Exercise 3.5

Q#1:

$$\int \frac{3x+1}{x^2-x-6}\,dx$$

$$\int \frac{3x+1}{x^2-3x+2x-6}\,dx$$

$$\int \frac{3x+1}{x(x-3)+2(x-3)}\,dx$$

$$\int \frac{3x+1}{(x-3)(x+2)}$$

Changing into partial fraction.

$$\frac{3x+1}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)}$$

$$3x+1 = A(x+2) + B(x-3)$$

$$= Ax + 2A + Bx - 3B$$

$$3x+1 = (A+B)x + (2A-3B)$$

By comparison we obtain

$$3 = A+B \qquad\qquad \text{---- i)}$$

$$1 = 2A - 3B \qquad\qquad \text{---- ii)}$$

$2(i) - ii) \Rightarrow$

$$2A + 2B = 6$$
$$2A - 3B = 1$$
$$\underline{\quad+\quad\quad+\quad\quad-\quad}$$

$$5B = 5$$

$$\Rightarrow \quad B = 1$$

Put value of B in i) we obtain

$$\Rightarrow \quad A + 1 = 3$$
$$\Rightarrow \quad A = 3 - 1$$
$$\Rightarrow \quad A = 2$$

So

$$\frac{3x+1}{(x-3)(x+2)} = \frac{2}{(x-3)} + \frac{1}{(x+2)}$$

$$\int \frac{3x+1}{(x-3)(x+2)}\,dx = \int \left(\frac{2}{(x-3)} + \frac{1}{(x+2)} \right) dx$$

$$= \int \frac{2}{(x-3)} \, dx + \int \frac{1}{(x+2)} \, dx$$

$$= 2 \int \frac{1}{(x-3)} \, dx + \int \frac{1}{(x+2)} \, dx$$

$$= 2 \ln|x-3| + \ln|x+2| + C$$

② $\int \frac{5x+8}{(x+3)(2x-1)} \, dx$

Solve as previous question.

③ $\int \frac{x^2 + 3x - 34}{x^2 + 2x - 15} \, dx$

$$\int \left(1 + \frac{(x-19)}{(x^2 + 2x - 15)} \right) dx$$

$$\begin{array}{r} 1 \\ x^2+2x-15 \overline{\smash{\big)}\, x^2+3x-34} \\ \underline{x^2 \pm 2x \mp 15} \\ x - 19 \end{array}$$

$$\int 1 \, dx + \int \frac{(x-19)}{(x^2+2x-15)} \, dx$$

$$x + \int \frac{(x-19)}{x^2+5x-3x-15} \, dx$$

$$x + \int \frac{(x-19)}{x(x+5)-3(x+5)} \, dx$$

$$x + \int \frac{(x-19)}{(x+5)(x-3)} \, dx \qquad\qquad ―― \text{i)}$$

Now solve $\int \frac{(x-19)}{(x+5)(x-3)} \, dx$

as previous question and then obtained answer write in i)

(4) $\int \dfrac{(a-b)\,x}{(x-a)(x-b)}\,dx$ $(a>b)$

Solve this question as question #1 is solved.

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(5) $\int \dfrac{3-x}{1-x-6x^2}\,dx$

$\int \dfrac{3-x}{-6x^2-x+1}\,dx$

$\int \dfrac{3-x}{-6x^2-3x+2x+1}$

$\int \dfrac{3-x}{-3x(2x+1)+1(2x+1)}\,dx$

$\int \dfrac{3-x}{(2x+1)(1-3x)}\,dx$

Now solve it according to previous question.

(6) $\int \dfrac{2x}{x^2-a^2}\,dx$ $(x>a)$

$\int \dfrac{2x}{(x+a)(x-a)}\,dx$

Solve this question according to previous question.

(7) $\int \dfrac{1}{6x^2+5x-4}\,dx$

$\int \dfrac{1}{6x^2+8x-3x-4}\,dx$

$\int \dfrac{1}{2x(3x+4)-1(3x+4)}\,dx$

$\int \dfrac{1}{(3x+4)(2x-1)}\,dx$

Solve this as previous questions are solved.

