0 & -1 & 0 & 1+\frac{1}{2\sqrt{2}} & & & & \\
0 & 0 & 0 & 0 & 0 & 1+\frac{1}{2\sqrt{2}} & & & \\
0 & -1 & 0 & 0 & 0 & \frac{1}{2\sqrt{2}} & 1+\frac{1}{\sqrt{2}} & & \\
\frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & 0 & \frac{-1}{2\sqrt{2}} & -1 & 0 & 1+\frac{1}{\sqrt{2}} & \\
\frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & 0 & -1 & \frac{1}{2\sqrt{2}} & 0 & 0 & 0 & 1+\frac{1}{\sqrt{2}}
\end{bmatrix}$$

$$[M]\frac{6}{ml}=\begin{bmatrix} 2(3+\sqrt{2}) & & & & & & & & \\ 0 & 2(3+\sqrt{2}) & & & & & & & \\ 1 & 0 & 6 & & & & sym. & & \\ 0 & 1 & 0 & 6 & & & & & \\ 0 & 0 & 1 & 0 & 2(1+\sqrt{2}) & & & & \\ 1 & 0 & 0 & 0 & 0 & 2(2+\sqrt{2}) & & & \\ 0 & 1 & 0 & 0 & 0 & 0 & 2(2+\sqrt{2}) & & \\ \sqrt{2} & 0 & 1 & 0 & \sqrt{2} & 1 & 0 & 4(1+\sqrt{2}) & \\ 0 & \sqrt{2} & 0 & 1 & 0 & 0 & 1 & 0 & 4(1+\sqrt{2}) \end{bmatrix}$$

The dynamic problem can be written as:

$$\left(\frac{ES}{L}(1+i\eta)[K]-w^2\frac{mL}{6}[M]\right)\{H\}=\{F\} \qquad [2.43]$$

with the Gaussian E of mean E_0 and standard deviation σ_E. E can be expressed as:

$$E=E_0+\sigma_E\zeta \qquad [2.44]$$

where ζ is a reduced centered normal variable.

Then:

$$\left(\frac{S}{L}(E_0+\sigma_E\zeta)[K](1+i\eta)-w^2\frac{mL}{6}[M]\right)\{H\}=\{F\} \qquad [2.45]$$

Equation [1.46] can be solved by expanding $\{H\}$ on a second-order chaos 2.

Figure 4.15 presents the mean value of the magnitude of the transfer function; the numerical values considered are a standard deviation of 1% and a damping of 4%. The calculations required for various methods are presented below. It can be easily noted that the time needed for the projection method is very short compared to the direct simulation method, knowing that the frequency domain of interest is divided into 401 points:

Direct Monte Carlo simulation, 2,000 drawings: 70.7400 seconds.

Projection on a second-order polynomial chaos: 0.8200 seconds.

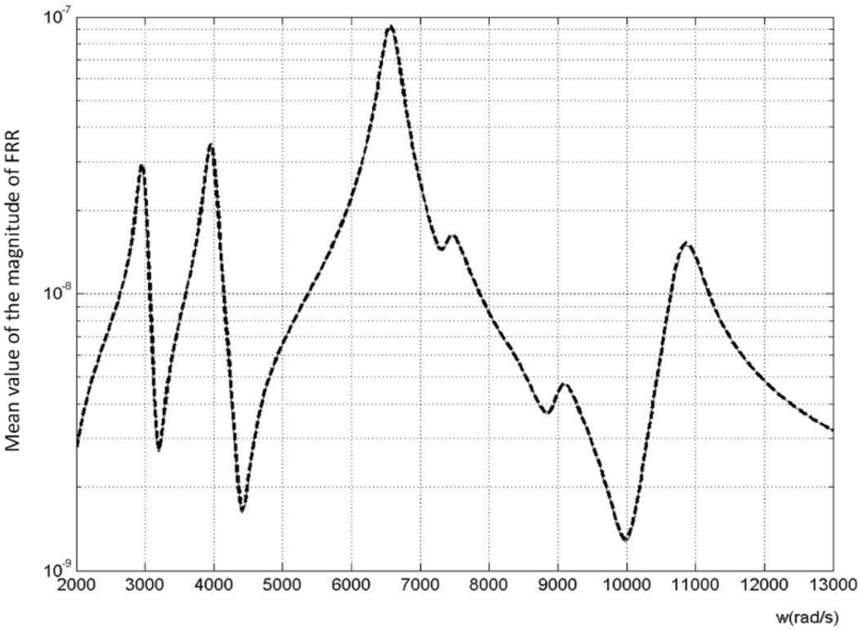

Figure 2.14. *Mean value of the module of the transfer function*

Figure 2.14 presents the mean value of the module of the transfer function located in (1,1) calculated by an expansion on a second-order polynomial chaos and a Monte Carlo simulation.

2.9.2. *Example of a linear oscillator*

Let us consider the application example of the forced response of a linear oscillator shown in Figure 2.8, which is governed by the following equation:

$$\ddot{x} + 2\xi\Omega\dot{x} + \Omega^2 x = f\sin(\omega t) \tag{2.46}$$

In this example, the equivalent viscous damping coefficient ξ and the undamped eigenfrequency Ω are considered as uncertain parameters that are described by truncated Gaussian probability densities that are statistically independent. Truncation is introduced because the values of uncertain parameters are physically bounded. It is particularly important to have positive values of Ω and ξ. For Gaussian variables, the truncation is at about three times the standard deviation about the mean value.

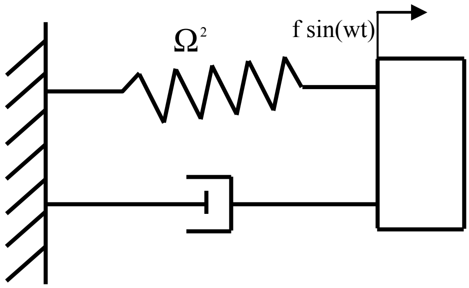

Figure 2.15. *Linear oscillator under study*

The equivalent viscous damping and the undamped eigenfrequency can be rewritten as:

$$\xi = \xi_0 + \varepsilon\xi_1 \tag{2.47}$$

$$\Omega = \Omega_0 + \varepsilon\Omega_1 \tag{2.48}$$

The parameter ε is a perturbation parameter that is assumed to be small.

The index 0 represents the mean value of the random quantity, while the index 1 represents the centered Gaussian random fluctuation.

Let us consider the variability of the module of the forced response during motion in the case of uncertain parameters.

The equivalent viscous damping coefficient ξ and the undamped eigenfrequency Ω have, respectively, mean values of 5% and 1 rad/s and standard deviations of 5% and 0.05 rad/s.

The standard deviation of the response was calculated by the Taguchi method with nine points of discretization for each random variable. The result is compared to that obtained by the MC method, using 10,000 simulations.

The result is shown in Figure 2.15. The random result shows a widening of the peak of deterministic resonance and a sensitive decrease of the resonance level. It is worth noting that the Taguchi method with nine points behaves very well compared to the Monte Carlo simulation.

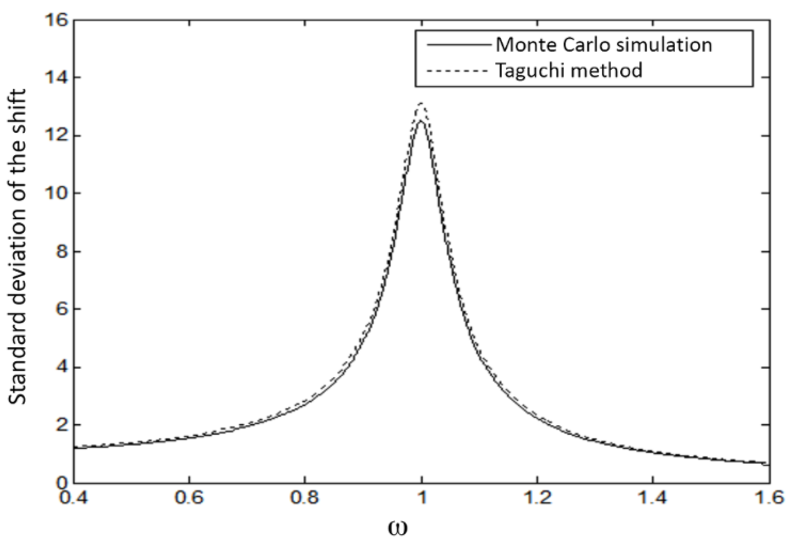

Figure 2.16. *Standard deviation of the module of the shift response*

2.10. Conclusion

This chapter presents several statistical methods for assessing the effect of uncertainties in the system response according to various approaches. If experimental data are described by probability laws, then the probabilistic approach is recommended. If the data are within an interval without any information on their variation, the interval-based algebraic approach is more convenient, but a problem of convergence may sometimes remain. When no probabilistic or statistical information are available, and there are unexpected ranges, fuzzy logic methods are more appropriate. There are several design of experiments methods that strongly reduce the dimensions of the problem being considered. The analysis of the main components can be used to determine one or two components with the strongest influence on the system with respect to given indicators.

3

Electromagnetic Waves and their Applications

3.1. Introduction

Electromagnetic waves can be used to study a physical–chemical (molecule, material, atmosphere, environment) or a technological system (nano-systems, MEMS) by analyzing their interactions. When the range of the electromagnetic waves is between UV and infrared, light–matter interaction occurs. The principle on which this is based is similar in other ranges of the electromagnetic spectrum. In general, the electromagnetic wave incident on an object is generated in a controlled state by the use of an adapted light source and filter. A detection system composed of adapted sensors and filters and embedded electronics is used for measuring the modifications undergone by the electromagnetic wave with respect to its initial state. By unraveling the physical processes by which the characteristics of the reflected, transmitted or scattered electromagnetic wave are modified with respect to the incident wave, the properties of the system can be assessed.

Depending on the device used to prepare the incident wave, and on the detection system, the volume of matter or material can be probed on the nanometer scale [DAH 16] (Figure 3.1). To assess the current state of these test techniques, a synthesis of the description of light phenomena is presented, starting with the ancient Greeks and up to modern times. In the first models of light properties, the ancient Greeks relied on the particle nature of light. For example, using geometrical optics (600 BC), Thales of Miletus measured the height of the pyramid of Cheops based on the length of

Applications and Metrology at Nanometer Scale 1:
Smart Materials, Electromagnetic Waves and Uncertainties,
First Edition. Pierre-Richard Dahoo, Philippe Pougnet and Abdelkhalak El Hami.

the shadow of a rod, with the sun as a source of light. Euclid (300 BC) and Ptolemy (280 BC), respectively, described in their books (“Optica” and “Optics”) the properties of light in the context of geometrical optics. From 1620 onwards, Snell and Descartes represented light by rays that form light beams that travel in a straight line in a homogeneous medium and follow the Snell–Descartes laws of reflection and refraction. Fermat principle (1657) on the time extremum of the optical path of a light ray led likewise to these results [BRU 65, HEC 05, MEI 15].

