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# Preliminaries

Our main goal is to introduce notation and terminology so that this book is self-contained.[1]

We assume you are familiar and comfortable with basic concepts about sets (e.g.: subsets, union, Cartesian product, cardinality, equivalence classes, quotients, etc.), functions (e.g.: injectivity, surjectivity, inverses, (pre)image, etc.), logic (e.g.: quantifiers, implication) and proofs (e.g.: you can write, read and understand proofs),[2] and we will not recall anything here. However, we need to have a little talk about foundations.

Several times in our coverage of category theory, we will use the term **collection** in order to avoid set-theoretical paradoxes. Collections are supposed to behave just like sets except that we will never consider collections containing other collections. We do not make it more formal because there are many ways to do it[3] and none of them are relevant to this course. However, you still need to know why we cannot use sets as is usual in all other courses.

In short, there exist collections of objects that cannot be sets.[4] In our case, we will need to talk about the collection of all sets and the collection of all groups (among others) and they cannot form sets. For the former, it is easy to see because if $S$ is the set of all sets, then it contains all its subsets and hence $\mathcal{P}(S) \subseteq S$, this leads to the contradiction $|\mathcal{P}(S)| \leq |S| < |\mathcal{P}(S)|$.

In the rest of this chapter, we cover the necessary background that we will use in the rest of the book. It is supposed to be a quick and (unfortunately) dry overview of stuff you may or may not have seen, so we will not dwell on explanations, intuitions and motivations.[5] You can safely skip these sections and come back whenever you click on a word or symbol that is defined here. We hope that this will save you from several trips to Wikipedia.

[1] Especially with the heavy use of the `knowledge` package, I felt it was necessary to cover enough background material in order to have the least amount of external links in the book.

[2] The very first things usually taught in early undergraduate mathematics courses.

[3] Most commonly, people use classes or Grothendieck universes. If this sticky point worries you, we suggest you keep it in the back of your mind and go read `https://arxiv.org/pdf/0810.1279.pdf` when you are a bit more comfortable with category theory.

[4] Famous examples include the collection of ordinal numbers which, by the Burali–Forti paradox, cannot be a set and the collection of all sets that do not contain themselves which, by the Russel paradox, cannot be a set.

[5] Contrarily to the other chapters of this book.

## Abstract Algebra

Here we recall definitions, examples and results you may have seen in classes on abstract algebra or linear algebra.[6]

[6] Monoids are not commonly covered, but they are simpler than groups and we need them at one point so we present them here.

## *Monoids*

**Definition 1** (Monoid)**.** A **monoid** is set $M$ equipped with a binary operation $\cdot : M \times M \to M$ called **multiplication** and an *identity* element[7] $1_M$ satisfying for all $x, y, z \in M$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \text{and} \quad 1_M \cdot x = x = x \cdot 1_M.$$

If it satisfies $\forall x, y \in M, x \cdot y = y \cdot x$, $M$ is a **commutative monoid**.

[7] Some authors call $1_M$ the **unity** or the **neutral** element.

Depending on the context, we will refer to a monoid either as $M$ or $(M, \cdot)$ or $(M, \cdot, 1_M)$.

*Remark* 2. We will quickly drop the $\cdot$ symbol and denote multiplication with plain juxtaposition (i.e.: $xy := x \cdot y$) for monoids and other algebraic structures with a multiplication.

**Examples 3.** 1. For any set $S$, the set of function from $S$ to itself form a monoid with the multiplication being composition and the identity being the identity map $s \mapsto s$.

2. The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ equipped with the operation of addition are all commutative monoids.

3. For any set $S$, the powerset $\mathcal{P}(S)$ has two simple monoid structures: one where the multiplication is $\cup$ and the identity if $\emptyset \subseteq S$ and the other where multiplication is $\cap$ and the identity is $S \subseteq S$.

**Definition 4** (Homomorphism)**.** Let $M$ and $N$ be two monoids, a **monoid homomorphism** from $M$ to $N$ is a function $f : M \to N$ satisfying the following property:

$$f(1_M) = 1_N \quad \text{and} \quad \forall x, y \in M, f(xy) = f(x)f(y).$$

When $f$ is a bijection, we call it a **monoid isomorphism**, say that $M$ and $N$ are **isomorphic** and denote $M \cong N$.

**Definition 5** (Submonoid)**.** Given a monoid $M$, a **submonoid** of $M$ is a subset $N \subseteq M$ containing $1_M$ that is closed under multiplication (i.e.: $\forall x, y \in N, x \cdot y \in N$).[8]

[8] This implies $N$ is also a monoid with the multiplication and identity inherited from $M$.

**Definition 6** (Kernel)**.** The **kernel** of a homomorphism $f : M \to N$ is the preimage of $1_N$: $\ker(f) := f^{-1}(1_N)$. For any homomorphism $f$, $\ker(f)$ is a submonoid of $M$.[9]

[9] Similarly, the image of a homomorphism is also a submonoid.

**Example 7.** The inclusions $(\mathbb{N}, +) \to (\mathbb{Z}, +) \to (\mathbb{Q}, +) \to (\mathbb{R}, +)$ are all monoid homomorphisms with trivial kernel.[10] This implies this is also a chain of inclusions as submonoids.

[10] i.e.: the kernel only contains the identity.

**Definition 8** (Monoid action)**.** Let $M$ be a monoid and $S$ a set, an (left) **action** of $M$ on $S$ is an operation $\star : M \times S \to S$ satisfying for all $x, y \in M$ and $s \in S$

$$(x \cdot y) \star s = x \star (y \star s) \quad \text{and} \quad 1_M \star s = s.$$

The data $(M, S, \star)$ will also be called an $M$**–set** and we may refer to it abusively with $S$.

Any monoid action has a **permutation representation** defined to be the map

$$\sigma_\star : M \to \Sigma_S = x \mapsto (s \mapsto x \star s).$$

Conversely, a map $\sigma : M \to \Sigma_S$ that satisfies $\sigma(1_M) = \mathrm{id}_S$ and $\sigma(xy) = \sigma(x) \circ \sigma(y)$ for any $x, y \in M$ gives rise to a monoid action $\star_\sigma$ defined by $x \star_\sigma s = \sigma(x)(s)$.[11]

[11] These are inverse operations, i.e.:

$$\sigma_{\star_\sigma} = \sigma \quad \text{and} \quad \star_{\sigma_\star} = \star.$$

**Example 9.** Any monoid $M$ has a canonical left action on itself defined by $x \star m = xm$ for all $x, m \in M$.

### Groups

**Definition 10** (Group)**.** A **group** is set $G$ equipped with a binary operation $\cdot : G \times G \to G$ called **multiplication**, an **inverse** operation $(-)^{-1} : G \to G$ and an **identity** element $1_G$ such that $(G, \cdot, 1_G)$ is a monoid and for all $x \in G$

$$x \cdot x^{-1} = 1_G = x^{-1} \cdot x.$$

If $(G, \cdot, 1_G)$ is a commutative monoid, we say that $G$ is an **abelian group**.

**Examples 11.** 1. For any set $S$, the set of bijections from $S$ to itself form a group with the multiplication being composition, the inverse being the set-theoretical inverse and the identity being the identity map $s \mapsto s$. We denote this group $\mathbf{\Sigma}_S$ and call it the group of **permutations** of $S$.[12]

2. The monoids on $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$ and $(\mathbb{R}, +)$ are also abelian groups with the inverse of $x$ being $-x$.

3.

[12] For $n \in \mathbb{N}$, we denote $\Sigma_n$ the group of permutations of $\{1, \ldots, n\}$.

**Definition 12** (Homomorphism)**.** Let $G$ and $H$ be two groups, a **group homomorphism** from $G$ to $H$ is a monoid homomorphism $f : G \to H$. It follows that[13]

$$\forall x \in G, f(x^{-1}) = f(x)^{-1}.$$

When $f$ is a bijection, we call it a **group isomorphism**, say that $G$ and $H$ are **isomorphic** and denote $G \cong H$.

[13] For this, you need to show that inverses are unique.

**Definition 13** (Subgroup)**.** Given a group $G$, a **subgroup** of $G$ is a submonoid $H$ of $G$ closed under taking inverses (i.e.: $\forall x \in H, x^{-1} \in H$).[14]

[14] This implies $H$ is also a group with the multiplication, inverse and identity inherited from $G$.

**Definition 14** (Quotient)**.** Let $G$ be a group and $H$ a subgroup of $G$, the *quotient* $G/H$ is the group whose elements are equivalence class of

**Definition 15** (Kernel)**.** The **kernel** of a homomorphism $f : G \to H$ is the preimage of $1_H$: $\ker(f) := f^{-1}(1_H)$. For any homomorphism $f$, $\ker(f)$ is a subgroup of $G$.[15]

[15] Similarly, the image of a homomorphism is also a subgroup.

**Definition 16** (Group action)**.** Let $G$ be a group and $S$ a set, an (left) **action** of $G$ on $S$ is a (left) monoid action of $G$ on $S$. A set $S$ equipped with action of $G$ is called a $G$**–set**.

**Example 17.** Any group $G$ has a canonical left action on itself defined by $x \star m = xm$ for all $x, m \in G$.

### Rings

### Fields

### Vector Spaces

## Order Theory

In this section, we briefly cover some early definitions and results from order theory. Since this subject is not usually taught in undergraduate courses, we spend a bit more time. In fact, we even introduce stuff we will not use later to make sure readers can get more familiar with the most important objects: posets and monotone functions.

**Definition 18** (Poset)**.** A **poset** (short for partially ordered set) is a pair $(A, \leq)$ comprising a set $A$ and a binary relation $\leq \subseteq A \times A$ that is

1. **reflexive** ($\forall x \in A, x \leq x$),
2. **transitive** ($\forall x, y, z \in A$ if $x \leq y$ and $y \leq z$ then $x \leq z$), and
3. **antisymmetric** ($\forall x, y \in A$ if $x \leq y$ and $y \leq x$ the $x = y$).

The relation is also called a **partial order**.[16]

[16] If antisymmetry is not satisfied, $\leq$ is called a **preorder**.

**Examples 19.** 1. The usual non-strict orders ($\leq$ and $\geq$) on $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are all partial orders. The strict orders do not satisfy reflexivity.

2. The divisibility relation $|$ on $\mathbb{N}$ satisfying $n \mid m$ whenever $n$ divides $m$ is a partial order.
3. For any set $S$, the powerset of $S$ $\mathcal{P}(S)$ is a poset when equipped with the $\subseteq$ relation.
4. Any subset of a poset inherits a poset structure by restricting the partial order.

For any monoid $M$, there are three preorders defined by the so-called Green's relations:

$$\forall x, y \in M x \leq_L y \Leftrightarrow \exists m \in M, x = my$$
$$\forall x, y \in M x \leq_R y \Leftrightarrow \exists m \in M, x = ym$$
$$\forall x, y \in M x \leq_J y \Leftrightarrow \exists m, m' \in M, x = mym'$$

**Definition 20** (Monotone)**.** A function $f : (A, \leq_A) \to (B, \leq_B)$ between posets is **monotone** (or **order-preserving**) if for any $a, a' \in A$, $a \leq a' \implies f(a) \leq f(a')$.

**Example 21.** You probably already know lots of monotone functions, but let us give two less intuitive examples. Let $f : S \to T$ be a function, the *image* map of $f$[17] is the function $\mathcal{P}(S) \to \mathcal{P}(T)$ defined by $S \supseteq X \mapsto f(X) := \{f(x) \mid x \in X\}$. When both powersets are equipped with the inclusion partial order, the image map is monotone because $X \subseteq X' \subseteq S$ implies $f(X) \subseteq f(X')$.

[17] Which we abusively denote $f$.

The *preimage* map is

$$f^{-1} : \mathcal{P}(T) \to \mathcal{P}(S) = T \supseteq Y \mapsto f^{-1}(Y) := \{y \in S \mid f(y) \in Y\}.$$

It is also order-preserving because $Y \subseteq Y' \subseteq T$ implies $f^{-1}(Y) \subseteq f^{-1}(Y')$.

**Fact 22.** *The composition of monotone functions between posets is monotone.*

**Definition 23** (Dual)**.** The **dual order**[18] of a poset $(A, \leq)$, denoted $(A, \leq)^{\mathrm{op}}$, is the same set equipped with the converse relation $\geq$ defined by

[18] This definition lets us avoid many symmetric arguments.

$$\forall x, y \in A, x \geq y \Leftrightarrow y \leq x.$$

**Definition 24.** Let $(A, \leq)$ be a poset and $S \subseteq A$, then $a \in A$ is an **upper bound** of $S$ if $\forall s \in S, s \leq a$. Moreover, $a \in A$ is a **supremum** of $S$, if it is a least upper bound, that is, $a$ is an upper bound of $S$ and for any upper bound $a'$ of $S$, $a \leq a'$. A supremum of $S$ is denoted $\vee S$, but when $S$ contains only two elements, we use the infix notation $s_1 \vee s_2$ and call this a **join**.

A **lower bound** (resp. **infimum/meet**) of $S$ is an upper bound (resp. supremum/join) of $S$ in the dual order $(A, \leq)^{\mathrm{op}}$.[19] An infimum of $S$ is denoted $\wedge S$ or $s_1 \wedge s_2$ in the binary case.

[19] Explicityly, $a \in A$ is a lower bound of $S$ if $\forall s \in S, a \leq s$. It is an infimum of $S$ if, in addition to being a lower bound of $S$, any lower bound $a'$ of $S$ satisfies $a' \leq a$.

**Proposition 25.** *Infimums and supremums are unique when they exist.*[20]

[20] This holds by antisymmetry.

**Definition 26.** A **complete lattice** comprises the data $(L, \wedge, \vee, \leq)$ where $(L, \leq)$ is a poset, and $\wedge, \vee : (\mathcal{P}(L), \subseteq) \to (L, \leq)$ are respectively infimum and supremum as defined above.[21] Observe that $L$ has a smallest element $\vee\emptyset$ and a largest element $\wedge\emptyset$ (they are usually called **top** and **bottom** respectively).

[21] Notice that, by definition, these are monotone maps when the domain $\mathcal{P}(L)$ is equipped with the inclusion order. Moreover, if these functions are defined on all of $\mathcal{P}(L)$, all supremums and infimums exist in $(L, \leq)$.

**Examples 27.** 1. For any set $S$, $(\mathcal{P}(S), \subseteq)$ is a complete lattice: the supremum of a family of subsets is their union and the infimum is their intersection.

2. Defining supremums and infimums on the poset $(\mathbb{N}, |)$ is subtle. When $S \subseteq \mathbb{N}$ is non-empty, $\wedge S$ is the greatest common divisor of all elements in $S$ and $\wedge\emptyset$ is $0$ because any integer divides $0$. For a finite and non-empty $S \subseteq \mathbb{N}$, $\vee S$ is the least common multiple of all elements in $S$. If $S$ is infinite, then $\vee S$ is $0$ and the supremum of the empty set is $1$ because $1$ divides any integer.

You might be wondering about possible posets where all infimums exist but not necessarily all supremums or vice-versa, it turns out that this is not possible as shown below.

**Lemma 28.** *Let $(L, \leq)$ be a poset, then the following are equivalent:*

*(i)* $(L, \wedge, \vee, \leq)$ *is a complete lattice.*

*(ii)* *Any $S \subseteq L$ has a supremum.*

*(iii)* *Any $S \subseteq L$ has an infimum.*

*Proof.* (i) $\implies$ (ii), (i) $\implies$ (iii) and (ii) + (iii) $\implies$ (i) are all trivial. Also, by using duality, we only need to prove (ii) $\implies$ (iii). For that, it suffices to note that for any $S \subseteq L$, $\wedge S = \bigvee\{a \in L \mid \forall s \in S, a \leq s\}$ is a suitable definition of the infimum.

Defined that way, $\wedge S$ is a lower bound of $S$ because if $s < \wedge S$, then $s < a$ for some lower bound $a$ of $S$[22], in particular $s \notin S$. Additionally, since we are taking the supremum over all lower bounds of $S$, no lower bound of $S$ can be greater and we conclude that $\wedge S$ is indeed the infimum of $S$. □

[22] Because $\wedge S$ was the least upper bound for lower bounds of $S$.

**Definition 29** (Fixpoints). Let $f : (L, \leq) \to (L, \leq)$, a **pre-fixpoint** of $L$ is an element $x \in L$ such that $f(x) \leq x$. A **post-fixpoint** is an element $x \in L$ such that $x \leq f(x)$. A **fixpoint** (or **fixed point**) of $f$ is a pre- and post-fixpoint.

**Theorem 30** (Knaester-Tarski). [23] *Let $(L, \wedge, \vee, \leq)$ be a complete lattice and $f : L \to L$ be monotone, then*

1. *The least fixpoint of $f$ is $\mu f := \wedge\{a \in L \mid f(a) \leq a\}$.*

2. *The greatest fixpoint of $f$ is $\nu f := \vee\{a \in L \mid a \leq f(a)\}$.*

[23] This is actually a weaker version of the Knaester-Tarski theorem which states that the fixpoints of a monotone $f$ form a complete lattice.

*Proof.* 1. Any fixpoint of $f$ is in particular a pre-fixpoint, thus $\mu f$, being a lower bound of all pre-fixpoints, is smaller than all fixpoints. Moreover, because for any pre-fixpoint $a \in L$, $f(\mu f) \leq f(a) \leq a$, $f(\mu f)$ is also a lower bound of the pre-fixpoints, so $f(\mu f) \leq \mu f$. We infer that $f(f(\mu f)) \leq f(\mu f)$, so $f(\mu f)$ is a pre-fixpoint and $\mu f \leq f(\mu f)$. We conclude that $\mu f$ is a fixpoint by antisymmetry.

2. Any fixpoint of $f$ is in particular a post-fixpoint, thus $\nu f$, being an upper bound of post-fixpoints, is bigger than all fixpoints. Moreover, because for any post-fixpoint $a \in L$, $a \leq f(a) \leq f(\nu f)$, $f(\nu f)$ is an upper bound of the post-fixpoints, so $\nu f \leq f(\nu f)$. We infer that $f(\nu f) \leq f(f(\nu f))$, so $f(\nu f)$ is a post-fixpoint and $f(\nu f) \leq \nu f$. We conclude that $\nu f$ is a fixpoint by antisymmetry.

The proof of the second item is the proof of the first item done in the dual order.

□

**Definition 31.** Let $(A, \leq)$ be a poset, a **closure operator** on $A$ is a map $c : A \to A$ that is

1. monotone,
2. **extensive** ($\forall x \in A, x \leq c(x)$), and
3. **idempotent** ($\forall x \in A, c(x) = c(c(x))$).[24]

[24] We will use this definition of idempotence in other contexts.

**Example 32.** The floor ($\lfloor - \rfloor$) and ceiling ($\lceil - \rceil$) operations are closure operators on $(\mathbb{R}, \geq)$ and $(\mathbb{R}, \leq)$ respectively.

**Definition 33.** Given two posets $(A, \leq)$ and $(B, \sqsubseteq)$, a **Galois connection** is a pair of monotone functions $l : A \to B$ and $r : B \to A$ such that for any $a \in A$ and $b \in B$,

$$l(a) \sqsubseteq b \Leftrightarrow a \leq r(b).$$

For such a pair, we write $l \dashv r : A \to B$.

**Lemma 34.** *Let $l \dashv r : A \to B$ be a Galois connection, then $l$ and $r$ are monotone.*

*Proof.* Assume towards a contradiction that $a < a'$ and $l(a) \not\sqsubseteq l(a')$, then because $l(a') \sqsubseteq l(a')$, we infer that $a' \leq r(l(a'))$ and thus, by transitivity, $a \leq r(l(a'))$. However, this contradicts the fact that $l(a) \not\sqsubseteq l(a')$ (using the $\Leftarrow$ of the Galois connection). We conclude that $l$ is monotone.

A symmetric argument works to show that $r$ is monotone. □

**Example 35.**

**Proposition 36.** *Let $l \dashv r : A \to B$ be a Galois connection, then $r \circ l : A \to A$ is a closure operator.*

*Proof.* Because $r$ and $l$ are monotone, $r \circ l$ is clearly monotone. Also, for any $a \in A$, $l(a) \sqsubseteq l(a)$ implying $a \leq r(l(a))$, so $r \circ l$ is extensive.

Now, in order to prove $r \circ l$ is idempotent, it is enough to show that[25]

$$r(l(a)) \geq r(l(r(l(a)))).$$

[25] The $\leq$ inequality follows by extensiveness.

Observe that since $r(b) \leq r(b)$ for any $b \in B$, we have $l(r(b)) \leq b$, thus in particular, with $b = l(a)$, we have $l(r(l(a))) \leq l(a)$. Applying $r$ which is monotone yields the desired inequality. $\square$

**Proposition 37.** *Let $l \dashv r : A \to B$ and $l' \dashv r : A \to B$ be Galois connections, then $l = l'$.*

**Proposition 38.** *Let $l \dashv r : A \to B$ and $l \dashv r' : A \to B$ be Galois connections, then $r = r'$.*

## *Topology*

In this section, we introduce the basic terminology of topological spaces. Again we go a bit further than needed to help readers that first learn about topology here. We end this section by recalling some definitions about metric spaces.

**Definition 39.** A **topological space** is a pair $(X, \tau)$, where $X$ is a set and $\tau \subseteq \mathcal{P}(X)$ is closed under arbitrary unions and finite intersections[26] whose elements are called **open sets** of $X$. We call $\tau$ a **topology** on $X$.

The **complement** of an open set $U$, denoted $U^c$, is said to be **closed**.[27]

In the sequel, fix a topological space $(X, \tau)$.

**Lemma 40.** *Let $(C_i)_{i \in I}$ be a family of closed sets of $X$, then $\cap_{i \in I} C_i$ is closed and if $I$ is finite, $\cup_{i \in I} C_i$ is also closed.*[28]

*Proof.* Both statements readily follow from DeMorgan's laws and the fact that the complement of a closed set is open and vice-versa. For the first one, DeMorgan's laws yield

$$\bigcap_{i \in I} C_i = \left( \bigcup_{i \in I} C_i^c \right)^c,$$

and the LHS is the complement of a union of opens, so it is closed. For the second one, DeMorgan's laws yield

$$\bigcup_{i \in I} C_i = \left( \bigcap_{i \in I} C_i^c \right)^c,$$

and the LHS is the complement of a finite intersection of opens, so it is closed. $\square$

**Lemma 41.** *A subset $A \subseteq X$ is open if and only if for any $x \in A$, there exists an open $U \subseteq A$ such that $x \in U$.*

*Proof.* ($\Rightarrow$) For any $x \in A$, set $U = A$.

($\Leftarrow$) For each $x \in X$, pick an open $U_x \subseteq A$ such that $x \in A$, then we claim $A = \cup_{x \in A} U_x$ which is open[29]. The $\subseteq$ inclusion follows because each $x \in A$ has a set $U_x$ in the union that contains $x$. The $\supseteq$ inclusion follows because each term of the union is a subset of $A$ by assumption. $\square$

**Lemma 42.** *A subset $A \subseteq X$ is closed if and only if for any $x \notin A$, there exists an open $U$ such that, $x \in U$ and $U \cap A = \emptyset$.*[30]

[26] For any family of open sets $\{U_i\}_{i \in I} \subseteq \tau$,

$$\bigcup_{i \in I} U_i \in \tau,$$

and if $I$ is finite,

$$\bigcap_{i \in I} U_i \in \tau.$$

[27] Observe that both the empty set and the whole space are open and closed (sometimes referred to as **clopen**) because

$$\emptyset = \bigcup_{U \in \emptyset} U \text{ and } X = \bigcap_{U \in \emptyset} U \text{ and } \emptyset = X^c.$$

[28] This lemma gives an alternative to the axioms of Definition 39. Indeed, it is sometimes more convenient to define a topological space by giving its closed sets, and you can show the axioms about open sets still hold.

[29] Arbitrary unions of opens are open.

[30] This result is simply a restatement of the last one by setting $A = A^c$.

**Definition 43.** Given $A \subseteq X$, the **closure** of $A$, denoted $A^-$ is the intersection of all closed sets containing $A$. One can show that $A^-$ is the smallest closed set containing $A$.[31] Then, it follows that $A$ is closed if and only if $A^- = A$.

[31] $A^-$ is closed because it is an intersection of closed sets and any closed sets containing $A$ also contains $A^-$ by definition.

Here are more easy results on the closure of a subset.

**Lemma 44.** *Given $A, B \subseteq X$ then the following statements hold:*

1. $A \subseteq B \implies A^- \subseteq B^-$
2. $A \subseteq A^-$
3. $A^{--} = A^-$
4. $\emptyset^- = \emptyset$
5. $(A \cup B)^- = A^- \cup B^-$

*Proof of Lemma 44.* 1. By definition, $B^-$ contains $B$, thus $A$, but $B^-$ is closed, so it must contain $A^-$.

2. By definition.
3. $A^-$ is closed, so its closure is itself.
4. 3 applied to $\emptyset$.
5. $\subseteq$ follows because the LHS is the smallest closed set containing $A \cup B$ and the RHS is closed and contains $A \cup B$.
   $\supseteq$: Since the RHS is closed, we have $(A^- \cup B^-)^- = A^- \cup B^-$ implying that the RHS is the smallest closed set containing $A^- \cup B^-$. Then, since the LHS is a closed set containing $A$ and $B$, it contains $A^-$ and $B^-$ and hence must contain the RHS.

□

*Remark* 45. If we view $\mathcal{P}(X)$ as partial order equipped with the inclusion relation, the previous lemma is about good properties of the function $(-)^- : \mathcal{P}(X) \to \mp(X)$. Namely, we showed in the first three points that it is a monotone, extensive and idempotent, and therefore it is a closure operator.[32]

[32] In fact, this is where the terminology comes from.

**Definition 46.** A subset $A \subseteq X$ is said to be **dense** (in $X$) if any non-empty open set intersects $A$ non-trivially, that is, $\forall \emptyset \neq U \in \tau, A \cap U \neq \emptyset$.

**Theorem 47** (Decomposition)**.** *Let $A \subseteq X$, then $A = A^- \cap (A \cup (A^-)^c)$, where $A^-$ is closed and $A \cup (A^-)^c$ is dense. This results says that any subset of $X$ can be decomposed into a closed and a dense set.*

*Proof.* The equality is clear[33] and $A^-$ is closed by definition. It is left to show that $A \cup (A^-)^c$ is dense. Let $U \neq \emptyset$ be an open set. If $U$ intersects $A$, we are done. Otherwise, we have the following equivalences:

$$U \cap A = \emptyset \Leftrightarrow A \subseteq U^c \Leftrightarrow A^- \subseteq U^c \Leftrightarrow U \subseteq (A^-)^c,$$

where the second $\Rightarrow$ holds because $U^c$ is closed. We conclude $U \cap (A^-)^c \neq \emptyset$. □

[33] We use (in this order) distributivity of $\cap$ over $\cup$, the fact that a set and its complement intersect trivially and the inclusion $A \subseteq A^-$:

$$\begin{aligned} A^- \cap (A \cup (A^-)^c) &= (A^- \cap A) \cup (A^- \cap (A^-)^c) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

**Lemma 48.** *A subset $A \subseteq X$ is dense if and only if $A^- = X$.*

*Proof.* ($\Rightarrow$) Since $(A^-)^c$ is open but it intersects trivially the dense set $A$, it must be empty, thus $A^-$ is the whole space.

($\Leftarrow$) Let $U$ be an open set such that $U \cap A = \emptyset$, then $A$ is contained in the closed set $U^c$, but this implies $A^- \subseteq U^c$,[34] thus $U$ is empty. □

[34] Recall that the closure of $A$ is the smallest closed set containing $A$.

**Definition 49.** Let $A \subseteq X$, the **interior** of $A$, denoted $A^\circ$ is the union of all open sets contained in $A$. Similarly to the closure, we can check that that $A^\circ$ is the largest open subset of $A$ and thus that $A$ is open if and only if $A = A^\circ$.[35]

[35] It also follows that $A \subseteq B \implies A^\circ \subseteq B^\circ$ and that $A^{\circ\circ} = A^\circ$.

We end this section by presenting a largely preferred way of defining a topology that avoid describing all open sets.

**Definition 50** (Base)**.** Let $X$ be a set, a **base** $B$ is a set $B \subseteq \mathcal{P}(X)$ such that $X = \cup_{U \in B} U$ and any finite intersection of sets in $B$ can be written as a union of sets in $B$.

**Lemma 51.** *Let $X$ and $B \subseteq \mathcal{P}(X)$. If $\tau$ is the set of all unions of sets in $B$, then it is a topology on $X$. We say that $\tau$ is the topology **generated** by $B$.*

*Proof.* By assumption, we know that unions of opens are open and finite intersections of sets in $B$ are open. It remains to show that finite intersections of unions of sets in $B$ are also open. Let $U = \cup_{i \in I} U_i$ and $V = \cup_{j \in J} V_j$ with $U_i \in B$ and $V_j \in B$, then by distributivity, we obtain

$$U \cap V = \cup_{i \in I} U_i \bigcap \cup_{j \in J} V_j = \bigcup_{i \in I, j \in J} U_i \cap V_j,$$

so $U \cap V$ is open.[36] The lemma then follows by induction. $\square$

[36] It is a union of opens.

In practice, instead of generating a topology from a base $B$, we start with any family $B_0 \subseteq \mathcal{P}(X)$ and let $B$ be its closure under finite intersections, which satisfies the axioms of a base. Such a $B_0$ is often called a **subbase** for the topology generated by $B$.

Another very useful way to define topological spaces is to consider the topology induced by a metric.

**Definition 52** (Metrics space)**.** A **metric space** $(X, d)$ is a set $X$ together with a function $d : X \times X \to \mathbb{R}$ called a **metric** with the following properties for $x, y, z \in X$:

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \Leftrightarrow x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, y) \leq d(x, z) + d(z, y)$

**Definition 53** (Non-expansive)**.** A function between metric spaces $f : (X, d_X) \to (Y, d_Y)$ is said to be **non-expansive**[37] if for all $x, x' \in X$,

$$d_Y(f(x), f(x')) \leq d_X(x, x').$$

[37] Also called 1–**Lipschitz** or **short**.

**Fact 54.** *The composition of any two non-expansive maps is non-expansive.*

**Definition 55** (Open ball)**.** Let $(X, d)$ be a metric space. Given a point $x \in X$ and a non-negative radius $r \in [0, \infty)$, the open ball of radius $r$ centered at $x$ is

$$B_r(x) := \{y \in X \mid d(x, y) < r.$$

**Definition 56** (Induced topology)**.** Any metric space $(X, d)$ has an *induced topology* generated by the set of all open balls of $X$.[38]

In this topology, a set $S \subseteq X$ is open if and only if every point $x \in S$ is contained in an open ball which is contained in $S$.[39]

[38] This topology is sometimes called the open ball topology.

[39] Equivalently, $\forall x \in S, \exists r > 0, B_r(x) \subseteq S$.

**Definition 57** (Convergence)**.** Let $(X, d)$ be a metric space, a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq X$ **converges** to $p \in X$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, d(p_n, p) < \varepsilon.$$

**Definition 58** (Cauchy sequence)**.** Let $(X, d)$ be a metric space, a sequence $\{p_n\}_{n\in\mathbb{N}} \subseteq X$ is called **Cauchy** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N \implies d(p_n, p_m) < \varepsilon.$$

**Definition 59** (Completeness)**.** A metric space in which every Cauchy sequence converges is called **complete**.

# *Categories and Functors*

As you will soon realize, many common mathematical objects can be viewed as categories or parts of a category, and often in several ways. Hence, there can be many starting points to motivate category theory even after restricting ourselves to the background of an undergraduate student in mathematics (see Chapter ). I do not want to spend much time in the realm of informal explanations, so we will start from the notion of directed graphs, quickly get to the definition of a category and begin an enumeration of examples which will carry on (implicitly) for the rest of the book. We will also define functors which are basically homomorphisms of categories and list a bunch of examples.

## *Categories*

**Definition 60** (Oriented graph)**.** An **directed graph** $G$ consists of a collection of **nodes/objects** denoted $G_0$ and a collection of **arrows/morphisms** denoted $G_1$ along with two maps $s, t : G_1 \to G_0$, so that each arrow $f \in G_1$ has a **source** $s(f)$ and a **target** $t(f)$.

We draw morphisms with arrows, the source being its tail and target being its head:

$$s(f) \xrightarrow{f} t(f)$$

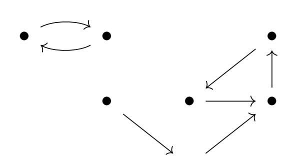
**Definition 61** (Paths)**.** A **path** in a directed graph $G$ is a sequence of arrows $(f_1, \dots, f_k)$ that are **composable** in the sense that $t(f_i) = s(f_{i-1})$ for $i = 2, \dots, k$ as drawn below in (1). The collection of paths of length $k$[40] in $G$ will be denoted $G_k$.

$$\bullet \xrightarrow{f_k} \bullet \xrightarrow{f_{k-1}} \bullet \cdots \bullet \xrightarrow{f_2} \bullet \xrightarrow{f_1} \bullet \qquad (1)$$

[40] The **length** of a path is the number of arrows in it.

Observe that the notation indicating the direction of the path does not correspond to the usual notation in graph theory. The motivation for this divergence will come shortly as the composition of arrows in a category is defined. The main idea is that, conceptually, arrows coincide more closely with functions between mathematical objects rather than arrows between nodes of a graph.

**Examples 62.** It is very simple to give an example of a directed graph by drawing a bunch of nodes and arrows between them as in (2), $G_0$ is the collection of nodes, $G_1$ is the collection of arrows and $s$ and $t$ can be inferred from looking at the head and tail of each arrow. Let us give more examples to motivate the next definition.



(2)

1. For any set $X$, there is a trivial directed graph with $X$ as its collection of nodes and no arrows. The source and target maps are the unique functions $\emptyset \to X$. You can represent it by drawing a node for each element of $X$.[41]

[41] This is a very uninteresting directed graph.

There is a slightly more complex directed graph whose nodes are the elements of $X$. For each pair $(x, x') \in X \times X$, we can add an arrow with source $x$ and target $x'$. Drawing it is still fairly simple[42]; you draw a node for each element of $X$ and an arrow from $x$ to $x'$ for each pair $(x, x')$.[43]

[42] Provided the set $X$ is finite

[43] Note that there are so-called *loops* which are arrows from a node to itself because $(x, x)$ is in $X \times X$.

2. Starting from a set $X$, we can define another directed graph by letting $X$ be its only node and the collection of arrows be the set of functions from $X$ to itself. The source and target maps are uniquely determined again, this time by their codomain that contains only the node $X$. This graph is already more interesting since the collection of arrows has a monoid structure. Indeed, the operation of composition of functions is associative and the identity function is the identity for this operation.

3. Taking inspiration from the previous examples, we define a directed graph **Set**. It contains one node for every set, i.e.: $\mathbf{Set}_0$ is the collection of all sets, [44] and one arrow with source $X$ and target $Y$ for every function $f : X \to Y$.

   Similarly to the last example, we recognize that the collection of arrows has a novel kind of structure induced by composition of functions and identity functions. It is not a monoid because you can only compose functions when one's source is the target of the other. Nonetheless, we still have associativity and identities that are at the core of the definition of a monoid. Since the theory of monoids is extremely rich and ubiquitous in mathematics, it is daring to study this seemingly more complex variant. We first need to make this structure abstract in the definition of a category.

[44] Notice how we could not have defined this graph if we required $G_0$ to be a set.

**Definition 63** (Category)**.** A directed graph $\mathbf{C}$ along with a **composition** map $\circ : \mathbf{C}_2 \to \mathbf{C}_1$ is a **category** if it satisfies the following properties:

1. For any $(f, g) \in \mathbf{C}_2$, $s(f \circ g) = s(g)$ and $t(f \circ g) = t(f)$. This is more naturally understood visually in (3).
2. For any $(f, g, h) \in \mathbf{C}_3$, $f \circ (g \circ h) = (f \circ g) \circ h$, namely, composition is associative. Again, the graphic representation in (4) may be more revealing.
3. For any object $A \in \mathbf{C}_0$, there exists an **identity** morphism $u_{\mathbf{C}}(A) \in \mathbf{C}_1$ with $A$ as its source and target that satisfies $u_{\mathbf{C}}(A) \circ f = f$ and $g \circ u_{\mathbf{C}}(A) = g$, for any $f, g \in \mathbf{C}_1$ where $t(f) = A$ and $s(g) = A$.

$$\begin{tikzcd} \bullet \arrow[r, "g"'] \arrow[rr, bend left, "f\circ g"] & \bullet \arrow[r, "f"'] & \bullet \end{tikzcd} \quad (3)$$

$$\begin{tikzcd} \bullet \arrow[r, "h"] \arrow[rr, bend right, "g\circ h"'] \arrow[rrr, bend left=50, "(f\circ g)\circ h"] \arrow[rrr, bend right=50, "f\circ(g\circ h)"'] & \bullet \arrow[r, "g"] \arrow[rr, bend left, "f\circ g"] & \bullet \arrow[r, "f"] & \bullet \end{tikzcd} \quad (4)$$

If the third property of Definition 63 is not satisfied, $\mathbf{C}$ will be referred to as a **semicategory**. Some authors choose to explicit when a category *does* satisfy this property, qualifying it as unital, but this term also has other meanings, hence our preference for the first convention.

*Remark* 64 (Notation). In general, we will denote categories with bold uppercase letters typeset with `\mathbf` ($\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, etc.), their objects with uppercase letters ($A$, $B$, $X$, $Y$, $Z$, etc.) and their morphisms with lowercase letters ($f$, $g$, $h$, etc.). When the category is clear from the context, we denote the identity morphisms $\mathrm{id}_A$ instead of $u_{\mathbf{C}}(A)$. We say that two morphisms are **parallel** if they have the same source and target. Let $f$ and $g$ be morphisms in a category, we say that $f$ **factors through** $g$ if there exists $h \in \mathbf{C}_1$ such that $f = g \circ h$ or $f = h \circ g$.

Observe that since $\circ$ is associative, it induces a unique composition map on paths of any finite lengths, which we abusively denote $\circ : \mathbf{C}_k \to \mathbf{C}_1$. This lets us write

$f_1 \circ f_2 \circ \cdots \circ f_k$ with no parentheses. Occasionally, we will refer to the image of the path under this map by the **composition of the path** or the **morphism that a path composes to**.

**Examples 65** (Boring examples)**.** It is really easy to construct a category by drawing its underlying directed graph and inferring the definition of the composition from it. Starting from the very simple graph depicted in (5), we can infer the definition of a category with a single object and its identity morphism. This category is denoted $\mathbf{1}$, the composition is trivial since $\mathrm{id}_\bullet \circ \mathrm{id}_\bullet = \mathrm{id}_\bullet$.

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd} \tag{5}$$

Similarly, we construct from the graph in (6) a category with two objects, their identity morphisms and nothing else. The composition is again trivial. This category will be denoted $\mathbf{1}+\mathbf{1}$.[45] More generally, for any collection $\mathbf{C}_0$, there is a category $\mathbf{C}$ whose collection of objects is $\mathbf{C}_0$ and whose collection of morphisms is $\mathbf{C}_1 := \{\mathrm{id}_X \mid X \in \mathbf{C}_0\}$. The composition map is completely determined by the third property in Definition 63.[46]

$$\begin{tikzcd} \bullet_1 \arrow[loop above] & \bullet_2 \arrow[loop above] \end{tikzcd} \tag{6}$$

[45] This notation is cleared up in Definition 165.

[46] i.e.: for any $X \in \mathbf{C}_0$, $\mathrm{id}_X \circ \mathrm{id}_X = \mathrm{id}_X$.

The graph in (7) corresponds to the category with objects $\{A, B\}$ and morphisms $\{\mathrm{id}_A, \mathrm{id}_B, f\}$.

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r, "f"] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd} \tag{7}$$

The composition map is then completely determined by the properties of identity morphisms.[47] This category is called the interval category or the **walking arrow**, and it is denoted $\mathbf{2}$. Note however that $\mathbf{1}+\mathbf{1} \neq \mathbf{2}$.

[47] i.e.: $f \circ \mathrm{id}_A = f$, $\mathrm{id}_B \circ f = f$, $\mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A$ and $\mathrm{id}_B \circ \mathrm{id}_B = \mathrm{id}_B$

Starting now, we will omit the identity morphisms from the diagrams (as is usual in the literature) for clarity reasons: they would hinder readability without adding information.

It is not always as straightforward to construct a category from a directed graph. For instance, if two distinct arrows have the same source and target, they must be explicitly drawn and the ambiguity in the composition must be dealt with. The graph in (8) is problematic in this way: it has two distinct paths of length two starting at the top-left corner and ending at the bottom-right corner. Since the composition of these paths can be equal to any of the two distinct morphisms between these corners, there is no obvious category corresponding to this graph.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \tag{8}$$

Still, we will often draw diagrams with nodes and arrows and let you infer the categorical structure (i.e.: what each path composes to) by stating that the diagram is **commutative**.

**Definition 66** (Commutativity)**.** Given a diagram representing objects and morphisms in a category, we say that it is **commutative** if the composition of any path of length bigger than one is equal to the composition of any other path with the same source and target. The morphism resulting from the composition may or may not be depicted.

*Remark* 67 (Convention). Reasoning with commutative diagrams is an acquired skill we will practice quite a lot in the following chapters. Yet there is no standard definition that everyone systematically uses.[48] For this reason, I decided to pick my

[48] This does not really lead to many misunderstandings anyway because what is meant by a diagram is usually made clear by the context.

favorite definition of commutativity which is uncommon[49]. In most cases, a diagram is called commutative when any two paths compose to the same morphism, but in practice, there are two exceptions handled by Definition 66:

[49] I have not seen the constraint on the length anywhere else.

1. Two parallel morphisms are not always equal in a commutative diagram.

2. An endomorphism[50] drawn in a commutative diagram is not the identity morphism (unless otherwise stated).

[50] An **endomorphism** is a morphism whose source and target coincide.

**Examples 68.** Arguably the most frequently used commutative diagram is the commutative square drawn in (9).

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \tag{9}$$

We say the square commutes when the bottom and top paths compose to the same (omitted in the diagram) morphism. The commutative square can also be seen as a category by inferring the missing morphism and the composition from commutativity. We can denote it $\mathbf{2} \times \mathbf{2}$.[51]

[51] This notation is explained in Definition 96.

Supposing that (10) commutes, we can infer that $f' \circ h = h' \circ f$, $g' \circ h' = h'' \circ g$, and $g' \circ f' \circ h = h'' \circ g \circ f$. Observe that the last equation can be derived from the first two which are equivalent to the commutativity of the two squares in (10). More generally, combining commutative diagrams in this way yields commutative diagrams, and this is the core of a powerful proof method called diagram paving that we introduce in Chapter .

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "h"'] & \bullet \arrow[r, "g"] \arrow[d, "h'"'] & \bullet \arrow[d, "h''"] \\ \bullet \arrow[r, "f'"'] & \bullet \arrow[r, "g'"'] & \bullet \end{tikzcd} \tag{10}$$

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \tag{11}$$

Stating that (11) commutes is equivalent to stating that $f \circ g = \mathrm{id}_A$ and $g \circ f = \mathrm{id}_B$. We will revisit this property in Definition 122.

It would be odd to require that (8) commutes. It would imply that the two parallel morphisms are equal because they are both equal to the composition of the bottom and top paths. We will avoid drawing parallel morphisms when they are supposed to be equal.

*Warning* 69. Diagrams are not commutative by default. We will always specify when a diagram is commutative. As our usage of commutative diagrams will ramp up in the following chapters, you have to try to remember that.

Before moving on to more interesting examples, we introduce the Hom notation.

**Definition 70** (Hom)**.** Let $\mathbf{C}$ be a category and $A, B \in \mathbf{C}_0$ be objects, the collection of all morphisms going from $A$ to $B$ is

$$\mathrm{Hom}_{\mathbf{C}}(A, B) := \{f \in \mathbf{C}_1 \mid s(f) = A \text{ and } t(f) = B\}.$$

This leads to an alternative way of defining the morphisms of $\mathbf{C}$, namely, one can describe $\mathrm{Hom}_{\mathbf{C}}(A, B)$ for all $A, B \in \mathbf{C}_0$ instead of describing all of $\mathbf{C}_1$ at once. Defining the morphisms this way also takes care of the source and target functions implicitly.

*Remark* 71 (Notation). Some authors choose to denote the collection of morphisms between $A$ and $B$ with $\mathbf{C}(A,B)$. We prefer to use the latter notation when working with 2–categories[52] to highlight the fact that $\mathbf{C}(A,B)$ has more structure. Other authors use hom with a lowercase "h", our choice here is arbitrary.

[52] c.f. Definition 265.

**Definition 72** (Smallness)**.** A category $\mathbf{C}$ is called **small** if the collections of objects and morphisms are sets. If for all objects $A, B \in \mathbf{C}_0$, $\mathrm{Hom}_{\mathbf{C}}(A,B)$ is a set, $\mathbf{C}$ is said to be **locally small** and $\mathrm{Hom}_{\mathbf{C}}(A,B)$ is called a **hom-set**. A category that is not small can be referred to as **large**.

**Example 73** (**Set**)**.** The category **Set** has the collection of sets as its objects and for any sets $X$ and $Y$, $\mathrm{Hom}_{\mathbf{Set}}(X,Y)$ is the set of all the functions from $X$ to $Y$. The composition map is given by composition of functions (which is associative) and the identity maps serve as the identity morphisms. This category is locally small but not small.[53]

[53] By our argument at the start of Chapter : the collection of all sets cannot be a set.

**Example 74.** Let $(X, \leq)$ be a partially ordered set, it can be viewed as a category with elements of $X$ as its objects. For any $x, y \in X$, the hom-set $\mathrm{Hom}_X(x,y)$ contains a single morphism if $x \leq y$ and is empty otherwise. The identity morphisms arise from the reflexivity of $\leq$. Since every hom-set contains at most one element and $\leq$ is transitive, the composition map is completely determined. Detailing this out, if $f : x \to y$ and $g : y \to z$ are morphisms, then we know that $x \leq y$ and $y \leq z$. Thus, transitivity implies that $x \leq z$ and there is a unique morphism $x \to z$, so it must be $g \circ f$.[54]

[54] Note that antisymmetry was not used in this argument, so one can more generally construct a category starting from a preorder. Such categories are called **thin** because each hom-set contains at most one morphism. It is straightforward to show the identities and composition ensure that any thin category $\mathbf{C}$ is constructed from the preorder $(\mathbf{C}_0, \leq)$ with

$$X \leq Y \Leftrightarrow \mathrm{Hom}_{\mathbf{C}}(X,Y) \neq \emptyset.$$

If a category corresponds to this construction for some poset, it is called **posetal**. In (12), we depict the posetal category associated to $(\mathbb{N}, \leq)$. The arrows between numbers $n$ and $n+k$ are omitted for $k > 1$ as they can be inferred by the composition $n \leq n+1 \leq n+2 \leq \cdots \leq n+k$.

$$\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \cdots \tag{12}$$

As a particular case of posetal categories, let $(X, \tau)$ be a topological space and note that the inclusion relation on open sets is a partial order on $\tau$. Thus, $X$ has a corresponding posetal category. More explicitly, the objects are open sets and for any $U, V \in \tau$, the hom-set $\mathrm{Hom}_X(U,V)$ contains the inclusion map $i_{UV}$ if $U \subseteq V$ and is empty otherwise. This category will be denoted $\mathcal{O}(X,\tau)$ or $\mathcal{O}(X)$.

**Example 75** (Single object categories)**.** If a category $\mathbf{C}$ has a single object $*$, then the only morphisms go from $*$ to $*$. In particular, $\mathbf{C}_1 = \mathrm{Hom}_{\mathbf{C}}(*,*)$ and $\mathbf{C}_2 = \mathbf{C}_1 \times \mathbf{C}_1$. Then, the associativity of $\circ$ and existence of $\mathrm{id}_*$ makes $(\mathbf{C}_1, \circ)$ into a monoid.

Conversely, a monoid $(M, \cdot)$ can be represented by a single object category $M$, where $\mathrm{Hom}_M(*,*) = M$ and the composition map is the monoid operation.

Since many algebraic structures have an associative operation with an identity element, this yields a fairly general construction. The single object category associated to a monoid or group $G$ will be denoted $\mathbf{B}G$ and referred to as the **delooping** of $G$.

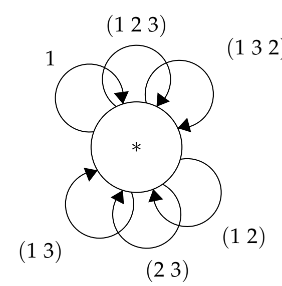



Figure 1: The delooping of the symmetric group $S_3$, a.k.a $\mathbf{B}S_3$.

The natural numbers can also be endowed with the monoid structure of addition, hence a particular instance of a single object category is the delooping of $(\mathbb{N}, +)$. Notice that this category is very different from the posetal category $(\mathbb{N}, \leq)$. In the former, $\mathbb{N}$ is in correspondence with the morphisms while in the latter, it is in correspondence with the objects.

Many simple examples of large categories arise as subcategories of **Set**.

**Definition 76** (Subcategory)**.** Let $\mathbf{C}$ be a category, a category $\mathbf{C}'$ is a **subcategory** of $\mathbf{C}$ if, the following properties are satisfied.

1. The objects and morphisms of $\mathbf{C}'$ are objects and morphisms of $\mathbf{C}$ (i.e.: $\mathbf{C}'_0 \subseteq \mathbf{C}_0$ and $\mathbf{C}'_1 \subseteq \mathbf{C}_1$).
2. The source and target maps of $\mathbf{C}'$ are the restrictions of the source and target maps of $\mathbf{C}$ on $\mathbf{C}'_1$ and for every morphism $f \in \mathbf{C}'_1$, $s(f), t(f) \in \mathbf{C}'_0$.
3. The composition map of $\mathbf{C}'$ is the restriction of the composition map of $\mathbf{C}$ on $\mathbf{C}'_2$ and for any $(f,g) \in \mathbf{C}'_2$, $f \circ_{\mathbf{C}'} g = f \circ_{\mathbf{C}} g \in \mathbf{C}'_1$.
4. The identity morphisms of objects in $\mathbf{C}'_0$ are the identity morphisms of objects in $\mathbf{C}_0$, i.e.: $u_{\mathbf{C}}(A) = u_{\mathbf{C}'}(A)$ when $A \in \mathbf{C}'_0$.

Intuitively, one can see $\mathbf{C}'$ as being obtained from $\mathbf{C}$ by removing some objects and morphisms, but making sure that no morphism is left with no source or no target and that no path is left without its composition.

**Exercise 77** (NOW!)**.** Find an example of a category $\mathbf{C}$ and a category $\mathbf{C}'$ that satisfy the first three conditions but not the fourth.

See solution.

**Definition 78** (Full and wide)**.** A subcategory $\mathbf{C}'$ of $\mathbf{C}$ is called **full** if for any objects $A, B \in \mathbf{C}'_0$, $\mathrm{Hom}_{\mathbf{C}'}(A,B) = \mathrm{Hom}_{\mathbf{C}}(A,B)$. It is called **wide** if $\mathbf{C}'_0 = \mathbf{C}_0$.[55]

[55] In words, a subcategory is full if the morphisms that were removed had their source or target removed as well and it is wide if no objects were removed.

**Examples 79** (Subcategories of **Set**)**.** We can selectively remove some objects and morphisms in **Set** to obtain the following categories.

1. Since the composition of injective functions is again injective, the restriction of morphisms in **Set** to injective functions yields a wide subcategory of **Set**, denoted **SetInj**. Unsurprisingly, **SetSurj** can be constructed similarly.
2. Removing all infinite sets from **Set** yields the full subcategory of finite sets denoted **FinSet**.[56]
3. Further removing sets from **FinSet** and keeping only $\emptyset$, $\{1\}$, $\{1,2\}$, $\{1,2,3\}$, etc., we obtain the category **FinOrd** which is a small full subcategory of **Set**.[57]
4. Since the composition of monotone maps is monotone and the identity function is monotone, we can view each set $\{1, \ldots, n\}$ as ordered with $\leq$ and remove all morphisms that are not monotone from **FinOrd**. The resulting category is the **simplex category** denoted $\mathbf{\Delta}$.

[56] This category is not small because there is no set of all finite sets.

[57] The name **FinOrd** is an abbreviation of finite ordinals, because we can also define **FinOrd** as the category of finite ordinals and functions between them.

**Examples 80** (Concrete categories)**.** This second list of examples contains so-called concrete categories, which, informally, are categories of sets with extra structure.[58]

[58] Formally, see Definition 92.

1. The category $\mathbf{Set}_*$ is the category of **pointed** sets. Its objects are sets with a distinguished element and its morphisms are functions that map distinguished elements to distinguished elements. In more details, $(\mathbf{Set}_*)_0$ is the collection of pairs $(X, x)$ where $X$ is a set and $x \in X$, and for any two pointed sets $(X, x)$ and $(Y, y)$,
$$\mathrm{Hom}_{\mathbf{Set}_*}((X,x),(Y,y)) = \{f : X \to Y \mid f(x) = y\}.$$
The identity morphisms and composition are defined as in **Set**, so the axioms of a category clearly hold after checking that if $f(X, x) \to (Y, y)$ satisfies $f(x) = y$ and $g : (Y, y) \to (Z, z)$ satisfies $g(y) = z$, then $(g \circ f)(x) = z$.

2. The category **Mon** is the category of monoids and their homomorphisms, let us uncover the structure of **Mon**.[59] The objects are monoids, so $\mathbf{Mon}_0$ is the collection of all monoids, and the morphisms are monoid homomorphisms, so for any $M, N \in \mathbf{Mon}_0$, $\mathrm{Hom}_{\mathbf{Mon}}(M, N)$ is the set of homomorphisms from $M$ to $N$. The composition in **Mon** is given by the composition of homomorphisms, we know it is well-defined because the composition of two homomorphisms is a homomorphism. Also, the composition is associative and the identity functions are homomorphisms, so we can define $u_{\mathbf{Mon}}(M) = \mathrm{id}_M$.

[59] These technicalities are essentially the same for the categories in the remainder of Example 80.

3. Similarly, the category of groups (resp. rings or fields) where the morphisms are group (resp. ring or field) homomorphisms is denoted **Grp** (resp. **Ring** or **Field**). The category of abelian groups (resp. commutative monoids or rings) is a full subcategory of **Grp** (resp. **Mon** or **Ring**) denoted **Ab** (resp. **CMon** or **CRing**).

4. Let $k$ be a fixed field, the category of vector spaces over $k$ where the morphisms are linear maps is denoted $\mathbf{Vect}_k$. The full subcategory of $\mathbf{Vect}_k$ consisting only of finite dimensional vector spaces is denoted $\mathbf{FDVect}_k$.

5. The category of partially ordered sets where morphisms are order-preserving functions is denoted **Poset**.

6. The category of topological spaces where morphisms are continuous functions is denoted **Top**.

Our last example is a large category which is neither a subcategory of **Set** nor a concrete category.

**Example 81 (Rel).** The category of sets and relations, denoted **Rel**, has as objects the collection of all sets, and for any sets $X$ and $Y$, $\mathrm{Hom}_{\mathbf{Rel}}(X, Y)$ is the set of relations between $X$ and $Y$, that is, the powerset of $X \times Y$. The composition of two relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is defined by
$$S \circ R = R; S := \{(x,z) \in X \times Z \mid \exists y \in Y, (x,y) \in R, (y,z) \in S\} \subseteq X \times Z.$$

*Remark* 82. You can view **Set** as the subcategory of **Rel** where you only take the relations $R \subseteq X \times Y$ satisfying for any $x \in X$,
$$|\{y \in Y \mid (x,y) \in R\}| = 1.$$

One can check that this composition is associative and that, for any set $X$, the **diagonal relation** $\Delta_X = \{(x,x) : x \in X\} \subseteq X \times X$ is the identity with respect to this composition.

## *Functors*

The list above is far from exhaustive; there are many more mathematical objects that can fit in a category and this is a main reason for studying this subject. Indeed, categories encapsulate a natural structure that accurately represents the heart of several mathematical theories from a global and abstract perspective. Still, a category is almost never studied on its own since the abstraction it provides can make the properties of its objects more obscure. For instance, stating and proving Lagrange's theorem in the framework of **Grp** is quite more involved than in the classical way. Nevertheless, we will get to see in subsequent chapters that some surprising links can arise between seemingly unrelated subjects through the study of how different categories relate. The central tool for exhibiting these relations is a functor.

As we will show, a functor is a morphism of categories, thus, to motivate the definition, we can look at other morphisms we have encountered. A clear similarity between categories like **Mon**, **Grp**, **Ring** or **Top** is that all the objects have some sort of structure that the morphisms preserve. In the first three categories, the structure on an object is the operations and identity elements that are preserved under homomorphisms, and in the last one, the structure on a topological space is the family of open sets which is preserved by continuous maps.[60] Hence, we want to define a morphism that preserves the structure of a category. Going back to Definition 63, we see that the structure of a category consists of the source and target maps, the composition and the identities.

[60] Recall that in topology, preserving the structure means the preimage of a continuous function sends opens to opens.

**Definition 83** (Functor)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories, a **functor** $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ is a pair of maps $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ and $F_1 : \mathbf{C}_1 \to \mathbf{D}_1$ such that diagrams (13), (14) and (15) commute.[61]

$$\begin{tikzcd} \mathbf{C}_0 \arrow[d, "F_0"'] & \mathbf{C}_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "F_1"] & \mathbf{C}_0 \arrow[d, "F_0"] \\ \mathbf{D}_0 & \mathbf{D}_1 \arrow[l, "s"] \arrow[r, "t"'] & \mathbf{D}_0 \end{tikzcd} \tag{13}$$

$$\begin{tikzcd} \mathbf{C}_2 \arrow[r, "F_2"] \arrow[d, "\circ_{\mathbf{C}}"'] & \mathbf{D}_2 \arrow[d, "\circ_{\mathbf{D}}"] \\ \mathbf{C}_1 \arrow[r, "F_1"'] & \mathbf{D}_1 \end{tikzcd} \tag{14}$$

$$\begin{tikzcd} \mathbf{C}_0 \arrow[r, "F_0"] \arrow[d, "u_{\mathbf{C}}"'] & \mathbf{D}_0 \arrow[d, "u_{\mathbf{D}}"] \\ \mathbf{C}_1 \arrow[r, "F_1"'] & \mathbf{D}_1 \end{tikzcd} \tag{15}$$

[61] $F_2$ is induced by the definition of $F_1$ with
$$F_2 = (f,g) \mapsto (F_1(f), F_1(g)).$$

*Remark* 84 (Digesting diagrams). Once again, we emphasize that commutative diagrams will be heavily employed to make clearer and more compact arguments,[62] and that it will take time to get used to them. For now, let us unpack the definition above to ease its comprehension.

A functor $F : \mathbf{C} \rightsquigarrow \mathbf{D}$[63] must satisfy the following properties.

[62] This is especially true when using a blackboard or pen and paper because it makes it easier to point at things. Sadly, I cannot point at things on this pdf you are reading.

[63] The $\rightsquigarrow$ (`\rightsquigarrow`) notation for functors is not that common, they are usually denoted with plain arrows because they are morphisms. Nonetheless, I feel it is useful to have a special treatment for functors until you get accustomed to them. The squiggly arrow notation is sometimes used for Kleisli morphisms which we cover in Chapter .

i. For any $A, B \in \mathbf{C}_0$ and $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, $F(f) \in \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$. This is equivalent to the commutativity of (13) which says $F_0(s(f)) = s(F_1(f))$ and $F_0(t(f)) = t(F_1(f))$.

ii. If $f, g \in \mathbf{C}_1$ are composable, then $F(f)$ and $F(g)$ are composable by i and $F(f \circ_{\mathbf{C}} g) = F(f) \circ_{\mathbf{D}} F(g)$ by commutativity of (14).

iii. If $A \in \mathbf{C}_0$, then $u_{\mathbf{D}}(F(A)) = F(u_{\mathbf{C}}(A))$ by commutativity of (15).[64]

[64] Alternatively, $\mathrm{id}_{F(A)} = F(\mathrm{id}_A)$.

The subscript on $F$ is often omitted, as is common in the literature, because it is always clear whether $F$ is applied to an object or a morphism. We will also denote application of $F$ with juxtaposition instead of parentheses, i.e.: we can write $FA$ and $Ff$ instead of $F(A)$ and $F(f)$.

**Examples 85** (Boring examples)**.** As usual, a few trivial constructions arise.

1. For any category $\mathbf{C}$, the **identity functor** $\mathrm{id}_{\mathbf{C}} : \mathbf{C} \rightsquigarrow \mathbf{C}$ is defined by letting $(\mathrm{id}_{\mathbf{C}})_0$ and $(\mathrm{id}_{\mathbf{C}})_1$ be identity maps on $\mathbf{C}_0$ and $\mathbf{C}_1$ respectively.

When the source and target of a functor coincide, we may refer to it as an **endofunctor**.

2. Let $\mathbf{C}$ be a category and $\mathbf{C}'$ a subcategory of $\mathbf{C}$, the **inclusion functor** $\mathcal{I} : \mathbf{C}' \rightsquigarrow \mathbf{C}$ is defined by letting $\mathcal{I}_0$ be the inclusion map $\mathbf{C}'_0 \hookrightarrow \mathbf{C}_0$ and $\mathcal{I}_1$ be the inclusion map $\mathbf{C}'_1 \hookrightarrow \mathbf{C}_1$.

3. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and $X$ be an object in $\mathbf{D}$, the **constant functor** $\Delta(X) : \mathbf{C} \rightsquigarrow \mathbf{D}$ is defined by letting $\Delta(X)_0(A) = X$ for any $A \in \mathbf{C}_0$ and $\Delta(X)_1(f) = \mathrm{id}_X$ for any $f \in \mathbf{C}_1$.

**Examples 86** (Less boring)**.** Functors with the source being one of $\mathbf{1}$, $\mathbf{2}$ or $\mathbf{2} \times \mathbf{2}$ (cf. Example 65) are a bit less boring. Let the target be a category $\mathbf{C}$ and let us analyze these functors.

- Let $F : \mathbf{1} \rightsquigarrow \mathbf{C}$, $F_0$ assigns to the single object $\bullet \in \mathbf{1}_0$ an object $F(\bullet) \in \mathbf{C}_0$. Then, by commutativity of (15), $F_1$ is completely determined by $\mathrm{id}_\bullet \mapsto \mathrm{id}_{F(\bullet)}$. We conclude that functors of this type are in correspondence with objects of $\mathbf{C}$.

- Let $F : \mathbf{2} \rightsquigarrow \mathbf{C}$, $F_0$ assigns to $A$ and $B$, two objects $FA, FB \in \mathbf{C}_0$ and $F_1$'s action on identities is fixed. Still, there is one choice to make for $F_1(f)$ which must be a morphism in $\mathrm{Hom}_{\mathbf{C}}(FA, FB)$. Therefore, $F$ sums up to a choice of two objects in $\mathbf{C}$ and a morphism between them. In other words, functors of this type are in correspondence with morphisms in $\mathbf{C}$.[65]

[65] After picking a morphism, the source and target are determined.

- Similarly (we leave the details as an exercise), functors of type $F : \mathbf{2} \times \mathbf{2} \rightsquigarrow \mathbf{C}$ are in correspondence with commutative squares inside the category $\mathbf{C}$.[66]

[66] i.e.: pairs of pairs of composable morphisms $((f, g), (f', g')) \in \mathbf{C}_2 \times \mathbf{C}_2$ satisfying $f \circ g = f' \circ g'$.

*Remark* 87 (Functoriality). We will use the term **functorial** as an adjective to qualify transformations that behave like functors and **functoriality** to refer to the property of behaving like a functor.

Throughout the rest of this book, the goal will essentially be to grow our list of categories and functors with more and more examples and perhaps exploit their properties wisely. Before pursuing this objective, we give important definitions analogous to injectivity and surjectivity of functions.

**Definition 88** (Full and faithful)**.** Let $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ be a functor. For $A, B \in \mathbf{C}_0$, denote the restriction of $F_1$ to $\mathrm{Hom}_{\mathbf{C}}(A, B)$ with

$$F_{A,B} : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(A), F(B)).$$

- If $F_{A,B}$ is injective for any $A, B \in \mathbf{C}_0$, then $F$ is **faithful**.
- If $F_{A,B}$ is surjective for any $A, B \in \mathbf{C}_0$, then $F$ is **full**.
- If $F_{A,B}$ is bijective for any $A, B \in \mathbf{C}_0$, then $F$ is **fully faithful**.

See solution.

**Exercise 89.** Show that the inclusion functor $\mathcal{I} : \mathbf{C}' \rightsquigarrow \mathbf{C}$ is faithful. Show it is full if and only if $\mathbf{C}'$ is a full subcategory.

*Remark* 90*.* While bijectivity is very strong to compare sets — it morally says that the elements of one set can be identified with the elements of another set — fully faithful functors are not as powerful. In fact, we can infer this by observing that the action of $F$ on objects is not restricted at all by full faithfulness. We will see later what properties ensure that a functor closely correlates its source with its target.

