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# Preliminaries

Our main goal is to introduce notation and terminology so that this book is self-contained.<sup>1</sup>

We assume you are familiar with basic concepts about sets (e.g.: subsets, union, Cartesian product, cardinality, equivalence classes, quotients, etc.), functions (e.g.: injectivity, surjectivity, inverses, (pre)image, etc.), logic (e.g.: quantifiers, implication) and proofs (e.g.: you can write, read and understand proofs),<sup>2</sup> and we will not recall anything here. However, we need to have a little talk about foundations.

Several times in our coverage of category theory, we will use the term **collection** in order to avoid set-theoretical paradoxes. **Collections** are supposed to behave just like sets except that we will never consider **collections** containing other **collections**. We do not make it more formal because there are many ways to do it<sup>3</sup> and none of them are relevant to this course. However, you still need to know why we cannot use sets as is usual in all other courses.

In short, there exist **collections** of objects that cannot be sets.<sup>4</sup> In our case, we will need to talk about the **collection** of all sets and the **collection** of all groups (among others) and they cannot form sets. For the former, it is easy to see because if  $S$  is the set of all sets, then it contains all its subsets and hence  $\mathcal{P}(S) \subseteq S$ , this leads to the contradiction  $|\mathcal{P}(S)| \leq |S| < |\mathcal{P}(S)|$ .

In the rest of this chapter, we cover the necessary background that we will use in the rest of the book. It is supposed to be a quick and (unfortunately) dry overview of stuff you may or may not have seen, so we will not dwell on explanations, intuitions and motivations.<sup>5</sup> You can safely skip these sections and come back whenever you click on a word or symbol that is defined here. We hope that this will save you from several trips to Wikipedia.

## Abstract Algebra

Here we recall definitions, examples and results you may have seen in classes on abstract algebra or linear algebra.<sup>6</sup>

### Monoids

**Definition 1** (Monoid). A **monoid** is set  $M$  equipped with a binary operation  $\cdot : M \times M \rightarrow M$  called **multiplication** and an **identity** element<sup>7</sup>  $1_M$  satisfying for all

<sup>1</sup> Especially with the heavy use of the **knowledge** package, I felt it was necessary to cover enough background material in order to have the least amount of external links in the book.

<sup>2</sup> The very first things usually taught in early undergraduate mathematics courses.

<sup>3</sup> Most commonly, people use classes or Grothendieck universes. If this sticky point worries you, we suggest you keep it in the back of your mind and go read <https://arxiv.org/pdf/0810.1279.pdf> when you are a bit more comfortable with category theory.

<sup>4</sup> Famous examples include the **collection** of ordinal numbers which, by the Burali-Forti paradox, cannot be a set and the **collection** of all sets that do not contain themselves which, by the Russel paradox, cannot be a set.

<sup>5</sup> Contrarily to the other chapters of this book.

<sup>6</sup> **Monoids** are not commonly covered, but they are simpler than **groups** and we need them at one point so we present them here.

<sup>7</sup> Some authors call  $1_M$  the **unity** or the **neutral** element.

$x, y, z \in M$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \text{and} \quad 1_M \cdot x = x = x \cdot 1_M.$$

□ If it satisfies  $\forall x, y \in M, x \cdot y = y \cdot x$ ,  $M$  is a **commutative monoid**.

*Remark 2.* We will quickly drop the  $\cdot$  symbol and denote **multiplication** with plain juxtaposition (i.e.:  $xy := x \cdot y$ ) for **monoids** and other algebraic structures with a multiplication.

**Examples 3.** 1. For any set  $S$ , the set of function from  $S$  to itself form a **monoid** with the **multiplication** being composition and the **identity** being the identity map  $s \mapsto s$ .

2. The sets  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$  and  $\mathbb{R}$  equipped with the operation of addition are all **commutative monoids**.

3. For any set  $S$ , the **powerset**  $\mathcal{P}(S)$  has two simple **monoid** structures: one where the **multiplication** is  $\cup$  and the **identity** if  $\emptyset \subseteq S$  and the other where **multiplication** is  $\cap$  and the **identity** is  $S \subseteq S$ .

□ **Definition 4** (Homomorphism). Let  $M$  and  $N$  be two **monoids**, a **monoid homomorphism** from  $M$  to  $N$  is a function  $f : M \rightarrow N$  satisfying the following property:

$$f(1_M) = 1_N \quad \text{and} \quad \forall x, y \in M, f(xy) = f(x)f(y).$$

□ When  $f$  is a bijection, we call it a **monoid isomorphism**, say that  $M$  and  $N$  are **isomorphic** and denote  $M \cong N$ .

□ **Definition 5** (Submonoid). Given a **monoid**  $M$ , a **submonoid** of  $M$  is a subset  $N \subseteq M$  containing  $1_M$  that is closed under **multiplication** (i.e.:  $\forall x, y \in N, x \cdot y \in N$ ).<sup>8</sup>

□ **Definition 6** (Kernel). The **kernel** of a **homomorphism**  $f : M \rightarrow N$  is the preimage of  $1_N$ :  $\ker(f) := f^{-1}(1_N)$ . For any **homomorphism**  $f$ ,  $\ker(f)$  is a **submonoid** of  $M$ .<sup>9</sup>

**Example 7.** The inclusions  $(\mathbb{N}, +) \rightarrow (\mathbb{Z}, +) \rightarrow (\mathbb{Q}, +) \rightarrow (\mathbb{R}, +)$  are all **monoid homomorphisms** with trivial **kernel**.<sup>10</sup> This implies this is also a chain of inclusions as **submonoids**.

□ **Definition 8** (Monoid action). Let  $M$  be a **monoid** and  $S$  a set, an (left) **action** of  $M$  on  $S$  is an operation  $\star : M \times S \rightarrow S$  satisfying for all  $x, y \in M$  and  $s \in S$

$$(x \cdot y) \star s = x \star (y \star s) \quad \text{and} \quad 1_M \star s = s.$$

□ Any **monoid action** has a **permutation representation** defined to be the map

$$\sigma_\star : M \rightarrow \Sigma_S = x \mapsto (s \mapsto x \star s).$$

Conversely, a map  $\sigma : M \rightarrow \Sigma_S$  that satisfies  $\sigma(1_M) = \text{id}_S$  and  $\sigma(xy) = \sigma(x) \circ \sigma(y)$  for any  $x, y \in M$  gives rise to a **monoid action**  $\star_\sigma$  defined by  $x \star_\sigma s = \sigma(x)(s)$ .<sup>11</sup>

**Example 9.** Any **monoid**  $M$  has a canonical **left action** on itself defined by  $x \star m = xm$  for all  $x, m \in M$ .

Depending on the context, we will refer to a **monoid** either as  $M$  or  $(M, \cdot)$  or  $(M, \cdot, 1_M)$ .

<sup>8</sup> This implies  $N$  is also a **monoid** with the **multiplication** and **identity** inherited from  $M$ .

<sup>9</sup> Similarly, the image of a **homomorphism** is also a **submonoid**.

<sup>10</sup> i.e.: the **kernel** only contains the **identity**.

□ The data  $(M, S, \star)$  will also be called an  **$M$ -set** and we may refer to it abusively with  $S$ .

<sup>11</sup> These are inverse operations, i.e.:

$$\sigma_{\star_\sigma} = \sigma \quad \text{and} \quad \star_{\sigma_\star} = \star.$$

## Groups

▮ **Definition 10** (Group). A **group** is set  $G$  equipped with a binary operation  $\cdot : G \times G \rightarrow G$  called **multiplication**, an **inverse** operation  $(-)^{-1} : G \rightarrow G$  and an **identity** element  $1_G$  such that  $(G, \cdot, 1_G)$  is a **monoid** and for all  $x \in G$

$$x \cdot x^{-1} = 1_G = x^{-1} \cdot x.$$

▮ If  $(G, \cdot, 1_G)$  is a **commutative monoid**, we say that  $G$  is an **abelian group**.

**Examples 11.** 1. For any set  $S$ , the set of bijections from  $S$  to itself form a **group** with the **multiplication** being composition, the **inverse** being the set-theoretical

▮ inverse and the **identity** being the identity map  $s \mapsto s$ . We denote this **group**  $\Sigma_S$  and call it the **group** of **permutations** of  $S$ .<sup>12</sup>

2. The **monoids** on  $(\mathbb{Z}, +)$ ,  $(\mathbb{Q}, +)$  and  $(\mathbb{R}, +)$  are also **abelian groups** with the **inverse** of  $x$  being  $-x$ .

3.

▮ **Definition 12** (Homomorphism). Let  $G$  and  $H$  be two **groups**, a **group homomorphism** from  $G$  to  $H$  is a **monoid homomorphism**  $f : G \rightarrow H$ . It follows that<sup>13</sup>

$$\forall x \in G, f(x^{-1}) = f(x)^{-1}.$$

▮ When  $f$  is a bijection, we call it a **group isomorphism**, say that  $G$  and  $H$  are **isomorphic** and denote  $G \cong H$ .