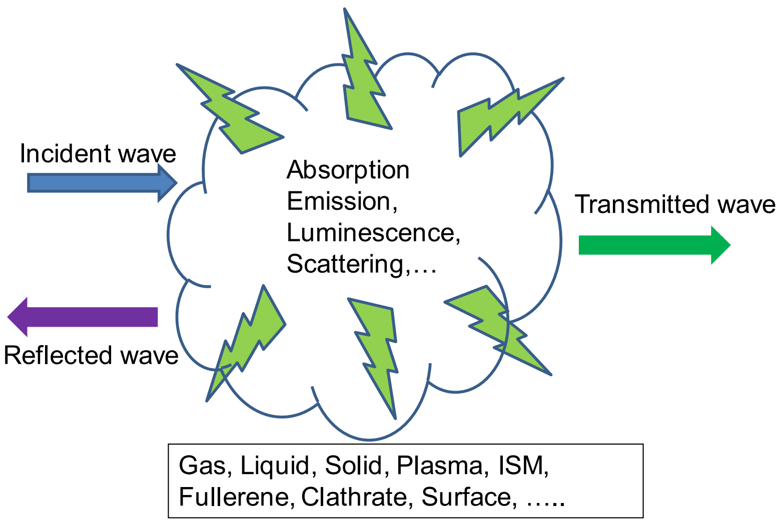

Figure 3.1. *Light–matter interaction. For a color version of this figure, see www.iste.co.uk/dahoo/metrology1.zip*

But light passing through a very small aperture undergoes diffraction. This phenomenon contradicts the theory of geometrical optics: the ray cannot be localized. In the same century, the wave-like properties were established on the basis of three postulates and the qualitative description of C. Huygens (1690). On this basis, A. Fresnel (1815) built a mathematical model for light propagation based on the existence of wave fronts. A point-source emits spherical wave fronts. Each point of the wave fronts behaves like a secondary point source emitting spherical waves in all directions. The secondary waves interfere so that the envelope of all the secondary wave fronts constitutes the new wave front. Due to this approach, light interference (T. Young experiments, 1801) and diffraction (F.M. Grimaldi experiments, 1664) could be interpreted. Furthermore, A. Fresnel assumed that the light wave is perpendicular to its direction of propagation (suggestion of A-M. Ampère, 1816). He proved that, when the incidence angle is non-zero, the coefficients of the reflection of light at the interface between two media have different expressions for an S wave (when the vibration is perpendicular on the plane of incidence) and a P wave (when light vibration is in the plane of incidence). Transverse polarization of light was thus formulated.

The approach developed by Huygens and Fresnel explains diffraction and interference phenomena. Approximately two centuries later (1865), James Clark Maxwell [MAX 54] unified the laws of electrostatics, magnetism and induction in a system of equations, simplified by Heaviside, which are known as Maxwell equations, in which electric and magnetic phenomena are considered in their field form. His equations led to postulating the existence of electromagnetic waves traveling with the speed of light ($3\ 10^8$ m/s or more exactly 2.99,729,458 x 10^8 m/s). Visible light covers only one part of the spectrum of frequencies from 500 kHz to 10^{14} GHz. The works of Maxwell proved the vector nature of light. The electric field that is perpendicular to the direction of propagation can be located by two different directions known as the polarization states of light.

The Newtonian particle theory of light does not explain interference and diffraction phenomena, but the corpuscular approach developed by Einstein (1905), according to which light is constituted of grains of energy, provides an interpretation of the photoelectric effect. Light interacts with matter in the form of energy quantum ($E = h\nu$), where ν is the frequency associated with the color of light. In the case of the photoelectric effect, induced absorption takes place. Modeling a black body as a source that radiates energy in discrete form, Planck solved the problem of the ultraviolet catastrophe of the black body emission. In 1915, Einstein explained the black body emission by introducing a symmetrical emission process in parallel with the absorption process that occurs in discrete form. Light can thus be considered as a wave or as a particle [BRO 68]. These approaches have been developed in specialized papers or books. The differences between classical sources (incoherent light) and quantum sources (coherent light) can be established from statistical theories of light. Notions of quantum mechanics are required to understand the particle nature of light in the form of photons [MES 64]

[COH 73] such as the Glauber formulation, for example. To describe the state of wave polarization in a simplified form, a gauge is used to obtain the equation of the propagation of the electromagnetic wave in the transverse representation. This gauge choice makes it possible to express the electromagnetic field as a sum of independent oscillators (using creation and annihilation operators). This naturally leads to a quantum description of light in terms of photons, grains of light proposed by Einstein to interpret the photoelectric effect. It can be shown that energy is associated with a frequency and that light has a discrete character. Finally, Glauber's approach connects the classical (continuous) and the quantum (discrete) descriptions and attributes a physical reality to coherent states with photons distributed according to a Poisson law.

Nowadays, the wave–particle duality of light can be verified experimentally. The non-locality phenomenon was evidenced in the experiments conducted by Alain Aspect on quantum entanglement [ASP 82].

This chapter is dedicated to various applications of electromagnetic waves. After having recalled the characteristics of these waves, applications are presented: the energy carried by a monochromatic plane wave, a rectangular waveguide, microwave antennas, the study of a wire antenna and antenna networks. These applications help understand the principles on which the fifth generation (5G) of mobile telecommunication systems are based.

3.2. Characteristics of the energy carried by an electromagnetic wave

Electromagnetic waves carry energy. If $\vec{E}$ is the electric field and $\overline{B}$ is the magnetic field of the electromagnetic wave, the direction of the wave is the direction of the vector $\vec{E} \times \vec{B}$.

The Poynting vector $\vec{S}$ is defined by:

$$\vec{S} = \frac{1}{0}\vec{E} \times \vec{B}$$

$\vec{S}$ is expressed in joule per second per unit surface $\left[J/\left(s.m^2\right)\right]$.

The norm of the Poynting vector represents the instantaneous power carried by the electromagnetic wave through the unit surface. The Poynting vector is perpendicular to the wave direction of propagation.

The Poynting vector can also be written in the following integral form (though denoted by the same letter, the energy E and the electric field $\vec{E}$ should not be confused):

$$P = \frac{dE}{dt} = \frac{1}{\mu_0}\oint_S \left\|\vec{E}\times\vec{B}\right\| dS = \frac{1}{\mu_0}\oint_S \left(\vec{E}\times\vec{B}\right)\circ d\vec{S} = c^2\varepsilon_0 \oint_S \left(\vec{E}\times\vec{B}\right)\circ \vec{n}dS$$

where $\vec{n}$ is the unit vector perpendicular to dS.

For a plane electromagnetic wave, the norm of the Poynting vector is:

$$S(x,t) = \frac{E(x,t)\cdot B(x,t)\cdot \sin(\pi/2)}{\mu_0} = \frac{\hat{E}\cdot\hat{B}}{\mu_0}\cdot\sin^2(kx-\omega t) = \frac{\hat{E}^2}{\mu_0\cdot c}\cdot\sin^2(kx-\omega t)$$

This quantity varies in time and space. In a given position, its average value is $\sin^2(\ )$ averaged over period T:

As

$$\frac{1}{T}\int_0^T \sin^2(\omega\cdot t)\,dt = \frac{1}{T}\left[\frac{1}{2}\int_0^T dt - \frac{1}{2}\int_0^T \cos(2\omega\cdot t)\,dt\right]$$
$$= \frac{1}{T}\left[\frac{T}{2} - \frac{1}{2}\left[\frac{1}{2\omega}\cdot\sin(2\omega\cdot t)\right]_0^T\right] = \frac{1}{2}$$

Hence:

$$\forall x \Rightarrow \overline{S} = \frac{\hat{E}\cdot\hat{B}}{2\mu_0} = \frac{\hat{E}^2}{2\mu_0 c} = \frac{c\hat{B}^2}{2\mu_0}$$

The average value of the Poynting vector of a plane electromagnetic wave is a constant that depends on neither position nor time.

The electric energy E_g of an ideal plate capacitor C (of section S, length L and electric permittivity ε) at a potential U generating plane electromagnetic waves is:

$$E_g = \frac{1}{2} C \cdot U = \frac{1}{2} \cdot \left(\frac{S \cdot \varepsilon}{L} \right) \cdot \left(\hat{E}^2 \cdot L^2 \right) = \frac{1}{2} \cdot S \cdot L \cdot \varepsilon \cdot \hat{E}^2 = \frac{1}{2} \cdot V \cdot \varepsilon \cdot \hat{E}^2$$

$$\Rightarrow \frac{E_g}{V} = \frac{1}{2} \cdot \varepsilon \cdot \hat{E}^2$$

The energy density is:

$$u_g = \frac{1}{2} \cdot \varepsilon \cdot \hat{E}^2$$

which leads us to:

$$u_g = \frac{1}{2\mu} \hat{B}^2$$

and the total energy u_{tot} carried by the electromagnetic wave in this specific case is:

$$u_{tot} = \frac{1}{2} \varepsilon \hat{E}^2 + \frac{1}{2\mu} \hat{B}^2 = \frac{1}{2} \varepsilon \hat{E}^2 + \frac{1}{2\mu c^2} \hat{E}^2 = \varepsilon \hat{E}^2 = \frac{1}{\mu} \hat{B}^2$$

The electric energy density of an electromagnetic wave is equal to its magnetic energy density.

Following this result, "the (average) intensity I of an electromagnetic wave" can be defined by the average value of its Poynting vector:

$$I = \overline{S}$$

It is therefore the average power carried by the wave on the unit surface. But given the above proven time average expression of the Poynting vector, it can be written as:

$$I = \overline{S} = \frac{\hat{E} \cdot \hat{B}}{2\mu_0}$$

Using the relation between energy and momentum:

$$p = \frac{E_{tot}}{c}$$

The momentum density of the electromagnetic wave is written as:

$$p = \frac{u_{tot}}{c} = \frac{\varepsilon \hat{E}^2}{c} = \frac{\hat{B}^2 c}{\mu}$$

As the direction of $\vec{E} \times \vec{B}$ is perpendicular to the wave front and therefore identical with the wave propagation direction, its module is:

$$\left\| \vec{E} \times \vec{B} \right\| = \hat{E} \cdot \hat{B} = \frac{1}{c} \hat{E}^2 = c\hat{B}^2$$

Therefore, the momentum density of the electromagnetic wave can be written as:

$$p = \varepsilon \left\| \vec{E} \times \vec{B} \right\| = \frac{1}{\mu c^2} \left\| \vec{E} \times \vec{B} \right\|$$

As the direction of the momentum is that of the propagation, the vector form of the relation is expressed as:

$$\vec{p} = \varepsilon \vec{E} \times \vec{B} = \frac{1}{\mu c^2} \vec{E} \times \vec{B}$$

If an electromagnetic wave has a momentum, it also has a density of the angular momentum. The angular momentum per unit volume is then:

$$\vec{b} = \vec{r} \times \vec{p} = \varepsilon \vec{r} \times \left(\vec{E} \times \vec{B} \right) = \frac{1}{\mu c^2} \vec{r} \times \left(\vec{E} \times \vec{B} \right)$$

Therefore, an electromagnetic wave carries a momentum and an angular momentum besides energy.