**Examples 91.** For all but the first example, we leave you to prove functoriality.[67] In the literature, a lot of functors are given only with their action on objects and the reader is supposed to figure out the action on morphisms. Not everyone has the same innate ability to do this, but I hope this book can give you enough experience to overcome this difficulty.

[67] It is an elementary task that is mostly relevant to the field of mathematics the functor comes from.

1. The **powerset functor** $\mathcal{P} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$ sends a set $X$ to its powerset $\mathcal{P}(X)$[68] and a function $f : X \to Y$ to the image map $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$. The latter sends a subset $S \subseteq X$ to
$$\mathcal{P}(f)(S) = f(S) := \{f(s) \mid s \in S\} \subseteq Y.$$
In order to prove that $\mathcal{P}$ is a functor, we need to show it makes diagrams (13), (14), and (15) commute. Equivalently, we can show it satisfies the three conditions in Remark 84.

   i. For any function $f : X \to Y$, the source and target of the image map $\mathcal{P}f$ are $\mathcal{P}X$ and $\mathcal{P}Y$ respectively as required.

   ii. Given two functions $f : X \to Y$ and $g : Y \to Z$, we can verify that $\mathcal{P}g \circ \mathcal{P}f = \mathcal{P}(g \circ f)$ by looking at the action of both sides on a subset $S \subseteq X$.
$$\begin{aligned}\mathcal{P}g(\mathcal{P}f(S)) &= \{g(y) \mid y \in \mathcal{P}f(S)\} \\ &= \{g(y) \mid y \in \{f(x) \mid x \in S\}\} \\ &= \{g(f(x)) \mid x \in S\}\end{aligned} \qquad \begin{aligned}\mathcal{P}(g \circ f)(S) &= \{(g \circ f)(x) \mid x \in S\} \\ &= \{g(f(x)) \mid x \in S\}\end{aligned}$$

   iii. Finally, the image map of $\mathrm{id}_X$ is the identity on $\mathcal{P}X$ because
$$\mathcal{P}\mathrm{id}_X(S) = \{\mathrm{id}_X(x) \mid x \in S\} = \{x \mid x \in S\} = S.$$

[68] The powerset of $X$ is the set of all subsets of $X$.

The powerset functor is faithful because the same image map cannot arise from two different functions[69], it is not full because lots of functions $\mathcal{P}(X) \to \mathcal{P}(Y)$ are not image maps. A cardinality argument suffices: when $|X|, |Y| \geq 2$,

$$|\mathrm{Hom}_{\mathbf{Set}}(X,Y)| = |Y|^{|X|} < |\mathcal{P}(Y)|^{|\mathcal{P}(X)|} = |\mathrm{Hom}_{\mathbf{Set}}(\mathcal{P}(X), \mathcal{P}(Y))|.$$

[69] Indeed, if $f(x) \neq g(x)$, then $f(\{x\}) \neq g(\{x\})$.

2. The concrete categories of Examples 80 are defined using a functor.

**Definition 92** (Concrete category)**.** We call a category **C concrete** if it is paired (generally implicitly) with a faithful functor $U : \mathbf{C} \rightsquigarrow \mathbf{Set}$. In most cases, $U$ is called the **forgetful functor** because it sends objects and morphisms of **C** to sets and functions by *forgetting* additional structure.

The forgetful functor $U : \mathbf{Grp} \rightsquigarrow \mathbf{Set}$ sends a group $(G, \cdot, 1_G)$ to its underlying set $G$, *forgetting about the operation and identity*. It sends a group homomorphism $f : G \to H$ to the underlying function, *forgetting about the homomorphism properties*. It is faithful since if two homomorphisms have the same underlying function, then they are equal.[70]

[70] We leave you the repetitive task to describe the forgetful functor for every concrete category in Examples 80.

3. It is also sometimes useful to consider *intermediate* forgetful functors. For example, $U : \mathbf{Ring} \rightsquigarrow \mathbf{Ab}$ sends a ring $(R, +, \cdot, 1_R, 0_R)$ to the abelian group $(R, +, 0_R)$, *forgetting about multiplication and* $1_R$. It sends a ring homomorphism $f : R \to S$ to the same underlying function seen as a group homomorphism.[71]

[71] It can do that because part of the requirements for ring homomorphisms is to preserve the underlying additive group structure.

4. In some cases, there is a canonical way to go in the opposite direction to the forgetful functor, it is called the free functor. For **Mon**, the free functor $F : \mathbf{Set} \rightsquigarrow \mathbf{Mon}$ sends a set $X$ to the free monoid generated by $X$ and a function $f : X \to Y$ to the unique group homomorphism $F(X) \to F(Y)$ that restricts to $f$ on the set of generators.[72]

   In Chapter , when covering adjunctions, we will study a strong relation between the forgetful functor $U$ and the free functor $F$ that will generalize to other mathematical structures.

[72] More details about free monoids are in Chapter .

5. Let $(X, \leq)$ and $(Y, \sqsubseteq)$ be posets, and $F : X \rightsquigarrow Y$ be a functor between their posetal categories. For any $a, b \in X$, if $a \leq b$, then $\mathrm{Hom}_X(a,b)$ contains a single element, thus $\mathrm{Hom}_Y(F(a), F(b))$ must contain a morphism as well,[73] or equivalently $F(a) \sqsubseteq F(b)$. This shows that $F_0$ is an order-preserving function on the posets.

   Conversely, any order-preserving function between $X$ and $Y$ will correspond to a unique functor as there is only one morphism in all the hom-sets.[74]

[73] The image of the element in $\mathrm{Hom}_X(a,b)$ under $F$.

[74] Given $f : (X, \leq) \to (Y, \sqsubseteq)$ order-preserving, the corresponding functor between the posetal categories of $X$ and $Y$ acts like $f$ of the objects and sends a morphism $a \to b$ to the unique morphism $f(a) \to f(b)$ which exists because $a \leq b \implies f(a) \sqsubseteq f(b)$.

**Exercise 93.** Let $A$ and $B$ be two sets, their powersets can be seen as posets with the order $\subseteq$. Thus, we can view $\mathcal{P}(A)$ and $\mathcal{P}(B)$ as posetal categories.

- Draw (using points and arrows) the category corresponding to $\mathcal{P}(\{0,1,2\})$.

See solution.

- Show that the image and preimage functions defined below are functors between these categories.[75]

$$f : \mathcal{P}(A) \to \mathcal{P}(B) = S \mapsto \{f(a) \mid a \in S\}$$
$$f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A) = S \mapsto \{a \in A \mid f(a) \in S\}$$

6. Let $G$ and $H$ be groups and $\mathbf{B}G$ and $\mathbf{B}H$ be their respective deloopings, then the functors $F : \mathbf{B}G \rightsquigarrow \mathbf{B}H$ are exactly the group homomorphisms from $G$ to $H$.[76] Let $F : \mathbf{B}G \rightsquigarrow \mathbf{B}H$ be a functor, the action of $F$ on objects is trivial since there is only one object in both categories. On morphisms, $F_1$ is a function from $G$ to $H$ which preserves composition and the identity morphism which, by definition, are the group multiplication and identity respectively. Thus, $F_1$ is a group homomorphism.

   Given a homomorphism $f : G \to H$, the reverse reasoning shows we obtain a functor $\mathbf{B}G \rightsquigarrow \mathbf{B}H$ by acting trivially on objects and with $f$ on morphisms.

7. For any group $G$, the functors $F : \mathbf{B}G \rightsquigarrow \mathbf{Set}$ are in correspondence with left actions of $G$. Indeed, if $S = F(*)$, then

$$F_1 : G = \mathrm{Hom}_{\mathbf{B}G}(*, *) \to \mathrm{Hom}_{\mathbf{Set}}(S, S)$$

   is such that $F(gh) = F(g) \circ F(h)$ for any $g, h \in G$ and $F(1_G) = \mathrm{id}_S$.[77] Moreover, since for any $g \in G$,

$$F(g^{-1}) \circ F(g) = F(g^{-1}g) = F(1_G) = \mathrm{id}_S = F(1_G) = F(gg^{-1}) = F(g) \circ F(g^{-1}),$$

   the function $F(g)$ is a bijection (its inverse if $F(g^{-1})$) and we conclude $F_1$ is the permutation representation of the group action defined by $g \star s = F(g)(s)$ for all $g \in G$ and $s \in S$.

   Given a group action on a set $S$, we leave you to show that letting $F_0 = * \mapsto S$ and $F_1$ be the permutation representation of the action yields a functor $F : \mathbf{B}G \rightsquigarrow \mathbf{Set}$.

8. In the previous example, replacing $\mathbf{Set}$ with $\mathbf{Vect}_k$, one obtains $k$-linear representations of $G$ instead of actions of $G$.[78]

*Remark* 94 (Non-examples). From this long (and yet hardly exhaustive) list, one might get the feeling that every important mathematical transformation is a functor. This is not the case, so we wanted to show where functoriality can fail and hopefully give you a bit of intuition about why they fail. Here are two instances showcasing the two most common ways (in my experience) you can decide that a mapping is not functorial.

For instance, let us define $F : \mathbf{FDVect}_k \rightsquigarrow \mathbf{Set}$ which assigns to any vector space over $k$ a choice of basis. There is no non-trivializing way to define an action of $F$ on linear maps which make $F$ into a functor. One informal reason for this failure is that we cannot choose bases globally, so $F$ is defined locally and its parts cannot be glued together.[79]

Another non-example is given by the center[80] of a group in $\mathbf{Grp}$. A homomor-

[75] i.e.: they are order-preserving functions.

[76] Similarly for the deloopings of monoids.

[77] This is because $gh$ is the composite of $g$ and $h$ in $\mathbf{B}G$ and $1_G$ is the identity morphism in $\mathbf{B}G$.

[78] You might not know about linear representations, we just mention them in passing.

[79] If you feel like you are making a non-canonical choice for every object, there is a good chance you are not dealing with a functor.

[80] The **center** of a group $G$, often denoted $Z(G)$, is the subset of $G$ containing elements that commute with all other elements, i.e.:

$$Z(G) = \{x \in G \mid \forall g \in G, xg = gx\}.$$

phism $H \to G$ does not necessarily send the center of $H$ in the center of $G$ (take for instance $S_2 \hookrightarrow S_3$), thus, we cannot easily define the function $Z(H) \to Z(G)$ induced by the homomorphism (unless we send everything to $1_G \in Z(G)$). This time, $Z$ is not a functor because it does not interact well with the morphisms of the category. Actually, if you decided to only keep group isomorphisms in the category, you could define the functor $Z$ because isomorphisms preserve the center of groups.

In this chapter, we introduced a novel structure, namely categories, that functors preserve.[81] Since we also introduced several categories where objects had some structure that morphisms preserve, it is reasonable to wonder whether categories and functors are also part of a category. In fact, the only missing ingredient is the composition of functors (we already know what the source and target of a functor is and every category has an identity functor). After proving the following proposition, we end up with the category **Cat** where objects are small categories and morphisms are functors.[82]

[81] We defined functors precisely so that they preserve the structure of categories.

[82] In order to avoid paradoxes of the Russel kind, it is essential to restrict **Cat** to contain only small categories.

**Proposition 95.** *Let $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ and $G : \mathbf{D} \rightsquigarrow \mathbf{E}$ be functors and $G \circ F : \mathbf{C} \rightsquigarrow \mathbf{E}$ be their* ***composition*** *defined by $G_0 \circ F_0$ on objects and $G_1 \circ F_1$ on morphisms. Then, $G \circ F$ is a functor.*

*Proof.* One could proceed with a really hands-on proof and show that $G \circ F$ satisfies the three necessary properties in a straightforward manner. This should not be too hard, but you will have to deal with notation for objects, morphisms and the composition from all three different categories. This can easily lead to confusion or worse, boredom!

Instead, we will use the diagrams we introduced in the first definition of a functor. From the functoriality of $F$ and $G$, we get two sets of three diagrams and combining them yields the diagrams for $G \circ F$.[83]

[83] Since $F$ is a functor, the top two squares of (16) and the left squares of (17) and (18) commute. Since $G$ is a functor, the bottom two squares (16) and the right squares of (17) and (18) commute.

$$\begin{tikzcd}
\mathbf{C}_0 \arrow[d, "F_0"'] & \mathbf{C}_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "F_1"] & \mathbf{C}_0 \arrow[d, "F_0"] \\
\mathbf{D}_0 \arrow[d, "G_0"'] & \mathbf{D}_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "G_1"] & \mathbf{D}_0 \arrow[d, "G_0"] \\
\mathbf{E}_0 & \mathbf{E}_1 \arrow[l, "s"'] \arrow[r, "t"] & \mathbf{E}_0
\end{tikzcd} \tag{16}$$

$$\begin{tikzcd}
\mathbf{C}_2 \arrow[r, "F_2"] \arrow[d, "\circ_{\mathbf{C}}"'] & \mathbf{D}_2 \arrow[r, "G_2"] \arrow[d, "\circ_{\mathbf{D}}"] & \mathbf{E}_2 \arrow[d, "\circ_{\mathbf{E}}"] \\
\mathbf{C}_1 \arrow[r, "F_1"'] & \mathbf{D}_1 \arrow[r, "G_1"'] & \mathbf{E}_1
\end{tikzcd} \tag{17}
\qquad
\begin{tikzcd}
\mathbf{C}_0 \arrow[r, "F_0"] \arrow[d, "u_{\mathbf{C}}"'] & \mathbf{D}_0 \arrow[r, "G_0"] \arrow[d, "u_{\mathbf{D}}"] & \mathbf{E}_0 \arrow[d, "u_{\mathbf{E}}"] \\
\mathbf{C}_1 \arrow[r, "F_1"'] & \mathbf{D}_1 \arrow[r, "G_1"'] & \mathbf{E}_1
\end{tikzcd} \tag{18}$$

To finish the proof, you need to convince yourself that combining commutative diagrams in this way yields commutative diagrams. We proceed with a proof by example. Take diagram (18), we know the left and right square are commutative because $F$ and $G$ are functors. To show that the rectangle also commutes, we need to show the top path and bottom path from $\mathbf{C}_0$ to $\mathbf{E}_1$ compose to the same function. Here is the derivation:[84]

[84] In this case, both the diagram and the derivation are fairly simple. This will not stay true in the rest of the book, but the complexity of diagrams will grow way slower than the complexity of derivations and we will mostly omit the latter for this reason.

$$\begin{aligned} G_1 \circ F_1 \circ u_{\mathbf{C}} &= G_1 \circ u_{\mathbf{D}} \circ F_0 && \text{left square commutes} \\ &= u_{\mathbf{E}} \circ G_0 \circ F_0 && \text{right square commutes} \end{aligned}$$

□

Since functors are also a new structure, one might expect that there are transformations between functors that preserve it. It is indeed the case, they are called natural transformations and they are the main subject of Chapter **??**. Moreover, although we will not cover it, there is a whole tower of abstraction that one could build in this way and it is the subject of study of higher category theory.

## *Products*

There is one last thing we want to mention to end this chapter. We have defined two new mathematical objects, categories and functors and presented several examples of each. By defining products, we give you access to an unlimited amount of new categories and functors you can construct from known ones.[85]

**Definition 96** (Product category)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be two categories, the **product** of $\mathbf{C}$ and $\mathbf{D}$, denoted $\mathbf{C} \times \mathbf{D}$, is the category whose objects are pairs of objects in $\mathbf{C}_0 \times \mathbf{D}_0$ and for any two pairs $(X, Y), (X', Y') \in (\mathbf{C} \times \mathbf{D})_0$,[86]

$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{D}}((X,Y),(X',Y')) := \mathrm{Hom}_{\mathbf{C}}(X,X') \times \mathrm{Hom}_{\mathbf{D}}(Y,Y').$$

The identity morphisms and the composition are defined componentwise, i.e.: $\mathrm{id}_{(X,Y)} = (\mathrm{id}_X, \mathrm{id}_Y)$ and if $(f, f') \in \mathbf{C}_2$ and $(g, g') \in \mathbf{D}_2$ are two composable pairs, then $(f,g) \circ (f',g') = (f \circ f', g \circ g')$.[87]

**Exercise 97.** Show that the assignment $\Delta_{\mathbf{C}} : \mathbf{C} \rightsquigarrow \mathbf{C} \times \mathbf{C} = X \mapsto (X, X)$ is functorial, i.e.: give its action on morphisms and show it satisfies the relevant axioms. We call $\Delta_{\mathbf{C}}$ the **diagonal functor**.

**Definition 98** (Product functor)**.** Let $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$ and $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$ be two functors, the **product** of $F$ and $G$, denoted $F \times G : \mathbf{C} \times \mathbf{D} \rightsquigarrow \mathbf{C}' \times \mathbf{D}'$, is defined componentwise on objects and morphisms, i.e.: for any $(X, Y) \in (\mathbf{C} \times \mathbf{D})_0$ and $(f, g) \in (\mathbf{C} \times \mathbf{D})_1$

$$(F \times G)(X,Y) = (FX, GY) \text{ and } (F \times G)(f,g) = (Ff, Gg).$$

Let us check this defines a functor.

i. By definition of $\mathbf{C}' \times \mathbf{D}'$, $(Ff, Gg)$ is a morphism from $(FX, GY)$ to $(FX', GY')$.

ii. For $(f, f') \in \mathbf{C}_2$ and $(g, g') \in \mathbf{D}_2$, we have

$$\begin{aligned} (F \times G)((f,g) \circ (f',g')) &= (F \times G)(f \circ f', g \circ g') \\ &= (F(f \circ f'), G(g \circ g')) \\ &= (Ff \circ Ff', Gg \circ Gg') \\ &= (Ff, Gg) \circ (Ff', Gg') \\ &= (F \times G)(f,g) \circ (F \times G)(f',g'). \end{aligned}$$

[85] This is akin to product of groups, diret sums of vector spaces, etc. In Chapter , we will see how all of these constructions are instances of a more general construction called (categorical) product.

[86] Explicitly, a morphism $(X, Y) \to (X', Y')$ is a pair of morphisms $X \to X'$ and $Y \to Y'$.

[87] We leave you to check that this defines the composition of all morphisms in $\mathbf{C} \times \mathbf{D}$. Namely, if $(f, g)$ and $(f', g')$ are composable, then $(f, f')$ and $(g, g')$ are composable.

See solution.

iii. Since $F$ and $G$ preserve identity morphisms, we have

$$(F \times G)(\mathrm{id}_{(X,Y)}) = (F \times G)(\mathrm{id}_X, \mathrm{id}_Y) = (F\mathrm{id}_X, G\mathrm{id}_Y) = (\mathrm{id}_{FX}, \mathrm{id}_{GY}) = \mathrm{id}_{(FX,GY)}.$$

**Exercise 99** (NOW!). Let $F : \mathbf{C} \times \mathbf{C}' \to \mathbf{D}$ be a functor. For $X \in \mathbf{C}_0$, we define $F(X, -) : \mathbf{C}' \rightsquigarrow \mathbf{D}$ on objects by $Y \mapsto F(X,Y)$ and on morphisms by $g \mapsto F(\mathrm{id}_X, g)$. Show that $F(X, -)$ is a functor. Define $F(-, Y)$ similarly.

See solution.

We will often use $-$ as a **placeholder** for an input so that the latter remains nameless. For instance, $f(-,-)$ means $f$ takes two inputs. The type of the inputs and outputs will be made clear in the context.

**Exercise 100.** Let $F : \mathbf{C} \times \mathbf{C}' \to \mathbf{D}$ be an action defined on objects and morphisms satisfying

$$F(f,g) = F(f, \mathrm{id}_{t(g)}) \circ F(\mathrm{id}_{s(f)}, g) = F(\mathrm{id}_{t(f)}, g) \circ F(f, \mathrm{id}_{s(g)}).$$

See solution.

Show that if for any $X \in \mathbf{C}_0$ and $Y \in \mathbf{C}'_0$, $F(X, -)$ and $F(-, Y)$ as defined above are functors, then $F$ is a functor. In other words, the functoriality of $F$ can be proven componentwise.

In the next chapters, we will present other interesting constructions, but we can stop here for now.

# *Duality*

The concept of duality is ubiquitous throughout mathematics. It can relate two perspectives of the same object as for dual vector spaces, two complementary optimization problems such as a maximization and a minimization linear program and even two seemingly unrelated fields like topology and logic (Stone duality). While this vague principle of duality is behind many groundbreaking results, the duality in question here is categorical duality and it is a bit more precise.

Informally, there is nothing more to say than "Take all the diagrams in a definition/theorem, reverse the arrows and reap the benefits of the dual concept/result."[88] The more formal version will follow after we first exhibit the principle in action.

[88] In my opinion, this is already a very good reason to learn category theory because we can basically get twice as much math as before by framing things with a categorical language.

Recall that, intuitively, a functor is a structure preserving transformation between categories. A simple example we have seen was functors between posets that were order-preserving functions. However, as a consequence, one might conclude that order-reversing functions impair the structure of a poset, which feels arbitrary. The same happens between deloopings of groups because anti-homomorphisms[89] cannot arise as functors between such categories.

[89] An **anti-homomorphism** $f : G \to H$ is a function satisfying $f(gg') = f(g')f(g)$ and $f(1_G) = f(1_H)$.

There are two options to remedy this discrepancy between intuition and formalism; both have duality as an underlying principle. In this chapter, we will describe the two options, dismiss one of them and showcase the strength of duality while exploring more basic category theory.

## *Contravariant Functors*

By modifying Definition 83 to require that $F(f)$ goes in the opposite direction, we obtain a contravariant functor. Incidentally, what we defined as a functor before is also called a **covariant** functor.

**Definition 101** (Contravariant functor)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories, a **contravariant functor** $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ is a pair of maps $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ and $F_1 : \mathbf{C}_1 \to \mathbf{D}_1$ making diagrams (19), (20) and (21) commute.[90]

[90] Where $F_2'$ is now induced by the definition of $F_1$ with $(f, g) \mapsto (F_1(g), F_1(f))$.

$$\begin{tikzcd} \mathbf{C}_0 \arrow[d, "F_0"'] & \mathbf{C}_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "F_1"] & \mathbf{C}_0 \arrow[d, "F_0"] \\ \mathbf{D}_0 & \mathbf{D}_1 \arrow[l, "t"] \arrow[r, "s"'] & \mathbf{D}_0 \end{tikzcd} \tag{19}$$

$$\begin{tikzcd} \mathbf{C}_2 \arrow[r, "F_2'"] \arrow[d, "\circ_{\mathbf{C}}"'] & \mathbf{D}_2 \arrow[d, "\circ_{\mathbf{D}}"] \\ \mathbf{C}_1 \arrow[r, "F_1"'] & \mathbf{D}_1 \end{tikzcd} \tag{20}$$

$$\begin{tikzcd} \mathbf{C}_0 \arrow[r, "F_0"] \arrow[d, "u_{\mathbf{C}}"'] & \mathbf{D}_0 \arrow[d, "u_{\mathbf{D}}"] \\ \mathbf{C}_1 \arrow[r, "F_1"'] & \mathbf{D}_1 \end{tikzcd} \tag{21}$$

In words, $F$ must satisfy the following properties.

i. For any $A, B \in \mathbf{C}_0$, if $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ then $F(f) \in \mathrm{Hom}_{\mathbf{D}}(F(B), F(A))$.

ii. If $f, g \in \mathbf{C}_1$ are composable, then $F(f \circ g) = F(g) \circ F(f)$.

iii. If $A \in \mathbf{C}_0$, then $u_{\mathbf{D}}(F(A)) = F(u_{\mathbf{C}}(A))$.

**Examples 102.** Just like their covariant counterparts, contravariant functors are quite numerous. Here are a few simple ones, we leave you to check that they satisfy the diagrams above.

1. Contravariant functors $F : (X, \leq) \rightsquigarrow (Y, \sqsubseteq)$ correspond to order-reversing functions between the posets $X$ and $Y$ and contravariant functors $F : \mathbf{B}G \rightsquigarrow \mathbf{B}H$ correspond to anti-homomorphisms between the groups $G$ and $H$.

2. The **contravariant powerset functor** $2^{-} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$ sends a set $X$ to its powerset $2^X$[91] and a function $f : X \to Y$ to the preimage map $2^f : \mathcal{P}(Y) \to \mathcal{P}(X)$, the latter sends a subset $S \subseteq Y$ to
$$2^f(S) = f^{-1}(S) := \{x \in X \mid f(x) \in S\} \subseteq X.$$

[91] We are using a different notation for the powerset

Next, there is a couple of functors that are key to understand the philosophy put forward by category theory.[92]

[92] We will talk more about it when covering the Yoneda lemma in Chapter **??**.

**Example 103** (Hom functors)**.** Let $\mathbf{C}$ be a locally small category and $A \in \mathbf{C}_0$ one of its objects.[93] We define the covariant and contravariant **Hom functors** from $\mathbf{C}$ to **Set**.

[93] We need local smallness so that each $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is a set and the functors land in **Set**.

1. The covariant Hom functor $\mathrm{Hom}_{\mathbf{C}}(A, -) : \mathbf{C} \rightsquigarrow \mathbf{Set}$ sends an object $B \in \mathbf{C}_0$ to the hom-set $\mathrm{Hom}_{\mathbf{C}}(A, B)$ and a morphism $f : B \to B'$ to the function
$$\mathrm{Hom}_{\mathbf{C}}(A, f) : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A, B') = g \mapsto f \circ g.$$
This function is called **post-composition by** $f$ and is denoted $f \circ (-)$.[94] Let us show $\mathrm{Hom}_{\mathbf{C}}(A, -)$ is a covariant functor.

[94] Some authors denote $f \circ (-)$ as $f^*$, we prefer to keep this notation for later (see pullbacks).

   i. For any $f \in \mathbf{C}_1$, it is clear from the definition that
   $$\mathrm{Hom}_{\mathbf{C}}(A, s(f)) = s(f \circ (-)) \text{ and } \mathrm{Hom}_{\mathbf{C}}(A, t(f)) = t(f \circ (-)).$$

   ii. For any $(f_1, f_2) \in \mathbf{C}_2$, we claim that
   $$\mathrm{Hom}_{\mathbf{C}}(A, f_1 \circ f_2) = \mathrm{Hom}_{\mathbf{C}}(A, f_1) \circ \mathrm{Hom}_{\mathbf{C}}(A, f_2).$$
   In the L.H.S., an element $g \in \mathrm{Hom}_{\mathbf{C}}(A, s(f_1 \circ f_2))$ is mapped to $(f_1 \circ f_2) \circ g$ and in the R.H.S., an element $g \in \mathrm{Hom}_{\mathbf{C}}(A, s(f_2))$ is mapped to $f_1 \circ (f_2 \circ g)$. Since $s(f_1 \circ f_2) = s(f_2)$ and composition is associative, we conclude that the two maps are the same.

iii. For any $B \in \mathbf{C}_0$, the post-composition by $u_{\mathbf{C}}(B)$ is defined to be the identity,[95] hence (15) also commutes.

[95] Namely, for any $f : A \to B$, $u_{\mathbf{C}}(B) \circ f = f$.

2. The contravariant Hom functor $\mathrm{Hom}_{\mathbf{C}}(-, A) : \mathbf{C} \rightsquigarrow \mathbf{Set}$ sends an object $B \in \mathbf{C}_0$ to the hom-set $\mathrm{Hom}_{\mathbf{C}}(B, A)$ and a morphism $f : B \to B'$ to the function

$$\mathrm{Hom}_{\mathbf{C}}(f, A) : \mathrm{Hom}_{\mathbf{C}}(B', A) \to \mathrm{Hom}_{\mathbf{C}}(B, A) = g \mapsto g \circ f.$$

This function is called **pre-composition by** $f$ and is denoted $(-) \circ f$.[96] Let us show $\mathrm{Hom}_{\mathbf{C}}(-, A)$ is a contravariant functor.

[96] Some authors denote $(-) \circ f$ as $f_*$, we prefer to keep this notation for later (see pushouts).

i. For any $f \in \mathbf{C}_1$, it is clear from the definition that

$$\mathrm{Hom}_{\mathbf{C}}(s(f), A) = t((-) \circ f)) \text{ and } \mathrm{Hom}_{\mathbf{C}}(t(f), A) = s((-) \circ f).$$

ii. For any $(f_1, f_2) \in \mathbf{C}_2$, we claim that

$$\mathrm{Hom}_{\mathbf{C}}(f_1 \circ f_2, A) = \mathrm{Hom}_{\mathbf{C}}(f_2, A) \circ \mathrm{Hom}_{\mathbf{C}}(f_1, A).$$

In the L.H.S., an element $g \in \mathrm{Hom}_{\mathbf{C}}(t(f_1 \circ f_2), A)$ is mapped to $g \circ (f_1 \circ f_2)$ and in the R.H.S., an element $g \in \mathrm{Hom}_{\mathbf{C}}(t(f_1), A)$ is mapped to $(g \circ f_1) \circ f_2$. Since $t(f_1 \circ f_2) = t(f_1)$ and composition is associative, we conclude that the two maps are the same.

iii. For any $B \in \mathbf{C}_0$, pre-composition by $u_{\mathbf{C}}(B)$ is defined to be the identity,[97] hence (21) also commutes.

[97] Namely, for any $f : B \to A$, $f \circ u_{\mathbf{C}}(B) = f$.

Right now, we only give one example of a contravariant Hom functor, but we will study them more in depth in Chapter .

**Example 104** (Dual vector space)**.** In the category $\mathbf{Vect}_k$, there is a special object $k$,[98] let us see what the contravariant functor $\mathrm{Hom}_{\mathbf{Vect}_k}(-, k)$ does. It assigns to any vector space $V$, the set of linear maps $V \to k$, that is the carrier set of the dual space $V^*$. It assigns to linear maps $T : V \to W$, the function

[98] We know it is special because we know some linear algebra. Still, $k$ has some interesting categorical properties that we will not cover here.

$$\mathrm{Hom}_{\mathbf{Vect}_k}(W, k) \ni \phi \mapsto \phi \circ T \in \mathrm{Hom}_{\mathbf{Vect}_k}(V, k).$$

We know that $\mathrm{Hom}_{\mathbf{Vect}_k}(V, k) = V^*$ can be seen as a vector space and it is easy to check that pre-composition by $T$ is a linear map $W^* \to V^*$. Therefore, we find that the assignment $V \mapsto V^* = \mathrm{Hom}_{\mathbf{Vect}_k}(-, k)$ is a contravariant functor $\mathbf{Vect}_k \rightsquigarrow \mathbf{Vect}_k$.

We will not dwell too long on contravariant functors as we will see right away how they can be avoided, but first, let us give a reason why we want to avoid them.

See solution.

**Exercise 105.** Let $F : \mathbf{C} \rightsquigarrow \mathbf{D}$, $G : \mathbf{D} \rightsquigarrow \mathbf{E}$ be contravariant functors, and $G \circ F : \mathbf{C} \rightsquigarrow \mathbf{E}$ be their composition defined by $G_0 \circ F_0$ on objects and $G_1 \circ F_1$ on morphisms. Show that $G \circ F$ is a *covariant* functor.[99] Using diagrams will be easier.

[99] We conclude that we cannot straightforwardly compose contravariant functors. This alone makes the following alternative more desirable: we want functors to be morphisms in a category, hence they must be composable.

## *Opposite Category*

Another way to deal with order-reversing maps $(X, \leq) \to (Y, \subseteq)$ is to consider the reverse order on $X$ and a covariant functor $(X, \geq) \rightsquigarrow (Y, \subseteq)$. This also works for anti-homomorphisms by constructing the opposite group $G^{\mathrm{op}}$ in which the operation is reversed, namely $g \cdot^{\mathrm{op}} h = hg$. The opposite category is a generalization of these constructions.

**Definition 106** (Opposite category)**.** Let $\mathbf{C}$ be a category, we denote the **opposite** category with $\mathbf{C}^{\mathrm{op}}$ and define it by[100]

$$\mathbf{C}_0^{\mathrm{op}} = \mathbf{C}_0,\ \mathbf{C}_1^{\mathrm{op}} = \mathbf{C}_1,\ s^{\mathrm{op}} = t,\ t^{\mathrm{op}} = s,\ u_{\mathbf{C}^{\mathrm{op}}} = u_{\mathbf{C}}$$

with the composition defined by $f^{\mathrm{op}} \circ^{\mathrm{op}} g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$.[101] This naturally leads to the following contravariant functor $(-)^{\mathrm{op}}_{\mathbf{C}} : \mathbf{C} \rightsquigarrow \mathbf{C}^{\mathrm{op}}$ which sends an object $A$ to $A^{\mathrm{op}}$ and a morphism $f$ to $f^{\mathrm{op}}$. It is called the **opposite functor**.

With this definition, one can see contravariant functors as covariant functors. Formally, let $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ be a contravariant functor, we can view $F$ as covariant functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}$ or from $\mathbf{C}$ to $\mathbf{D}^{\mathrm{op}}$ via the compositions $F \circ (-)^{\mathrm{op}}_{\mathbf{C}^{\mathrm{op}}}$ and $(-)^{\mathrm{op}}_{D} \circ F$ respectively.[102]

In the rest of this book, we choose to work with functors of type $\mathbf{C}^{\mathrm{op}} \rightsquigarrow \mathbf{D}$ instead of contravariant functors.[103]

**Examples 107.** 1. As hinted at before, the category corresponding to $(X, \geq)$ is the opposite category of $(X, \leq)$ and $(\mathbf{B}G)^{\mathrm{op}}$ is the category corresponding to the opposite group of $G$, i.e.: $(\mathbf{B}G)^{\mathrm{op}} = \mathbf{B}(G^{\mathrm{op}})$.

2. We have seen that functors $\mathbf{B}G \rightsquigarrow \mathbf{Set}$ correspond to left actions of a group $G$. You can check that functors $\mathbf{B}G^{\mathrm{op}} \rightsquigarrow \mathbf{Set}$ correspond to right actions of $G$.

3. The two Hom functors defined in Example 103 are now written

$$\mathrm{Hom}_{\mathbf{C}}(A, -) : \mathbf{C} \rightsquigarrow \mathbf{Set} \text{ and } \mathrm{Hom}_{\mathbf{C}}(-, A) : \mathbf{C}^{\mathrm{op}} \rightsquigarrow \mathbf{Set}.$$

By Exercise 100, they can be combined into a functor

$$\mathrm{Hom}_{\mathbf{C}}(-, -) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \rightsquigarrow \mathbf{Set}$$

acting on objects as $(A, B) \mapsto \mathrm{Hom}_{\mathbf{C}}(A, B)$ and on morphisms as $(f, g) \mapsto (g \circ - \circ f)$. The condition in Exercise 100 is satisfied because[104]

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{C}}(f, g) &= g \circ - \circ f \\
&= \mathrm{id}_{t(g)} \circ (g \circ - \circ \mathrm{id}_{t(f)}) \circ f = \mathrm{Hom}_{\mathbf{C}}(f, \mathrm{id}_{t(g)}) \circ \mathrm{Hom}_{\mathbf{C}}(\mathrm{id}_{t(f)}, g) \\
&= g \circ (\mathrm{id}_{s(g)} \circ - \circ f) \circ \mathrm{id}_{s(f)} = \mathrm{Hom}_{\mathbf{C}}(\mathrm{id}_{s(f)}, g) \circ \mathrm{Hom}_{\mathbf{C}}(f, \mathrm{id}_{s(g)}).
\end{aligned}$$

This will be called the Hom **bifunctor**.

**Exercise 108.** Let $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ be a functor, show that its dual $F^{\mathrm{op}}$ defined by $A^{\mathrm{op}} \mapsto (FA)^{\mathrm{op}}$ on objects and $f^{\mathrm{op}} \mapsto (Ff)^{\mathrm{op}}$ on morphisms is a functor $\mathbf{C}^{\mathrm{op}} \rightsquigarrow \mathbf{D}^{\mathrm{op}}$.

See solution.

[100] Intuitively, we reverse the direction of all morphisms in $\mathbf{C}$ and reverse the order of composition as well.

[101] Note that the $-^{\mathrm{op}}$ notation here is just used to distinguish elements in $\mathbf{C}$ and $\mathbf{C}^{\mathrm{op}}$ but the class of objects and morphisms are the same.

[102] Recall from Exercise 105 that these compositions are covariant.

[103] We still had to introduce the notion because you might encounter contravariant functors in the wild.

[104] Looking at where the source and target functions are applied, these equalities do not match exactly what is in Exercise 100 since $\mathrm{Hom}_{\mathbf{C}}(-, -)$ is contravariant in the second component.

*Remark* 109. It is sometimes useful to compose the Hom bifunctor with other functors as follows. Given two functors $F, G : \mathbf{C} \rightsquigarrow \mathbf{D}$, there is a functor $\mathrm{Hom}_{\mathbf{D}}(F-, G-) : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \rightsquigarrow \mathbf{D}$ acting on objects by $(X, Y) \mapsto \mathrm{Hom}_{\mathbf{D}}(FX, GY)$ and on morphisms by $(f, g) \mapsto Gg \circ (-) \circ Ff$. One can check functoriality by showing

$$\mathrm{Hom}_{\mathbf{D}}(F-, G-) = \mathrm{Hom}_{\mathbf{D}}(-, -) \circ (F^{\mathrm{op}} \times G).$$

## *Duality in Action*

Let us start illustrating how duality can be useful with some simple definitions and results.

**Definition 110** (Monomorphism)**.** Let $\mathbf{C}$ be a category, a morphism $f \in \mathbf{C}_1$ is said to be **monic** (or a **monomorphism**) if for any parallel morphisms $g$ and $h$ such that $(f, g), (f, h) \in \mathbf{C}_2$, $f \circ g = f \circ h$ implies $g = h$. Equivalently, $f$ is monic if $g = h$ whenever the following diagram commutes.[105]

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & \bullet \arrow[r, "f"] & \bullet \end{tikzcd} \tag{22}$$

[105] According to Definition 66, this diagram commutes if and only of $f \circ g = f \circ h$ because the paths $(f, g)$ and $(f, h)$ are the only paths of length bigger than one.

Standard notation for a monomorphism is $\bullet \hookrightarrow \bullet$ (`\hookrightarrow`).

**Proposition 111.** *Let $\mathbf{C}$ be a category and $f : A \to B$ a morphism, if there exists $f' : B \to A$ such that $f' \circ f = \mathrm{id}_A$,*[106] *then $f$ is a monomorphism.*

[106] We say that $f'$ is a **left inverse** of $f$.

*Proof.* If $f \circ g = f \circ h$, then $f' \circ f \circ g = f' \circ f \circ h$ implying $g = h$. □

Not all monomorphisms have a left inverse, those that do are called **split monomorphisms**.

**Proposition 112.** *Let $\mathbf{C}$ be a category and $(f_1, f_2) \in \mathbf{C}_2$, if $f_1 \circ f_2$ is a monomorphism, then $f_2$ is a monomorphism.*

*Proof.* Let $g, h \in \mathbf{C}_1$ be such that $f_2 \circ g = f_2 \circ h$, we readily get that $(f_1 \circ f_2) \circ g = (f_1 \circ f_2) \circ h$. Since $f_1 \circ f_2$ is a monomorphism, this implies $g = h$. □

The last two results hint at the fact that monomorphisms are analogues to injective functions and we will see that they are exactly the same in the category **Set**, but first let us introduce the dual concept after the formal definition of duality.

**Definition 113** (Duality)**.** Given a definition or statement in an arbitrary category $\mathbf{C}$, one could view this concept inside the category $\mathbf{C}^{\mathrm{op}}$ and obtain a similar definition or statement where all morphisms and the order of composition are reversed, this is called the **dual** concept. For a definition or result where multiple *arbitrary* categories are involved, the dual version is obtained by taking the opposite of all categories.[107] It is common to refer to a dual notion with the prefix "co" (e.g.: presheaf and copresheaf).

[107] Note the emphasis on the word "arbitrary". For instance, a **presheaf** is a functor $F : \mathbf{C}^{\mathrm{op}} \rightsquigarrow$ **Set** and the dual concept is a **copresheaf**, a functor $F : \mathbf{C} \rightsquigarrow$ **Set**; we did not take the opposite of **Set**.

Dualizing the definition of a monomorphism yields an epimorphism.

**Definition 114** (Epimorphism)**.** Let $\mathbf{C}$ be a category, a morphism $f \in \mathbf{C}_1$ is said to be **epic** (or an **epimorphism**) if for any two parallel morphisms $g$ and $h$ such that $(g, f), (h, f) \in \mathbf{C}_2$, $g \circ f = h \circ f$ implies $g = h$. Equivalently, $f$ is epic if $g = h$ whenever the following diagram commutes.[108]

$$\begin{tikzcd} \bullet \arrow[r, "f"] & \bullet \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & \bullet \end{tikzcd} \qquad (23)$$

Standard notation for an epimorphism is $\bullet \twoheadrightarrow \bullet$ (`\twoheadrightarrow`).

The dual versions of Propositions 111 and 112 also hold. Although translating our previous proofs to the dual case is straightforward, we will do the two next proofs relying on duality to convey the general sketch that works anytime a dual result needs to be proven.

**Proposition 115.** *Let $\mathbf{C}$ be a category and $f : A \to B$ a morphism, if there exists $f' : B \to A$ such that $f \circ f' = \mathrm{id}_B$, then $f$ is epic.*[109]

*Proof.* Observe that $f$ is epic in $\mathbf{C}$ if and only if $f^{\mathrm{op}}$ is monic in $\mathbf{C}^{\mathrm{op}}$ (reverse the arrows in the definition).[110] Moreover, by definition,

$$f'^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ f')^{\mathrm{op}} = \mathrm{id}_B{}^{\mathrm{op}} = \mathrm{id}_{B^{\mathrm{op}}},$$

so by the result for monomorphisms, $f^{\mathrm{op}}$ is monic and hence $f$ is epic. □

Not all epimorphisms have a right inverse, those that do are called **split epimorphisms**.

**Proposition 116.** *Let $\mathbf{C}$ be a category and $(f_1, f_2) \in \mathbf{C}_2$, if $f_1 \circ f_2$ is epic, then $f_1$ is epic.*

*Proof.* Since $f_2{}^{\mathrm{op}} \circ f_1{}^{\mathrm{op}} = (f_1 \circ f_2)^{\mathrm{op}}$ is monic, the result for monomorphisms implies $f_1{}^{\mathrm{op}}$ is monic and hence $f_1$ is epic. □

**Example 117** (**Set**)**.** We mentioned that monomorphisms are like generalizations of injective functions, and you may have guessed that epimorphisms are, in the same sense, generalizations of surjective functions. Let us make this precise.

- A function $f : A \to B$ is a monomorphism in **Set** if and only if it is injective:[111]

  ($\Leftarrow$) Since $f$ is injective, it has a left inverse, so it is monic by Proposition 111.

  ($\Rightarrow$) Given $a \in A$, let $g_a : \{*\} \to A$ be the function sending $*$ to $a$. For any $a_1 \neq a_2 \in A$, the functions $g_{a_1}$ and $g_{a_2}$ are different, hence $f \circ g_{a_1} \neq f \circ g_{a_2}$. Therefore, $f(a_1) \neq f(a_2)$ and since $a_1$ and $a_2$ were arbitrary, $f$ is injective.

- A function $f : A \to B$ is an epimorphism if and only if it is surjective:[112]

  ($\Leftarrow$) Since $f$ is surjective, it has a right inverse, so it is epic by Proposition 115.

  ($\Rightarrow$) Let $h : B \to \{0, 1\}$ be the constant function at 1 and $g : B \to \{0, 1\}$ be the indicator function of $\mathrm{Im}(f) \subseteq B$, namely,

  $$g(x) = \begin{cases} 1 & \exists a \in A, x = f(a) \\ 0 & \text{o/w} \end{cases}.$$

[108] Seeing the diagrams make it clearer that the concepts are dual. Reversing the arrows in (22) yields (23) and vice-versa.

[109] We say that $f'$ is a **right inverse** of $f$.

[110] This is another way to see that two concepts are dual.

[111] As a consequence, since all injective functions have a left inverse, all the monomorphisms in **Set** are split monic.

[112] If you assume the axiom of choice, all surjective functions have a right inverse and thus all epimorphisms in **Set** are split epic.

We see that $g \circ f = h \circ f \equiv 1$, and $f$ being epic implies $g = h$. Thus, any element of $B$ is in the image of $f$, that is, $f$ is surjective.

**Example 118 (Mon).** Inside the category **Mon**, the monomorphisms correspond exactly to injective homomorphisms.

($\Rightarrow$) Let $f : M \to M'$ be an injective homomorphisms and $g_1, g_2 : N \to M$ be two parallel homomorphisms. Suppose that $f \circ g_1 = f \circ g_2$, then for all $x \in N$, $f(g_1(x)) = f(g_2(x))$, so by injectivity of $f$, $g_1(x) = g_2(x)$. Therefore, $g_1 = g_2$ and since $g_1$ and $g_2$ were arbitrary, $f$ is a monomorphism.

($\Leftarrow$) Let $f : M \to M'$ be a monomorphism. Let $x, y \in M$ and define $p_x : (\mathbb{N}, +) \to M$ by $k \mapsto x^k$ and similarly for $p_y$. It is easy to show that $p_x$ and $p_y$ are homomorphisms.[113] If $f(x) = f(y)$, then, by the homomorphism property, for all $k \in \mathbb{N}$

$$f(p_x(k)) = f(x^k) = f(x)^k = f(y)^k = f(y^k) = f(p_y(k)).$$

In other words, we get $f \circ p_x = f \circ p_y$, so $p_x = p_y$ and $x = y$. This direction follows.

Conversely, an epimorphism is not necessarily surjective. For example, the inclusion homomorphism $i : (\mathbb{N}, +) \to (\mathbb{Z}, +)$ is clearly not surjective, but it is an epimorphism. Indeed, let $g, h : (\mathbb{Z}, +) \to M$ be two monoid homomorphisms satisfying $g \circ i = h \circ i$. In particular, $g(n) = h(n)$ for any $n \in \mathbb{N} \subset \mathbb{Z}$. It remains to show that also $g(-n) = h(-n)$: we have

$$\begin{aligned} h(n)g(-n) = g(n)g(-n) = g(n-n) = g(0) = 1_M \\ h(-n)h(n) = h(-n+n) = h(0) = 1_M, \end{aligned}$$

but then $g(-n) = h(-n)h(n)g(-n) = h(-n)$.

[113] It follows from the definition of $x^k$ which is $x \overset{k}{\cdots} x$.

**Exercise 119.** Show that monomorphism in **Cat** is a functor that is faithful and injective on objects, it is called an **embedding**.[114]

See solution.

[114] Finding a nice characterization of epimorphisms in **Cat** is an open question as far as I know.

**Exercise 120.** Show that a morphism $f \in \mathbf{C}_1$ is monic if and only if the function $\mathrm{Hom}_{\mathbf{C}}(A, f)$ is injective for all $A \in \mathbf{C}_0$. Dually, show that $f$ is epic if and only if the function $\mathrm{Hom}_{\mathbf{C}}(f, A)$ is injective for all $A \in \mathbf{C}_0$.

See solution.

*Remark* 121. These alternate definitions of monomorphisms and epimorphisms are more categorical in nature. In fact, in the setting of enriched category theory they are preferable because they generalize easily.

**Definition 122** (Isomorphism)**.** Let **C** be a category, a morphism $f : A \to B$ is said to be an **isomorphism** if there exists a morphism $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.[115]

[115] Then $f^{-1}$ is called the **inverse** of $f$. One can check that if $f'$ is a left inverse of $f$ and $f''$ is a right inverse, then $f' = f'' = f^{-1}$. Hence, the inverse is unique.

**Exercise 123.** Show that the property of being monic/epic/an isomorphism is invariant under composition, i.e., if $f$ and $g$ are composable monomorphisms, then $f \circ g$ is monic and similarly for epimorphisms and isomorphisms.

See solution.

*Remark* 124. The results shown about monic and epic morphisms[116] imply that any isomorphism is monic and epic. However, the converse is not true as witnessed by

[116] Proposition 111 and 115.

the inclusion morphism $i$ described in Example 118.[117] A category where all monic and epic morphisms are isomorphisms (e.g.: **Set**) is called **balanced**. If there exists an isomorphism between two objects $A$ and $B$, then they are **isomorphic**, denoted $A \cong B$. Isomorphic objects are also isomorphic in the opposite category,[118] that is, the concept of **isomorphism** is **self-dual**.

[117] This is not akin to the situation in **Set** because, there, all monomorphisms and epimorphisms are split (assuming the axiom of choice).

[118] Because the left inverse becomes the right inverse and vice-versa.

For most intents and purposes, we will not distinguish between isomorphic objects in a category because all the properties we care about will hold for one if and only if they hold for the other. This attitude should be somewhat familiar if you have done a bit of abstract algebra because it is natural to substitute the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ for $\mathbb{Z}/6\mathbb{Z}$ or $k^n$ for an $n$–dimensional vector space over $k$. It is slightly less natural in **Set** because, for instance, it equates the sets $\{0,1\}$ and $\{a,b\}$ which may be too coarse-grained for our intuition.

**Example 125** (**Set**)**.** A function $f : X \to Y$ in $\mathbf{Set}_1$ has an inverse $f^{-1}$ if and only if $f$ is bijective, thus isomorphisms in **Set** are bijections. As a consequence, we have $A \cong B$ if and only if $|A| = |B|$.[119]

[119] This is in fact the definition of cardinality.

**Example 126** (**Cat**)**.** An isomorphism in **Cat** is a functor $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ with an inverse $F^{-1} : \mathbf{D} \rightsquigarrow \mathbf{C}$. This implies that $F_0$ and $F_1$ are bijections[120] because $F_0^{-1}$ is the inverse of $F_0$ and $F_1^{-1}$ is the inverse of $F_1$.

[120] Note that $F_1$ being a bijection implies that $F$ is fully faithful.

Conversely, if $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ is a functor whose components on objects and morphisms are bijective, we check that defining $F^{-1} : \mathbf{D} \rightsquigarrow \mathbf{C}$ with $F_0^{-1} := (F_0)^{-1}$ and $F_1^{-1} = (F_1)^{-1}$ yields a functor.

i. Let $f \in \mathrm{Hom}_{\mathbf{D}}(A,B)$, by bijectivity of $F_0$ and $F_1$, there are $X, Y \in \mathbf{C}_0$ and $g : X \to Y$ such that $FX = A$, $FY = B$ and $Fg = f$. Then, by definition,
$$s(F^{-1}f) = s(g) = X = F^{-1}FX = F^{-1}A, \text{ and}$$
$$t(F^{-1}f) = t(g) = Y = F^{-1}FY = F^{-1}B.$$

ii. For any $(f, f') \in \mathbf{D}_2$ with $f = Fg$ and $f' = Fg'$, we find
$$F^{-1}(f \circ f') = F^{-1}(Fg \circ Fg') = F^{-1}F(g \circ g') = g \circ g' = F^{-1}Fg \circ F^{-1}Fg' = Ff \circ Ff'.$$

iii. For any $A \in \mathbf{D}_0$ with $A = FX$, we find
$$F^{-1}\mathrm{id}_A = F^{-1}\mathrm{id}_{FX} = F^{-1}F\mathrm{id}_X = \mathrm{id}_X = \mathrm{id}_{F^{-1}FX} = \mathrm{id}_{F^{-1}A}.$$

We conclude that isomorphisms are precisely the fully faithful functors which are bijective on objects.

**Examples 127** (Concrete categories)**.** 1. It is a simple exercise in an algebra class to show that isomorphisms in the categories **Mon**, **Grp**, **Ring**, **Field** and $\mathbf{Vect}_k$ are the isomorphisms in their respective theory, namely, bijective homomorphisms.

2. In **Poset**, isomorphisms are bijective order-preserving functions.

3. In **Top**, it is not enough to have a bijective continuous function, we need to require that it has a continuous inverse. Such functions are called *homeomorphisms*.

**Definition 128** (Initial object)**.** Let $\mathbf{C}$ be a category, an object $A \in \mathbf{C}_0$ is said to be **initial** if for any $B \in \mathbf{C}_0$, $|\mathrm{Hom}_{\mathbf{C}}(A, B)| = 1$, namely there are no two parallel morphisms with source $A$ and every object has a morphism coming from $A$. The[121] initial object of a category, if it exists, is denoted $\emptyset$ and the *unique* morphism from $\emptyset$ to $X \in \mathbf{C}_0$ is denoted $() : \emptyset \to X$.

[121] We will soon see why we can use *the* instead of *an*.

**Definition 129** (Terminal object)**.** Let $\mathbf{C}$ be a category, an object $A \in \mathbf{C}_0$ is said to be **terminal** (or final) if for any $B \in \mathbf{C}_0$, $|\mathrm{Hom}_{\mathbf{C}}(B, A)| = 1$, namely there are no two parallel morphisms with target $A$ and every object has a morphism going to $A$. The terminal object of a category, if it exists, is denoted $\mathbf{1}$ and the *unique* morphism from $X \in \mathbf{C}_0$ into $\mathbf{1}$ is denoted $[] : X \to \mathbf{1}$.

*Remark* 130 (Notation)*.* The motivation behind the notations $\emptyset$ and $\mathbf{1}$ is given shortly, but the notations for the morphisms will be explained in Chapter .

An object is initial in a category $\mathbf{C}$ if and only if it is terminal in $\mathbf{C}^{\mathrm{op}}$. Also, if an object is initial and terminal, we say it is a **zero** object and usually denote it $\mathbf{0}$.

**Example 131** (**Set**)**.** Let $X$ be a set, there is a unique function from the empty set into $X$, it is the empty function.[122] We infer that the empty set is the initial object in **Set**, hence the notation $\emptyset$. For the terminal object, we observe that there is a unique function $X \to \{*\}$ sending all elements of $X$ to $*$, thus $\{*\}$ is terminal in **Set**.

[122] Recall (or learn here) that a function $f : A \to B$ is defined via subset of $f \subseteq A \times B$ that satisfies $\forall a \in A, \exists! b \in B, (a, b) \in f$. When $A$ is empty, $A \times B$ is empty and the unique subset of $\emptyset \subseteq A \times B$ satisfies the condition vacuously. In passing, when $B$ is empty but $A$ is not, the unique subset of $A \times B$ does not satisfy the condition, so there is no function $A \to \emptyset$.

In this example, we could have chosen any singleton to show it is terminal. However, that choice is irrelevant to a good category theorist since, as any two singletons are isomorphic (because they have the same cardinality), any two terminal objects are isomorphic.

**Proposition 132.** *Let $\mathbf{C}$ be a category and $A, B \in \mathbf{C}_0$ be initial, then $A \cong B$.*

*Proof.* Let $f$ be the single element in $\mathrm{Hom}_{\mathbf{C}}(A, B)$ and $f'$ be the single element in $\mathrm{Hom}_{\mathbf{C}}(B, A)$. Since the identity morphisms are the only elements of $\mathrm{Hom}_{\mathbf{C}}(A, A)$ and $\mathrm{Hom}_{\mathbf{C}}(B, B)$, $f' \circ f$ and $f \circ f'$, belonging to these sets, must be the identities. In other words $f$ and $f'$ are inverses, thus $A \cong B$. □

**Corollary 133** (Dual)**.** *Let $\mathbf{C}$ be a category and $A, B \in \mathbf{C}_0$ be terminal, then $A \cong B$.*[123]

[123] From now on, I let you prove many dual results on your own — I will try to continue doing the complicated ones. They are not necessarily great exercises, but you can certainly do them if you want to follow this book at a slower pace.

Rewording the last two results, we can say that initial (resp. terminal) objects are unique up to isomorphisms. However, the situation is quite nicer. Initial (resp. terminal) objects are unique up to *unique* isomorphisms. Indeed, if there is an isomorphism $f : A \to B$ and $A$ and $B$ are initial (resp. terminal), then, by definition, $f$ is the unique morphism in $\mathrm{Hom}_{\mathbf{C}}(A, B)$.

See solution.

**Exercise 134.** Show that in **Cat**, the initial object is the empty category (no objects and no morphisms) and the terminal object is $\mathbf{1}$ (hence the notation).[124]

[124] **Hint**: the unique functor $[] : \mathbf{C} \to \mathbf{1}$ is the constant functor at the object $\bullet \in \mathbf{1}_0$.

**Example 135** (**Grp**)**.** Similarly to **Set**, the trivial group with one element is terminal in **Grp**. Moreover, note that there are no empty group (because there is no identity element), but any group homomorphism from the trivial group $\{1\}$ into a group $G$ must send $1$ to $1_G$, which completely determines the homomorphism. Therefore, the trivial group is also initial in **Grp**, it is the zero object.

See solution.

**Exercise 136.** Find a category with only two objects $X$ and $Y$ such that

(a) $X$ is initial but not terminal and $Y$ is terminal but not initial.

(b) $X$ is initial but not terminal and $Y$ neither terminal nor initial.

(c) $X$ is terminal but not initial and $Y$ is neither terminal nor initial.

(d) $X$ is initial and terminal and $Y$ is neither terminal nor initial.

**Examples 137.** Here are more examples of categories where initial and terminal objects may or may not exist.

1. $\exists$ terminal, $\nexists$ initial: Consider the poset $(\mathbb{N}, \geq)$ represented by diagram (24). It is clear that $0$ is terminal and no element can be initial because $0 \geq x$ implies $x = 0$.

$$\overset{0}{\bullet} \longleftarrow \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \cdots \tag{24}$$

2. $\nexists$ terminal, $\exists$ initial:[125] Recall the category **SetInj** of finite sets and injective functions. The empty set is still initial but the singletons are not terminal because a function from a set $S$ into $\{*\}$ is never injective when $|S| > 1$.

[125] Of course, you could take the opposite of $(\mathbb{N}, \geq)$, that is $(\mathbb{N}, \leq)$, but that is not fun.

3. $\nexists$ terminal, $\nexists$ initial: Let $G$ be a non-trivial group, the delooping of $G$ has no terminal and no initial objects. The category $\mathbf{B}G$ has a single object $*$ with $\mathrm{Hom}_{\mathbf{B}G}(*, *) = G$, so $*$ cannot be initial nor terminal when $|G| > 1$.

   For a more interesting example, consider the category **Field**. Its underlying directed graph is disconnected[126] because there are no field homomorphisms between fields of different characteristic. Therefore, **Field** has no initial nor terminal objects.

[126] There are objects with no morphisms between them.

4. $\exists$ terminal, $\exists$ initial: The empty set is both initial and terminal in the category **Rel** because a relation between $\varnothing$ and $A$ is either a subset of $\varnothing \times A$ or $A \times \varnothing$, and the latter both have a unique subset for all sets $A$.

   For an example with no zero object, let $X$ be a non-empty topological space where $\tau$ is the collection of open sets.[127] The category of open sets $\mathcal{O}(X)$ satisfies

$$\mathrm{Hom}_{\mathcal{O}(X)}(U, V) = \begin{cases} \{i_{U,V}\} & U \subseteq V \\ \varnothing & U \not\subseteq V \end{cases}$$

   Since the empty set is contained in every open set, it is an initial object. Since the full set $X$ contains every open set, it is a terminal object. No other set can be initial as it cannot be contained in $\varnothing$ nor be terminal as it cannot contain $X$. Moreover, note that the two objects are not isomorphic because $X \not\subseteq \varnothing$.

[127] Recall that it must contain $\varnothing$ and $X$.

See solution.

**Exercise 138.** Let $\mathbf{C}$ be a category with a terminal object $\mathbf{1}$. Show any morphism $f : \mathbf{1} \to X$ is monic. State and prove the dual statement.

**Example 139.** For our last application of duality in this section,[128] let $X$ be a set and consider the posetal category $(\mathcal{P}(X), \subseteq)$. We would like to define the union

[128] Don't worry, we will have plenty of opportunities to use duality later.

of two subsets of $X$ in this category. The usual definition $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$ is not suitable because the data in the posetal category $\mathcal{P}(X)$ never refers to elements of $X$. In particular, the subsets $A, B \subseteq X$ are simply objects in the category and it is not clear to us how we can determine what elements are in $A$ and $B$ with our categorical tools (objects and morphisms).

We propose another characterization of the union of $A$ and $B$. First, what is obvious, $A \cup B$ contains $A$ and it contains $B$. Second, $A \cup B$ is the smallest subset of $X$ containing $A$ and $B$. Indeed, if $Y \subseteq X$ contains all element in $A$ and $B$, then it also contains $A \cup B$. Using the order $\subseteq$ (or equivalently, the morphisms in the category $\mathcal{P}(X)$), we have $A, B \subseteq A \cup B$ and $\forall Y$ s.t. $A, B \subseteq Y$ then $A \cup B \subseteq Y$.[129] This yields a definition of $\cup$ within the category $\mathcal{P}(X)$, which means we can dualize it.

[129] We leave it as an exercise to show that $A \cup B$ is the only subset of $X$ satisfying this property.

The dual of this property (reversing all inclusions) is as follows.[130]

[130] The symbol $\square$ is a placeholder for the operation which we will find to be dual to union.

$$A \square B \subseteq A, B \text{ and } \forall Y \text{ s.t. } Y \subseteq A, B \text{ then } Y \subseteq A \square B$$

Putting this in words, $A \square B$ is the largest subset of $X$ which is contained in $A$ and $B$. That is, of course, the intersection $A \cap B$. In this way, union and intersection are dual operations. If you search your memory for properties about union and intersection that you proved when you first learned about sets, you will find that they usually come in pairs; the first property being the dual of the second.[131]

[131] e.g.:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

## *More Vocabulary*

In the next chapter, we will start heavily using diagrams, so before going further, we need to define the formal notion that we will use[132] and practice diagram paving. We also introduce a couple of new concepts and their dual to keep practicing with the fundamental notions of this chapter.

[132] In fact, we will now refrain from referring to every picture on the page as a diagram and keep this terminology for the formal use (without necessarily making it explicitly formal).

**Definition 140** (Diagram)**.** A **diagram** in $\mathbf{C}$ is a functor $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ where $\mathbf{J}$ is usually a small or even finite category. We say that $J$ is the **shape** of the diagram $F$.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \qquad (25)$$

*Remark* 141. Diagrams are usually represented by (partially) drawing the image of $F$. All the diagrams drawn up to this point define the domain of the functor implicitly. For instance, when considering a commutative square in $\mathbf{C}$, what is actually considered is the image from a functor with codomain $\mathbf{C}$ and domain the category $\mathbf{2} \times \mathbf{2}$ represented in (**??**).

Since diagrams are defined as functors, they interact well with other functors. Actually, if $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ is a diagram of shape $\mathbf{J}$ in $\mathbf{C}$ and $G : \mathbf{C} \rightsquigarrow \mathbf{D}$ is a functor, then $G \circ F$ is a diagram of shape $\mathbf{J}$ in $\mathbf{D}$. Some functors interact even more nicely with diagrams.

**Definition 142.** Let $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$ be a functor and $P$ a property[133] of diagrams.

- We say that $F$ **preserves** diagrams with property $P$ if for any diagram $D : \mathbf{J} \rightsquigarrow \mathbf{C}$, if $D$ has property $P$, then $F \circ D$ has property $P$.

- We say that $F$ **reflects** diagrams with property $P$ if for any diagram $D : \mathbf{J} \rightsquigarrow \mathbf{C}$, if $F \circ D$ has property $P$, then $D$ has property $P$.

[133] This is intentionally a vague term. In Chapter , we will have a more formal but less general definition of preserving and reflecting.

*Warning* 143. Preserving and reflecting a property $P$ are not dual notions. The dual of preserving (resp. reflecting) $P$ is preserving (resp. reflecting) the dual of $P$.

It follows easily from functoriality that functors preserve commutative diagrams. The following two exercises are a quick investigation in preservation and reflection of simple properties we have seen in this chapter.

See solution.

**Exercise 144.** 1. Find an example of functor which does not preserve monomorphisms.[134]

2. Show that if $f \in \mathbf{C}_1$ is a split monomorphism, then $F(f)$ is also a split monomorphism, i.e.: any functor preserves split monomorphisms.

3. State and prove the dual statement.

4. Infer that all functors preserve isomorphisms, in particular functors send isomorphic objects to isomorphic objects.

[134] We can see a morphism as a diagram of shape $\mathbf{2}$. Indeed, a functor $\mathbf{2} \rightsquigarrow \mathbf{C}$ amounts to a choice of a morphism in $\mathbf{C}_1$. Therefore, a functor $F$ preserves monomorphisms if whenever $f$ is monic, $F(f)$ also is.

See solution.

**Exercise 145.** 1. Find an example of functor which does not reflect monomorphisms.[135]

2. Show that if $F$ is faithful, then $F$ reflects monomorphisms.

3. State and prove the dual statement.

[135] A functor reflects monomorphisms if whenever $Ff$ is monic, $f$ also is.

The next set theoretical notion we categorify is subsets. A subset $I \subseteq S$ can be identified with the inclusion function $I \hookrightarrow S$, and since the latter is injective, we may want to consider monomorphisms with target $S$ to be some kind of generalized subset. Observe however that an injection $I \hookrightarrow S$ is not necessarily an inclusion function. This does not matter because, in reality, we are interested in the image of this injection. We run into another obstacle because if two injections into $S$ have the same image, they represent the same subset. We overcome this using the following exercise.