▮ **Definition 13** (Subgroup). Given a **group**  $G$ , a **subgroup** of  $G$  is a **submonoid**  $H$  of  $G$  closed under taking **inverses** (i.e.:  $\forall x \in H, x^{-1} \in H$ ).<sup>14</sup>

▮ **Definition 14** (Quotient). Let  $G$  be a **group** and  $H$  a **subgroup** of  $G$ , the **quotient**  $G/H$  is the **group** whose elements are equivalence class of

▮ **Definition 15** (Kernel). The **kernel** of a **homomorphism**  $f : G \rightarrow H$  is the preimage of  $1_H$ :  $\ker(f) := f^{-1}(1_H)$ . For any **homomorphism**  $f$ ,  $\ker(f)$  is a **subgroup** of  $G$ .<sup>15</sup>

▮ **Definition 16** (Group action). Let  $G$  be a **group** and  $S$  a set, an (left) **action** of  $G$  on  $S$  is a (left) **monoid action** of  $G$  on  $S$ .

**Example 17.** Any **group**  $G$  has a canonical **left action** on itself defined by  $x \star m = xm$  for all  $x, m \in G$ .

<sup>12</sup> For  $n \in \mathbb{N}$ , we denote  $\Sigma_n$  the **group** of **permutations** of  $\{1, \dots, n\}$ .

<sup>13</sup> For this, you need to show that **inverses** are unique.

<sup>14</sup> This implies  $H$  is also a **group** with the **multiplication**, **inverse** and **identity** inherited from  $G$ .

<sup>15</sup> Similarly, the image of a **homomorphism** is also a **subgroup**.

## Rings

## Fields

## Vector Spaces

## Order Theory

In this section, we briefly cover some early definitions and results from order theory. Since this subject is not usually taught in undergraduate courses, we spend

a bit more time. In fact, we even introduce stuff we will not use later to make sure readers can get more familiar with the most important objects: **posets** and **monotone** functions.

▮ **Definition 18** (Poset). A **poset** (short for **partially ordered set**) is a pair  $(A, \leq)$  comprising a set  $A$  and a binary relation  $\leq \subseteq A \times A$  that is

- ▮ 1. **reflexive** ( $\forall x \in A, x \leq x$ ),
- ▮ 2. **transitive** ( $\forall x, y, z \in A$  if  $x \leq y$  and  $y \leq z$  then  $x \leq z$ ), and
- ▮ 3. **antisymmetric** ( $\forall x, y \in A$  if  $x \leq y$  and  $y \leq x$  then  $x = y$ ).

The relation is also called a **partial order**.<sup>16</sup>

**Examples 19.** 1. The usual non-strict orders ( $\leq$  and  $\geq$ ) on  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  are all **partial orders**. The strict orders do not satisfy **reflexivity**.

2. The divisibility relation  $|$  on  $\mathbb{N}$  satisfying  $n | m$  whenever  $n$  divides  $m$  is a **partial order**.
3. For any set  $S$ , the **powerset** of  $S$   $\mathcal{P}(S)$  is a **poset** when equipped with the  $\subseteq$  relation.
4. Any subset of a **poset** inherits a **poset** structure by restricting the **partial order**.

▮ **Definition 20** (Monotone). A function  $f : (A, \leq_A) \rightarrow (B, \leq_B)$  between **posets** is **monotone** (or **order-preserving**) if for any  $a, a' \in A$ ,  $a \leq a' \implies f(a) \leq f(a')$ .

**Example 21.** You probably already know lots of **monotone** functions, but let us give two less intuitive examples. Let  $f : S \rightarrow T$  be a function, the **image** map of  $f$ <sup>17</sup> is the function  $\mathcal{P}(S) \rightarrow \mathcal{P}(T)$  defined by  $S \supseteq X \mapsto f(X) := \{f(x) \mid x \in X\}$ . When both **powersets** are equipped with the inclusion **partial order**, the **image** map is **monotone** because  $X \subseteq X' \subseteq S$  implies  $f(X) \subseteq f(X')$ .

▮ The **preimage** map is

$$f^{-1} : \mathcal{P}(T) \rightarrow \mathcal{P}(S) = T \supseteq Y \mapsto f^{-1}(Y) := \{y \in S \mid f(y) \in Y\}.$$

It is also **order-preserving** because  $Y \subseteq Y' \subseteq T$  implies  $f^{-1}(Y) \subseteq f^{-1}(Y')$ .

**Fact 22.** The composition of **monotone** functions between **posets** is **monotone**.

▮ **Definition 23** (Dual). The **dual order**<sup>18</sup> of a **poset**  $(A, \leq)$ , denoted  $(A, \leq)^{\text{op}}$ , is the same set equipped with the converse relation  $\geq$  defined by

$$\forall x, y \in A, x \geq y \Leftrightarrow y \leq x.$$

▮ **Definition 24.** Let  $(A, \leq)$  be a **poset** and  $S \subseteq A$ , then  $a \in A$  is an **upper bound** of  $S$  if  $\forall s \in S, s \leq a$ . Moreover,  $a \in A$  is a **supremum** of  $S$ , if it is a least **upper bound**, that is,  $a$  is an **upper bound** of  $S$  and for any **upper bound**  $a'$  of  $S$ ,  $a \leq a'$ . A **supremum** of  $S$  is denoted  $\bigvee S$ , but when  $S$  contains only two elements, we use the infix notation  $s_1 \vee s_2$  and call this a **join**.

▮<sup>16</sup> If **antisymmetry** is not satisfied,  $\leq$  is called a **preorder**.

For any **monoid**  $M$ , there are three **preorders** defined by the so-called Green's relations:

$$\begin{aligned} \forall x, y \in M, x \leq_L y &\Leftrightarrow \exists m \in M, x = my \\ \forall x, y \in M, x \leq_R y &\Leftrightarrow \exists m \in M, x = ym \\ \forall x, y \in M, x \leq_J y &\Leftrightarrow \exists m, m' \in M, x = mym' \end{aligned}$$

<sup>17</sup> Which we abusively denote  $f$ .

<sup>18</sup> This definition lets us avoid many symmetric arguments.

⌈ A **lower bound** (resp. **infimum/meet**) of  $S$  is an **upper bound** (resp. **supremum/join**) of  $S$  in the **dual order**  $(A, \leq)^{\text{op}}$ .<sup>19</sup> An **infimum** of  $S$  is denoted  $\bigwedge S$  or  $s_1 \wedge s_2$  in the binary case.

**Proposition 25.** *Infimums and supremums are unique when they exist.*<sup>20</sup>

⌈ **Definition 26.** A **complete lattice** comprises the data  $(L, \wedge, \vee, \leq)$  where  $(L, \leq)$  is a **poset**, and  $\wedge, \vee : (\mathcal{P}(L), \subseteq) \rightarrow (L, \leq)$  are respectively **infimum** and **supremum** as defined above.<sup>21</sup> Observe that  $L$  has a smallest element  $\bigvee \emptyset$  and a largest element  $\bigwedge \emptyset$  (they are usually called **top** and **bottom** respectively).

**Examples 27.** 1. For any set  $S$ ,  $(\mathcal{P}(S), \subseteq)$  is a **complete lattice**: the **supremum** of a family of subsets is their union and the **infimum** is their intersection.

2. Defining **supremums** and **infimums** on the **poset**  $(\mathbb{N}, |)$  is subtle. When  $S \subseteq \mathbb{N}$  is non-empty,  $\bigwedge S$  is the greatest common divisor of all elements in  $S$  and  $\bigwedge \emptyset$  is 0 because any integer divides 0. For a finite and non-empty  $S \subseteq \mathbb{N}$ ,  $\bigvee S$  is the least common multiple of all elements in  $S$ . If  $S$  is infinite, then  $\bigvee S$  is 0 and the **supremum** of the empty set is 1 because 1 divides any integer.

You might be wondering about possible **posets** where all **infimums** exist but not necessarily all **supremums** or vice-versa, it turns out that this is not possible as shown below.

**Lemma 28.** *Let  $(L, \leq)$  be a **poset**, then the following are equivalent:*

- (i)  $(L, \wedge, \vee, \leq)$  is a **complete lattice**.
- (ii) Any  $S \subseteq L$  has a **supremum**.
- (iii) Any  $S \subseteq L$  has an **infimum**.

*Proof.* (i)  $\implies$  (ii), (i)  $\implies$  (iii) and (ii) + (iii)  $\implies$  (i) are all trivial. Also, by using duality, we only need to prove (ii)  $\implies$  (iii). For that, it suffices to note that for any  $S \subseteq L$ ,  $\bigwedge S = \bigvee \{a \in L \mid \forall s \in S, a \leq s\}$  is a suitable definition of the **infimum**.

Defined that way,  $\bigwedge S$  is a **lower bound** of  $S$  because if  $s < \bigwedge S$ , then  $s < a$  for some **lower bound**  $a$  of  $S$ <sup>22</sup>, in particular  $s \notin S$ . Additionally, since we are taking the **supremum** over all **lower bounds** of  $S$ , no **lower bound** of  $S$  can be greater and we conclude that  $\bigwedge S$  is indeed the **infimum** of  $S$ .  $\square$

⌈ **Definition 29** (Fixpoints). Let  $f : (L, \leq) \rightarrow (L, \leq)$ , a **pre-fixpoint** of  $L$  is an element  $x \in L$  such that  $f(x) \leq x$ . A **post-fixpoint** is an element  $x \in L$  such that  $x \leq f(x)$ .  
⌈ A **fixpoint** (or **fixed point**) of  $f$  is a **pre-** and **post-fixpoint**.