This is not surprising. An electromagnetic interaction between two electrical charges involves energy and momentum exchange between the charges. This takes place through the electromagnetic field that carries the exchanged density of energy and momentum.

3.3. The energy of a plane monochromatic electromagnetic wave

A plane monochromatic electromagnetic wave travels in a vacuum along direction Oz. The electric components of this wave in the directions x, y and z are expressed by:

$$E_x = E_{ox} \cos(kz - \omega t + \Phi_1)$$

$$E_y = E_{oy} \cos(kz - \omega t + \Phi_2)$$

$$E_z = 0$$

1) Determine the curve described by the tip of the electric field vector in the following cases:

– $\Phi_1 = \Phi_2$

– $E_{ox} = E_{oy}$ and $\Phi_1 - \Phi_2 = \pi/2$

2) Determine the dispersion relation.

3) Determine the relation between E and B. Write the explicit expression of $B(x, y, z, t)$.

4) Calculate the electromagnetic energy density $u(r, t)$ at any point.

5) Determine in various ways the Poynting vector $R(r, t)$, equal to the current density $j_u(r, t)$ of the electromagnetic energy:

a) using a general expression of R as a function of E and B;

b) using the relation $j_u = u\, v_u$;

c) using the conservation equation that reflects the conservation of the electromagnetic energy in a vacuum. The direction of R should first be determined by symmetry.

6) Calculate the average power through the unit surface S that is normal to the propagation direction.

3.3.1. *Answer to question 1*

3.3.1.1. *First case*

Since $\Phi_1 = \Phi_2$

$$\frac{E_x}{E_{0x}} = \frac{E_y}{E_{0y}}$$

For example, in the plane z=0, this means that: $\forall t, \tan\ \theta = \dfrac{E_{0y}.\cos\ (\omega t)}{E_{0x}.\cos\ (\omega t)} = \dfrac{E_{0y}}{E_{0x}} = \dfrac{E_y}{E_x}$, hence $\tan\ \theta$ is constant.

As angle θ is constant, the tip of the electric field vector describes a straight line, which corresponds to a state of rectilinear polarization. The same is applicable to any plane z = constant.

3.3.1.2. *Second case*

Since E_{ox}, = E_{oy} and Φ_1 - $\Phi_2 = \pi/2$, the components of the electric field which are expressed by:

$$\begin{pmatrix} E_x = E_{0x} \cos\left(kz - \omega t + \Phi_1\right) \\ E_y = E_{0x} \cos\left(kz - \omega t + \Phi_1 - \dfrac{\pi}{2}\right) = E_{0x} \sin\left(kz - \omega t + \Phi_1\right) \\ E_z = 0 \end{pmatrix}$$

verify the following equation:

$\left(\dfrac{E_x}{E_{0x}}\right)^2 + \left(\dfrac{E_y}{E_{0x}}\right)^2 = 1$, which is the equation of a circle of radius E_{0x}.

In this case, the tip of the electric field vector describes a circle; the wave is circularly polarized.

3.3.2. *Answer to question 2*

The dispersion equation is established using the wave equation that links the second-order time derivative of the wave to its second-order space derivative:

$$k^2 = -\mu_0 \times \varepsilon_0 \times \omega^2 = 0$$

$$k^2 = \mu_0 \times \varepsilon_0 \times \omega^2$$

3.3.3. *Answer to question 3*

Using the relation: $\overrightarrow{rot}\ \vec{E} = \dfrac{-\partial \vec{B}}{\partial t}$ in the case of a plane wave, the nabla operators and the partial derivative with respect to time can be replaced by multiplication by *j k* and *-j ω*.

Considering the case $\Phi_1 = \Phi_2 = 0$

$$\begin{pmatrix} E_x = E_{0x}.\cos\ (kz - \omega t + \Phi_1) \\ E_y = E_{0y}.\cos\ (kz - \omega t + \Phi_2) \\ E_z = 0 \end{pmatrix} = \begin{pmatrix} E_x = E_{0x}.\cos\ (kz - \omega t) \\ E_y = E_{0y}.\cos\ (kz - \omega t) \\ E_z = 0 \end{pmatrix}$$

$$j\vec{k} \wedge \begin{pmatrix} E_x = E_{0x}.\exp(j(kz - \omega t)) \\ E_y = E_{0y}.\exp(j(kz - \omega t)) \\ E_z = 0 \end{pmatrix} = -j\omega \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

$$j\begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix} \wedge \begin{pmatrix} E_x = E_{0x}.\exp(j(kz - \omega t)) \\ E_y = E_{0y}.\exp(j(kz - \omega t)) \\ E_z = 0 \end{pmatrix} = \begin{pmatrix} -jkE_{0y}.\exp(j(kz - \omega t)) \\ +jkE_{0x}.\exp(j(kz - \omega t)) \\ 0 \end{pmatrix}$$

$$-j\omega\begin{pmatrix}B_x\\B_y\\B_z\end{pmatrix}=\begin{pmatrix}-jkE_{0y}.\exp(j(kz-\omega t))\\+jkE_{0x}.\exp(j(kz-\omega t))\\0\end{pmatrix}$$

$$\begin{pmatrix}B_x\\B_y\\B_z\end{pmatrix}=\begin{pmatrix}\frac{k}{\omega}E_{0y}.\exp(j(kz-\omega t))\\-\frac{k}{\omega}E_{0x}.\exp(j(kz-\omega t))\\0\end{pmatrix}=\begin{pmatrix}\frac{k}{\omega}E_{0y}.\cos((kz-\omega t))\\-\frac{k}{\omega}E_{0x}.\cos((kz-\omega t))\\0\end{pmatrix}$$

3.3.4. *Answer to question 4*

The electromagnetic energy density is given by the formula:

$$u=\frac{\varepsilon_0}{2}\vec{E}^2+\frac{1}{2\mu_0}\vec{B}^2$$

Hence, the following expression is based on the expressions of E and B:

$$du\ =\frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial z}dz$$

For a constant energy density, $du = 0$, hence:

$$U_\varepsilon=\frac{-\frac{\partial u}{\partial t}}{\frac{\partial u}{\partial z}}$$

3.3.5. *Answer to question 5*

The general expression of R as a function of E and B is applied.

For the calculation of the Poynting vector: $\vec{R} = \dfrac{\vec{E} \wedge \vec{B}}{\mu_0}$; the real part of E and B must be considered.

If E and B are perpendicular, it can be readily written:

$$\left|\vec{R}\right| = \frac{\left|\vec{E}\right| \cdot \left|\vec{B}\right|}{\mu_0} = \frac{\vec{E}^2}{c\mu_0}$$

The energy density is given by u, with $\mu_0 \varepsilon_0 c^2 = 1$.

Applying the relation $\boldsymbol{j_u} = u\ v_u$, the following relation is obtained: $\left|\vec{j}_u\right| = uc = c\varepsilon_0 E^2 = \dfrac{E^2}{\mu_0 . c} = \left|\vec{R}\right|$

where “c” is the speed of propagation of light.

The conservation equation, which reflects the conservation of electromagnetic energy in a vacuum, is used. This method requires the prior specification by symmetry of the direction of R.

By analogy with the charge conservation equation, the electromagnetic energy conservation equation can be written straightforward. Using the charge conservation equation, the electromagnetic energy conservation equation can thus be obtained.

From the charge conservation equation: $\dfrac{\partial}{\partial t}\rho + div\, j = 0$ with $\vec{j} = \rho \vec{V}$.

The electromagnetic energy conservation equation can be written as:

$$\frac{\partial}{\partial t} U + div\, \vec{R} = 0$$

Since:

$$div\,\vec{R} = \frac{\partial}{\partial z}R$$

and

$$\frac{\partial}{\partial t}U = 2\varepsilon_0\omega(E_{ox}^2\cos(\phi_x)\sin(\phi_x) + E_{oy}^2\cos(\phi_y)\sin(\phi_y))$$
$$\phi_x = kz - \omega t + \varphi_1 \; and \; \phi_y = kz - \omega t + \varphi_2$$

and

$$R = \int 2\varepsilon_0\omega(E_{ox}^2\cos(\phi_x)\sin(\phi_x)dz + \int E_{oy}^2\cos(\phi_y)\sin(\phi_y))dz$$
$$\phi_x = kz - \omega t + \varphi_1 \; and \; \phi_y = kz - \omega t + \varphi_2$$

By integration, it can be obtained:

$$\left|\vec{R}\right| = \varepsilon_0 c\,\vec{E}^2 = \frac{\vec{E}^2}{c\mu_0}$$

3.3.6. *Answer to question 6*

The power is given by: $P(t) = \iint \vec{R}.\vec{dS} = R.S$

The average power is equal to: $\langle P(t)\rangle = \frac{1}{2}\frac{\varepsilon_0}{\mu_0}E^2.S$

3.4. Rectangular waveguide as a high-pass frequency filter

The purpose of this exercise is to show that a given volume of finite dimensions cannot contain all the photons of the electromagnetic spectrum, in other words, all the possible wavelengths (respectively, frequencies).

The speed of light in a vacuum, (c), depends on the properties of the vacuum, characterized by its electric permittivity $\varepsilon_0 = 8.86 \times 10^{-12}$ F.m^{-1} and its magnetic permeability $\mu_o = 4\pi 10^{-7}$ H m^{-1} and is given by the well-known relation: $\varepsilon_0\mu_o c^2 = 1$.

A hollow waveguide, of rectangular cross section, of height a along Ox, of width b along Oy, of infinite length along Oz is considered (Figure 3.2). The walls of this waveguide are made of a perfect conductor of negligible thickness.

The objective is to study whether a wave expressed by:

$\vec{E}(M,t) = \vec{E}_0(y)e^{i(\omega t - kz)}$, with $\vec{E}_0(y) = E_0(y)\ \vec{u}_x$, where $E_0(y)$ is real.

and

$\vec{B}(M,t) = \vec{B}_0(y)e^{i(\omega t - kz)}$, with $\vec{B}_0(y) = B_{0y}(y)\ \vec{u}_y + B_{0z}(y)\ \vec{u}_z$

can propagate in this waveguide.

Consider $k_c^2 = \dfrac{\omega^2}{c^2} - k^2$, where k_c is real and positive.