See solution.

**Exercise 146.** Let $\mathbf{C}$ be a category and $X \in \mathbf{C}_0$, we define the relation $\sim$ on monomorphisms with target $X$ by

$$m \sim m' \Leftrightarrow \exists \text{ isomorphism } i, m = m' \circ i.$$

Show that $\sim$ is an equivalence relation.

**Definition 147** (Subobject)**.** Let $\mathbf{C}$ be a category, a **subobject** of $X \in \mathbf{C}_0$ is an equivalence class of the relation $\sim$ defined above. We will often abusively refer to a subobject simply with a monomorphism $Y \hookrightarrow X$. The collection of subobjects of $X$ is denoted $\mathrm{Sub}_{\mathbf{C}}(X)$. If for any $X \in \mathbf{C}_0$, $\mathrm{Sub}_{\mathbf{C}}(X)$ is a set, we say that $\mathbf{C}$ is **well-powered**.

**Example 148 (Set).** Let $X \in \mathbf{Set}_0$, subobjects of $X$ correspond to subsets of $X$.[136] Indeed, any subset $I \subseteq X$ has an inclusion function $i : I \hookrightarrow X$ which is injective, hence monic. For the other direction, we can show that $i : I \hookrightarrow X$ and $j : J \hookrightarrow X$ are in the same equivalence class in $\mathrm{Sub}_{\mathbf{Set}}(X)$ if and only if $\mathrm{Im}(i) = \mathrm{Im}(j)$.[137] We conclude that the correspondence between $\mathrm{Sub}_{\mathbf{Set}}(X)$ and $\mathcal{P}(X)$ sends $[i]$ to the image of $i$ and $I \subseteq X$ to the equivalence class of the inclusion $i : I \hookrightarrow X$.

[136] The notation $\mathrm{Sub}_{\mathbf{Set}}(X)$ is perfect!

[137] ($\Rightarrow$) If $i \sim j$, then there exists a bijection $f$ such that $i = j \circ f$. It follows that the image of $j$ is the image of $i$.

($\Leftarrow$) Suppose $\mathrm{Im}(i) = \mathrm{Im}(j)$, we define $f : I \to J = x \mapsto j^{-1}(i(x))$, where $j^{-1}$ is the left inverse of $j$. It is clear that $i = j \circ f$ and a quick computation shows $f$ is an isomorphism with inverse $x \mapsto i^{-1}(j(x))$, where $i^{-1}(x)$ is the left inverse of $i$.

The next exercise generalizes the poset $(\mathcal{P}(X), \subseteq)$.

**Exercise 149.** Let $\mathbf{C}$ be a category and $X \in \mathbf{C}_0$, we define the relation $\leq$ on $\mathrm{Sub}_{\mathbf{C}}(X)$:

See solution.

$$[m] \leq [m'] \Leftrightarrow \exists \text{ morphism } k, m = m' \circ k.$$

Show that $\leq$ is a well-defined partial order.

We can use duality to obtain (for free) the notion of quotient objects.

**Definition 150** (Quotients)**.** Let $\mathbf{C}$ be a category and $X \in \mathbf{C}_0$, there is an equivalence relation $\sim$ on epimorphisms with source $X$ defined by

$$q \sim q' \Leftrightarrow \exists \text{ isomorphism } i, q = i \circ q'.$$

A **quotient object** (or simply quotient) of $X$ is an equivalence class of the relation $\sim$ defined above.[138] The collection of quotients of $X$ is denoted $\mathrm{Quot}_{\mathbf{C}}(X)$. If for any $X \in \mathbf{C}_0$, $\mathrm{Quot}_{\mathbf{C}}(X)$ is a set, we say that $\mathbf{C}$ is **co-well-powered**. There is a partial order $\leq$ on $\mathrm{Quot}_{\mathbf{C}}(X)$ defined by

[138] We will often abusively refer to a quotient simply with an epimorphism $X \twoheadrightarrow Y$.

$$[q] \leq [q'] \Leftrightarrow \exists \text{ morphism } k, q = k \circ q'.$$

# *Limits and Colimits*

The unifying power of categorical abstraction is arguably its biggest benefit. Indeed, it is often the case that many mathematical objects or results from different fields fit under the same categorical definition or fact. In my opinion, category theory is at its peak of elegance when a complex idea becomes close to trivial when viewed categorically, and when this same view helps link together the intuitions behind many ideas throughout mathematics.

The next two chapters concern one particular instance of this power, that is, the use of universal properties to define mathematical constructions. This chapter will cover limits and colimits which are specific cases of universal constructions. The term *universal* is somewhat delicate to define, therefore, we postpone its definition to next chapter and for a while, I recommend you try to recognize *universality* as the thing that all definitions of (co)limits given below have in common.

The first section presents several examples; each of its subsection is dedicated to one kind of limit or colimit of which a detailed example in **Set** is given along with a couple of interesting examples in other categories. It is not straightforward to build intuition about all kinds of (co)limits due to their innumerable applications. For now, I think it is fine if you are comfortable with the intuition in **Set** as it transposes well to concrete categories, but if you persist in learning category theory, you will get to see examples with other flavors. The second section gives a formal framework to talk about all the examples previously explored as well as a few general results. In the sequel, **C** denotes a category.

## *Examples*

Before giving the definition of (co)limits which is very abstract, we present a lot of examples of how they are used. These are very interesting on their own because they show you how many things mathematicians care about in different contexts can be seen as the same abstract construction. Still, keep in mind that, after adding another level of abstraction, we will bring all these examples together as instances of (co)limits.

### *Product*

Given two sets $S$ and $T$, the most common construction of the Cartesian product $S \times T$ is conceptually easy: you take all pairs of elements $S$ and $T$, that is,

$$S \times T := \{(s,t) \mid s \in S, t \in T\}.$$

However, this does not have a nice categorical analog because it requires picking out elements in $S$ and $T$. If one hopes to generalize products to other categories, the construction must only involve objects and morphisms.

**Question 151.** *What are significant functions (morphisms in* **Set***) to consider when studying $S \times T$?*

*Answer.* Projection maps. They are functions $\pi_1 : S \times T \to S$ and $\pi_2 : S \times T \to T$,[139] but that is not enough to define the product. Indeed, there are projection maps $\pi_1' : S \times T \times S \to S$ and $\pi_2' : S \times T \times S \to T$, but $S \times T \times S$ is not always isomorphic to $S \times T$. □

[139] The projections are defined by $\pi_1(s,t) = s$ and $\pi_2(s,t) = t$ for all $(s,t) \in S \times T$.

**Question 152.** *What is unique[140] about $S \times T$ with the projections $\pi_1$ and $\pi_2$?*

[140] Always up to isomorphism of course.

*Answer.* For one, $\pi_1$ and $\pi_2$ are surjective, and while they are not injective, they have an invertible-like property. Namely, given $s \in S$ and $t \in T$, the pair $(s,t)$ is completely determined from $\pi_1^{-1}(s) \cap \pi_2^{-1}(t)$. □

Again, in order to get rid of the references to specific elements, another point of view is needed. Let $X$ be a set of *choices* of pairs, an element $x \in X$ chooses elements in $S$ and $T$ via functions $c_1 : X \to S$ and $c_2 : X \to T$ (similar to the projections). Now, the *almost-inverse* defined above yields a function

$$! : X \to S \times T = x \mapsto \pi^{-1}(c_1(x)) \cap \pi^{-1}(c_2(x)).$$

This function maps $x \in X$ to an element in $S \times T$ that makes the same choice as $x$, and it is the only one that does so. Categorically, $!$ is the unique morphism in $\mathrm{Hom}_{\mathbf{C}}(X, S \times T)$ satisfying $\pi_i \circ ! = c_i$ for $i = 1, 2$. Later, we will see that this property completely determines $S \times T$. For now, enjoy the power we gain from generalizing this idea.

**Definition 153** (Binary product)**.** Let $A, B \in \mathbf{C}_0$. A (categorical) **binary product** of $A$ and $B$ is an object, denoted $A \times B$, along with two morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ called **projections** that satisfy the following universal property[141]: for every object $X \in \mathbf{C}_0$ with morphisms $f_A : X \to A$ and $f_B : X \to B$, there is a unique morphism $! : X \to A \times B$ making diagram (26) commute.[142]

[141] Remember that the word universal is not yet defined, we are trying to get an idea of what it means with these examples.

[142] We will often denote $! = \langle f_A, f_B \rangle$.

$$\begin{tikzcd} & X \arrow[dl, "f_A"'] \arrow[d, dashed, "!"] \arrow[dr, "f_B"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd} \tag{26}$$

**Example 154** (**Set**)**.** Cleaning up the argument above, we show that the Cartesian product $A \times B$ with the usual projections is a binary product in **Set**. To show that it satisfies the universal property, let $X$, $f_A$ and $f_B$ be as in the definition. A function $! : X \to A \times B$ that makes (26) commute must satisfy

$$\forall x \in X, \pi_A(!(x)) = f_A(x) \text{ and } \pi_B(!(x)) = f_B(x).$$

Equivalently, $!(x) = (f_A(x), f_B(x))$. Since this uniquely determines $!$, $A \times B$ is indeed the binary product.

**Examples 155.** Most of the constructions throughout mathematics with the name *product* can also be realized with a categorical product. Examples include the product of groups, rings or vector spaces, the product of topologies, etc. The fact that all these constructions are based on the Cartesian product of the underlying sets is a corollary of a deeper result about the forgetful functors that all these categories have in common.[143]

In another flavor, let $X$ be a topological space and $\mathcal{O}(X)$ be the category of opens. If $A, B \subseteq X$ are open, what is their product? Following Definition 153, the existence of $\pi_A$ and $\pi_B$ imply that $A \times B$[144] is included in both sets, or equivalently $A \times B \subseteq A \cap B$.

Moreover, for any open set $X$ included in $A$ and $B$ (via $f_A$ and $f_B$), $X$ should be included in $A \times B$ (via !).[145] In particular, $X$ can be $A \cap B$ (it is open by definition of a topology), thus $A \cap B \subseteq A \times B$. In conclusion, the product of two open sets is their intersection. In an arbitrary poset, the same argument is used to show the product is the greatest lower bound/infimum/meet.

[143] We show in Chapter  that these forgetful functors are right adjoints and thus they preserve binary products (Proposition 316).

[144] Recall that $\times$ denotes the categorical product, not the Cartesian product of sets.

[145] Notice that uniqueness of ! is already given in a posetal category.

*Remark* 156*.* Given two objects in an arbitrary category, their product does not necessarily exist. Nevertheless, when it exists, one can (and we will) show that it is unique up to unique isomorphism.[146] Thus, in the sequel, we will speak of *the* product of two objects and similarly for other constructions presented in this chapter. Moreover, we will often refer to the object $A \times B$ alone (without the projections) as the product.

[146] The uniqueness of the isomorphism is under the condition that it preserves the structure of the product. We will clear up this subtlety in Remark 205.

**Exercise 157.** Let $A$ and $B$ be two sets, show that their product exists in the category **Rel** and find what it is.

See solution.

Before reaching even more generality, it is sane to check that we can prove some properties of the Cartesian product using the categorical definition. This would ensure that we are not venturing in useless abstract nonsense. We prove the harder one and leave you two easier ones as exercises.

**Proposition 158.** *Let $A, B, C \in \mathbf{C}_0$ be such that $A \times B$ and $B \times C$ exist. If $A \times (B \times C)$ exists, then $(A \times B) \times C$ exists and both products are isomorphic. In other words, the binary product is associative.*[147]

[147] Just like the Cartesian product is associative (up to isomorphism). The existence hypothesis is not necessary in **Set** because the Cartesian product of any two sets always exists.

*Proof.* We will show that $A \times (B \times C)$ satisfies the definition of the product $(A \times B) \times C$ with projections defined below. This means $(A \times B) \times C$ exists and the fact that $A \times (B \times C) \cong (A \times B) \times C$ follows trivially (we defined them to be the same object).[148]

First, we need two projections $\pi_{A\times B} : A \times (B \times C) \to A \times B$ and $\pi_C : A \times (B \times C) \to C$. In the diagram below, we show how to obtain them.[149]

$$
\begin{tikzcd}
 & & A \times (B \times C) \arrow[dl, dotted, "\pi_{A\times B}"] \arrow[dr, "\pi_{B\times C}"] \arrow[ddll, bend right, "\pi_A"'] \arrow[ddrr, dotted, bend left, "\pi_C"] & & \\
 & A \times B \arrow[dl, "\pi_A"] \arrow[dr, "\pi_B"] & & B \times C \arrow[dl, "\pi_B"'] \arrow[dr, "\pi_C"] & \\
A & & B & & C
\end{tikzcd}
\tag{27}
$$

[148] In any case, as we will prove in Proposition 204, if you had another construction for $(A \times B) \times C$, it would be isomorphic to ours.

[149] We overload the notation and rely on the source and target of the morphisms to avoid confusion

The dotted arrow $\pi_C$ is simply the composition $\pi_C \circ \pi_{B\times C}$. The dotted arrow $\pi_{A\times B}$ is obtained via the property of the product $A \times B$ and the morphisms $\pi_A : A \times (B \times C) \to A$ and $\pi_B \circ \pi_{B\times C} : A \times (B \times C) \to B$. It is the unique morphism making (27) commute, that is, $\pi_{A\times B} = \langle \pi_A, \pi_B \circ \pi_{B\times C} \rangle$.

Suppose there is an object $X$ and morphisms $p_{A\times B} : X \to A \times B$ and $p_C : X \to C$. We need to find $! : X \to A \times (B \times C)$ that makes (**??**) commute and is unique with that property. By post-composing with the appropriate projections, we can see how ! acts from the point of view of $A$, $B$ and $C$:

$$
\begin{aligned}
\pi_A \circ ! = \pi_A \circ \langle \pi_A, \pi_B \circ \pi_{B\times C} \rangle \circ ! &= \pi_A \circ p_{A\times B} \\
\pi_B \circ \pi_{B\times C} \circ ! = \pi_B \circ \langle \pi_A, \pi_B \circ \pi_{B\times C} \rangle \circ ! &= \pi_B \circ p_{A\times B} \\
\pi_C \circ \pi_{B\times C} \circ ! &= p_C.
\end{aligned}
$$

By the universal property of $B \times C$, we find that $\pi_{B\times C} \circ ! = \langle \pi_B \circ p_{A\times B}, p_C \rangle$ and then by the universal property of $A \times (B \times C)$, we find that $! = \langle \pi_A \circ p_{A\times B}, \langle \pi_B \circ p_{A\times B}, p_C \rangle \rangle$. The two uses of universal properties ensures that we found the unique possible choice for !. □

See solution.

**Exercise 159.** Let $A, B \in \mathbf{C}_0$. If $A \times B$ exists, then $B \times A$ exists and both products are isomorphic. In other words, the binary product is commutative.[150]

[150] Just like the Cartesian product is commutative (up to isomorphism).

See solution.

**Exercise 160.** Let $\mathbf{1}$ be the terminal object in $\mathbf{C}$. Show that for any $A \in \mathbf{C}_0$, the product of $\mathbf{1}$ and $A$ is $A$.[151]

[151] This property is expected because in **Set**, $\mathbf{1} = \{*\}$ and

$$\{*\} \times A = \{(*, a) \mid a \in A\} \cong A.$$

To generalize the categorical product to more than two objects, one can, for instance, define the product of a finite family of sets recursively with the binary product.[152] This is well-defined thanks to the associativity and commutativity of $\times$, but the proofs above are messy and they will not generalize to the infinite case. In contrast, generalizing the universal property illustrated in (26) yields a simpler definition that works even for arbitrary families.

[152] For a family $\{X_1, \ldots, X_n\} \subseteq \mathbf{C}_0$:

$$\prod_{i=1}^{n} X_i = \begin{cases} X_1 & n = 1 \\ \left(\prod_{i=1}^{n-1} X_i\right) \times X_n & n > 1 \end{cases}$$

**Definition 161** (Product)**.** Let $\{X\}_{i\in I}$ be an $I$–indexed family of objects of $\mathbf{C}$. The **product** of this family is an object, denoted $\prod_{i\in I} X_i$ along with projections $\pi_j : \prod_{i\in I} X_i \to X_j$ for all $j \in I$ satisfying the following universal property: for any object

$X$ with morphisms $\{f_j : X \to X_j\}_{j \in I}$, there is a unique morphism $! : X \to \prod_{i \in I} X_i$ making (28) commute for all $j \in I$.[153]

$$\begin{tikzcd} X \arrow[d, dashed, "!"'] \arrow[dr, "f_j"] & \\ \prod_{i \in I} X_i \arrow[r, "\pi_j"'] & X_j \end{tikzcd} \tag{28}$$

[153] Analogously to the binary case, we may write $! = \langle f_j \rangle_{j \in I}$ or, in the finite case, $! = \langle f_1, \dots, f_n \rangle$.

*Warning* 162. In a lot of cases, the arbitary product will be a straightforward generalization of the binary product,[154] but that is not true in all cases. For instance, in the category of open subsets of a topological space, the arbitrary product is not always the intersection. This is because arbitrary intersections of open sets are not necessarily open. To resolve this problem, it suffices to take the interior of the intersection which is open by definition.

[154] e.g.: in **Set**, the Cartesian product of an arbitrary family of sets is still the set of ordered tuples (instead of pairs) of elements in the sets.

Here are three more properties of Cartesian products that generalize to categorical products.

See solution.

**Exercise 163** (NOW!)**.** Let $\{f_i : X_i \to Y_i\}_{i \in I}$ be a family of morphisms in $\mathbf{C}$, show that there is a unique morphism $\prod_{i \in I} f_i : \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ making the following square commute for all $j \in I$.

$$\begin{tikzcd} \prod_{i \in I} X_i \arrow[r, "\prod_{i \in I} f_i"] \arrow[d, "\pi_j"'] & \prod_{i \in I} Y_i \arrow[d, "\pi_j"] \\ X_j \arrow[r, "f_j"'] & Y_j \end{tikzcd} \tag{29}$$

We call $\prod_{i \in I} f_i$ the **product of morphisms**. In the finite case, we will write $f_1 \times \cdots \times f_n$.

See solution.

**Exercise 164.** Let $X$, $Y$ and $\{X_i\}_{i \in I}$ be objects of $\mathbf{C}$ such that $\prod_{i \in I} X_i$ exists. For any family $f_i : X \to X_i$ and $g : Y \to X$ show that $\langle f_i \rangle_{i \in I} \circ g = \langle f_i \circ g \rangle_{i \in I}$. Conclude that for families $\{f_i : X_i \to Y_i\}_{i \in I}$ and $\{g_i : Z_i \to X_i\}_{i \in I}$, $(\prod f_i) \circ (\prod g_i) = \prod (f_i \circ g_i)$.[155]

[155] It may be useful to restate this in the binary case. For any $f : X \to Y$, $f' : X' \to Y'$, $g : Z \to X$ and $g' : Z' \to X'$, we have
$$(f \times f') \circ (g \times g') = (f \circ g) \times (g \circ g').$$

A family of objects in $\mathbf{C}$ is also called a **discrete** diagram,[156] we will call the product of this family the limit of this diagram. The big takeaway from last chapter is that each time we read a new definition, it is worth to dualize it. Thus, we ask: what is the colimit of a discrete diagram?

[156] Because it corresponds to a functor from a discrete category (one with no non-identity morphisms) into $\mathbf{C}$ (recall that a diagram is a functor into $\mathbf{C}$).

*Coproduct*

**Definition 165** (Coproduct)**.** Let $\{X\}_{i \in I}$ be an $I$–indexed family of objects in $\mathbf{C}$, its **coproduct** is an object, denoted $\coprod_{i \in I} X_i$ (or $X_1 + X_2$ in the binary case), along with morphisms $\kappa_j : X_j \to \coprod_{i \in I} X_i$ for all $j \in I$ called **coprojections** satisfying the following universal property: for any object $X$ with morphisms $\{f_j : X_j \to X\}_{j \in I}$, there is a unique morphism $! : \coprod_{i \in I} X_i \to X$ making (30) commute for all $j \in I$.[157]

[157] We may denote $! = [f_j]_{j \in I}$ or, in the finite case, $! = [f_1, \dots, f_n]$.

$$\begin{tikzcd} X_j \arrow[r, "\kappa_j"] \arrow[dr, "f_j"'] & \coprod_{i\in I} X_i \arrow[d, dashed, "!"] \\ & X \end{tikzcd} \tag{30}$$

Let us find out what coproducts of sets are.

**Example 166** (**Set**). Let $\{X_i\}_{i\in I}$ be a family of sets, first note that if $X_j = \emptyset$ for $j \in I$, then there is only one morphism $X_j \to X$ for any $X$.[158] In particular, (30) commutes no matter what $\coprod_{i\in I} X_i$ and $X$ are. Therefore, removing $X_j$ from this family does not change how the coproduct behaves, hence no generality is loss from assuming all $X_i$s are non-empty.

[158] Because $\emptyset$ is initial.

Second, for any $j \in I$, let $X = X_j$, $f_j = \mathrm{id}_{X_j}$ and for any $j' \neq j$, let $f_{j'}$ be any function in $\mathrm{Hom}(X_{j'}, X_j)$.[159] Commutativity of (30) implies $\kappa_j$ has a left inverse because $! \circ \kappa_j = f_j = \mathrm{id}_{X_j}$, so all coprojections are injective.

[159] One exists because $X_j$ is non-empty.

Third, we claim that for any $j \neq j' \in I$, $\mathrm{Im}(\kappa_j) \cap \mathrm{Im}(\kappa_{j'}) = \emptyset$. Assume towards a contradiction that there exists $j \neq j' \in I$, $x \in X_j$ and $x' \in X_{j'}$ such that $\kappa_j(x) = \kappa_{j'}(x')$. Then, let $X = \{0,1\}$, $f_j \equiv 0$, $f_{j'} \equiv 1$ and the other morphisms be chosen arbitrarily. The universal property implies that $! \circ \kappa_j \equiv 0$ and $! \circ \kappa_{j'} \equiv 1$, but it contradicts $!(\kappa_j(x)) = !(\kappa_{j'}(x'))$.

Finally, the previous point says that $\coprod_{i\in I} X_i$ contains distinct copies of the images of all coprojections. Furthermore, the $\kappa_j$s being injective, their image can be identified with the $X_j$s to obtain[160]

[160] The symbol $\sqcup$ denotes the disjoint union of sets.

$$\bigsqcup_{i\in I} X_i \subseteq \coprod_{i\in I} X_i.$$

For the converse inclusion, in (30), let $X$ be the disjoint union and the $f_j$s be the inclusions. Assume there exists $x$ in the R.H.S. that is not in the L.H.S., then we can define $!' : \coprod_{i\in I} X_i \to \bigsqcup_{i\in I} X_i$ that only differs from $!$ at $x$. Since $x$ is not in the image of any coprojection, the diagrams still commute and this contradicts the uniqueness of $!$.

In conclusion, the coproduct in **Set** is the disjoint union and the coprojections are the inclusions.[161]

[161] We recover the intuition for why empty sets can be ignored. This is a general fact proven in Exercise 170.

*Remark* 167. If this example looks more complicated than the product of sets, it is because we started knowing nothing concrete about coproducts of sets and gradually discovered what properties they had using specific objects and morphisms we know exist in **Set**. In contrast, we knew what products of sets were, and we just had to show they satisfied the universal property.[162]

[162] One might argue that coming up with this universal property was the hard part in that case.

In general, the hard part is to find what construction satisfies a universal property, proving it does is easier.

**Examples 168. In the category of open sets of** $(X,\tau)$**:** let $\{U_i\}_{i\in I}$ be a family of open sets and suppose $\coprod_i U_i$ exists. The coprojections yield inclusions $U_j \subseteq \coprod_i U_i$ for all $j \in I$, so $\coprod_i U_i$ must contain all $U_j$s and thus $\cup_i U_i$. Moreover, in (30), letting $f_j$ be the inclusion $U_j \hookrightarrow \cup_i U_i$ for all $j \in I$,[163] the existence of $!$ yields an inclusion

[163] These morphisms are in $\mathcal{O}(X)$ because $\cup_i U_i$ is open.

$\coprod_i U_i \subseteq \cup_i U_i$. We conclude that the coproduct in this category is the union. In an arbitrary poset, the same argument is used to show the coproduct is the least upper bound/supremum/join.

**In $\mathbf{Vect}_k$:** the coproduct, also called the direct sum, is defined by[164]

$$\coprod_{i\in I} V_i = \bigoplus_{i\in I} V_i := \left\{ v \in \prod_{i\in I} V_i \mid v(i) \neq 0 \text{ for finitely many } i\text{'s} \right\},$$

where $\kappa_j : V_j \hookrightarrow \coprod_i V_i$ sends $v$ to $\bar{v} \in \prod_i V_i$ with $\bar{v}_j = v$ and $\bar{v}_{j'} = 0$ whenever $j \neq j'$. To verify this, let $\{f_j : V_j \to X\}_{j\in I}$ be a family of linear maps. We can construct ! by defining it on basis elements of the direct sum, which are just the basis elements of all $V_j$s seen as elements of the sum (via the coprojections).[165] Indeed, if $b$ is in the basis of $V_j$, we let $!(\bar{b}) = f_j(b)$. Extending linearly yields a linear map $! : \coprod_i V_i \to X$. Uniqueness is clear because if $h : \coprod_i V_i \to X$ differs from ! on one of the basis elements, it does not make (30) commute.

[164] Here, the symbol $\prod$ denotes the Cartesian product of the $V_i$s as sets. The categorical product of vector spaces is also the direct sum, where the projections are the usual ones.

[165] It is necessary to require finitely many nonzero entries, otherwise the basis of the coproduct would not be the union of all bases of the $V_j$s.

**Exercise 169.** Let $A$ and $B$ be two sets, show that their coproduct exists in the category **Rel** and find what it is.

See solution.

**Exercise 170.** Show that products are dual to coproducts, namely, if a product of a familiy $\{X_i\}_{i\in I}$ exists in $\mathbf{C}$, then this object and the projections are the coproduct of this family and the coprojections in $\mathbf{C}^{\mathrm{op}}$ and vice-versa. Conclude that you can define the **coproduct of morphisms** dually to Exercise 163, we denote them $\coprod_{i\in I} f_i$ or $f_1 + \cdots + f_n$ in the finite case.

See solution.

**Exercise 171.** Dually to Exercise 164, show that if $X$, $Y$ and $\{X_i\}_{i\in I}$ are objects of $\mathbf{C}$ such that $\prod_{i\in I} X_i$ exists, then for any family $f_i : X_i \to X$ and $g : X \to Y$ show that $g \circ [f_i]_{i\in I} = [g \circ f_i]_{i\in I}$.

See solution.

**Exercise 172.** Let $\mathbf{C}$ have a terminal object $\mathbf{1}$. Show that the assignment $X \mapsto X + \mathbf{1}$ is functorial, i.e.: define the action of $(- + \mathbf{1})$ on morphisms and show it satisfies the axioms of a functor.[166]

See solution.

[166] We call $(- + \mathbf{1})$ the **maybe functor**.

In a very similar way to the product and coproduct, we will define various constructions in **Set** as limits or colimits.[167]

[167] We will follow more closely the section on coproducts where we started with the definition of the (co)limit and then detailed an example in **Set**.

### *Equalizer*

We briefly mentioned that a product (resp. coproduct) is a limit (resp. colimit) of a discrete diagram. The rest of the examples before generalizing will be (co)limits of small diagrams that contain morphisms.

**Definition 173** (Fork)**.** A **fork** in $\mathbf{C}$ is a diagram of shape (31) or (32) that commutes.[168]

$$O \xrightarrow{o} A \underset{g}{\overset{f}{\rightrightarrows}} B \qquad (31) \qquad\qquad A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{o} O \qquad (32)$$

[168] Once again, we make use of our convention that commutativity does not make parallel morphisms equal.

Since these are dual notions, we will prefer to call (32) a **cofork**. If (31) commutes, we say that $o$ **equalizes** $f$ and $g$. If (32) commutes, we say that $o$ **coequalizes** $f$ and $g$.

**Definition 174** (Equalizer)**.** Let $A, B \in \mathbf{C}_0$ and $f, g : A \to B$ be parallel morphisms. The **equalizer** of $f$ and $g$ is an object $E$ and a morphism $e : E \to A$ satisfying $f \circ e = g \circ e$ with the following universal property: for any morphism $o : O \to A$ equalizing $f$ and $g$, there is a unique $! : O \to E$ making (33) commute.[169]

[169] Try to look for a common pattern in this definition and the definition of a product (both are instances of limits).

$$\begin{tikzcd} O \arrow[d, dashed, "!"'] \arrow[dr, "o"] & & \\ E \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd} \tag{33}$$

A common notation for $e$ is $\mathrm{Eq}(f, g)$. There is also a straightforward generalization of equalizers to more than two morphisms.[170]

[170] If $\{f_i\}_{i \in I}$ is a family of parallel morphisms, their equalizer is a morphism $e \in \mathbf{C}_1$ such that
$$\forall i \neq j, f_i \circ o = f_j \circ o,$$
and every $o$ with this property factors through $e$ in a unique way.

**Example 175** (**Set**)**.** Let $f, g : A \to B$ be two functions and suppose their equalizer exists and it is $e : E \to A$. By associativity, for any $h : O \to E$, the composite $e \circ h$ is a candidate for $o$ in diagram (33) because $f \circ (e \circ h) = g \circ (e \circ h)$. What is more, if $h'$ is such that $e \circ h = e \circ h'$, then $h = h'$ or it would contradict the uniqueness of $!$. In other words, $e$ is monic/injective.[171]

[171] This argument was independent of the category, hence we can conclude that an equalizer of parallel morphisms is always monic.

This implies $E$ can be identified with its image under $e$. Since $f \circ e = g \circ e$, the image of $e$ is contained in the subset $\{a \in A \mid f(a) = g(a)\}$. Now, by the universal property of the equalizer, letting $O$ be this subset and $o$ be the inclusion, there is an injection[172] $! : \{a \in A \mid f(a) = g(a)\} \hookrightarrow E$, thus both sets are equal. In conclusion, the equalizer of two parallel functions is the subset $E$ in which they are coincide and $e : E \hookrightarrow A$ is the inclusion.

[172] The fact that $!$ is an injection comes from the fact that the inclusion $o$ is an injection and $e \circ ! = o$.

**Examples 176. In a posetal category**: hom-sets are singletons, so it must be the case that $f = g$ whenever $f$ and $g$ are parallel. Therefore, any $o : O \to A$ satisfies $f \circ o = g \circ o$. Written using the order notation, the universal property is then equivalent to the fact that $E \leq A$ and $O \leq A$ implies $O \leq E$. In particular, if $O = A$, then $A \leq E$, so $A = E$ by antisymmetry.

**In Ab, Ring or Vect$_k$**: For the same reason that the Cartesian product of the underlying sets is the underlying set of the product,[173] the construction of equalizers is as in **Set**. However, since each of these categories have a notion of additive inverse for morphisms, the equalizer of $f$ and $g$ has a cooler name, that is, $\ker(f - g)$.[174]

[173] We explain this in Chapter .

[174] The equalizer of $f$ and $g$ is the subset of $A$ where $f$ and $g$ are equal, or equivalently, where $f - g$ is 0 (when $f - g$ and 0 are defined).

**Definition 177** (Idempotents)**.** A morphism $f : A \to A \in \mathbf{C}_1$ is called **idempotent** when $f \circ f = f$. It is called **split idempotent** if there exist morphisms $s : E \to A$ and $r : A \to E$ such that $s \circ r = f$ and $r \circ s = \mathrm{id}_E$.[175]

[175] We can show that split idempotents are idempotent because
$$f \circ f = s \circ r \circ s \circ r = s \circ \mathrm{id}_E \circ r = f.$$

**Proposition 178.** *An idempotent morphism $f : A \to A \in \mathbf{C}_1$ is split idempotent if and only if the equalizer of $f$ and $\mathrm{id}_A$ exists.*

*Proof.* ($\Rightarrow$) Let $f = s \circ r$ be such that $r \circ s = \mathrm{id}_E$, we claim that $s$ is the equalizer. First, we can check that $s$ equalizes $f$ and $\mathrm{id}_A$ because $f \circ s = s \circ r \circ s = \mathrm{id}_E \circ s = s = s \circ \mathrm{id}_A$. Next, given $o : O \to A$ making (34) commute, we need to find a morphism $!$ that fits in the diagram. Its uniqueness is given by $s$ being monic (it has a left inverse). Noticing that $o = f \circ o = s \circ r \circ o$, we find $! = r \circ o$.

$$\begin{tikzcd} O \arrow[d, dashed, "!"'] \arrow[dr, "o"] & & \\ E \arrow[r, "s"'] & A \arrow[r, shift left, "\mathrm{id}_A"] \arrow[r, shift right, "f"'] & A \end{tikzcd} \tag{34}$$

($\Leftarrow$) If $e : E \to A$ is the coequalizer of $f$ and $\mathrm{id}_A$, then since $f$ equalizes $f$ and $\mathrm{id}_A$, there exists $! : A \to E$ such that $e \circ ! = f$. By monicity of $e$, we find that $e \circ (! \circ e) = f \circ e = e$ implies $! \circ e = \mathrm{id}_A$, so $f$ is a split idempotent (let $s := e$ and $r := !$). $\square$

The equalizer of $f$ and $g$ is the limit of the diagram containing only the two parallel morphisms, we define its colimit in the next section.

*Coequalizer*

**Definition 179** (Coequalizer)**.** Let $A, B \in \mathbf{C}_0$ and $f, g : A \to B$ be parallel morphisms. The **coequalizer** of $f$ and $g$ is an object $D$ and a morphism $d : B \to D$ satisfying $d \circ f = d \circ g$ with the following universal property: for any morphism $o : B \to O$ coequalizing $f$ and $g$, there is a unique $! : D \to O$ making (35) commute.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "d"] \arrow[dr, "o"'] & D \arrow[d, dashed, "!"] \\ & & O \end{tikzcd} \tag{35}$$

**Example 180** (**Set**)**.** Let $f, g : A \to B$ be two functions and suppose $d : B \to D$ is their coequalizer. Similarly to the dual case, one can show that $d$ is epic/surjective. Since $d \circ f = d \circ g$, for any $b, b' \in B$,

$$\left(\exists a \in A, f(a) = b \text{ and } g(a) = b'\right) \implies d(b) = d(b'). \tag{$*$}$$

Denoting $\sim$ to be the relation in the L.H.S. of ($*$), the implication is $b \sim b' \implies d(b) = d(b')$. Note that $\sim$ is not necessarily an equivalence relation but $=$ is, thus, the converse implication does not always hold.[176]

Consequently, it makes sense to consider the equivalence relation generated by $\sim$,[177] denoted $\simeq$. As noted above, the forward implication $b \simeq b' \implies d(b) = d(b')$ still holds. For the converse, in (35), let $O := B/\simeq$ and $o : B \to B/\simeq$ be the quotient map, by post-composing with !, we have

$$d(b) = d(b') \implies o(b) = o(b') \implies b \simeq b'.$$

In conclusion, $D = B/\simeq$ and $d : B \to D$ is the quotient map.

**Examples 181. In a posetal category:** an argument dual to the one for equalizers shows the coequalizer of $f, g : A \to B$ is $B$.

**In Ab, Ring or Vect$_k$:** Let $f, g : A \to B$ be homomorphisms and suppose $d : B \to D$ is their coequalizers. Consider the homomorphism $f - g$, since $d$ coequalizes $f$ and $g$, $d \circ (f - g) = d \circ f - d \circ g = 0$, or equivalently, $\mathrm{Im}(f - g) \subseteq \ker(d)$. Now, consider diagram (36) as an instance of (35), where $q$ is the quotient map.[178]

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "d"] \arrow[dr, "q"'] & D \arrow[d, dashed, "!"] \\ & & B/\mathrm{Im}(f - g) \end{tikzcd} \tag{36}$$

[176] For instance, when $b \sim b' \sim b''$, $d(b) = d(b'')$, but it might not be the case that $b \sim b''$.

[177] In this case, it is simply the transitive closure.

[178] It is commutative because $q \circ (f - g) = 0$ by definition of $q$.

We claim that ! has an inverse, implying that $D \cong B/\mathrm{Im}(f-g)$.[179] Indeed, for $[x] \in B/\mathrm{Im}(f-g)$, we must have

$$!^{-1}([x]) = !^{-1}(q(x)) = !^{-1}(!(d(x))) = d(x),$$

and it is only left to show $!^{-1}$ is well-defined because the inverse of a homomorphism is a homomorphism. This follows because if $[x] = [x']$, then there exists $y \in \mathrm{Im}(f-g)$ such that $x = x' + y$, so

$$!^{-1}(x) = d(x) = d(x'+y) = d(x') + d(y) = d(x') + 0 = !^{-1}(x').$$

In the special case that $g \equiv 0$, $B/\mathrm{Im}(f)$ is called the *cokernel* of $f$, denoted $\mathrm{coker}(f)$.

[179] This is not enough to say that $B/\mathrm{Im}(f-g)$ with the quotient map is the coequalizer, we leave you the task to complete the proof using this isomorphism that crucially satisfies $! \circ d = q$.

**Exercise 182.** Show that an idempotent morphism $f : A \to A \in \mathbf{C}_1$ is split idempotent if and only if the coequalizer of $f$ and $\mathrm{id}_A$ exists.

See solution.

*Pullback*

**Definition 183** (Cospan)**.** A **cospan** in $\mathbf{C}$ comprises three objects $A, B, C$ and two morphisms $f$ and $g$ as in (37).[180]

$$A \xrightarrow{f} C \xleftarrow{g} B \tag{37}$$

[180] Just like forks, coforks and spans that we introduce later, cospan is simply a name that we give to a certain shape of diagram that occurs quite often.

**Definition 184** (Pullback)**.** Let $A \xrightarrow{f} C \xleftarrow{g} B$ be a cospan in $\mathbf{C}$. Its **pullback** is an object, denoted $A \times_C B$, along with morphisms $p_A : A \times_C B \to A$ and $p_B : A \times_C B \to B$ such that $f \circ p_A = g \circ p_B$ and the following universal property holds: for any object $X$ and morphisms $s : X \to A$ and $t : X \to B$ satisfying $f \circ s = g \circ t$, there is a unique morphism $! : X \to A \times_C B$ making (38) commute.[181]

$$\begin{tikzcd} X \arrow[drr, bend left, "t"] \arrow[dr, dashed, "!"] \arrow[ddr, bend right, "s"'] & & \\ & A \times_C B \arrow[r, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd} \tag{38}$$

[181] The $\lrcorner$ symbol is a standard convention to specify that a square is not only commutative, but also a pullback square.

We will call $p_A$ the pullback of $g$ **along** $f$ and sometimes denote it $f^*(g)$. Symmetrically, $p_B$ is the pullback of $f$ along $g$, denoted $g^*(f)$.

A drawback of the notation $A \times_C B$ is that it does not refer to the morphisms $f$ and $g$ which are crucial in the definition. An alternative notation is $f \times_C g$ (I learned about it here). An argument supporting this notation is in Exercise 243.

**Example 185 (Set).** Let $A \xrightarrow{f} C \xleftarrow{g} B$ be a cospan in **Set** and suppose that its pullback is $A \xleftarrow{p_A} A \times_C B \xrightarrow{p_B} B$. Observe that $p_A$ and $p_B$ look like projections, and in fact, by the universality of the product $A \times B$, there is a map $h : A \times_C B \to A \times B$ such that $h(x) = (p_A(x), p_B(x))$ ((39) commutes). Consider the image of $h$, if $(a,b) \in \mathrm{Im}(h)$, then there exists $x \in A \times_C B$ such that $p_A(x) = a$ and $p_B(x) = b$. Moreover, the commutativity of the square in (39) implies $f(a) = g(b)$, hence

$$\begin{tikzcd} A \times_C B \arrow[rr, "p_B"] \arrow[dr, dashed, "h"] \arrow[dd, "p_A"'] & & B \arrow[dd, "g"] \\ & A \times B \arrow[ur, "\pi_B"'] \arrow[dl, "\pi_A"] & \\ A \arrow[rr, "f"'] & & C \end{tikzcd} \tag{39}$$

$$\mathrm{Im}(h) \subseteq \{(a,b) \in A \times B \mid f(a) = g(b)\} =: E.$$

Now, letting $X = E$, $s = \pi_A$ and $t = \pi_B$, by definition, $f \circ s = g \circ t$ hence, there is a unique $! : E \to A \times_C B$ satisfying $p_A \circ ! = \pi_A$ and $p_B \circ ! = \pi_B$. Viewing $h$ as going in the opposite direction to $!$,[182] it is easy to see that for any $(a,b) \in E$,[183]

$$(h \circ !)(a,b) = (p_A(!(a,b)), p_B(a,b)) = (\pi_A(a,b), \pi_B(a,b)) = (a,b),$$

thus $!$ has a left inverse and is injective. Assume towards a contradiction that it is not surjective, then let $y \in A \times_C B$ not be in the image of $!$ and denote $x = !(p_A(y), p_B(y))$. Define $!'$ as acting exactly like $!$ except on $(p_A(y), p_B(y))$ where it goes to $y$ instead of $x$. This ensure that $!'$ still makes the diagram commutes, but this contradicts the uniqueness of $!$.

[182] We just saw that the image of $h$ is contained in $E$, so we can see $h$ as a function $h : A \times_C B \to E$.

[183] We use the fact that $\pi_A \circ h \circ ! = p_A \circ !$ and similarly for $B$.

As a particular case, when one function in the cospan is an inclusion, say, $B \subseteq C$ and $g : B \hookrightarrow C$, the pullback is the preimage of $B$ under $f$ since

$$\{(a,b) \in A \times B \mid f(a) = g(b) = b\} \cong \{a \mid f(a) \in B\} = f^{-1}(B).$$

You can also check that $p_A$ is the inclusion $f^{-1}(B) \hookrightarrow A$ and $p_B$ is $f$ restricted to $f^{-1}(B)$. As a particular case of that, if the cospan consists of two inclusions $A \hookrightarrow C \hookleftarrow B$, then its pullback is the intersection $A \cap B$ with $p_A$ and $p_B$ being the inclusions.

$$\begin{tikzcd} A \cap B \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook] \\ A \arrow[r, hook] & C \end{tikzcd}$$

**Examples 186. In a posetal category**, the commutativity of the square in (38) does not depend on the morphisms, thus the universal property is equivalent to the property of being a product.

**The composition of relations** $R$ and $S$ can be defined using pullbacks in **Set**. Given relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, we can apply the projections to subsets to obtain (40). Then, taking the pullback of the cospan in the middle and using the characterization of the pullback in **Set** from Example 185, we obtain

$$\begin{tikzcd} & R \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & & S \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_Z"] & \\ X & & Y & & Z \end{tikzcd} \quad (40)$$

$$R \times_Y S = \left\{ ((x,y),(y',z)) \in R \times S \mid y = y' \right\}.$$

Observe in (41) that we have functions from $R \times_Y S$ to $X$ and $Z$: $\pi_X \circ p_R$ and $\pi_Z \circ p_S$. Thus, by the universal property of the product $X \times Z$, there is a function $! : R \times_Y S \to X \times Z$. After a bit of computations, recalling that $p_R((x,y),(y',z)) = (x,y)$ and $p_S((x,y),(y',z)) = (y',z)$, we find that the image of $!$ is precisely the composite relation[184]

$$S \circ R = \{(x,z) \mid \exists y, (x,y) \in R, (y,z) \in S\}.$$

[184] Our argument here heavily relies on working with sets and functions, but there is a way to generalize relations in other nice enough categories using this idea.

$$\begin{tikzcd} & & R \times_Y S \arrow[dl, "p_R"'] \arrow[dr, "p_S"] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\ & R \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & & S \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_Z"] & \\ X & & Y & & Z \end{tikzcd} \quad (41)$$

**Exercise 187.** Let $f : X \to Y$ be a morphism in **C**. Show $f$ is monic if and only if the square in (42) is a pullback.[185]

See solution.

[185] This result and its dual will sometimes be used to treat monomorphisms (resp. epimorphisms) as limits (resp. colimits). In most of these cases, it will be crucial that this limit (resp. colimit) only involves the monomorphism (resp. epimorphism) and the identity morphism which is preserved by any functor.

$$
\begin{tikzcd}
X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\tag{42}
$$

**Exercise 188.** Supposing (43) commutes, show that if the right square is a pullback and $i$ and $j$ are isomosphisms, then the rectangle is a pullback.

See solution.

$$
\begin{tikzcd}
X \arrow[r, leftrightarrow, "i"] \arrow[d] & A \times_C B \arrow[r, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
Y \arrow[r, leftrightarrow, "j"'] & A \arrow[r, "f"'] & C
\end{tikzcd}
\tag{43}
$$

Supposing (44) commutes, show that if the left square is a pullback and $i$ and $j$ are isomorphisms, then the rectangle is a pullback.

$$
\begin{tikzcd}
A \times_C B \arrow[r, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, leftrightarrow, "i"] \arrow[d, "g"] & X \arrow[d] \\
A \arrow[r, "f"'] & C \arrow[r, leftrightarrow, "j"'] & Y
\end{tikzcd}
\tag{44}
$$

*Pushout*

**Definition 189** (Span)**.** A **span** in **C** comprises three objects $A, B, C$ and two morphisms $f$ and $g$ as in (45).

$$
A \xleftarrow{f} C \xrightarrow{g} B \tag{45}
$$

**Definition 190** (Pushout)**.** Let $A \xleftarrow{f} C \xrightarrow{g} B$ be a span in **C**. Its **pushout** is an object, denoted $A +_C B$, along with morphisms $k_A : A \to A +_C B$ and $k_B : B \to A +_C B$ such that $k_A \circ f = k_B \circ g$ and the following universal property holds: for any object $X$ and morphisms $s : A \to X$ and $t : B \to X$ satisfying $s \circ f = t \circ g$, there is a unique morphism $! : A +_C B \to X$ making (46) commute.[186]

[186] The $\ulcorner$ symbol is a standard convention to specify that the square is not only commutative, but also a pushout square.

$$
\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "k_B"] \arrow[ddr, bend left, "t"] & \\
A \arrow[r, "k_A"'] \arrow[drr, bend right, "s"'] \arrow[ur, phantom, "\ulcorner", very near end] & A +_C B \arrow[dr, dashed, "!"] & \\
& & X
\end{tikzcd}
\tag{46}
$$

We will call $k_A$ the pushout of $g$ **along** $f$ and sometimes denote it $f_*(g)$. Symmetrically, $k_B$ is the pushout of $f$ along $g$, denoted $g_*(f)$.

**Example 191** (**Set**)**.** Let $A \xleftarrow{f} C \xrightarrow{g} B$ be a span in **Set** and suppose its pushout is $A \xrightarrow{k_A} A +_C B \xleftarrow{k_B} B$. Similarly to above, observe that $k_A$ and $k_B$ are like coprojections, so there is a unique map $! : A + B \to A +_C B$ such that $!(a) = k_A(a)$ and $!(b) = k_B(b)$. Furthermore, for any $c \in C$, $!(f(c)) = !(g(c))$, thus

$$\exists c \in C, f(c) = a \text{ and } g(c) = b \implies !(a) = !(b).$$

This is very similar to what happened for coequalizers and after working everything out, we obtain that $! : A + B \to A +_C B$ is the coequalizer of $\kappa_A \circ f$ and $\kappa_B \circ g$. This is a general fact that does not only apply in **Set** but in every category with binary coproducts and coequalizers.

As a particular case, if $C = A \cap B$ and $f$ and $g$ are simply inclusions, then $A +_C B = A \cup B$ (the *non-disjoint* union).

See solution.

**Exercise 192.** Show that if (47) is a pushout square, then $d$ is the coequalizer of $f$ and $g$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "d"] \\ B \arrow[r, "d"'] \arrow[ur, phantom, "\ulcorner"] & D \end{tikzcd} \tag{47}$$

**Example 193** (Rewriting)**.** The categorical approach to graph rewriting is full of uses of pushouts. In this example, we will try to give a flavor of a particular method called double-pushout rewriting (DPO) in an almost trivial setting using words instead of graphs. □.

## *Generalization*

There exists many other examples of (co)limits but these six examples give quite a good idea of what it is to be a limit or colimit. More precisely, we will see in Theorem 217 and Exercise 224 that any limit can be built out of products and equalizers or pullbacks and a terminal object. Dually, we can build colimits out of coproducts and coequalizers or pushouts and an initial object.

Let us try to informally spell out the general pattern in the definitions of each example.

- We start with a shape for a diagram $D$ (i.e.: a discrete diagram, two parallel morphisms, a span, a cospan, etc.).
- The limit (resp. colimit) of $D$ is an object $L$ along with morphisms from $L$ to every object in the diagram (resp. in the opposite direction) such that combining $D$ with these morphisms yields a commutative diagram.
- These morphisms satisfy a universal property. More specifically, for any object $L'$ with morphisms from $L'$ to every object in the diagram (resp. in the opposite direction) that commute with $D$, there is a unique $! : L' \to L$ (resp. $L \to L'$) such that combining all the morphisms with $D$ yields a commutative diagram.

We have already formalized the first step when we defined diagrams. For the second and third step, notice that the morphisms given for $L$ and $L'$ have the same conditions, they form a cone (resp. cocone).

*Definitions*

We start by formalizing limits.

**Definition 194** (Cone)**.** Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram. A cone from $X$ to $F$ is an object $X \in \mathbf{C}_0$, called the **tip**, along with a family of morphisms $\{\psi_Y : X \to F(Y)\}$ indexed by objects $Y \in \mathbf{J}_0$ such that for any morphism $a : Y \to Z$ in $\mathbf{J}_1$, $F(a) \circ \psi_Y = \psi_Z$, i.e.: diagram (48) commutes.

$$\begin{tikzcd} & X \arrow[dl, "\psi_Y"'] \arrow[dr, "\psi_Z"] & \\ F(Y) \arrow[rr, "F(a)"'] & & F(Z) \end{tikzcd} \tag{48}$$

Often, the terminology cone over $F$ is used.

Next, the fact that the morphism ! keeps everything commutative can be generalized. We say that ! is a morphism of cones.

**Definition 195** (Morphism of cones)**.** Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram and $\{\psi_Y : A \to F(Y)\}_{Y \in \mathbf{J}_0}$ and $\{\phi_Y : B \to F(Y)\}_{Y \in \mathbf{J}_0}$ be two cones over $F$. A **morphism of cones** from $A$ to $B$ is a morphism $g : A \to B$ in $\mathbf{C}_1$ such that for any $Y \in \mathbf{J}_0$, $\phi_Y \circ g = \psi_Y$, i.e.: (49) commutes.

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "\psi_Y"'] & & B \arrow[dl, "\phi_Y"] \\ & F(Y) & \end{tikzcd} \tag{49}$$

After verifying that morphisms can be composed, the last two definitions give rise to the category of cones over a diagram $F$ which we denote $\mathrm{Cone}(F)$. Finally, the universal property can be stated in terms of cones, thus giving the general definition of a limit. Indeed, the limit of a diagram $D$ is a cone $L$ over $D$ such that for every cone $L'$ over $D$, there is a unique cone morphism $! : L' \to L$ called the **mediating morphism**. Equivalently, $L$ is the terminal object of $\mathrm{Cone}(F)$.

**Definition 196** (Limit)**.** Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram, the **limit** of $F$, if it exists, is the terminal object of $\mathrm{Cone}(F)$. It is denoted $\lim_{\mathbf{J}} F$ or $\lim F$.

*Remark* 197. Often, $\lim F$ also designates the tip of the cone as an object in $\mathbf{C}$ rather than the whole cone.[187] We may also refer to the whole cone as the **limit cone**.

[187] This can sometimes be a source of confusion because many authors implicitly omit parts of the proof involving the rest of the cone and the reader is expected to reconstruct the missing parts.

**Examples 198.** While you can play around with the three examples of limits we have already given and make them fit in this general definition, we add to this list three examples in increasing order of complexity.

1. Consider an empty diagram in $\mathbf{C}$, that is, the functor $\emptyset$ from the empty category to $\mathbf{C}$. A cone from $X$ to $\emptyset$ is just an object $X \in \mathbf{C}_0$ as there are no objects in the diagram. Consequently, a morphism in $\mathrm{Cone}(\emptyset)$ is simply a morphism in $\mathbf{C}$, so $\mathrm{Cone}(\emptyset)$ is the same as the original category $\mathbf{C}$ and $\lim\emptyset$ is the terminal object of $\mathbf{C}$ if it exists.[188]

[188] Alternatively, we can say that the terminal object is the product of an empty family.

2. Given a group $G$, recall from Example 91.7 that a $G$–set can be seen as a diagram in **Set**, i.e.: a functor $\mathbf{B}G \rightsquigarrow \mathbf{Set}$. We claim that the limit of this diagram is the set $\mathsf{Fix}(S)$ of fixed points of the action (an element $s$ of a $G$–set is a **fixed point** if $g \cdot s = s$).[189] Let $F : \mathbf{B}G \rightsquigarrow \mathbf{Set}$ be a $G$–set with $F(*) = S$, a cone from $F$ is a set $P$ along with a function $p : P \to S$ such that for any $g \in G$, (50) commutes.

$$\begin{tikzcd} & P \arrow[dl, "p"'] \arrow[dr, "p"] & \\ S \arrow[rr, "F(g)=g\cdot-"'] & & S \end{tikzcd} \tag{50}$$

We infer from this diagram that the image of $p$ is contained in the set of fixed points.[190] Therefore, $p$ factors uniquely through the inclusion $\mathsf{Fix}(S) \hookrightarrow S$. We conclude that the coned formed by $\mathsf{Fix}(S) \hookrightarrow S$ is the limit cone.

[189] Recall that the limit of two parallel morphisms was called an equalizer. In this example, we are taking the limit of several parallel morphisms. Thus, one can also see the limit of $F$ as the generalized equalizer of all the morphisms $g \cdot -$ with $g \in G$.

[190] For any $x \in P$, we have $g \cdot p(x) = p(x)$.

3. Let $x$ denote an indeterminate variable and $k$ be a field, $k[x]$ denotes the ring of polynomials over $x$.[191] We will show that $k[\![x]\!]$, the ring of **formal power series** over $x$, can be defined as a limit.

   Let $I = \langle x \rangle$ be the ideal generated by $x$, it contains all the polynomials with no constant terms, and denote $I^n = \langle x^n \rangle$. In the sequel, we view elements of $k[x]/I^n$ as polynomials with degree at most $n-1$.[192] The following three key properties are satisfied (we leave the proof to the interested readers).

   a) For any $n \leq m \in \mathbb{N}$ and $p \in k[x]/I^m$, forgetting about all terms in $p$ of degree at least $n$ yields a ring homomorphism $\pi_{m,n} : k[x]/I^m \to k[x]/I^n$.[193]

   b) For any $n \in \mathbb{N}$, we can do the same thing for power series to obtain a homomorphism $\pi_{\infty,n} : k[\![x]\!] \to k[x]/I^n$.

   c) Any composition of the homomorphisms above can be seen as a single homomorphism above. Namely, $\forall n \leq m \leq l \in \mathbb{N} \cup \infty$,

   $$\pi_{m,n} \circ \pi_{l,m} = \pi_{l,n}.$$

   Consider the posetal category $(\mathbb{N}, \geq)$, a) and c) imply that $F(n) := k[x]/I^n$ and $F(m \geq n) := \pi_{m,n}$ defines a functor $F : (\mathbb{N}, \geq) \to \mathbf{Ring}$. This is the diagram represented in (51).

   $$\cdots \longrightarrow k[x]/I^n \xrightarrow{\pi_{n,n-1}} \cdots \longrightarrow k[x]/I^2 \xrightarrow{\pi_{2,1}} k[x]/I \xrightarrow{\pi_{1,0}} k[x]/I^0 = \mathbf{0} \tag{51}$$

   Now, using b) and c), we see that $k[\![x]\!]$ along with $\{\pi_{\infty,n}\}_{n \in \mathbb{N}}$ is a cone over the diagram $F$. It is in fact the terminal cone. Let $\{p_n : R \to k[x]/I^n\}_{n \in \mathbb{N}}$ be another cone over $F$ and $! : R \to k[\![x]\!]$ a morphism of cones. By commutativity, for any $m \leq n$, the coefficients for $x^m$ of $!(r)$ and $p_n(r)$ must agree. Now, by commutativity of the cone $\{p_n\}_{n \in \mathbb{N}}$, $p_n(r)$ and $p_{n-1}(r)$ have the same coefficients except for $x^n$, thus we can compactly define $!$ by

   $$!(r) := p_0(r) + \sum_{n>0} (p_n(r) - p_{n-1}(r)).$$

[191] In Chapter , we will describe a nice categorical definition of $k[x]$, but, for now, let us assume you know what polynomials are and how they can be added and multiplied together. You can skip this example if you are not familiar with rings.

[192] More accurately, $k[x]/I^n$ contains equivalence classes of polynomials, but their representatives are exactly the polynomials of degree at most $n-1$. Since $I^0 = k[x]$, the quotien $k[x]/I^0$ is the trivial ring, i.e.: the zero object in **Ring**.

[193] Note that $\pi_{m,m}$ is the identity.

This completely determines $!$, so it is unique.[194]

The construction of this diagram from quotienting different powers of the same ideal is used in different contexts, it is called the **ring completion** of $k[x]$ with respect to $I$. For instance, one can define the $p$–adic integers with base ring $\mathbb{Z}$ and the ideal generated by $p$ for any prime $p$.

[194] Existence follows from the same equation.

### *Codefinitions*

Put simply, a colimit in $\mathbf{C}$ is a limit in $\mathbf{C}^{\mathrm{op}}$. I suggest you spend a bit of time trying to dualize all of the previous section on your own, but it is done below for completeness.

**Definition 199** (Cocone)**.** Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram. A **cocone** from $F$ to $X$ is an object $X \in \mathbf{C}_0$ along with a family of morphisms $\{\psi_Y : F(Y) \to X\}$ indexed by objects of $Y \in \mathbf{J}_0$ such that for any morphism $a : Y \to Z$ in $\mathbf{J}$, $\psi_Z \circ F(a) = \psi_Y$, i.e.: (52) commutes.

$$\begin{tikzcd} F(Y) \arrow[rr, "F(a)"] \arrow[dr, "\psi_Y"'] & & F(Z) \arrow[dl, "\psi_Z"] \\ & X & \end{tikzcd} \tag{52}$$

**Definition 200** (Morphism of cocones)**.** Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram and $\{\psi_Y : F(Y) \to A\}_{Y \in \mathbf{J}_0}$ and $\{\phi_Y : F(Y) \to B\}_{Y \in \mathbf{J}_0}$ be two cocones. A **morphism of cocones** from $A$ to $B$ is a morphism $g : A \to B$ in $\mathbf{C}$ such that for any $Y \in \mathbf{J}_0$, $g \circ \psi_Y = \phi_Y$, i.e.: (53) commutes.

$$\begin{tikzcd} & F(Y) \arrow[dl, "\psi_Y"'] \arrow[dr, "\phi_Y"] & \\ A \arrow[rr, "g"'] & & B \end{tikzcd} \tag{53}$$

The category of cocones from $F$[195] is denoted $\mathrm{Cocone}(F)$.

[195] Some authors call them **cones under** $F$.

**Definition 201** (Colimit)**.** Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram, the colimit of $F$ denoted $\mathrm{colim} F$, if it exists, is the initial object of $\mathrm{Cocone}(F)$.

**Examples 202.** We dualize two examples from the previous section.

1. Dually to Example 198.1, $\mathrm{colim}\varnothing$ is the is the initial object of $\mathbf{C}$ if it exists.[196]

2. Dually to Example 198.2, we claim that the colimit of the diagram corresponding to a group action is the set of its orbits. Let $F : \mathbf{B}G \rightsquigarrow \mathbf{Set}$ be a $G$–set with $F(*) = S$, a cocone from $F$ is a set $Q$ along with a function $q : S \to Q$ such that for any $g \in G$, (54) commutes.

$$\begin{tikzcd} S \arrow[rr, "F(g)=g\cdot -"] \arrow[dr, "q"'] & & S \arrow[dl, "q"] \\ & Q & \end{tikzcd} \tag{54}$$

[196] Alternatively, the initial object is the coproduct of an empty family.

One can also see the colimit of $F$ as the (generalized) coequalizer of all the morphisms $g \cdot -$ with $g \in G$.

We infer that if there exists $g \in G$ such that $g \cdot s = s'$, then $q(s) = q(s')$. Denoting $o(s) := \{g \cdot s \mid g \in G\}$ to be the orbit of $s \in S$, the set of orbits of $S$

$$O := \{o(s) \mid s \in S\}$$

along with the map $o : S \to O$ forms a cocone from $F$ since $o(g \cdot -) = o$.[197] This cocone is the colimit since for any $q : S \to Q$ as in (54), any $! : O \to Q$ making (55) commute is completely determined by $!(o(s)) = q(s)$ (which is well-defined since $o(s) = o(s') \implies \exists g \in G, g \cdot s = g \cdot s' \implies q(s) = q(s')$).

[197] Since the orbits are, by definition, stable under the action of $G$.

$$\begin{tikzcd} S \arrow[rr, "g\cdot -"] \arrow[dr, "o"] \arrow[ddr, bend right, "q"'] & & S \arrow[dl, "o"'] \arrow[ddl, bend left, "q"] \\ & O \arrow[d, dashed, "!"] & \\ & Q & \end{tikzcd} \tag{55}$$

See solution.

**Exercise 203** (Trivial (co)limits)**.** Show the following (co)limits always exist and find what they are.

1. The limit of a diagram with only one morphism.
2. The colimit of a diagram with only one morphism.
3. The limit of a span.
4. The colimit of a cospan.

*Results*

**Proposition 204** (Uniqueness)**.** *Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram, the limit (resp. colimit) of $F$, if it exists, is unique up to unique isomorphism.*

*Proof.* This follows from the uniqueness of terminal (resp. initial) objects.[198] □

[198] Corollary 133 (resp. Proposition 132).

*Remark* 205*.* The isomorphism between two limits (also colimits) is unique when viewed as a morphism of cone. There might exists an isomorphism between the tips that is not a morphism of cone. For instance, let $A$, $B$ and $C$ be finite sets. One can check that both $A \times (B \times C)$ and $(A \times B) \times C$ are products of $\{A, B, C\}$ (with the usual projection maps). Thus, there is an isomorphism between them. One can check that, for it to be a morphism of cones, it must send $(a, (b, c))$ to $((a, b), c)$, but any other bijection between them is an isomorphism in **Set**.

For this reason, the limit really consists of the whole cone, and not just of the object at the tip. Unfortunately, this subtlety is not well cared for in the literature and it can and has led to errors.

Recall the definition of preserve and reflect we gave in Definition 142, with the framework of (co)limits, we can give more formal related definitions.

Let $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$ be a functor and $D : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram. We define a functor $F_D : \mathrm{Cone}(D) \rightsquigarrow \mathrm{Cone}(F \circ D)$ that sends a cone $\{\psi_X : A \to DX\}_{X \in \mathbf{J}_0}$ to $\{F\psi_X : FA \to FDX\}_{X \in \mathbf{J}_0}$,[199] and sends a morphism $g : \{\psi_X\}_{X \in \mathbf{J}_0} \to \{\phi_X\}_{X \in \mathbf{J}_0}$ to $Fg : \{F\psi_X\}_{X \in \mathbf{J}_0} \to \{F\phi_X\}_{X \in \mathbf{J}_0}$.[200] In simple terms, $F_D$ takes a cone over $D$ and applies $F$ to every object and morphism in it to obtain a cone over $F \circ D$. We leave you to define the very similar functor $F^D : \mathrm{Cocone}(D) \rightsquigarrow \mathrm{Cocone}(F \circ D)$.

[199] The family $\{F\psi_X\}_{X \in \mathbf{J}_0}$ is a cone over $F \circ D$ since $Da \circ \psi_X = \psi_Y$ implies $FDa \circ F\psi_X = F\psi_Y$ for any $a : X \to Y \in \mathbf{J}_1$.

[200] Again, the fact that $Fg$ is a morphism of cones follows straightforwardly from

$$\phi_X \circ g = \psi_X \implies F\phi_X \circ Fg = F\psi_X.$$

**Definition 206.** Let $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$ be a functor and $\mathbf{J}$ be a category.