**Theorem 30** (Knaester-Tarski).<sup>23</sup> *Let  $(L, \wedge, \vee, \leq)$  be a **complete lattice** and  $f : L \rightarrow L$  be **monotone**, then*

1. The least **fixpoint** of  $f$  is  $\mu f := \bigwedge \{a \in L \mid f(a) \leq a\}$ .
2. The greatest **fixpoint** of  $f$  is  $\nu f := \bigvee \{a \in L \mid a \leq f(a)\}$ .

<sup>19</sup> Explicitly,  $a \in A$  is a **lower bound** of  $S$  if  $\forall s \in S, a \leq s$ . It is an **infimum** of  $S$  if, in addition to being a **lower bound** of  $S$ , any **lower bound**  $a'$  of  $S$  satisfies  $a' \leq a$ .

<sup>20</sup> This holds by **antisymmetry**.

<sup>21</sup> Notice that, by definition, these are **monotone** maps when the domain  $\mathcal{P}(L)$  is equipped with the inclusion order. Moreover, if these functions are defined on all of  $\mathcal{P}(L)$ , all **supremums** and **infimums** exist in  $(L, \leq)$ .

<sup>22</sup> Because  $\bigwedge S$  was the least **upper bound** for **lower bounds** of  $S$ .

<sup>23</sup> This is actually a weaker version of the Knaester-Tarski theorem which states that the **fixpoints** of a **monotone**  $f$  form a **complete lattice**.

*Proof.* 1. Any **fixpoint** of  $f$  is in particular a **pre-fixpoint**, thus  $\mu f$ , being a **lower bound** of all **pre-fixpoints**, is smaller than all **fixpoints**. Moreover, because for any **pre-fixpoint**  $a \in L$ ,  $f(\mu f) \leq f(a) \leq a$ ,  $f(\mu f)$  is also a **lower bound** of the **pre-fixpoints**, so  $f(\mu f) \leq \mu f$ . We infer that  $f(f(\mu f)) \leq f(\mu f)$ , so  $f(\mu f)$  is a **pre-fixpoint** and  $\mu f \leq f(\mu f)$ . We conclude that  $\mu f$  is a **fixpoint** by **antisymmetry**.

2. Any **fixpoint** of  $f$  is in particular a **post-fixpoint**, thus  $\nu f$ , being an **upper bound** of **post-fixpoints**, is bigger than all **fixpoints**. Moreover, because for any **post-fixpoint**  $a \in L$ ,  $a \leq f(a) \leq f(\nu f)$ ,  $f(\nu f)$  is an **upper bound** of the **post-fixpoints**, so  $\nu f \leq f(\nu f)$ . We infer that  $f(\nu f) \leq f(f(\nu f))$ , so  $f(\nu f)$  is a **post-fixpoint** and  $f(\nu f) \leq \nu f$ . We conclude that  $\nu f$  is a **fixpoint** by **antisymmetry**.  $\square$

The proof of the second item is the proof of the first item done in the **dual order**.

Definition 31. Let  $(A, \leq)$  be a **poset**, a **closure operator** on  $A$  is a map  $c : A \rightarrow A$  that is

1. **monotone**,

2. **extensive** ( $\forall x \in A, x \leq c(x)$ ), and

3. **idempotent** ( $\forall x \in A, c(x) = c(c(x))$ ).<sup>24</sup>

**Example 32.** The floor ( $\lfloor - \rfloor$ ) and ceiling ( $\lceil - \rceil$ ) operations are **closure operators** on  $(\mathbb{R}, \geq)$  and  $(\mathbb{R}, \leq)$  respectively.

Definition 33. Given two **posets**  $(A, \leq)$  and  $(B, \sqsubseteq)$ , a **Galois connection** is a pair of **monotone** functions  $l : A \rightarrow B$  and  $r : B \rightarrow A$  such that for any  $a \in A$  and  $b \in B$ ,

$$l(a) \sqsubseteq b \Leftrightarrow a \leq r(b).$$

For such a pair, we write  $l \dashv r : A \rightarrow B$ .

**Lemma 34.** Let  $l \dashv r : A \rightarrow B$  be a **Galois connection**, then  $l$  and  $r$  are **monotone**.

*Proof.* Assume towards a contradiction that  $a < a'$  and  $l(a) \not\sqsubseteq l(a')$ , then because  $l(a') \sqsubseteq l(a')$ , we infer that  $a' \leq r(l(a'))$  and thus, by transitivity,  $a \leq r(l(a'))$ . However, this contradicts the fact that  $l(a) \not\sqsubseteq l(a')$  (using the  $\Leftarrow$  of the **Galois connection**). We conclude that  $l$  is **monotone**.

A symmetric argument works to show that  $r$  is **monotone**.  $\square$

**Example 35.**

**Lemma 36.** Let  $l \dashv r : A \rightarrow B$  be a **Galois connection**, then  $r \circ l : A \rightarrow A$  is a **closure operator**.

*Proof.* Because  $r$  and  $l$  are **monotone**,  $r \circ l$  is clearly **monotone**. Also, for any  $a \in A$ ,  $l(a) \sqsubseteq l(a)$  implying  $a \leq r(l(a))$ , so  $r \circ l$  is **extensive**.

Now, in order to prove  $r \circ l$  is **idempotent**, it is enough to show that<sup>25</sup>

$$r(l(a)) \geq r(l(r(l(a)))).$$

Observe that since  $r(b) \leq r(b)$  for any  $b \in B$ , we have  $l(r(b)) \leq b$ , thus in particular, with  $b = l(a)$ , we have  $l(r(l(a))) \leq l(a)$ . Applying  $r$  which is **monotone** yields the desired inequality.  $\square$

<sup>24</sup> We will use this definition of **idempotence** in other contexts.

<sup>25</sup> The  $\leq$  inequality follows by **extensiveness**.



## Topology

In this section, we introduce the basic terminology of **topological spaces**. Again we go a bit further than needed to help readers that first learn about **topology** here. We end this section by recalling some definitions about **metric spaces**.

Definition 37. A **topological space** is a pair  $(X, \tau)$ , where  $X$  is a set and  $\tau \subseteq \mathcal{P}(X)$  is closed under arbitrary unions and finite intersections<sup>26</sup> whose elements are called **open sets** of  $X$ . We call  $\tau$  a **topology** on  $X$ .

The **complement** of an **open set**  $U$ , denoted  $U^c$ , is said to be **closed**.<sup>27</sup>

In the sequel, fix a **topological space**  $(X, \tau)$ .

Lemma 38. Let  $(C_i)_{i \in I}$  be a family of **closed sets** of  $X$ , then  $\bigcap_{i \in I} C_i$  is **closed** and if  $I$  is finite,  $\bigcup_{i \in I} C_i$  is also **closed**.<sup>28</sup>

Proof. Both statements readily follow from DeMorgan's laws and the fact that the **complement** of a **closed set** is **open** and vice-versa. For the first one, DeMorgan's laws yield

$$\bigcap_{i \in I} C_i = \left( \bigcup_{i \in I} C_i^c \right)^c,$$

and the LHS is the **complement** of a union of **opens**, so it is **closed**. For the second one, DeMorgan's laws yield

$$\bigcup_{i \in I} C_i = \left( \bigcap_{i \in I} C_i^c \right)^c,$$

and the LHS is the **complement** of a finite intersection of **opens**, so it is **closed**.  $\square$

Lemma 39. A subset  $A \subseteq X$  is **open** if and only if for any  $x \in A$ , there exists an **open**  $U \subseteq A$  such that  $x \in U$ .

Proof. ( $\Rightarrow$ ) For any  $x \in A$ , set  $U = A$ .

( $\Leftarrow$ ) For each  $x \in X$ , pick an open  $U_x \subseteq A$  such that  $x \in U_x$ , then we claim  $A = \bigcup_{x \in A} U_x$  which is **open**<sup>29</sup>. The  $\subseteq$  inclusion follows because each  $x \in A$  has a set  $U_x$  in the union that contains  $x$ . The  $\supseteq$  inclusion follows because each term of the union is a subset of  $A$  by assumption.  $\square$

Lemma 40. A subset  $A \subseteq X$  is **closed** if and only if for any  $x \notin A$ , there exists an **open**  $U$  such that,  $x \in U$  and  $U \cap A = \emptyset$ .<sup>30</sup>

Definition 41. Given  $A \subseteq X$ , the **closure** of  $A$ , denoted  $A^-$  is the intersection of all **closed sets** containing  $A$ . One can show that  $A^-$  is the smallest **closed set** containing  $A$ .<sup>31</sup> Then, it follows that  $A$  is **closed** if and only if  $A^- = A$ .

Here are more easy results on the **closure** of a subset.

Lemma 42. Given  $A, B \subseteq X$  then the following statements hold:

<sup>26</sup> For any family of **open sets**  $\{U_i\}_{i \in I} \subseteq \tau$ ,

$$\bigcup_{i \in I} U_i \in \tau,$$

and if  $I$  is finite,

$$\bigcup_{i \in I} U_i \in \tau.$$

<sup>27</sup> Observe that both the empty set and the whole **space** are **open** and **closed** (sometimes referred to as **clopen**) because

$$\emptyset = \bigcup_{U \in \emptyset} U \text{ and } X = \bigcap_{U \in \emptyset} U \text{ and } \emptyset = X^c.$$

<sup>28</sup> This lemma gives an alternative to the axioms of Definition 37. Indeed, it is sometimes more convenient to define a **topological space** by giving its **closed sets**, and you can show the axioms about **open sets** still hold.