⑧ $\int \dfrac{2x^3 - 3x^2 - x - 7}{2x^2 - 3x - 2}\, dx$

$$\begin{array}{r} x \\ 2x^2 - 3x - 2 \,\overline{\big)\, 2x^3 - 3x^2 - x - 7} \\ \underline{2x^3 \mp 3x^2 \mp 2x} \\ x - 7 \end{array}$$

$\int \left[x + \dfrac{(x-7)}{(2x^2 - 3x - 2)} \right] dx$

$\int x\, dx + \int \dfrac{(x-7)}{(2x^2 - 3x - 2)}\, dx$

$\dfrac{x^2}{2} + \int \dfrac{(x-7)}{2x^2 - 4x + x - 2}\, dx$

$\dfrac{x^2}{2} + \int \dfrac{(x-7)}{2x(x-2) + 1(x-2)}\, dx$

$\dfrac{x^2}{2} + \int \dfrac{(x-7)}{(x-2)(2x+1)}\, dx \quad\quad \longleftarrow \text{i)}$

Solve $\int \dfrac{(x-7)}{(x-2)(2x+1)}\, dx$ according to

Previous question and put answer in i)

Here we introduce another method to solve question. choice is yours.

⑨ $\int \dfrac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}\, dx$

changing into partial fraction

$\dfrac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-3)}$

$3x^2 - 12x + 11 = A(x-2)(x-3) + B(x-1)(x-3) + c(x-1)(x-2) \quad -\text{i}$

Put $x = 1$ in i)

$3 - 12 + 11 = A(1-2)(1-3) + B(1-1)(1-3) + c(1-1)(1-2)$

$-9 + 11 = A(-1)(-2)$

$$2 = 2A$$
$$1 = A$$
$$\Rightarrow A = 1$$

Now Put $x = 2$ in i)

$$3(2)^2 - 12(2) + 11 = B(2-1)(2-3)$$
$$3(4) - 12(2) + 11 = B(1)(-1)$$
$$12 - 24 + 11 = -B$$
$$-1 = -B$$
$$1 = B$$
$$\Rightarrow B = 1$$

Now Put $x = 3$ in i)

$$3(3)^2 - 12(3) + 11 = C(3-1)(3-2)$$
$$3(9) - 12(3) + 11 = C(2)(1)$$
$$27 - 36 + 11 = 2C$$
$$2 = 2C$$
$$\Rightarrow C = 1$$

So

$$\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} + \frac{1}{(x-2)} + \frac{1}{(x-3)}$$

$$\int \frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} dx = \int \left[\frac{1}{(x-1)} + \frac{1}{(x+2)} + \frac{1}{(x-3)} \right] dx$$

$$= \int \frac{1}{(x-1)} dx + \int \frac{1}{(x+2)} dx + \int \frac{1}{(x-3)} dx$$

$$= \ln|x-1| + \ln|x+2| + \ln|x-3| + C$$

(10) $\int \dfrac{2x-1}{x(x-1)(x-3)} dx$

Solve according to previous question.

⑪ $\int \dfrac{5x^2 + 9x + 6}{(x^2-1)(2x+3)} \, dx$

$\int \dfrac{5x^2 + 9x + 6}{(x+1)(x-1)(2x+3)} \, dx$

Solve according to previous question.

⑫ $\int \dfrac{4+7x}{(1+x)^2(2+3x)} \, dx$

changing into partial fraction

$$\dfrac{4+7x}{(1+x)^2(2+3x)} = \dfrac{A}{(2+3x)} + \dfrac{B}{(1+x)} + \dfrac{C}{(1+x)^2}$$

multiply by $(1+x)^2(2+3x)$ in both sides
we obtain

$$4 + 7x = A(1+x)^2 + B(1+x) + C(2+3x) \quad\text{——i)}$$
$$\text{Put} \quad x = -\dfrac{2}{3} \quad \text{in i)}$$

$$4 + 7\left(-\dfrac{2}{3}\right) = A\left(1 - \dfrac{2}{3}\right)^2$$
$$4 - \dfrac{14}{3} = A\left(\dfrac{1}{3}\right)^2$$

$$-\dfrac{2}{3} = A\left(\dfrac{1}{9}\right)$$
$$A = -\dfrac{2}{3} \times 9$$
$$A = -6$$

Now

$$\text{Put} \quad x = -1 \quad \text{in} \quad \text{i)}$$
$$4 + 7(-1) = C(2 + 3(-1))$$
$$4 - 7 = C(2 - 3)$$
$$-3 = -C$$
$$3 = C$$
$$\Rightarrow \quad C = 3$$

Equating the Coefficient of x^2 on both
sides of i)

$$A + 3B = 0 \qquad \text{where} \quad A = -6$$
$$-6 + 3B = 0$$
$$3B = 6$$
$$B = 2$$