- We say that $F$ **preserves** limits of shape $\mathbf{J}$ if for any diagram $D : \mathbf{J} \rightsquigarrow \mathbf{C}$, if $\{\psi_X\}_{X \in \mathbf{J}_0}$ is the limit cone over $D$, then $\{F\psi_X\}_{X \in \mathbf{J}_0}$ is the limit cone over $F \circ D$. In other words, for any $D$, $F_D$ preserves (in the sense of Definition 142) terminal objects.[201]
- We say that $F$ **reflects** limits of shape $\mathbf{J}$ if for any diagram $D : \mathbf{J} \rightsquigarrow \mathbf{C}$, if $\{\psi_X\}_{X \in \mathbf{J}_0}$ is a cone over $D$ and $\{F\psi_X\}_{X \in \mathbf{J}_0}$ is the limit cone over $F \circ D$, then $\{\psi_X\}_{X \in \mathbf{J}_0}$ is also the limit cone over $D$. In other words, for any $D$, $F_D$ reflects (in the sense of Definition 142) terminal objects.
- We say that $F$ **creates** limits of shape $\mathbf{J}$ if for any diagram $D : \mathbf{J} \rightsquigarrow \mathbf{C}$, if $\{\phi_X\}_{X \in \mathbf{J}_0}$ is a limit cone over $F \circ D$, then there exists a unique cone over $D$ $\{\psi_X\}_{X \in \mathbf{J}_0}$ such that $F\psi_X = \phi_X$ and $\{\psi_X\}_{X \in \mathbf{J}_0}$ is a limit cone.

We leave to you the dualization of this definition.[202]

[201] We will often be less rigorous and write something like $\lim(F \circ D) = F(\lim_{\mathbf{J}} D)$. For instance, we will say that $F$ preserves binary products if $FX \times FY = F(X \times Y)$.

[202] Replace cone by cocone and limit by colimit. See solution.

**Exercise 207.** Fix $A \in \mathbf{C}_0$, show that the functor $\mathrm{Hom}_{\mathbf{C}}(A, -)$ preserves binary products. Namely, if $X, Y \in \mathbf{C}_0$ and $X \times Y$ exists, then

$$\mathrm{Hom}_{\mathbf{C}}(A, X \times Y) \cong \mathrm{Hom}_{\mathbf{C}}(A, X) \times \mathrm{Hom}_{\mathbf{C}}(A, Y).$$

**Corollary 208** (Dual)**.** *Fix $A \in \mathbf{C}_0$, show the functor $\mathrm{Hom}_{\mathbf{C}}(-, A)$ preserves binary coproducts.*

## *Diagram chasing*

We show four results in increasing order of complexity to demonstrate diagram chasing through examples.

**Theorem 209.** *Consider the pullback square in* (56)*.*

$$\begin{tikzcd} A \times_C B \arrow[r, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{56}$$

*If $g$ is monic, then $p_A$ also is. Symmetrically, if $f$ is monic, then $p_B$ also is.*[203]

*Proof.* Let $h_1, h_2 : X \to A \times_C B$ be such that $p_A \circ h_1 = p_A \circ h_2$, we need to show that $h_1 = h_2$. First, observe that $h_1$ and $h_2$ yield two cones over the cospan $A \xrightarrow{f} C \xleftarrow{g} B$ as depicted in (57).

$$\begin{tikzcd} X \arrow[dr, shift left, "h_2"] \arrow[dr, shift right, "h_1"'] \arrow[drr, bend left, "p_B \circ h_1"] \arrow[drr, bend left=40, "p_B \circ h_2"] \arrow[ddr, bend right, "p_A \circ h_1 = p_A \circ h_2"'] & & \\ & A \times_C B \arrow[r, "p_B"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd} \tag{57}$$

[203] This is commonly stated simply as: "The pullback of a monomorphism is a monomorphism."

The two cones are

$$\begin{tikzcd} X \arrow[r, "p_B \circ h_1"] \arrow[d, "p_A \circ h_1"'] & B \\ A & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "p_B \circ h_2"] \arrow[d, "p_A \circ h_2"'] & B \\ A & \end{tikzcd}$$

They make the squares commute because the original pullback square commutes.

Furthermore, $h_1$ and $h_2$ are cone morphisms between $X$ and $A \times_C B$ and since the pullback is the terminal cone over this cospan, they are unique. Now, we already have that the projections onto $A$ is the same for both new cones, but we claim this is also true for the projections onto $B$. Indeed, because $g$ is monic and the square commutes, we have the following implications.

$$\begin{aligned}
p_A \circ h_1 = p_A \circ h_2 &\implies & f \circ p_A \circ h_1 = f \circ p_A \circ h_2 \\
&\implies & g \circ p_B \circ h_1 = g \circ p_B \circ h_2 \\
&\implies & p_B \circ h_1 = p_B \circ h_2
\end{aligned}$$

In other words, the two new cones are in fact the same cones, hence $h_1$ and $h_2$ are the same morphisms by uniqueness, which concludes our proof. □

**Corollary 210.** *The pushout of an epimorphism is an epimorphism.*

**Proposition 211.** *Let $\{f_i, g_i : X_i \to Y_i\}_{i \in I}$ be a familiy of parallel morphisms in* $\mathbf{C}$ *such that for any $i \in I$,* (58) *is an equalizer, then* (59) *is an equalizer.*

$$E_i \xrightarrow{e_i} X_i \underset{g_i}{\overset{f_i}{\rightrightarrows}} Y_i \tag{58}$$

$$\prod_{i \in I} E_i \xrightarrow{\prod_{i \in I} e_i} \prod_{i \in I} X_i \underset{\prod_{i \in I} g_i}{\overset{\prod_{i \in I} f_i}{\rightrightarrows}} \prod_{i \in I} Y_i \tag{59}$$

**Corollary 212** (Dual)**.**

**Theorem 213** (Pasting Lemma)**.** *Consider diagram* (60)*, where the right square is a pullback. This result is called the* ***pasting lemma****.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'
\end{tikzcd} \tag{60}$$

*If* (60) *commutes, the left square is a pullback if and only if the rectangle is.*

*Proof.* ($\Rightarrow$) Explicitly, we have to show that $\alpha : A' \leftarrow A \to C : g \circ f$ is the pullback of $g' \circ f' : A' \to C' \leftarrow C : \gamma$. The commutativity $g' \circ f' \circ \alpha = \gamma \circ g \circ f$ implies this is already a cone over the cospan we just described. Now, suppose there is another cone over this cospan, namely, there exist morphisms $p_{A'} : X \to A'$ and $p_C : X \to C$ satisfying $g' \circ f' \circ p_{A'} = \gamma \circ p_C$ as depicted in (61).

$$\begin{tikzcd}
X \arrow[dr, dashed, "!_A"'] \arrow[drr, dashed, "!_B"] \arrow[drrr, bend left, "p_C"] \arrow[ddr, bend right, "p_{A'}"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "g"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\gamma"] \\
& A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'
\end{tikzcd} \tag{61}$$

Notice that composing $p_{A'}$ with $f'$, we obtain a cone over the cospan in the right square and by universality of $B$, this yields a unique morphism $!_B : X \to B$ satisfying

$g \circ !_B = p_C$ and $\beta \circ !_B = f' \circ p_{A'}$. This second equality yields cone over the cospan in the left square, thus we get a unique morphism $!_A : X \to A$ satisfying $\alpha \circ !_A = p_{A'}$ and $f \circ !_A = !_B$. Composing the last equality with $g$, we get

$$g \circ f \circ !_A = g \circ !_B = p_C,$$

showing that $!_A$ is a morphism of cones over the rectangular cospan.

What is more, any other morphism $m : X \to A$ of cones over this cospan must satisfy

$$g \circ f \circ m = p_C \text{ and } \beta \circ f \circ m = f' \circ \alpha \circ m = f' \circ p_{A'},$$

and thus, $f \circ m$ is a morphism of cones over the cospan in the right rectangle. By uniqueness, $f \circ m = !_B$, so $m$ is also a morphism of cones over the cospan in the left square, and by universality of $A$, $m = !_A$.

($\Leftarrow$) Explicitly, we have to show that $\alpha : A' \leftarrow A \to B : f$ is the pullback of $f' : A' \to B \leftarrow B : \beta$.

$$\begin{tikzcd}
X \arrow[dr, dashed, "!_A"] \arrow[drr, bend left, "p_B"] \arrow[ddr, bend right, "p_{A'}"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\gamma"] \\
& A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C'
\end{tikzcd} \tag{62}$$

Let $p_{A'} : A' \leftarrow X \to B : p_B$ be a cone over the cospan of the left square (i.e.: $\beta \circ p_B = f' \circ p_{A'}$). The commutativity of (60) implies $p_{A'} : A' \leftarrow X \to C : g \circ p_B$ is a cone over the rectangle cospan, then by universality of $A$, there exists a unique $!_A : X \to A$ such that $g \circ f \circ !_A = g \circ p_B$ and $\alpha \circ !_A = p_A$. Moreover, with the commutativity of the left square, we find that $f \circ !_A$ is a morphism of cones over the right cospan satisfying $\beta \circ f \circ !_A = f' \circ \alpha \circ !_A = f' \circ p_{A'} = \beta \circ p_B$ and $g \circ f \circ !_A = g \circ p_B$. But since our hypothesis on $p_{A'}$ and $p_B$ implies $p_B$ is a morphism of cones satisfying the same equations, by universality of $B$, $p_B = f \circ !_A$. Therefore, $!_A$ is a morphism of cone over the left cospan.

Finally, if $m : X \to A$ also satisfies $\alpha \circ m = p_{A'}$ and $f \circ m = p_B$. We find in particular that $m$ is a morphism of cones over the rectangle cospan, hence by universality of $A$, $m = !_A$. □

**Corollary 214.** *In diagram* (60) *where the right square is not necessarily a pullback but the left square is a pushout, the right square is a pushout if and only if the rectangle is.*

See solution.

**Exercise 215.** Show that (63) is a pullback square. Let $i : A' \to A$ be an isomorphism, show that (64) is a pullback square.[204]

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_B"'] & B
\end{tikzcd} \tag{63} \qquad \begin{tikzcd}
A' \arrow[r, "i"] \arrow[d, "f \circ i"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_B"'] & B
\end{tikzcd} \tag{64}$$

[204] We can summarize the first square by saying that the pullback of any morphism along the identity gives back the original morphism. The second square is basically a converse to the statement "pullbacks are unique up to isomorphism" in this very special case.

**Definition 216** ((Co)completeness)**.** A category is said to be **(co)complete** (resp. **finitely** (co)complete) if any small (resp. finite) diagram has a (co)limit.

**Theorem 217.** *Suppose that a category* $\mathbf{C}$ *has all products and equalizers then* $\mathbf{C}$ *has all limits, i.e.:* $\mathbf{C}$ *is complete.*

*Proof.* Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram, we will show that the limit of $F$ is obtained from the equalizer of two morphisms[205]

$$u_1, u_2 : \prod_{X \in \mathbf{J}_0} F(X) \to \prod_{a \in \mathbf{J}_1} F(t(a)),$$

which are defined below. The equalizer and the products it involves exist by hypothesis.

[205] Recall that $s$ and $t$ denote the sources and targets of morphisms.

First, let us try to explain the intuition behind this construction. The limit of $F$ is the terminal cone over $F$. In particular, it is a cone over $F$, namely, a family of morphisms $\psi_X : \lim F \to FX$ indexed by $X \in \mathbf{J}_0$ such that for any $a : X \to Y \in \mathbf{J}_1$, $Fa \circ \psi_X = \psi_Y$. Since $\mathbf{C}$ has products, we can also specify the morphisms in the cone by a single morphism $\psi : \lim F \to \prod_{X \in \mathbf{J}_0} FX$.[206]

[206] The family $\{\psi_X\}$ gives rise to $\psi$ by the universal property of the product and $\psi$ gives rise to the family by post-composing with the projections $\pi_X : \prod_{X \in \mathbf{J}_0} FX \to FX$.

$$\psi_X = \pi_X \circ \psi$$

The additional property of the cone is now $\forall a : X \to Y \in \mathbf{J}_1, Fa \circ \pi_X \circ \psi = \pi_Y \circ \psi$. Replacing the objects $X$ and $Y$ with $s(a)$ and $t(a)$ respectively, we obtain two families of morphisms

$$\{Fa \circ \pi_{s(a)} : \prod_{X \in \mathbf{J}_0} FX \to Ft(a) \mid a \in \mathbf{J}_1\} \quad \text{and} \quad \{\pi_{t(a)} : \prod_{X \in \mathbf{J}_0} FX \to Ft(a) \mid a \in \mathbf{J}_1\}.$$

The universal property of products yields two parallel morphisms $u_1, u_2 : \prod_{X \in \mathbf{J}_0} FX \to \prod_{a \in \mathbf{J}_1} Ft(a)$ making (65) commute.

$$\begin{tikzcd}
\prod_{X \in \mathbf{J}_0} FX \arrow[dr, dashed, "u_1"] \arrow[drr, bend left, "Fa \circ \pi_{s(a)}"] & & \\
& \prod_{a \in \mathbf{J}_1} Ft(a) \arrow[r, "\pi_a"] & Ft(a) \\
\prod_{X \in \mathbf{J}_0} FX \arrow[ur, dashed, "u_2"'] \arrow[urr, bend right, "\pi_{t(a)}"'] & &
\end{tikzcd} \tag{65}$$

We find that $\psi$ equalizes $u_1$ and $u_2$,[207] and since we did not use the fact that $\psi$ is terminal, we infer that any cone over $F$ yields a morphism from the tip to the product $\prod_{X \in \mathbf{J}_0} FX$ that equalizes $u_1$ and $u_2$. Notice that this process can be reversed, hence any morphism that equalizes $u_1$ and $u_2$ corresponds to a cone over $F$.

[207] We check that $u_1 \circ \psi = u_2 \circ \psi$ by post-composing with $\pi_{t(a)}$ for every $a \in \mathbf{J}_1$. Indeed, we have

$$\begin{aligned} \pi_a \circ u_1 \circ \psi &= Fa \circ \pi_{s(a)} \circ \psi \\ &= \pi_{t(a)} \circ \psi && \text{(def. of } \psi) \\ &= \pi_a \circ u_2 \circ \psi, \end{aligned}$$

and the universal property of $\prod_{a \in \mathbf{J}_1} Ft(a)$ implies $u_1 \circ \psi = u_2 \circ \psi$.

We are on a good track because we have shown that cones over $F$ are in correspondence with cones over the parallel morphisms $u_1$ and $u_2$. If we can show there is also a correspondence between the morphisms of such cones, we will be able to conclude that the terminal cone over $u_1$ and $u_2$ (i.e.: their equalizer) is the terminal cone over $F$ (i.e.: the limit of $F$).[208]

[208] More abstractly, we show there is an isomorphism between the categories $\mathrm{Cone}(F)$ and $\mathrm{Cone}(U)$, where $U$ is the diagram with only two parallel morphisms sent to $u_1$ and $u_2$. One can check that isomorphisms of categories preserve terminal objects, so the equalizer of $u_1$ and $u_2$ is the limit of $F$.

Let $\{\psi_X, \phi_X : A \to FX\}_{X \in \mathbf{J}_0}$ be two cones over $F$, $g : A \to B$ be a morphism of cones, and $\psi$ and $\phi$ be the corresponding morphism that equalize $u_1$ and $u_2$. We will show that (66) commutes. By definition of $g$, we have $\phi_X \circ g = \psi_X$ for any $X \in \mathbf{J}_0$, which we can rewrite in $\pi_X \circ \phi \circ g = \pi_X \circ \psi$. By the universal property of the product $\prod_{X \in \mathbf{J}_0} FX$, we conclude that $\phi \circ g = \psi$.

$$\begin{tikzcd}
A \arrow[dr, "\psi"] \arrow[dd, "g"'] & & \\
& \prod_{X \in \mathbf{J}_0} FX \arrow[r, dashed, shift left, "u_1"] \arrow[r, dashed, shift right, "u_2"'] & \prod_{a \in \mathbf{J}_1} Ft(a) \\
B \arrow[ur, "\phi"'] & &
\end{tikzcd} \tag{66}$$

Conversely, given $g$ that makes (66), it is clear that $g$ is a morphism of cones because for any $X \in \mathbf{J}_0$, $\phi_X \circ g = \pi_X \circ \phi \circ g = \pi_X \circ \psi = \psi_X$.

In conclusion, let $\psi : L \to \prod_{X \in \mathbf{J}_0}$ be the equalizer of $u_1$ and $u_2$, the limit of $F$ is the cone $\{\pi_X \circ \psi_X\}_{X \in \mathbf{J}_0}$.

$\square$

*Remark* 218. The same proof yields a more general statement: For any cardinal $\kappa$, if a category $\mathbf{C}$ has all products of size less than $\kappa$ and equalizers, then it has limits of any diagram with less than $\kappa$ objects and morphisms.

**Corollary 219** (Dual)**.** *If a category* $\mathbf{C}$ *has all coproducts of size less than* $\kappa$ *and coequalizers, then it has colimits of any diagram with less than* $\kappa$ *objects and morphisms.*

**Definition 220.** A functor $\mathbf{C} \rightsquigarrow \mathbf{D}$ is said to be (**finitely**) **(co)continuous** if it preserves all (finite) (co)limit.

See solution.

**Exercise 221.** Show that a functor is continuous if and only if it preserves products and equalizers. State and prove the dual statement.

**Theorem 222.** *Fix* $A \in \mathbf{C}_0$*, the functor* $\mathrm{Hom}_{\mathbf{C}}(A, -)$ *is continuous.*

*Proof.* We could use Exercises 207 and 221 and then show that $\mathrm{Hom}_{\mathbf{C}}(A, -)$ also preserves equalizers, but the direct proof is not very long and it lets us get even more familiar with cones.

Let $D : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram and $\{\psi_X : \lim D \to DX\}_{X \in \mathbf{J}_0}$ be the limit cone, we need to show that $\{\psi_X \circ - : \mathrm{Hom}_{\mathbf{C}}(A, \lim D) \to \mathrm{Hom}_{\mathbf{C}}(A, DX)\}_{X \in \mathbf{J}_0}$ is a limit cone.

$$\begin{tikzcd} & \mathrm{Hom}_{\mathbf{C}}(A, DX) \arrow[dd, "Da\circ -"] \\ \mathrm{Hom}_{\mathbf{C}}(A, \lim D) \arrow[ur, "\psi_X \circ -"] \arrow[dr, "\psi_Y \circ -"'] & \\ & \mathrm{Hom}_{\mathbf{C}}(A, DY) \end{tikzcd} \tag{67}$$

First, for any $a : X \to Y \in \mathbf{J}_1$, we have $Da \circ \psi_X = \psi_Y$, which implies (67) commutes. Hence, $\{\psi_X \circ -\}_{X \in \mathbf{J}_0}$ is a cone over $\mathrm{Hom}_{\mathbf{C}}(A, D-)$.

Next, if $\{\phi_X : T \to \mathrm{Hom}_{\mathbf{C}}(A, DX)\}_{X \in \mathbf{J}_0}$ is another cone over $\mathrm{Hom}_{\mathbf{C}}(A, D-)$, then observe that any $t \in T$ gives rise to a cone over $D$ $\{\phi_X(t) : A \to DX\}_{X \in \mathbf{J}_0}$. Indeed, we have

$$Df \circ \phi_X(t) = ((Df \circ -) \circ \phi_X)(t) = \phi_Y(t).$$

$$\begin{tikzcd} A \arrow[dr, "\phi_X(t)"] \arrow[dd, "g(t)"'] & \\ & DX \\ \lim D \arrow[ur, "\psi_X"'] & \end{tikzcd} \tag{68}$$

We obtain a unique morphism of cones $g(t) : A \to \lim D$ making (68) commute for all $X \in \mathbf{J}_0$. This is a function $g : T \to \mathrm{Hom}_{\mathbf{C}}(A, \lim D)$ that is a morphism of cones because combining (68) for every $t \in T$ yields $(\psi_X \circ -) \circ g = \phi_X$.

If $g' : T \to \mathrm{Hom}_{\mathbf{C}}(A, \lim D)$ is another morphism of cones, then we must have that $g'(t)$ also makes (68) for all $X \in \mathbf{J}_0$.[209] Therefore, $g'(t) : A \to \lim D$ is a morphism of cones and since $\lim D$ is terminal, we conclude $g'(t) = g(t)$ and $g' = g$. $\square$

[209] We have

$$\psi_X \circ g'(t) = ((\psi \circ -) \circ g')(t) = \phi_X(t).$$

**Corollary 223** (Dual)**.** *Fix* $A \in \mathbf{C}_0$*, the functor* $\mathrm{Hom}_{\mathbf{C}}(-, A)$ *is continuous.*[210]

[210] More concisely, the Hom bifunctor is continuous in each argument.

See solution.

**Exercise 224.** Show that a category with all pullbacks and a terminal object is finitely complete.

**Corollary 225** (Dual)**.** *A category with all pushouts and an initial object is finitely cocomplete.*

*Remark* 226. We can conclude[211] that a functor is finitely continuous if and only if it preserves pullbacks and the terminal object and it is finitely coconituous if and only if it preserves pushouts and the initial object.

[211] Similarly to Exercise 221.

# *Universal Properties*

## *Examples*

### *Free Monoid*

The construction of a *free* object is common to different fields of mathematics and the example we will carry out in **Mon** can be carried out in many other categories like **Grp**, **Ab**, **Ring**, $\mathbf{Mod}_R$ (we will do this one in the next section). In fact, one way to view this construction comes from the forgetful functor to **Set** that all these categories have in common. In Chapter , we will cover adjoints and recover the free constructions from $U$.

We choose **Mon** because the concrete characterization of a free monoid is the simplest.

**Definition 227** (Classical)**.** A monoid $M$ is said to be **free** if it can be presented by a set of generators without any relations, i.e. $M = \langle A \mid \emptyset \rangle$. In this case, $M$ is called the **free monoid on** $A$ and denoted $A^*$.

It is easy to check that $A^*$ is the set of finite words with symbols in $A$ with the operation being concatenation and identity being the empty word (denoted ″). In order to give a categorical characterization, we need to look at homomorphisms from or into the free monoid. Notice that any homomorphism $h^* : A^* \to M$ is completely determined by where $h^*$ sends elements of $A$. Indeed, in order to satisfy the homomorphism property, we must have for any $a_1, a_2 \in A$,

$$h^*(a_1 a_2) = h^*(a_1) \cdot h^*(a_2) \text{ and } h^*(\varepsilon) = 1_M.$$

In general, the unique homomorphism sending $a \in A$ to $h(a)$ can be defined recursively:

$$h^*(w) = \begin{cases} h(a) \cdot h^*(w') & a \in A, w \in A^*, w = aw' \\ 1_M & w = \varepsilon \end{cases}.$$

Now, suppose that a monoid $N$ contains $A$ and satisfies the same property, that is for any (set-theoretic) function $h : A \to M$, there is a unique homomorphism $h^* : N \to M$ with $h^*(a) = h(a)$.

If we take $M = A^*$, and $h : A \to A^* = a \mapsto a$, then we get a homomorphism $h^*_N : N \to A^*$. Moreover, taking $M = N$ and $i : A \hookrightarrow N$ be the inclusion, the property of $A^*$ means there is a unique homomorphism $i^* : A^* \to N$. Note that $h_N^* \circ i^* : A^* \to A^*$ is a homomorphism satisfying $a \mapsto a$, so it must be the identity by uniqueness. We conclude that $N$ and $A^*$ are isomorphic.

**Definition 228** (Categorical)**.** The free monoid of a set $A$ is an object $A^*$ in **Mon** along with a *canonical inclusion* $i : A \to U(A^*)$ that satisfies the following universal property: for any monoid $M$ and function $h : A \to U(M)$, there exists a unique homomorphism $h^* : A^* \to M$ such that $U(h^*) \circ i = h$, namely, $h^*(i(a)) = h(a)$. This is summarized in (69), where we omit the $U$ as the underlying set of a monoid is often denoted with the same symbol as the monoid.

$$\begin{array}{ccc} \text{in } \mathbf{Set} & & \text{in } \mathbf{Mon} \\ \begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "h"'] & A^* \arrow[d, dashed, "h^*"] \\ & M \end{tikzcd} & \xleftarrow{\text{forgetful}} & \begin{tikzcd} A^* \arrow[d, dashed, "h^*"] \\ M \end{tikzcd} \end{array} \tag{69}$$

*Abelianization*

**Definition 229** (Classical)**.** Let $G$ be a group, the **abelianization** of $G$, denoted $G^{\mathrm{ab}}$, is the quotient of $G$ with $G' := \{xyx^{-1}y^{-1} \mid x, y \in G\} \leq G$, called the *commutator subgroup*, that is $G^{\mathrm{ab}} := G/G'$.

Let us get insight into this definition. The abelianization is supposed to be the *biggest* abelian quotient of $G$. To see why, note that if $A$ is an abelian group, any homomorphism $h : G \to A$ must satisfy $h(xyx^{-1}y^{-1}) = 1_A$ for any $x, y \in G$. Hence, $G'$ is contained in the kernel of $h$. This yields a factorization $h = G \xrightarrow{\pi} G/G' \xrightarrow{h^*} A$ with $h^*$ unique, where $\pi$ is the canonical quotient map.

Moreover, since **Ab** is a full subcategory of **Grp**, $h^*$ is also unique as a morphism in **Ab**. Using the fact that $G/G'$ is abelian, we conclude the following categorical definition of $G^{\mathrm{ab}}$.

**Definition 230** (Categorical)**.** Let $G$ be a group, the abelianization of $G$ is an abelian group $G^{\mathrm{ab}}$ with a map $\pi : G \to G^{\mathrm{ab}}$ satisfying the following universal property: for any homomorphism $h : G \to A$ where $A$ is abelian, there is a unique homomorphism $h^* : G^{\mathrm{ab}} \to A$ such that $h^* \circ \pi = h$. This is summarized in (70).

$$\begin{array}{ccc} \text{in } \mathbf{Grp} & & \text{in } \mathbf{Ab} \\ \begin{tikzcd} G \arrow[r, "\pi"] \arrow[dr, "h"'] & G^{\mathrm{ab}} \arrow[d, dashed, "h^*"] \\ & A \end{tikzcd} & \xleftarrow{\text{forgetful}} & \begin{tikzcd} G^{\mathrm{ab}} \arrow[d, dashed, "h^*"] \\ A \end{tikzcd} \end{array} \tag{70}$$

*Vector Space Basis*

**Definition 231** (Classical)**.** Let $V$ be a vector space over a field $k$, a **basis** for $V$ is a subset $S \subseteq V$ that is linearly independent and generates $V$, namely, any $v \in V$ can be expressed as a linear combination of elements in $S$ and any $s \in S$ cannot be expressed as a linear combination of elements in $S \setminus \{s\}$.

Once again, we would like to get rid of the content of this definition talking about elements, so we focus on what this means for linear maps coming out of $V$. Let $S$ be a basis of $V$, $W$ be another vector space over $k$ and $T : V \to W$ be a linear map. By linearity, $T$ is completely determined by where it sends the elements of $S$. Indeed, for any $v \in V$, write $v$ as a linear combination $\sum_{s \in S} \lambda_s s$ with $\lambda_s \in k$ (only finitely many of the coefficients are non-zero), then $T(v) = \sum_{s \in S} \lambda_s T(s)$. We conclude that any (set-theoretic) function $t : S \to W$ extends to a unique linear map $T : V \to W$.

We claim that this property completely characterizes bases of $V$. Indeed, let $S \subseteq V$ be such that for any $t : S \to W$, there is a unique linear map $T : V \to W$ extending $t$. We will show that $S$ is generating and linearly independent.

1. Assume towards a contradiction that $S$ is not generating, that is, there exists $v \in V$ that is not a linear combination of vectors in $S$. Equivalently, if $U$ is the subspace generated by $S$, then $V/U$ is not 0. Now, let $t : S \to V/U$ be the 0 map, both the quotient map $\pi : V \to V/U$ and the 0 map $0 : V \to V/U$ extend $t$, and since $V/U$ is not trivial, they are different maps.

2. Assume towards a contradiction that $S$ is not linearly dependent, that is, there exists $v \in S$ is such that $v = \sum_{s \in S - v} \lambda_s s$. Consider the function

$$t : S \to V \oplus V = \begin{cases} (s, 0) & s \neq v \\ (0, v) & s = v \end{cases}.$$

   There cannot exist a linear map $T : V \to V \oplus V$ extending $t$ because by linearity, we can show

$$(0, v) = t(v) = T(v) = T(\sum_{s \in S - v} \lambda_s s) = \sum_{s \in S - v} \lambda_s T(s) = \sum_{s \in S - v} \lambda_s (s, 0),$$

   which is absurd.

In conclusion, we have the following alternate definition of a vector space basis.

**Definition 232** (Categorical)**.** Let $V$ be a vector space, a basis of $V$ is a set $S$ along with an inclusion $i : S \to V$ satisfying the following universal property: for any function $t : S \to W$ where $W$ is a vector space, there is a unique linear map $T : V \to W$ such that $T \circ i = t$. This is summarized in (71).

$$\begin{tikzcd} & \text{in } \mathbf{Set} & & \text{in } \mathbf{Vect}_k \\ S \arrow[r, "i"] \arrow[ddr, "t"'] & V \arrow[dd, dashed, "T"] & & V \arrow[dd, dashed, "T"] \\ & & \xleftarrow{\text{forgetful}} & \\ & W & & W \end{tikzcd} \tag{71}$$

*Exponential Objects*

This section and the following two are motivated by important constructions in **Set** that we want to define categorically. Going further in this direction amounts to doing topos theory, namely, studying categories which look a lot like **Set**.

**Exercise 233.** Let $\mathbf{C}$ be a category and $X \in \mathbf{C}_0$ be such that for any $Y \in \mathbf{C}_0$, $Y \times X$ exists. Show that $- \times X$ is a functor $\mathbf{C} \rightsquigarrow \mathbf{C}$.

See solution.

Let $A$ and $X$ be sets, $A^X$ commonly denotes the set of functions $X \to A$. In hope to generalize this construction to other categories, let us study morphisms into $A^X$. Given a set $B$ and a morphism $f : B \to A^X$, there is a natural operation called **uncurrying** that takes $f$ to $\lambda^{-1} f : B \times X \to A$ which basically evaluates both $f$ and its output at the same time. Namely, $\lambda^{-1} f(b, x) = f(b)(x)$.

As a particular case, we consider the identity function $A^X \to A^X$. Uncurrying yields the **evaluation** function $\mathsf{ev} : A^X \times X \to A$ that evaluates the function in the first coordinate at the second coordinate: $\mathsf{ev}(f, x) = f(x)$.

Now, as the name suggests, uncurrying has an inverse operation called **currying** which takes $g : B \times X \to A$ to $\lambda g : B \to A^X$ defined by $\lambda g(b) = x \mapsto g(b, x)$. Morally, $\lambda g$ delays the evaluation of $g$ to later.[212] Moreover, notice that the currying of $g$ satisfies $\mathsf{ev}(\lambda g(b), x) = g(b, x) \in A$ for any $b \in B$ and $x \in X$. This along with the fact that currying and uncurrying are bijective operations[213] leads to a universal property that $\mathsf{ev}$ satisfies. It is summarized in (72).

[212] For computer scientists, this is also related to the concept of *continuations*.

[213] Check that $\lambda \lambda^{-1} g = g$ and $\lambda^{-1} \lambda g = g$.

$$\text{in } \mathbf{Set} \qquad \begin{tikzcd} A & A^X \times X \arrow[l, "\mathsf{ev}"'] \\ & B \times X \arrow[ul, "g"] \arrow[u, dashed, "\lambda g \times \mathrm{id}_X"'] \end{tikzcd} \xleftarrow{- \times X} \qquad \text{in } \mathbf{Set} \qquad \begin{tikzcd} A^X \\ B \arrow[u, dashed, "\lambda g"'] \end{tikzcd} \tag{72}$$

This is entirely categorical, so we can define exponential objects as follows.

**Definition 234** (Exponential)**.** Let $\mathbf{C}$ be a category and $X \in \mathbf{C}_0$ be such that $- \times X$ is a functor.[214] For $A \in \mathbf{C}_0$, the **exponential** $A^X$ (if it exists) is an object $A^X$ along with a morphism $\mathsf{ev} : A^X \times X \to A$ such that for all $g : B \times X \to A$, there is a unique $\lambda g : B \to A^X$ making (72) commute.

[214] i.e.: all binary products with $X \in \mathbf{C}_0$ exist.

*Subobject Classifier*

**Exercise 235.** Let $\mathbf{C}$ be a well-powered category with all pullbacks. We define $\mathrm{Sub}_{\mathbf{C}}$ on morphisms: it sends $f : X \to Y$ to $f^*(-) : \mathrm{Sub}_{\mathbf{C}}(Y) \to \mathrm{Sub}_{\mathbf{C}}(X)$ sending $m : I \to Y$ to $f^*(m)$ (the pullback of $m$ along $f$ as depicted in (73)). Show that this is well-defined and makes $\mathrm{Sub}_{\mathbf{C}}$ into a functor $\mathbf{C}^{\mathrm{op}} \rightsquigarrow \mathbf{Set}$.

See solution.

$$\begin{tikzcd} J \arrow[r] \arrow[d, hook, "f^*(m)"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[d, hook, "m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{73}$$

In $\mathbf{Set}$, recall that subobjects are subsets. Hence, letting $\Omega = \{\bot, \top\}$ there is a correspondence between $\mathrm{Sub}_{\mathbf{Set}}(X)$ and $\mathrm{Hom}_{\mathbf{Set}}(X, \Omega)$, it sends $I \subseteq X$ to the characteristic function $\chi_I : X \to \Omega$,[215] and in the other direction $f : X \to \Omega$ is sent to $f^{-1}(\top) \subseteq X$. Furthermore, recall that the preimage can be seen as a pullback, so we can define $\chi_I$ as the unique function making (74) into a pullback square.

[215] The characteristic function $\chi_I$ is defined by
$$\chi_I(x) = \begin{cases} \top & x \in I \\ \bot & x \notin I \end{cases}.$$

Uniqueness holds because this pullback implies $I = \chi_I^{-1}(\top)$.

$$\begin{tikzcd} I \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, hook, "\top"] \\ X \arrow[r, "\chi_I"'] & \Omega \end{tikzcd} \tag{74}$$

The role played by the two element set $\{\bot, \top\}$ can now be generalized to other categories.

**Definition 236** (Subobject classifier)**.** Let $\mathbf{C}$ be a category with a terminal object $\mathbf{1}$. A **subobject classifier** is a morphism $\top : \mathbf{1} \to \Omega \in \mathbf{C}_1$ such that for any monomorphism $I \hookrightarrow X$ there is a unique morphism $\chi_m : X \to \Omega$ such that (74) is a pullback square. We call $\chi_I$ the **characteristic map** of $I \hookrightarrow X$.

$$\begin{tikzcd} I' \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & I \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{1} \arrow[d, hook, "\top"] \\ X \arrow[r, dashed, "\mathrm{id}_X"] \arrow[rr, bend right, "\chi_{I'}"'] & X \arrow[r, dashed, "\chi_I"] & \Omega \end{tikzcd} \tag{75}$$

Before drawing a diagram like those above to summarize the universal property of a subobject classifier, we need to make sure that the characteristic maps of two monomorphisms in the same equivalence class in $\mathrm{Sub}_{\mathbf{C}}(X)$ are equal. Looking at (75), the right square is a pullback by hypothesis and the left square is a pullback by Exercise 215. Therefore, the rectangle is a pullback by the pasting lemma and we see that $\chi_{I'} = \chi_I \circ \mathrm{id}_X$ by uniqueness of the characteristic map.

Now, in a well-powered category $\mathbf{C}$ has a terminal object and all pullbacks,[216] a subobject classifier $\top : \mathbf{1} \to \Omega$ is such that for any subobject $m$ of $X$, which we identify as a morphism $m : \mathbf{1} \to \mathrm{Sub}_{\mathbf{C}}(X)$, there is a unique morphism $\chi_m : X \to \Omega$ such that $\chi_m{}^*(\top) = m$. This is summarized in (76).

[216] The definition of subobject classifier does not need the well-poweredness and the existence of all pullbacks, but they are necessary to have a universal property because it uses the functor $\mathrm{Sub}_{\mathbf{C}}$. In any case, subobject classifiers are usually used when these conditions are satisfied.

$$\begin{tikzcd} \text{in } \mathbf{Set} & & \text{in } \mathbf{C} \\ \mathbf{1} \arrow[r, "\top"] \arrow[dr, "m"'] & \mathrm{Sub}_{\mathbf{C}}(\Omega) \arrow[d, dashed, "\chi_m{}^*(-)"] & \Omega \\ & \mathrm{Sub}_{\mathbf{C}}(X) & X \arrow[u, dashed, "\chi_m"'] \end{tikzcd} \quad \xleftarrow{\mathrm{Sub}_{\mathbf{C}}} \tag{76}$$

Notice that the dashed arrow gets reversed because $\mathrm{Sub}_{\mathbf{C}}$ is contravariant. We could also write "in $\mathbf{C}^{\mathrm{op}}$" and not reverse the arrow.

*Power Objects*

Let $X$ be a set, the powerset of $X$, $\mathcal{P}X$ is the set of all subsets of $X$.

$$\begin{tikzcd} \text{in } \mathbf{Set} & & \text{in } \mathbf{C} \\ \mathbf{1} \arrow[r, "\ni_A"] \arrow[dr, "m"'] & \mathrm{Sub}_{\mathbf{C}}(\mathfrak{P}X \times X) \arrow[d, dashed, "\chi_m{}^*(-\times \mathrm{id}_X)"] & \mathfrak{P}X \\ & \mathrm{Sub}_{\mathbf{C}}(Y \times X) & Y \arrow[u, dashed, "\chi_m"'] \end{tikzcd} \quad \xleftarrow{\mathrm{Sub}_{\mathbf{C}}(-\times X)} \tag{77}$$

A finitely complete category where every object has a power object is called an **(elementary) topos**. Topos theory is a vast subject concerned with properties and uses of toposes.

## *Generalization*

**Definition 237** (Comma category)**.** Given two functors $\mathbf{D} \overset{F}{\rightsquigarrow} \mathbf{C} \overset{G}{\leftsquigarrow} \mathbf{E}$ , there is a category $F \downarrow G$,[217] called the **comma category**, whose objects are triples $(X, Y, \alpha)$

[217] Some authors denote this category $F/G$.

with $X \in \mathbf{D}_0$, $Y \in \mathbf{E}_0$ and $\alpha : F(X) \to G(Y)$ (in $\mathbf{C}_1$) and morphisms between $(X_1, Y_1, \alpha)$ and $(X_2, Y_2, \beta)$ are pairs of morphisms $(f, g) \in \mathrm{Hom}_{\mathbf{D}}(X_1, X_2) \times \mathrm{Hom}_{\mathbf{E}}(Y_1, Y_2)$ yielding a commutative square as in (78).

$$\begin{tikzcd} F(X_1) \arrow[r, "F(f)"] \arrow[d, "\alpha"'] & F(X_2) \arrow[d, "\beta"] \\ G(Y_1) \arrow[r, "G(g)"'] & G(Y_2) \end{tikzcd} \tag{78}$$

**Definition 238** (Arrow category)**.** In the setting of Definition 237, if $F = G = \mathrm{id}_{\mathbf{C}}$, then $\mathrm{id}_{\mathbf{C}} \downarrow \mathrm{id}_{\mathbf{C}}$ is called the **arrow category** of $\mathbf{C}$ and denoted $\mathbf{C}^{\to}$. Its objects are morphisms in $\mathbf{C}$ and its morphisms are commutative squares in $\mathbf{C}$.[218]

[218] Less concisely, a morphism $\phi : f \to g$ between morphisms $f : X \to Y$ and $g : X' \to Y'$ is a pair of morphisms $\phi_X : X \to X'$ and $\phi_Y : Y \to Y'$ making (**??**) commute.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"'] & Y \arrow[d, "\phi_Y"] \\ X' \arrow[r, "g"'] & Y' \end{tikzcd} \tag{79}$$

**Exercise 239.** Let $\mathbf{C}$ be a category (note the change of font to distinguish the functors from their action).

1. Show that $\mathsf{id} : \mathbf{C} \rightsquigarrow \mathbf{C}^{\to}$ sending $X \in \mathbf{C}_0$ to $\mathrm{id}_X$ is functorial.
2. Show that $\mathsf{s} : \mathbf{C}^{\to} \rightsquigarrow \mathbf{C}$ sending $f \in \mathbf{C}_0^{\to}$ to $s(f)$ is functorial.
3. Show that $\mathsf{t} : \mathbf{C}^{\to} \rightsquigarrow \mathbf{C}$ sending $f \in \mathbf{C}_0^{\to}$ to $t(f)$ is functorial.

See solution.

**Definition 240** (Slice category)**.** In the setting of Definition 237, if $F = \mathrm{id}_{\mathbf{C}}$ and $G = X : \mathbf{1} \rightsquigarrow \mathbf{C}$ is a constant functor selecting one object $G(\bullet) = X \in \mathbf{C}_0$, then $\mathrm{id}_{\mathbf{C}} \downarrow X$ is called the **slice category** over $X$ and denoted $\mathbf{C}/X$.[219] Its objects are morphisms in $\mathbf{C}$ with target $X$ and its morphisms are commutative triangles with $X$ as a tip as in (80).

[219] Some authors call this category $\mathbf{C}$ over $X$.

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & X & \end{tikzcd} \tag{80}$$

**Definition 241** (Coslice category)**.** In the setting of Definition 237, if $G = \mathrm{id}_{\mathbf{C}}$ and $F = X : \mathbf{1} \rightsquigarrow \mathbf{C}$ is a constant functor selecting one object $F(\bullet) = X \in \mathbf{C}_0$, then $X \downarrow \mathrm{id}_{\mathbf{C}}$ is called the **coslice category** under $X$ and denoted $X/\mathbf{C}$.[220] Its objects are morphisms in $\mathbf{C}$ with source $X$ and its morphisms are commutative triangles with $X$ as a tip as in (80).

[220] Some authors call this category $\mathbf{C}$ under $X$.

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ A \arrow[rr] & & B \end{tikzcd} \tag{81}$$

See solution.

**Exercise 242.** Show that for any category $\mathbf{C}$ and object $X \in \mathbf{C}_0$, the slice category $\mathbf{C}/X$ has a terminal object. State and prove the dual statement.

See solution.

**Exercise 243.** Show that the product of $f : A \to X$ and $g : B \to X$ in $\mathbf{C}/X$ exists if and only if the pullback of $A \xrightarrow{f} X \xleftarrow{g} B$ exists in $\mathbf{C}$. State and prove the dual statement.

Back to universal properties.

**Definition 244** (Universal morphism)**.** If $F : \mathbf{D} \rightsquigarrow \mathbf{C}$ is a functor and $X \in \mathbf{C}_0$. A **universal morphism** from $X$ to $F$ is an initial object in $X \downarrow F$. Namely, it is a morphism $a : X \to F(A)$ such that for any other morphism $b : X \to F(B)$, there is unique commutative triangle as in (82).

$$\begin{tikzcd} & X \arrow[dl, "a"'] \arrow[dr, "b"] & \\ F(A) \arrow[rr, dashed, "F(f)"'] & & F(B) \end{tikzcd} \tag{82}$$

Notice that equivalently, one could say that for any $b : X \to F(B)$, there is a unique morphism $f : A \to B$ in $\mathbf{D}$ such that $F(f) \circ a = b$, which is summarized in (83).

$$\begin{tikzcd} & \text{in } \mathbf{C} & & \text{in } \mathbf{D} \\ X \arrow[r, "a"] \arrow[dr, "b"'] & FA \arrow[d, dashed, "Ff"] & & A \arrow[d, dashed, "f"] \\ & FB & & B \end{tikzcd} \quad Ff \xleftarrow{F} f \tag{83}$$

The dual notion is a universal morphism from $F$ to $X$, it is a terminal object in $F \downarrow X$. The dual of (83) is depicted below.

$$\begin{tikzcd} & \text{in } \mathbf{C} & & \text{in } \mathbf{D} \\ A & FA \arrow[l, "a"'] & & A \\ & FB \arrow[ul, "b"] \arrow[u, dashed, "Ff"'] & & B \arrow[u, dashed, "f"'] \end{tikzcd} \quad Ff \xleftarrow{F} f \tag{84}$$

**Definition 245** (Universal property)**.** A **universal property** is the property of being a universal morphism.

**Examples 246.** Here we translate all the examples of this chapter into the general language.

1. The free monoid on a set $A$ is the universal morphism from $A$ to the forgetful functor $\mathbf{Mon} \rightsquigarrow \mathbf{Set}$.

2. The abelianization of a group $G$ is the universal morphism from $G$ to the forgetful functor $\mathbf{Ab} \rightsquigarrow \mathbf{Grp}$.

3. The set $S \subseteq V$ is a basis for the vector space $V$ when the inclusion $S \hookrightarrow V$ is the universal morphism from $S$ to the forgetful functor $\mathbf{Vect}_k \rightsquigarrow \mathbf{Set}$.

4. An exponential object is an object $A^X$ along with a universal morphism from the functor $- \times X$ to $A$.

5. A subobject classifier is a morphism $\top : \mathbf{1} \hookrightarrow \Omega$ such that the corresponding function $\top : \mathbf{1} \to \mathrm{Sub}_{\mathbf{C}}(\Omega)$ is a universal morphism from $\mathbf{1}$ to the functor $\mathrm{Sub}_{\mathbf{C}}$.

6. A power object of $X$ is an object $\mathfrak{P}X$ along with a universal morphism $\ni_X$ from $\mathbf{1}$ to $\mathrm{Sub}_{\mathbf{C}}(- \times X)$.

We will not bother applying this general definition anymore because the formalism is not crucial to the study of universal properties.

We have to postpone to Chapter  showing that, as we have claimed, any (co)limit satisfies a universal property. Still, you might have noticed that our definition of universal property also uses a special case of (co)limits, that is, initial and terminal objects. What is more, in the following chapters, we will introduce a couple more concepts which often coincide[221] with the concepts of (co)limits and universal properties.

[221] By *coincide*, we mean that one is a special case of the other or vice-versa or both directions.

# *Natural Transformations*

## *Natural Transformations*

Natural transformations are admittedly what made mathematicians want to study category theory in the first place. In short, they are morphisms between functors, i.e.: transformations that preserve the structure of functors.

The abstract structure of a category is very familiar because it resembles what is found in algebraic structures such as groups, rings or vector spaces. That is to say, it consists of the data of one or more sets with one or more operations satisfying one or more properties. In contrast, the definition of a functor is more opaque and by itself, the structure of a functor is not obvious. A functor is effectively a morphism between categories, hence a natural transformation will be a *morphism between morphisms*. Before moving on, one might find it enlightening to look for a satisfying definition of morphism between two group homomorphisms $f, g : G \to H$ and then observe its meaning when $f$ and $g$ are seen as functors $\mathbf{B}G \rightsquigarrow \mathbf{B}H$.

For the general case, let $F, G : \mathbf{C} \rightsquigarrow \mathbf{D}$ be functors. Morally, the structure of $F$ and $G$ is encapsulated in the following diagrams for every arrow, $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$.

$$\begin{tikzcd} A \arrow[r, "F_0"] \arrow[d, "f"'] & F(A) \arrow[d, "F_1(f)"] \\ B \arrow[r, "F_0"'] & F(B) \end{tikzcd} \quad (85) \qquad \begin{tikzcd} A \arrow[r, "G_0"] \arrow[d, "f"'] & G(A) \arrow[d, "G_1(f)"] \\ B \arrow[r, "G_0"'] & G(B) \end{tikzcd} \quad (86)$$

Thus, a morphism between $F$ and $G$ should fit in this picture by sending diagram (85) to diagram (86) in a commutative way.

**Definition 247** (Natural transformation)**.** Let $F, G : \mathbf{C} \rightsquigarrow \mathbf{D}$ be two (covariant) functors, a **natural transformation** $\phi : F \Rightarrow G$ is a map $\phi : \mathbf{C}_0 \to \mathbf{D}_1$ that satisfies $\phi(A) \in \mathrm{Hom}_{\mathbf{D}}(F(A), G(A))$ for all $A \in \mathbf{C}_0$ and makes (87) commute for any $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$:[222]

$$\begin{tikzcd} F(A) \arrow[r, "\phi(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\phi(B)"'] & G(B) \end{tikzcd} \tag{87}$$

Each $\phi(A)$ will be called a **component** of $\phi$ and may also be denoted $\phi_A$.

[222] When doing proofs relying on naturality (i.e.: the property of being natural), we will use (87) where we instantiate $\phi$, $F$, $G$, $A$, $B$ and $f$ with the natural transformation, functors, objects and morphism that is needed in the proof. In order to make this instantiation less painful, we will use the shorthand $\mathsf{NAT}(\phi, A, B, f)$ and instantiate the parameters (we can omit $F$ and $G$ because they should be known from the context).

As usual, there are trivial examples of natural transformations such as the **identity transformation** $\mathbb{1}_F : F \Rightarrow F$ that sends every object $A$ to the identity map $\mathrm{id}_{F(A)}$, but let us go back to the group case. Although very specific to single object categories, it is simple enough to quickly digest.

**Example 248.** Let $f, g : \mathbf{B}G \rightsquigarrow \mathbf{B}H$ be functors (i.e.: group homomorphisms), both send the unique object $*$ in $\mathbf{B}G$ to $*$ in $\mathbf{B}H$. Thus, a natural transformation $\phi : f \Rightarrow g$ has a single component $\phi(*) : * \to *$ in $H$, which is simply an element $\phi \in H$. The commutativity condition is then exhibited by diagram (88) (which lives in $\mathbf{B}H$) for any $x \in G$.

$$\begin{tikzcd} * \arrow[r, "\phi"] \arrow[d, "f(x)"'] & * \arrow[d, "g(x)"] \\ * \arrow[r, "\phi"'] & * \end{tikzcd} \tag{88}$$

Recall that composition in $\mathbf{B}H$ is just multiplication in $H$, so naturality of $\phi$ says that for any $x \in G$, $\phi \cdot f(x) = g(x) \cdot \phi$. Equivalently, $\phi f(x) \phi^{-1} = g(x)$. Therefore, $g = c_\phi \circ f$ where $c_\phi$ denotes conjugation by $\phi$.[223] In short, natural transformations between group homomorphisms correspond to factorizations through conjugations.

[223] In a group $(H, \cdot)$, **conjugation** by an element $h \in H$ is the homomorphism $c_h$ defined $x \mapsto hxh^{-1}$.

Next, an example closer to the general idea of a natural transformation.

**Example 249.** Fix some $n \in \mathbb{N}$ and define the functor $\mathrm{GL}_n : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$ by[224]

$$R \mapsto \mathrm{GL}_n(R) \text{ for any commutative ring } R \text{ and}$$
$$f \mapsto \mathrm{GL}_n(f) \text{ for any ring homomorphism } f.$$

[224] The map $\mathrm{GL}_n(f)$ is just the extension of $f$ on $\mathrm{GL}_n(R)$ by applying $f$ to every element of the matrices.

The second functor is $(-)^\times : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$ which sends a commutative ring $R$ to its group of units $R^\times$ and a ring homomorphism $f$ to $f^\times$, its restriction on $R^\times$. Checking these mappings define two covariant functors is left as an (simple) exercise, but one might expect these to be functors as they play nicely with the structure of the objects involved.

A natural transformation between these two functors is $\det : \mathrm{GL}_n \Rightarrow (-)^\times$ which maps a commutative ring $R$ to $\det_R$, the function calculating the determinant of a matrix in $\mathrm{GL}_n(R)$. The first thing to check is that $\det_R \in \mathrm{Hom}_{\mathbf{Grp}}(\mathrm{GL}_n(R), R^\times)$ which is clear because the determinant of an invertible matrix is always a unit, $\det_R(I_n) = 1$ and $\det_R$ is a multiplicative map.[225] The second thing is to verify that diagram (89) commutes for any $f \in \mathrm{Hom}_{\mathbf{CRing}}(R, S)$:

[225] i.e.: $\det_R(AB) = \det_R(A)\det_R(B)$.

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\det_R"] \arrow[d, "\mathrm{GL}_n(f)"'] & R^\times \arrow[d, "f^\times = f|_{R^\times}"] \\ \mathrm{GL}_n(S) \arrow[r, "\det_S"'] & S^\times \end{tikzcd} \tag{89}$$

We will check the claim for $n = 2$, but the general proof should only involve more notation to write the bigger expressions, no novel idea. Let $a, b, c, d \in R$, we have

$$(\det_S \circ \mathrm{GL}_2(f)) \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \det_S \left( \begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix} \right)$$

$$
\begin{aligned}
&= f(a)f(d) - f(b)f(c) \\
&= f(ad - bc) \\
&= f^{\times}(ad - bc) \\
&= (f^{\times} \circ \det_R)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right).
\end{aligned}
$$

We conclude that the diagram commutes and that det is indeed a natural transformation.[226]

[226] Modulo the cases $n > 2$. See solution.

**Exercise 250.** Let $F, G : \mathbf{C} \times \mathbf{C}' \rightsquigarrow \mathbf{D}$ be two functors. Show that a family

$$\{\phi_{X,Y} : F(X,Y) \to G(X,Y) \mid X \in \mathbf{C}_0, Y \in \mathbf{C}'_0\}$$

is a natural transformation if and only if for any $X \in \mathbf{C}_0$ and $Y \in \mathbf{C}'_0$, both[227]

[227] Recall the definition of $F(X,-)$ and $F(-,Y)$ from Exercise 99.

$$\phi_{X,-} : F(X,-) \Rightarrow G(X,-) \text{ and } \phi_{-,Y} : F(-,Y) \Rightarrow G(-,Y)$$

are natural.

Now, in order to talk about a category of functors, it remains to describe the composition of natural transformations.

**Definition 251** (Vertical composition)**.** Let $F, G, H : \mathbf{C} \rightsquigarrow \mathbf{D}$ be parallel functors and $\phi : F \Rightarrow G$ and $\eta : G \Rightarrow H$ be two natural transformations. Then, the **vertical composition** of $\phi$ and $\eta$, denoted $\eta \cdot \phi : F \Rightarrow H$ is defined by $(\eta \cdot \phi)(A) = \eta(A) \circ \phi(A)$ for all $A \in \mathbf{C}_0$. If $f : A \to B$ is a morphism in $\mathbf{C}$, then diagram (90) commutes by naturality of $\phi$ and $\eta$, showing that $\eta \cdot \phi$ is a natural transformation from $F$ to $H$.

The notation $\cdot$ is not widespread, most authors use $\circ$ because vertical composition is the composition in a functor category. We believe the distinction is helpful as you learn this material.

$$
\begin{tikzcd}
F(A) \arrow[r, "\phi(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\eta(A)"] \arrow[d, "G(f)"'] & H(A) \arrow[d, "H(f)"'] \\
F(B) \arrow[r, "\phi(B)"'] & G(B) \arrow[r, "\eta(B)"'] & H(B)
\end{tikzcd}
\tag{90}
$$

The meaning of *vertical* will come to light when horizontal composition is introduced in a bit.

**Definition 252** (Functor categories)**.** For any two categories $\mathbf{C}$ and $\mathbf{D}$, there is a functor category denoted $[\mathbf{C}, \mathbf{D}]$.[228] Its objects are functors from $\mathbf{C}$ to $\mathbf{D}$, its morphisms are natural transformations between such functors and the composition is the vertical composition defined above. One can check that associativity of $\cdot$ follows from associativity of composition in $\mathbf{D}$ and that the identity morphism for a functor $F$ is $\mathbb{1}_F$.

[228] Some authors denote it $\mathbf{D}^{\mathbf{C}}$, analogously to the exponential of sets.

**Example 253.** Recall that a left action of a group $G$ on a set $S$ is just a functor $\mathbf{B}G \rightsquigarrow \mathbf{Set}$. Now, between two such functors $F, F' \in [\mathbf{B}G, \mathbf{Set}]$, a natural transformation is a single map $\sigma : F(*) \to F'(*)$ such that $\sigma \circ F(g) = F'(g) \circ \sigma$ for any $g \in G$. In other words, denoting $\cdot$ for both group actions on $F(*)$ and on $F'(*)$, $\sigma$ satisfies

$\sigma(g \cdot x) = g \cdot (\sigma(x))$ for any $g \in G$ and $x \in F(*)$. In group theory, such a map is called $G$–**equivariant**.

Therefore, the category $[\mathbf{B}G, \mathbf{Set}]$ can be identified as the category of $G$–sets (sets equipped with an action of $G$) with $G$–equivariant maps as the morphisms.

**Exercise 254** (NOW!)**.** Isomorphisms in a functor category are called **natural isomorphisms**. Show that they are precisely the natural transformations whose components are all isomorphisms.

See solution.

Functors that are naturally isomorphic are essentially the same functor; they send the same object to isomorphic objects and the same morphism to morphisms that are well-behaved under composition with isomorphisms between the source and targets.

**Examples 255.** We can recover constructions we have seen before by studying categories of functors with a simple domain.

1. The terminal category $\mathbf{1}$ has a single object $\bullet$ and no morphism other than the identity. Notice that for any category $\mathbf{C}$, a functor $F : \mathbf{1} \rightsquigarrow \mathbf{C}$ is a simply a choice of object $F(\bullet) \in \mathbf{C}_0$ because $F(\mathrm{id}_\bullet) = \mathrm{id}_{F(\bullet)}$. If $F, G \in [\mathbf{1}, \mathbf{C}]$, then a natural transformation $\phi : F \Rightarrow G$ is simply a choice of morphism $\phi : F(\bullet) \to G(\bullet)$ because the naturality square (91) for the only morphism $\mathrm{id}_\bullet$ is trivially commutative. We conclude that $[\mathbf{1}, \mathbf{C}]$ can be identified with the category $\mathbf{C}$ itself.

$$\begin{tikzcd} F(\bullet) \arrow[r, "F(\mathrm{id}_\bullet)"] \arrow[d, "\phi"'] & F(\bullet) \arrow[d, "\phi"] \\ G(\bullet) \arrow[r, "G(\mathrm{id}_\bullet)"'] & G(\bullet) \end{tikzcd} \tag{91}$$

2. Similarly, we can see a functor $F : \mathbf{1} + \mathbf{1} \rightsquigarrow \mathbf{C}$[229] as a choice of two objects $F(\bullet_1)$ and $F(\bullet_2)$ (not necessarily distinct) and a natural transformation $\phi : F \Rightarrow G$ between two such functors as a choice of two morphisms $\phi_1 : F(\bullet_1) \to G(\bullet_1)$ and $\phi_2 : F(\bullet_2) \to G(\bullet_2)$. Therefore, we infer that $[\mathbf{1} + \mathbf{1}, \mathbf{C}]$ can be identified with $\mathbf{C} \times \mathbf{C}$.

[229] Recall $\mathbf{1} + \mathbf{1}$ is the category depicted in (6).

3. Let us go one level harder. A functor $F : \mathbf{2} \rightsquigarrow \mathbf{C}$[230] is a choice of two objects $FA$ and $FB$ as well as a morphism $Ff : FA \to FB$. It can also be seen as a single choice of morphism $Ff$ because $FA$ and $FB$ are determined to be the source and target of $Ff$ respectively. A natural transformation $\phi : F \Rightarrow G$ between two such functors is *not* simply a choice of two morphisms $\phi_A : FA \to GA$ and $\phi_B : FB \to GB$ because, while the naturality squares for $\mathrm{id}_A$ and $\mathrm{id}_B$ trivially commute, the naturality square (92) for $f$ is an additional constraint on $\phi$. Namely, it says $(\phi_A, \phi_B)$ makes a commutative square with $Ff$ and $Gf$, hence we can identify $[\mathbf{2}, \mathbf{C}]$ with the arrow category $\mathbf{C}^{\to}$.

[230] Recall $\mathbf{2}$ is the category depicted in (7).

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\phi_A"'] & FB \arrow[d, "\phi_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd} \tag{92}$$

**Exercise 256.** Show that the opposite of $[\mathbf{C}, \mathbf{D}]$ is $[\mathbf{C}^{\mathrm{op}}, \mathbf{D}^{\mathrm{op}}]$.

See solution.

It is now time to build intuition for the horizontal composition of natural transformations which will ultimately lead to the notion of a 2–category.

**Definition 257** (The left action of functors)**.** Let $F, F' : \mathbf{C} \rightsquigarrow \mathbf{D}$, $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$ be functors and $\phi : F \Rightarrow F'$ a natural transformation as summarized in (93).[231]

$$\begin{tikzcd} \mathbf{C} \arrow[r, squiggly, bend left, "F"{name=U}] \arrow[r, squiggly, bend right, "F'"'{name=L}] & \mathbf{D} \arrow[r, squiggly, "G"] & \mathbf{D}' \arrow[from=U, to=L, Rightarrow, "\phi"] \end{tikzcd} \tag{93}$$

[231] Using squiggly arrows for functors in diagrams is very non-standard, but I believe it helps remember what kind of objects we are dealing with. Moreover, since these diagrams are not commutative, it makes a good contrast with the plain arrow notation which was mostly used for commutative diagrams.

The functor $G$ acts on $\phi$ by sending it to $G\phi := A \mapsto G(\phi(A)) : \mathbf{C}_0 \to \mathbf{D}'_1$. Showing that (94) commutes for any $f \in \mathrm{Hom}_{\mathbf{C}}(A,B)$ will imply that $G\phi$ is a natural transformation from $G \circ F$ to $G \circ F'$ .

$$\begin{CD}
(G\circ F)(A) @>{G\phi(A)}>> (G\circ F')(A)\\
@V{(G\circ F)(f)}VV @VV{(G\circ F')(f)}V\\
(G\circ F)(B) @>>{G\phi(B)}> (G\circ F')(B)
\end{CD} \tag{94}$$

Consider this diagram after removing all applications of $G$, by naturality of $\phi$, it is commutative. Since functors preserve commutativity, the diagram still commutes after applying $G$, hence $G\phi : G \circ F \Rightarrow G \circ F'$ is indeed natural.[232]

[232] More concisely, we apply $G$ to $\mathsf{NAT}(\phi, A, B, f)$ to obtain (94).

We leave you to check this constitutes a left action, namely, for any $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$, $G' : \mathbf{D}' \rightsquigarrow \mathbf{D}''$ and $\phi : F \Rightarrow F'$,

$$\mathrm{id}_{\mathbf{D}}\phi = \phi \text{ and } G'(G\phi) = (G' \circ G)\phi.$$

**Definition 258** (The right action of functors)**.** Let $F, F' : \mathbf{C} \rightsquigarrow \mathbf{D}$, $H : \mathbf{C}' \rightsquigarrow \mathbf{C}$ be functors and $\phi : F \Rightarrow F'$ a natural transformation as summarized in (95).

$$\begin{tikzcd}
\mathbf{C}' \arrow[r, rightsquigarrow, "H"] & \mathbf{C} \arrow[r, rightsquigarrow, bend left, "F"] \arrow[r, rightsquigarrow, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathbf{D}
\end{tikzcd} \tag{95}$$

The functor $H$ acts on $\phi$ by sending it to $\phi H := A \mapsto \phi(H(A)) : \mathbf{C}'_0 \to \mathbf{D}_1$. Showing that (96) commutes for any $f \in \mathrm{Hom}_{\mathbf{C}'}(A,B)$ will imply that $\phi H$ is a natural transformation from $F \circ H$ to $F' \circ H$.

$$\begin{CD}
(F\circ H)(A) @>{\phi H(A)}>> (F'\circ H)(A)\\
@V{(F\circ H)(f)}VV @VV{(F'\circ H)(f)}V\\
(F\circ H)(B) @>>{\phi H(B)}> (F'\circ H)(B)
\end{CD} \tag{96}$$

Commutativity of (96) follows by naturality of $\phi$: change $f$ in diagram (87) with the morphism $H(f) : H(A) \to H(B)$, i.e.: (96) is $\mathsf{NAT}(\phi, HA, HB, Hf)$.

We leave you to check this constitutes a right action, namely, for any $H : \mathbf{C}' \rightsquigarrow \mathbf{C}$, $H' : \mathbf{C}'' \rightsquigarrow \mathbf{C}'$ and $\phi : F \Rightarrow F'$,

$$\phi\mathrm{id}_{\mathbf{C}} = \phi \text{ and } (\phi H)H' = \phi(H \circ H').$$

**Proposition 259.** *The two actions commute, i.e.: in the setting of* (97), $G(\phi H) = (G\phi)H$.[233]

[233] For this reason and the associativity of the two actions, we will drop all the parentheses from such expressions. We will also drop the $\circ$ for composition of functors. All in all, expect to find expressions like $G'G\phi HH'$ and infer the natural transformation $A \mapsto G'(G(\phi(H(H'(A)))))$.

$$\begin{tikzcd}
\mathbf{C}' \arrow[r, rightsquigarrow, "H"] & \mathbf{C} \arrow[r, rightsquigarrow, bend left, "F"] \arrow[r, rightsquigarrow, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathbf{D} \arrow[r, "G"] & \mathbf{D}'
\end{tikzcd} \tag{97}$$

*Proof.* In both the L.H.S. and the R.H.S., an object $A \in \mathbf{C}_0$ is sent to $G(\phi(H(A)))$. $\square$

A very useful result following from the properties of these two actions is that for any commutative diagram in $[\mathbf{C}, \mathbf{D}]$, we can pre-compose and post-compose with any functors and still obtain a commutative diagram. For instance, if (98) commutes in $[\mathbf{C}, \mathbf{D}]$, then for any functors $H : \mathbf{C}' \rightsquigarrow \mathbf{C}$ and $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$, then (99) commutes.[234]

[234] We will often use this property by writing things like "apply $G(-)H$ to (98)" to use the commutativity of (99) in a proof.