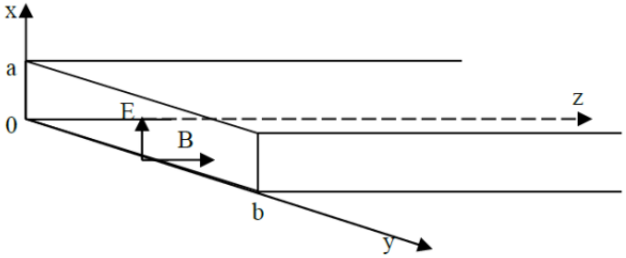

Figure 3.2. *Rectangular waveguide*

1) Does the form given for the wave verify Maxwell equations?

2) Based on the equation of propagation of the electric field E, find the differential equation verified by $E_0(y)$.

3) Using the boundary conditions of the electric field E in $y = 0$ and $y = b$, prove that k_c is an integer multiple of π/b.

4) Deduce the expression of $E_0(y)$.

5) Using ***rot E***, determine the components of $B_0(y)$.

6) Express k as a function of ω, c, b and n (n is an integer).

7) Deduce the existence of a cutoff frequency f_c below which propagation is no longer possible.

8) Calculate f_c for $b = 2$ cm.

9) Determine the wave phase velocity as a function of ω, c, n and b.

10) Determine the wave group velocity as a function of ω, c, n and b.

11) What is the relation between phase velocity and group velocity?

12) Calculate the average value in time of the Poynting vector at a point M of the waveguide.

13) Deduce the average power (P) transmitted by a section of the waveguide.

14) Calculate the time average electromagnetic energy (W) in the waveguide per unit length.

15) Define in this case the speed of energy propagation and prove that it is simply expressed as a function of $<P>$ and $<W>$. Comment.

3.4.1. *Answer to question 1*

There are four Maxwell equations:

$$div\ \vec{E} = \frac{\rho}{\varepsilon_0},\ div\ \vec{B} = 0,\ rot\ \vec{E} = \frac{-\partial \vec{B}}{\partial t} \text{ and } rot\ \vec{B} = \left(\mu_0\ \vec{j}\right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The electric field $\vec{E}$ has only one component directed along $\vec{x}$, which leads, after the application of the div operator, to the following result:

$$div(\vec{E}(M,t)) = \frac{\partial}{\partial x} E_0\left(y\right) e^{i(\omega t - kz)} = 0$$

This means that charges are necessarily absent. This condition is verified, since in the waveguide, there are no free charges and the walls being metallic, there are no charges in the volume of a perfect metal, of infinite conductivity.

For the magnetic induction $\vec{B}$ field, there are two components along $\vec{y}$ and $\vec{z}$, which leads after the application of the operator div to the following result:

$$div(\vec{B}(M,t)) = e^{i(\omega t-kz)}\frac{\partial}{\partial y}E_0\left(y\right)+E_0\left(y\right)\frac{\partial}{\partial z}e^{i(\omega t-kz)}$$

$$=\left(\frac{dE_0\left(y\right)}{dy}+ikE_0\left(y\right)\right)e^{i(\omega t-kz)}$$

This relation is not possible unless $div\ \vec{B}=0$, which means $\left(\frac{dE_0\left(y\right)}{dy}+ikE_0\left(y\right)\right)=0$.

According to Maxwell's third equation:

$$\vec{rot}\ \ \vec{E}=\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix}\wedge\begin{pmatrix}E_x\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \partial/\partial z\,E_x\\ -\partial/\partial y\,E_x\end{pmatrix}=\begin{pmatrix}0\\ -ikE_0\left(y\right)e^{i(\omega t-kz)}\\ -\dfrac{\partial E_0\left(y\right)}{\partial y}e^{i(\omega t-kz)}\end{pmatrix}$$

$$=i\omega\begin{pmatrix}0\\ B_{0y}\left(y\right)e^{i(\omega t-kz)}\\ B_{0z}\left(y\right)e^{i(\omega t-kz)}\end{pmatrix}$$

This means that the amplitudes of the components in y and z of the magnetic induction field depend on those of field E or on its derivative with respect to y.

According to Maxwell's fourth equation:

$$\vec{rot}\ \vec{B} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \wedge \begin{pmatrix} 0 \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial B_{0z}(y)}{\partial y} e^{i(\omega t - kz)} + ikB_{0y}(y) e^{i(\omega t - kz)} \\ 0 \\ 0 \end{pmatrix}$$

$$= \mu_0 \vec{j} - i\omega\mu_0\varepsilon_0 \begin{pmatrix} E_0(y) e^{i(\omega t - kz)} \\ 0 \\ 0 \end{pmatrix}$$

This means that the current j is necessarily directed along Ox and the amplitudes of the component in y and of the derivative with respect to y of the component in z of the magnetic induction field depend on those of the field E and on the amplitude of the current.

There is no current in a vacuum, with the possible exception of the walls.

3.4.2. *Answer to question 2*

The expression of the equation of propagation of the electric field $\vec{E}$ is:

$$\Delta \vec{E} - \frac{\partial}{\partial t}\left(\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} \right) = 0$$

The electric field $\vec{E}$ being linearly polarized along $\vec{x}$, only one equation remains, and its form, being expanded with respect to component x, is given by:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E_0(y) e^{i(\omega t - kz)} = 0$$

Consequently, after calculating the derivatives, the following equation is obtained:

$$\left(\frac{\partial^2}{\partial y^2}E_0\left(y\right)-k^2E_0\left(y\right)+\frac{\omega^2}{c^2}E_0\left(y\right)\right)e^{i(\omega t-kz)}=0$$

which is the equation verified by the amplitude $E_0\left(y\right)$ expressed as:

$$\frac{\partial^2}{\partial y^2}E_0\left(y\right)+k_c{}^2E_0\left(y\right)=0$$

where: $k_c{}^2=\frac{\omega^2}{c^2}-k^2$.

3.4.3. *Answer to question 3*

The boundary conditions correspond to the conservation of the tangential component of $\vec{E}$, on the metal walls. Since $\vec{E}=\vec{0}$ in a metal (considered as perfect, of infinite conductivity), then $\vec{E}=\vec{0}$ on the walls in $x=0$ and $x=a$ and on the walls in $y=0$ and $y=b$. This condition applies only to the tangential component of $\vec{E}$. Since the electric field $\vec{E}$ is tangent to the wall in y, at the interface between the two media, metal and air, the following relation is obtained:

$$\vec{E}_{t(air)}=\vec{E}_{t(metal)}$$

Hence: $\vec{E}\left(y=0\right)=\vec{0}$ and $\vec{E}\left(y=b\right)=\vec{0}$.

3.4.4. *Answer to question 4*

The objective is to solve the differential equation:

$\frac{\partial^2}{\partial y^2}E_0\left(y\right)+k_c{}^2E_0\left(y\right)=0$ with the boundary conditions: $E_0(b)=E_0(0)=0$.

The solution having the form e^{ry}, the differential equation leads to the characteristic equation: $\left(r^2 + k_c^2\right) = 0$, which yields the complex solution: $E_0\left(y\right) = Ae^{jk_c y} + Be^{-jk_c y}$,

where A and B are complex conjugates or in cosine form, as: $E_0\left(y\right) = C\cos\left(k_c y + \phi\right)$, where C and φ are related to the coefficients A and B of the complex solution.

Since: $E_0\left(0\right) = 0$, then: $C\cos\left(0 + \phi\right) = 0$.

which yields $\phi = \pm \pi/2$, leading to $E_0\left(y\right) = C\sin\left(k_c y\right)$.

The second boundary condition $E_0\left(b\right) = 0$ requires $E_0\left(b\right) = C.\sin\left(k_c b\right) = 0$.

If $C \neq 0$, then: $k_c b = p\pi$ and p being an integer.

The boundary condition $y = b$ leads to:

$k_c = p\pi/b$, which leads to the following expression for the amplitude of the field:

$$E_0\left(y\right) = A.\sin\left(\frac{p\pi}{b}y\right)$$

3.4.5. *Answer to question 5*

The expression of $\vec{E}\left(y,z\right) = C\sin\left(\frac{p\pi}{b}y\right)e^{i(\omega t - kz)}\vec{u}_x$ and the relation $\vec{rot}\,\vec{E} = \frac{-\partial\vec{B}}{\partial t}$ lead to:

$$\vec{rot}\ \vec{E} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \wedge \begin{pmatrix} E_x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \partial/\partial z\, E_x \\ -\partial/\partial y\, E_x \end{pmatrix} = \begin{pmatrix} 0 \\ -ikC\sin\left(\frac{p\pi}{b}y\right)e^{i(\omega t - kz)} \\ \frac{-p\pi}{b}C\cos\left(\frac{p\pi}{b}y\right)e^{i(\omega t-kz)} \end{pmatrix}$$

$$\vec{rot}\ \vec{E} = \frac{-\partial\vec{B}}{\partial t} = -i\omega\vec{B}$$

This leads to: $\vec{B} = \frac{-1}{i\omega}.\vec{rot}\ \vec{E}$

Hence, the expression: $\vec{B} = \begin{pmatrix} 0 \\ \frac{k}{\omega}C\sin\left(\frac{p\pi}{b}y\right)e^{i(\omega t-kz)} \\ \frac{-ip\pi}{b\omega}C\cos\left(\frac{p\pi}{b}y\right)e^{-i(\omega t-kz)} \end{pmatrix}$

$$B_{0y}(y) = \frac{k}{\omega}C\sin\left(\frac{p\pi}{b}y\right)$$

and $B_{0z}(y) = \frac{-ip\pi}{b\omega}C\cos\left(\frac{p\pi}{by}\right)$.

3.4.6. *Answer to question 6*

Since $k^2 = \frac{\omega^2}{c^2} - k_c{}^2$, with the angular frequency $\omega = 2\pi f$ and $k_c = {n\pi}/{b}$,

hence $k^2 = \frac{\omega^2}{c^2} - \left(\frac{n\pi}{b}\right)^2$.

3.4.7. *Answer to question 7*

The relation between the frequency f and the angular frequency is $\omega = 2\pi f$.

Thus: $\dfrac{\omega^2}{c^2} - k_c^{\ 2} = \dfrac{\omega^2}{c^2} - \left(\dfrac{n\pi}{b}\right)^2 = k^2 \geq 0$,

If: $\dfrac{\omega^2}{c^2} \geq k_c^2$ then $\dfrac{2\pi f}{c} \geq \dfrac{n\pi}{b}$

$f \geq \dfrac{nc}{2b}$ $\quad f \geq f_C = \dfrac{c}{2b}$ for n=1

Therefore, frequency f must be above $f_C = \dfrac{c}{2b}$.

3.4.8. *Answer to question 8*

$f_C = \dfrac{c}{2b}$;

f_c= (3 10^{10}/4) = 7.5 GHz (domain of centimeter waves).

No wave of wavelength above 4 cm can propagate in the waveguide.