So

$$\frac{4+7x}{(2+3x)(1+x^2)} = \frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}$$

$$\int \frac{4+7x}{(2+3x)(1+x)^2} dx = \int \left[\frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2} \right] dx$$

$$= \int \frac{-6}{2+3x} dx + \int \frac{2}{1+x} dx + \int \frac{3}{(1+x)^2} dx$$

$$= -2 \int \frac{3}{2+3x} dx + 2 \int \frac{1}{1+x} dx + 3 \int (1+x)^{-2} dx$$

$$= -2 \ln|2+3x| + 2 \ln|1+x| + 3 \frac{(1+x)^{-2+1}}{-2+1} + C$$

$$= -2 \ln|2+3x| + 2 \ln|1+x| - \frac{3}{(1+x)} + C$$

⑬ $\int \dfrac{2x^2}{(x-1)(x+1)} dx$

Solve as question # 12 solved –

⑭ $\int \dfrac{1}{(x-1)(x+1)^2} dx$

Solve as question # 12 is solved.

⑮ $\int \dfrac{x+4}{x^3-3x^2+4} dx$

Factorizing x^3-3x^2+4 by synthetic division

$$= \begin{array}{r|rrrr} & 1 & -3 & 0 & 4 \\ -1 & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$(x+1)(x^2-4x+4)$

$(x+1)(x-2)^2$

it can be written as

$$\int \frac{x+4}{(x^3-3x^2+4)} dx = \int \frac{x+4}{(x+1)(x-2)^2} dx$$

Now solve it by previous method.

(16) $\int \frac{x^3 - 6x^2 + 25}{(x+1)^2 (x-2)^2} dx$

this question is also solved by previous method.

(17) $\int \frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} dx$

changing into partial fraction

$$\frac{x^3 + 22x^2 + 14x - 17}{(x-3)(x+2)^3} = \frac{A}{(x-3)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

Now solve it by previous method.

(18) $\int \frac{x-2}{(x+1)(x^2+1)} dx$

changing into partial fraction

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)}$$

$$x-2 = A(x^2+1) + (Bx+C)(x+1) \quad \text{---- i)}$$

Put $x = -1$ in i)

$$-1-2 = A[(1)^2 + 1]$$
$$-3 = A(1+1)$$
$$-3 = 2A$$
$$\Rightarrow A = -\frac{3}{2}$$

From equation i)

$$x-2 = A(x^2+1) + (Bx+C)(x+1)$$

$$= A(x^2+1) + Bx^2 + Bx + Cx + C$$

Equating coefficient of like powers of x

$\Rightarrow \qquad A+B = 0 \qquad \text{—— I}$

$\qquad\qquad B+C = 1 \qquad\text{——— II}$

Put $A = -\dfrac{3}{2}$ in I

$\qquad -\dfrac{3}{2} + B = 0$

$\Rightarrow \quad B = \dfrac{3}{2}$

Put the value of B in II

$\qquad \dfrac{3}{2} + C = 1$

$\qquad C = 1 - \dfrac{3}{2}$

$\qquad C = -\dfrac{1}{2}$

So

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{-\dfrac{3}{2}}{x+1} + \frac{\dfrac{3}{2}x - \dfrac{1}{2}}{x^2+1}$$

$$= -\frac{3}{2(x+1)} + \frac{(3x-1)}{2(x^2+1)}$$

$$\int \frac{x-2}{(x+1)(x^2+1)}\, dx = \int \left[\frac{-3}{2(x+1)} + \frac{(3x-1)}{2(x^2+1)} \right] dx$$

$$= \int \frac{-3}{2(x+1)}\, dx + \int \frac{(3x-1)}{2(x^2+1)}\, dx$$

$$= -\frac{3}{2} \int \frac{1}{(x+1)}\, dx + \frac{1}{4}\int \frac{2(3x-1)}{(x^2+1)}\, dx$$

$$= -\frac{3}{2} \ln |x+1| + \frac{1}{4} \int \frac{(6x-2)}{(x^2+1)} \, dx$$

$$= -\frac{3}{2} \ln |x+1| + \frac{1}{4} \left[\int \frac{6x}{(x^2+1)} \, dx - \int \frac{2}{(x^2+1)} \, dx \right]$$

$$= -\frac{3}{2} \ln |x+1| + \frac{1}{4} \left[3 \int \frac{2x}{(x^2+1)} \, dx - 2 \int \frac{1}{(x^2+1)} \, dx \right]$$

$$= -\frac{3}{2} \ln |x+1| + \frac{1}{4} \left[3 \ln |x^2+1| - 2 \tan^{-1} x + C \right]$$

$$= -\frac{3}{2} \ln |x+1| + \frac{3}{4} \ln |x^2+1| - \frac{2}{4} \tan^{-1} x + \frac{1}{4} C$$

$$= -\frac{3}{2} \ln |x+1| + \frac{3}{4} \ln |x^2+1| - \frac{1}{2} \tan^{-1} x + C'$$