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\phi"'] & Y \arrow[d, "\phi'"] \\ X' \arrow[r, "\eta'"'] & Y' \end{tikzcd} \qquad (98) \qquad \begin{tikzcd} G \circ X \circ H \arrow[r, "G\eta H"] \arrow[d, "G\phi H"'] & G \circ Y \circ H \arrow[d, "G\phi' H"] \\ G \circ X' \circ H \arrow[r, "G\eta' H"'] & G \circ Y' \circ H \end{tikzcd} \qquad (99)$$

We will refer to these two actions as the **biaction** of functors on natural transformations and they will motivate the definition of another way to compose natural transformations.

Let $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ be categories, $H, H' : \mathbf{C} \rightsquigarrow \mathbf{D}$ and $G, G' : \mathbf{D} \rightsquigarrow \mathbf{E}$ be functors and $\phi : H \Rightarrow H'$ and $\eta : G \Rightarrow G'$ be natural transformations. This is summarized in (100).

$$\begin{tikzcd}[column sep=huge] \mathbf{C} \arrow[r, squiggly, bend left, "H"{name=U}] \arrow[r, squiggly, bend right, "H'"'{name=L}] & \mathbf{D} \arrow[r, squiggly, bend left, "G"{name=U2}] \arrow[r, squiggly, bend right, "G'"'{name=L2}] & \mathbf{E} \arrow[from=U, to=L, Rightarrow, "\phi"] \arrow[from=U2, to=L2, Rightarrow, "\eta"] \end{tikzcd} \qquad (100)$$

The ultimate goal is to obtain a composition of $\phi$ and $\eta$ that is a natural transformation $G \circ H \Rightarrow G' \circ H'$. Note that the biaction defined above yields four other natural transformations:

$$G\phi : G \circ H \Rightarrow G \circ H' \qquad \eta H : G \circ H \Rightarrow G' \circ H$$
$$G'\phi : G' \circ H \Rightarrow G' \circ H' \qquad \eta H' : G \circ H' \Rightarrow G' \circ H'.$$

All of the functors involved go from $\mathbf{C}$ to $\mathbf{E}$, so all four natural transformations fit in diagram (101) that lives in the functor category $[\mathbf{C}, \mathbf{E}]$.

$$\begin{tikzcd} G \circ H \arrow[r, "G\phi"] \arrow[d, "\eta H"'] & G \circ H' \arrow[d, "\eta H'"] \\ G' \circ H \arrow[r, "G'\phi"'] & G' \circ H' \end{tikzcd} \qquad (101)$$

At first glance, this suggests two different definitions for the horizontal composition, that is, the composition of the top path $(\eta H' \cdot G\phi)$ or the composition of the bottom path $(G'\phi \cdot \eta H)$. Surprisingly, both definitions coincide as shown in the next result.

**Lemma 260.** *Diagram* (101) *commutes, i.e.:* $\eta H' \cdot G\phi = G'\phi \cdot \eta H$.[235]

[235] Similarly to NAT, we will refer to the commutativity of (101) with $\mathsf{HOR}(\phi, \eta)$. We use HOR because this lemma is crucial in the definition of horizontal composition.

*Proof.* Fix an object $A \in \mathbf{C}_0$. Under $\eta H' \cdot G\phi$, it is sent to $\eta(H'(A)) \circ G(\phi(A))$ and under $G'\phi \cdot \eta H$, it is sent to $G'(\phi(A)) \circ \eta(H(A))$. Thus, the proposition is equivalent to saying diagram (102) is commutative (in $\mathbf{E}$) for all $A \in \mathbf{C}_0$.

$$\begin{tikzcd}
(G \circ H)(A) \arrow[r, "G(\phi(A))"] \arrow[d, "\eta(H(A))"'] & (G \circ H')(A) \arrow[d, "\eta(H'(A))"] \\
(G' \circ H)(A) \arrow[r, "G'(\phi(A))"'] & (G' \circ H')(A)
\end{tikzcd} \tag{102}$$

This follows from $\mathsf{NAT}(\eta, HA, H'A, \phi(A))$. $\square$

**Definition 261** (Horizontal composition)**.** In the setting described in (100), we define the **horizontal composition** of $\eta$ and $\phi$ by $\eta \diamond \phi = \eta H' \cdot G\phi = G'\phi \cdot \eta H$.[236]

[236] The $\diamond$ notation is not standard but there are no widespread symbol denoting horizontal composition. I have mostly seen $*$ or plain juxtaposition. Hopefully, you will encounter papers/books clear enough that you can typecheck to find what composition is being used.
See solution.

One crucial point we have made in earlier chapters is that a notion of composition must satisfy associativity and have identities. We will show the former right after you show the latter.

**Exercise 262.** Let $H : \mathbf{C}' \rightsquigarrow \mathbf{C}$, $F, F' : \mathbf{C} \rightsquigarrow \mathbf{D}$ and $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. Show that $\phi \diamond \mathbb{1}_H = \phi H$ and $\mathbb{1}_G \diamond \phi = G\phi$. Infer that $\mathbb{1}_{\mathrm{id}_{\mathbf{C}}}$ is the identity at $\mathbf{C}$ for $\diamond$.

**Proposition 263.** *In the setting of* (103), $\psi \diamond (\eta \diamond \phi) = (\psi \diamond \eta) \diamond \phi$.

$$\begin{tikzcd}
\mathbf{C} \arrow[r, bend left, squiggly, "H"] \arrow[r, bend right, squiggly, "H'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathbf{D} \arrow[r, bend left, squiggly, "G"] \arrow[r, bend right, squiggly, "G'"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{E} \arrow[r, bend left, squiggly, "K"] \arrow[r, bend right, squiggly, "K'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathbf{F}
\end{tikzcd} \tag{103}$$

*Proof.* Similarly to how we constructed diagram (101) previously, we can use the biaction of functors and composition of functors to obtain the following diagram in $[\mathbf{C}, \mathbf{F}]$.[237]

[237] All $\circ$'s are left out for simplicity.

$$\begin{tikzcd}
 & K'GH \arrow[rr, "K'G\phi"] \arrow[dd, "K'\eta H"] & & K'GH' \arrow[dd, "K'\eta H'"] \\
KGH \arrow[ur, "\psi GH"] \arrow[rr, "KG\phi"] \arrow[dd, "K\eta H"'] & & KGH' \arrow[ur, "\psi GH'"'] \arrow[dd, "K\eta H'"] & \\
 & K'G'H \arrow[rr, "K'G'\phi"] & & K'G'H' \\
KG'H \arrow[ur, "\psi G'H"] \arrow[rr, "KG'\phi"'] & & KG'H' \arrow[ur, "\psi G'H'"'] &
\end{tikzcd} \tag{104}$$

Here is how each face commutes.

**Top:** $\mathsf{HOR}(\psi, G\eta)$

**Bottom:** $\mathsf{HOR}(\psi, G'\eta)$

**Left:** $\mathsf{HOR}(\psi, \eta H)$

**Right:** $\mathsf{HOR}(\psi, \eta H')$

**Front:** $\mathsf{HOR}(K\eta, \phi)$

**Back:** $\mathsf{HOR}(K'\eta, \phi)$

As detailed in the margin, this commutes because each face of the cube corresponds to a variant of diagram (101) (with some substitutions and application of a functor) and combining commutative diagrams yields commutative diagrams. Then, it follows that $\diamond$ is associative because[238] $\psi \diamond (\eta \diamond \phi)$ is the diagonal of the front face followed by the bottom right arrow and $(\psi \diamond \eta) \diamond \phi$ is the top front arrow followed by the diagonal of the right face. $\square$

[238] We could have drawn only the front and right face, but the cube is cooler.

There is one last thing to conclude that **Cat** is a 2–category, namely, that the vertical and horizontal compositions interact nicely.

**Proposition 264** (Interchange identity)**.** *In the setting of* (106)*, the* ***interchange identity*** *holds:*

$$(\eta' \cdot \eta) \diamond (\phi' \cdot \phi) = (\eta' \diamond \phi') \cdot (\eta \diamond \phi). \tag{105}$$

It is in the drawing of (106) that the intuition behind the terms vertical and horizontal is taken.

$$\begin{tikzcd}
\mathbf{C} \arrow[r, squiggly, bend left=50, "H"{name=H}] \arrow[r, squiggly, "H'"{name=Hp}] \arrow[r, squiggly, bend right=50, "H''"'{name=Hpp}] \arrow[from=H, to=Hp, Rightarrow, "\phi"] \arrow[from=Hp, to=Hpp, Rightarrow, "\phi'"] & \mathbf{D} \arrow[r, squiggly, bend left=50, "G"{name=G}] \arrow[r, squiggly, "G'"{name=Gp}] \arrow[r, squiggly, bend right=50, "G''"'{name=Gpp}] \arrow[from=G, to=Gp, Rightarrow, "\eta"] \arrow[from=Gp, to=Gpp, Rightarrow, "\eta'"] & \mathbf{E}
\end{tikzcd} \tag{106}$$

*Proof.* Akin to the other proofs, this is a matter of combining the right diagrams. After combining the diagrams in $[\mathbf{C}, \mathbf{E}]$ corresponding to $\eta \diamond \phi$ and $\eta' \diamond \phi'$, it is easy to see that the R.H.S. of (105) is the morphism going from $G \circ H$ to $G'' \circ H''$ (see (107)).

$$\begin{tikzcd}
G \circ H \arrow[r, "G\phi"] \arrow[d, "\eta H"'] & G \circ H' \arrow[d, "\eta H'"] & \\
G' \circ H \arrow[r, "G'\phi"'] & G' \circ H' \arrow[r, "G'\phi'"] \arrow[d, "\eta' H'"'] & G' \circ H'' \arrow[d, "\eta' H''"] \\
& G'' \circ H' \arrow[r, "G''\phi'"'] & G'' \circ H''
\end{tikzcd} \tag{107}$$

$$\begin{tikzcd}
G \circ H \arrow[r, "G\phi"] \arrow[d, "\eta H"'] & G \circ H' \arrow[r, "G\phi'"] & G \circ H'' \arrow[d, "\eta H''"] \\
G' \circ H \arrow[d, "\eta' H"'] & & G' \circ H'' \arrow[d, "\eta' H''"] \\
G'' \circ H \arrow[r, "G''\phi"'] & G'' \circ H' \arrow[r, "G''\phi'"'] & G'' \circ H''
\end{tikzcd} \tag{108}$$

Moreover, observe that the diagram corresponding to the L.H.S. can be factored with the following equations (it also yields the factored diagram in (108)).

$$(\eta' \cdot \eta)H = \eta' H \cdot \eta H \qquad (\eta' \cdot \eta)H'' = \eta' H'' \cdot \eta H''$$
$$G(\phi' \cdot \phi) = G\phi' \cdot G\phi \qquad G''(\phi' \cdot \phi) = G''\phi' \cdot G''\phi$$

Combining the factored diagram with (107), we obtain (109) from which the interchange identity readily follows.[239]

$$\begin{tikzcd}
G \circ H \arrow[r, "G\phi"] \arrow[d, "\eta H"'] & G \circ H' \arrow[r, "G\phi'"] \arrow[d, "\eta H'"] & G \circ H'' \arrow[d, "\eta H''"] \\
G' \circ H \arrow[r, "G'\phi"'] \arrow[d, "\eta' H"'] & G' \circ H' \arrow[r, "G'\phi'"] \arrow[d, "\eta' H'"'] & G' \circ H'' \arrow[d, "\eta' H''"] \\
G'' \circ H \arrow[r, "G''\phi"'] & G'' \circ H' \arrow[r, "G''\phi'"'] & G'' \circ H''
\end{tikzcd} \tag{109}$$

□

[239] The top right and bottom left square commute by $\mathsf{HOR}(\eta, \phi')$ and $\mathsf{HOR}(\eta', \phi)$ respectively. This implies all of (109) commutes and we have seen that the path from $G \circ H$ to $G'' \circ H''$ can be seen as the R.H.S. of (105) by looking at (107) or the L.H.S. by looking at (108). Thus, we infer the equality of (105).

**Definition 265** (Strict 2–cateory)**.** A **strict 2–category** consists of

- a category $\mathbf{C}$,
- for every $A, B \in \mathbf{C}_0$ a category $\mathbf{C}(A, B)$ with $\mathrm{Hom}_{\mathbf{C}}(A, B)$ as its objects (composition is denoted $\cdot$ and identities $\mathbb{1}$) and morphisms are called 2–**morphisms**,

- a category with $\mathbf{C}_0$ as its objects, where the morphisms are pairs of parallel morphisms of $\mathbf{C}$ along with a 2–morphism between them. A morphism in this category is also called a 2**–cell**. The identity 2–cell at $A \in \mathbf{C}_0$ is the pair $(\mathrm{id}_A, \mathrm{id}_A)$ and the 2–morphism $\mathbb{1}_{\mathrm{id}_A}$ and composition of 2–cells is denoted $\diamond$),

such that the interchange identity (105) holds.[240]

[240] The interchange identity does not come out of nowhere, it is equivalent to the composition $\circ$ being a functor $\mathbf{C}(B,C) \times \mathbf{C}(A,B) \rightsquigarrow \mathbf{C}(A,C)$ that acts on 2–morphisms by $\diamond$ for every $A, B, C \in \mathbf{C}_0$. We leave you to show this in the special case of the 2–category of categories in Exercise 267.

This book is not the place to further study 2–categories, but we can say a few interesting things about them. There are notions of morphisms between 2–categories (called 2–functors) and morphisms between them (called 2–natural transformations). The latter can be composed in three different ways (analog to vertical and horizontal composition for 2–morphisms) and all possible compositions interact well together. In particular,[241] there is a unique 2–natural transformation that is the composition of all 2–natural transformations in (110) (there are multiple ways to obtain it, depending on what compositions you do in what order, but as in the interchange identity, we require them to lead to the same 2–natural transformation).

[241] There several so-called coherence axioms that describe how all compositions interact, but we state only one of them.

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=4ex, "\Downarrow \to \Downarrow \to \Downarrow"] \arrow[r, phantom, shift right=4ex, "\Downarrow \to \Downarrow \to \Downarrow"] & \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=4ex, "\Downarrow \to \Downarrow \to \Downarrow"] \arrow[r, phantom, shift right=4ex, "\Downarrow \to \Downarrow \to \Downarrow"] & \bullet \end{tikzcd} \tag{110}$$

The category of 2–categories with 2–functors and 2–natural transformations is now an instance of a 3–category. The field of *higher category theory* studies the generalizations of this to $n$–categories for any $n$ (even $n = \infty$!). However, most of higher category theory drops the *strict* part of our definition of 2–category because this condition is too strong. Very briefly, they allow the properties of composition, namely associativity and identities, to hold up to natural isomorphisms.

There is a relatively simple way to define strict $n$–categories using *enriched category theory*.[242] The definition of a locally small category can be seen as entirely taking place in the category **Set**. From this point of view, a locally small category is a collection $\mathbf{C}_0$ of objects equipped with

[242] I hope you can indulge this continued digression. Higher categories and enriched categories are not an indispensible tool in mathematics as I believe categories are to some extent. Still, I think these two little teasers might inspire some readers.

- a set $\mathrm{Hom}_{\mathbf{C}}(A, B) \in \mathbf{Set}_0$ for every $A, B \in \mathbf{C}_0$,
- a function $\circ_{A,B,C} \in \mathrm{Hom}_{\mathbf{Set}}\left(\mathrm{Hom}_{\mathbf{C}}(B,C) \times \mathrm{Hom}_{\mathbf{C}}(A,B), \mathrm{Hom}_{\mathbf{C}}(A,C)\right)$ for every $A, B, C \in \mathbf{C}_0$,
- and a function $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{Set}}\left(\mathbf{1}, \mathrm{Hom}_{\mathbf{C}}(A,A)\right)$,

with conditions that can be stated as commutative diagrams in **Set**. Commutativity of (111) and (112) means that the identity morphisms are neutral with respect to composition and commutativity of (113) means composition is associative.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{C}}(B,C) \times \mathbf{1} \arrow[r, "\mathrm{id} \times \mathrm{id}_B"] \arrow[dr, "\pi_{\mathrm{Hom}_{\mathbf{C}}(B,C)}"'] & \mathrm{Hom}_{\mathbf{C}}(B,C) \times \mathrm{Hom}_{\mathbf{C}}(B,B) \arrow[d, "\circ_{B,B,C}"] \\ & \mathrm{Hom}_{\mathbf{C}}(B,C) \end{tikzcd} \tag{111}$$

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{C}}(B,B) \times \mathrm{Hom}_{\mathbf{C}}(A,B) \arrow[d, "\circ_{A,B,B}"'] & \mathrm{Hom}_{\mathbf{C}}(A,B) \times \mathbf{1} \arrow[l, "\mathrm{id}_B \times \mathrm{id}"'] \arrow[dl, "\pi_{\mathrm{Hom}_{\mathbf{C}}(A,B)}"] \\ \mathrm{Hom}_{\mathbf{C}}(A,B) & \end{tikzcd} \tag{112}$$

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{C}}(C,D) \times \mathrm{Hom}_{\mathbf{C}}(B,C) \times \mathrm{Hom}_{\mathbf{C}}(A,B) \arrow[r, "\circ_{B,C,D} \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \circ_{A,B,C}"'] & \mathrm{Hom}_{\mathbf{C}}(B,D) \times \mathrm{Hom}_{\mathbf{C}}(A,B) \arrow[d, "\circ_{A,B,D}"] \\ \mathrm{Hom}_{\mathbf{C}}(C,D) \times \mathrm{Hom}_{\mathbf{C}}(A,C) \arrow[r, "\circ_{A,C,D}"'] & \mathrm{Hom}_{\mathbf{C}}(A,D) \end{tikzcd} \tag{113}$$

It turns out we can abstract the properties of $\mathbf{1}$ and $\times$ that ensure we can do category theory: we say that $(\mathbf{Set}, \times, \mathbf{1})$ is a **monoidal category**.[243] Now, *enriched category theory* is done by replacing **Set** with another category that has a monoidal structure.

[243] The specific properties are not too relevant for us right now, but know that $\times$ and $\mathbf{1}$ are called the **tensor** and **unit** of the monoidal category.

**Examples 266.** 1. The category $\mathbf{1}$ is a monoidal category with the tensor and unit being trivial (there is only one object, so there is no choice). A category enriched in $\mathbf{1}$ is simply a collection $\mathbf{C}_0$ because there is no choice when defining $\mathrm{Hom}_{\mathbf{C}}(A,B) \in \mathbf{1}_0$, $\circ_{A,B,C} \in \mathbf{1}_1$ and $\mathrm{id}_A \in \mathbf{1}_1$.

2.

3. The category **Cat** of small categories is monoidal with the tensor being $\times$ and the unit being $\mathbf{1}$. A category enriched in **Cat** is a strict 2–category. For instance, the 2–category of categories is a collection $\mathbf{Cat}_0$ of objects, a category $\mathbf{Cat}(\mathbf{C},\mathbf{D}) = [\mathbf{C},\mathbf{D}]$ for every $\mathbf{C},\mathbf{D} \in \mathbf{Cat}_0$, a functor $\mathrm{id}_{\mathbf{C}} : \mathbf{1} \rightsquigarrow [\mathbf{C},\mathbf{C}]$ that picks the identity functor and, as you will show in Exercise 267, a morphism

$$\circ_{\mathbf{C},\mathbf{D},\mathbf{E}} \in \mathrm{Hom}_{\mathbf{Cat}}([\mathbf{D},\mathbf{E}] \times [\mathbf{C},\mathbf{D}], [\mathbf{C},\mathbf{E}]).$$

The diagrams corresponding to (111), (112), and (113) (now they live in **Cat**) commute by results we have shown in this chapter.

4. Generalizing the previous item, a strict $n$–category is a category enriched in the category of strict $(n-1)$–categories.

**Exercise 267** (NOW!)**.** Show that there is a functor $[\mathbf{D},\mathbf{E}] \times [\mathbf{C},\mathbf{D}] \rightsquigarrow [\mathbf{C},\mathbf{E}]$ whose action on objects is $(F,G) \mapsto F \circ G$.

See solution.

## *Equivalences*

Recall that an isomorphism of categories is an isomorphism in the category **Cat**, namely, a functor $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ with an inverse $G : \mathbf{D} \rightsquigarrow \mathbf{C}$ such that $F \circ G = \mathrm{id}_{\mathbf{D}}$ and $G \circ F = \mathrm{id}_{\mathbf{C}}$. As is typical in mathematics, one cannot distinguish between isomorphic categories as they only differ in notations and terminology.

**Examples 268.**

Another example for readers who know a bit of advanced algebra. Let $k$ be a field and $G$ a finite group, the categories of $k[G]$–modules ($k[G]$ is the group ring of $k$ over $G$) and of $k$–linear representations of $G$ are isomorphic.

1. It was already shown in Example 253 (the details were implicit) that for a group $G$, the category $[\mathbf{B}G, \mathbf{Set}]$ is isomorphic to the category of $G$–sets with $G$–equivariant maps as morphisms.

2. In Example 255, three other isomorphisms were implicitly given:

$$[\mathbf{1}, \mathbf{C}] \cong \mathbf{C} \qquad [\mathbf{1}+\mathbf{1}, \mathbf{C}] \cong \mathbf{C} \times \mathbf{C} \qquad [\mathbf{2}, \mathbf{C}] \cong \mathbf{C}^{\rightarrow}.$$

3. The category $\mathbf{Rel}$ of sets with relations is isomorphic to $\mathbf{Rel}^{\mathrm{op}}$.[244] The functor $\mathbf{Rel} \rightsquigarrow \mathbf{Rel}^{\mathrm{op}}$ is the identity on objects and sends a relation $R \subseteq X \times Y$ to the opposite relation $\text{Я} \subseteq Y \times X$ (which is a morphism $X \to Y$ in $\mathbf{Rel}^{\mathrm{op}}$) defined by $(y, x) \in \text{Я} \Leftrightarrow (x, y) \in R$. The inverse is defined similarly.

[244] An arbitrary category $\mathbf{C}$ is not always isomorphic to its opposite. While the opposite functors $(-)^{\mathrm{op}}_{\mathbf{C}} : \mathbf{C} \rightsquigarrow \mathbf{C}^{\mathrm{op}}$ and $(-)^{\mathrm{op}}_{\mathbf{C}^{\mathrm{op}}} : \mathbf{C}^{\mathrm{op}} \rightsquigarrow \mathbf{C}$ are inverses of each other, they are contravariant functors.

4. Given three categories $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$, there is an isomorphism[245]

$$[\mathbf{C} \times \mathbf{D}, \mathbf{E}] \cong [\mathbf{C}, [\mathbf{D}, \mathbf{E}]].$$

[245] You might recognize a similarity with exponentials which rely on an isomorphism $\mathrm{Hom}_{\mathbf{C}}(B \times X, A) \cong \mathrm{Hom}_{\mathbf{C}}(B, A^X)$. The example here is more than an instance of exponentials of categories because the isomorphism is not only as sets but as categories.

The construction of the isomorphism follows the intuition of currying and uncurrying of functions, so the definitions are straightforward. Still, you will see that verifying the straightforward defintions are well-typed is cumbersome (but simple) because there are several levels of functors and natural transformations

Let $F : \mathbf{C} \times \mathbf{D} \rightsquigarrow \mathbf{E}$, the currying of $F$ is $\Lambda F : \mathbf{C} \rightsquigarrow [\mathbf{D}, \mathbf{E}]$ defined as follows. For $X \in \mathbf{C}_0$, the functor $\Lambda F(X)$ sends $Y \in \mathbf{D}_0$ to $F(X, Y)$ and $g \in \mathbf{D}_1$ to $F(\mathrm{id}_X, g)$. We showed in Exercise 99 that $\Lambda F(X) := F(X, -)$ is a functor. For $f \in \mathrm{Hom}_{\mathbf{C}}(X, X')$, we define the natural transformation $\Lambda F(f) : F(X, -) \Rightarrow F(X', -)$ by

$$\Lambda F(f)_Y = F(f, \mathrm{id}_Y) : F(X, Y) \Rightarrow F(X', Y).$$

The naturality square (114) is commutative because, by functoriality of $F$, the top and bottom path are equal to $F(f, g)$. We also have to show $\Lambda F$ is a functor, namely $\Lambda F(\mathrm{id}_X) = \mathbb{1}_{F(X,-)}$ and $\Lambda F(f \circ f') = \Lambda F(f) \cdot \Lambda F(f')$. We can verify this componentwise using functoriality of $F$.

$$\begin{tikzcd} F(X,Y) \arrow[r, "F(\mathrm{id}_X,g)"] \arrow[d, "F(f,\mathrm{id}_Y)"'] & F(X,Y') \arrow[d, "F(f,\mathrm{id}_{Y'})"] \\ F(X',Y) \arrow[r, "F(\mathrm{id}_{X'},g)"'] & F(X',Y') \end{tikzcd} \tag{114}$$

$$\Lambda F(\mathrm{id}_X)_Y = F(\mathrm{id}_X, \mathrm{id}_Y) = \mathrm{id}_{F(X,Y)}$$
$$\Lambda F(f \circ f')_Y = F(f \circ f', \mathrm{id}_Y) = F(f, \mathrm{id}_Y) \circ F(f', \mathrm{id}_Y) = \Lambda F(f)_Y \circ \Lambda F(f')_Y.$$

It remains to define $\Lambda-$ on morphisms. Given a natural transformation $\phi : F \Rightarrow F'$, we define $\Lambda\phi : \Lambda F \Rightarrow \Lambda F'$ at component $X \in \mathbf{C}_0$ by the natural transformation:

$$\Lambda\phi(X) = \phi_{X,-} : F(X, -) \Rightarrow F'(X, -).$$

We showed in Exercise 250 that $\phi_{X,-}$ is natural. Finally, we can check that $\Lambda-$ is a functor with the following derivations.[246]

$$\Lambda\mathbb{1}_F(X) = \mathbb{1}_{FX,-} = \mathbb{1}_{F(X,-)}$$
$$\Lambda(\phi \cdot \eta)(X) = (\phi \cdot \eta)_{X,-} = \phi_{X,-} \cdot \eta_{X,-} = \Lambda\phi \cdot \Lambda\eta$$

[246] The second inequality on the second line can be verified componentwise, i.e.: for every $Y \in \mathbf{D}_0$, we have

$$(\phi \cdot \eta)_{X,Y} = \phi_{X,Y} \circ \eta_{X,Y}(\phi_{X,-} \cdot \eta_{X,-})_Y.$$

Conversely, let $F : \mathbf{C} \rightsquigarrow [\mathbf{D}, \mathbf{E}]$, the uncurrying of $F$ is $\Lambda^{-1}F : \mathbf{C} \times \mathbf{D} \rightsquigarrow \mathbf{E}$ defined as follows. We use Exercise 100 to define $\Lambda^{-1}F$ componentwise. Fixing $X \in \mathbf{C}_0$, we know that $F(X)$ is a functor, so we set $\Lambda^{-1}F(X, -) = F(X)$. Fixing $Y \in \mathbf{D}_0$, we define $\Lambda^{-1}F(-, Y)$ on objects by sending $X \in \mathbf{C}_0$ to $F(X)(Y)$ and $f \in \mathbf{C}_1$ to $F(f)_Y$. To show $\Lambda^{-1}F(-, Y)$ is a functor, we use the functoriality of $F$ as follows.

$$\Lambda^{-1}F(\mathrm{id}_X, Y) = F(\mathrm{id}_X)_Y = \mathbb{1}_{F(X)_Y} = \mathrm{id}_{F(X)(Y)}$$
$$\Lambda^{-1}F(f \circ f', Y) = F(f \circ f')_Y = (F(f) \cdot F(f'))_Y = F(f)_Y \circ F(f')_Y.$$

Now, for every $f : X \to X'$ and $g : Y \to Y'$, the naturality of $F(f)$ implies the commutativity of (115). This means we can define

$$\begin{tikzcd} F(X)(Y) \arrow[r, "F(X)(g)"] \arrow[d, "F(f)_Y"'] & F(X)(Y') \arrow[d, "F(f)_{Y'}"] \\ F(X')(Y) \arrow[r, "F(X')(g)"'] & F(X')(Y') \end{tikzcd} \tag{115}$$

$$\Lambda^{-1}F(f, g) := \Lambda^{-1}F(X', g) \circ \Lambda^{-1}F(f, Y) = \Lambda^{-1}F(f, Y') \circ \Lambda^{-1}F(X, g),$$

and conclude by Exercise 100 that $\Lambda^{-1}F : \mathbf{C} \times \mathbf{D} \rightsquigarrow \mathbf{E}$ is a functor.

Given a natural transformation $\phi : F \Rightarrow F'$, we define $\Lambda^{-1}\phi : \Lambda^{-1}F \Rightarrow \Lambda^{-1}F'$ by $\Lambda^{-1}\phi_{X,Y} := (\phi_X)_Y$. By Exercise 250, it is enough to show naturality in one component at a time. Fix $X \in \mathbf{C}_0$, by hypothesis ($\phi_X$ is a morphism in $[\mathbf{D}, \mathbf{E}]$), $\phi_X : F(X) \Rightarrow F'(X)$ is natural in $Y$. Fix $Y \in \mathbf{D}_0$, we need to show the following square commutes.

$$\begin{tikzcd} F(X)(Y) \arrow[r, "\Lambda^{-1}F(f,Y)"] \arrow[d, "(\phi_X)_Y"'] & F(X')(Y) \arrow[d, "(\phi_{X'})_Y"] \\ F'(X)(Y) \arrow[r, "\Lambda^{-1}F'(f,Y)"'] & F'(X')(Y) \end{tikzcd} \tag{116}$$

Recalling that $\Lambda^{-1}F(f, Y) = F(f)_Y$ and $\Lambda^{-1}F'(f, Y) = F'(f)_Y$, we recognize this square as $\mathsf{NAT}(\phi, X, X', f)$ evaluated at $Y$. Finally, we can check that $\Lambda^{-1}-$ is a functor with the following derivations.

$$(\Lambda^{-1}\mathbb{1}_F)_{X,Y} = ((\mathbb{1}_F)_X)_Y = \mathrm{id}_{F(X)(Y)} = (\mathbb{1}_{\Lambda^{-1}F})_{X,Y}$$
$$(\Lambda^{-1}\phi \cdot \eta)_{X,Y} = ((\phi \cdot \eta)_X)_Y = (\phi_X)_Y \circ (\eta_X)_Y = (\Lambda^{-1}\phi)_{X,Y} \cdot (\Lambda^{-1}\eta)_{X,Y}$$

The last step (I promise) of this proof is to show that $\Lambda-$ and $\Lambda^{-1}-$ are inverses of each other. The mindless computations below suffice.

$$\Lambda\Lambda^{-1}F(X)(Y) = \Lambda^{-1}F(X, Y) = F(X)(Y)$$
$$\Lambda\Lambda^{-1}F(f)_Y = \Lambda^{-1}F(f, Y) = F(f)_Y$$

$$\Lambda^{-1}\Lambda F(X, Y) = \Lambda F(X)(Y) = F(X, Y)$$
$$\Lambda^{-1}\Lambda F(f, g) = \Lambda F(X')(g) \circ \Lambda F(f)_Y = F(\mathrm{id}_{X'}, g) \circ F(f, \mathrm{id}_Y) = F(f, g)$$

Although there are other interesting instances of isomorphic categories, natural transformations lead to a more nuanced (and often more useful) equality between two categories, that is, equivalence.

**Definition 269** (Equivalence)**.** A functor $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ is an **equivalence** of categories if there exists a functor $G : \mathbf{D} \rightsquigarrow \mathbf{C}$ such that $F \circ G \cong \mathrm{id}_{\mathbf{D}}$ and $G \circ F \cong \mathrm{id}_{\mathbf{C}}$.[247] This is clearly symmetric, so we say two categories $\mathbf{C}$ and $\mathbf{D}$ are **equivalent**, denoted $\mathbf{C} \simeq \mathbf{D}$, if there is an equivalence between them. Moreover, we say that $G$ is a **quasi-inverse** of $F$ and vice-versa.

[247] Recall that $\cong$ between functors stands for natural isomorphisms.

In order to gain more intuition on how equivalences equate two categories, let us observe what properties this forces on the functor $F$. For any morphism $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$, the following square commutes where $\phi(A)$ and $\phi(B)$ are isomorphisms.[248]

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, shift left, "\phi(A)"] & B \arrow[d, shift right, "\phi(B)"'] \\ GF(A) \arrow[r, "GF(f)"'] \arrow[u, shift left, "\phi(A)^{-1}"] & GF(B) \arrow[u, shift right, "\phi(B)^{-1}"'] \end{tikzcd} \tag{117}$$

[248] Naturality of $\phi$ only gives us $GF(f) \circ \phi(A) = \phi(B) \circ f$, but by composing with $\phi(A)^{-1}$ or $\phi(B)^{-1}$, we obtain the commutativity of all of (117). In particular, we have $GF(f) = \phi(B) \circ f \circ \phi(A)^{-1}$.

This implies that the map $f \mapsto GF(f) : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(GF(A), GF(B))$ is a bijection. Indeed, pre-composition by $\phi(A)^{-1}$ and post-composition by $\phi(B)$ are both bijections,[249] so

$$f \mapsto \phi(B) \circ f \circ \phi(A)^{-1} = GF(f)$$

is a bijection. Since $A$ and $B$ are arbitrary, $G \circ F$ is a fully faithful functor and a symmetric argument shows $F \circ G$ is also fully faithful. Then, it is easy to conclude that $F$ and $G$ must be fully faithful as well.

[249] Recall the definitions of monomorphisms and epimorphisms and the fact that isomorphisms are monic and epic.

What is more, the existence of an isomorphism $\eta(A) : A \to FG(A)$ for any object $A$ implies $F$ (symmetrically $G$) has the following property.

**Definition 270** (Essentially surjective)**.** A functor $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ is **essentially surjective** if for any $X \in \mathbf{D}_0$, there exists $Y \in \mathbf{C}_0$ such that $X \cong F(Y)$.

We will show that these two properties (full faithfulness and essential surjectivity) are necessary and sufficient for $F$ to be an equivalence.

**Theorem 271.** *A functor $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ is an equivalence of categories if and only if $F$ is fully faithful and essentially surjective.*

*Proof.* ($\Rightarrow$) Shown above.

($\Leftarrow$) We construct a functor $G : \mathbf{D} \rightsquigarrow \mathbf{C}$ such that $G \circ F \cong \mathrm{id}_{\mathbf{C}}$ and $F \circ G \cong \mathrm{id}_{\mathbf{D}}$. Since $F$ is essentially surjective, for any $A \in \mathbf{D}_0$, there exists an object $G(A) \in \mathbf{C}_0$ and an isomorphism $\phi(A) : F(G(A)) \cong A$. Hence, $A \mapsto G(A)$ is a good candidate to describe the action of $G$ on objects.

Next, similarly to the converse direction, note that for any $A, B \in \mathbf{D}_0$, the map

$$f \mapsto \phi(B) \circ f \circ \phi(A)^{-1}$$

is a bijection from $\mathrm{Hom}_{\mathbf{D}}(A, B)$ to $\mathrm{Hom}_{\mathbf{D}}(FG(A), FG(B))$. Moreover, since the functor $F$ is fully faithful, it induces a bijection

$$F_{A,B} : \mathrm{Hom}_{\mathbf{C}}(G(A), G(B)) \to \mathrm{Hom}_{\mathbf{D}}(FG(A), FG(B))$$

which in turns yields a bijection

$$G_{A,B} : \mathrm{Hom}_{\mathbf{D}}(A,B) \to \mathrm{Hom}_{\mathbf{C}}(G(A),G(B)) = f \mapsto F_1^{-1}(\phi(B) \circ f \circ \phi(A)^{-1}).$$

This is the action of $G$ on morphisms. Observe that the construction of $G$ ensures that $F \circ G \cong \mathrm{id}_{\mathbf{D}}$ through the natural transformation $\phi$. It remains to show that $G$ is indeed a functor and find a natural isomorphism $\eta : G \circ F \cong \mathrm{id}_{\mathbf{C}}$.

For any composable morphisms $(f,g)$, it is easy to verify that

$$F(G(f) \circ G(g)) = FG(f) \circ FG(g) = FG(f \circ g),$$

so functoriality of $G$ follows after applying $F_1^{-1}$. To find $\eta$, recall that the definition of $G$ yields commutativity of (118) for any $f \in \mathrm{Hom}_{\mathbf{C}}(A,B)$.

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, leftrightarrow, "\phi(F(A))"'] & F(B) \arrow[d, leftrightarrow, "\phi(F(B))"] \\ FGF(A) \arrow[r, "FGF(f)"'] & FGF(B) \end{tikzcd} \tag{118}$$

Then, because $F$ is fully faithful, the following square also commutes in $\mathbf{C}$ where $\eta = X \mapsto F_1^{-1}(\phi(F(X)))$ and we conclude that $\eta$ is a natural isomorphism $\mathrm{id}_{\mathbf{C}} \cong G \circ F$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, leftrightarrow, "\eta(A)"'] & B \arrow[d, leftrightarrow, "\eta(B)"] \\ GF(A) \arrow[r, "GF(f)"'] & GF(B) \end{tikzcd} \tag{119}$$

□

The insight to extract from this argument is that two categories are equivalent if they describe the same objects and morphisms with the only relaxation that isomorphic objects can appear any number of times in either category. In contrast, categories can only be isomorphic if they have exactly the same objects and morphisms.

*Remark* 272. We used the axiom of choice to construct the quasi-inverse of $F$.

We will detail a couple of *easy* examples of equivalences and briefly mention a few *harder* ones. Of course, all the isomorphisms of categories we saw earlier are examples of equivalences where the natural isomorphisms are identities.

**Examples 273** (Easy)**.** 1. Consider the full subcategory of **FinSet** consisting only of the sets $\emptyset, \{1\}, \{1,2\}, \ldots, \{1,\ldots,n\}, \ldots$, denote it **FinOrd**.[250] The inclusion functor is fully faithful by definition and we claim it is essentially surjective. Indeed, any set $X \in \mathbf{FinSet}_0$ has a finite cardinality $n$, so $X \cong \{1,\ldots,n\} \in \mathbf{FinOrd}_0$.

2. In a very similar fashion, an early result in linear algebra says that any finite dimensional vector space over a field $k$ is isomorphic to $k^n$ for some $n \in \mathbb{N}$.

[250] The name **FinOrd** is an abbreviation of finite ordinals, because we can also define **FinOrd** as the category of finite ordinals and functions between them.

Thus, the category whose objects are $k^n$ for all $n \in \mathbb{N}$ and morphisms are $m \times n$ matrices with entries in $k$,[251] which we denote $\mathbf{Mat}(k)$, is equivalent to the category of finite dimensional vector spaces.

[251] After making a choice of basis for all $k^n$, an $m \times n$ matrix with entries in $k$ corresponds to a linear map $k^n \to k^m$.

3. A **partial** function $f : X \rightharpoonup Y$ is a function that may not be defined on all of $X$.[252] There is category $\mathbf{Par}$ of sets and partial functions where identity morphism and composition are defined straightforwardly.[253] We can view a partial function $f : X \rightharpoonup Y$ as a total function $f' : X \to Y + \mathbf{1}$ which assigns to every $x$ where $f(x)$ is undefined the value $* \in \mathbf{1}$. Further extending $f'$ to $[f', \mathrm{id}_{\mathbf{1}}] : X + \mathbf{1} \to Y + \mathbf{1}$, we can see any partial function as a function between pointed sets where the distinguished element corresponds to being undefined.

   We claim that this yields a fully faithful functor $\mathbf{Par} \rightsquigarrow \mathbf{Set}_*$ sending $X$ to $(X + \mathbf{1}, *)$ and $f : X \rightharpoonup Y$ to $[f', \mathrm{id}_{\mathbf{1}}]$.

[252] In this context, a *normal* function defined on all of $X$ is called **total**.

[253] You can view $\mathbf{Par}$ as the subcategory of $\mathbf{Rel}$ where you only take the relations $R \subseteq X \times Y$ satisfying for any $x \in X$ (cf. Remark 82),

$$|\{y \in Y \mid (x,y) \in R\}| \leq 1.$$

The first two examples and many other simple examples of equivalences are examples of skeletons. They are morally a subcategory where all the isomorphic copies are removed.

**Definition 274** (Skeleton)**.** A category is called **skeletal** if there it contains no two isomorphic objects. A **skeleton** of a category is an equivalent skeletal category.

**Examples 275.** We have shown that $\mathbf{FinOrd} \simeq \mathbf{FinSet}$ and $\mathbf{Mat}(k) \simeq \mathbf{FDVect}_k$ and we leave to you the easy task to check that these are examples of skeletons.[254]

[254] Namely, you should show that no two sets in **FinOrd** are isomorphic and no two spaces in $\mathbf{Mat}(k)$ are isomorphic.

A category always has a skeleton if you assume the axiom of choice and the next result justifies us calling it *the* skeleton of a category.

**Exercise 276.** Show that all skeletons of a category are isomorphic.

See solution.

Here are other more interesting examples of equivalent categories.

**Example 277** (Medium)**.** Let $\mathbf{C}$ be a category, there is a functor $F : \mathbf{C} \rightsquigarrow \mathbf{C}^{\to}$ sending $X$ to $\mathrm{id}_X$ and $f : X \to Y$ to the commutative square in (120). This functor is an equivalence if and only if all morphisms in $\mathbf{C}$ are isomorphisms.[255] It is clearly fully faithful, so it is left to show $F$ is essentially surjective if and only if $\mathbf{C}$ is a groupoid.

[255] Such a category is called a **groupoid**.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd} \tag{120}$$

($\Rightarrow$) For any $f : X \to Y \in \mathbf{C}_1$, by hypothesis, there exists $A \in \mathbf{C}_0$ such that $\mathrm{id}_A \cong f$ in $\mathbf{C}^{\to}$. Let $(s : A \to X, t : A \to Y)$ be the isomorphism, its inverse must be $(s^{-1}, t^{-1})$. Looking at the chain of commutative squares in (121), we can infer that $s \circ t^{-1}$ is the inverse of $f$.[256]

$$\begin{tikzcd} Y \arrow[r, "t^{-1}"] \arrow[d, "\mathrm{id}_Y"'] & A \arrow[r, "s"] \arrow[d, "\mathrm{id}_A"'] & X \arrow[r, "s^{-1}"] \arrow[d, "f"'] & A \arrow[r, "s"] \arrow[d, "\mathrm{id}_A"'] & X \arrow[d, "\mathrm{id}_X"] \\ Y \arrow[r, "t^{-1}"'] & A \arrow[r, "t"'] & Y \arrow[r, "t^{-1}"'] & A \arrow[r, "s"'] & X \end{tikzcd} \tag{121}$$

[256] The composition $f \circ s \circ t^{-1}$ is the top path of the combined two leftmost squares, the bottom path is $t \circ t^{-1} \circ \mathrm{id}_Y = \mathrm{id}_Y$. The composition $s \circ t^{-1} \circ f$ is the bottom path of the combined two rightmost squares, the top path is $\mathrm{id}_X \circ s \circ s^{-1} = \mathrm{id}_X$.

($\Leftarrow$) Let $f : X \to Y$ be an object of $\mathbf{C}^{\to}$, the inverse of $f$ satisfies $f \circ f^{-1} = \mathrm{id}_Y$ and $f^{-1} \circ f = \mathrm{id}_X$, so the squares in (122) are isomorphisms in $\mathbf{C}^{\to}$ (they are inverses of each other). Thus, we find that $f$ is isomorphic to $\mathrm{id}_X$ which is in the image of $F$.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[d, "f^{-1}"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd} \tag{122}$$

**Examples 278** (Hard)**.** Examples of significant equivalences are all over the place in higher mathematics. However, they require a bit of work to describe them, thus let us only say a few words on them.

1. The equivalence between the category of affine schemes and the opposite of the category of commutative rings is a seminal result in scheme theory, a huge part of modern algebraic geometry.

2. The equivalence between Boolean lattices and Stone spaces is again seminal in the theory of Stone-type dualities. These can lead to deep connections between topology and logic. One application in particular is the study of the behavior of computer programs through formal semantics.

**Exercise 279.** Show that equivalence of categories is an equivalence relation. See solution.

**Exercise 280.** Show that $\mathbf{C} \simeq \mathbf{C}'$ and $\mathbf{D} \simeq \mathbf{D}'$ implies $[\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}', \mathbf{D}']$. See solution.

# *Yoneda Lemma*

## *Representable Functors*

Throughout this chapter, let $\mathbf{C}$ be a locally small category. Recall that for an object $A \in \mathbf{C}_0$, there are two Hom functors from $\mathbf{C}$ to $\mathbf{Set}$. The covariant one, $\mathrm{Hom}_{\mathbf{C}}(A, -)$, sends an object $B \in \mathbf{C}_0$ to $\mathrm{Hom}_{\mathbf{C}}(A, B)$ and a morphism $f : B \to B'$ to $f \circ (-)$. The contravariant one, $\mathrm{Hom}_{\mathbf{C}}(-, A)$, sends an object $B \in \mathbf{C}_0$ to $\mathrm{Hom}_{\mathbf{C}}(B, A)$ and a morphism $f : B \to B'$ to $(-) \circ f$. In order to lighten the notation, we denote these functors $H^A$ and $H_A$ respectively.[257]

Although these functors are sometimes interesting on their own, their full power is unleashed when they are related to other functors through natural transformations. Before doing that, let us investigate how nice Hom functors are. For instance, many Hom functors can be described in simpler terms.

[257] It might seem like this contradicts the notation used so far because $H^A$ is covariant and $H_A$ contravariant. However, this is not their *variance* in the parameter $A$, and we will show that in fact, the *variances* in $A$ are opposites.

**Examples 281.**

1. Let $\mathbf{1} = \{*\}$ be the terminal object in $\mathbf{Set}$, then what is the action of $H^{\mathbf{1}}$? For any object $B$,
$$H^{\mathbf{1}}(B) = \mathrm{Hom}_{\mathbf{Set}}(\mathbf{1}, B)$$
is easy to describe because for any element $b \in B$, there is a unique function $f : \mathbf{1} \to B = * \mapsto b$. Hence, there is an isomorphism from $H^{\mathbf{1}}(B)$ to $B$ for any $B \in \mathbf{C}_0$, it sends $f$ to $f(*)$ and its inverse sends $b \in B$ to the map $* \mapsto b$. Moreover, these isomorphisms are natural in $B$ because (123) clearly commutes for any $f : B \to B'$, yielding a natural isomorphism $H^{\mathbf{1}} \cong \mathrm{id}_{\mathbf{C}}$.

$$\begin{tikzcd} H^{\mathbf{1}}(B) \arrow[r, "f\circ(-)"] \arrow[d, leftrightarrow] & H^{\mathbf{1}}(B') \arrow[d, leftrightarrow] \\ B \arrow[r, "f"'] & B' \end{tikzcd} \tag{123}$$

2. Consider again the terminal object but in the category $\mathbf{Grp}$, namely, the group $\mathbf{1}$ only containing an identity element. Then, for any group $G$, the set $H^{\mathbf{1}}(G)$ is a singleton because any homomorphism $f : \mathbf{1} \to G$ must send the identity to the identity and no other choice can be made. Therefore, unlike in $\mathbf{Set}$, $H^{\mathbf{1}}$ is very uninteresting and acts like the constant functor $\mathbf{1} : \mathbf{Grp} \rightsquigarrow \mathbf{Set}$.

3. A better choice of object to mimic the behavior of $\mathrm{id}_{\mathbf{Grp}}$ is the additive group $\mathbb{Z}$. Indeed, for any $g \in G$, there is a unique homomorphism $f : \mathbb{Z} \to G$ sending $0$ to the identity and $1$ to $g$.[258] A very similar argument as above yields a natural isomorphism $H^{\mathbb{Z}} \cong \mathrm{id}_{\mathbf{Grp}}$.

[258] Note that $f$ is completely determined by $f(1)$ because the homomorphism properties imply that $f(n) = f(1) + \overset{n}{\cdots} + f(1)$, $f(-n) = f(n)^{-1}$, and $f(0)$ must be the identity.

4. The terminal object in **Cat** is the category **1** with a single object • and no morphism other than the identity. Observe that for any category **C**, a functor $\mathbf{1} \rightsquigarrow \mathbf{C}$ is just a choice of object. Therefore, the same argument will show that $H^{\mathbf{1}} \cong (-)_0$, where $(-)_0$ sends a category to its set[259] of objects and a functor to its action restricted on objects.

   In order to obtain a similar way to extract morphisms, consider the category **2** with two objects and a single morphism between them. One obtains a natural isomorphism $H^{\mathbf{2}} \cong (-)_1$.[260]

[259] Recall that **Cat** only contains small categories.

[260] You can prove this as we did for $H^{\mathbf{1}} \cong (-)_0$ or use Example 255.3.

These examples suggest that functors that are naturally isomorphic to Hom functors have nice properties,[261] they are said to be representable.

[261] In fact, we already know that the Hom functors are continuous (Theorem 222 and Corollary 223).

**Definition 282** (Representable functor)**.** A covariant functor $F : \mathbf{C} \rightsquigarrow \mathbf{Set}$ is **representable** if there is an object $X \in \mathbf{C}_0$ such that $F$ is naturally isomorphic to $\mathrm{Hom}_{\mathbf{C}}(X, -)$. If $F$ is contravariant, then it is representable if it is naturally isomorphic to $\mathrm{Hom}_{\mathbf{C}}(-, X)$.

**Examples 283.** Let us give examples of the contravariant kind.

1. The contravariant powerset functor $2^{-} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$ sends a set $X$ to its powerset $\mathcal{P}(X)$ and a function $f : X \to Y$ to the inverse image $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$. It is common to identify subsets of a given set with functions from this set into $2 = \{0, 1\}$. Formally, this is an isomorphism $2^X \cong H_2(X) = 2^X$ for any $X$, it maps $S \subseteq X$ to the characteristic function $\chi_S$.[262] In the reverse direction, it sends a function $g : X \to \{0, 1\}$ to $g^{-1}(1)$. It is easy to check that for any $f : X \to Y$, the isomorphisms make (124) commute, so $2^{-} \cong H_2$.

$$\begin{tikzcd} H_2(X) \arrow[r, "f\circ(-)"] \arrow[d, leftrightarrow] & H_2(Y) \arrow[d, leftrightarrow] \\ 2^X \arrow[r, "f^{-1}"'] & 2^Y \end{tikzcd} \tag{124}$$

[262] It sends $x \in X$ to 1 if $x \in S$ and to 0 otherwise.

2. In functional programming, it is often useful to transform a function taking multiple arguments so that it ends up taking a single argument but outputs another function. For instance, the multiplication function $\mathsf{mult} : \mathsf{int} \times \mathsf{int} \to \mathsf{int}$ that takes two numbers as inputs and outputs their product can be rewritten as $\mathsf{multc} : \mathsf{int} \to (\mathsf{int} \to \mathsf{int})$. The function $\mathsf{multc}$ takes a number as input and outputs a function that outputs the product of its input and the initial input of $\mathsf{multc}$. For example $\mathsf{multc}(3)$ is a function that outputs $3 \cdot n$ when $n$ is the input. This new function $\mathsf{multc}$ is said to be the curried version of $\mathsf{mult}$ in honor of Haskell Curry. This leads to a more general argument in **Set**.

   Fix two sets $A$ and $B$. The functor $\mathrm{Hom}(- \times A, B)$ maps a set $X$ to $\mathrm{Hom}(X \times A, B)$ and a function $f : X \to Y$ to the function $(-) \circ (f \times \mathrm{id}_A)$.[263] As suggested by the currying process for $\mathsf{mult}$, for any set $X$, there is a bijection $\mathrm{Hom}(X \times A, B) \cong \mathrm{Hom}(X, B^A)$. The image of $f : X \times A \to B$ is denoted $\lambda f$ and it satisfies

[263] You can see it as the composition $H_B \circ (- \times A)$.

$f(x,a) = \lambda f(x)(a)$ for any $x \in X$ and $a \in A$. It is easy to check that this is a bijection and also that it is natural in $X$ because (125) commutes for any $f : X \to Y$, so $\mathrm{Hom}(- \times A, B) \cong \mathrm{Hom}(-, B^A)$.

$$\begin{tikzcd} \mathrm{Hom}(X \times A, B) \arrow[r, "(-)\circ(f\times \mathrm{id}_A)"] \arrow[d, leftrightarrow] & \mathrm{Hom}(Y \times A, B) \arrow[d, leftrightarrow] \\ \mathrm{Hom}(X, B^A) \arrow[r, "(-)\circ f"'] & \mathrm{Hom}(Y, B^A) \end{tikzcd} \tag{125}$$

In the first item of Examples 281 and 283, we made an arbitrary choice of set. That is, we could have taken any singleton in the first case and any set with two elements in the second. More generally, one can show that if $A \cong B$, then $H_A \cong H_B$ and $H^A \cong H^B$.

**Exercise 284.** Let $A, B \in \mathbf{C}_0$ be isomorphic objects. Show that $H^A \cong H^B$. Dually, show that $H_A \cong H_B$.

See solution.

Surprisingly, the converse is also true and it will follow from the Yoneda lemma, but we prove it on its own first as a warm-up for the proof of the lemma.

**Proposition 285.** *Let $A, B \in \mathbf{C}_0$ be such that $H^A \cong H^B$, then $A \cong B$.*

*Proof.* The natural isomorphism gives two natural transformations $\phi : H^A \Rightarrow H^B$ and $\eta : H^B \Rightarrow H^A$ such that for any object $X \in \mathbf{C}_0$,

$$\eta_X \circ \phi_X : H^A(X) \to H^A(X) \quad \text{and} \quad \phi_X \circ \eta_X : H^B(X) \to H^B(X)$$

are identities. In order to show $A \cong B$, we will find two morphisms $f : B \to A$ and $g : A \to B$ such that $f \circ g = \mathrm{id}_A$ and $g \circ f = \mathrm{id}_B$. With the given data, there is no freedom to construct $f$ and $g$. Since $\mathbf{C}$, $A$ and $B$ are arbitrary, there are only two morphisms that are required to exist, $\mathrm{id}_A$ and $\mathrm{id}_B$. Next, we note that $\mathrm{id}_A \in H^A(A)$ and $\mathrm{id}_B \in H^B(B)$, hence, we can set $f := \phi_A(\mathrm{id}_A)$ and $g := \eta_B(\mathrm{id}_B)$.[264]

Now, $\phi_A(\mathrm{id}_A)$ is a morphism from $B$ to $A$, so (126) commutes by naturality of $\eta$.

$$\begin{tikzcd} H_B(A) \arrow[r, "\eta_A"] & H_A(A) \\ H_B(B) \arrow[u, "\phi_A(\mathrm{id}_A)\circ(-)"] \arrow[r, "\eta_B"'] & H_A(B) \arrow[u, "\phi_A(\mathrm{id}_A)\circ(-)"'] \end{tikzcd} \tag{126}$$

We conclude, by starting with $\mathrm{id}_B$ in the bottom left, that

$$g \circ f = \phi_A(\mathrm{id}_A) \circ \eta_B(\mathrm{id}_B) = \eta_A(\phi_A(\mathrm{id}_A)) = \mathrm{id}_A.$$

A dual argument shows that

$$f \circ g = \eta_B(\mathrm{id}_B) \circ \phi_A(\mathrm{id}_A) = \phi_B(\eta_B(\mathrm{id}_B)) = \mathrm{id}_B,$$

and we have shown $A \cong B$. □

[264] To emphasize the point about *no freedom*, try to convince yourself that any morphisms of type $B \to A$ and $A \to B$ that we can construct from $\mathrm{id}_A$, $\mathrm{id}_B$, $\phi$ and $\eta$ (the only data we have) must be equal to $f$ and $g$ as we defined them.

For every $A \in \mathbf{C}_0$, there are two functors $H^A$ and $H_A$, they are objects of $[\mathbf{C}, \mathbf{Set}]$ and $[\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ respectively. It is then reasonable to expect that the assignments $A \mapsto H^A$ and $A \mapsto H_A$ are functorial.

**Definition 286 (Yoneda embeddings).** The contravariant embedding $H^{(-)} : \mathbf{C}^{\mathrm{op}} \rightsquigarrow [\mathbf{C}, \mathbf{Set}]$ sends $A \in \mathbf{C}_0$ to the Hom functor $H^A$ and a morphism $f : A' \to A$ to the natural transformation $H^f : H^A \Rightarrow H^{A'}$ defined by $H^f_B := \mathrm{Hom}_{\mathbf{C}}(f, B) = (-) \circ f$ for every $B \in \mathbf{C}_0$. The naturality of $H^f$ follows because (127) commutes (by associativity) for any $g : B \to B'$.

$$\begin{tikzcd} H^A(B) \arrow[r, "(-)\circ f"] \arrow[d, "g\circ(-)"'] & H^{A'}(B) \arrow[d, "g\circ(-)"] \\ H^A(B') \arrow[r, "(-)\circ f"'] & H^{A'}(B') \end{tikzcd} \tag{127}$$

The covariant embedding $H_{(-)} : \mathbf{C} \rightsquigarrow [\mathbf{C}^{\mathrm{op}}, \mathbf{Set}]$ sends $B \in \mathbf{C}_0$ to the Hom functor $H_B$ and a morphism $f : B \to B'$ to the natural transformation $H_f : H_B \Rightarrow H_{B'}$ defined by $H^A_f = \mathrm{Hom}_{\mathbf{C}}(A, f) = f \circ (-)$ for any $A \in \mathbf{C}_0$. Naturality follows from a similar argument.

Functoriality is left for the reader to check.[265] The embeddings are called like that because both functors are fully faithful and injective on objects as will follow from the Yoneda lemma.

[265] A quick proof is to recognize the embeddings as the curried Hom bifunctor, i.e.:

$$H_{(-)} = \Lambda\mathrm{Hom}_{\mathbf{C}}(-,-).$$

## *Yoneda Lemma*

We have understood how an object $A \in \mathbf{C}_0$ sees the category $\mathbf{C}$ through representables, but since a representable is an object of another category, it is daring to study what representables see and how it relates to the object it represents. More formally, what is the functor $\mathrm{Hom}_{[\mathbf{C},\mathbf{Set}]}(H^A, -)$ describing. For simplicity, we denote it $\mathrm{Nat}(H^A, -)$ because, for a functor $F : \mathbf{C} \rightsquigarrow \mathbf{Set}$, $\mathrm{Nat}(H^A, F)$ is the collection[266] of natural transformations from $H^A$ to $F$.

[266] Even if $\mathbf{C}$ is locally small, there is no guarantee that $[\mathbf{C}, \mathbf{Set}]$ is locally small. Nevertheless, one consequence of the Yoneda lemma is that $\mathrm{Nat}(F, G)$ is a set whenever $F$ is representable.

The surprising relation that the Yoneda lemma describes is that $\mathrm{Nat}(H^A, F)$ is isomorphic to $F(A)$ naturally in $F$ and $A$. We first show the isomorphism and then explain the naturality.

**Lemma 287** (Yoneda lemma I). *For any $A \in \mathbf{C}_0$ and $F : \mathbf{C} \rightsquigarrow \mathbf{Set}$,*

$$\mathrm{Nat}(H^A, F) \cong F(A).$$

*Proof.* Fix $A$ and $F$, let $\phi_{A,F} : \mathrm{Nat}(H^A, F) \to F(A)$ be defined by $\alpha \mapsto \alpha_A(\mathrm{id}_A)$ (check that the types match). Let $\eta_{A,F} : F(A) \to \mathrm{Nat}(H^A, F)$ send an element $a \in F(A)$ to the natural transformation that has components $\eta_{A,F}(a)_B : f \mapsto F(f)(a) : \mathrm{Hom}_{\mathbf{C}}(A, B) \to F(B)$ for any $B \in \mathbf{C}_0$. Checking (128) commutes for any $g : B \to B'$

shows that $\eta_{A,F}(a)$ is a natural transformation.

$$
\begin{tikzcd}
H^A(B) \arrow[r, "F(-)(a)"] \arrow[d, "g\circ(-)"'] & F(B) \arrow[d, "F(g)"] \\
H^A(B') \arrow[r, "F(-)(a)"'] & F(B')
\end{tikzcd}
\tag{128}
$$

We now check that $\phi_{A,F}$ and $\eta_{A,F}$ are inverses. First, $(\eta \circ \phi)_{A,F}$ sends $\alpha \in \mathrm{Nat}(H^A, F)$ to $\eta_{A,F}(\alpha_A(\mathrm{id}_A))$, and at any $B \in \mathbf{C}_0$, we have

$$
\begin{aligned}
\eta_{A,F}(\alpha_A(\mathrm{id}_A))_B(f) &= F(f)(\alpha_A(\mathrm{id}_A)) && \text{def of } \eta \\
&= \alpha_B(f \circ \mathrm{id}_A) && \text{naturality of } \alpha \\
&= \alpha_B(f),
\end{aligned}
$$

thus $\alpha = (\eta \circ \phi)_{A,F}(\alpha)$.

Conversely, $(\phi \circ \eta)_{A,F}$ sends $a \in F(A)$ to $\eta_{A,F}(a)_A(\mathrm{id}_A) = F(\mathrm{id}_A)(a) = a$.

We conclude that $\eta_{A,F}$ and $\phi_{A,F}$ are inverses. □

What this results first tells us is that $\mathrm{Nat}(H^A, F)$ is a set (because it is isomorphic to $F(A)$ which is a set). This lets us define two new functors to understand the second part of the Yoneda lemma.

The assignment $(A, F) \mapsto \mathrm{Nat}(H^A, F)$ is a functor $\mathbf{C} \times [\mathbf{C}, \mathbf{Set}] \rightsquigarrow \mathbf{Set}$. We denote it $\mathrm{Nat}(H^{(-)}, -)$, it sends a morphism $(g, \mu) : (A, F) \to (A', F')$ to $\mu \cdot (-) \cdot H^g : \mathrm{Nat}(H^A, F) \to \mathrm{Nat}(H^{A'}, F')$.[267]

The assignment $(A, F) \mapsto F(A)$ is another functor of the same type. We denote it $\mathsf{Ev}$ (for evaluation), it sends a morphism $(g, \mu) : (A, F) \to (A', F')$ to $F'(g) \circ \mu_A : F(A) \to F'(A')$.[268]

[267] If $g : A \to A'$, $\mu : F \Rightarrow F'$, and $\eta \in \mathrm{Nat}(H^A, F)$, we have

$$H^{A'} \overset{H^g}{\Longrightarrow} H^A \overset{\eta}{\Longrightarrow} F \overset{\mu}{\Longrightarrow} F' \in \mathrm{Nat}(H^{A'}, F').$$

We leave you to finish checking functoriality.

[268] You can check this functor is the uncurrying of the identity functor on $[\mathbf{C}, \mathbf{Set}]$, i.e.: $\mathsf{Ev} = \Lambda^{-1}\mathrm{id}_{[\mathbf{C},\mathbf{Set}]}$

**Lemma 288** (Yoneda lemma II)**.** *There is a natural isomorphism* $\mathrm{Nat}(H^{(-)}, -) \cong \mathsf{Ev}$.

*Proof.* The components of this isomorphism are the ones described in the first part of the result. It remains to show that $\phi$ is natural in $(A, F)$. For any $(g, \mu) : (A, F) \to (A', F')$, we need to show the following square commutes.

$$
\begin{tikzcd}
\mathrm{Nat}(H^A, F) \arrow[r, "\phi_{A,F}"] \arrow[d, "\mu\cdot(-)\cdot H^g"'] & F(A) \arrow[d, "F'(g)\circ\mu_A"] \\
\mathrm{Nat}(H^{A'}, F') \arrow[r, "\phi_{A',F'}"'] & F'(A')
\end{tikzcd}
\tag{129}
$$

Starting with a natural transformation $\alpha \in \mathrm{Nat}(H^A, F)$ the lower path sends it to $(\mu \cdot \alpha \cdot H^g)_{A'}(\mathrm{id}_{A'})$ and the upper path sends it to $(F'(g) \circ \mu_A)(\alpha_A(\mathrm{id}_A))$. The following derivation shows they are equal.

$$
\begin{aligned}
(\mu \cdot \alpha \cdot H^g)_{A'}(\mathrm{id}_{A'}) &= (\mu_{A'} \circ \alpha_{A'})(H^g_{A'}(\mathrm{id}_{A'})) && \text{def of composition} \\
&= (\mu_{A'} \circ \alpha_{A'})(g) && \text{def of } H^g_{A'} \\
&= (\mu_{A'} \circ \alpha_{A'})(H^A_g(\mathrm{id}_A)) && \text{def of } H^A_g
\end{aligned}
$$

$$
\begin{aligned}
&= (\mu_{A'} \circ \alpha_{A'} \circ H^A_g)(\mathrm{id}_A) \\
&= (\mu_{A'} \circ F(g) \circ \alpha_A)(\mathrm{id}_A) && \text{naturality of } \alpha \\
&= (F'(g) \circ \mu_A)(\alpha_A(\mathrm{id}_A)) && \text{naturality of } \mu
\end{aligned}
$$

□

**Corollary 289.** *The Yoneda embeddings $H^{(-)}$ and $H_{(-)}$ are fully faithful.*

*Proof.* Left as an exercise. □

**Example 290** (Cayley's theorem with the Yoneda lemma)**.** Cayley's theorem states that any group is isomorphic to the subgroup of a permutation group. We will use the Yoneda lemma to show that.