3.4.9. *Answer to question 9*

The expression of the phase speed V_ϕ is: $\quad V_\phi = \omega/k$

The expression of the wave vector (question 6) is: $\quad k = \sqrt{\dfrac{\omega^2}{c^2} - \left(\dfrac{n\pi}{b}\right)^2}$

Replacing the expression of the wave vector in the expression of the phase velocity: $V_\phi = \omega/k = \dfrac{\omega}{\sqrt{\dfrac{\omega^2}{c^2} - \left(\dfrac{n\pi}{b}\right)^2}}$

3.4.10. *Answer to question 10*

The expression of the group velocity V_g is: $V_g = d\omega/dk$.

From the expression: $\dfrac{\omega^2}{c^2} = k^2 + \left(\dfrac{n\pi}{b}\right)^2$, it is deduced that: $\dfrac{2\omega}{c^2}.\dfrac{d\omega}{dk} = 2k$

hence: $V_g = d\omega/dk = \dfrac{k}{\omega}c^2 = c^2\dfrac{\sqrt{\dfrac{\omega^2}{c^2} - \left(\dfrac{n\pi}{b}\right)^2}}{\omega} = \dfrac{c}{\omega}\sqrt{\omega^2 - \dfrac{n^2\pi^2c^2}{b^2}} = \dfrac{c^2}{V_\phi}$.

3.4.11. *Answer to question 11*

The relation between phase velocity and group velocity is: $V_\phi.V_g = c^2$.

3.4.12. *Answer to question 12*

The Poynting vector is given by: $\vec{R} = \dfrac{1}{\mu_0}\mathrm{Re}\left(\vec{E}\right) \wedge \mathrm{Re}\left(\vec{B}\right)$.

According to equation [4.23], page 80 of *Nanometer-scale Defect Detection Using Polarized Light*:

$$R_e\left(\vec{E}\right) = \begin{pmatrix} A\sin\left(\dfrac{n\pi}{b}y\right)\cos\left(\omega t - kz\right) \\ 0 \\ 0 \end{pmatrix}$$, A being a real number.

and: $R_e\left(\vec{B}\right)=\begin{pmatrix} 0 \\ A\dfrac{k}{\omega}\sin\left(\dfrac{n\pi}{b}y\right)\cos\left(\omega t-kz\right) \\ -\dfrac{An\pi}{\omega}\cos\left(\dfrac{n\pi}{b}y\right)\sin\left(\omega t-kz\right) \end{pmatrix}$, A being a real number.

Hence:

$$\vec{R}=\frac{1}{\mu_0}\begin{pmatrix} 0 \\ \dfrac{A^2 n\pi}{\omega}\sin\left(\dfrac{n\pi}{b}y\right)\cos\left(\dfrac{n\pi}{b}y\right)\cos\left(\omega t-kz\right).\sin\left(\omega t-kz\right) \\ \dfrac{A^2 k}{\omega}\sin^2\left(\dfrac{n\pi}{b}y\right)\cos^2\left(\omega t-kz\right) \end{pmatrix}$$

3.4.13. *Answer to question 13*

$$\langle P\rangle=\left\langle\vec{R}\cdot s\vec{u}_z\right\rangle=\frac{ab}{2}\frac{A^2k}{\mu_0\omega}\left\langle\sin^2\left(\frac{n\pi}{b}y\right)\right\rangle=\frac{1}{2}\frac{ab}{2}\frac{A^2k}{\mu_0\omega} \text{ with } s=ab$$

3.4.14. *Answer to question 14*

The expression of the average energy density is:

$$u=\left\langle\frac{1}{2}\varepsilon_0\left(R_e\left(\vec{E}\right)\right)^2+\frac{1}{2}\left(R_e\left(\vec{B}\right)\right)^2\times\frac{1}{\mu_0}\right\rangle=\varepsilon_0\left\langle\vec{E}^2\right\rangle=\frac{\varepsilon_0A^2}{2}\left\langle\sin^2\left(\frac{n\pi y}{b}\right)\right\rangle=\frac{1}{2}\frac{\varepsilon_0A^2}{2}$$

Multiplying by the volume of the unit length, abx1, the average electromagnetic energy is given by:

$$<W>=<u>\cdot s\times 1=ab\,\frac{\varepsilon_0A^2}{2}\left\langle\sin^2\left(\frac{n\pi y}{b}\right)\right\rangle=\frac{1}{2}ab\,\frac{\varepsilon_0A^2}{2}$$

3.4.15. *Answer to question 15*

The speed of energy propagation is given by the ratio between the module of the Poynting vector in the direction perpendicular to the surface crossed and the energy density: $V_e = \left\langle \frac{|\vec{R}|}{u} \right\rangle$

$$\text{As: } \langle \vec{R} \cdot \vec{u}_z \rangle = \frac{1}{2}\frac{A^2 k}{\mu_0 \omega}\left\langle \sin^2\left(\frac{n\pi}{b}y\right)\right\rangle = \frac{1}{2}\frac{A^2}{\mu_0 c}\left\langle \sin^2\left(\frac{n\pi}{b}y\right)\right\rangle$$

and

$$u = \left\langle \frac{1}{2}\varepsilon_0\left(R_e\left(\vec{E}\right)\right)^2 + \frac{1}{2}\left(R_e\left(\vec{B}\right)\right)^2 \times \frac{1}{\mu_0}\right\rangle = \varepsilon_0\left\langle \vec{E}^2 \right\rangle = \frac{\varepsilon_0 A^2}{2}\left\langle \sin^2\left(\frac{n\pi y}{b}\right)\right\rangle$$

$$V_e = \left(\frac{1}{2}\frac{A^2}{\mu_0 c}\left\langle \sin^2\left(\frac{n\pi}{b}y\right)\right\rangle\right) \Big/ \left(\frac{1}{2}\varepsilon_0 A^2\left\langle \sin^2\left(\frac{n\pi}{b}y\right)\right\rangle\right) = \frac{1}{\varepsilon_0 \mu_0 c} = c$$

$$\text{As: } \langle P \rangle = \langle \vec{R} \cdot s\vec{u}_z \rangle = \frac{ab}{2}\frac{A^2 k}{\mu_0 \omega}\left\langle \sin^2\left(\frac{n\pi}{b}y\right)\right\rangle$$

$$< W > = < u > \cdot s \times 1 = ab\frac{\varepsilon_0 A^2}{2}\left\langle \sin^2\left(\frac{n\pi y}{b}\right)\right\rangle$$

hence: $V_e = \frac{< P >}{< W >} = \frac{1}{\mu_0}.\frac{k}{\omega}.\frac{1}{\varepsilon_0} = \frac{1}{c\mu_0\varepsilon_0} = c$.

3.5. Characteristics of microwave antennas

Microwaves correspond to the spectrum of sub-meter waves down to millimeter waves, which is the frequency range between 300 MHz and 300 GHz. Microwave antennas are devices that convert electric energy into electromagnetic energy and are used in a wide frequency range (30 kHz to 300 GHz) in communication systems: radio, television, wireless telephony, geo-localization, satellite observation or communication, radar detection, remote sensing [SIL 49], [MAR 90], [BRA 78]. Microwave antennas are also used for the characterization of matter–electromagnetic radiation interaction and for the detection of nanometer-scale defects.

3.5.1. *Introduction to antennas*

A transmission antenna is a device that converts electromagnetic fields propagating through a waveguide or a line into radiation in the environment surrounding the antenna. The phenomena involved in an antenna are reversible. If an electromagnetic wave front is intercepted by the radiative elements of a receiving antenna, the latter picks up the energy of the incident electromagnetic wave and transfers the corresponding electromagnetic field to the transmission line. In practice, an antenna behaves simultaneously as an emitter and a receiver.

Antennas are composed of radiative elements of various designs or forms: wire, dipole, loop, helix, spiral, monopole, cone (pyramid, circular cone), rectangular waveguide, micro-strip, with parabolic reflector, etc. Depending on their design, antennas can be omnidirectional or directional. Omnidirectional antennas can be fixed. Directional antennas can be motor-driven and oriented towards the emitter or the area to cover. The systems using antennas vary greatly. Depending on the applications, there are passive antennas, active antennas with preamplifier, which are used more often in reception, and antennas arranged in networks or matrices.

The structure of an antenna is composed of conductive elements or resonant cavities. In the emission mode, these elements convert the electric currents at their surface into electromagnetic radiation. Conversely, in reception mode, the electromagnetic waves picked up by the conductive elements or by the resonant structures are converted into an electric signal. At the resonance frequency of the antenna circuit, the electric impedance of the antenna becomes real and therefore purely resistive. The condition for an antenna to be efficient is that its input resistance is adapted to the impedance of the waveguide or of the transmission line of the emitting circuit.

An antenna is often characterized by its polarization direction, its input impedance, its bandwidth, its efficiency, its gain and its directivity or its radiation diagram.

The polarization of a receiving antenna is the direction of the electric field that maximizes reception. The direction of polarization of a transmission antenna is the direction of the radiated electric field. When the radiated electric field is parallel to the ground, the polarization of the antenna is vertical. When the radiated electric field is perpendicular (90^{o}) to the ground, polarization is horizontal. There are antennas with circular polarization.

An antenna often operates in a quite low range of frequencies. The bandwidth of an antenna is represented by the two limits of frequencies between which this antenna can operate without efficiency loss. In this range of frequencies, the impedance of the antenna is real. Beyond this bandwidth, the antenna behaves as a series resonant circuit whose impedance (Z_a) has a real part (R_a) and an imaginary part (X_a). The impedance of the antenna is:

$$Z_a = R_a + iX_a$$

To avoid reflection phenomena, the impedance of an emission antenna must be adapted to the impedance of the environment in which the waves propagate (in a vacuum, the impedance is 377 Ohm).

The impedance of a receiving antenna must be adapted to the impedance of the environment in which the waves propagate and to the impedance of the circuit that amplifies the electric signal picked up by the antenna. If these conditions are met, there is no reflection of electromagnetic waves and the maximum radiated power can be transferred.

If $Z_r = R_r + jX_r$ is the impedance of the receiving circuit and $Z_a = R_a + jX_a$ is the impedance of the receiving antenna, these relations are expressed by: $Z_a = Z_r^*$

$$Z^*_r = R_r - jX_r = R_a + jX_a$$

or:

$$R_r = R_a$$
$$jX_r = -jX_a$$

Let us consider an emission antenna, located at the center O of a spherical reference system (Figure 3.3). Oz axis corresponds to the vertical axis of the reference system. The horizontal plane corresponds to the set of points where $\theta = \frac{\pi}{2}$ and ϕ is arbitrary. The sets of points defined by an angle θ between 0 and π, and an angle ϕ constant (respectively, -ϕ), in a spherical reference system, are in a vertical plane.