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(19) $\displaystyle \int \frac{x}{(x-1)(x^2+1)} \, dx$ Solve from

Solved question.

(20) $\displaystyle \int \frac{9x-7}{(x+3)(x^2+1)} \, dx$

Solve this question according to the previous questions are solved.

(21) $\displaystyle \int \frac{1+4x}{(x-3)(x^2+4)} \, dx$

Solve it according to previous questions.

(22) $\quad\displaystyle\int \frac{12}{x^3+8}\, dx$

$$\int \frac{12}{(x)^3+(2)^3}\, dx$$

$$\int \frac{12}{(x+2)(x^2-2x+4)}\, dx$$

change into partial fraction.

$$\frac{12}{(x+2)(x^2-2x+4)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2-2x+4)}$$

multiply by $(x+2)(x^2-2x+4)$ in both sides.

$$12 = A(x^2-2x+4) + (Bx+c)(x+2) \;\;—i)$$

Put $\quad x=-2 \quad$ in i)

$$12 = A[(-2)^2 - 2(-2) + 4]$$

$$12 = A\,12$$

$$1 = A$$

$$\Rightarrow A = 1$$

Equating coefficient of like power of x from i)

$$A + B = 0 \qquad\qquad —ii)$$

$$-2A + 2B + C = 0 \;\;— iii)$$

$$\text{Put} \quad A = 1$$

ii) $\Rightarrow \qquad 1 + B = 0$

$\qquad \Rightarrow \qquad B = -1$

iii) $\Rightarrow \qquad -2(1) + 2(-1) + C = 0$

$$-2 - 2 + C = 0$$

$$C = 4$$

So

$$\frac{12}{(x+2)(x^2-2x+4)} = \frac{1}{(x+2)} + \frac{(-x+4)}{x^2-2x+4}$$

$$= \frac{1}{(x+2)} - \frac{(x-4)}{x^2-2x+4}$$

$$\int \frac{12}{(x+2)(x^2-2x+4)}\,dx = \int\left[\frac{1}{x+2} - \frac{(x-4)}{x^2-2x+4}\right]dx$$

$$= \int\frac{1}{x+2}\,dx - \int\frac{(x-4)}{x^2-2x+4}\,dx$$

$$= \ln|x+2| - \frac{1}{2}\int\frac{2(x-4)}{x^2-2x+4}\,dx$$

$$= \ln|x+2| - \frac{1}{2}\left[\int\frac{2x-8}{x^2-4x+4}\right]dx$$

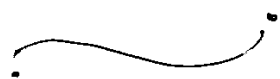$$= \ln|x+2| - \frac{1}{2}\left[\int\frac{2x-2-6}{x^2-2x+4}\,dx\right]$$

$$= \ln|x+2| - \frac{1}{2}\left[\int\frac{(2x-2)}{x^2-2x+4}\,dx - \int\frac{6}{x^2-2x+4}\,dx\right]$$

$$= \ln|x+2| - \frac{1}{2}\left[\ln|x^2-2x+4| - 6\int\frac{1}{(x-2)^2}\,dx\right]$$

$$= \ln|x+2| - \frac{1}{2}\left[\ln|x^2-2x+4| + 6\frac{1}{(x-2)} + C\right]$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \frac{6}{2(x-2)} + \frac{C}{2}$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \frac{3}{(x-2)} + C'$$

(23) $\displaystyle\int \frac{9x+6}{x^3-8}\, dx$

$\displaystyle\int \frac{9x+6}{(x)^3-(2)^3}$

$\displaystyle\int \frac{9x+6}{(x-2)(x^2+2x+4)}\, dx$

Solve according previous question.

(24) $\displaystyle\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}\, dx$

change into partial fraction

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+4)}$$

Now solve it according to previous experience.

(25) $\displaystyle\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}\, dx$

Solve this question according to previous experience.

㉖ $\displaystyle\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}\, dx$

change into partial fraction.