Recall the first part of the Yoneda lemma which states that for a category $\mathbf{C}$, a functor $F : \mathbf{C} \rightsquigarrow \mathbf{Set}$ and an object $A \in \mathbf{C}_0$, we have

$$\mathrm{Nat}(\mathrm{Hom}(A, -), F) \cong F(A).$$

Moreover, we know the explicit maps, namely, a natural transformation $\phi$ in the L.H.S. is mapped to $\phi_A(\mathrm{id}_A)$ and an element $u \in F(A)$ is mapped to the natural transformation $\{\phi_B = f \mapsto F(f)(u) \mid B \in \mathbf{C}_0\}$.

Let us apply this to $\mathbf{C}$ being the delooping of $G$. Recall that any functor $F : \mathbf{B}G \rightsquigarrow \mathbf{Set}$ sends $*$ to a set $S$ and any $g \in G$ to a permutation of $S$, it corresponds to an action of $G$ on $S$.

To use the Yoneda lemma, our only choice of object for $A$ is $*$ and we will choose for $F$ the functor it represents, i.e.: $F = \mathrm{Hom}(*, -)$. The Yoneda lemma yields

$$\mathrm{Nat}(\mathrm{Hom}(*, -), \mathrm{Hom}(*, -)) \cong \mathrm{Hom}(*, *).$$

We already know what the R.H.S. is $G$,[269] but we have to do a bit of work to understand the L.H.S. First, observe that a natural transformation $\phi : \mathrm{Hom}(*, -) \Rightarrow \mathrm{Hom}(*, -)$ is just one morphism $\phi_* : \mathrm{Hom}(*, *) \to \mathrm{Hom}(*, *)$. Namely, it is a map from $G$ to $G$. Second, recalling that $\mathrm{Hom}(*, g) = g \circ (-)$ and that $*$ is the only object in $\mathbf{C}_0$, we get that $\phi_*$ must only make (130) commute.

[269] By definition of $\mathbf{B}G$.

$$
\begin{tikzcd}
G \arrow[r, "\phi_*"] \arrow[d, "g\circ(-)"'] & G \arrow[d, "g\circ(-)"] \\
G \arrow[r, "\phi_*"'] & G
\end{tikzcd}
\tag{130}
$$

This is equivalent to $\phi_*(g \cdot h) = g \cdot \phi_*(h)$, and we get that each $\phi_*$ is a $G$–equivariant map. Denote the set of $G$–equivariant maps $\mathrm{Hom}_G(G, G)$. We obtain that, as sets,

$$\mathrm{Hom}_G(G, G) \cong G.$$

Now, we can check that $\mathrm{Hom}_G(G, G)$ is a subgroup of $\Sigma_G$ (the group of permutations of the set $G$) and that the bijection is in fact an group isomorphism. Cayley's theorem follows.

To check that $\mathrm{Hom}_G(G,G) < \Sigma_G$, we have to show that $\mathrm{id}_G$ is $G$–equivariant, that $G$–equivariant maps are bijective and that they are stable under composition and taking inverse. First, we have $\mathrm{id}_G(g \cdot h) = g \cdot h = g \cdot \mathrm{id}_G(h)$, so $\mathrm{id}_G \in \mathrm{Hom}_G(G,G)$. Second, let $f$ be a $G$–equivariant map. For any $g \in G$, we have $f(g) = f(g \cdot 1) = g \cdot f(1)$. Thus, $f$ is determined only by where it sends the identity. Additionally, sice for any choice of $f(1)$, $g \cdot f(1)$ ranges over $G$ when $g$ ranges over $G$, $f$ is bijective. Therefore, if $f$ and $f'$ are both $G$–equivariant map, then

$$(f \circ f')(g \cdot h) = f(f'(g \cdot h)) = f(g \cdot f'(h)) = g \cdot (f \circ f')(h),$$

hence $f \circ f'$ is $G$–equivariant. Finally, $f^{-1}$ is the $G$–equivariant map sending 1 to $f(1)^{-1}$ and we conclude that $\mathrm{Hom}_G(G,G)$ is a subgroup of $\Sigma_G$.

The final check is that the Yoneda bijection $G \to \mathrm{Hom}_G(G,G)$ sending $g$ to $(-) \cdot g$ is a group homomorphism.[270] It is clear that it sends the identity to the identity and for any $g,h \in G$

$$(-) \cdot gh = ((-) \cdot g) \cdot h = ((-) \cdot h) \circ ((-) \cdot g),$$

so this is a group homomorphism.

[270] isomorphism follows because it is a bijection.

## *Universality as Representability*

Representability is one of the two ways to describe universal constructions that we hinted at at the end of Chapter . In this section, we will explore how any universal property is equivalent to representability of some functor. Since (co)limits and universal morphisms are initial or terminal objects in some category, there is a first trivial way to express universality as representability.

**Exercise 291** (NOW!). Let $\mathbf{C}$ be a category, $X \in \mathbf{C}_0$ and $\mathbf{1} : \mathbf{C} \rightsquigarrow \mathbf{Set}$ be the constant functor at the singleton $\mathbf{1} = \{\star\}$. Show that $\mathrm{Hom}_{\mathbf{C}}(X,-) \cong \mathbf{1}$ if and only if $X$ is initial. Dually, $\mathrm{Hom}_{\mathbf{C}}(-,X) \cong \mathbf{1}$ if and only if $X$ is terminal.[271]

See solution.

[271] In the dual statement, the domain of $\mathbf{1}$ is $\mathbf{C}^{\mathrm{op}}$.

It turns out this result is not very useful.

**Proposition 292.** *Let $X, Y \in \mathbf{C}_0$. The product of $X$ and $Y$ exists if and only if there exists $P \in \mathbf{C}_0$ such that* $\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta_{\mathbf{C}}(-),(X,Y)) \cong \mathrm{Hom}_{\mathbf{C}}(-,P)$. *The product is $P$.*

*Proof.* ($\Rightarrow$) Let $P = X \times Y$, for any $A \in \mathbf{C}_0$, there is an isomorphism

$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((A,A),(X,Y)) \cong \mathrm{Hom}_{\mathbf{C}}(A, X \times Y)$$

which sends the pair $(f : A \to X, g : A \to Y)$ to $\langle f, g \rangle : A \to X \times Y$.[272] In the other direction, $p : A \to X \times Y$ is sent to the pair $(\pi_X \circ p, \pi_Y \circ p)$. Let us show it is natural in $A$. For any $m : A' \to A$, (131) commutes because the top path sends the pair $(f,g)$ to the morphism $\langle f, g \rangle$ then to $\langle f, g \rangle \circ m = \langle f \circ m, g \circ m \rangle$ and the bottom path sends $(f,g)$ to $(f,g) \circ (m,m) = (f \circ m, g \circ m)$ which is then sent to $\langle f \circ m, g \circ m \rangle$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((A,A),(X,Y)) \arrow[r, "\sim"] \arrow[d, "-\circ(m,m)"'] & \mathrm{Hom}_{\mathbf{C}}(A, X \times Y) \arrow[d, "-\circ m"] \\
\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((A',A'),(X,Y)) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathbf{C}}(A', X \times Y)
\end{tikzcd} \tag{131}$$

[272] Recall that $\langle f, g \rangle$ is the unique morphism satisfying $\pi_X \circ \langle f, g \rangle = f$ and $\pi_Y \circ \langle f, g \rangle = g$. Be careful not to confuse it with a pair of morphisms.

($\Leftarrow$) First, we define $\pi_X$ and $\pi_Y$ to be the pair of morphisms corresponding to $\mathrm{id}_P$ under the isomorphism $\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((P,P),(X,Y)) \cong \mathrm{Hom}_{\mathbf{C}}(P,P)$. Given two morphisms $f : A \to X$ and $g : A \to Y$, the isomorphism

$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((A,A),(X,Y)) \cong \mathrm{Hom}_{\mathbf{C}}(A,P)$$

yields a unique morphism $! : A \to P$. To see that $\pi_X \circ ! = f$ and $\pi_Y \circ ! = g$ we start with $\mathrm{id}_P$ in the top right of (132) which commutes by hypothesis.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((P,P),(X,Y)) \arrow[d, "-\circ(!,!)"'] & \mathrm{Hom}_{\mathbf{C}}(P,P) \arrow[l, "\sim"'] \arrow[d, "-\circ !"] \\
\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((A,A),(X,Y)) & \mathrm{Hom}_{\mathbf{C}}(A,P) \arrow[l, "\sim"]
\end{tikzcd} \tag{132}$$

□

**Corollary 293** (Dual)**.** *Let $X, Y \in \mathbf{C}_0$. The coproduct of $X$ and $Y$ exists if and only if there exists $S \in \mathbf{C}_0$ such that $\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((X,Y),\Delta_{\mathbf{C}}(-)) \cong \mathrm{Hom}_{\mathbf{C}}(S,-)$. The coproduct is $S$.*[273]

[273] We implicitly use the fact that $(\mathbf{C}\times\mathbf{C})^{\mathrm{op}} \cong \mathbf{C}^{\mathrm{op}} \times \mathbf{C}^{\mathrm{op}}$.

In order to generalize these two results to arbitrary (co)limits, we define the generalized version of $\Delta_{\mathbf{C}}$.

**Definition 294** (Generalized diagonal functor)**.** Let $\mathbf{J}$ and $\mathbf{C}$ be categories, the **generalized diagonal functor** $\Delta^{\mathbf{J}}_{\mathbf{C}} : \mathbf{C} \rightsquigarrow [\mathbf{J},\mathbf{C}]$ sends an object $X \in \mathbf{C}_0$ to the constant functor at $X$ and a morphism $f : X \to Y \in \mathbf{C}_1$ to the natural transformation whose components are all $f : X \to Y$.

We have $\Delta^{\mathbf{J}}_{\mathbf{C}}(f) : X \Rightarrow Y$ because for any $a \in \mathbf{J}_1$, the square below commutes.

$$\begin{tikzcd}
X \arrow[r, "X(a)=\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\
Y \arrow[r, "Y(a)=\mathrm{id}_Y"'] & Y
\end{tikzcd}$$

*Remark* 295. This is a generalization of the diagonal functor $\Delta_{\mathbf{C}} : \mathbf{C} \rightsquigarrow \mathbf{C}\times\mathbf{C}$ because, with the isomorphism $[\mathbf{1}+\mathbf{1},\mathbf{C}] \cong \mathbf{C}\times\mathbf{C}$ described in Example 255.2, we can identify $\Delta_{\mathbf{C}}$ with $\Delta^{\mathbf{1}+\mathbf{1}}_{\mathbf{C}}$.

**Proposition 296.** *Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram. The limit of $F$ exists if and only if there is an object $L \in \mathbf{C}_0$ such that $\mathrm{Nat}(\Delta^{\mathbf{J}}_{\mathbf{C}}(-),F) \cong \mathrm{Hom}_{\mathbf{C}}(-,L)$.*[274] *The tip of the limit cone is $L$.*

[274] Recall that

$$\mathrm{Nat}(\Delta^{\mathbf{J}}_{\mathbf{C}}(-),F) = \mathrm{Nat}(-,F)\circ\Delta^{\mathbf{J}}_{\mathbf{C}}.$$

*Proof.* First, we note that for any $X \in \mathbf{C}_0$, a natural transformation $\psi : \Delta^{\mathbf{J}}_{\mathbf{C}}(X) \Rightarrow F$ is a cone over $F$ with tip $X$. Indeed, for any $a : A \to B \in \mathbf{J}_1$, the naturality square in (133) is commutative.

$$\begin{tikzcd}
X \arrow[r, "X(a)=\mathrm{id}_X"] \arrow[d, "\psi_A"'] & X \arrow[d, "\psi_B"] \\
FA \arrow[r, "F(a)"'] & FB
\end{tikzcd} \tag{133}$$

This is equivalent to $\{\psi_A : X \to FA\}_{A\in\mathbf{J}_0}$ being a cone over $F$. Furthermore, a morphism of cones $\phi \to \psi$ is a morphism $f$ between the tips such that $\forall A \in \mathbf{J}_0, \phi_A = \psi_A \circ f$. By looking at (134), we see this condition is equivalent to $\phi = \psi \circ \Delta^{\mathbf{J}}_{\mathbf{C}}(f)$.

$$\begin{tikzcd}
& Y \arrow[rr, "\mathrm{id}_Y"] \arrow[dl, "f"'] \arrow[dd, "\phi_A"' near end] & & Y \arrow[dl, "f"'] \arrow[dd, "\phi_B"] \\
X \arrow[rr, "\mathrm{id}_X" near end, crossing over] \arrow[dr, "\psi_A"'] & & X \arrow[dr, "\psi_B"] & \\
& FA \arrow[rr, "F(a)"'] & & FB
\end{tikzcd} \tag{134}$$

($\Rightarrow$) Let $\{\psi_A : L \to FA\}_{A\in\mathbf{J}_0}$ be the terminal cone over $F$ and see it as a natural transformation $\psi : \Delta^{\mathbf{J}}_{\mathbf{C}}(L) \Rightarrow F$. We need to define a natural isomorphism $\mathrm{Nat}(\Delta^{\mathbf{J}}_{\mathbf{C}}(-),F) \cong \mathrm{Hom}_{\mathbf{C}}(-,L)$. Similarly to the proofs of the previous section, we will see that we only need to see where $\mathrm{id}_L$ is sent to and the rest of the natural transformation will *construct itself*. Our only choice for the cone corresponding to $\mathrm{id}_L$ is $\psi$ (it is the only cone we know exists).

$$\begin{tikzcd}
\mathrm{Nat}(\Delta^{\mathbf{J}}_{\mathbf{C}}(L),F) \arrow[r, leftrightarrow] \arrow[d, "-\circ\Delta^{\mathbf{J}}_{\mathbf{C}}(f)"'] & \mathrm{Hom}_{\mathbf{C}}(L,L) \arrow[d, "-\circ f"] \\
\mathrm{Nat}(\Delta^{\mathbf{J}}_{\mathbf{C}}(X),F) \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathbf{C}}(X,L)
\end{tikzcd} \tag{135}$$

Indeed, for any $f : X \to L$ the naturality square in (135) means the cone corresponding to $f : X \to L$ is $\{\psi_A \circ f : X \to FA\}_{A \in \mathbf{J}_0}$ by starting with $\mathrm{id}_L$ in the top right. Now, since $\psi$ is the terminal cone, for any cone $\{\phi_A : X \to FA\}_{A\in\mathbf{J}_0}$, there is a unique morphism of cones $f : X \to L$ which satisfies $\forall A \in \mathbf{J}_0, \psi_A \circ f = \phi_A$. We conclude that $f \mapsto \psi \circ \Delta^{\mathbf{J}}_{\mathbf{C}}(f)$ is a natural isomorphism.

($\Leftarrow$) Let $\psi : \Delta^{\mathbf{J}}_{\mathbf{C}}(L) \Rightarrow F$ be the cone corresponding to $\mathrm{id}_L \in \mathrm{Hom}_{\mathbf{C}}(L, L)$ under the natural isomorphism, we will show it is terminal. By the commutativity of (135) and bijectivity of the horizontal arrows, for any cone $\phi : \Delta^{\mathbf{J}}_{\mathbf{C}}(X) \Rightarrow F$, there is a unique morphism $f : X \to L$ such that $\phi = \psi \circ \Delta^{\mathbf{J}}_{\mathbf{C}}(f)$. By the first paragraph of the proof, this is the unique morphism of cones showing $\psi$ is terminal. □

**Corollary 297** (Dual)**.** *Let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram. The colimit of $F$ exists if and only if there is an object $L \in \mathbf{C}_0$ such that $\mathrm{Nat}(F, \Delta^{\mathbf{J}}_{\mathbf{C}}(-)) \cong \mathrm{Hom}_{\mathbf{C}}(L, -)$. The tip of the colimit cone is $L$.*

**Proposition 298.** *Let $U : \mathbf{Mon} \rightsquigarrow \mathbf{Set}$ be the forgetful functor, $A$ be a set and $A^*$ be the free monoid on $A$, we have $\mathrm{Hom}_{\mathbf{Set}}(A, U-) \cong \mathrm{Hom}_{\mathbf{Mon}}(A^*, -)$.*

*Proof.* We have already shown before Definition 228 that sending $h : A \to M$ to $h^* : A^* \to M$ is a bijection.[275] Now, we need to show it is natural in $M$. For any monoid homomorphism $f : M \to N$, (136) commutes (we omitted applications of $U$) because starting with $h : A \to M$, we have $(f \circ h)^* = f \circ h^*$.[276]

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{Set}}(A, M) \arrow[r, "\sim"] \arrow[d, "f\circ-"'] & \mathrm{Hom}_{\mathbf{Mon}}(A^*, M) \arrow[d, "f\circ-"] \\ \mathrm{Hom}_{\mathbf{Set}}(A, N) \arrow[r, "\sim"'] & \mathrm{Hom}_{\mathbf{Mon}}(A^*, N) \end{tikzcd} \tag{136}$$

□

[275] In the other direction, $h : A^* \to M$ is sent to $U(h) \circ i$ where $i : A \hookrightarrow A^*$ is the inclusion.

[276] To check this, let $w = a_1 \cdots a_n \in A^*$, we have
$$\begin{aligned}(f \circ h)^*(w) &= fh(a_1) \cdots fh(a_n)\\ &= f(h(a_1) \cdots h(a_n))\\ &= f(h(w)).\end{aligned}$$

In the next Proposition, we will generalize this result to see how any universal morphism corresponds to some kind of representability and we will even give a converse direction. The generalizations of the proof is straightforward, so we suggest you try to get familiar with a specific case in the next exercise.

See solution.

**Exercise 299.** Let $\mathbf{C}$ be a category and $X \in \mathbf{C}_0$ be such that $- \times X$ is a functor. An object $A \in \mathbf{C}_0$ has an exponential $A^X \in \mathbf{C}_0$ if and only if $\mathrm{Hom}_{\mathbf{C}}(- \times X, A) \cong \mathrm{Hom}_{\mathbf{C}}(-, A^X)$.

**Proposition 300.** *Let $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ be a functor and $X \in \mathbf{D}_0$. There is a universal morphism from $X$ to $F$ if and only if there exists $A \in \mathbf{C}_0$ such that $\mathrm{Hom}_{\mathbf{D}}(X, F-) \cong \mathrm{Hom}_{\mathbf{C}}(A, -)$.*

*Proof.* ($\Rightarrow$) Let $a : X \to FA$ be a universal morphism, by definition, for any $b : X \to FB$, there is a unique morphism $\phi_B(b) : A \to B$ such that $F(\phi_B(b)) \circ a = b$. In the other direction, $\phi_B^{-1}$ sending $f : A \to B$ to $Ff \circ a$ is the inverse of $\phi_B$.[277] Let us now check that $\phi_B$ is natural. For any $m : B \to B'$, (137) commutes because when starting with $f : A \to B$ in the top right, the top path sends it to $Ff \circ a$ then to $Fm \circ Ff \circ a$

[277] We check they are inverses:
$$\phi_B^{-1}(\phi_B(b)) = F(\phi_B(b)) \circ a = b$$
$$\phi_B(\phi_B^{-1}(f)) = \phi_B(Ff \circ a) = f.$$

and the bottom path sends it to $m \circ f$ then to $F(m \circ f) \circ a$.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{C}}(X, FB) \arrow[d, "Fm\circ-"'] & \mathrm{Hom}_{\mathbf{D}}(A, B) \arrow[l, "\sim"'] \arrow[d, "m\circ-"] \\ \mathrm{Hom}_{\mathbf{C}}(X, FB') & \mathrm{Hom}_{\mathbf{D}}(A, B') \arrow[l, "\sim"] \end{tikzcd} \tag{137}$$

($\Leftarrow$) Let $a : X \to FA$ be the image of $\mathrm{id}_A : A \to A$ under the isomorphism $\mathrm{Hom}_{\mathbf{C}}(X, FA) \cong \mathrm{Hom}_{\mathbf{D}}(A, A)$, we claim that $a$ is a universal morphism from $X$ to $F$. Given $b : X \to FB$, let $\phi_B(b)$ be its image under the isomorphism $\mathrm{Hom}_{\mathbf{C}}(X, FB) \cong \mathrm{Hom}_{\mathbf{D}}(A, B)$, it satisfies $F(\phi_B(b)) \circ a = b$ because (138) commutes (start with $\mathrm{id}_A$ in the top right corner). The morphism $\phi_B(b)$ is unique with this property because any other $f : A \to B$ is the image of some $b' \neq b$ under $\phi_B$ yielding $Ff \circ a = b' \neq b$. $\square$

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{C}}(X, FA) \arrow[d, "F(\phi_B(b))\circ-"'] & \mathrm{Hom}_{\mathbf{D}}(A, A) \arrow[l, "\sim"'] \arrow[d, "\phi_B(b)\circ-"] \\ \mathrm{Hom}_{\mathbf{C}}(X, FB) & \mathrm{Hom}_{\mathbf{D}}(A, B) \arrow[l, "\sim"] \end{tikzcd} \tag{138}$$

**Corollary 301** (Dual). *Let $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ be a functor and $X \in \mathbf{D}_0$. There is a universal morphism from $F$ to $X$ if and only if there exists $A \in \mathbf{C}_0$ such that $\mathrm{Hom}_{\mathbf{D}}(F-, X) \cong \mathrm{Hom}_{\mathbf{C}}(-, A)$.*

Comparing Propositions 296 and 301 and their duals, we infer that (co)limits satisfy universal properties.

**Theorem 302.** *Let $F \in [\mathbf{J}, \mathbf{C}]_0$ be a diagram.*

- *The limit of $F$ exists if and only if there is a universal morphism from $\Delta^{\mathbf{J}}_{\mathbf{C}}$ to $F$.*
- *The colimit of $F$ exists if and only if there is a universal morphism from $F$ to $\Delta^{\mathbf{J}}_{\mathbf{C}}$.*

In the next chapter, we will lift these correspondence to a more global version. Namely, we will see how to assemble the universal morphisms for all diagrams of shape $\mathbf{J}$ into a powerful object.

# Adjunctions

We start with a universal morphism $\eta_X : X \to RLX$ for all $X \in \mathbf{C}_0$ and develop a lot of things. First, we show that $L$ is functorial. For any $f : X \to Y$, the universality of $\eta_X$ yields a unique morphism $Lf : LX \to LY$ satisfying $RLf \circ \eta_X = \eta_Y \circ f$ as summarized in (139).

$$\begin{tikzcd}[row sep=small] & \textbf{in } \mathbf{C} & & & \textbf{in } \mathbf{D} \\ X \arrow[r, "\eta_X"] \arrow[d, "f"'] \arrow[dr, dashed, "\eta_Y \circ f" description] & RLX \arrow[d, dashed, "RLf"] & & & LX \arrow[d, dashed, "Lf"] \\ Y \arrow[r, "\eta_Y"'] & RLY & & & LY \arrow[lll, phantom, ""] \end{tikzcd} \qquad \xleftarrow{\;R\;} \tag{139}$$

The functoriality follows from the following equalities showing that $L(\mathrm{id}_X) = \mathrm{id}_{LX}$ and $L(g \circ f) = Lg \circ Lf$ because these morphisms make the relevant diagrams commute:

$$R(\mathrm{id}_{LX}) \circ \eta_X = \mathrm{id}_{RLX} \circ \eta_X = \eta_X = \eta_X \circ \mathrm{id}_X$$
$$R(Lg \circ Lf) \circ \eta_X = RLg \circ RLf \circ \eta_X = RLg \circ \eta_Y \circ f = \eta_Z \circ (g \circ f).$$

Note that the definition of $L$ on morphisms gives us that $\eta$ is a natural transformation $\mathrm{id}_{\mathbf{C}} \Rightarrow RL$. Next, we will define a natural transformation $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathbf{D}}$. For $X \in \mathbf{D}_0$, we let $\varepsilon_X$ be the unique morphism given by the universality of $\eta_{RX}$ such that $R(\varepsilon_X) \circ \eta_{RX} = \mathrm{id}_{RX}$ (see (140)).

$$\begin{tikzcd}[row sep=small] & \textbf{in } \mathbf{C} & & \textbf{in } \mathbf{D} \\ RX \arrow[r, "\eta_{RX}"] \arrow[dr, "\mathrm{id}_{RX}"'] & RLRX \arrow[d, dashed, "R\varepsilon_X"] & & LRX \arrow[d, dashed, "\varepsilon_X"] \\ & RX & & X \end{tikzcd} \qquad \xleftarrow{\;R\;} \tag{140}$$

Let us show that $\varepsilon_X : LRX \to X$ is a universal morphism from $L$ to $X$. For any $f : LA \to X$, if $g : A \to RX \in \mathbf{C}_1$ is such that $f = \varepsilon_X \circ Lg$, then applying $R$ and pre-composing with $\eta_A$, we obtain

$$\begin{aligned} Rf \circ \eta_A &= R\varepsilon_X \circ RLg \circ \eta_A \\ &= R\varepsilon_X \circ \eta_{RX} \circ g && \mathsf{NAT}(\eta, A, RX, g) \\ &= \mathrm{id}_{RX} \circ g && \text{definition of } \varepsilon_X \\ &= g. \end{aligned}$$

We conclude that $g := Rf \circ \eta_A$ is the unique morphism such that $f = \varepsilon_X \circ Lg$, hence $\varepsilon_X$ is universal. Next, we show that $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathbf{D}}$ is natural. For any $f : X \to Y \in \mathbf{D}_1$, by universality, there is a unique morphism $g : RX \to RY$ such that $f \circ \varepsilon_X = \varepsilon_Y \circ Lg$ (see (141)) and by our derivation above, $g = Rf \circ R\varepsilon_X \circ \eta_{RX} = Rf$. Thus, we find that $f \circ \varepsilon_X = \varepsilon_Y \circ LRf$, namely $\varepsilon$ is natural.

$$\begin{tikzcd}[row sep=small] & \textbf{in } \mathbf{D} & & \textbf{in } \mathbf{C} \\ Y & LRY \arrow[l, "\varepsilon_Y"'] & & RY \\ X \arrow[u, "f"] & LRX \arrow[l, "\varepsilon_X"] \arrow[u, dashed, "Lg"'] \arrow[ul, dashed, "\varepsilon_X \circ f" description] & & RX \arrow[u, dashed, "g"'] \end{tikzcd} \qquad \xleftarrow{\;L\;} \tag{141}$$

Second to last thing, we show that $\eta$ and $\varepsilon$ satisfy the the **triangle identities** shown in (142) and (143) (they are commutative in $[\mathbf{C}, \mathbf{D}]$ and $[\mathbf{D}, \mathbf{C}]$ respectively).

$$\begin{tikzcd} L \arrow[r, "L\eta"] \arrow[dr, "\mathbb{1}_L"'] & LRL \arrow[d, "\varepsilon L"] \\ & L \end{tikzcd} \tag{142}$$

$$\begin{tikzcd} RLR \arrow[d, "R\varepsilon"'] & R \arrow[l, "\eta R"'] \arrow[dl, "\mathbb{1}_R"] \\ R & \end{tikzcd} \tag{143}$$

The second one holds by definition of $\varepsilon_X$ (for any $X \in \mathbf{D}_0$, $R\varepsilon_X \circ \eta_{RX} = \mathrm{id}_{RX}$). For the first one, by universality there is a unique morphism $g : X \to RLX$ such that $\mathrm{id}_{LX} = \varepsilon_{LX} \circ Lg$ (see (144)) and by our derivation above, $g = R(\mathrm{id}_{LX}) \circ \eta_X = \eta_X$. We find that $\varepsilon_{LX} \circ L\eta_X = \mathrm{id}_{LX}$ as desired.

$$\begin{tikzcd} \text{in } \mathbf{D} & & & \text{in } \mathbf{C} \\ LX & LRLX \arrow[l, "\varepsilon_{LX}"'] & & RLX \\ & LX \arrow[ul, "\mathrm{id}_{LX}"] \arrow[u, dashed, "Lg"'] & & X \arrow[u, dashed, "g"'] \arrow[ll, "L"'] \end{tikzcd} \tag{144}$$

Finally, we now show that there is a natural isomorphisms

$$\Phi : \mathrm{Hom}_{\mathbf{C}}(-, R-) \cong \mathrm{Hom}_{\mathbf{D}}(L-, -) : \Phi^{-1}.$$

For $g : X \to RY$, we define $\Phi_{X,Y}(g) = \varepsilon_Y \circ Lg$ and for $f : LX \to Y$, we define $\Phi^{-1}_{X,Y}(f) = Rf \circ \eta_X$.[278] The derivations below show these are inverses:

[278] Note because it will be useful that $\Phi_{X,Y}(\mathrm{id}_{RX}) = \varepsilon_X$ and $\Phi^{-1}_{X,Y}(\mathrm{id}_{LX}) = \eta_X$.

$$\Phi^{-1}_{X,Y}(\Phi_{X,Y}(g)) = R\varepsilon_Y \circ RLg \circ \eta_X = R\varepsilon_Y \circ \eta_{RY} \circ g = g$$
$$\Phi_{X,Y}(\Phi^{-1}_{X,Y}(f)) = \varepsilon_Y \circ LRf \circ L\eta_X = f \circ \varepsilon_{LX} \circ L\eta_X = f.$$

To show that $\Phi$ is natural, we need to show that (145) commutes for any $x : X' \to X$ and $y : Y \to Y'$. Starting with $g : X \to RY$ in the top left, the bottom path sends it to $Ry \circ g \circ x$ then to $\varepsilon_{Y'} \circ LRy \circ Lg \circ Lx$ and the top path sends $g$ to $\varepsilon_Y \circ Lg$ then to $y \circ \varepsilon_Y \circ Lg \circ Lx$. The end results are equal by $\mathsf{NAT}(\varepsilon, Y, Y', y)$.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{C}}(X, RY) \arrow[r, leftrightarrow, "\Phi_{X,Y}"] \arrow[d, "Ry\circ-\circ x"'] & \mathrm{Hom}_{\mathbf{D}}(LX, Y) \arrow[d, "y\circ-\circ Lx"] \\ \mathrm{Hom}_{\mathbf{C}}(X', RY') \arrow[r, leftrightarrow, "\Phi_{X',Y'}"'] & \mathrm{Hom}_{\mathbf{D}}(LX', Y') \end{tikzcd} \tag{145}$$

**Definition 303** (Adjunction)**.** An **adjunction** between a functor $L : \mathbf{C} \rightsquigarrow \mathbf{D}$ and $R : \mathbf{D} \rightsquigarrow \mathbf{C}$ is the following data:[279]

[279] While this data is always part of an adjunction, we will prove in the next theorem that it is not necessary to specify all this data to obtain an adjunction. Moreover, this definition is not exhaustive in the sense that there is more things that you could construct and more properties you can derive from an adjunction. Still, we have to limit ourselves to a finite list and we mentioned the parts of an adjunction that are most commonly used. One notable omission is that of adjunctions as Kan extensions.

- A natural transformation $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow RL$ called the **unit** such that $\eta_X$ is initial in $X \downarrow R$ for each $X \in \mathbf{C}_0$.
- A natural transformation $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathbf{D}}$ called the **counit** such that $\varepsilon_X$ is terminal in $L \downarrow X$ for each $X \in \mathbf{D}_0$.
- The unit $\eta$ and counit $\varepsilon$ satisfy the triangle identities.
- A natural isomorphism $\Phi : \mathrm{Hom}_{\mathbf{C}}(-, R-) \cong \mathrm{Hom}_{\mathbf{D}}(L-, -) : \Phi^{-1}$ such that $\Phi_{RX,X}(\mathrm{id}_{RX}) = \varepsilon_X$ and $\Phi^{-1}_{X,LX}(\mathrm{id}_{LX}) = \eta_X$.

We denote $\mathbf{C} : L \dashv R : \mathbf{D}$ when there is an adjunction between $L : \mathbf{C} \rightsquigarrow \mathbf{D}$ and $R : \mathbf{D} \rightsquigarrow \mathbf{C}$ and we call $L$ the **left adjoint** and $R$ the **right adjoint**.[280]

[280] When they are clear from the context or irrelevant, we omit the categories from the notation and write $L \dashv R$.

**Example 304** (Boring)**.** The identity functor on any category is self-adjoint: $\mathrm{id}_{\mathbf{C}} \dashv \mathrm{id}_{\mathbf{C}}$. Both the unit and counit are $\mathbb{1}_{\mathrm{id}_{\mathbf{C}}}$.[281]

[281] You can prove this easily but it also follows from Proposition 313 and the fact that $\mathrm{id}_{\mathbf{C}}$ is its own inverse.
See solution.

**Exercise 305.** Show that if $\mathbf{C} : L \dashv R : \mathbf{D}$ is an adjunction and $R \cong R'$, then $L \dashv R'$. State the dual statement and prove it.

Giving all this data in order to define an adjunction is cumbersome and turns out not to be necessary.

**Theorem 306.** *Two functors $L : \mathbf{C} \rightsquigarrow \mathbf{D}$ and $R : \mathbf{D} \rightsquigarrow \mathbf{C}$ are adjoints if at least one of the following holds.*

*i. There is a natural transformation $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow RL$ such that $\eta_X$ is initial in $X \downarrow R$ for each $X \in \mathbf{C}_0$.*

ii. *There is a natural transformation $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathbf{D}}$ such that $\varepsilon_X$ is terminal in $L \downarrow X$ for each $X \in \mathbf{D}_0$.*

iii. *There are two natural transformations $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow RL$ and $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathbf{D}}$ that satisfy the triangle identities.*[282]

iv. *There is a natural isomorphism $\Phi : \mathrm{Hom}_{\mathbf{C}}(-, R-) \cong \mathrm{Hom}_{\mathbf{D}}(L-, -) : \Phi^{-1}$.*

[282] They satisfy

$$\varepsilon L \cdot L\eta = \mathbb{1}_L \qquad R\varepsilon \cdot \eta R = \mathbb{1}_R.$$

*Proof.* We have already shown that (i) gives rise to an adjunction at the start of the chapter.

For (ii), we can use duality. Indeed, taking the dual of Definition 303, we see that $L \dashv R$ if and only if $R^{\mathrm{op}} \dashv L^{\mathrm{op}}$ and $\eta$ and $\varepsilon$ swap their roles as unit and counit. Hence, from $\varepsilon$, we can derive an adjunction $R^{\mathrm{op}} \dashv L^{\mathrm{op}}$ as we did at the start of the chapter and duality yields $L \dashv R$.

For (iii), it is enough to show $\eta_X$ is initial in $X \downarrow R$ and use (i).[283]Recall from our construction of $\Phi$ and $\Phi^{-1}$ above that for any $g : X \to RY \in \mathbf{C}_1$, there is a unique morphism $\Phi_{X,Y}(g) = \varepsilon_Y \circ Lg$ such that $R(\Phi_{X,Y}(g)) \circ \eta_X = \Phi^{-1}_{X,Y}(\Phi_{X,Y}(g)) = g$. Thus, $\eta_X$ is a universal morphism as required.

[283] As before note that the triangle identities ensure that the adjunction constructed from (i) will have $\varepsilon$ as a counit.

For (iv), we will construct a unit satisfying (i). Fix $X \in \mathbf{C}_0$, we have a natural isomorphism $\Phi_{X,-} : \mathrm{Hom}_{\mathbf{C}}(X, R-) \cong \mathrm{Hom}_{\mathbf{D}}(LX, -)$. By Proposition 300, there is a universal morphism $\eta_X : X \to RLX$ from $X$ to $R$.[284] This yields a natural transformation $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow RL$ because for any $f : X \to Y$, the commutativity of (146) implies (by starting with $\mathrm{id}_{LX}$ and $\mathrm{id}_{LY}$ in the top left and top right corners respectively) $RLf \circ \eta_X = \Phi^{-1}_{X,LY}(Lf) = \eta_Y \circ f$.

[284] From the proof of Proposition 300, we recover $\eta_X = \Phi^{-1}_{X,LX}(\mathrm{id}_{LX})$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(LX, LX) \arrow[r, "Lf\circ -"] \arrow[d, leftrightarrow, "\Phi_{X,LX}"'] & \mathrm{Hom}_{\mathbf{D}}(LX, LY) \arrow[d, leftrightarrow, "\Phi_{X,LY}"'] & \mathrm{Hom}_{\mathbf{D}}(LY, LY) \arrow[l, "-\circ Lf"'] \arrow[d, leftrightarrow, "\Phi_{Y,LY}"] \\
\mathrm{Hom}_{\mathbf{C}}(X, RLX) \arrow[r, "RLf\circ -"'] & \mathrm{Hom}_{\mathbf{C}}(X, RLY) & \mathrm{Hom}_{\mathbf{C}}(Y, RLY) \arrow[l, "-\circ f"]
\end{tikzcd} \tag{146}$$

□

Each points of Theorem 306 can be seen as a definition of adjunctions.[285] We would like to spend a bit more time on point (iv) which is, in our opinion, the hardest definition to internalize and yet the easiest one to use in concrete contexts. The definition of an adjunction according to (iv) can be stated as follows.

[285] In fact, that is how most textbooks present it.

Two functors $L : \mathbf{C} \rightsquigarrow \mathbf{D}$ and $R : \mathbf{D} \rightsquigarrow \mathbf{C}$ are adjoint if there is a natural isomorphism[286]

$$\mathrm{Hom}_{\mathbf{C}}(-, R-) \cong \mathrm{Hom}_{\mathbf{D}}(L-, -).$$

[286] We use Remark 109 to define

$$\mathrm{Hom}_{\mathbf{C}}(-, R-) := \mathrm{Hom}_{\mathbf{C}}(-,-) \circ (\mathrm{id}_{\mathbf{C}^{\mathrm{op}}} \times R)$$
$$\mathrm{Hom}_{\mathbf{D}}(L-, -) := \mathrm{Hom}_{\mathbf{D}}(-,-) \circ (L^{\mathrm{op}} \times \mathrm{id}_{\mathbf{D}})$$

Less concisely, for any $X \in \mathbf{C}_0$ and $Y \in \mathbf{D}_0$, there is an isomorphism $\Phi_{X,Y}$ : $\mathrm{Hom}_{\mathbf{C}}(X, RY) \cong \mathrm{Hom}_{\mathbf{D}}(LX, Y)$ such that for any $f : X \to X' \in \mathbf{C}_1$ and $g : Y \to Y' \in \mathbf{D}_1$, (147) commutes. We split the naturality in two squares because we will often use one square on its own[287] as we did on both sides of (146).

[287] This is possible by Exercise 250.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}}(X', RY) \arrow[r, "-\circ f"] \arrow[d, leftrightarrow, "\Phi_{X',Y}"'] & \mathrm{Hom}_{\mathbf{C}}(X, RY) \arrow[r, "Rg\circ-"] \arrow[d, leftrightarrow, "\Phi_{X,Y}"'] & \mathrm{Hom}_{\mathbf{C}}(X, RY') \arrow[d, leftrightarrow, "\Phi_{X,Y'}"] \\
\mathrm{Hom}_{\mathbf{D}}(LX', Y) \arrow[r, "-\circ Lf"'] & \mathrm{Hom}_{\mathbf{D}}(LX, Y) \arrow[r, "g\circ-"'] & \mathrm{Hom}_{\mathbf{D}}(LX, Y')
\end{tikzcd} \tag{147}$$

Our main point in the introduction to this chapter was that grouping universal morphisms together as we did into an adjunction yields a notion of *global* universal construction. In particular, we can characterize when a category has all (co)limits of shape $\mathbf{J}$.

**Theorem 307.** *A category* $\mathbf{C}$ *has all limits of shape* $\mathbf{J}$ *if (and only if)*[288] *the functor* $\Delta^{\mathbf{J}}_{\mathbf{C}}$ *has a right adjoint.*

[288]

*Proof.* ($\Rightarrow$) For each diagram $F : \mathbf{J} \rightsquigarrow \mathbf{C}$, we pick (with the axiom of choice) a limit $\lim_{\mathbf{J}} F$ given by completeness and a universal morphism $\Delta^{\mathbf{J}}_{\mathbf{C}} \to F$ given by Theorem 302. By our argument at the start of the chapter, we get an adjunction $\Delta^{\mathbf{J}}_{\mathbf{C}} \dashv \lim_{\mathbf{J}}$.

($\Leftarrow$) Suppose $\mathbf{C} : \Delta^{\mathbf{J}}_{\mathbf{C}} \dashv L : [\mathbf{J}, \mathbf{C}]$ with unit $\eta$ and let $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ be a diagram. By definiton, $\eta_F : \Delta^{\mathbf{J}}_{\mathbf{C}} L(F) \to F$ is a universal morphism from $\Delta^{\mathbf{J}}_{\mathbf{C}}$ to $F$. Thus, by Theorem 302, $L(F)$ is the limit of $F$. $\square$

**Corollary 308** (Dual)**.** *A category* $\mathbf{C}$ *has all colimits of shape* $\mathbf{J}$ *if and only if the functor* $\Delta^{\mathbf{J}}_{\mathbf{C}}$ *has a left adjoint.*

In the rest of this chapter, we will see many examples of adjunctions and results about adjoint functors and try to have a balance between the different definitions we use.[289] We start with a long list of examples.

[289] We try to care about which definition is easiest to use but it is not always possible.

**Examples 309** (Old stuff)**.** Let us revisit some of the universal morphisms from Example 246 and see what adjunction may arise from them.

1. For every set $A$, there is a free monoid $A^*$ and an inclusion $A \hookrightarrow A^*$ that is a universal morphism from $A \to U(A^*)$, where $U : \mathbf{Mon} \rightsquigarrow \mathbf{Set}$ is the forgetful functor. Thus, $U$ has a left adjoint $(-)^* : \mathbf{Set} \rightsquigarrow \mathbf{Mon}$.[290]

[290] It sends $A$ to $A^*$ and $f : A \to B$ to the unique homomorphism $f^* : A^* \rightsquigarrow B^*$ satisfying $f^*(a) = f(a)$ for all $a \in A$.

2. Fixing a field $k$, every set $S$ is the basis of the vector space $k[S]$, so the forgetful functor $\mathbf{Vect}_k \rightsquigarrow \mathbf{Set}$ has a left adjoint $k[-] : \mathbf{Set} \rightsquigarrow \mathbf{Vect}_k$.

3. Fix $X \in \mathbf{C}_0$ such that $- \times X$ is a functor. If for every $A$, the exponential object $A^X$ exists, then $- \times X$ has a right adjoint $-^X : \mathbf{C} \rightsquigarrow \mathbf{C}$.

**Example 310.** Recall from Exercise 172 the maybe functor $- + \mathbf{1}$. Denote $\mathbf{1} = \{*\}$ for the terminal object of $\mathbf{Set}$. We consider a very similar functor $- + \mathbf{1} : \mathbf{Set} \rightsquigarrow \mathbf{Set}_*$ sending a set $X$ to $(X + \mathbf{1}, *)$ and $f : X \to Y$ to $f + \mathrm{id}_{\mathbf{1}} : X + \mathbf{1} \to Y + \mathbf{1}$. In the other direction, we have the forgetful functor $U : \mathbf{Set}_* \rightsquigarrow \mathbf{Set}$ that forgets about the distinguished element of a pointed set. We claim that $- + \mathbf{1} \dashv U$.

First, for every set $X$, we need to define $\eta_X : X \to U((X + \mathbf{1}, *)) = X + \mathbf{1}$. The only obvious choice is to let $\eta_X$ be the inclusion of $X$ in $X + \mathbf{1}$ and one can check it makes $\eta$ into a natural transformation $\mathrm{id}_{\mathbf{Set}} \Rightarrow U(- + \mathbf{1})$.

Check $\eta$ and $\varepsilon$ are natural:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "f"'] & X + \mathbf{1} \arrow[d, "f + \mathrm{id}_{\mathbf{1}}"] \\
Y \arrow[r, "\eta_Y"'] & Y + \mathbf{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
(X, x) \arrow[r, "\varepsilon_{(X,x)}"] \arrow[d, "f"'] & (X + \mathbf{1}, *) \arrow[d, "f + \mathrm{id}_{\mathbf{1}}"] \\
(Y, y) \arrow[r, "\varepsilon_{(Y,y)}"'] & (Y + \mathbf{1}, *)
\end{tikzcd}$$

Second, for every pointed set $(X, x)$, we need to define $\varepsilon_{(X,x)} : (X + \mathbf{1}, *) \to (X, x)$. Again, there is one clear choice, i.e.: acting like the identity on $X$ and sending $*$ to $x$, we will denote $\varepsilon_{(X,x)} = [\mathrm{id}_X, * \mapsto x]$.

Finally, after checking the triangle identities which we instantiate below,[291] we conclude that $- + \mathbf{1} \dashv U$.

$$\begin{tikzcd}
(X+\mathbf{1},*) \arrow[r, "\eta_X+\mathrm{id}_{\mathbf{1}}"] \arrow[dr, "\mathrm{id}_{X+\mathbf{1}}"'] & ((X+\mathbf{1})+\mathbf{1},\star) \arrow[d, "{[\mathrm{id}_{X+\mathbf{1}},\star\mapsto *]}"] \\
& (X+\mathbf{1},*)
\end{tikzcd} \tag{148}$$

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & X+\mathbf{1} \arrow[d, "{[\mathrm{id}_X,*\mapsto x]}"] \\
& X
\end{tikzcd} \tag{149}$$

A good exercise in categorical thinking is to generalize this example to an arbitrary category $\mathbf{C}$ with binary coproducts and a terminal object.[292]

[291] When dealing with a set $(X + \mathbf{1}) + \mathbf{1}$, we will denote $*$ for the element of the inner $\mathbf{1}$ and $\star$ for the outer one.
In (149), $X = U(X, x)$.

[292] See ... for a solution.

**Example 311** (**Top**). Let $U : \mathbf{Top} \rightsquigarrow \mathbf{Set}$ be the forgetful functor sending a topological space to its underlying set. We will find a left and a right adjoint to $U$.

**Left adjoint:** Fix a topological space $(X, \tau)$ and a set $Y$. We need to find a topological space $(LY, \lambda)$ so that continuous functions $(LY, \lambda) \to (X, \tau)$ are in correspondence with functions $Y \to X$. It turns out there is a trivial topology that we can put on $Y$ that makes any function $f : Y \to X$ continuous, it is called the **discrete** topology and contains all the subsets of $Y$.[293] We can check that any function $f : Y \to X$ is continuous relative to the discrete topology because for any open set $U \in \tau$, $f^{-1}(U)$ is a subset of $Y$ and hence it is open in $(Y, \mathcal{P}(Y))$. After checking that sending $Y$ to $(Y, \mathcal{P}(Y))$ and $f : Y \to Y'$ to $f : (Y, \mathcal{P}(Y)) \to (Y', \mathcal{P}(Y'))$ is a functor, we denote it disc, we find can conclude that disc $\dashv U$.

[293] It is clear that the set of all subsets of $Y$ is a topology because any union or intersection of subsets is still a subset.

**Right adjoint:** Fix a topological space $(X, \tau)$ and a set $Y$. We need to find a topological space $(LY, \lambda)$ so that continuous functions $(X, \tau) \to (LY, \lambda)$ are in correspondence with functions $X \to Y$. Again, there is a trivial topology that we can put on $Y$ that makes any function $f : X \to Y$ continuous, it is called the *codiscrete* topology and contains only the empty set and the full space $Y$.[294] We can check that any function $f : X \to Y$ is continuous relative to the codiscrete topology because the $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$ must be open by the definition of a topology. After checking that sending $Y$ to $(Y, \{\varnothing, Y\})$ and $f : Y \to Y'$ to $f : (Y, \{\varnothing, Y\}) \to (Y', \{\varnothing, Y'\})$ is a functor, we denote it codisc, we can conclude that $U \dashv$ codisc.

[294] Since $\varnothing \cap Y = \varnothing$ and $\varnothing \cup Y$, we conclude that $\{\varnothing, Y\}$ is closed under any union and intersection, hence it is a topology.

We found our first chain of adjunctions disc $\dashv U \dashv$ codisc. Another interesting one is $\mathrm{colim}_{\mathbf{J}} \dashv \Delta^{\mathbf{J}}_{\mathbf{C}} \dashv \lim_{\mathbf{J}}$ in a category $\mathbf{C}$ with all limits of shape $\mathbf{J}$. A less interesting one is $\cdots \dashv \mathrm{id}_{\mathbf{C}} \dashv \mathrm{id}_{\mathbf{C}} \dashv \mathrm{id}_{\mathbf{C}} \dashv \cdots$. Here is a chain of five adjunctions.

**Exercise 312.** Let $\mathbf{C}$ be a category and $\mathsf{id}, \mathsf{s}, \mathsf{t}$ be the functors described in Exercise 239. Show they are related by the adjunctions $\mathsf{t} \dashv \mathsf{id} \dashv \mathsf{s}$. Suppose furthermore that $\mathbf{C}$ has an initial object $\varnothing$ and a terminal object $\mathbf{1}$. Show that the constant functor at $\mathrm{id}_{\varnothing}$ is left adjoint to $\mathsf{t}$ and the constant functor at $\mathrm{id}_{\mathbf{1}}$ is right adjoint to $\mathsf{s}$.

See solution.

As a final example, we show that any equivalence gives rise to two adjunctions. In this sense[295], one can see a left (resp. right) adjoint to a functor $F$ as an approxi-

[295] And in another sense related to Kan extensions.

mation to a left (resp. right) inverse that is even coarser than a quasi-inverse.[296]

**Proposition 313.** *Let $L : \mathbf{C} \rightsquigarrow \mathbf{D}$ and $R : \mathbf{D} \rightsquigarrow \mathbf{C}$ be quasi-inverses, then $L \dashv R$ and $R \dashv L$.*

*Proof.* It is enough to show $L \dashv R$ as the definition of quasi-inverses is symmetric. □

Let us now turn to the many great properties of adjoint functors.

**Proposition 314.** *A left adjoint is unique up to natural isomorphism. Namely, if $L \dashv R$ and $L' \dashv R$, then $L \cong L'$.*

*Proof.* For any $X \in \mathbf{C}_0$, we define $\phi_X : LX \to L'X$ to be the image of $\mathrm{id}_{L'X} \in \mathrm{Hom}_{\mathbf{D}}(L'X, L'X)$ under the composition of the natural isomorphisms

$$\mathrm{Hom}_{\mathbf{D}}(L'X, L'X) \cong \mathrm{Hom}_{\mathbf{C}}(X, RL'X) \cong \mathrm{Hom}_{\mathbf{D}}(LX, L'X).$$

Then, for any $f : X \to Y$, the naturality squares in (150) imply $L'f \circ \phi_X = \phi_Y \circ Lf$.[297]

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(L'X, L'X) \arrow[r, "L'f\circ -"] \arrow[d, leftrightarrow] & \mathrm{Hom}_{\mathbf{D}}(L'X, L'Y) \arrow[d, leftrightarrow] & \mathrm{Hom}_{\mathbf{D}}(L'Y, L'Y) \arrow[l, "-\circ L'f"'] \arrow[d, leftrightarrow] \\
\mathrm{Hom}_{\mathbf{C}}(X, RL'X) \arrow[r, "RL'f\circ -"] \arrow[d, leftrightarrow] & \mathrm{Hom}_{\mathbf{C}}(X, RLY) \arrow[d, leftrightarrow] & \mathrm{Hom}_{\mathbf{C}}(Y, RLY) \arrow[l, "-\circ f"'] \arrow[d, leftrightarrow] \\
\mathrm{Hom}_{\mathbf{D}}(LX, L'X) \arrow[r, "L'f\circ -"'] & \mathrm{Hom}_{\mathbf{D}}(LX, L'Y) & \mathrm{Hom}_{\mathbf{D}}(LY, L'Y) \arrow[l, "-\circ Lf"]
\end{tikzcd} \tag{150}$$

We conclude that $\phi : L \Rightarrow L'$ is natural. With a symmetric argument, we construct $\phi^{-1} : L' \Rightarrow L$[298] and we check that they are inverses with (151) and (152).

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(LX, LX) \arrow[r, "\phi_X\circ -"] \arrow[d, leftrightarrow] & \mathrm{Hom}_{\mathbf{D}}(LX, L'X) \arrow[d, leftrightarrow] \\
\mathrm{Hom}_{\mathbf{D}}(L'X, LX) \arrow[r, "\phi_X\circ -"'] & \mathrm{Hom}_{\mathbf{D}}(L'X, L'X)
\end{tikzcd} \tag{151}$$

Starting with $\mathrm{id}_{LX}$ in the top left of (151) and reaching the top right, we find that the image of $\phi_X \circ \phi_X^{-1}$ under the isomorphism is $\phi_X$ which is the image of $\mathrm{id}_{L'X}$, thus $\phi_X \circ \phi_X^{-1} = \mathrm{id}_{L'X}$. We proceed with a symmetric argument for (152). □

**Corollary 315** (Dual)**.** *If $L \dashv R$ and $L \dashv R'$, then $R \cong R'$.*

**Proposition 316.** *Let $\mathbf{C} : L \dashv R : \mathbf{D}$ be adjoint functors and $X, Y \in \mathbf{D}_0$. If $X \times Y$ exists, then $R(X \times Y)$ with the projections $R(\pi_X)$ and $R(\pi_Y)$ is the product $R(X) \times R(Y)$.*[299]

*Proof.* Let $p_X : A \to RX$ and $p_Y : A \to RY$ be such that (153) commutes.

$$\begin{tikzcd}
 & A \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\
RX & R(X \times Y) \arrow[l, "R\pi_X"] \arrow[r, "R\pi_Y"'] & RY
\end{tikzcd} \tag{153}$$

[296] Furthermore, it follows from Proposition 314 (resp. Corollary 315) that the left (resp. right) adjoint of $F$ is the left (resp. right) inverse or quasi-inverse when the latter exists.

[297] Start with $\mathrm{id}_{L'X}$ and $\mathrm{id}_{L'Y}$ at the top left and top right respectively and compare the results at the bottom middle.

[298] i.e.: $\phi_X^{-1}$ is the image of $\mathrm{id}_{LX}$ under

$$\mathrm{Hom}_{\mathbf{D}}(LX, LX) \cong \mathrm{Hom}_{\mathbf{C}}(X, RLX) \cong \mathrm{Hom}_{\mathbf{D}}(L'X, LX$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{D}}(L'X, L'X) \arrow[r, "\phi_X^{-1}\circ -"] \arrow[d, leftrightarrow] & \mathrm{Hom}_{\mathbf{D}}(L'X, LX) \arrow[d, leftrightarrow] \\
\mathrm{Hom}_{\mathbf{D}}(LX, L'X) \arrow[r, "\phi_X^{-1}\circ -"'] & \mathrm{Hom}_{\mathbf{D}}(LX, LX)
\end{tikzcd} \tag{152}$$

[299] In other words, right adjoints preserve binary products.

We need to show there is a unique mediating morphism $A \to R(X \times Y)$. First, we will get rid of the applications of $R$ at the bottom, in order to use the universal property of the product $X \times Y$. To do this, we apply $L$ to (153) and use the counit $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathbf{D}}$ to obtain (154).

$$\begin{tikzcd} & LA \arrow[dl, "Lp_X"'] \arrow[dr, "Lp_Y"] & \\ LRX \arrow[d, "\varepsilon_X"'] & LR(X \times Y) \arrow[l, "LR\pi_X"] \arrow[r, "LR\pi_Y"'] \arrow[d, "\varepsilon_{X\times Y}"'] & LRY \arrow[d, "\varepsilon_Y"'] \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd} \tag{154}$$

$$\begin{tikzcd} & LA \arrow[dl, "Lp_X"'] \arrow[dr, "Lp_Y"] \arrow[dd, dashed, "!"'] & \\ LRX \arrow[d, "\varepsilon_X"'] & & LRY \arrow[d, "\varepsilon_Y"'] \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

The universal property of $X \times Y$ tells us there is a unique $! : LA \to X \times Y$ such that $\pi_X \circ ! = \varepsilon_X \circ Lp_X$ and $\pi_Y \circ ! = \varepsilon_Y \circ Lp_Y$. We claim that $!^{\mathrm{t}}$ is the mediating morphism of (153), i.e.: $R\pi_X \circ !^{\mathrm{t}} = p_X$ and $R\pi_Y \circ !^{\mathrm{t}} = p_Y$. Using the adjunction $L \dashv R$, we obtain the following commutative square.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D}}(LA, X \times Y) \arrow[r, leftrightarrow] \arrow[d, "\pi_X \circ -"'] & \mathrm{Hom}_{\mathbf{C}}(A, R(X \times Y)) \arrow[d, "R\pi_X \circ -"] \\ \mathrm{Hom}_{\mathbf{D}}(LA, X) \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathbf{C}}(A, RX) \end{tikzcd} \tag{155}$$

Now, starting with ! on the top left corner, we obtain the following derivation.

$$\begin{aligned} p_X &= {p_X}^{\mathrm{t}\,\mathrm{t}} \\ &= (\varepsilon_X \circ Lp_X)^{\mathrm{t}} \\ &= (\pi_X \circ !)^{\mathrm{t}} && \text{definition of } ! \\ &= R\pi_X \circ !^{\mathrm{t}} && \text{commutativity of (155)} \end{aligned}$$

Replacing $X$ with $Y$ in the previous argument shows $!^{\mathrm{t}}$ makes (156) commute. For the uniqueness, note that if $m : A \to R(X \times Y)$ can replace $!^{\mathrm{t}}$, then (157) commutes which implies by uniqueness of ! that $m^{\mathrm{t}} = \varepsilon_{X \times Y} \circ Lm = !$. Transposing yields $!^{\mathrm{t}} = m$.

$$\begin{tikzcd} & A \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] \arrow[d, "!^{\mathrm{t}}"'] & \\ RX & R(X \times Y) \arrow[l, "R\pi_X"] \arrow[r, "R\pi_Y"'] & RY \end{tikzcd} \tag{156}$$

$$\begin{tikzcd} & LA \arrow[dl, "Lp_X"'] \arrow[dr, "Lp_Y"] \arrow[d, "Lm"'] & \\ LRX \arrow[d, "\varepsilon_X"'] & LR(X \times Y) \arrow[l, "LR\pi_X"] \arrow[r, "LR\pi_Y"'] \arrow[d, "\varepsilon_{X\times Y}"'] & LRY \arrow[d, "\varepsilon_Y"'] \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd} \tag{157}$$

$\square$

**Corollary 317** (Dual)**.** *Let* $\mathbf{C} : L \dashv R : \mathbf{D}$ *be adjoint functors and* $A, B \in \mathbf{C}_0$*. If* $A + B$ *exists, then* $L(A + B)$ *with the coprojections* $L\kappa_A$ *and* $L\kappa_B$ *is the coproduct* $LA \times LB$*.*[300]

**Proposition 318.** *Let* $\mathbf{C} : L \dashv R : \mathbf{D}$ *be adjoint functors. If* $g : X \to Y \in \mathbf{D}_1$ *is monic, then* $R(g)$ *is monic.*[301]

[300] In other words, left adjoints preserve binary coproducts.

[301] In other words, right adjoints preserve monomorphisms.

*Proof.* Let $h_1, h_2 : Z \to R(X)$ be such that $R(g) \circ h_1 = R(g) \circ h_2$, we need to show that $h_1 = h_2$. Since $L \dashv R$, we have the following commutative square.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{C}}(Z, RX) \arrow[r, leftrightarrow] \arrow[d, "Rg\circ-"'] & \mathrm{Hom}_{\mathbf{D}}(LZ, X) \arrow[d, "g\circ-"] \\ \mathrm{Hom}_{\mathbf{C}}(Z, RY) \arrow[r, leftrightarrow] & \mathrm{Hom}_{\mathbf{D}}(LZ, Y) \end{tikzcd} \tag{158}$$

Starting with $h_1$ and $h_2$ in the top left corner, we find that[302]

$$g \circ {h_1}^{\mathrm{t}} = (Rg \circ h_1)^{\mathrm{t}} = (Rg \circ h_2)^{\mathrm{t}} = g \circ {h_2}^{\mathrm{t}},$$

which, by monicity of $g$ implies ${h_1}^{\mathrm{t}} = {h_2}^{\mathrm{t}}$. This in turn means that $h_1 = h_2$ because $(-)^{\mathrm{t}}$ is a bijection. $\square$

[302] The first and last equality follow from commutativity of (158) and the middle equality is a hypothesis.

**Corollary 319** (Dual)**.** *Let $\mathbf{C} : L \dashv R : \mathbf{D}$ be adjoint functors. If $f : A \to B \in \mathbf{C}_1$ is epic, then $L(f)$ is epic.*[303]

[303] In other words, left adjoints preserve epimorphisms.

*Remark* 320. We want to put the emphasis on a crucial step in the proof above which was to derive $g \circ {h_1}^{\mathrm{t}} = (Rg \circ h_1)^{\mathrm{t}}$ from (158).[304] By varying the arguments slightly (i.e.: going around the square in another direction or considering the naturality square involving pre-composition), we cook up four similar equations that can be helpful.

[304] It was also a crucial step in the proof of Proposition 316, we used (155) to derive $(\pi_X \circ !)^{\mathrm{t}} = R\pi_X \circ !^{\mathrm{t}}$.

$$\forall g : X \to Y, f : Z \to RX, \qquad g \circ f^{\mathrm{t}} = (Rg \circ f)^{\mathrm{t}} \tag{159}$$
$$\forall g : X \to Y, f : LZ \to X, \qquad (g \circ f)^{\mathrm{t}} = Rg \circ f^{\mathrm{t}} \tag{160}$$
$$\forall g : LX \to Y, f : Z \to X, \qquad g^{\mathrm{t}} \circ f = (g \circ Lf)^{\mathrm{t}} \tag{161}$$
$$\forall g : X \to RY, f : Z \to X, \qquad (g \circ f)^{\mathrm{t}} = g^{\mathrm{t}} \circ Lf \tag{162}$$

**Theorem 321.** *Right adjoints are continuous.*

*Proof.* Let $\mathbf{C} : L \dashv R : \mathbf{D}$ be an adjunction and $F : \mathbf{J} \rightsquigarrow \mathbf{D}$ be a diagram in $\mathbf{D}$ whose limit cone is $\{\ell_X : \lim F \to FX\}_{X \in \mathbf{J}_0}$. We claim that $\{R\ell_X : R\lim F \to RFX\}_{\mathbf{J}_0}$ is the limit cone of $R \circ F$. For any other cone making (163) commute for any $f : X \to Y \in \mathbf{J}_1$, we can apply transposition to the $c_X$'s to obtain (**??**) which commutes by (159).[305]

[305] In (159), putting $g := Ff$ and $f := c_X$, we obtain

$${c_Y}^{\mathrm{t}} = (RFf \circ c_X)^{\mathrm{t}} = Ff \circ {c_X}^{\mathrm{t}}.$$

$$\begin{tikzcd} & C \arrow[ddl, bend right, "c_X"'] \arrow[ddr, bend left, "c_Y"] & \\ & R\lim F \arrow[dl, "R\ell_X"'] \arrow[dr, "R\ell_Y"] & \\ RFX \arrow[rr, "RFf"'] & & RFY \end{tikzcd} \tag{163}$$

$$\begin{tikzcd} & LC \arrow[ddl, bend right, "{c_X}^{\mathrm{t}}"'] \arrow[ddr, bend left, "{c_Y}^{\mathrm{t}}"] & \\ & \lim F \arrow[dl, "\ell_X"'] \arrow[dr, "\ell_Y"] & \\ FX \arrow[rr, "Ff"'] & & FY \end{tikzcd} \tag{164}$$

By the universal property of $\lim F$, there is a unique mediating morphism $! : LC \to \lim F$ making (165) commute. Transposing ! yields a mediating morphism making (166) commutes by (160).[306]

[306] In (160), putting $g := \ell_X$ and $f := !$, we obtain

$$c_X = ({c_X}^{\mathrm{t}})^{\mathrm{t}} = (\ell_X \circ !)^{\mathrm{t}} = R\ell_X \circ !^{\mathrm{t}}.$$

Symmetrically, we have

$$c_Y = ({c_Y}^{\mathrm{t}})^{\mathrm{t}} = (\ell_Y \circ !)^{\mathrm{t}} = R\ell_Y \circ !^{\mathrm{t}}.$$

$$
\begin{tikzcd}
& LC \arrow[d, dashed, "!"] \arrow[ddl, bend right, "{c_X}^{\mathrm{t}}"'] \arrow[ddr, bend left, "{c_Y}^{\mathrm{t}}"] & \\
& \lim F \arrow[dl, "\ell_X"'] \arrow[dr, "\ell_Y"] & \\
FX \arrow[rr, "Ff"'] & & FY
\end{tikzcd}
\tag{165}
$$

$$
\begin{tikzcd}
& C \arrow[d, dashed, "!^{\mathrm{t}}"] \arrow[ddl, bend right, "c_X"'] \arrow[ddr, bend left, "c_Y"] & \\
& R\lim F \arrow[dl, "R\ell_X"'] \arrow[dr, "R\ell_Y"] & \\
RFX \arrow[rr, "RFf"'] & & RFY
\end{tikzcd}
\tag{166}
$$

Finally, $!^{\mathrm{t}}$ is the only mediating morphism that fits in (166) because if $m : C \to R\lim F$ fits, then $m^{\mathrm{t}} : LC \to \lim F$ fits in (165)[307] and by uniqueness of $!$, $m^{\mathrm{t}} = !$ which further implies $m = !^{\mathrm{t}}$. □

[307] Suppose $R\ell_X \circ m = c_X$, then we use (159) to conclude

$$ {c_X}^{\mathrm{t}} = (R\ell_X \circ m)^{\mathrm{t}} = \ell_X \circ m^{\mathrm{t}}, $$

and similarly for $Y$.