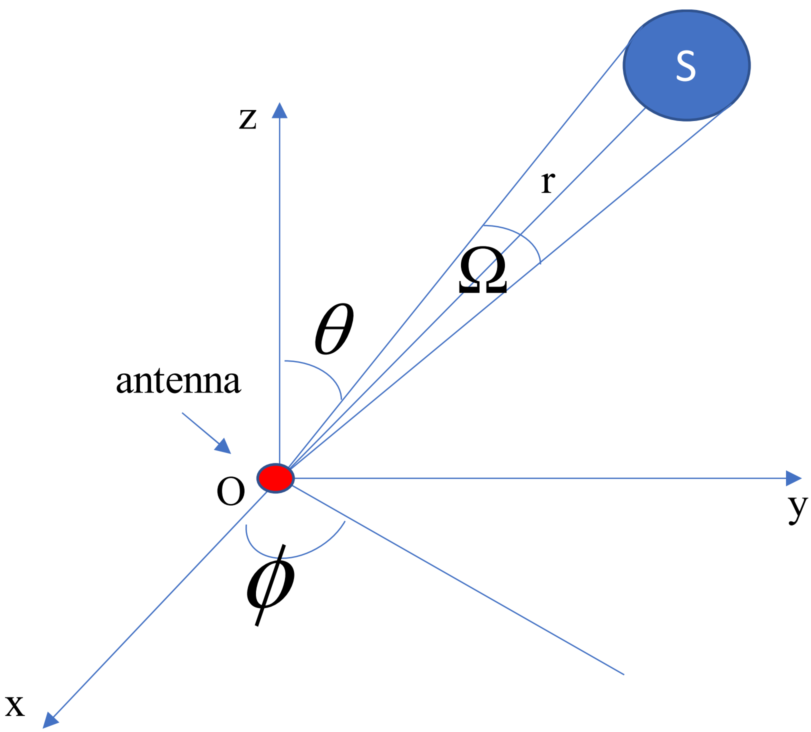

Figure 3.3. *Antenna at the center of a spherical reference system: definition of angles θ and ϕ. The solid angle of the pick-up surface at point M is $\Omega = \frac{S}{r^2}$. For a color version of this figure, see www.iste.co.uk/dahoo/metrology1.zip*

The power P_A radiated in a direction defined by the angles (θ,ϕ) can be expressed in a solid angle Ω by the relation:

$$P(\theta,\phi,\Omega)=\frac{P_A}{\Omega}$$

The power density radiated through an elementary surface at a distance r is:

$$p(\theta,\phi,r)=\frac{P_A}{\Omega r^2}$$

The total radiated power P_r is:

$$P_r=\int_\theta\int_\phi P(\theta,\phi)\,d\theta d\phi$$

For the specific case of an isotropic lossless antenna, the power per unit solid angle is: $P(\theta,\phi)=\dfrac{P_A}{4\pi}$, where P_A is the electric power supplied, and the power density per unit solid angle $p(\theta,\phi,r)$ at a distance r is:

$$p(\theta,\phi,r)=\frac{P_A}{4\pi r^2}$$

The power received by an antenna is expressed as a function of the effective reception surface A_{eff} $\left[m^2\right]$ and of the intensity of the received power $I(\theta,\phi)$ $\left[W/m^2\right]$ by the relation:

$$P_r=I(\theta,\phi)\,A_{eff}\,[W]$$

The effective reception surface depends on (θ,ϕ) and can be described by $A_{eff}(\theta,\phi)$.

In the case of an antenna of length dl, the picked-up voltage V_r is $V_r = E.dl$.

If $R_r = R_a$, the received power P_r is:

$$P_r = \frac{V_{eff}^2}{4R_r} = \frac{E^2 dl^2}{4R_r}$$

The received power is:

$$P_r = A_{eff} \frac{E_{eff}^2}{\sqrt{\frac{\mu_0}{\varepsilon_0}}}$$

Since the resistance of radiation in vacuum a is:

$$R_r = \frac{2\pi}{3} Z_0 \frac{dl^2}{\lambda^2}$$

Then:

$$A_{eff} = \frac{dl^2 \lambda^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}}{4\left(\frac{2\pi}{3} dl^2 \sqrt{\frac{\mu_0}{\varepsilon_0}}\right)} = \frac{3\lambda^2}{8\pi}$$

An antenna is often characterized by its gain. The gain G in a direction of solid angle defined by (θ, ϕ) is the ratio between the power P_A radiated per unit solid angle and the power supplied to the antenna divided by 4π.

$$G(\theta, \phi) = \frac{\left(\frac{P_A(\theta, \phi)}{\Omega}\right)}{\left(\frac{Supplied\ Power}{4\pi}\right)}$$

This definition of gain involves the comparison between the power radiated by a unit solid angle and that of an isotropic antenna of the same power. Examples of an isotropic antenna are an emission–reception WIFI antenna in a residential building or a radar antenna for airplane detection in an airport. The gain of a real antenna is often compared to the gain of an isotropic antenna, such as a dipole or a wire antenna. The gain is then expressed in decibels (dB).

$$G = 10\log_{10}\frac{P}{P_{ideal}}(dB)$$

If the reference ideal antenna is a dipole, the unit is the dBd (decibel with respect to dipole). In the case of an isotropic reference antenna, it is the dBi (decibel with respect to the isotropic antenna). The difference between dBi and dBd is 2.15 decibels, which means that a dipole whose length is half a wavelength has a gain of 2.15 dBi.

For certain applications, it is interesting to channel the emitted radiation in a privileged direction, so that a high gain emission/detection direction is available. It is, for example, the case of a radar detection antenna. The gain of a directive antenna G_D in a direction (θ,ϕ) is defined by:

$$G_D(\theta,\phi) = \frac{(P(\theta,\phi))}{\left(\frac{Total\ radiated\ power}{4\pi}\right)}$$

The gain of a lossless directive antenna whose power is concentrated in a single beam (characterized by a cone of solid angle Ω_s) in which power density is evenly distributed is:

$$G_D = \frac{4\pi}{\Omega_s}$$

A directive antenna has one or two lobes clearly more significant than the others referred to as “main lobes”. The narrower the most directive lobe of an antenna is, the more directive is this antenna. The antenna beamwidth characterizes the width of the main lobe. The beamwidth at 3 dB represents the portion of space in which most of the power is radiated. It is the angle between the two directions of the main lobe where the radiated power is equal to half the power radiated in the direction of the maximal radiation. Directivity corresponds to the width of the main lobe principal, between the attenuation angles at 3 dB.

The area (S) illuminated by the beam emitted by an ideal antenna is the product of the solid angle (Ω_R) and the distance (r) between the antenna and the radiated area to the square.

The area illuminated (S) by the beam emitted by an ideal antenna is the product of the solid angle (Ω_R) and the squared distance (r) between the antenna and the radiated area:

$$S = r^2 \Omega_R$$

If the beam emitted by the antenna has a circular section, then the width of the beam θ_R is given by the relation:

$$\Omega_R = \frac{\pi}{4} \theta_R{}^2$$

The radiation diagram of an antenna is a characteristic function of the radiation depending on angles (θ, ϕ), which varies along the direction between 0 and 1. A three-dimensional display of antenna lobes is thus possible, in the horizontal plane or in the vertical plane including the most significant lobe. The power radiation diagram represents the distribution of power per unit solid angle in the direction of the solid angle. The radiation diagram is determined based on the power emitted or received per unit solid angle in the direction of the solid angle.

An antenna is characterized by its near field diagram (in the proximity of the antenna, in the zone where the electric field varies inversely with distance) and by its far field diagram (far from the antenna in the zone where the electric field varies inversely with distance to the square).

The diagram of the near field radiated in the proximity of the antenna can be used to evaluate the impedance of the antenna and its reactive power.

A microwave transmission antenna converts the waves propagated in a line into waves radiated in the environment. Experimentally, a radiation is generated by a time variable electric current or by a charge oscillation.

By definition, the density of electric current in a conductor is defined by an electric charge density ρ_s traveling with a speed V through a section (S) of the conductor:

$$J_S = \rho_s V \ \left(A/m^2\right)$$

In a perfect electric wire, the electric charge density ρ_s can be assumed to be constant. The current in the wire I_s is then expressed as a function of free charges ρ through the wire and the speed V of these charges by the relation:

$$I_s = J_S.S = \left(\rho_s S\right)V \ = \rho V \ \left(A\right)$$

If the current varies in time, it can be written as:

$$\frac{dJ_s}{dt} = \rho \ \frac{dV}{dt}\left(A/m^2\ /\ s\right)$$

According to this equation, an electromagnetic radiation can be created by the variation in time of an electric current or by the variation of the acceleration of electric charges. In summary, if a charge does not move, there is no radiation. If a charge moves at constant speed, there is no radiation if the wire is straight and infinitely long, but if the wire has discontinuities or curves, folds, radiation is then possible. Antenna design uses charge oscillations for transient effects and pulses and variable currents for harmonic variations in time.

3.5.2. *Radiation of a wire antenna*

A current of periodic amplitude $I = I_0 e^{j\omega t}\ \left(A\right)$ travels through a wire antenna of length (l):

1) Calculate the magnetic potential generated at a point at a distance r from the antenna (r>> l).

2) Deduce the radiated magnetic and electric fields.

3) Calculate the radiated far electric field. Plot the diagram of the radiated electric field. Calculate the ratio of the modules of the electric field and the radiated magnetic field.

4) Calculate the average radiated power. Plot the diagram of the power radiated by the antenna in far field depending on the angle θ in the vertical plane xoz. Find the beamwidth of the antenna at 3 dB.

5) If the antenna has a resistance R, calculate the power dissipated in the antenna and deduce the antenna resistance (which for a current I through the antenna would dissipate a power equivalent to the radiated power).

6) Calculate the antenna efficiency, its gain G and its directivity G_D.

7) Calculate the electric and magnetic fields radiated in the near field. Calculate the Poynting vector and determine the corresponding radiated power in the near field. Find the equivalent Thévenin impedance $Z_A(\omega)$ of the wire antenna.

3.5.2.1. *Answer to question 1*

For symmetry reasons, the origin O of the Cartesian reference system is chosen at the bottom of the antenna wire and the Oz axis of the reference system is directed along the antenna wire.

The vector potential at time t and a point M at a distance r from the antenna is generated by a current in the antenna at a previous time t'.