<sup>29</sup> Arbitrary unions of **opens** are **open**.

<sup>30</sup> This result is simply a restatement of the last one by setting  $A = A^c$ .

<sup>31</sup>  $A^-$  is **closed** because it is an intersection of **closed sets** and any **closed set** containing  $A$  also contains  $A^-$  by definition.

1.  $A \subseteq B \implies A^- \subseteq B^-$
2.  $A \subseteq A^-$
3.  $A^{--} = A^-$
4.  $\emptyset^- = \emptyset$
5.  $(A \cup B)^- = A^- \cup B^-$

*Remark 43.* If we view  $\mathcal{P}(X)$  as **partial order** equipped with the inclusion relation, the previous lemma is about good properties of the function  $(-)^- : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ . Namely, we showed in the first three points that it is a **monotone**, **extensive** and **idempotent**, and therefore it is a **closure operator**.<sup>32</sup>

Definition 44. A subset  $A \subseteq X$  is said to be **dense** (in  $X$ ) if any non-empty **open set** intersects  $A$  non-trivially, that is,  $\forall \emptyset \neq U \in \tau, A \cap U \neq \emptyset$ .

**Theorem 45** (Decomposition). Let  $A \subseteq X$ , then  $A = A^- \cap (A \cup (A^-)^c)$ , where  $A^-$  is **closed** and  $A \cup (A^-)^c$  is **dense**. This results says that any subset of  $X$  can be decomposed into a **closed** and a **dense** set.

*Proof.* The equality is clear<sup>33</sup> and  $A^-$  is **closed** by definition. It is left to show that  $A \cup (A^-)^c$  is **dense**. Let  $U \neq \emptyset$  be an **open set**. If  $U$  intersects  $A$ , we are done. Otherwise, we have the following equivalences:

$$U \cap A = \emptyset \Leftrightarrow A \subseteq U^c \Leftrightarrow A^- \subseteq U^c \Leftrightarrow U \subseteq (A^-)^c,$$

where the second  $\Rightarrow$  holds because  $U^c$  is **closed**. We conclude  $U \cap (A^-)^c \neq \emptyset$ .  $\square$

**Lemma 46.** A subset  $A \subseteq X$  is **dense** if and only if  $A^- = X$ .

*Proof.*  $(\Rightarrow)$  Since  $(A^-)^c$  is **open** but it intersects trivially the **dense** set  $A$ , it must be empty, thus  $A^-$  is the whole **space**.

$(\Leftarrow)$  Let  $U$  be an **open set** such that  $U \cap A = \emptyset$ , then  $A$  is contained in the **closed set**  $U^c$ , but this implies  $A^- \subseteq U^c$ ,<sup>34</sup> thus  $U$  is empty.  $\square$

Definition 47. Let  $A \subseteq X$ , the **interior** of  $A$ , denoted  $A^\circ$  is the union of all **open sets** contained in  $A$ . Similarly to the **closure**, we can check that that  $A^\circ$  is the largest **open** subset of  $A$  and thus that  $A$  is **open** if and only if  $A = A^\circ$ .<sup>35</sup>

We end this section by presenting a largely preferred way of defining a **topology** that avoid describing all **open sets**.

Definition 48 (Base). Let  $X$  be a set, a **base**  $B$  is a set  $B \subseteq \mathcal{P}(X)$  such that  $X = \bigcup_{U \in B} U$  and any finite intersection of sets in  $B$  can be written as a union of sets in  $B$ .

**Lemma 49.** Let  $X$  and  $B \subseteq \mathcal{P}(X)$ . If  $\tau$  is the set of all unions of sets in  $B$ , then it is a **topology** on  $X$ . We say that  $\tau$  is the **topology generated** by  $B$ .

*Proof of Lemma 42.* 1. By definition,  $B^-$  contains  $B$ , thus  $A$ , but  $B^-$  is **closed**, so it must contain  $A^-$ .

2. By definition.

3.  $A^-$  is **closed**, so its **closure** is itself.

4. 3 applied to  $\emptyset$ .

5.  $\subseteq$  follows because the LHS is the smallest **closed set** containing  $A \cup B$  and the RHS is **closed** and contains  $A \cup B$ .

$\supseteq$ : Since the RHS is **closed**, we have  $(A^- \cup B^-)^- = A^- \cup B^-$  implying that the RHS is the smallest **closed set** containing  $A^- \cup B^-$ . Then, since the LHS is a **closed set** containing  $A$  and  $B$ , it contains  $A^-$  and  $B^-$  and hence must contain the RHS.  $\square$

<sup>32</sup> In fact, this is where the terminology comes from.

<sup>33</sup> We use (in this order) distributivity of  $\cap$  over  $\cup$ , the fact that a set and its **complement** intersect trivially and the inclusion  $A \subseteq A^-$ :

$$\begin{aligned} A^- \cap (A \cup (A^-)^c) &= (A^- \cap A) \cup (A^- \cap (A^-)^c) \\ &= A \cup \emptyset \\ &= A \end{aligned}$$

<sup>34</sup> Recall that the **closure** of  $A$  is the smallest **closed set** containing  $A$ .

<sup>35</sup> It also follows that  $A \subseteq B \implies A^\circ \subseteq B^\circ$  and that  $A^{\circ\circ} = A^\circ$ .

*Proof.* By assumption, we know that unions of **opens** are **open** and finite intersections of sets in  $B$  are **open**. It remains to show that finite intersections of unions of sets in  $B$  are also **open**. Let  $U = \cup_{i \in I} U_i$  and  $V = \cup_{j \in J} V_j$  with  $U_i \in B$  and  $V_j \in B$ , then by distributivity, we obtain

$$U \cap V = \cup_{i \in I} U_i \cap \cup_{j \in J} V_j = \bigcup_{i \in I, j \in J} U_i \cap V_j,$$

so  $U \cap V$  is **open**.<sup>36</sup> The lemma then follows by induction.  $\square$

<sup>36</sup> It is a union of **opens**.

In practice, instead of **generating** a **topology** from a **base**  $B$ , we start with any family  $B_0 \subseteq \mathcal{P}(X)$  and let  $B$  be its closure under finite intersections, which satisfies the axioms of a **base**. Such a  $B_0$  is often called a **subbase** for the **topology generated** by  $B$ .

Another very useful way to define **topological spaces** is to consider the **topology** induced by a **metric**.

▮ **Definition 50** (Metrics space). A **metric space**  $(X, d)$  is a set  $X$  together with a function  $d : X \times X \rightarrow \mathbb{R}$  called a **metric** with the following properties for  $x, y, z \in X$ :

1.  $d(x, y) \geq 0$
2.  $d(x, y) = 0 \Leftrightarrow x = y$
3.  $d(x, y) = d(y, x)$
4.  $d(x, y) \leq d(x, z) + d(z, y)$

▮ **Definition 51** (Non-expansive). A function between **metric spaces**  $f : (X, d_X) \rightarrow (Y, d_Y)$  is said to be **non-expansive**<sup>37</sup> if for all  $x, x' \in X$ ,

$$d_Y(f(x), f(x')) \leq d_X(x, x').$$

<sup>37</sup> Also called 1-Lipschitz or short.

**Fact 52.** The composition of any two **non-expansive** maps is **non-expansive**.

▮ **Definition 53** (Open ball). Let  $(X, d)$  be a **metric space**. Given a point  $x \in X$  and a non-negative radius  $r \in [0, \infty)$ , the **open ball** of radius  $r$  centered at  $x$  is

$$B_r(x) := \{y \in X \mid d(x, y) < r\}.$$

▮ **Definition 54** (Induced topology). Any **metric space**  $(X, d)$  has an **induced topology generated** by the set of all **open balls** of  $X$ .<sup>38</sup>

In this **topology**, a set  $S \subseteq X$  is **open** if and only if every point  $x \in S$  is contained in an **open ball** which is contained in  $S$ .<sup>39</sup>

<sup>38</sup> This **topology** is sometimes called the **open ball topology**.

<sup>39</sup> Equivalently,  $\forall x \in S, \exists r > 0, B_r(x) \subseteq S$ .

▮ **Definition 55** (Convergence). Let  $(X, d)$  be a **metric space**, a sequence  $\{p_n\}_{n \in \mathbb{N}} \subseteq X$  **converges** to  $p \in X$  if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, d(p_n, p) < \varepsilon.$$

▮ **Definition 56** (Cauchy sequence). Let  $(X, d)$  be a **metric space**, a sequence  $\{p_n\}_{n \in \mathbb{N}} \subseteq X$  is called **Cauchy** if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n \geq N \implies d(p_n, p_m) < \varepsilon.$$

▮ **Definition 57** (Completeness). A **metric space** in which every **Cauchy sequence converges** is called **complete**.

# Categories and Functors

## Categories

Definition 58 (Oriented graph). An **oriented graph**  $G$  consists of a **collection** of **nodes/objects** denoted  $G_0$  and a **collection** of **arrows/morphisms** denoted  $G_1$  along with two maps  $s, t : G_1 \rightarrow G_0$ , so that each **arrow**  $f \in G_1$  has a **source**  $s(f)$  and a **target**  $t(f)$ .