**Corollary 322** (Dual)**.** *Left adjoints are cocontinuous.*

*Remark* 323.

**Theorem 324.** *If* $\mathbf{C} : L \dashv R : \mathbf{D}$ *and* $\mathbf{D} : L' \dashv R' : \mathbf{E}$ *are two adjunctions, then* $\mathbf{C} : L'L \dashv RR' : \mathbf{E}$ *is an adjunction.*[308]

[308] This theorem is often referred to as *adjunctions can be composed*.

*Proof.* Let $\eta$ and $\varepsilon$ be the unit and counit of the first adjunction and $\eta'$ and $\varepsilon'$ be the unit and counit of the second one. We define the following unit and counit for the composite adjunction:

$$
\begin{aligned}
\widehat{\eta} &= R\eta' L \cdot \eta : \mathrm{id}_{\mathbf{C}} \Rightarrow RR'L'L \\
\widehat{\varepsilon} &= \varepsilon' \cdot L'\varepsilon R' : L'LRR' \Rightarrow \mathrm{id}_{\mathbf{E}}.
\end{aligned}
$$

The following diagrams show the triangle identities.

$$
\begin{tikzcd}
L'L \arrow[r, "L'L\eta"] \arrow[rr, bend left, "L'L\widehat{\eta}"] \arrow[dr, "\mathbb{1}_{L'L}"'] \arrow[ddrr, bend right, "\mathbb{1}_{L'L}"'] & L'LRL \arrow[r, "L'LR\eta' L"] \arrow[d, "L'\varepsilon L"] \arrow[dl, phantom, "\text{(a)}"] \arrow[dr, phantom, "\text{(b)}"] & L'LRR'L'L \arrow[d, "L'\varepsilon R'L'L"] \arrow[dd, bend left, "\widehat{\varepsilon}L'L"] \\
& L'L \arrow[r, "L'\eta' L"] \arrow[dr, "\mathbb{1}_{L'L}"'] \arrow[r, phantom, shift right=4ex, "\text{(c)}"] & L'R'L'L \arrow[d, "\varepsilon' L'L"] \\
& & L'L
\end{tikzcd}
\tag{167}
$$

Showing (167) commutes:

(a) Apply $L'(-)$ to the left triangle identity of $\eta$ and $\varepsilon$.

(b) Apply $L'(-)L$ to $\mathsf{HOR}(\varepsilon, \eta')$.

(c) Apply $(-)L$ to the left triangle identity of $\eta'$ and $\varepsilon'$.

$$
\begin{tikzcd}
RR'L'LRR' \arrow[d, "RR'L'\varepsilon R'"] \arrow[dd, bend right, "RR'\widehat{\varepsilon}"'] & RLRR' \arrow[l, "R\eta' LRR'"'] \arrow[d, "R\varepsilon R'"] \arrow[dl, phantom, "\text{(a)}"] & RR' \arrow[l, "\eta RR'"'] \arrow[ll, bend right, "\widehat{\eta}RR'"'] \arrow[dl, "\mathbb{1}_{RR'}"] \arrow[ddll, bend left, "\mathbb{1}_{RR'}"] \arrow[l, phantom, shift right=4ex, "\text{(b)}"] \\
RR'L'R' \arrow[d, "RR'\varepsilon'"] & RR' \arrow[l, "R\eta' R'"'] \arrow[dl, "\mathbb{1}_{RR'}"] \arrow[l, phantom, shift right=4ex, "\text{(c)}"] & \\
RR' & &
\end{tikzcd}
\tag{168}
$$

Showing (168) commutes:

(a) Apply $R(-)R'$ to $\mathsf{HOR}(\eta', \varepsilon)$.

(b) Apply $(-)R'$ to the right triangle identity of $\eta$ and $\varepsilon$.

(c) Apply $R(-)$ to the right triangle identity of $\eta'$ and $\varepsilon'$.

□

**Proposition 325.** *If* $\mathbf{D} : L \dashv R : \mathbf{E}$ *is an adjunction, then there is an adjunction* $[\mathbf{C}, \mathbf{D}] : L- \dashv R- : [\mathbf{C}, \mathbf{E}]$.

*Proof.* First, we can see that $L-$ and $R-$ are functors by Exercise 267.[309] Composing them yields $RL- : [\mathbf{C}, \mathbf{D}] \rightsquigarrow [\mathbf{C}, \mathbf{D}]$ and $LR- : [\mathbf{C}, \mathbf{E}] \rightsquigarrow [\mathbf{C}, \mathbf{E}]$. Let $\eta : \mathrm{id}_{\mathbf{D}} \Rightarrow RL$ and $\varepsilon : LR \Rightarrow \mathrm{id}_{\mathbf{E}}$ be the unit and counit of $L \dashv R$. We claim that $\eta- = F \mapsto \eta F$ and $\varepsilon- = G \mapsto \varepsilon G$ are the unit and counit of an adjunction $L- \dashv R-$.

To see that $\eta-$ and $\varepsilon-$ are natural transformations of the right type, we can recognize them in the image of $\Lambda(- \circ -)$ (noting that $\mathrm{id}_{\mathbf{D}}- = \mathrm{id}_{[\mathbf{C},\mathbf{D}]}$ and $\mathrm{id}_{\mathbf{E}}- = \mathrm{id}_{[\mathbf{C},\mathbf{E}]}$):

$$\begin{aligned} \eta- &= \Lambda(- \circ -)(\eta) : \mathrm{id}_{[\mathbf{C},\mathbf{D}]} \Rightarrow RL- \\ \varepsilon- &= \Lambda(- \circ -)(\varepsilon) : LR- \Rightarrow \mathrm{id}_{[\mathbf{C},\mathbf{E}]}. \end{aligned}$$

It is left to show the triangle identities hold assuming they hold for $\eta$ and $\varepsilon$. In the following derivations, we use three simple facts:[310]

- the biaction of $F-$ and $G-$ on $\phi-$ yields $(F\phi G)-$,
- $\phi - \cdot \phi' - = (\phi \cdot \phi')-$, and
- $(\mathbb{1}_F)- = \mathbb{1}_{F-}$.

Now, the triangle identities hold by:

$$\begin{aligned} (\varepsilon-)(L-) \cdot (L-)(\eta-) &= (\varepsilon L-) \cdot (L\eta-) = (\varepsilon L \cdot L\eta)- = (\mathbb{1}_L)- = \mathbb{1}_{L-} \\ (R-)(\varepsilon-) \cdot (\eta-)(R-) &= (R\varepsilon-) \cdot (\eta R-) = (R\varepsilon \cdot \eta R)- = (\mathbb{1}_R)- = \mathbb{1}_{R-}. \end{aligned}$$

□

**Corollary 326** (Dual). *If* $\mathbf{D} : L \dashv R : \mathbf{E}$ *is an adjunction, then there is an adjunction* $[\mathbf{C}, \mathbf{D}] : -L \dashv -R : [\mathbf{C}, \mathbf{E}]$.

**Theorem 327.** *Let* $\mathbf{D}$ *be a category with all limits of shape* $\mathbf{J}$. *For any category* $\mathbf{C}$, *the functor category* $[\mathbf{C}, \mathbf{D}]$ *has all limits of shape* $\mathbf{J}$ *and the limit of any diagram* $F : \mathbf{J} \rightsquigarrow [\mathbf{C}, \mathbf{D}]$ *satisfies for any* $X \in \mathbf{C}_0$, $(\lim_{\mathbf{J}} F)(X) = \lim_{\mathbf{J}}(F(-)(X))$.[311]

*Proof.* From previous results, we have the following chain of adjunctions.

$$\begin{tikzcd} {[\mathbf{C},\mathbf{D}]} \arrow[r, shift left, "\Delta^{\mathbf{J}}_{\mathbf{D}}\circ-"] \arrow[r, phantom, "\perp"] & {[\mathbf{C},[\mathbf{J},\mathbf{D}]]} \arrow[l, shift left, "\lim_{\mathbf{J}}\circ-"] \arrow[r, shift left, "\Lambda^{-1}"] \arrow[r, phantom, "\perp"] & {[\mathbf{C}\times\mathbf{J},\mathbf{D}]} \arrow[l, shift left, "\Lambda"] \arrow[r, shift left, "-\circ\mathsf{swap}"] \arrow[r, phantom, "\perp"] & {[\mathbf{J}\times\mathbf{C},\mathbf{D}]} \arrow[l, shift left, "-\circ\mathsf{swap}^{-1}"] \arrow[r, shift left, "\Lambda"] \arrow[r, phantom, "\perp"] & {[\mathbf{J},[\mathbf{C},\mathbf{D}]]} \arrow[l, shift left, "\Lambda^{-1}"] \end{tikzcd} \qquad (169)$$

From left to right. The first adjunction is induced by Proposition 325 and the adjunction $\Delta^{\mathbf{J}}_{\mathbf{D}} \dashv \lim_{\mathbf{J}}$ given by completeness of $\mathbf{D}$. The second adjunction is obtained from Proposition 313 and the fact that $\Lambda$ and $\Lambda^{-1}$ are inverses. The third adjunction is induced by Corollary 326 and the canonical isomorphism $\mathsf{swap} : \mathbf{C} \times \mathbf{J} \rightsquigarrow \mathbf{J} \times \mathbf{C}$.[312] The fourth adjunction is similar to the second one.

[309] They are compositions:

$$\begin{aligned} L- &= (- \circ -) \circ (L \times \mathrm{id}_{[\mathbf{C},\mathbf{D}]}) \\ R- &= (- \circ -) \circ (R \times \mathrm{id}_{[\mathbf{C},\mathbf{E}]}). \end{aligned}$$

Alternatively, we can use Example 268.4 where we described currying for functors. In that setting, we have

$$\begin{aligned} L- &= \Lambda(- \circ -)(L) \\ R- &= \Lambda(- \circ -)(R). \end{aligned}$$

These functors send a natural transformation $\phi : F \Rightarrow G$ to $L\phi$ and $R\phi$ respectively.

[310] They can be shown by proving the equality at each component.

[311] In other words (that you will often hear), limits in functor categories are taken pointwise.

[312] One could also see that $- \circ \mathsf{swap}$ and $- \circ \mathsf{swap}^{-1}$ are inverses.

There is a simpler way to describe the composition of the three rightmost adjunctions. If we view a functor $F : \mathbf{C} \rightsquigarrow [\mathbf{J}, \mathbf{D}]$ as taking two arguments and write it $F(-_1)(-_2)$, the composition $\Lambda \circ (- \circ \mathsf{swap}) \circ \Lambda^{-1}$ (the top path) swaps the order of the arguments to yield the functor $F(-_2)(-_1) : \mathbf{J} \rightsquigarrow [\mathbf{C}, \mathbf{D}]$. The bottom path swaps back the arguments.

Next, we show that the composition of the top path is $\Delta^{\mathbf{J}}_{[\mathbf{C},\mathbf{D}]}$. Starting with a functor $F : \mathbf{C} \rightsquigarrow \mathbf{D}$, the first left adjoint sends it to $\Delta^{\mathbf{J}}_{\mathbf{D}} \circ F$ which sends $X \in \mathbf{C}_0$ to the constant functor at $FX$ and $f : X \to Y \in \mathbf{C}_1$ to the natural transformation whose components are all $Ff : FX \to FY$. Applying the three other left adjoints, we obtain a functor which sends any $j \in \mathbf{J}_0$ to the functor $F$ and any $m : j \to j' \in \mathbf{J}_1$ to $\mathbb{1}_F$. We conclude that the top path sends $F$ to the constant functor at $F$.

We obtain a right adjoint to $\Delta^{\mathbf{J}}_{[\mathbf{C},\mathbf{D}]}$ by composing all the adjunctions in 169 with Theorem 324 and thus $[\mathbf{C}, \mathbf{D}]$ has all limits of shape $\mathbf{J}$. To compute them, we can compose the right adjoints in 169 to find $(\lim_{\mathbf{J}} F)(X) = \lim_{\mathbf{J}}(F(-)(X))$. $\square$

**Corollary 328** (Dual)**.** *Let* $\mathbf{D}$ *be a category with all colimits of shape* $\mathbf{J}$*. For any category* $\mathbf{C}$*, the functor category* $[\mathbf{C}, \mathbf{D}]$ *has all colimits of shape* $\mathbf{J}$ *and the colimit of any diagram* $F : \mathbf{J} \rightsquigarrow [\mathbf{C}, \mathbf{D}]$ *satisfies for any* $X \in \mathbf{C}_0$, $(\operatorname{colim}_{\mathbf{J}} F)(X) = \operatorname{colim}_{\mathbf{J}}(F(-)(X))$.[313]

[313] In other words, colimits are taken pointwise. You can use Exercise 256 or draw a similar chain of adjunctions as in (169).

**Corollary 329.** *If a category* $\mathbf{D}$ *is (finitely) complete or cocomplete, then so is* $[\mathbf{C}, \mathbf{D}]$ *for any category* $\mathbf{C}$.

**Exercise 330.** Let $\mathbf{C}$ have all limits of shape $\mathbf{J}$ and $\mathbf{C} : L \dashv R : \mathbf{D}$ be an adjunction. Using Theorem 307, Corollary 315, Theorem 324 and Proposition 325, show that $R$ preserves all limits of shape $\mathbf{J}$.

See solution.

# Monads and Algebras

## POV: Category Theory

We will start from the concept of an adjunction which, as we hope was made clear in the previous chapter, is ubiquitous and powerful throughout mathematics. However, we will start with a great oversimplification; we will assume the categories concerned are posetal.

An adjunction between posets $(P, \leq)$ and $(Q, \sqsubseteq)$ is a pair of order-preserving functions $L : P \to Q$ and $R : Q \to P$ satisfying for any $p \in P$ and $q \in Q$, $L(p) \sqsubseteq q \iff p \leq R(q)$. You might recognize this as a Galois connection from Chapter , this explains the notation $L \dashv R$ we introduced back then.

Let us derive again the properties of the composite $R \circ L$ using what we know about adjoints.[314]

[314] Recall that we showed $R \circ L$ was a closure operator in Proposition 36.

It is of course a monotone function but we can derive a couple of additional properties. First, the existence of the unit $\eta : \mathrm{id}_P \Rightarrow RL$ means that for any $p \in P$, there is $\eta_p : p \to RL(p)$, so $RL$ is extensive.[315] Second, the existence of the counit $\varepsilon : RL \Rightarrow \mathrm{id}_P$ means that for any $p \in P$, there is $R(\varepsilon_{L(p)}) : RLRL(p) \to RL(p)$ and $RL(\eta_p) : RL(p) \to RLRL(p)$, so $RL$ is idempotent (i.e.: $\forall p \in P, RL(p) = RLRL(p)$). This means $RL$ is a closure operator.

[315] i.e.: $\forall p \in P, p \leq RL(p)$.

We will generalize this discussion to arbitrary categories now. Let $\mathbf{C} : L \dashv R : \mathbf{D}$ be an adjoint pair, we have two natural transformations $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow RL$ and $R\varepsilon L : RLRL \Rightarrow RL$ that interact well together due to the triangle identities. Applying $R(-)$ to (142) and $(-)L$ to (143) yields two diagrams that we combine into (170). We can add to the diagram coming from $\mathsf{HOR}(\varepsilon, \varepsilon)$ which act on by $R(-)L$ to obtain (171).

$$\begin{tikzcd} RL \arrow[r, "RL\eta"] \arrow[dr, "\mathbb{1}_{RL}"'] & RLRL \arrow[d, dashed, "R\varepsilon L"] & RL \arrow[l, "\eta RL"'] \arrow[dl, "\mathbb{1}_{RL}"] \\ & RL & \end{tikzcd} \qquad (170)$$

$$\begin{tikzcd} RLRLRL \arrow[r, "R\varepsilon LRL"] \arrow[d, "RLR\varepsilon L"'] & RLRL \arrow[d, "R\varepsilon L"] \\ RLRL \arrow[r, "R\varepsilon L"'] & RL \end{tikzcd} \qquad (171)$$

These diagrams are precisely what is required to define a monad.

**Definition 331** (Monad)**.** A **monad** is a triple comprised of an endofunctor $M : \mathbf{C} \rightsquigarrow \mathbf{C}$ and two natural transformations $\eta : \mathrm{id}_{\mathbf{C}} \Rightarrow M$ and $\mu : M^2 \Rightarrow M$ called the **unit** and **multiplication** respectively that make (172) and (173) commute in $[\mathbf{C}, \mathbf{C}]$.

$$\begin{tikzcd} M \arrow[r, "M\eta"] \arrow[dr, "\mathbb{1}_M"'] & M^2 \arrow[d, dashed, "\mu"] & M \arrow[l, "\eta M"'] \arrow[dl, "\mathbb{1}_M"] \\ & M & \end{tikzcd} \tag{172}$$

$$\begin{tikzcd} M^3 \arrow[r, "\mu M"] \arrow[d, "M\mu"'] & M^2 \arrow[d, "\mu"] \\ M^2 \arrow[r, "\mu"'] & M \end{tikzcd} \tag{173}$$

**Examples 332.** Our discussion above tells us that any adjoint pair $L \dashv R$ corresponds to a monad $(RL, \eta, R\varepsilon L)$, so all the examples of adjunctions you have seen correspond to suitable examples of monads. For instance, all closure operators are monads. Here are more examples described from adjunctions in Chapter .

1. The adjunction $\mathbf{Set} : (-)^* \dashv U : \mathbf{Mon}$ yields the free monoid monad abusively denoted $(-)^* : \mathbf{Set} \rightsquigarrow \mathbf{Set}$ sending a set $A$ to the underlying set of the free monoid on $A$. The unit sends $a \in A$ to the word $a \in A^*$ by inclusion and the multiplication sends a finite word over finite words over $A$ to the concatenation of the words.[316]

2. Similarly to the previous example, there is monad $k[-]$ on $\mathbf{Set}$ sending $A$ to the underlying set of the vector space $k[A]$.[317]

3. 

4. Both adjunctions with the forgetful functor $\mathbf{Top} \rightsquigarrow \mathbf{Set}$ induce the identity monad.

[316] e.g.: it sends `(aa)(ab)(bb)` to `aaabbb`.

[317] We leave you to figure out the unit and multiplication depending on your preferred way to construct $k[A]$ (either as polynomials over variables in $A$ or functions from $A$ to $k$).

**Examples 333.** Here, we describe three simple yet very useful examples and let you ponder on the adjunctions they might or might not originate from.

1. Suppose $\mathbf{C}$ has (binary) coproducts and a terminal object $\mathbf{1}$, then $(- + \mathbf{1}) : \mathbf{C} \rightsquigarrow \mathbf{C}$ is a monad.[318] We write $\mathsf{inl}^{X+Y}$ (resp. $\mathsf{inr}^{X+Y}$) for the coprojection of $X$ (resp. $Y$) into $X + Y$.[319] First, note that for a morphism $f : X \to Y$,

$$f + \mathbf{1} = [\mathsf{inl}^{Y+\mathbf{1}} \circ f, \mathsf{inr}^{Y+\mathbf{1}}] : X + \mathbf{1} \to Y + \mathbf{1}.$$

The components of the unit are given by the coprojections, i.e.: $\eta_X = \mathsf{inl}^{X+\mathbf{1}} : X \to X + \mathbf{1}$, and the components of the multiplication are

$$\mu_X = [\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}] : X + \mathbf{1} + \mathbf{1} \to X + \mathbf{1}.$$

Checking that (172) commutes, we have for any $X \in \mathbf{C}$:

$$\begin{aligned} \mu_X \circ (\eta_X + \mathbf{1}) &= [\mu_X \circ \mathsf{inl}^{(X+\mathbf{1})+\mathbf{1}} \circ \eta_X, \mu_X \circ \mathsf{inr}^{(X+\mathbf{1})+\mathbf{1}}] \\ &= [[\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}] \circ \mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}] \\ &= [\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}] \\ &= \mathrm{id}_{X+\mathbf{1}} \\ &= [\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}] \\ &= \mu_X \circ \mathsf{inl}^{(X+\mathbf{1})+\mathbf{1}} \\ &= \mu_X \circ \eta_{X+\mathbf{1}} \end{aligned}$$

[318] It is called the **maybe monad**. It is a generalization of the maybe functor defined in Exercise 172 and you may want to generalize the adjunction described in Example 310 to this setting before going to the next section.

[319] These notations are very common in the community of programming language research, they stand for *injection left* (resp. *right*). We may omit the superscript in case it is too cumbersome.

For (173), we have for any $X \in \mathbf{C}$:

$$\begin{aligned}
\mu_X \circ (\mu_X + \mathbf{1}) &= [\mu_X \circ \mathsf{inl}^{(X+\mathbf{1})+\mathbf{1}} \circ \mu_X, \mu_X \circ \mathsf{inr}^{(X+\mathbf{1})+\mathbf{1}}] \\
&= [[\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}] \circ \mu_X, \mathsf{inr}^{X+\mathbf{1}}] \\
&= [[\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}], \mathsf{inr}^{X+\mathbf{1}}] \\
&= [\mu_X, \mathsf{inr}^{X+\mathbf{1}}] \\
&= [[\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}], \mathsf{inr}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}] \\
&= [\mu_X \circ \mathsf{inl}^{(X+\mathbf{1})+\mathbf{1}}, \mu_X \circ \mathsf{inr}^{(X+\mathbf{1})+\mathbf{1}}, \mu_X \circ \mathsf{inr}^{(X+\mathbf{1})+\mathbf{1}}] \\
&= \mu_X \circ \mu_{X+\mathbf{1}}
\end{aligned}$$

2. The covariant powerset functor $\mathcal{P} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$ is a monad with the following unit and multiplication:

$$\eta_X : X \to \mathcal{P}(X) = x \mapsto \{x\} \text{ and } \mu_X : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X) = F \mapsto \bigcup_{s \in F} s.$$

Checking that (172) commutes, we have for any $S \subseteq \mathcal{P}(X)$:

$$\begin{aligned}
\mu_X(\mathcal{P}(\eta_X)(S)) &= \mu_X\left(\{\{x\} \mid x \in S\}\right) \\
&= \bigcup_{x \in S} \{x\} \\
&= S \\
&= \bigcup \{S\} \\
&= \mu_X(\{S\}) \\
&= \mu_X(\eta_{\mathcal{P}(X)}(S))
\end{aligned}$$

For (173), we have for any $\mathcal{F} \in \mathcal{P}(\mathcal{P}(\mathcal{P}(X))$:

$$\begin{aligned}
\mu_X(\mu_{\mathcal{P}(X)}(\mathcal{F})) &= \mu_X\left(\bigcup_{F \in \mathcal{F}} F\right) \\
&= \bigcup_{\substack{s \in \mathcal{P}(X) \\ \exists F \in \mathcal{F}, s \in F}} s \\
&= \{x \in X \mid \exists s \in \mathcal{P}(X), x \in s \text{ and } \exists F \in \mathcal{F}, s \in F\} \\
&= \bigcup_{F \in \mathcal{F}} \bigcup_{s \in F} s \\
&= \mu_X\left(\left\{\bigcup_{s \in F} s \mid F \in \mathcal{F}\right\}\right) \\
&= \mu_X(\mathcal{P}(\mu_X)(\mathcal{F}))
\end{aligned}$$

3. The functor $\mathcal{D} : \mathbf{Set} \to \mathbf{Set}$ sends a set $X$ to the set of finitely supported distributions on $X$, i.e.:

$$\mathcal{D}(X) := \{\varphi \in [0,1]^X \mid \sum_{x \in X} \varphi(x) = 1 \text{ and } \varphi(x) \neq 0 \text{ for finitely many } x\text{'s}\}.$$

It sends a function $f : X \to Y$ to the function between distributions

$$\lambda \varphi^{\mathcal{D}(X)}.\lambda y^Y.\varphi(f^{-1}(y)).$$

More verbosely, the weight of $\mathcal{D}(f)(\varphi)$ at point $y$ is equal to the total weight of $\varphi$ on the preimage of $y$ under $f$. It is a monad with unit $\eta_X = x \mapsto \delta_x$, where $\delta_x$ is the Dirac distribution at $x$ (all the weight is at $x$), and multiplication

$$\mu_X = \Phi \mapsto \lambda x^X. \sum_{\phi \in \text{supp}(\Phi)} \Phi(\phi) \cdot \phi(x),$$

where $\text{supp}(\Phi)$ is the support of $\Phi$, i.e.: $\text{supp}(\Phi) := \{\varphi \mid \Phi(\varphi) \neq 0\}$.

After looking long enough for adjunctions giving rise to the monads in Examples 333, two questions dare to be asked. Does every monad arise from an adjunction in the same way as above? If yes, is that adjunction unique?

The second question might not be as natural to novices in category theory but it is almost as important as the first one. Indeed, uniqueness is a very strong property and if every monad had a unique corresponding adjunction, one might expect it to be fairly easy to find. This is part of the beauty of category theory. We are working with very little data $M$, $\eta$ and $\mu$ so if it completely determined an adjunction $L \dashv R$ with its unit and counit and the natural isomorphism $\text{Hom}(L-,-) \cong \text{Hom}(-,R-)$, it could not do so in a very convoluted way merely because there is not that many ways to manipulate the original data.

In any case, we will respectively give a positive and negative answer to these questions. Fortunately, while we might not benefit from the power of uniqueness, there are two special adjunctions arising from a monad whose descriptions are fairly straightforward. In the order we present them, the first is due to Kleisli and the second to Eilenberg and Moore. In the rest of this section, $(M, \eta, \mu)$ will be a monad on a category $\mathbf{C}$.

### *Kleisli Category* $\mathbf{C}_M$

An intuitive way to think about monads is through the idea of **generalized elements**.[320] Given an object $A \in \mathbf{C}_0$, we can view $MA$ as extending $A$ with more *general* or *structured* elements built from $A$.

In this picture, the morphisms $\eta_A : A \to MA$ give a way to understand anything inside $A$ trivially as a general element of $A$. The morphisms $\mu_A : M^2A \to MA$ imply that higher order structures can be collapsed so that generalized elements over generalized elements of $A$ are generalized elements of $A$. The functoriality of $M$ implies that the new structures in $MA$ are somewhat independent of $A$. Indeed, for every morphisms $f : A \to B$, there is a morphism $Mf : MA \to MB$ which, by naturality of $\eta$ ($Mf(\eta_A) = \eta_B(f)$), acts just like $f$ on the trivial generalization of elements in $A$. Commutativity of (172) says that the trivial generalization[321] of a generalized element is indeed trivial, namely, after collapsing via $\mu$, we end up with what we started with. Finally, the associativity of $\mu$ (i.e.: commutativity of (173))

[320] This is not a formal term.

[321] There are two ways to do it corresponding to the L.H.S. and R.H.S. of (172).

corresponds to the fact that in higher order of generalizations, one can collapse the structure at every level in any order and end up with the same thing.

Now, we can also consider **generalized morphisms**. Let us say we were given an ill-defined morphism $f : A \to B$ that sends some of the stuff in $A$ outside of $B$. One way to fix this might be to consider general elements of $B$ and see $f$ as a morphism $A \to MB$. We will call such morphisms **Kleisli morphisms** and write $f : A \nrightarrow B$ for $f : A \to MB$.[322]

With an arbitrary functor $F$, you might have a hard time to come up with a way to compose two Kleisli morphisms $A \to FB$ and $B \to FC$ or even define the identity Kleisli morphism $A \to FA$, but the data of a monad lets you do just that. Indeed, given $f : A \nrightarrow B$ and $g : B \nrightarrow C$, while $g$ is not composable with $f$, $Mg$ is so we have $Mg \circ f : A \to MMC$ and it suffices to apply the multiplication $\mu_C$ to obtain $\mu_C \circ Mg \circ f : A \nrightarrow C$. We denote $g \circ_M f := \mu_C \circ Mg \circ f$ and call it the **Kleisli composition**. Also, for any $A \in \mathbf{C}_0$, the component of the unit at $A$ yields a Kleisli morphism $\eta_A : A \nrightarrow A$. Let us check that $\circ_M$ is associative and that $\eta_A$ behaves like the identity with respect to $\circ_M$.

Let $f : A \nrightarrow B$, $g : B \nrightarrow C$ and $h : C \nrightarrow D$ be Kleisli morphisms, the compositions $h \circ_M (g \circ_M f)$ and $(h \circ_M g) \circ_M f$ are respectively the bottom and top path of the following commutative diagram, so we conclude that $\circ_M$ is associative.

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[d, "f"] \arrow[dddd, bend right, "g \circ_M f"'] & & MB \arrow[d, "Mg"'] \arrow[ddd, bend left, "M(h \circ_M g)"] \\
MB \arrow[d, "Mg"] & \text{(a)} & MMC \arrow[d, "MMh"'] \\
MMC \arrow[rr, "MMh"] \arrow[dd, "\mu_C"] & & MMMD \arrow[d, dashed, "M\mu_D"] \arrow[ddl, "\mu_{MD}"'] \\
& \text{(b)} & MMD \arrow[d, "\mu_D"] \\
MC \arrow[r, "Mh"'] & MMD \arrow[r, "\mu_D"'] \arrow[ur, phantom, "\text{(c)}"] & MD
\end{tikzcd} \tag{174}$$

Showing (174) commutes:

(a) Trivial.

(b) $\mathsf{NAT}(\mu, C, MD, h)$.

(c) Components of (173) at $D$.

We show that $\eta_B \circ_M f = f$ and $f \circ_M \eta_A = f$ with the following derivations.

$$\begin{aligned}
\eta_B \circ_M f &= \mu_B \circ M\eta_B \circ f \\
\text{by L.H.S. of (172)} &= \mathrm{id}_{MB} \circ f \\
&= f
\end{aligned}
\qquad
\begin{aligned}
f \circ_M \eta_A &= \mu_B \circ Mf \circ \eta_A \\
\text{by } \mathsf{NAT}(\eta, A, MB, f) &= \mu_B \circ \eta_{MB} \circ f \\
\text{by R.H.S. of (172)} &= \mathrm{id}_{MB} \circ f \\
&= f
\end{aligned}$$

This leads to the definition of the category $\mathbf{C}_M$.[323]

**Definition 334** ($\mathbf{C}_M$)**.** Let $\mathbf{C}$ be a category and $(M, \eta, \mu)$ a monad on $\mathbf{C}$. The **Kleisli category** of $M$, denoted $\mathbf{C}_M$[324], has the same objects as $\mathbf{C}$ and the morphisms in $\mathrm{Hom}_{\mathbf{C}_M}(A, B)$ are the elements of $\mathrm{Hom}_{\mathbf{C}}(A, MB)$. The identity for $A \in \mathbf{C}_0$ is $\eta_A : A \to MA$ and composition is $\circ_M$.

**Examples 335.** We describe the Kleisli category for the monads in Examples 333.

[322] Another common notation for Kleisli morphisms is $f : A \rightsquigarrow B$ but this clashes with our notation for functors.

[323] Notice that we had to use all the data from the monad: the naturality of $\eta$ and $\mu$, the commutativity of both diagrams (172) and (173) as well as functoriality of $M$ (the latter was used implicitly).

[324] Some authors denote it $\mathrm{Kl}(M)$.

1. By identifying a Kleisli morphism $f : A \nrightarrow B$ with a partial function $A \rightharpoonup B$ as we did in Example 273.3, we can show that $\mathbf{Set}_{-+1} \cong \mathbf{Par}$.

2. In $\mathbf{Set}_{\mathcal{P}}$, objects are sets and morphisms are functions $r : X \to \mathcal{P}(Y)$. Viewing the latter as a relation $R \subseteq X \times Y$ defined by $(x,y) \in R \Leftrightarrow y \in r(x)$, we can verify that composition of relations corresponds to Kleisli composition in $\mathbf{Set}_{\mathcal{P}}$.[325]

   Let $r : X \to \mathcal{P}(Y)$ and $s : Y \to \mathcal{P}(Z)$ be Kleisli morphisms, $R$, $S$ and $SR$ be the relations corresponding to $r$, $s$ and $s \circ_{\mathcal{P}} r$. We need to show $SR = S \circ R$. Fix $x \in X$, we have

$$(s \circ_{\mathcal{P}} r)(x) = (\mu_Z^{\mathcal{P}} \circ \mathcal{P}(s) \circ r)(x) = \bigcup \mathcal{P}(s)(r(x)) = \{z \in Z \mid \exists y \in r(x), z \in s(y)\}.$$

   Since $y \in r(x) \Leftrightarrow (x,y) \in R$ and $z \in s(y) \Leftrightarrow (y,z) \in S$, we conclude that

$$(x,z) \in SR \Leftrightarrow z \in (s \circ_{\mathcal{P}} r)(x) \Leftrightarrow (x,z) \in S \circ R.$$

   After a bit more administrative arguments, one finds that $\mathbf{Set}_{\mathcal{P}} \cong \mathbf{Rel}$.

3.

Since we can view any object of $\mathbf{C}$ as an object of $\mathbf{C}_M$, we may wonder if we can do the same with morphisms to obtain a functor $\mathbf{C} \rightsquigarrow \mathbf{C}_M$. The key idea is to view $f : A \to B$ as a generalized morphism by trivially generalizing its target, that is, by post-composing with $\eta_B$. We claim that $F_M : \mathbf{C} \rightsquigarrow \mathbf{C}_M$ acting as identity on objects and post-composing by components of $\eta$ on morphisms is a functor.[326] Indeed, $F_M(\mathrm{id}_A) = \eta_A$ is the identity on $A$ in $\mathbf{C}_M$ and

$$\begin{aligned}
F_M(g \circ f) &= \eta_C \circ g \circ f && \\
&= Mg \circ \eta_B \circ f && \mathsf{NAT}(\eta, B, C, g)\\
&= Mg \circ \mu_B \circ M(\eta_B) \circ \eta_B \circ f && \text{by (172)}\\
&= \mu_C \circ MMg \circ M(\eta_B) \circ \eta_B \circ f && \mathsf{NAT}(\mu, B, C, g)\\
&= \mu_C \circ M(\eta_C) \circ Mg \circ \eta_B \circ f && M\mathsf{NAT}(\eta, B, C, g)\\
&= F_M(g) \circ_M F_M(f). && \text{def. of } \circ_M
\end{aligned}$$

We will now construct a right adjoint $U_M : \mathbf{C}_M \rightsquigarrow \mathbf{C}$ to $F_M$. Given $A$ and $B$ objects of both $\mathbf{C}$ and $\mathbf{C}_M$, the Kleisli morphisms from $F_M A$ to $B$ are precisely the morphisms in $\mathbf{C}$ from $A$ to $MB$, thus we infer that the identity function is an isomorphism $\mathrm{Hom}_{\mathbf{C}_M}(F_M A, B) \cong \mathrm{Hom}_{\mathbf{C}}(A, MB)$. This implies $U_M$ sends $B$ to $MB$ and we can define $U_M$ on morphisms by imposing the naturality of the aforementioned isomorphism. Given $g : A \nrightarrow B$, starting with $\eta_A$ on the top left of (175), we find that $U_M g \circ \eta_A = g$ which implies $U_M g = \mu_B \circ Mg$.[327]

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}_M}(A, A) \arrow[r, leftrightarrow, "\mathrm{id}"] \arrow[d, "g \circ_M (-)"'] & \mathrm{Hom}_{\mathbf{C}}(A, MA) \arrow[d, "U_M g \circ (-)"] \\
\mathrm{Hom}_{\mathbf{C}_M}(A, B) \arrow[r, leftrightarrow, "\mathrm{id}"'] & \mathrm{Hom}_{\mathbf{C}}(A, MB)
\end{tikzcd} \tag{175}$$

[325] Composition of relations was defined in Example 81.

[326] Explicitly, for any $A \in \mathbf{C}_0$, $F_M(A) = A$ and for any $f : A \to B$, $F_M(f) = \eta_B \circ f$.

[327] This implication is subtle. While it is true that we do not yet know if another $f$ satisfies $f \circ \eta_A = g$. Once we know (in a few moments) defining $U_M g = \mu_B \circ Mg$ yields an adjunction $F_M \dashv U_M$ whose unit is $\eta$, we know that $\eta_A$ is universal and uniqueness of $U_M g$ follows.

As a sanity check (and for a bit of practice), let us verify $U_M$ is a functor. For any $A \in \mathbf{C}_{M0}$, $U_M(\eta_A) = \mu_A \circ M(\eta_A) = \mathrm{id}_A$ by the L.H.S. of (172) and for any for any $f : A \nrightarrow B$ and $g : B \nrightarrow C$,

$$
\begin{aligned}
U_M(g \circ_M f) &= U_M(\mu_C \circ Mg \circ f) \\
&= \mu_C \circ M(\mu_C \circ Mg \circ f) \\
&= \mu_C \circ M(\mu_C) \circ MMg \circ Mf \\
&= \mu_C \circ \mu_{MC} \circ MMg \circ Mf && \text{by (173)} \\
&= \mu_C \circ Mg \circ \mu_B \circ Mf && \text{by naturality of } \mu \\
&= U_M(g) \circ U_M(f).
\end{aligned}
$$

Let us now verify that $F_M \dashv U_M$. Let $A, B \in \mathbf{C}_0$ (we view $B$ as an object of $\mathbf{C}_M$), we saw that the identity function is an isomorphism $\mathrm{Hom}_{\mathbf{C}_M}(F_M A, B) \cong \mathrm{Hom}_{\mathbf{C}}(A, U_M B)$ and we now check it is natural. We need to show (176) commutes for any $f : A' \to A$ and $g : B \nrightarrow B'$. It follows from this derivation starting with $k : A \nrightarrow B$ in the top left.

$$
\begin{tikzcd}
\mathrm{Hom}_{\mathbf{C}_M}(A,B) \arrow[r, leftrightarrow, "\mathrm{id}"] \arrow[d, "g\circ(-)\circ_M F_M f"'] & \mathrm{Hom}_{\mathbf{C}}(A,MB) \arrow[d, "U_M g\circ(-)\circ f"] \\
\mathrm{Hom}_{\mathbf{C}_M}(A',B') \arrow[r, leftrightarrow, "\mathrm{id}"'] & \mathrm{Hom}_{\mathbf{C}}(A',MB')
\end{tikzcd}
\tag{176}
$$

$$
\begin{aligned}
g \circ_M k \circ_M F_M f &= \mu_{B'} \circ M(g) \circ \mu_B \circ M(k) \circ \eta_A \circ f \\
&= \mu_{B'} \circ M(g) \circ \mu_B \circ \eta_{MB} \circ k \circ f && \text{by naturality of } \eta \\
&= \mu_{B'} \circ M(g) \circ \mathrm{id}_{MB} \circ k \circ f && \text{by (172)} \\
&= \mu_{B'} \circ M(g) \circ k \circ f \\
&= U_M g \circ k \circ f
\end{aligned}
$$

Finally, in order to achieve our initial goal of finding an adjunction that induces the original monad, we need to make sure the monad arising from $F_M \dashv U_M$ is $(M, \eta, \mu)$. First, we check that $U_M F_M = M$. On objects, it is clear. On a morphism $f : A \to B$, we have

$$U_M(F_M(f)) = U_M(\eta_B \circ f) = \mu_B \circ M(\eta_B) \circ Mf \overset{(172)}{=} Mf.$$

Next, as $\eta_A$ is the image of the identity on $A$ in $\mathbf{C}_M$ under the natural isomorphismcomponent, the unit of the adjunction is the unit of the monad. The counit of the adjunction at $A$ is $\varepsilon_A = \mathrm{id}_{MA}$, thus $(U_M \varepsilon F_M)_A = U_M(\mathrm{id}_{F_M A}) = \mu_A \circ M(\mathrm{id}_{MA}) = \mu_A$.

Recall that we claimed $F_M \dashv U_M$ was special in some way and that this was the (informal) reason why it was relatively easy to find, the next proposition will make this precise.

**Definition 336** ($\mathrm{Adj}_M$)**.** Let $\mathbf{C}$ be a category and $(M, \eta, \mu)$ a monad on $\mathbf{C}$. The **category of adjunctions inducing** $M$ is denoted $\mathrm{Adj}_M$. Its objects are adjoint pairs $L \dashv R$ with unit $\eta$ and counit $\varepsilon$ sastisfying $R \circ L = M$ $R\varepsilon L = \mu$. Its morphisms $L \dashv R \to L' \dashv R'$) are functors $K$ satisfying $K \circ L = L'$ and $R' \circ K = R$ as in (177).

$$
\begin{tikzcd}
\mathbf{D} \arrow[rr, "K"] \arrow[dr, "R", bend left=15] \arrow[dr, phantom, "\top"] & & \mathbf{D}' \arrow[dl, "R'", bend left=15] \arrow[dl, phantom, "\perp"] \\
& \mathbf{C} \arrow[ul, "L", bend left=15] \arrow[ur, "L'", bend left=15] &
\end{tikzcd}
\tag{177}
$$

We can restate the end result of the discussion above as $F_M \dashv U_M$ being an object of $\mathrm{Adj}_M$. It is special because it is initial.

**Proposition 337.** *The adjunction $F_M \dashv U_M$ is initial in* $\mathrm{Adj}_M$.

*Proof.* Let $\mathbf{C} : L \dashv R : \mathbf{D} \in \mathrm{Adj}_M$ with unit $\eta$ and counit $\varepsilon$, we claim there is a unique functor $K : \mathbf{C}_M \rightsquigarrow \mathbf{D}$ satisfying $K \circ F_M = L$ and $R \circ K = U_M$ as in (178).

$$\begin{tikzcd} \mathbf{C}_M \arrow[rr, dashed, "K"] \arrow[dr, bend left, "U_M"] & & \mathbf{D} \arrow[dl, bend left, "R"] \\ & \mathbf{C} \arrow[ul, bend left, "F_M"] \arrow[ur, bend left, "L"] & \end{tikzcd} \tag{178}$$

On objects, $K$ is determined by $KA = KF_MA = LA$. To a morphism $f : A \nrightarrow B$, we need to assign a morphism in $Kf \in \mathrm{Hom}_{\mathbf{D}}(LA, LB)$ such that $RKf = U_Mf = \mu_B \circ Mf = R\varepsilon_{LB} \circ RLf$. It is clear that $Kf = \varepsilon_{LB} \circ Lf$ is a candidate but to show it is unique, we consider the following naturality square coming from the adjunction $L \dashv R$.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D}}(LA, LA) \arrow[r, "R - \circ \eta_A"] \arrow[d, "Kf \circ (-)"'] & \mathrm{Hom}_{\mathbf{C}}(A, RLA) \arrow[d, "RKf \circ (-)"] \\ \mathrm{Hom}_{\mathbf{D}}(LA, LB) & \mathrm{Hom}_{\mathbf{C}}(A, RLB) \arrow[l, "\varepsilon_{LB} \circ L-"] \end{tikzcd} \tag{179}$$

Starting with $\mathrm{id}_{LA}$ in the top left and reaching the bottom left, we find

$$\begin{aligned} Kf &= \varepsilon_{LB} \circ LRKf \circ L\eta_A \\ &= \varepsilon_{LB} \circ LR\varepsilon_{LB} \circ LRLf \circ L\eta_A && \text{hypothesis on } RKf \\ &= \varepsilon_{LB} \circ LR\varepsilon_{LB} \circ L\eta_{RLB} \circ Lf && \mathsf{NAT}(\eta, A, RLB, f) \\ &= \varepsilon_{LB} \circ \varepsilon_{LRLB} \circ L\eta_{RLB} \circ Lf && \mathsf{HOR}(\varepsilon, \varepsilon)L \\ &= \varepsilon_{LB} \circ \varepsilon_{LMB} \circ L\eta_{MB} \circ Lf && RL = M \\ &= \varepsilon_{LB} \circ \mathrm{id}_{MB} \circ Lf && \text{triangle identity} \\ &= \varepsilon_{LB} \circ Lf \end{aligned}$$

To finish the proof, let us verify $K$ is functorial.

$$K(u_{\mathbf{C}_M}(A)) = K(\eta_A) = \varepsilon_{LB} \circ L(\eta_A) \stackrel{(142)}{=} \mathrm{id}_A$$

$$\begin{aligned} K(g \circ_M f) &= K(\mu_C \circ RLg \circ f) \\ &= \varepsilon_{LC} \circ L(\mu_C) \circ LRLg \circ Lf \\ &= \varepsilon_{LC} \circ LR\varepsilon_{LC} \circ LRLg \circ Lf && \text{by hypothesis on } \varepsilon \\ &= \varepsilon_{LC} \circ \varepsilon_{LRLC} \circ LRLg \circ Lf && \mathsf{HOR}(\varepsilon, \varepsilon)L \\ &= \varepsilon_{LC} \circ Lg \circ \varepsilon_{LB} \circ Lf && \mathsf{NAT}(\varepsilon, LB, LRLC, Lg) \\ &= Kg \circ Kf \end{aligned}$$

□

See solution.

**Exercise 338.** Let $K : L \dashv R \to L' \dashv R'$ be a morphism in $\mathrm{Adj}_M$, $\varepsilon$ and $\varepsilon'$ be the counits of the source and target respectively. Show that $K\varepsilon = \varepsilon' K$.

*Eilenberg–Moore Category* $\mathbf{C}^M$

For the second solution to the problem of finding an adjunction inducing a given monad, we look at the more structural side of monads.

**Definition 339** ($M$–algebra)**.** Let $(M, \eta, \mu)$ be a monad, an **Eilenberg–Moore algebra** for $M$ or simply $M$–algebra is a pair $(A, \alpha)$ consisting of an object $A \in \mathbf{C}_0$ and a morphism $\alpha : MA \to A$ such that (180) and (181) commute.

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & MA \arrow[d, "\alpha"] \\ & A \end{tikzcd} \quad (180) \qquad \begin{tikzcd} M^2A \arrow[r, "\mu_A"] \arrow[d, "M\alpha"'] & MA \arrow[d, "\alpha"] \\ MA \arrow[r, "\alpha"'] & A \end{tikzcd} \quad (181)$$

We will often denote an $M$–algebra using only its underlying object or its underlying morphism.

**Definition 340** (Homomorphism)**.** Let $(M, \eta, \mu)$ be a monad and $(A, \alpha)$ and $(B, \beta)$ be two $M$–algebras. An $M$–algebra **homomorphism** or simply $M$–homomorphism from $(A, \alpha)$ to $(B, \beta)$ is a morphism $h : A \to B$ making (182) commute.

$$\begin{tikzcd} MA \arrow[r, "Mh"] \arrow[d, "\alpha"'] & MB \arrow[d, "\beta"] \\ A \arrow[r, "h"'] & B \end{tikzcd} \tag{182}$$

After checking that the composition of two $M$–homomorphisms is an $M$–homomorphism and $\mathrm{id}_A$ is an $M$–homomorphism from $(A, \alpha)$ to itself whenever $\alpha$ is an $M$–algebra, we get a category of $M$–algebras and $M$–homomorphism called the **Eilenberg–Moore category** of $M$ and denoted $\mathbf{C}^M$.

Since $\mathbf{C}^M$ was built from objects and morphisms in $\mathbf{C}$, there is an obvious forgetful functor $U^M : \mathbf{C}^M \rightsquigarrow \mathbf{C}$ sending an $M$–algebra $(A, \alpha)$ to its underlying object $A$ and an $M$–homomorphism to its underlying morphism. We will now find a left adjoint $F^M : \mathbf{C} \rightsquigarrow \mathbf{C}^M$ to $U^M$. Since we want this adjunction to induce the monad $M$, we require that $U^M F^M = M$. It means $F^M$ must send $A \in \mathbf{C}_0$ to an $M$–algebra on $MA$ and $h \in \mathbf{C}_1$ to $Mh$. There is straightforward choice given to us by the data of $M$, that is, $F^M A = (MA, \mu_A : MMA \to MA)$ and it turns out naturality of $\mu$ yields commutativity of

$$\begin{tikzcd} M^2A \arrow[r, "M^2h"] \arrow[d, "\mu_A"'] & M^2B \arrow[d, "\mu_B"] \\ MA \arrow[r, "Mh"'] & MB \end{tikzcd}, \tag{183}$$

which implies $Mh$ is indeed an $M$–homomorphism. Because $M$ is a functor, we immediately obtain that $F^M$ is a functor. We now show that $F^M \dashv U^M$ with unit $\eta$ and counit $\varepsilon$ satisfying $U^M \varepsilon F^M = \mu$.

Let us define the counit and verify the triangle identities. For an $M$–algebra $\alpha : MA \to A$, we want an $M$–homomorphism $\varepsilon_\alpha : F^M U^M A = (MA, \mu_A) \to (A, \alpha)$. Again, we have a straightforward choice since $\alpha$, being an $M$–algebra, satisfies $\alpha \circ$

$\mu_A = \alpha \circ M\alpha$, hence we can set $\varepsilon_\alpha = \alpha$. The following derivations show the triangle identities hold.

$$\varepsilon_{F^M A} \circ F^M \eta_A = \varepsilon_{\mu_A} \circ M\eta_A = \mu_A \circ M\eta_A = \mathrm{id}_{MA} = \mathrm{id}_{F^M A}$$
$$U^M \varepsilon_\alpha \circ \eta_{U^M(A,\alpha)} = \alpha \circ \eta_A = \mathrm{id}_A = \mathrm{id}_{U^M(A,\alpha)}$$

Lastly, we verify
$$U^M(\varepsilon_{F^M A}) = U^M(\varepsilon_{\mu_A}) = U^M(\mu_A) = \mu_A,$$
and we conclude $F^M \dashv U^M$ is an object of $\mathrm{Adj}_M$.

Dually to Proposition 337, we show that this adjunction is special in a precise way.

**Proposition 341.** *The adjunction $(F^M, U^M)$ is terminal in* $\mathrm{Adj}_M$*.*

*Proof.* Let $\mathbf{C} : L \dashv R : \mathbf{D} \in \mathrm{Adj}_M$ with unit $\eta$ and counit $\varepsilon$, we claim there is a unique functor $K : \mathbf{D} \rightsquigarrow \mathbf{C}^M$ satisfying $K \circ L = F^M$ and $U^M \circ K = R$ as in (184).

$$\begin{tikzcd} \mathbf{D} \arrow[rr, dashed, "K"] \arrow[dr, "R", shift left] & & \mathbf{C}^M \arrow[dl, "U^M", shift left] \\ & \mathbf{C} \arrow[ul, "L", shift left] \arrow[ul, phantom, "\top"] \arrow[ur, "F^M", shift left] \arrow[ur, phantom, "\perp"] & \end{tikzcd} \tag{184}$$

As before, we can determine $K$ by the equation $U^M K = R$ which means it sends $A \in \mathbf{D}_0$ to an $M$–algebra on $RA$ and $f : A \to B \in \mathbf{D}_1$ to an $M$–homomorphism $Rf : KA \to KB$. The only missing piece of this puzzle is the algebra structure on $KA$. We have two clues. First, $Rf$ is an $M$–homomorphism, i.e.: denoting $KA = (RA, \alpha_A)$ and $KB = (RB, \alpha_B)$, we must ensure (185) commutes. Second, $(KA, \alpha_A)$ is an $M$–algebra, so (186) and (187) commute.

$$\begin{tikzcd} MRA \arrow[r, "MRf"] \arrow[d, "\alpha_A"'] & MRB \arrow[d, "\alpha_B"] \\ RA \arrow[r, "Rf"'] & RB \end{tikzcd} \tag{185}$$

$$\begin{tikzcd} RA \arrow[r, "\eta_{RA}"] \arrow[dr, "\mathrm{id}_{RA}"'] & MRA \arrow[d, "\alpha_A"] \\ & RA \end{tikzcd} \tag{186}$$

$$\begin{tikzcd} MMRA \arrow[r, "\mu_A"] \arrow[d, "M\alpha_A"'] & MRA \arrow[d, "\alpha_A"] \\ MRA \arrow[r, "\alpha_A"'] & RA \end{tikzcd} \tag{187}$$

Replacing $M$ with $RL$, we recognize the first diagram as a naturality square showing $\alpha$ is a natural transformation $RLR \Rightarrow R$ and the two other diagrams yield

$$\alpha \cdot \eta R = \mathbb{1}_R \quad \text{and} \quad \alpha \cdot RL\alpha = \alpha \cdot \mu.$$

Moreover, we can see that $\alpha_A = R\varepsilon_A$ makes (186) commute by a triangle identity. This candidate also makes (185) commute because $R\varepsilon_A$ is a natural transformation and (187) commute because

$$\begin{aligned} R\varepsilon_A \circ \mu_A &= R\varepsilon_A \circ R\varepsilon_{LA} && R\varepsilon L = \mu \\ &= R(\varepsilon_A \circ \varepsilon_{LA}) && \text{functoriality of } R \\ &= R(\varepsilon_A \circ LR(\varepsilon_A)) && \mathsf{HOR}(\varepsilon, \varepsilon) \\ &= R\varepsilon_A \circ MR\varepsilon_A && RL = M. \end{aligned}$$

To verify uniqueness, recall that the counit of the adjunction $F^M \dashv U^M$ sends an $M$–algebra $(X, x)$ to the $M$–homomorphism $x : (MX, \mu_X) \to (X, x)$. Thus, $\alpha_A$ is the result of applying the counit to $KA$ and by Exercise 338, we have $\alpha_A = K\varepsilon_A = R\varepsilon_A$. As $K$ acts like $R$ on morphisms, it is obviously functorial. $\square$

The following picture summarizes the last two sections.

$$
\begin{tikzcd}
 & \mathbf{D} \arrow[d, shift left, "R"] \arrow[dr, dashed] & \\
\mathbf{C}_M \arrow[r, shift left, "U_M"] \arrow[ur, dashed] \arrow[r, phantom, "\top"] & \mathbf{C} \arrow[l, shift left, "F_M"] \arrow[u, shift left, "L"] \arrow[u, phantom, "\dashv"] \arrow[r, shift left, "F^M"] \arrow[r, phantom, "\perp"] & \mathbf{C}^M \arrow[l, shift left, "U^M"]
\end{tikzcd}
\tag{188}
$$

With the following two results, one can see the Kleisli category inside the Eilenberg–Moore category as the full subcategory of free algebras.

See solution.

**Exercise 342.** Show that the unique morphism $F_M \dashv U_M \to F^M \dashv U^M$ is the functor $\mathbf{C}_M \rightsquigarrow \mathbf{C}^M$ sending $A \in \mathbf{C}_0$ to $(MA, \mu_A)$ and $f : A \twoheadrightarrow B$ to $\mu_B \circ Mf$.

**Proposition 343.** *The functor $\mathbf{C}_M \rightsquigarrow \mathbf{C}^M$ of Exercise 342 is fully faithful.*

*Proof.* **Full:** Suppose $g : MA \to MB$ is such that $g \circ \mu_A = \mu_B \circ Mg$, then

$$\mu_B \circ M(g \circ \eta_A) = \mu_B \circ Mg \circ M\eta_A = g \circ \mu_A \circ M\eta_A = g,$$

so $g$ is the image of $g \circ \eta_A$ in $\mathbf{C}_M$.

**Faithful:** Suppose $\mu_B \circ Mg = \mu_B \circ Mf$, then pre-composing with $\eta_A$, we find that $f = f \circ_M \eta_A = g \circ_M \eta_A = g$. $\square$

## *POV: Universal Algebra*

In this section, we will highlight the link between algebraic structures as you have encountered them in other classes with the Eilenberg–Moore algebras discussed above. We will only work over the category **Set**.[328] We start by developing an example.

[328] The ideas of universal algebra have be developed in other settings like enriched categories.

**Example 344** ($\mathcal{P}_{\mathrm{ne}}$)**.** Consider the non-empty finite powerset functor $\mathcal{P}_{\mathrm{ne}}$ sending $X$ to $\{S \in \mathcal{P}(X) \mid S \text{ is finite and non-empty}\}$. The same unit and multiplication as defined for $\mathcal{P}$ make $\mathcal{P}_{\mathrm{ne}}$ into a monad.[329] A $\mathcal{P}_{\mathrm{ne}}$–algebra is a function $\alpha : \mathcal{P}_{\mathrm{ne}}(A) \to A$ satisfying the equations $\alpha\{a\} = a$ and $\alpha(\mathcal{P}_{\mathrm{ne}}(\alpha)(S)) = \alpha(\bigcup S)$. From this, we can extract a binary operation $\oplus_\alpha : A \times A \to A$ by defining $x \oplus_\alpha y = \alpha\{x, y\}$. This operation is clearly commutative and idempotent,[330] but it is also associative by the following derivation.

[329] It is easy to see as the $\eta$ and $\mu$ restrict to finite and non-empty.

[330] i.e.: $x \oplus_\alpha y = y \oplus_\alpha y$ and $x \oplus_\alpha x = x$.

$$
\begin{aligned}
(x \oplus_\alpha y) \oplus_\alpha z &= \alpha\{x, y\} \oplus_\alpha z \\
&= \alpha\{\alpha\{x, y\}, z\} \\
&= \alpha\{\alpha\{x, y\}, \alpha\{z\}\}
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha\{\mathcal{P}_{\text{ne}}\alpha\{\{x,y\},\{z\}\}\} \\
&= \alpha\{\mu_A\{\{x,y\},\{z\}\}\} \\
&= \alpha\{x,y,z\}.
\end{aligned}
$$

Since a $\mathcal{P}_{\text{ne}}$–homomorphism $h : (A,\alpha) \to (B,\beta)$ commutes with $\alpha$ and $\beta$ it also commutes with $\oplus_\alpha$ and $\oplus_\beta$.[331]

[331] i.e.: $h(a \oplus_\alpha a') = h(a) \oplus_\beta h(a')$.

Conversely, if $\oplus$ is an idempotent, associative and commutative binary operation on $A$, we can define $\alpha_\oplus$ on non-empty finite sets of $A$ by iterating $\oplus$. Namely,

$$\alpha_\oplus\{x\} = x \oplus x \quad \text{and} \quad \alpha_\oplus\{x_1,\ldots,x_n\} = x_1 \oplus x_2 \oplus \cdots \oplus x_n.$$

It is well-defined by associativity and commutativity and we can check that it is the inverse of the operation described in the previous paragraph. That is to say, we can check that $\alpha_{\oplus_\alpha} = \alpha$ and $\oplus_{\alpha_\oplus} = \oplus$. For the former, it is clear for singleton sets and for any $n > 1$, we have the following derivation.

$$
\begin{aligned}
\alpha_{\oplus_\alpha}\{x_1,\ldots,x_n\} &= x_1 \oplus_\alpha \cdots \oplus_\alpha x_n \\
&= \alpha\{x_1, x_2 \oplus_\alpha \cdots \oplus_\alpha x_n\} \\
&= \vdots \\
&= \alpha\{x_1, \alpha\{x_2, \alpha\{\cdots, \alpha\{x_n\}\}\}\} \\
\text{using } \alpha \circ \mathcal{P}_{\text{ne}}(\alpha) = \alpha \circ \mu_A &= \alpha\{x_1, x_2, \alpha\{\cdots, \alpha\{x_n\}\}\} \\
&= \vdots \\
&= \alpha\{x_1,\ldots,x_n\}
\end{aligned}
$$

For the latter, we have

$$x \oplus_{\alpha_\oplus} y = \alpha_\oplus\{x,y\} = x \oplus y.$$

A set equipped with an idempotent, commutative and associative binary operation is called a **semilattice**[332] and we have shown above that $\mathcal{P}_{\text{ne}}$–algebras are in correspondence with semilattices. Through the introduction of basic notions in universal algebra, we will explain how this correspondence is functorial and generalize the core idea behind it.

[332] A semilattice can also be called a sup-semilattice, join-semilattice, inf-semilattice or meet-semilattice. This is because a semilattice can also be defined as a poset where all supremums/joins (resp. , infimums/meets) exist.

**Definition 345** (Algebraic theory)**.** An **algebraic signature**[333] is a set $\Sigma$ of operation symbols along with **arities** in $\mathbb{N}$, we denote $f : n \in \Sigma$ for an $n$–ary operation symbol $f$ in $\Sigma$. Given a set $X$, one constructs the set of $\Sigma$**–terms** with variables in $X$, denoted $T_\Sigma(X)$ by iterating operations symbols:

[333] Also called algebraic similarity type.

$$
\begin{gathered}
\forall x \in X,\, x \in T_\Sigma(X) \\
\forall t_1,\ldots,t_n \in T_\Sigma(X), f : n \in \Sigma,\, f(t_1,\ldots,t_n) \in T_\Sigma(X).
\end{gathered}
$$

An **equation**[334] $E$ over $\Sigma$ is a pair of $\Sigma$–terms over a set of dummy variables which we usually denote with an equality sign (e.g.: $s = t$ for $s,t \in T_\Sigma(X)$ and $X$ is the set of dummy variables). We will call the tuple $(\Sigma, E)$ an **algebraic theory**.

[334] Also called axiom.

**Example 346.** The algebraic theory of semilattices contains a single binary operation $\Sigma_S = \{\oplus : 2\}$ and the following equations in $E_S$:[335]

$$x \oplus x = x \qquad I\text{: idempotence}$$
$$x \oplus y = y \oplus x \qquad C\text{: commutativity}$$
$$(x \oplus y) \oplus z = x \oplus (y \oplus z). \qquad A\text{: associativity}$$

Let $X = \{x, y, z\}$, the set of $\Sigma$–terms contains infinitely many terms, e.g.: $x \oplus y$, $x \oplus (y \oplus z)$, $(x \oplus x) \oplus (y \oplus z) \oplus (z \oplus x)$, etc.[336]

**Definition 347** ($(\Sigma, E)$–algebras)**.** Given an algebraic theory $(\Sigma, E)$, a $(\Sigma, E)$**–algebra** is a set $A$ along with operations $f^A : A^n \to A$ for all $f : n \in \Sigma$ such that the pairs of terms in $E$ are always equal when the operation symbols and dummy variables are instantiated in $A$.[337] We usually denote $\Sigma^A$ for the set operations $f^A$.

**Examples 348.** As is suggested by the terminology, the common algebraic structures can be defined with simple algebraic theories.

1. We can define a monoid as an algebra for the signature $\{\cdot : 2, 1 : 0\}$ and the equations $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $1 \cdot x = x$, $x \cdot 1 = x$. We will say that this is the algebraic theory of monoids.

2. Adding the unary operation $(-)^{-1}$ and the equations $x \cdot x^{-1} = 1$ and $x^{-1} \cdot x = 1$, we obtain the theory of groups.

3. Adding the equation $x \cdot y = y \cdot x$ yields the theory of abelian groups.

4. With the signature $\{+ : 2, \cdot : 2, 1 : 0, 0 : 0\}$, we can add the abelian group equations for the operation $+$ (identity is 0), the monoid equations for $\cdot$ (identity is 1) and the distributivity equation $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and thus obtain the theory of rings.

5. The theory of semilattices has this named because a $(\Sigma_S, E_S)$–algebra is a semilattice.

We also have homomorphisms between $(\Sigma, E)$–algebras.

**Definition 349** ($(\Sigma, E)$–algebra homomorphisms)**.** Given two $(\Sigma, E)$–algebras $A$ and $B$, a **homomorphism** between them is a map $h : A \to B$ commuting with all operations in $\Sigma$, that is $\forall f : n \in \Sigma, h \circ f^A = f^B \circ h^n$.[338]

The category of $(\Sigma, E)$–algebras and their homomorphisms (with the obvious composition and identities) is denoted $\mathrm{Alg}(\Sigma, E)$.

**Example 350** ($\Sigma_S, E_S$)**.** Recall from Example 344 that $\mathcal{P}_{\mathrm{ne}}$–algebras correspond to semilattices. Up to a couple of missing functoriality arguments, we have shown that the categories $\mathbf{Set}^{\mathcal{P}_{\mathrm{ne}}}$ and $\mathrm{Alg}(\Sigma_S, E_S)$ are isomorphic. We say that $(\Sigma_S, E_S)$ is an **algebraic presentation** of the monad $\mathcal{P}_{\mathrm{ne}}$ or that the theory of semilattices presents the monad $\mathcal{P}_{\mathrm{ne}}$.

[335] It will be made clear why this is the theory of semilattices shortly.

[336] The parentheses are here to denote the order in which the operation symbols was applied. While in semilattices, the operation $\oplus$ satisfies the equations making the parentheses and order irrelevant, when describing terms over the signature, we cannot remove them.