The difference $(t-t')$ is the duration required for the electromagnetic wave to propagate from the antenna to the point M. $(t-t')$ is a delay equal to the distance between the antenna and the point M divided by the speed of light, such that:

$$(t-t') = \frac{r}{c}$$

If $\vec{j}$ is the current density in the antenna wire, the vector potential $\vec{A}$ at the point M is obtained from the expression of the retarded potential integrated over the entire volume (V) of the wire antenna:

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\int_{\infty} \frac{\vec{j}\left(\vec{r}',t-\dfrac{\left|\vec{r}-\vec{r}'\right|}{c}\right)}{\left|\vec{r}-\vec{r}'\right|}d^3\vec{r}'$$

The point M being distant (far field hypothesis with the distance r >> λ the wavelength), the difference in distance being $\left|\vec{r}-\vec{r}'\right| \approx r$, the expression of A is therefore:

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi r}\int_V \vec{j}\left(t-\frac{r}{c}\right)dV$$

If the antenna current flows through a, the element of cross-section S, then the element of volume current $\vec{j}dV$ is expressed by the relation:

$$\vec{j}dV = (\vec{j}.d\vec{S}).d\vec{l}$$

$$\int_V \vec{j}dV = \int_S \vec{j}.d\vec{S}\int_l d\vec{l} = I\vec{l}$$

Since the current I in the antenna is periodic, it can be expressed in complex notation by (see § 4.2 of Chapter 4 [DAH 16]): $I = I_0 e^{j\omega t}$

The vector potential at the point M is: $\vec{A}(r,t) = \frac{\mu_0}{4\pi}\frac{\vec{l}}{r}I_0 e^{j\omega\left(t-\frac{r}{c}\right)}$

Since the wave vector k of electromagnetic waves is related to the angular frequency ω by the relation $k = \frac{\omega}{c}$, the vector potential in the Cartesian reference system $\left(\hat{x},\hat{y},\hat{z}\right)$ is expressed by:

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} I_0 l e^{j\omega t}\frac{e^{-jkr}}{r}\hat{z}$$

To simplify the expressions of the radiated electric and magnetic fields, the vector potential $\vec{A}$(M) is expressed in spherical coordinates $\left(\hat{r},\hat{\theta},\hat{\phi}\right)$ (Figure 3.4).

Hence:

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} I_0 l e^{j\omega t}\frac{e^{-jkr}}{r}\hat{z} = \frac{\mu_0}{4\pi} I_0 l e^{j\omega t}\frac{e^{-jkr}}{r}\left(cos\,\theta\,\hat{r} - sin\,\theta\,\hat{\theta}\right)$$

$\vec{A}$ has two components A_r and A_θ that are directed along $\hat{r}$ and $\hat{\theta}$.

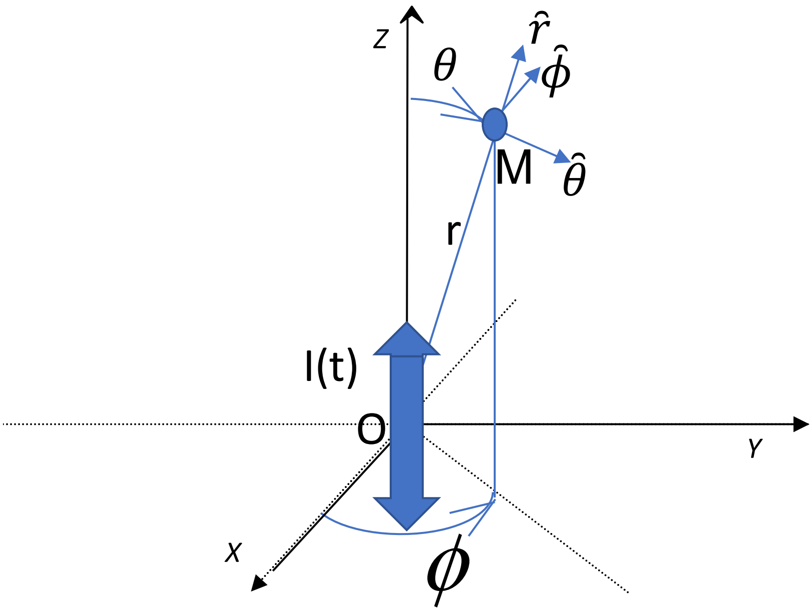

Figure 3.4. *Wire antenna. For a color version of this figure, see www.iste.co.uk/dahoo/metrology1.zip*

3.5.2.2. *Answer to question 2*

The magnetic field is obtained by the relation:

$$\vec{B}(\vec{r},t) = rot\vec{A}(\vec{r},t)$$

In spherical coordinates, this gives:

$$\overrightarrow{rot}\vec{A}(\vec{r},t) = \begin{pmatrix} \dfrac{\hat{r}}{r^2 \sin\theta} & \dfrac{\hat{\theta}}{r\sin\theta} & \dfrac{\hat{\phi}}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{pmatrix}$$

Since $\vec{A}$ has no component along $\widehat{\phi}$ and is independent of position ϕ :

$$A_\phi = 0 \text{ and } \frac{\partial}{\partial \phi} = 0$$

$$rot\vec{A}(r,t) = \frac{\mu_0}{4\pi r} I_0 l \left(\frac{1}{r} \left(\frac{\partial}{\partial r} \left(e^{j(\omega t - kr)} \right) (-\sin\theta) - \frac{\partial}{\partial \theta} \left(\frac{1}{r} e^{j(\omega t - kr)} \right) (\cos\theta) \right) \right) \widehat{\phi}$$

$$\vec{B}(r,t) = rot\vec{A}(r,t) = -\frac{\mu_0}{4\pi} I_0 l \sin\theta \left(\frac{1}{r^2} + \frac{jk}{r} \right) e^{j(\omega t - kr)} \widehat{\phi}$$

The magnetic field $\vec{H}(r,t)$ is therefore written as:

$$\vec{H}(r,t) = \frac{\vec{B}(r,t)}{\mu_0} = -\frac{I_0 l \sin\theta}{4\pi} \left(\frac{1}{r^2} + \frac{jk}{r} \right) e^{j(\omega t - kr)} \widehat{\phi}$$

The magnetic field $\vec{H}(r,t)$ is directed along the direction $\widehat{\phi}$ and is therefore perpendicular to the direction of propagation of the radiated wave.

The radiated magnetic field is equal to the real part of $\vec{H}(r,t)$:

$$Re\left(\vec{H}(r,t)\right) = -Re\left(\frac{I_0 l \sin\theta}{4\pi} \left(\frac{jk}{r} \right) e^{j(\omega t - kr)} \widehat{\phi} \right) = -\frac{I_0 lk \sin\theta}{4\pi r} \sin(\omega t - kr) \widehat{\phi}$$

At the point M, there is no current. The radiated electric field in M is obtained by Ampère's law:

$$\frac{1}{c^2} \frac{\partial}{\partial t} \left(\vec{E}(r,t) \right) = rot\vec{B}(r,t)$$

The only possible dependence of $\vec{E}(r,t)$ with respect to time is from the factor $e^{j\omega t}$ of the current oscillation such that:

$$\vec{E}(r,t) = \vec{E}(r) e^{j\omega t}$$

The magnetic induction radiated at the point M is such that:

$$rot\vec{B}(r,t) = \frac{1}{c^2}\frac{\partial}{\partial t}\left(\vec{E}(r,t)e^{j\omega t}\right) = j\frac{\omega}{c^2}\vec{E} = j\frac{k}{c}\vec{E}$$

This leads to:

$$jk\vec{E} = c\frac{\mu_0}{4\pi}I_0 l\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\left(\frac{jk}{r}+\frac{1}{r^2}\right)\sin^2\theta\, e^{j(\omega t-kr)}\right)\hat{r} + \left(\frac{1}{r}\left(-\frac{\partial}{\partial r}\left(r\left(\frac{jk}{r}+\frac{1}{r^2}\right)\sin\theta\, e^{-jkr}\right)e^{j\omega t}\right)\right)\hat{\theta}\right)$$

$$jk\vec{E} = c\frac{\mu_0}{4\pi}I_0 l\left(\left(\frac{1}{r}\right)\left(\frac{jk}{r}+\frac{1}{r^2}\right)(2\cos\theta)e^{j(\omega t-kr)}\hat{r} + \left(\frac{1}{r}\left(\left(\frac{1}{r^2}\sin\theta\, e^{-j(\omega t-kr)}\right) + (jk\sin\theta)\left(jk+\frac{1}{r}\right)e^{j(\omega t-kr)}\right)\right)\hat{\theta}\right)$$

$$jk\vec{E} = c\frac{\mu_0}{4\pi}I_0 l\left(\left(\frac{jk}{r^2}+\frac{1}{r^3}\right)(2\cos\theta)e^{j(\omega t-kr)}\hat{r} + \left(\left(\left(\frac{1}{r^3}\sin\theta\, e^{-j(\omega t-kr)}\right) + (jk\sin\theta)\left(j\frac{k}{r}+\frac{1}{r^2}\right)e^{j(\omega t-kr)}\right)\right)\hat{\theta}\right)$$

$$\vec{E} = -jc\frac{\mu_0}{4\pi k}I_0 l\left(\left(\frac{1}{r^3}+\frac{jk}{r^2}\right)(2\cos\theta)e^{j(\omega t-kr)}\hat{r} + \left(\sin\theta\left(\frac{1}{r^3}-\frac{k^2}{r}+j\frac{k}{r^2}\right)e^{-j(\omega t-kr)}\right)\hat{\theta}\right)$$

Since the point M is far from the antenna, the distance kr is such that $kr \gg 1$. If λ is the wavelength of the radiation, this means that the point M is at a distance of several wavelengths from the antenna (or $r \gg \frac{\lambda}{2\pi}$) and the expressions in $\frac{1}{r^2}$ and $\frac{1}{r^3}$ of the radiated electric field can be neglected. The electric field can then be written as:

$$\vec{E}\left(r,t\right)) = jc\frac{\mu_0}{4\pi}I_0 l \sin\theta \frac{k}{r} e^{-j(\omega t - kr)}\widehat{\theta}$$

The radiated electric field is the real part of $\vec{E}(r,t)$:

$$\mathrm{Re}\left(\vec{E}\left(r,t\right)\right) = \mathrm{Re}\left(jc\frac{\mu_0 k}{4\pi r}I_0 l \sin\theta e^{j(\omega t - kr)}\right)\widehat{\theta} = -c\frac{\mu_0 k}{4\pi r}I_0 l \sin\theta \sin(\omega t - kr)\widehat{\theta}$$

Therefore, the expressions of the radiated electric and magnetic fields are given by:

$$\vec{H}(r,t) = -\frac{k}{4\pi r}I_0 l\, sin\,\theta\, sin(\,\omega t - kr)\hat{\phi}$$

$$\vec{E}(r,t) = -\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{k}{4\pi r}I_0 l\, sin\,\theta\, sin(\,\omega t - kr)\hat{\theta}$$

The radiated electric and magnetic fields are in phase and transverse to the direction of propagation. The electric field is directed along $\widehat{\theta}$. The magnetic field is directed along $\widehat{\phi}$, which is orthogonal to $\widehat{\theta}$. Polarization is thus rectilinear. Radiation is symmetrical with respect to ϕ but not with respect to θ. For the far field, radial components are negligible.