Definition 59 (Paths). A **path** in an **oriented graph**  $G$  is a sequence of **arrows**  $(f_1, \dots, f_k)$  that are **composable** in the sense that  $t(f_i) = s(f_{i+1})$  for  $i = 1, \dots, k-1$ , as drawn below in (1). The **collection** of **paths** of **length**  $k$ <sup>40</sup> in  $G$  will be denoted  $G_k$ .

$$\bullet \xrightarrow{f_k} \bullet \xrightarrow{f_{k-1}} \bullet \dots \bullet \xrightarrow{f_2} \bullet \xrightarrow{f_1} \bullet \quad (1)$$

Observe that the notation indicating the direction of the **path** does not correspond to the usual notation in graph theory. The motivation for this divergence will come shortly as the **composition** of **arrows** in a **category** is defined. The main idea is that, conceptually, **arrows** coincide more closely with functions between mathematical objects rather than **arrows** between nodes of a graph.

Definition 60 (Category). An **oriented graph**  $\mathbf{C}$  along with a **composition** map  $\circ : \mathbf{C}_2 \rightarrow \mathbf{C}_1$  is a **category** if it satisfies the following properties:

1. For any  $(f, g) \in \mathbf{C}_2$ ,  $s(f \circ g) = s(g)$  and  $t(f \circ g) = t(f)$ . This is more naturally understood visually in (2).
2. For any  $(f, g, h) \in \mathbf{C}_3$ ,  $f \circ (g \circ h) = (f \circ g) \circ h$ , namely, composition is **associative**.
3. For any **object**  $A \in \mathbf{C}_0$ , there exists an **identity** morphism  $u_{\mathbf{C}}(A) \in \mathbf{C}_1$  with  $A$  as its **source** and **target** that satisfies  $u_{\mathbf{C}}(A) \circ f = f$  and  $g \circ u_{\mathbf{C}}(A) = g$ , for any  $f, g \in \mathbf{C}_1$  where  $t(f) = A$  and  $s(g) = A$ .

Remark 61 (Notation). In general, we will denote **categories** with uppercase letters typeset with  $\mathbf{C}, \mathbf{D}, \mathbf{E}$ , etc.), their **objects** with uppercase letters ( $A, B, X, Y, Z$ , etc.) and their **morphisms** with lowercase letters ( $f, g, h$ , etc.). When the **category** is clear from the context, we denote the **identity morphism**  $\text{id}_A$  instead of  $u_{\mathbf{C}}(A)$ .

We say that two **morphisms** are **parallel** if they have the same **source** and **target**.

<sup>40</sup> The **length** of a **path** is the number of **arrows** in it.

$$\bullet \xrightarrow[g]{f \circ g} \bullet \quad (2)$$

If the third property of Definition 60 is not satisfied,  $\mathbf{C}$  will be referred to as a **semicategory**. Some authors choose to explicit when a category *does* satisfy this property, qualifying it as **unital**, but this term also has other meanings, hence our preference for the first convention.

Observe that since  $\circ$  is **associative**, it induces a unique **composition** map on paths of any finite lengths, which we abusively denote  $\circ : \mathbf{C}_k \rightarrow \mathbf{C}_1$ . This lets us write  $f_1 \circ f_2 \circ \dots \circ f_k$  with no parentheses. Occasionally, we will mention the **composition of a path** or the **morphism that a path composes to** to mean the image of the path under this map.

**Examples 62** (Boring examples). It is really easy to construct a **category** by drawing its underlying **oriented graph** and inferring the definition of the **composition** from it. Starting from the very simple **graph** depicted in (3), we can infer the definition of a **category** with a single **object** and its **identity morphism**. This **category** is denoted **1**, the **composition** is trivial since  $\text{id}_\bullet \circ \text{id}_\bullet = \text{id}_\bullet$ .

Similarly, we construct from the **graph** in (4) a **category** with two **objects**, their **identity morphisms** and nothing else. The **composition** is again trivial. This category will be denoted **1 + 1**.<sup>41</sup> More generally, for any **collection**  $\mathbf{C}_0$ , there is a **category**  $\mathbf{C}$  whose **collection** of **objects** is  $\mathbf{C}_0$  and whose **collection** of **morphisms** is  $\mathbf{C}_1 := \{\text{id}_X \mid X \in \mathbf{C}_0\}$ . The **composition map** is completely determined by the third property in Definition 60.<sup>42</sup>

The **graph** in (5) corresponds to the **category** with **objects**  $\{A, B\}$  and **morphisms**  $\{\text{id}_A, \text{id}_B, f\}$ .

$$\text{id}_A \curvearrowright A \xrightarrow{f} B \curvearrowleft \text{id}_B \quad (5)$$

The **composition map** is then completely determined by the properties of **identity morphisms**.<sup>43</sup> This category is denoted **2**, note however that **1 + 1**  $\neq$  **2**. Starting now, we will omit the **identity morphisms** from the diagrams (as is usual in the literature) for clarity reasons; they would hinder readability without adding information.

It is not always as straightforward to construct a **category** from an **oriented graph**. For instance, if two distinct **arrows** have the same **source** and **target**, they must be explicitly drawn and the ambiguity in the **composition** must be dealt with. The **graph** in (6) is problematic: it has two distinct **paths** of **length** two starting at the top-left corner and ending at the bottom-right corner. Since the **composition** of these **paths** can be equal to any of the two distinct **morphisms** between these corners, there is no obvious **category** corresponding to this **graph**.

Diagram (21) shows a very important example of a simple **category** that *handles* this problem.

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \quad (7)$$

It is implicitly stating that the bottom and top **paths compose to** the same **morphism**, the latter is thus absent of the diagram. This **category** is called the **commutative square** and is denoted **2 × 2**.<sup>44</sup>

**Definition 63** (Commutativity). The term **commutative** is generalized to arbitrary diagrams, it means that any two **paths** of **length** bigger than 1 which have the same

$$\bullet \quad (3)$$

$$\begin{array}{cc} \bullet_1 & \bullet_2 \end{array} \quad (4)$$

<sup>41</sup> This name will be explained in

<sup>42</sup> i.e.: for any  $X \in \mathbf{C}_0$ ,  $\text{id}_X \circ \text{id}_X = \text{id}_X$ .

<sup>43</sup> i.e.:  $f \circ \text{id}_A = f$ ,  $\text{id}_B \circ f = f$ ,  $\text{id}_A \circ \text{id}_A = \text{id}_A$  and  $\text{id}_B \circ \text{id}_B = \text{id}_B$

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & \searrow & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array} \quad (6)$$

<sup>44</sup> This notation will be explained in Definition 89.

source and target must compose to the same morphism.

*Warning 64.* Diagrams are not commutative by default. As our usage of commutative diagrams will ramp up in the following chapters,

*Remark 65 (Convention).* As you will see in this book, commutative diagrams are used quite a lot in category theory, yet there is no standard definition that everyone systematically uses.<sup>45</sup> For this reason, I decided to pick my favorite definition of commutativity which is somewhat uncommon (the constraint on the length is not usual).

<sup>45</sup> This does not really lead to many misunderstandings anyway because what is meant by a diagram is made clear from the context.

Before moving on to more interesting examples, we introduce the  $\text{Hom}$  notation.

▮ **Definition 66 (Hom).** Let  $\mathbf{C}$  be a category and  $A, B \in \mathbf{C}_0$  be objects, the collection of all morphisms going from  $A$  to  $B$  is

$$\text{Hom}_{\mathbf{C}}(A, B) := \{f \in \mathbf{C}_1 \mid s(f) = A \text{ and } t(f) = B\}.$$

This leads to an alternative way of defining the morphisms of  $\mathbf{C}$ , namely, one can describe  $\text{Hom}_{\mathbf{C}}(A, B)$  for all  $A, B \in \mathbf{C}_0$  instead of describing all of  $\mathbf{C}_1$  at once. Defining the morphisms this way also means takes care of the source and target functions implicitly.

*Remark 67 (Notation).* Some authors choose to denote the collection of morphisms between  $A$  and  $B$  with  $\mathbf{C}(A, B)$ . We prefer to use the latter notation when working with 2-categories<sup>46</sup> to highlight the fact that  $\mathbf{C}(A, B)$  has more structure. Other authors use  $\text{hom}$  with a lowercase “h”, our choice here is arbitrary.

<sup>46</sup> c.f. Definition 223.

▮ **Definition 68 (Smallness).** A category  $\mathbf{C}$  is called **small** if the collections of objects and morphisms are sets. If for all objects  $A, B \in \mathbf{C}_0$ ,  $\text{Hom}_{\mathbf{C}}(A, B)$  is a set,  $\mathbf{C}$  is said to be **locally small** and  $\text{Hom}_{\mathbf{C}}(A, B)$  is called a **hom-set**. A category that is not small can be referred to as **large**.

▮ **Example 69 (Set).** The category **Set** has the collection of sets as its objects and for any sets  $X$  and  $Y$ ,  $\text{Hom}_{\mathbf{Set}}(X, Y)$  is the set of all the functions from  $X$  to  $Y$ . The composition map is given by composition of functions, the associativity follows from the definition and the identity maps serve as the identity morphisms. This category is locally small but not small.<sup>47</sup>

<sup>47</sup> By Russel’s paradox.