[337] The operation symbol $f$ is always instantiated by $f^A$ and a dummy variable can be instantiated by any element of $A$. For instance, suppose $(A, f^A, g^A)$ is a $(\Sigma, E)$–algebra and $f(x, g(y)) = g(y)$ is an equation in $E$, then for any $a, b \in A$, $f^A(a, g^A(b)) = g^A(b)$.

[338] We write $h^n$ for componentwise application of the map $h$ to vectors in $A^n$, i.e.: $h^n(a_1, \ldots, a_n) = (h(a_1), \ldots, h(a_n))$.

It turns out all algebraic theories present at least one monad.

**Definition 351** (Term monad)**.** Let $(\Sigma, E)$ be an algebraic theory, one can assign to any set $X$, the set $T_{\Sigma,E}(X)$ of terms in $T_\Sigma(X)$ modulo the equations in $E$.[339] This can be extended to functions $f : X \to Y$, by variable substitution, i.e.: $T_\Sigma(f)$ acts on a term $t$ by replacing all occurrences of $x \in X$ with $f(x) \in Y$ and $T_{\Sigma,E}(f)$ acts on equivalence classes by $[t] \mapsto [T_\Sigma(f)(t)]$. We obtain a functor $T_{\Sigma,E}$ on which we can put a monad structure.

The unit is obvious because any element of $X$ is a $\Sigma$–term, thus $\eta_X : X \to T_{\Sigma,E}(X)$ maps $x$ to the equivalence class containing the term $x$. The multiplication is derived from the fact that applying operations in $\Sigma$ to $\Sigma$–terms yields $\Sigma$–terms. More explicitly, $\mu_X$ is a *flattening* operation defined recursively by

$$\forall t \in T_\Sigma(X), \mu_X([[t]]) = [t]$$
$$\forall f : n \in \Sigma, t_1, \ldots, t_n \in T_\Sigma T_{\Sigma,E}(X), \mu_X([f(t_1, \ldots, t_n)]) = [f(\mu_X([t_1]), \ldots, \mu_X([t_n]))]$$

One can show that $\mathbf{Set}^{T_{\Sigma,E}}$ is the category of $(\Sigma, E)$–algebras.

[339] Let us not waste time here to make this more formal as there is a lot to say that is not relevant to the rest of this story. We say that two terms $s$ and $t$ are equal modulo $E$ if we can rewrite $s$ using the equations in $E$ and obtain $t$. The informal notion of *rewriting* is good enough for us (we hope you got a sense of what rewriting means when learning about high school algebra).

Unfortunately, the term monads are not very simple to work with[340] and it is often desirable to find other simpler monads which are presented by the same theory or conversely to find an algebraic presentation for a given monad.

[340] In fact, you might have realized we chose to not even bother.

**Examples 352.** 1. The algebraic theory presenting $\mathcal{D}$ is called the theory of **convex algebras** and is denoted $(\Sigma_{\mathsf{CA}}, E_{\mathsf{CA}})$, it consists of a binary operation $+_p : 2$ for any $p \in (0, 1)$ which is meant to represent a choice between the two terms in the operation, the left one being chosen with probability $p$ and the second one with probability $1 - p$. There are three equations in the theory that morally ensure that terms representing the same probabilistic choice are equal.[341]

$$\begin{aligned} x +_p x &= x && I_p\text{: idempotence} \\ x +_p y &= y +_{\overline{p}} x && C_p\text{: skew-commutativity} \\ (x +_q y) +_p z &= x +_{pq} (y +_{\frac{p\overline{q}}{\overline{pq}}} z) && A_p\text{: skew-associativity} \end{aligned}$$

These equations are necessary for every distribution in $\mathcal{D}X$ to correspond uniquely to an equivalence class in $T_{\Sigma_{\mathsf{CA}}, E_{\mathsf{CA}}}(X)$.

[341] For $x \in [0, 1]$, we denote $\overline{x} := 1 - x$.

2. The monad $(- + \mathbf{1})$ is particular because it is really simple and combines very well with other monads.

**Proposition 353.** *For any monad $M$, there is a monad structure on the composition $M(- + \mathbf{1})$. Moreover, if M is presented by $(\Sigma, E)$ the monad $M(- + \mathbf{1})$ is presented by $(\Sigma \cup \{* : 0\}, E)$, that is, the new theory only has an additional constant*[342] *which is neutral with respect to the operation symbols.*

[342] A 0–ary opeartion is more commonly called a constant.

*Proof.* Postponed to Exercise 359. □

We often qualify theories with an added constant as **pointed**. For instance, the theories presented by $\mathcal{P}_{\text{ne}}(- + \mathbf{1})$ and $\mathcal{D}(- + \mathbf{1})$ are those of **pointed semilattices** and **pointed convex algebras** respectively.

*Remark* 354 (Lawvere's way). There is another way to do universal algebra *more categorically* still very much linked to monads: Lawvere theories. Algebras over a Lawvere theory[343] are defined more abstractly using the categorical language and, on this account, they enjoy straightforward generalization through enrichment or lifting to higher order categories.

[343] They are called models of the theory.

## *POV: Computer Programs*

In this section, we will develop on an original idea by Eugenio Moggi that monads are suitable models for a general notion of *computation*. In the sequel, we will use the terms *type* and *set* interchangeably.

Moggi gave a justification for using monads in computer science (particularly in programming semantics) via the informal intuition of *computational types*. For a type $A$, the computational type of $A$ should contain all computations which return a value of type $A$. It is intended for the interpretation of *computation* to be made explicit by an instance of a monad. In most cases, it can be thought of as a piece of code which returns some value, but for now, we start by building the intuition in an abstract sense.

Let $MA$ denote the computational type of $A$ and $MMA$ the computational type of $MA$, that is computations returning values which are themselves computations of type $A$. The following items should coincide with our intuition of computation.

1. For any $x \in A$, there is a trivial computation $\mathsf{return}\ x \in MA$.
2. For any $C \in MMA$, we can reduce $C$ to $\mathsf{flatten}(C) \in MA$ which executes $C$ and the computation returned by $C$ to obtain a final return value of type $A$.
3. If $C \in MA$, then $\mathsf{flatten}(\mathsf{return}\ C) = C$.
4. If $C \in MA$ and $C' \in MMA$ does the same computation as $C$ but instead of returning a value $x$, it returns the computation $\mathsf{return}\ x$, then $\mathsf{flatten}(C') = C$.
5. If $MMMA$ is the computational type of $MMA$ and $C \in MMMA$, then there are two ways to flatten $C$. First, there is the computation $C_1$ which executes $C$ and executes the returned computation (of type $MMA$) to obtain a final value of type $MA$, hence $C_1 \in MMA$ and $\mathsf{flatten}(C_1) \in MA$. Second, $C_2$ executes $C$ and flattens the returned computation to obtain a final value of type $MA$, $C_2$ is also of type $MMA$ and $\mathsf{flatten}(C_2) \in MA$. These two operations should yield the same result.

Now, a monad $M$ is a description of computational types that is general, namely, for any type $A$, the monad $M$ gives a type $MA$ behaving as expected. You can check that $x \mapsto \mathsf{return}\ x$ is the unit of this monad and $\mathsf{flatten}$ is the multiplication.

**Examples 355.** Here, we list more examples commonly used in computer science.

**List monad**: For any set $X$, let $L(X)$ denote the set of all finite lists whose elements are chosen in $X$. This is a functor that sends a function $f : X \to Y$ to its extension on lists $L(f) : L(X) \to L(Y)$ which applies $f$ to all elements on the list (in lots of programming languages, one writes $L(f) := \mathtt{map}(f, -)$).Then, we can put a monad structure on $L$. The unit maps send an element $x \in X$ to the list containing only that element: $\eta_X = x \mapsto [x]$. The multiplication maps concatenate all the lists in a lists of lists: $\mu_X = [\ell_1, \ldots, \ell_n] \mapsto \ell_1 \ell_2 \cdots \ell_n$. It is easy to check diagrams (172) to (173) commute.

**Termination:** In order to model computations that might terminate with no output, the monad $(- + \mathbf{1})$ is often used. For any type $X$, the type $X + \mathbf{1}$ has all the values of type $X$ and an additional termination value denoted $*$. The behavior of the unit and multiplication of the monad can be interpreted as the fact that the stage of the computation that leads to a termination is irrelevant. This monad is also known as the Maybe monad.

**Non-deterministic choice:** The model for nondeterministic choice is given by the monad $\mathcal{P}_{\mathrm{ne}}$. The elements of $S \in \mathcal{P}_{\mathrm{ne}}(X)$ are seen as the possible outcomes of a nondeterministic choice. The unit is basically viewing a deterministic choice as a nondeterministic choice. The multiplication reduces the number of choices without changing the behavior. For instance, consider a process that nondeterministically chooses between two boxes containing two coins each and then chooses a coin in the box. By simply observing the final choice, we would not be able to distinguish it from a process that nondeterministically chooses between the four coins from the start.

**Probabilistic choice:** In the same vein, probabilistic choice can be interpreted with the monad $\mathcal{D}$ of finitely supported distributions.

**Exceptions:** As a generalization of termination, we can put a monad structure on the functor $(\cdot + E)$ where $E$ is a set of exceptions that the computation can raise.

This view sheds light on one important features of monads we have not yet explored. If $M$ and $\widehat{M}$ are monads describing computational effects, it is natural to ask for a way to combine them. Indeed, it does not seem too ambitious to have a model for programs which, for instance, make nondeterministic choices and also might terminate with no output. It turns out there is a very useful tool to deal with this at the level of monads.

**Definition 356** (Monad distributive law)**.** Let $(M, \eta, \mu)$ and $(\widehat{M}, \widehat{\eta}, \widehat{\mu})$ be two monads on $\mathbf{C}$, a natural transformation $\lambda : M\widehat{M} \Rightarrow \widehat{M}M$ is called a **monad distributive law of** $M$ **over** $\widehat{M}$ if it makes (189), (190) commute.

$$\begin{tikzcd}
M \arrow[r, Rightarrow, "M\widehat{\eta}"] \arrow[dr, Rightarrow, "\widehat{\eta}M"'] & M\widehat{M} \arrow[d, Rightarrow, "\lambda"] & \widehat{M} \arrow[l, Rightarrow, "\eta\widehat{M}"'] \arrow[dl, Rightarrow, "\widehat{M}\eta"] \\
& \widehat{M}M &
\end{tikzcd} \tag{189}$$

$$\begin{tikzcd}
MM\widehat{M} \arrow[rr, Rightarrow, "\mu\widehat{M}"] \arrow[d, Rightarrow, "M\lambda"'] & & M\widehat{M} \arrow[d, Rightarrow, "\lambda"] & & M\widehat{M}\widehat{M} \arrow[ll, Rightarrow, "M\widehat{\mu}"'] \arrow[d, Rightarrow, "\lambda\widehat{M}"] \\
M\widehat{M}M \arrow[r, Rightarrow, "\lambda M"'] & \widehat{M}MM \arrow[r, Rightarrow, "\widehat{M}\mu"'] & \widehat{M}M & \widehat{M}\widehat{M}M \arrow[l, Rightarrow, "\widehat{\mu}M"] & \widehat{M}M\widehat{M} \arrow[l, Rightarrow, "\widehat{M}\lambda"]
\end{tikzcd} \tag{190}$$

**Proposition 357.** *If $\lambda : M\widehat{M} \Rightarrow \widehat{M}M$ is a monad distributive law, then the composite $\overline{M} = \widehat{M}M$ is a monad with unit $\overline{\eta} = \widehat{\eta} \diamond \eta$ and multiplication $\overline{\mu} = (\widehat{\mu} \diamond \mu) \cdot \widehat{M}\lambda M$.*

*Proof.* We have to show that the following instances of (172) and (173) commute.

$$\begin{tikzcd}
\overline{M} \arrow[r, Rightarrow, "\overline{M}(\widehat{\eta}\diamond\eta)"] \arrow[ddr, Rightarrow, "\mathbb{1}_{\overline{M}}"'] & \overline{M}^2 \arrow[d, Rightarrow, "\widehat{M}\lambda M"] & \overline{M} \arrow[l, Rightarrow, "(\widehat{\eta}\diamond\eta)\overline{M}"'] \arrow[ddl, Rightarrow, "\mathbb{1}_{\overline{M}}"] \\
& \widehat{M}^2M^2 \arrow[d, Rightarrow, "\widehat{\mu}\diamond\mu"] & \\
& \overline{M} &
\end{tikzcd} \qquad (191)$$

$$\begin{tikzcd}
\overline{M}^3 \arrow[r, Rightarrow, "\overline{M}\widehat{M}\lambda M"] \arrow[d, Rightarrow, "\widehat{M}\lambda M\overline{M}"'] & \overline{M}\widehat{M}^2M^2 \arrow[r, Rightarrow, "\overline{M}(\widehat{\mu}\diamond\mu)"] & \overline{M}^2 \arrow[d, Rightarrow, "\widehat{M}\lambda M"] \\
\widehat{M}^2M^2\overline{M} \arrow[d, Rightarrow, "(\widehat{\mu}\diamond\mu)\overline{M}"'] & & \widehat{M}^2M^2 \arrow[d, Rightarrow, "\widehat{\mu}\diamond\mu"] \\
\overline{M}^2 \arrow[r, Rightarrow, "\widehat{M}\lambda M"'] & \widehat{M}^2M^2 \arrow[r, Rightarrow, "\widehat{\mu}\diamond\mu"'] & \overline{M}
\end{tikzcd} \qquad (192)$$

For the left part of (191), we have the following diagram, the justifications of each part is given below with what diagram has to be considered and what functors should be applied to it (recall that acting on the diagrams does not affect commutativity). The notation (172).L (resp. .R) means only the left (resp. right) part of the diagram is considered.

$$\begin{tikzcd}
\overline{M} \arrow[r, "\overline{M}\widehat{\eta}"] \arrow[rrr, Rightarrow, bend left, "\overline{M}(\widehat{\eta}\diamond\eta)"] \arrow[dr, "\overline{M}\widehat{\eta}"'] \arrow[dddrrr, bend right, "\widehat{M}\widehat{\eta}M"'] \arrow[ddddrrr, Rightarrow, bend right=60, "\mathbb{1}_{\overline{M}}=\mathbb{1}_{\widehat{M}}M"'] \arrow[ddr, phantom, "\text{(a)}"] & \overline{M}\widehat{M} \arrow[rr, "\overline{M}\widehat{M}\eta"] \arrow[dr, "\overline{M}\eta\widehat{M}"] \arrow[dd, "\widehat{M}\mathbb{1}_M\widehat{M}"'] & & \overline{M}^2 \arrow[dd, Rightarrow, "\widehat{M}\lambda M"] \\
& & \overline{M}M\widehat{M} \arrow[ur, "\overline{M}\lambda"'] \arrow[dl, "\widehat{M}\mu\widehat{M}"] & \\
& \overline{M}\widehat{M} \arrow[ddrr, "\widehat{M}\lambda"] & & \widehat{M}^2M^2 \arrow[d, "\widehat{M}^2\mu"] \arrow[dd, Rightarrow, bend left, "\widehat{\mu}\diamond\mu"] \\
& & & \widehat{M}^2M \arrow[d, "\widehat{\mu}M"] \\
& & & \overline{M}
\end{tikzcd} \qquad (193)$$

(b) (c) (d) (e) (f)

(a) $\widehat{M}\mathbb{1}_M\widehat{M}$ is the identity transformation.

(b) Act on (172).L with $\widehat{M}$ on the left and right.

(c) Act on (189).R with $\overline{M}$ on the left.

(d) Act on (190).L with $\widehat{M}$ on the left.

(e) Act on (189).L with $\widehat{M}$ on the left.

(f) Act on (172) with $M$ on the right.

Without a diagram, the derivation is this (we use ; to denote the opposite of $\circ$, i.e.: composition in the order read on the diagram):

$$
\begin{aligned}
\overline{M}(\widehat{\eta} \diamond \eta); \widehat{M}\lambda M; \widehat{\mu} \diamond \mu &= \overline{M}\widehat{\eta}; \overline{M}\widehat{M}\eta; \widehat{M}\lambda M; \widehat{M}^2\mu; \widehat{\mu}M && \text{def of } \diamond \\
&= \overline{M}\widehat{\eta}; \overline{M}\eta\widehat{M}; \overline{M}\lambda; \widehat{M}\lambda M; \widehat{M}\widehat{M}\mu; \widehat{\mu}M && \overline{M}(189).\mathrm{R} \\
&= \overline{M}\widehat{\eta}; \overline{M}\eta\widehat{M}; \widehat{M}\mu\widehat{M}; \widehat{M}\lambda; \widehat{\mu}M && \widehat{M}(190).\mathrm{L} \\
&= \overline{M}\widehat{\eta}; \widehat{M}\mathbb{1}_M\widehat{M}; \widehat{M}\lambda; \widehat{\mu}M && \widehat{M}(172).\mathrm{L}\widehat{M} \\
&= \overline{M}\widehat{\eta}; \widehat{M}\lambda; \widehat{\mu}M && \\
&= \widehat{M}\widehat{\eta}M; \widehat{\mu}M && \widehat{M}(189).\mathrm{R} \\
&= \mathbb{1}_{\widehat{M}}M = \mathbb{1}_{\overline{M}} && (172).\mathrm{L}M
\end{aligned}
$$

For the right part of (191), the derivation is very similar.

For (192), we do the same thing.

$$
\begin{tikzcd}
\overline{M}^3 \arrow[rr, Rightarrow, "\overline{M}\widehat{M}\lambda M"] \arrow[d, Rightarrow, "\widehat{M}\lambda M\overline{M}"'] \arrow[drr, phantom, "(a)"] & & \overline{M}\widehat{M}^2M^2 \arrow[rr, "\overline{M}\widehat{M}^2\mu"] \arrow[d, "\widehat{M}\lambda\widehat{M}M^2"'] \arrow[rrrr, Rightarrow, bend left, "\overline{M}(\widehat{\mu}\diamond\mu)"] \arrow[drr, phantom, "(b)"] & & \overline{M}\widehat{M}^2M \arrow[rr, "\overline{M}\widehat{\mu}M"] \arrow[d, "\widehat{M}\lambda\widehat{M}M"] \arrow[ddrr, phantom, "(c)"] & & \overline{M}^2 \arrow[dd, Rightarrow, "\widehat{M}\lambda M"] \\
\widehat{M}^2M^2\overline{M} \arrow[rr, "\widehat{M}^2M\lambda M"] \arrow[dd, "\widehat{\mu}M^2\overline{M}"] \arrow[dddd, Rightarrow, bend right, "(\widehat{\mu}\diamond\mu)\overline{M}"'] \arrow[dr, phantom, "(d)"] & & \widehat{M}^2M\widehat{M}M^2 \arrow[rr, "\widehat{M}^2M\widehat{M}\mu"] \arrow[dl, "\widehat{\mu}M\widehat{M}M^2"'] \arrow[dr, "\widehat{M}^2\lambda M^2"] \arrow[dd, phantom, "(e)"] \arrow[drr, phantom, "(f)"] & & \widehat{M}^2M\overline{M} \arrow[d, "\widehat{M}^2\lambda M"] & & \\
& \overline{M}\widehat{M}M^2 \arrow[dr, "\widehat{M}\lambda M^2"] & & \widehat{M}^3M^3 \arrow[r, "\widehat{M}^3M\mu"] \arrow[dl, "\widehat{\mu}\widehat{M}M^3"'] \arrow[dd, "\widehat{M}^3\mu M"] \arrow[ddr, phantom, "(j)"] & \widehat{M}^3M^2 \arrow[rr, "\widehat{M}\widehat{\mu}M^2"] \arrow[dd, "\widehat{M}^3\mu"'] \arrow[dr, "\widehat{\mu}\widehat{M}M^2"] \arrow[drr, phantom, "(g)"] \arrow[dddr, phantom, "(k)"] & & \widehat{M}^2M^2 \arrow[d, "\widehat{\mu}M^2"] \arrow[ddd, Rightarrow, bend left, "\widehat{\mu}\diamond\mu"] \\
\widehat{M}M^2\overline{M} \arrow[ur, "\widehat{M}M\lambda M"'] \arrow[dd, "\widehat{M}\mu\overline{M}"] \arrow[ddrr, phantom, "(h)"] & & \widehat{M}^2M^3 \arrow[dd, "\widehat{M}^2\mu M"] \arrow[dr, phantom, "(i)"] & & & \widehat{M}^2M^2 \arrow[r, "\widehat{\mu}M^2"'] \arrow[dd, "\widehat{M}^2\mu"'] \arrow[ddr, phantom, "(m)"] & \widehat{M}M^2 \arrow[dd, "\widehat{M}\mu"] \\
& & & \widehat{M}^3M^2 \arrow[r, "\widehat{M}^3\mu"] \arrow[dl, "\widehat{\mu}\widehat{M}M^2"] \arrow[dr, phantom, "(l)"] & \widehat{M}^3M \arrow[dr, "\widehat{\mu}\widehat{M}M"'] & & \\
\overline{M}^2 \arrow[rr, Rightarrow, "\widehat{M}\lambda M"'] & & \widehat{M}^2M^2 \arrow[rrr, "\widehat{M}^2\mu"'] \arrow[rrrr, Rightarrow, bend right, "\widehat{\mu}\diamond\mu"'] & & & \widehat{M}^2M \arrow[r, "\widehat{\mu}M"'] & \overline{M}
\end{tikzcd}
$$

(a) Def of $\widehat{M}\lambda \diamond \lambda M$.

(b) Def of $\widehat{M}\lambda\widehat{M} \diamond \mu$.

(c) Apply $\widehat{M}(\cdot)M$ to (190).R.

(d) Def of $\widehat{\mu} \diamond M\lambda M$.

(e) Def of $\widehat{\mu} \diamond \lambda M^2$.

(f) Def of $\widehat{M}^2\lambda \diamond \mu$.

(g) Apply $(\cdot)M^2$ to associativity of $\widehat{\mu}$ (173).

(h) Apply $\widehat{M}(\cdot)M$ to (190).L.

(i) Def of $\widehat{\mu}\widehat{M} \diamond \mu M$.

(j) Apply $\widehat{M}^3$ to associativity of $\mu$ (173).

(k) Def of $\widehat{\mu}\widehat{M} \diamond \mu$.

(l) Same as (k): Def of $\widehat{\mu}\widehat{M} \diamond \mu$.

(m) Def of $\widehat{\mu} \diamond \mu$.

□

**Corollary 358.** *If* $\mathbf{C}$ *has (binary) coproducts and a terminal object* $\mathbf{1}$ *and* $M$ *is a monad, then* $M(-+\mathbf{1})$ *is also monad.*

*Proof.* We will exhibit a monad distributive law of $M$ over $(-+\mathbf{1})$. We claim

$$\iota_X : MX + \mathbf{1} \to M(X+1) = [M(\mathsf{inl}^{X+\mathbf{1}}), \eta_{X+\mathbf{1}} \circ \mathsf{inr}^{X+\mathbf{1}}]$$

is a monad distributive law $\iota : (-+\mathbf{1})M \Rightarrow M(-+\mathbf{1})$. Then, it follows by Proposition 357. $\square$

**Exercise 359.** Show Proposition 353 with the monad structure on $M(-+\mathbf{1})$ given in Corollary 358.

See solution.

**Example 360** (Rings)**.** Consider the term monads for the theory of monoids and abelian groups $T_{\mathbf{Mon}}$ and $T_{\mathbf{Ab}}$. You can check that they are the monads induced by the free-forgetful adjunctions between **Mon** and **Set** and **Ab** and **Set**. Also, $T_{\mathbf{Mon}}$ is the same thing as the list monad. Call the binary operation of $T_{\mathbf{Mon}}$ and $T_{\mathbf{Ab}}$ the product and sum respectively.

Then, by identifying products of sums (elements of $T_{\mathbf{Mon}}T_{\mathbf{Ab}}X$) with sums of products (elements of $T_{\mathbf{Ab}}T_{\mathbf{Mon}}X$) by *distributing* the product over the sum as we are used to do with, say, real numbers, we obtain a monad distributive law of $T_{\mathbf{Mon}}$ over $T_{\mathbf{Ab}}$. The resulting composite monad $T_{\mathbf{Ab}}T_{\mathbf{Mon}}$ is the term monad for the theory of rings. The term distributive law comes from this example.

*Remark* 361*.* It is not always possible to combine monads in such a natural way. For instance, it was shown that no distributive law exist between $\mathcal{P}_{\text{ne}}$ and $\mathcal{D}$ and even that no monad structure can exist on $\mathcal{P}_{\text{ne}}\mathcal{D}$ or $\mathcal{D}\mathcal{P}_{\text{ne}}$. Thus, modelling combined probabilistic and nondeterministic effects has been quite a hard endeavor and is still an active area of research I discovered in an internship with Matteo Mio and Valeria Vignudelli at ENS de Lyon last summer.

If you are looking for more applications of this perspective on monads and especially if you enjoyed the assignment on Brzozowski's algorithm, I suggest you look into the paper *Generalizing Determinization From Automata to Coalgebras* available at `https://arxiv.org/abs/1302.1046`.

## *Exercises*

1. Show that the triple $(\mathcal{D}, \eta, \mu)$ described in Example 333.3 is a monad.
2. Show that the Kleisli category of the powerset monad is the category **Rel** of relations.
3. Show that $\iota$ defined in the proof of Corollary 358 is a monad distributive law.
4. Show Proposition 353 with the monad structure on $M(-+\mathbf{1})$ given in Corollary 358.

# *Solutions to Exercises*

## *Solutions to Chapter*

*Solution to Exercise 77.* Take any monoid $M$ with an idempotent element $x \neq 1_M$ (it satisfies $x \cdot x = x$). Letting $\mathbf{C}$ be $\mathbf{B}M$ and $\mathbf{C}'$ contain the object $*$ and only the morphism $x$ yields a suitable example because the identity in $\mathbf{C}'$ is $x$. □

*Solution to Exercise 97.* On morphisms, we define $\Delta_{\mathbf{C}}(f) = (f, f)$. The functoriality properties hold because everything in $\mathbf{C} \times \mathbf{C}$ is done componentwise.

i. For $f : X \to Y$, we have $(f, f) : (X, X) \to (Y, Y)$.

ii. For $f : X \to Y$ and $g : Y \to Z$, we have $(g, g) \circ (f, f) = (g \circ f, g \circ f)$.

iii. For any $X \in \mathbf{C}_0$, we have $\Delta_{\mathbf{C}}(\mathrm{id}_X) = (\mathrm{id}_X, \mathrm{id}_X) = \mathrm{id}_{(X,X)}$.

□

*Solution to Exercise 99.* A quick way to show $F(X, -)$ is a functor is to recognize it as the composition of $F$ with $X \times \mathrm{id}_{\mathbf{C}'}$, where $X$ is the constant functor at $X$. Similarly, $F(-, Y) := F \circ (\mathrm{id}_{\mathbf{C}} \times Y)$. □

*Solution to Exercise 100.* Let us show the three properties of functoriality.

i. For any $(f, g) : (X, X') \to (Y, Y')$, by hypothesis, we have the following commutative square showing $F(f, g)$ has the right source and target.

$$\begin{tikzcd} F(X,X') \arrow[r, "F(\mathrm{id}_X, g)"] \arrow[d, "F(f,\mathrm{id}_{X'})"'] \arrow[dr, "F(f,g)"] & F(X,Y') \arrow[d, "F(f,\mathrm{id}_{Y'})"] \\ F(Y,X') \arrow[r, "F(\mathrm{id}_Y, g)"'] & F(Y,Y') \end{tikzcd}$$

ii. Let us have two morphisms $(f, g) : (X, X') \to (Y, Y')$ and $(f', g') : (Y, Y') \to (Z, Z')$ in $\mathbf{C} \times \mathbf{C}'$. The hypothesis on $F(-, -)$ gives the four commutative squares below and the functoriality of $F$ in each component gives the commutativity of the parts denoted by $*$.

$$
\begin{tikzcd}
F(X,X') \arrow[r, "F(\mathrm{id}_X,g)"] \arrow[d, "F(f,\mathrm{id}_{X'})"] \arrow[dr, "F(f,g)"] \arrow[rr, bend left, "F(\mathrm{id}_X,g'\circ g)", "*"'] \arrow[dd, bend right, "F(f'\circ f,\mathrm{id}_{X'})"', "*"] & F(X,Y') \arrow[r, "F(\mathrm{id}_X,g')"] \arrow[d, "F(f,\mathrm{id}_{Y'})"] \arrow[dr, "F(f,g')"] & F(X,Z') \arrow[d, "F(f,\mathrm{id}_{Z'})"] \arrow[dd, bend left, "F(f'\circ f,\mathrm{id}_{Z'})", "*"'] \\
F(Y,X') \arrow[r, "F(\mathrm{id}_Y,g)"] \arrow[d, "F(f',\mathrm{id}_{X'})"] \arrow[dr, "F(f',g)"] & F(Y,Y') \arrow[r, "F(\mathrm{id}_Y,g')"] \arrow[d, "F(f',\mathrm{id}_{Y'})"] \arrow[dr, "F(f',g')"] & F(Y,Z') \arrow[d, "F(f',\mathrm{id}_{Z'})"] \\
F(Z,X') \arrow[r, "F(\mathrm{id}_Z,g)"] \arrow[rr, bend right, "F(\mathrm{id}_Z,g'\circ g)"', "*"] & F(Z,Y') \arrow[r, "F(\mathrm{id}_Z,g')"] & F(Z,Z')
\end{tikzcd}
$$

We conclude from the commutativity of the whole diagram that $F(f',g') \circ F(f,g) = F(f' \circ f, g' \circ g)$.

iii. For any $(A,B) \in (\mathbf{C} \times \mathbf{C}')_0$, the functoriality of either component yields

$$F(\mathrm{id}_{(A,B)}) = F(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{F(A,B)}.$$

$\square$

## *Solutions to Chapter*

*Solution to Exercise 123.* Let us have two morphisms $f : X \to Y$ and $g : Y \to Z$.

- Suppose $f$ and $g$ are monic. For any $h_1, h_2 : Z \to Z'$ satisfying $h_1 \circ g \circ f = h_2 \circ g \circ f$, monicity of $f$ implies $h_1 \circ g = h_2 \circ g$ which in turn, by monicity of $g$ imply $h_1 = h_2$. Thus, $g \circ f$ is monic.
- We apply duality. Suppose $f$ and $g$ are epic, then $f^{\mathrm{op}}$ and $g^{\mathrm{op}}$ are monic so $(g \circ f)^{\mathrm{op}} = f^{\mathrm{op}} \circ g^{\mathrm{op}}$ is monic, thus $g \circ f$ is epic.
- If $f$ and $g$ are isomorphisms, then it is easy to check that $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$, implying $g \circ f$ is an isomorphism.

$\square$

*Solution to Exercise 136.* We draw the categories with all the morphisms and we let you infer the composition[344] and show that they fit the requirement (by counting morphisms).

$$
\begin{tikzcd}
X \arrow[loop above, "\mathrm{id}_X"] \arrow[r] & Y \arrow[loop above, "\mathrm{id}_Y"]
\end{tikzcd}
$$

(a)

$$
\begin{tikzcd}
X \arrow[loop above, "\mathrm{id}_X"] \arrow[r, bend left] & Y \arrow[loop above, "\mathrm{id}_Y"] \arrow[l, bend left] \arrow[l, bend left=60]
\end{tikzcd}
$$

(b)

[344] The categories (a) and (b) have a uniquely determined composition. For (c) and (d), composing the non-identity endomorphism with itself can yield either itself or $\mathrm{id}_Y$.

$$\begin{tikzcd} X \arrow[loop above, "\mathrm{id}_X"] & Y \arrow[l] \arrow[loop above, "\mathrm{id}_Y"] \arrow[loop below] \end{tikzcd} \quad \text{(c)} \qquad \begin{tikzcd} X \arrow[loop above, "\mathrm{id}_X"] \arrow[r, bend left] & Y \arrow[l, bend left] \arrow[loop above, "\mathrm{id}_Y"] \arrow[loop below] \end{tikzcd} \quad \text{(d)}$$ □

*Solution to Exercise 144.* 1. Let $f : A \to B$ be the only non-identity morphism in $\mathbf{2}$, it is a monomorphism vacuously because there is only one morphism with target $A$ ($\mathrm{id}_A$). Now, for any morphism $m : X \to Y \in \mathbf{C}_1$, we can define $F : \mathbf{2} \rightsquigarrow \mathbf{C}$ by $FA = X$, $FB = Y$ and $Ff = m$ and it will be a functor. Thus, choosing $m$ that is not monic yields the required example.

2. If $f$ is split monic, it has a right inverse $f'$. This implies $Ff'$ is the right inverse of $Ff$ because $Ff \circ Ff' = F(f \circ f') = F(\mathrm{id}) = \mathrm{id}$. We conclude that $Ff$ is split monic.

3. We need to show that functors preserve split epimorphisms. By duality, if $f$ is split epic, then $f^{\mathrm{op}}$ is split monic, thus it is preserved by the functor $F^{\mathrm{op}}$. And $Ff = (F^{\mathrm{op}}(f^{\mathrm{op}}))^{\mathrm{op}}$ is split epic.

4. Functors preserve isomorphisms because a morphism is an isomorphism if and only if it is split epic and split monic.[345] If $A \cong B$ and $i : A \to B$ is an isomorphism, then $Fi : FA \to FB$ is an isomorphism, so $FA \cong FB$.

[345] Because split epic is equivalent to having a left inverse and split monic is equivalent to having a right inverse.

□

*Solution to Exercise 145.* 1. Let $\mathbf{C}$ be a category with at least one morphism $f$ that is not monic, the only functor $[] : \mathbf{C} \rightsquigarrow \mathbf{1}$ sends $f$ to $\mathrm{id}_\bullet$ which is monic.

2. Suppose that $F(f)$ is monic and let $g$ and $h$ be such that $f \circ g = f \circ h$. By monicity of $F(f)$, $F(f) \circ F(g) = F(f \circ g) = F(f \circ h) = F(f) \circ F(h)$ implies $F(g) = F(h)$. Since $F$ is faithful, $g = h$.

3. We need to show faithful functors reflect epimorphisms.

□

*Solution to Exercise 146.* Let us have three monomorphisms $m : Y \hookrightarrow X$, $n : Z \hookrightarrow X$ and $o : W \hookrightarrow X$.

**Reflexivity:** We have $m \circ \mathrm{id}_Y = m$ thus $m \sim m$.

**Symmetry:** Suppose that $m \sim n$, namely, there is an isomorphism $i : Y \to X$ such that $m = n \circ i$. Then, pre-composing with the isomorphism $i^{-1}$ yields $m \circ i^{-1} = n$ which implies $n \sim m$.

**Transitivity:** If $m \sim n$ and $n \sim o$, then there exist isomorphisms $i : Y \to Z$ and $i' : W \to Z$ satisfying $m = n \circ i$ and $n = o \circ i'$. Therefore, we have $m = o \circ i' \circ i$ which implies $m \sim o$.[346] □

[346] Recall that the composition of two isomorphisms is an isomorphism.

*Solution to Exercise 149.* Let us have five monomorphisms $m : Y \hookrightarrow X$, $m : Y' \hookrightarrow X$, $n : Z \hookrightarrow X$, $n' : Z' \hookrightarrow X$ and $o : W \hookrightarrow X$.[347]

[347] Recall that we often use $m$ to refer to $[m]$.

**Well-defined:** Suppose that $m \leq n$, $m' \sim m$ and $n \sim n'$, namely, there is a morphism $k : Y \to Z$ and isomorphisms $i : Y \circ Y'$ and $i' : Z' \to Z$ such that $m =$

$n \circ k$, $m' = m \circ i$ and $n = n' \circ i'$. Combining these equalities yields $m' = n' \circ i' \circ k \circ i$ which witnesses $m' \leq n'$.

**Reflexivity:** We have $m \circ \mathrm{id}_Y = m$ thus $m \leq m$.

**Antisymmetry:** If $m \leq n$ and $n \leq m$, then there exist morphisms $k : Y \to Z$ and $k' : Z \to Y$ satisfying $m = n \circ k$ and $n = m \circ k'$. Combining these two equalities yield $m = m \circ k' \circ k$ and $n = n \circ k \circ k'$. Therefore, since $m$ and $n$ are monic, we infer that $k' \circ k = \mathrm{id}_Y$ and $k \circ k' = \mathrm{id}_Z$. This means $k$ is an isomorphism and $m \sim n$ (so $[m] = [n]$).

**Transitivity:** If $m \leq n$ and $n \leq o$, then there exist morphisms $k : Y \to Z$ and $k' : W \to Z$ satisfying $m = n \circ k$ and $n = o \circ k'$. Therefore, we have $m = o \circ k' \circ k$ which implies $m \leq o$. $\square$

## *Solutions to Chapter*

*Solution to Exercise 159.* As we have said that binary products are unique up to isomorphism, it is enough to show that $A \times B$ satisfies the same universal property as $B \times A$. Let $\pi_A$ and $\pi_B$ be the projections of $A \times B$, we claim that $B \xleftarrow{\pi_B} A \times B \xrightarrow{\pi_A} A$ is the product of $B$ and $A$. Indeed, for any $B \xleftarrow{p_B} X \xrightarrow{p_A} A$, we use the original universal property of $A \times B$ to find a unique mediating morphism $! : X \to A \times B$ such that $\pi_B \circ ! = p_B$ and $\pi_A \circ ! = p_A$. $\square$

*Solution to Exercise 160.* $\square$

*Solution to Exercise 163.* The existence and uniqueness of $\prod_{i \in I} f_i$ is given by the universal property of the product $\prod_{i \in I} Y_i$ with for each $j \in I$, the morphism $f_j \circ \pi_j : \prod_{i \in I} X_i \to Y_j$. $\square$

*Solution to Exercise 187.* ($\Rightarrow$) Suppose $f : X \to Y$ is monic, commutativity of (42) is trivial. For any $X \xleftarrow{g} Z \xrightarrow{h} X$ satisfying $f \circ g = f \circ h$, we have $g = h$. Thus $g = h$ is the mediating morphism $!$ of (194), it is unique because $\mathrm{id}_X \circ m = g$ implies $m = g$.

$$\begin{tikzcd} Z \arrow[drr, bend left, "h"] \arrow[dr, dashed, "!"] \arrow[ddr, bend right, "g"'] & & \\ & X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd} \qquad (194)$$

($\Leftarrow$) For any $g, h : Z \to X$ satisfying $f \circ g = f \circ h$, the universal property of the pullback tells us there is a unique $! : Z \to X$ making (194) commute. Since $!$ satisfies $g = \mathrm{id}_X \circ ! = h$, we conclude $g = ! = h$, thus $f$ is a monomorphism.

The dual statement is that $f : X \to Y$ is epic if and only if (195) is a pushout. We leave the proof to you. $\square$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\mathrm{id}_X"] \\ Y \arrow[r, "\mathrm{id}_X"'] \arrow[ur, phantom, "\ulcorner", very near end] & Y \end{tikzcd} \qquad (195)$$

*Solution to Exercise 215.* Let $p_A : X \to A$ and $p_B : X \to B$ be such that (196) commutes. A mediating morphism $! : X \to A$ must satisfy $\mathrm{id}_A \circ ! = p_A$ and $f \circ ! = p_B$. The first equality ensures $! = p_A$ is unique and satisfies the second equality because the outer square commuting yields $f \circ p_A = p_B$.

$$\begin{tikzcd} X \arrow[drr, bend left, "p_A"] \arrow[ddr, bend right, "p_B"'] & & \\ & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ & B \arrow[r, "\mathrm{id}_B"'] & B \end{tikzcd} \tag{196}$$

$$\begin{tikzcd} X \arrow[drr, bend left, "p_A"] \arrow[ddr, bend right, "p_B"'] & & \\ & A' \arrow[r, "i"] \arrow[d, "f\circ i"'] & A \arrow[d, "f"] \\ & B \arrow[r, "\mathrm{id}_B"'] & B \end{tikzcd} \tag{197}$$

Let $p_A : X \to A$ and $p_B : X \to B$ be such that (197) commutes. A unique mediating morphism $! : X \to A$ must satisfy $i\circ ! = p_A$ and $f \circ i \circ ! = p_B$. Post-composing the first equality by $i^{-1}$ implies $! = i^{-1} \circ p_A$ is unique and satisfies the second equality because $f \circ i \circ i^{-1} \circ p_A = f \circ p_A = p_B$. □

*Solution to Exercise 224.* We will show that if $\mathbf{C}$ has all pullbacks and a terminal object, then it has all finite products and equalizers. This implies, using Remark 218, that $\mathbf{C}$ is finitely complete.

For finite products, recall that it is enough to show that $\mathbf{C}$ has all binary products as it already has the empty product (the terminal object). We claim that the pullback of $A \xrightarrow{[]} \mathbf{1} \xleftarrow{[]} B$ is the binary product $A \times B$.

$$\begin{tikzcd} A \times_{\mathbf{1}} B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "{[]}"] \\ A \arrow[r, "{[]}"'] & \mathbf{1} \end{tikzcd} \tag{198}$$

Indeed, for any $A \xleftarrow{p_A} X \xrightarrow{p_B} B$, we have $[] \circ p_A = [] \circ p_B$, thus, there is a unique morphism $! : X \to A \times_{\mathbf{1}} B$ making (199) commute. Since the commutativity of the squares always hold, this is equivalent to the unviersal property of the binary product. Hence $A \times B \cong A \times_{\mathbf{1}} B$.

$$\begin{tikzcd} X \arrow[drr, bend left, "p_B"] \arrow[ddr, bend right, "p_A"'] \arrow[dr, dashed, "!"] & & \\ & A \times_{\mathbf{1}} B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "{[]}"] \\ & A \arrow[r, "{[]}"'] & \mathbf{1} \end{tikzcd} \tag{199}$$

□

## Solutions to Chapter

*Solution to Exercise 233.* We define $- \times X$ on morphisms by sending $f : Y \to Y' \in \mathbf{C}_1$ to $f \times \mathrm{id}_X : Y \times X \to Y' \times X$. Functoriality follows from the definition of $\times$ on morphisms. Indeed, $\mathrm{id}_Y \times \mathrm{id}_X$ is the only morphism making (200) commute and $(g \circ f) \times \mathrm{id}_X$ is the only morphism making (201) commute.

Recall that if $f : A \to A'$ and $g : B \to B'$, $f \times g : A \times B \to A' \times B'$ is the unique morphism making the diagram below commute:

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] \arrow[d, dashed, "f\times g"] & B \arrow[d, "g"] \\ A' & A' \times B' \arrow[l, "\pi_{A'}"] \arrow[r, "\pi_B"'] & B' \end{tikzcd}$$

$$\begin{tikzcd} Y & Y \times X \arrow[l, "\pi_Y"'] \arrow[d, dashed, "\mathrm{id}_Y \times \mathrm{id}_X"] \\ Y \arrow[u, phantom] & Y \times X \arrow[l, "\pi_Y"] \end{tikzcd} \quad \begin{tikzcd} Y \arrow[d, "\mathrm{id}_Y"'] \\ Y \end{tikzcd} \tag{200}$$

$$\begin{tikzcd} Y \arrow[d, dashed, "f"] \arrow[dd, bend right, "g \circ f"'] & Y \times X \arrow[l, "\pi_Y"'] \arrow[d, dashed, "f \times \mathrm{id}_X"] \arrow[dd, dashed, bend left, "(g \circ f) \times \mathrm{id}_X"] \\ Y' \arrow[d, dashed, "g"] & Y' \times X \arrow[l, "\pi_{Y'}"] \arrow[d, dashed, "g \times \mathrm{id}_X"] \\ Y'' & X \arrow[l, "\pi_{Y''}"] \end{tikzcd} \tag{201}$$

□

*Solution to Exercise 235.* First, we know that the pullback of the monomorphism $m$ along $f$ is monic by Theorem 209. Next, for $n : I' \hookrightarrow X \in \mathrm{Sub}_{\mathbf{C}}(Y)$, we need to show $[m] = [n]$ implies $[f^*(m)] = [f^*(n)]$.[348] In (202), we need to show there is an isomorphism $i' : J \to J'$ making everything commute.

[348] Recall that $[m] = [n]$ when there is an isomorphism $i$ satifying $n = m \circ i$.

$$\begin{tikzcd} & J' \arrow[rr, "j'"] \arrow[d, hook, "f^*(n)"] & & I' \arrow[d, hook, "n"] \\ J \arrow[r, phantom] \arrow[rr, "j"] \arrow[dr, hook, "f^*(m)"'] & X \arrow[rr, "f"'] & I \arrow[ur, "i"] \arrow[r, hook, "m"] & Y \end{tikzcd} \tag{202}$$

By the pullback property of $J'$, there is a unique mediating morphism $i' : J \to J'$ commuting with (202).[349] Similarly, the pullback property of $J$, there is a unique mediating morphism $i'^{-1} : J' \to J$ commuting with (202).[350] The fact that $i'$ and $i'^{-1}$ are inverses follows from viewing $i'^{-1} \circ i'$ as a mediating morphism from the pullback $J$ to itself which must be the identity by uniqueness. Similarly for $i' \circ i'^{-1}$.

[349] Use the fact that $n \circ i^{-1} \circ j = m \circ j = f \circ f^*(m)$.

[350] Use the fact that $m \circ i \circ j' = n \circ j' = f \circ f^*(n)$.

For functoriality of $\mathrm{Sub}_{\mathbf{C}}$, we need to show $\mathrm{id}^*(m) = m$ and $g^*(f^*(m)) = f \circ g^*(m)$. The first equality follows from Exercise 215 and the second from the pasting lemma. □

*Solution to Exercise 239.* 1. On morphisms, id sends $f : X \to Y$ to the commutative square $f : \mathrm{id}_X \to \mathrm{id}_Y$ depicted in (**??**). Since the identity of $\mathrm{id}_X \in \mathbf{C}_0^{\to}$ is $\mathrm{id}_X : \mathrm{id}_X \to \mathrm{id}_X$ and the composition of commutative squares is done by composing the left part and right part independently, we conclude that $\mathrm{id}(f \circ g) = f \circ g = \mathrm{id}(f) \circ \mathrm{id}(g)$. Thus, id is a functor.

2. On morphisms, $s$ sends a commutative square $\phi : f \to g$ to the morphism $s(f) \to s(g)$ in the square, we denote it $s(\phi)$. In other words, we send a commutative square to its left part. Again, since the composition in $\mathbf{C}^{\to}$ is done independently on the left and right part, we find that $s(\phi \circ \psi) = s(\phi) \circ s(\psi)$, thus $s$ is a functor (see (204) for a visual aid).

3. On morphisms, $t$ sends a commutative square $\phi : f \to g$ to the morphism $t(f) \to t(g)$ in the square, we denote it $t(\phi)$. With a similar argument to the second point, we conclude that $t$ is a functor.

□

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd} \tag{203}$$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "s(\psi)"'] & \bullet \arrow[d, "t(\psi)"] \\ \bullet \arrow[r, "g"] \arrow[d, "s(\phi)"'] & \bullet \arrow[d, "t(\phi)"] \\ \bullet \arrow[r, "h"'] & \bullet \end{tikzcd} \tag{204}$$

*Solution to Exercise 242.* The terminal object of $\mathbf{C}/X$ is the identity morphism $\mathrm{id}_X : X \to X$. For any object of the slice category $f : A \to X$, we have the commutative triangle (205) with $! = f$. Uniqueness of ! follows from $\mathrm{id}_X \circ ! = f \implies ! = f$.

$$\begin{tikzcd} A \arrow[rr, dashed, "f"] \arrow[dr, "f"'] & & X \arrow[dl, "\mathrm{id}_X"] \\ & X & \end{tikzcd} \tag{205}$$

The dual statement is that $\mathrm{id}_X$ is the initial object of $X/\mathbf{C}$. $\square$

## Solutions to Chapter

*Solution to Exercise 250.* ($\Rightarrow$) For any $g : Y \to Y'$, the naturality of $\phi$ yields this commutative square.

$$\begin{tikzcd} F(X,Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "F(X,g)=F(\mathrm{id}_X,g)"'] & G(X,Y) \arrow[d, "G(\mathrm{id}_X,g)=G(X,g)"] \\ F(X,Y') \arrow[r, "\phi_{X,Y'}"'] & G(X,Y') \end{tikzcd} \tag{206}$$

We conclude that $\phi_{X,-}$ is a natural transformation $F(X,-)$. A symmetric argument works for $\phi_{-,Y}$ (see (207)).

$$\begin{tikzcd} F(X,Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "F(f,\mathrm{id}_Y)"'] & G(X,Y) \arrow[d, "G(f,\mathrm{id}_Y)"] \\ F(X',Y) \arrow[r, "\phi_{X',Y}"'] & G(X',Y) \end{tikzcd} \tag{207}$$

($\Leftarrow$) For any $(f,g) : (X,Y) \to (X',Y')$, we note that, by functoriality, $F(f,g) = F(f,\mathrm{id}_{Y'}) \circ F(\mathrm{id}_X, g)$ and similarly for $G$. Thus, we can combine the naturality of $\phi_{X,-}$ and $\phi_{-,Y}$ to obtain the commutativity of $\phi_{X,Y}$ as shown in (208).

$$\begin{tikzcd} F(X,Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "F(\mathrm{id}_X,g)"] \arrow[dd, bend right=60, "F(f,g)"'] & G(X,Y) \arrow[d, "G(\mathrm{id}_X,g)"'] \arrow[dd, bend left=60, "G(f,g)"] \\ F(X,Y') \arrow[r, "\phi_{X,Y'}"] \arrow[d, "F(f,\mathrm{id}_{Y'})"] & G(X,Y') \arrow[d, "G(f,\mathrm{id}_{Y'})"'] \\ F(X',Y') \arrow[r, "\phi_{X',Y'}"'] & G(X',Y') \end{tikzcd} \tag{208}$$

$\square$

*Solution to Exercise 254.* Let $F, G : \mathbf{C} \rightsquigarrow \mathbf{D}$ be functors.

($\Rightarrow$) If $\phi : F \Rightarrow G$ is a natural isomorphism, then it has an inverse $\phi^{-1} : G \Rightarrow F$ which satisfies $\phi \cdot \phi^{-1} = \mathbb{1}_G$ and $\phi^{-1} \cdot \phi = \mathbb{1}_F$. Looking at each components, we find $\phi_X \circ (\phi^{-1})_X = \mathrm{id}_X$ and $(\phi^{-1})_X \circ \phi_X = \mathrm{id}_X$, hence they are isomorphisms.

($\Leftarrow$) Let $\phi : F \Rightarrow G$ be a natural transformation such that $\phi_X$ is an isomorphism for each $X \in \mathbf{C}_0$. We claim that the family $\phi_X^{-1}$ is the inverse of $\phi$. After we show that this family is a natural transformation $G \Rightarrow F$, the construction implies it is the inverse of $\phi$. For any $f : X \to Y \in \mathbf{C}_1$, the naturality of $\phi$ implies $\phi_Y \circ F(f) = G(f) \circ \phi_X$. Pre-composing with $\phi_X^{-1}$, we have $G(f) = \phi_Y \circ F(f) \circ \phi_X^{-1}$ and therefore

$$\phi_Y^{-1} \circ G(f) = \phi_Y^{-1} \circ \phi_Y \circ F(f) \circ \phi_X^{-1} = F(f) \circ \phi_X^{-1}$$

yields the naturality of $\phi^{-1}$. $\square$

*Solution to Exercise* **??**. We have already seen in Exercise 108 that we can take the dual of a functor $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ to obtain a functor $F^{\mathrm{op}} : \mathbf{C}^{\mathrm{op}} \rightsquigarrow \mathbf{D}^{\mathrm{op}}$. It remains to check that a natural transformation $F \Rightarrow G$ can be identified with a natural transformation $G^{\mathrm{op}} \Rightarrow F^{\mathrm{op}}$. This follows from observing that the naturality square (209) in $\mathbf{D}$ corresponds to the naturality square (210) in $\mathbf{D}^{\mathrm{op}}$.[351]

[351] i.e.: (209) commutes if and only if (210) commutes.

$$\begin{tikzcd} FX \arrow[r, "\phi_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\phi_Y"'] & GY \end{tikzcd} \qquad (209) \qquad \begin{tikzcd} G^{\mathrm{op}}Y \arrow[r, "\phi_Y"] \arrow[d, "G^{\mathrm{op}}f"'] & F^{\mathrm{op}}Y \arrow[d, "F^{\mathrm{op}}f"] \\ G^{\mathrm{op}}X \arrow[r, "\phi_X"'] & F^{\mathrm{op}}X \end{tikzcd} \qquad (210)$$ □

*Solution to Exercise 267.* On morphisms, this functor must send a pair of natural transformations $\eta : F \Rightarrow F'$ and $\phi : G \Rightarrow G'$ to a natural transformation $FG \Rightarrow F'G'$. This is exactly what horizontal composition does.

To see that horizontal composition is functorial, first note that $\mathbb{1}_F \diamond \mathbb{1}_G = \mathbb{1}_{FG}$. Next, the fact that horizontal composition commutes with composition of functors is exactly the interchange identity. □

*Solution to Exercise 279.* We need to show that $\simeq$ is reflexive, symmetric and transitive. Symmetry is trivial because the definition of $\mathbf{C} \simeq \mathbf{D}$ is symmetric. Reflexivity follows from the fact that the identity functor on any category is fully faithful and essentially surjective.

For transitivity, given the categories and functors represented in (211) with natural isomorphisms $\phi : FG \Rightarrow \mathrm{id}_{\mathbf{D}}$, $\psi : GF \Rightarrow \mathrm{id}_{\mathbf{C}}$, $\phi' : F'G' \Rightarrow \mathrm{id}_{\mathbf{E}}$ and $\psi' : G'F' \Rightarrow \mathrm{id}_{\mathbf{D}}$, we claim that the composition $G \circ G'$ is the quasi-inverse of $F' \circ F$.

$$\begin{tikzcd} \mathbf{C} \arrow[r, rightsquigarrow, bend left, "F"] & \mathbf{D} \arrow[l, rightsquigarrow, bend left, "G"] \arrow[r, rightsquigarrow, bend left, "F'"] & \mathbf{E} \arrow[l, rightsquigarrow, bend left, "G'"] \end{tikzcd} \qquad (211)$$

Since the biaction of functors preserves natural isomorphisms,[352] we have two natural isomorphisms

[352] This holds because acting on the left or right with a functor is a functor, part of this is shown in the next solution and it also follows from the previous exercise.

$$\phi' \cdot (F'\phi G') : F'FGG' \Rightarrow \mathrm{id}_{\mathbf{E}} \text{ and } \psi \cdot (G\psi' F) : GG'F'F \Rightarrow \mathrm{id}_{\mathbf{C}},$$

which shows $\mathbf{C} \simeq \mathbf{E}$. □

*Solution to Exercise 280.* We will show the following two implications

$$\begin{aligned} \forall \mathbf{D} \quad \mathbf{C} \simeq \mathbf{C}' &\implies [\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}', \mathbf{D}] \\ \forall \mathbf{C} \quad \mathbf{D} \simeq \mathbf{D}' &\implies [\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}, \mathbf{D}'] \end{aligned}$$

and infer that $\mathbf{C} \simeq \mathbf{C}'$ and $\mathbf{D} \simeq \mathbf{D}'$ implies

$$[\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}', \mathbf{D}] \simeq [\mathbf{C}', \mathbf{D}'].$$

For the first implication, let $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$ and $G : \mathbf{C}' \rightsquigarrow \mathbf{C}$ be quasi-inverses. We define the functor $(-)F : [\mathbf{C}', \mathbf{D}] \rightsquigarrow [\mathbf{C}, \mathbf{D}]$ that acts on functors by pre-composition and on natural transformations by the right action in Definition 258.[353] Similarly, we define the functor $(-)G : [\mathbf{C}, \mathbf{D}] \rightsquigarrow [\mathbf{C}', \mathbf{D}]$. We claim that $(-)F$ and $(-)G$ are quasi-inverses.

[353] i.e.: $H : \mathbf{C} \rightsquigarrow \mathbf{D}$ is mapped to $HF = H \circ F$ and $\phi : H \Rightarrow H'$ is mapped to $\phi F$. Functoriality follows from the properties of the right action.

Another way to show functoriality is to recall that $\phi F = \phi \diamond \mathbb{1}_F$ and hence $(-)F$ is the composition of the functor

$$\mathrm{id}_{[\mathbf{C}', \mathbf{D}]} \times F : [\mathbf{C}', \mathbf{D}] \times \mathbf{1} \rightsquigarrow [\mathbf{C}', \mathbf{D}] \times [\mathbf{C}, \mathbf{C}']$$

with the horizontal composition functor defined in Exercise 267.

Let $\Phi : GF \Rightarrow \mathrm{id}_{\mathbf{C}}$ be a natural isomorphism witnessing $F$ and $G$ being quasi-inverses, then $(-)\Phi$ is a natural isomorphism from $(-)GF$ to $\mathrm{id}_{[\mathbf{C}, \mathbf{D}]}$. Indeed, for any $\phi : H \Rightarrow H' \in [\mathbf{C}, \mathbf{D}]_1$, (212) commutes as the top path and bottom path are both equal to $\phi \diamond \Phi$ and $H\Phi$ is an isomorphism because $\Phi$ is and functors preserve isomorphisms.

$$\begin{tikzcd} HGF \arrow[r, "H\Phi"] \arrow[d, "\phi GF"'] & H \arrow[d, "\phi"] \\ H'GF \arrow[r, "H'\Phi"'] & H' \end{tikzcd} \tag{212}$$

We leave to you the symmetric argument showing $(-)FG \cong \mathrm{id}_{[\mathbf{C}',\mathbf{D}]}$ and the similar argument for the second implication. □

## *Solutions to Chapter*

*Solution to Exercise 291.* ($\Rightarrow$) Suppose there is a natural isomorphism $\phi : \mathrm{Hom}_{\mathbf{C}}(X, -) \Rightarrow \mathbf{1}$, then for any object $Y \in \mathbf{C}_0$, there is a bijection $\mathrm{Hom}_{\mathbf{C}}(X, Y) \cong \{\star\}$. Hence, there is a unique morphism $X \to Y$.

($\Leftarrow$) Suppose that $X$ is initial, then for any $Y \in \mathbf{C}_0$, we have an isomorphism $\phi_Y : \mathrm{Hom}_{\mathbf{C}}(X, Y) \to \mathbf{1}(Y)$ which sends the unique morphism $X \to Y$ to $\star$. We need to show this family is natural in $Y$. Let $f : Y \to Y' \in \mathbf{C}_1$, (213) clearly commutes because all sets are singletons.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{C}}(X,Y) \arrow[r, "\phi_Y"] \arrow[d, "f\circ-"'] & \mathbf{1}(Y) \arrow[d, "\mathrm{id}_{\mathbf{1}}"] \\ \mathrm{Hom}_{\mathbf{C}}(X,Y') \arrow[r, "\phi_{Y'}"'] & \mathbf{1}(Y') \end{tikzcd} \tag{213}$$

□

## *Solutions to Chapter*

*Solution to Exercise 312.* We will proceed by defining the units and counits because, as you will see, they are practically given and then we will verify they satisfy the triangle identities. We denote $(\phi_X, \phi_Y)$ for a commutative square with $\mathsf{s}(\phi_X, \phi_Y) = \phi_X$ and $\mathsf{t}(\phi_X, \phi_Y) = \phi_Y$

($\mathsf{t} \dashv \mathsf{id}$) The component of the unit at $f \in \mathbf{C}_0^{\to}$ is a commutative square from $f$ to $\mathsf{id}(\mathsf{t}(f)) = \mathrm{id}_{t(f)}$. You should convince yourself that (214) is the only such square that is guaranteed to exist no matter what $\mathbf{C}$ is, we have $\eta_f = (f, \mathrm{id}_{t(f)})$. The component of the counit at $X \in \mathbf{C}_0$ is a morphism from $t(\mathrm{id}_X) = X$ to $X$. Again, the only possible choice is $\varepsilon_X = \mathrm{id}_X$. We check in the following derivations that the triangle identities hold.

$$\begin{tikzcd} s(f) \arrow[r, "f"] \arrow[d, "f"'] & t(f) \arrow[d, "\mathrm{id}_{t(f)}"] \\ t(f) \arrow[r, "\mathrm{id}_{t(f)}"'] & t(f) \end{tikzcd} \tag{214}$$

$$\varepsilon_{\mathsf{t}(f)} \circ \mathsf{t}(\eta_f) = \mathrm{id}_{t(f)} \circ \mathrm{id}_{t(f)} = \mathrm{id}_{\mathsf{t}(f)}$$
$$\mathsf{id}(\varepsilon_X) \circ \eta_{\mathsf{id}(X)} = (\mathrm{id}_X, \mathrm{id}_X) \circ (\mathrm{id}_X, \mathrm{id}_X) = (\mathrm{id}_X, \mathrm{id}_X) = \mathrm{id}_{\mathsf{id}(X)}.$$

($\mathsf{id} \dashv \mathsf{s}$) The component of the unit at $X \in \mathbf{C}_0$ is a morphism from $X$ to $\mathsf{s}(\mathsf{id}(X)) = X$, thus $\eta_X = \mathrm{id}_X$. The component of the counit at $f \in \mathbf{C}_0^{\to}$ is a commutative square from $\mathsf{id}(\mathsf{s}(f)) = \mathrm{id}_{s(f)}$ to $f$. Again, there is only once choice: $\varepsilon_f = (\mathrm{id}_{s(f)}, f)$ depicted in (215). The following derivations show the triangle identities hold.

$$\begin{tikzcd} s(f) \arrow[r, "\mathrm{id}_{s(f)}"] \arrow[d, "\mathrm{id}_{s(f)}"'] & s(f) \arrow[d, "f"] \\ s(f) \arrow[r, "f"'] & t(f) \end{tikzcd} \tag{215}$$

$$\varepsilon_{\mathsf{id}(X)} \circ \mathsf{id}(\eta_X) = (\mathrm{id}_X, \mathrm{id}_X) \circ (\mathrm{id}_X, \mathrm{id}_X) = (\mathrm{id}_X, \mathrm{id}_X) = \mathrm{id}_{\mathsf{id}(X)}$$
$$\mathsf{s}(\varepsilon_f) \circ \eta_{\mathsf{s}(f)} = \mathrm{id}_{s(f)} \circ \mathrm{id}_{s(f)} = \mathrm{id}_{\mathsf{s}(f)}.$$

(? ⊣ t) If t has a left adjoint ?, then there is a isomorphism $\mathrm{Hom}_{\mathbf{C}^{\to}}(?X, f) \cong \mathrm{Hom}_{\mathbf{C}}(X, t(f))$ that is natural in $X$ and $f$. □

*Solution to Exercise 330.* Using Theorem 307, Theorem 324 and Proposition 325, we can obtain two chains of adjunctions.

$$\begin{tikzcd} \mathbf{C} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, shift left, "R"] \arrow[r, shift left, "\Delta^{\mathbf{J}}_{\mathbf{D}}"] \arrow[r, phantom, "\perp"] & {[\mathbf{J}, \mathbf{D}]} \arrow[l, shift left, "\lim_{\mathbf{J}}"] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C} \arrow[r, shift left, "\Delta^{\mathbf{J}}_{\mathbf{C}}"] \arrow[r, phantom, "\perp"] & {[\mathbf{J}, \mathbf{C}]} \arrow[l, shift left, "\lim_{\mathbf{J}}"] \arrow[r, shift left, "L-"] \arrow[r, phantom, "\perp"] & {[\mathbf{J}, \mathbf{D}]} \arrow[l, shift left, "R-"] \end{tikzcd}$$

Then, observing that both composite left adjoints are equal,[354] we conclude by Corollary 315 that $R\lim_{\mathbf{J}} \cong \lim_{\mathbf{J}}(R-)$. □

[354] Both $\Delta^{\mathbf{J}}_{\mathbf{D}} \circ L$ and $L\Delta^{\mathbf{J}}_{\mathbf{C}}$ send $X \in \mathbf{C}_0$ to the constant functor at $LX$.

## *Solutions to Chapter*

*Solution to Exercise* **??**. By the universal property of $\eta'$ and one of the triangle identities, $\varepsilon'_{KA}$ is the unique morphism such that $R'\varepsilon'_{KA} \circ \eta'_{R'KA} = \mathrm{id}_{R'KA}$ (see (216)).

$$\begin{tikzcd} R'KA \arrow[r, "\eta'_{R'KA}"] \arrow[dr, "\mathrm{id}_{R'KA}"'] & R'L'R'KA \arrow[d, dashed, "R'\varepsilon'_{KA}"] & & L'R'KA \arrow[d, dashed, "\varepsilon'_{KA}"] \arrow[ll, "R"'] \\ & R'KA & & KA \end{tikzcd} \tag{216}$$

We claim that $K\varepsilon_A$ also fits in the place of $\varepsilon'_{KA}$ in (216) which means they are equal by uniqueness. We need to show $R'K\varepsilon_A \circ \eta'_{R'KA} = \mathrm{id}_{R'KA}$. Recalling that $\eta' = \eta$ and $R'K = R$, we rewrite the equality as $R\varepsilon_A \circ \eta_{RA} = \mathrm{id}_{RA}$ which holds by a triangle identity. □