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{(Ax+B)}{(4x^2+1)} + \frac{(Cx+D)}{(x^2-x+1)}$$

multiply by $(4x^2+1)(x^2-x+1)$ in both sides.

$3x+1 = (Ax+B)(x^2-x+1) + (Cx+D)(4x^2+1)$

$= Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + 4Cx^3 + Cx + 4Dx^2 + D$

$= (A+4C)x^3 + (-A+B+4D)x^2 + (A-B+C)x + B+D$

Equating coefficient of like powers of x

$A + 4C = 0$ ——— I

$-A + B + 4D = 0$ ——— II

$A - B + C = 3$ ——— III

$B + D = 1$ ——— IV

adding II and III

$4D + C = 3$

$\Rightarrow \quad C = 3 - 4D$ ——— V

i) $\Rightarrow \quad A = -4C$

$\Rightarrow \quad A = -4(3-4D)$

$A = -12 + 16D$ ——— VI

IV $\Rightarrow \quad B = 1 - D$ ——— VII

Put value of A, B & C in III

$$-12 + 16D - 1 + D + 3 - 4D = 3$$

$$13D - 10 = 3$$

$$13D = 13$$

$$D = 1$$

$$\text{VII} \implies \quad B = 1 - D$$

$$B = 1 - 1$$

$$B = 0$$

$$\text{V} \implies \quad C = 3 - 4D$$

$$C = 3 - 4(1)$$

$$= 3 - 4$$

$$C = -1$$

$$\text{VI} \implies A = -12 + 16D$$

$$= -12 + 16(1)$$

$$= -12 + 16$$

$$A = 4$$

So

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{4x+0}{4x^2+1} + \frac{(-x+1)}{x^2-x+1}$$

$$= \frac{4x}{4x^2+1} - \frac{(x-1)}{(x^2-x+1)}$$

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}\,dx = \int \left[\frac{4x}{4x^2+1} - \frac{(x-1)}{(x^2-x+1)} \right] dx$$

$$= \int \frac{4x}{4x^2+1} \, dx - \int \frac{(x-1)}{x^2-x+1} \, dx$$

$$= \frac{1}{2} \int \frac{8x}{4x^2+1} \, dx - \frac{1}{2} \int \frac{2(x-1)}{x^2-x+1} \, dx$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \left[\int \frac{2x-2}{x^2-x+1} \, dx \right]$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \left[\int \frac{(2x-1)-1}{x^2-x+1} \, dx \right]$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \left[\int \frac{(2x-1)}{x^2-x+1} \, dx - \int \frac{1}{x^2-x+1} \, dx \right]$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \left[\ln |x^2-x+1| - \int \frac{1}{x^2-x+1+\frac{1}{4}-\frac{1}{4}} \, dx \right]$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \left[\ln |x^2-x+1| - \int \frac{1}{(x^2-x+\frac{1}{4})+(1-\frac{1}{4})} \, dx \right]$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \left[\ln |x^2-x+1| - \int \frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \, dx \right]$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \left[\ln |x^2-x+1| - \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{(x-\frac{1}{2})}{\frac{\sqrt{3}}{2}} \right]$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \left[\ln |x^2-x+1| - \frac{2}{\sqrt{3}} \tan^{-1} \frac{(2x-1)}{\sqrt{3}} \right]$$

$$= \frac{1}{2} \ln |4x^2+1| - \frac{1}{2} \ln |x^2-x+1| + \frac{1}{\sqrt{3}} \tan^{-1} \frac{(2x-1)}{\sqrt{3}}$$

(27) $\qquad \displaystyle\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}\, dx$

Solve this according to previous question.

(28) $\qquad \displaystyle\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}\, dx$

Solve this question as question # 26 is solved.

(29) $\qquad \displaystyle\int \cdot \frac{-2}{\llcorner 1}\, dx$

$\qquad \displaystyle\int \frac{\wedge^{\ 2} -}{(x\qquad \cdot)}\, \frac{}{}\, dx$

Now solve   . sum by previous sum.

(30) $\qquad \displaystyle\int \frac{3x-}{(x^2-x+2)(\quad +x+2)}\, dx$

Solve this sum with the help of previous sum.

(31) $\qquad \displaystyle\int \frac{3x^3 \cdot \ ^2 + 9x + 5}{(x^2 + \ldots +1)(x^2+2x+3)}\, dx$

Solve this sum with the help of
Previous sum.   www. mathcity.org