**Example 70.** Let  $(X, \leq)$  be a partially ordered set, then  $X$  can be viewed as a category with elements of  $X$  as its objects. For any  $x, y \in X$ , the hom-set  $\text{Hom}_X(x, y)$  contains a single morphism if  $x \leq y$  and is empty otherwise. The identity morphisms arise from the reflexivity of  $\leq$ . Since every hom-set contains at most one element and  $\leq$  is transitive, the composition map is completely determined. Detailing this out, if  $f : x \rightarrow y$  and  $g : y \rightarrow z$  are morphisms, then we know that  $x \leq y$  and  $y \leq z$ . Thus, transitivity implies that  $x \leq z$  and there is a unique morphism  $x \rightarrow z$  so it must be  $g \circ f$ .

▮ If a category corresponds to this construction for some poset, it is called **posetal**. In (8), we depict the posetal category associated to  $(\mathbb{N}, \leq)$ . The arrows between

numbers  $n$  and  $n + k$  are omitted for  $k > 1$  as they can be inferred by the composition  $n \leq n + 1 \leq n + 2 \leq \dots \leq n + k$ .

$$\overset{0}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \dots \quad (8)$$

As a particular case of [posetal categories](#), let  $(X, \tau)$  be a [topological space](#) and note that the inclusion of [open sets](#) is a [partial order](#) on  $\tau$ . Thus  $X$  has a corresponding [posetal category](#). More explicitly, the [objects](#) are [open sets](#) and for any  $U, V \in \tau$ , the [hom-set](#)  $\text{Hom}_X(U, V)$  contains the inclusion map  $i_{UV}$  if  $U \subseteq V$  and is empty otherwise. This category will be denoted  $\mathcal{O}(X, \tau)$  or  $\mathcal{O}(X)$ .

**Example 71** (Single object categories). If a [category](#)  $\mathbf{C}$  has a single [object](#)  $*$ , then the only [morphisms](#) go from  $*$  to  $*$ . In particular,  $\mathbf{C}_1 = \text{Hom}_{\mathbf{C}}(*, *)$  and  $\mathbf{C}_2 = \mathbf{C}_1 \times \mathbf{C}_1$ . Then, the [associativity](#) of  $\circ$  and existence of  $\text{id}_*$  makes  $(\mathbf{C}_1, \circ)$  into a [monoid](#).

Conversely, a [monoid](#)  $(M, \cdot)$  can be represented by a single [object category](#)  $M$ , where  $\text{Hom}_M(*, *) = M$  and the [composition map](#) is the [monoid operation](#).

Since many algebraic structures have an [associative](#) operation with an identity element, this yields a fairly general construction. The single [object category](#) associated to a [monoid](#) or [group](#)  $G$  will be denoted  $\mathbf{B}G$  and referred to as the [delooping](#) of  $G$ .

The natural numbers can also be endowed with the [monoid](#) structure of addition, thus a particular instance of a single object [category](#) is the [delooping](#) of  $(\mathbb{N}, +)$ . Notice that this [category](#) is very different from the [posetal category](#)  $(\mathbb{N}, \leq)$ . In the former,  $\mathbb{N}$  is in correspondence with the [morphisms](#) while in the latter, it is in correspondence with the [objects](#).

A lot of simple examples of [large categories](#) arise as [subcategories](#) of [Set](#).

**Definition 72** (Subcategory). Let  $\mathbf{C}$  be a [category](#), a [category](#)  $\mathbf{C}'$  is a [subcategory](#) of  $\mathbf{C}$  if, the following properties are satisfied.

1. The [objects](#) and [morphisms](#) of  $\mathbf{C}'$  are [objects](#) and [morphisms](#) of  $\mathbf{C}$  (i.e.:  $\mathbf{C}'_0 \subseteq \mathbf{C}_0$  and  $\mathbf{C}'_1 \subseteq \mathbf{C}_1$ ).
2. The [source](#) and [target](#) maps of  $\mathbf{C}'$  are the restrictions of the [source](#) and [target](#) maps of  $\mathbf{C}$  on  $\mathbf{C}'_1$  and for every morphism  $f \in \mathbf{C}'_1$ ,  $s(f), t(f) \in \mathbf{C}'_0$ .
3. The [composition map](#) of  $\mathbf{C}'$  is the restriction of the [composition map](#) of  $\mathbf{C}$  on  $\mathbf{C}'_2$  and for any  $(f, g) \in \mathbf{C}'_2$ ,  $f \circ_{\mathbf{C}'} g = f \circ_{\mathbf{C}} g \in \mathbf{C}'_1$ .
4. The [identity morphisms](#) of [objects](#) in  $\mathbf{C}'_0$  are the [identity morphisms](#) of [objects](#) in  $\mathbf{C}_0$ , i.e.:  $u_{\mathbf{C}}(A) = u_{\mathbf{C}'}(A)$  when  $A \in \mathbf{C}'_0$ .

Intuitively, one can see  $\mathbf{C}'$  as being obtained from  $\mathbf{C}$  by removing some [objects](#) and [morphisms](#), but making sure that no [morphism](#) is left with no [source](#) or no [target](#) and that no [path](#) is left without its [composition](#).

**Exercise 73 (NOW!).** Find an example of a [category](#)  $\mathbf{C}$  and a [category](#)  $\mathbf{C}'$  that satisfy the first three conditions but not the fourth.

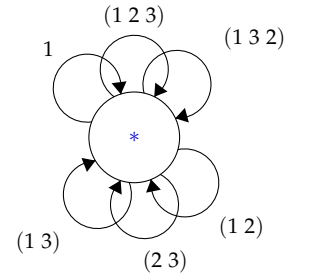


Figure 1: The [delooping](#) of the symmetric group  $S_3$ , aka  $\mathbf{B}S_3$ .

See solution.



- Definition 74 (Full and wide). A subcategory  $C'$  of  $C$  is called **full** if for any objects  $A, B \in C'_0$ ,  $\text{Hom}_{C'}(A, B) = \text{Hom}_C(A, B)$ . It is called **wide** if  $C'_0 = C_0$ .

Examples 75 (Subcategories of **Set**). One can view most of the theory studied in the first year of a typical mathematics curriculum through the lens of category theory as witnessed by the following list.

1. Since the composition of injective functions is again injective, the restriction of morphisms in **Set** to injective functions yields a **wide subcategory** of **Set**, denoted **SetInj**. Unsurprisingly, **SetSurj** can be constructed similarly.
2. Removing all infinite sets from **Set** yields the **full subcategory** of finite sets denoted **FinSet**.
- 3.

Example 76 (Concrete categories). This second list of examples contains so-called **concrete categories**, which, informally, are **categories** of sets with extra structure.<sup>48</sup>

<sup>48</sup> A formal definition is given in

1. The category **Set<sub>\*</sub>** is the **category** of **pointed** sets. Its **objects** are sets with a distinguished element and its **morphisms** are **functions** that map distinguished elements to distinguished elements. More formally, **Set<sub>\*</sub>0** is the **collection** of pairs  $(X, x)$  where  $X$  is a set and  $x \in X$ . Moreover, for any two **pointed** sets  $(X, x)$  and  $(Y, y)$ ,

$$\text{Hom}_{\text{Set}_*}((X, x), (Y, y)) = \{f : X \rightarrow Y \mid f(x) = y\}.$$

The **identity morphisms** and **composition** are defined as in **Set**, so the axioms of a **category** clearly hold after checking that if  $f : (X, x) \rightarrow (Y, y)$  satisfies  $f(x) = y$  and  $g : (Y, y) \rightarrow (Z, z)$  satisfies  $g(y) = z$ , then  $(g \circ f)(x) = z$ .

2. The category **Mon** is the **category** of **monoids** and their **homomorphisms**, let us detail the structure of **Mon**.<sup>49</sup> The **objects** are **monoids**, so **Mon<sub>0</sub>** is the **collection** of all **monoids**, and the **morphisms** are **monoid homomorphisms**, so for any  $M, N \in \text{Mon}_0$ ,  $\text{Hom}_{\text{Mon}}(M, N)$  is the set of **homomorphisms** from  $M$  to  $N$ . The **composition** in **Mon** is given by the composition of **homomorphisms**, we know it is well-defined because the composition of two **homomorphisms** is a **homomorphism**. Also, the composition is **associative** and the identity functions are **homomorphisms**, so we can define  $\mu_{\text{Mon}}(M) = \text{id}_M$ .
3. Similarly, the **category** of **groups** (resp. **rings** or **fields**) where the **morphisms** are **group** (resp. **ring** or **field**) **homomorphisms** is denoted **Grp** (resp. **Ring** or **Field**). The **category** of **abelian groups** (resp. commutative **monoids** or **rings**) is a **full subcategory** of **Grp** (resp. **Mon** or **Ring**) denoted **Ab** (resp. **CMon** or **CRing**).
4. Let  $k$  be a fixed **field**, the **category** of **vector spaces** over  $k$  where the **morphisms** are **linear maps** is denoted **Vect<sub>k</sub>**. The **full subcategory** of **Vect<sub>k</sub>** consisting only of finite dimensional **vector spaces** is denoted **FDVect<sub>k</sub>**.

<sup>49</sup> These details are essentially the same for the **categories** in the rest of Example 76.