3.5.2.3. *Answer to question 3*

The module of the radiated electric field varies with $\sin\theta$:

$$\vec{E}(r) = \sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{k}{4\pi r}I_0 l \sin\theta = E_r \sin\theta$$

The diagram of the radiated far field is obtained by plotting the function $f(\theta) = E_r \sin\theta$. In polar coordinates, this curve is a circle (Figure 3.5). The radiated field is maximum for $\theta = 90^\circ$ and zero along the axis of the antenna.

The ratio of the module of the radiated electric field to the module of the radiated magnetic field $\left|\frac{\vec{E}(r,t)}{\vec{H}(r,t)}\right|$ is equal to the impedance of the medium in which the wave propagates. In a vacuum, this impedance is given by: $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ or 377 Ohm.

The electromagnetic radiation has the orthogonality and proportionality properties $\left|\frac{\vec{E}(r,t)}{\vec{H}(r,t)}\right|$ of a uniform plane wave. Nevertheless, the amplitude of the magnetic and the electric far fields decreases with $\frac{1}{r}$, which is not the case for a plane wave, and is thus an approximation of the wave front surface by the plane tangent to the point considered.

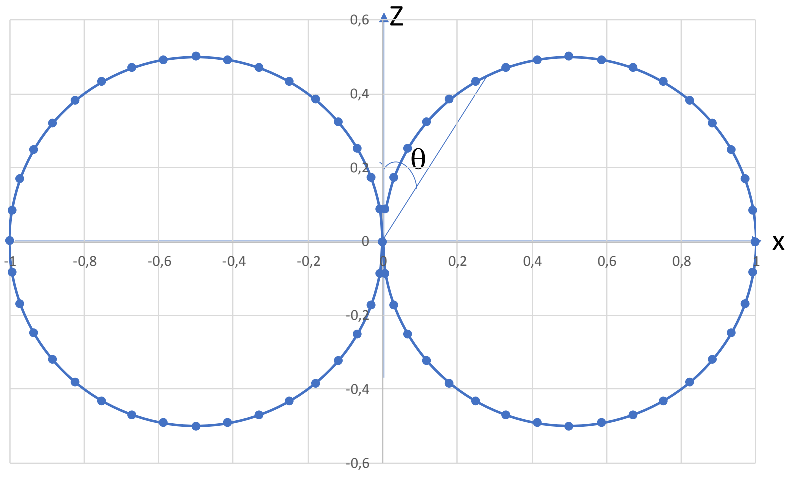

Figure 3.5. *Electric field strength radiated by the wire antenna in the plane defined by the vertical axis z and the horizontal axis x. For a color version of this figure, see www.iste.co.uk/dahoo/metrology1.zip*

3.5.2.4. *Answer to question 4*

The radiated electromagnetic field being a plane wave, the Poynting vector $\vec{S}(\vec{r},t)$ (energy flux per unit surface and time) is expressed as:

$$\vec{S}(\vec{r},t)=\left(\vec{E}(\vec{r},t)\times\vec{H}^*(\vec{r},t)\right)=Z_0\left(\frac{klI_0}{4\pi r}\right)^2\sin^2\theta\sin^2(\omega t-kr)\hat{r}$$

The temporal average value of the radiated flux is written as:

$$<\vec{S}(t)>=0.5\,Re\left(\vec{S}(\vec{r},t)\right)=0.5\,Re\left(\vec{E}(\vec{r},t)\times\vec{H}^*(\vec{r},t)\right)$$

$$<\vec{S}(t)>=Z_0\left(\frac{klI_{eff}}{4\pi r}\right)^2\sin^2\theta\quad\hat{r}\quad\left[W/m^2\right]$$

with $I_0^2=2I_{eff}^2$.

I_{eff} is the effective current in the antenna wire.

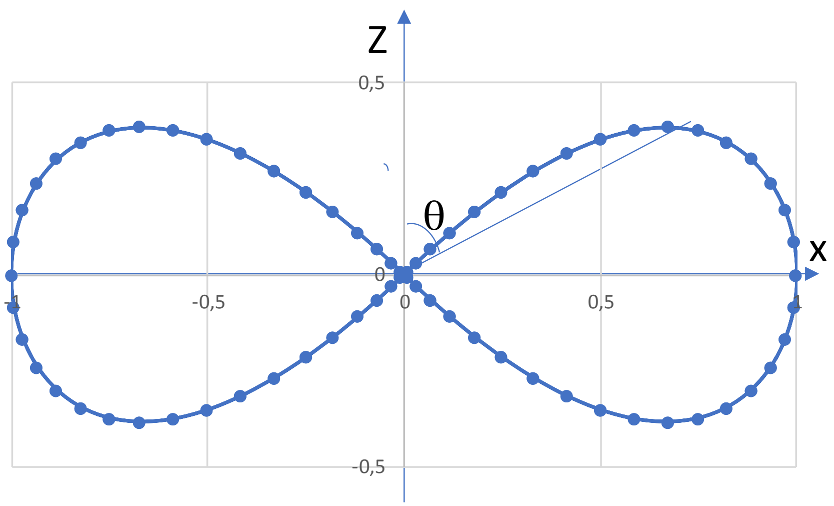

Figure 3.6. *Power radiated by the wire antenna in the plane defined by the vertical axis Z and the horizontal axis X. For a color version of this figure, see www.iste.co.uk/dahoo/metrology1.zip*

The radiated power varies as a function of $\left(\frac{klI_{eff}}{4\pi r}\right)^2 \sin^2\theta$. The diagram of the power radiated by the antenna in far field is represented in Figure 3.6 by a curve of revolution about axis $\hat{z}$.

If the energy flux is integrated over the entire surface of the sphere of radius r, the power radiated per second is given by:

$$P_r = Z_0 \oint < S(r,\theta) > r^2 d\Omega$$

$$d\Omega = \sin\theta d\theta d\phi$$

$$P_r = \int_0^{2\pi} d\phi \int_0^{\pi} < S(t,r,\theta) > \sin\theta d\theta = \pi Z_0 \left(\frac{kI_{eff}l}{4\pi}\right)^2 \int_0^{\pi} \sin^3\theta d\theta$$

$$P_r = \int_0^{2\pi} d\phi \int_0^{\pi} < S(t,r,\theta) > sin\,\theta\, d\theta$$
$$= \pi Z_0 \left(\frac{kI_{eff}l}{4\pi}\right)^2 \int_0^{\pi} sin^3\,\theta\, d\theta$$

$$P_r = \pi Z_0 \left(\frac{kIl}{4\pi}\right)^2 \left(\frac{2}{3}\right) = \frac{2\pi}{3} Z_0 \left(\frac{kIl}{4\pi}\right)^2 = \frac{2\pi}{3} Z_0 \left(\frac{2\pi Il}{4\pi\lambda}\right)^2$$
$$= \frac{2\pi}{3} Z_0 \left(\frac{Il}{2\lambda}\right)^2$$

$$P_r = \frac{2\pi\mu_0 c l^2}{3\lambda^2}\frac{I_0^2}{2} = \frac{2\pi}{3}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{I_0^2 l^2}{2\lambda^2} = \frac{2\pi}{3}\left(Z_0 I_{eff}^2\right)\frac{l^2}{\lambda^2}\ [W]$$

The radiated power (in the far field region) does not depend on distance; radiation extends to infinity. The radiated power is proportional to $\left(\frac{I_{eff}l}{\lambda}\right)^2$ and is therefore inversely proportional to the wavelength to the square.

In the horizontal plane, the radiated power is omnidirectional.

In the vertical plane, the beamwidth at 3 dB is the angle θ_{-3dB} for which $\dfrac{P_r(\theta_{-3dB},\phi)}{P_{\max}} = \dfrac{1}{2}$ 90°.

$$\theta_{-3dB} = \frac{\pi}{2}$$. This antenna is not directive.

3.5.2.5. *Answer to question 5*

If the antenna has a resistance R, the power dissipated in the antenna is: $P_R = \left(RI_{eff}^2\right)[W]$

The resistance for which the antenna current I would dissipate a power equivalent to the radiated power is known as radiation resistance R_r. This resistance is equal to:

$$R_r = \frac{2\pi}{3} Z_0 \frac{l^2}{\lambda^2} \; [\Omega]$$

3.5.2.6. *Answer to question 6*

The efficiency η of the antenna is expressed as:

$$\eta = \frac{P_r}{P_r + P_R} = \frac{R_r}{R_r + R_R}$$

The gain G is written as:

$$G(\theta,\phi) = \frac{\left(\dfrac{P_A(\theta,\phi)}{\Omega}\right)}{\left(\dfrac{Supplied\,Power}{4\pi}\right)}$$

The directivity G_D is given by:

$$G_D(\theta,\phi) = \frac{\left(\dfrac{P_A(\theta,\phi)}{\Omega}\right)}{\dfrac{Total\ radiated\ power}{4\pi}}$$

$$G_D = \frac{Z_0\left(\dfrac{2\pi l I_{eff}}{4\pi r\lambda}\right)^2 r^2 \sin^2\theta}{\dfrac{2\pi}{3}\left(Z_0 I_{eff}^2\right)\dfrac{l^2}{\lambda^2}\dfrac{1}{4\pi}} = \frac{3}{2}\sin^2\theta$$

It can be noted that for a wire antenna, the following relation holds:

$$G = G_D\, \text{t}$$

3.5.2.7. *Answer to question 7*

In the near field, the predominant terms vary as $\dfrac{1}{r^3}$

$$\vec{E} \cong -jc\frac{\mu_0}{4\pi k} I_0 l\left(\left(\frac{1}{r^3}\right)(2\cos\theta)e^{j(\omega t-kr)}\widehat{r} + \left(\sin\theta\left(\frac{1}{r^3}\right)e^{-j(\omega t-kr)}\right)\widehat{\theta}\right)$$

$$\vec{E} \cong -jc^2\frac{I_0 l}{4\pi\varepsilon_0\omega r^3}\left((2\cos\theta)e^{j(\omega t-kr)}\widehat{r} + \left(\sin\theta e^{-j(\omega t-kr)}\right)\widehat{\theta}\right)$$

$$\vec{E} = -jc\frac{\mu_0}{4\pi k} I_0 l\left(\begin{array}{l}\left(\dfrac{1}{r^3}+\dfrac{jk}{r^2}\right)(2\cos\theta)e^{j(\omega t-kr)}\widehat{r} \\ +\left(\sin\theta\left(\dfrac{1}{r^3}-\dfrac{k^2}{r}+j\dfrac{k}{r^2}\right)e^{-j(\omega t-kr)}\right)\widehat{\theta}\end{array}\right)$$

The dominant term in near field for $\vec{H}(r,t)$ is given by:

$$\vec{H}(r,t) = -\frac{I_0 l\sin\theta}{4\pi}\left(\frac{1}{r^2}\right)e^{j(\omega t-kr)}\widehat{\phi}$$