- 5. The **category** of **partially ordered sets** where **morphisms** are **order-preserving functions** is denoted **Poset**.
- 6. The **category** of **topological spaces** where **morphisms** are **continuous functions** is denoted **Top**.

Our last example is a **large category** which is not a **subcategory** of **Set**.

- Example 77 (Rel).** The **category** of sets and relations, denoted **Rel**, has as **objects** the **collection** of all sets and for any sets  $X$  and  $Y$ ,  $\text{Hom}_{\mathbf{Rel}}(X, Y)$  is the set of relations between  $X$  and  $Y$ , that is, the powerset of  $X \times Y$ . The composition of two relations  $R \subseteq X \times Y$  and  $S \subseteq Y \times Z$  is defined by

$$S \circ R = R; S := \{(x, z) \in X \times Z \mid \exists y \in Y, (x, y) \in R, (y, z) \in S\} \subseteq X \times Z.$$

- One can check that this composition is **associative** and that, for any set  $X$ , the **diagonal relation**  $\Delta_X = \{(x, x) : x \in X\} \subseteq X \times X$  is the identity with respect to this composition.

*Remark 78.* You can view **Set** as the **subcategory** of **Rel** where you only take the relations  $R \subseteq X \times Y$  satisfying for any  $x \in X$ ,

$$|\{y \in Y \mid (x, y) \in R\}| = 1.$$

## Functors

The list above is far from exhaustive; there are many more mathematical objects that can fit in a **category** and this is a main reason for studying this subject. Indeed, **categories** encapsulate a natural structure that accurately represents the heart of several mathematical theories from a global and abstract perspective. Still, a **category** is almost never studied on its own since the abstraction it provides can make the properties of its objects more obscure. For instance, stating and proving Lagrange's theorem in the framework of **Grp** is quite more involved than in the classical way. Nevertheless, we will get to see in subsequent chapters that some surprising links can arise between seemingly unrelated subjects through the study of how different **categories** relate. The central tool for exhibiting these relations is a **functor**.

As we will show, a **functor** is a **morphism** of **categories**, thus, to motivate the definition, we can look at other **morphisms** we have encountered. A clear similarity between **categories** like **Set**, **Grp**, **Ring** or **Top** is that all the **objects** have some sort of structure that the **morphisms** preserve. Hence, we want to define a **morphism** that preserves the structure of a **category**, the latter being given by the **source** and **target** maps, the **composition** and the **identities**.

- Definition 79 (Functor).** Let  $\mathbf{C}$  and  $\mathbf{D}$  be **categories**,<sup>50</sup> a **functor**  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  is a pair of maps  $F_0 : \mathbf{C}_0 \rightarrow \mathbf{D}_0$  and  $F_1 : \mathbf{C}_1 \rightarrow \mathbf{D}_1$  such that diagrams (9), (10) and (11) commute.<sup>51</sup>

$$\begin{array}{ccccc} \mathbf{C}_0 & \xleftarrow{s} & \mathbf{C}_1 & \xrightarrow{t} & \mathbf{C}_0 \\ F_0 \downarrow & & \downarrow F_1 & & \downarrow F_0 \\ \mathbf{D}_0 & \xleftarrow{s} & \mathbf{D}_1 & \xrightarrow{t} & \mathbf{D}_0 \end{array} \quad (9)$$

<sup>50</sup> We call  $\mathbf{C}$  the **domain** of  $F$  and  $\mathbf{D}$  its **codomain**.

<sup>51</sup>  $F_2$  is induced by the definition of  $F_1$  with  $(f, g) \mapsto (F_1(f), F_1(g))$ .

$$\begin{array}{ccc} \mathbf{C}_2 & \xrightarrow{F_2} & \mathbf{D}_2 \\ \circ_{\mathbf{C}} \downarrow & & \downarrow \circ_{\mathbf{D}} \\ \mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1 \end{array} \quad (10)$$

$$\begin{array}{ccc} \mathbf{C}_0 & \xrightarrow{F_0} & \mathbf{D}_0 \\ u_{\mathbf{C}} \downarrow & & \downarrow u_{\mathbf{D}} \\ \mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1 \end{array} \quad (11)$$

*Remark 80* (Digesting diagrams). **Commutative diagrams** will be heavily employed to make clearer and more compact arguments.<sup>52</sup> However, it is an acquired skill to quickly grasp their meaning and make effective use of their advantages. Unpacking the above definition will help to understand it as well as getting better with manipulating diagrams.

A **functor**  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$ <sup>53</sup> must satisfy the following properties.

- i. For any  $A, B \in \mathbf{C}_0$  and  $f \in \text{Hom}_{\mathbf{C}}(A, B)$ ,  $F(f) \in \text{Hom}_{\mathbf{D}}(F(A), F(B))$ . This is equivalent to the **commutativity** of (9) which says  $F_0(s(f)) = s(F_1(f))$  and  $F_0(t(f)) = t(F_1(f))$ .
- ii. If  $f, g \in \mathbf{C}_1$  are **composable**, then  $F(f \circ_{\mathbf{C}} g) = F(f) \circ_{\mathbf{D}} F(g)$  by **commutativity** of (10).
- iii. If  $A \in \mathbf{C}_0$ , then  $u_{\mathbf{D}}(F(A)) = F(u_{\mathbf{C}}(A))$  (alternatively,  $\text{id}_{F(A)} = F(\text{id}_A)$ ) by **commutativity** of (11).

The subscript on  $F$  is often omitted, as is common in the literature, because it is always clear whether  $F$  is applied to an **object** or a **morphism**.

**Examples 81** (Boring examples). As usual, a few trivial constructions arise.

1. For any **category**  $\mathbf{C}$ , the **identity functor**  $\text{id}_{\mathbf{C}} : \mathbf{C} \rightsquigarrow \mathbf{C}$  is defined by letting  $(\text{id}_{\mathbf{C}})_0$  and  $(\text{id}_{\mathbf{C}})_1$  be identity maps on  $\mathbf{C}_0$  and  $\mathbf{C}_1$  respectively.
2. Let  $\mathbf{C}$  be a **category** and  $\mathbf{C}'$  a **subcategory** of  $\mathbf{C}$ , the **inclusion functor**  $\mathcal{I} : \mathbf{C}' \rightsquigarrow \mathbf{C}$  is defined by letting  $\mathcal{I}_0$  be the inclusion map  $\mathbf{C}'_0 \hookrightarrow \mathbf{C}_0$  and  $\mathcal{I}_1$  be the inclusion map  $\mathbf{C}'_1 \hookrightarrow \mathbf{C}_1$ .
3. Let  $\mathbf{C}$  and  $\mathbf{D}$  be **categories** and  $X$  be an object in  $\mathbf{D}$ , the **constant functor**  $X : \mathbf{C} \rightsquigarrow \mathbf{D}$  is defined by letting  $X_0(A) = X$  for any  $A \in \mathbf{C}_0$  and  $X_1(f) = \text{id}_X$  for any  $f \in \mathbf{C}_1$ .

**Examples 82** (Less boring). **Functors** with the domain being one of **1**, **2** or **2 × 2** (cf. Example 62) are a bit less boring. Let the codomain be a **category**  $\mathbf{C}$  and let us analyze these **functors**.

- Let  $F : \mathbf{1} \rightsquigarrow \mathbf{C}$ ,  $F_0$  assigns to the single **object**  $\bullet \in \mathbf{1}_0$  an **object**  $F(\bullet) \in \mathbf{C}_0$ . Then, by **commutativity** of (11),  $F_1$  is completely determined by  $\text{id}_{\bullet} \mapsto \text{id}_{F(\bullet)}$ . We conclude that **functors** of this type are in correspondence with **objects** of  $\mathbf{C}$ .
- Let  $F : \mathbf{2} \rightsquigarrow \mathbf{C}$ ,  $F_0$  assigns to  $A$  and  $B$ , two **objects**  $FA, FB \in \mathbf{C}_0$  and  $F_1$ 's action on **identities** is fixed. Still, there is one choice to make for  $F_1(f)$  which must be a **morphism** in  $\text{Hom}_{\mathbf{C}}(FA, FB)$ . Therefore,  $F$  sums up to a choice of two **objects** in  $\mathbf{C}$  and a **morphism** between them. In other words, **functors** of this type are in correspondence with **morphisms** in  $\mathbf{C}$ .<sup>54</sup>

<sup>52</sup> This is especially true when using a blackboard or pen and paper because it makes it easier to point at things. Sadly, I cannot point at things on this pdf you are reading.

<sup>53</sup> The  $\rightsquigarrow$  notation for **functors** is not that common, they are usually denoted with plain arrows because they are **morphisms**. Still, we feel it is useful to have a special treatment for **functors** until you get accustomed to them. The squiggly arrow notation is sometimes used for **Kleisli morphisms** which we cover in.

When the **domain** and **codomain** of a **functor** coincide, we may refer to it as an **endofunctor**.

<sup>54</sup> After picking a **morphism**, the **source** and **target** are determined.

- Similarly (we leave the details as an exercise), functors of type  $F : \mathbf{2} \times \mathbf{2} \rightsquigarrow \mathbf{C}$  are in correspondence with **commutative** squares inside the **category**  $\mathbf{C}$ .<sup>55</sup>

Remark 83 (Functoriality). We will use the term **functorial** as an adjective to qualify transformations that behave like **functors** and **functoriality** to refer to the property of behaving like a **functor**.

Throughout the rest of this book, the goal will essentially be to grow our list of **categories** and **functors** with more and more interesting examples and perhaps exploit their properties wisely. Before pursuing this objective, we give important definitions analogous to injectivity and surjectivity of functions.

**Definition 84** (Full and faithful). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  be a **functor**. For  $A, B \in \mathbf{C}_0$ , denote the restriction of  $F_1$  to  $\mathbf{Hom}_{\mathbf{C}}(A, B)$  with

$$F_{A,B} : \mathbf{Hom}_{\mathbf{C}}(A, B) \rightarrow \mathbf{Hom}_{\mathbf{D}}(F(A), F(B)).$$

- If  $F_{A,B}$  is injective for any  $A, B \in \mathbf{C}_0$ , then  $F$  is **faithful**.
- If  $F_{A,B}$  is surjective for any  $A, B \in \mathbf{C}_0$ , then  $F$  is **full**.
- If  $F_{A,B}$  is bijective for any  $A, B \in \mathbf{C}_0$ , then  $F$  is **fully faithful**.

Remark 85. These notions never mention the action of  $F$  on **objects**, so they cannot lead to a notion of **isomorphism** of **categories**.

**Examples 86.** For the following examples, we leave to the reader the easy and irrelevant to category theory task of proving they are actually functors.

1. The **powerset functor**  $\mathcal{P} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$  sends a set  $X$  to its **powerset**  $\mathcal{P}(X)$  and a function  $f : X \rightarrow Y$  to the image map  $\mathcal{P}(f) : \mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ , the latter sends a subset  $S \subseteq X$  to

$$\mathcal{P}(f)(S) = f(S) := \{f(s) \mid s \in S\} \subseteq Y.$$

The **powerset functor** is **faithful** because the same image map cannot arise from two different functions<sup>56</sup>, it is not **full** because lots of functions  $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$  are not image maps. One can also argue by cardinality because (when  $|X|, |Y| \geq 2$ )

$$|\mathbf{Hom}_{\mathbf{Set}}(X, Y)| = |Y|^{|X|} < |\mathcal{P}(Y)|^{|\mathcal{P}(X)|} = |\mathbf{Hom}_{\mathbf{Set}}(\mathcal{P}(X), \mathcal{P}(Y))|.$$

2. While in general the **inclusion functor** of a **subcategory** is not interesting,<sup>57</sup> there are some distinguished cases. For instance, when considering **subcategories** of **Set** such as those mentioned in Example 75, the **inclusion functor** gets a fancier denomination, namely, the **forgetful functor** (denoted  $U$  for underlying). This is because, morally, the **functor** is forgetting about the inner structure of the **objects** and **morphisms** and it outputs the underlying sets and functions. The **forgetful functor**  $U : \mathbf{Grp} \rightsquigarrow \mathbf{Set}$  sends a group  $(G, \cdot, 1_G)$  to its underlying set  $G$ , *forgetting about the operation and identity*. It sends a **group homomorphism**  $f : G \rightarrow H$  to the underlying function, *forgetting about the homomorphism properties*.

<sup>55</sup> i.e.: pairs of pairs of **composable** morphisms  $((f, g), (f', g')) \in \mathbf{C}_2 \times \mathbf{C}_2$  satisfying  $f \circ g = f' \circ g'$ .

<sup>56</sup> Indeed, if  $f(x) \neq g(x)$ , then  $f(\{x\}) \neq g(\{x\})$ .

<sup>57</sup> Still, one can check the **inclusion functor** is always **faithful** and it is **full** if and only if the **subcategory** is **full**.

3. It is also sometimes useful to consider *intermediate forgetful functors*. For example,  $U : \mathbf{Ring} \rightsquigarrow \mathbf{Ab}$  sends a ring  $(R, +, \cdot, 1_R, 0_R)$  to the abelian group  $(R, +, 0_R)$ , *forgetting about multiplication and  $1_R$* . It sends a *ring homomorphism*  $f : R \rightarrow S$  to the same underlying function seen as a *group homomorphism*.<sup>58</sup>
4. In some cases, there is a canonical way to go in the opposite direction of the *forgetful functor*, that is, the *free functor*. For  $\mathbf{Grp}$ , the *free functor*  $F : \mathbf{Set} \rightsquigarrow \mathbf{Grp}$  sends a set to the *free group* generated by this set and a function  $f : X \rightarrow Y$  to the unique *group homomorphism*  $F(X) \rightarrow F(Y)$  that restricts to  $f$  on the set of generators.

Later in the book, when covering *adjunctions*, we will study a strong relation between the *forgetful functor*  $U$  and the *free functor*  $F$  that will generalize to other mathematical structures.

5. Let  $(X, \leq)$  and  $(Y, \sqsubseteq)$  be *posets*, and  $F : X \rightsquigarrow Y$  be a functor between their *posetal categories*. For any  $a, b \in X$ , if  $a \leq b$ , then  $\text{Hom}_X(a, b)$  contains a single element, thus  $\text{Hom}_Y(F(a), F(b))$  must contain a *morphism* as well,<sup>59</sup> or equivalently  $F(a) \sqsubseteq F(b)$ . This shows that  $F_0$  is an *order-preserving* function on the *posets*.

Conversely, any *order-preserving* function between  $X$  and  $Y$  will correspond to a unique *functor* since there is only one *morphism* in all the *hom-sets*.<sup>60</sup>

6. Let  $G$  and  $H$  be *groups* and  $\mathbf{BG}$  and  $\mathbf{BH}$  be their respective *deloopings*, then the *functors*  $F : \mathbf{BG} \rightsquigarrow \mathbf{BH}$  are exactly the *group homomorphisms* from  $G$  to  $H$ .<sup>61</sup>
7. For any *group*  $G$ , the *functors*  $F : \mathbf{BG} \rightsquigarrow \mathbf{Set}$  are in correspondence with *left actions* of  $G$ . Indeed, if  $S = F(*)$ , then

$$F_1 : G = \text{Hom}_{\mathbf{BG}}(*, *) \rightarrow \text{Hom}_{\mathbf{Set}}(S, S)$$

is such that  $F(gh) = F(g) \circ F(h)$  for any  $g, h \in G$  and  $F(1_G) = \text{id}_S$ .<sup>62</sup> Moreover, since for any  $g \in G$ ,

$$F(g^{-1}) \circ F(g) = F(g^{-1}g) = F(1_G) = \text{id}_S = F(1_G) = F(gg^{-1}) = F(g) \circ F(g^{-1}),$$

the function  $F(g)$  is a bijection (its inverse is  $F(g^{-1})$ ) and we conclude  $F_1$  is the *permutation representation* of the *group action* defined by  $g \star s = F(g)(s)$  for all  $g \in G$  and  $s \in S$ .

8. In the previous example, replacing  $\mathbf{Set}$  with  $\mathbf{Vect}_k$ , one obtains  $k$ -linear representations of  $G$  instead of *actions* of  $G$ .<sup>63</sup>

*Remark 87.* From this long (and yet hardly exhaustive) list, one might get the feeling that every important mathematical transformation is a *functor*. This is not the case so we wanted to show where *functoriality* can fail and hopefully give you a bit of intuition about why they fail.

For instance, let us define  $F : \mathbf{FDVect}_k \rightsquigarrow \mathbf{Set}$  which assigns to any *vector space* over  $k$  a choice of basis. There is no non-trivializing way to define an action of  $F$

<sup>58</sup> It can do that because part of the requirements for *ring homomorphisms* is to preserve the underlying additive *group* structure.

<sup>59</sup> The image of the element in  $\text{Hom}_X(a, b)$  under  $F$ .

<sup>60</sup> Given  $f : (X, \leq) \rightarrow (Y, \sqsubseteq)$  *order-preserving*, the corresponding *functor* between the *posetal categories* of  $X$  and  $Y$  acts like  $f$  of the *objects* and sends a *morphism*  $a \rightarrow b$  to the unique morphism  $f(a) \rightarrow f(b)$  which exists because  $a \leq b \implies f(a) \sqsubseteq f(b)$ .

<sup>61</sup> Similarly for the *deloopings* of *monoids*.

<sup>62</sup> This is because  $gh$  is the composite of  $g$  and  $h$  in  $\mathbf{BG}$  and  $1_G$  is the *identity morphism* in  $\mathbf{BG}$ .

<sup>63</sup> You might know about linear representations, we just mention them in passing.

on linear maps which make  $F$  into a functor. Another example of a non-functor is given by the center of a group in **Grp**: a morphism of group  $H \rightarrow G$  does not necessarily send the center of  $H$  in the center of  $G$  (take for instance  $S_2 \hookrightarrow S_3$ ).

In this chapter, we introduced a novel structure, namely **categories**, that **functors** preserve.<sup>64</sup> Since we also introduced several categories where **objects** had some structure that **morphisms** preserve, it is reasonable to wonder whether **categories** are also part of a **category**. In fact, the only missing ingredient is the **composition** of **functors** (we already know what the **source** and **target** of a **functor** is and every **category** has an **identity functor**). After proving the following proposition, we end up with the **category Cat** where **objects** are **categories** and **morphisms** are **functors**. In order to avoid paradoxes of the Russel kind, it is essential to restrict **Cat** to contain only **small categories**.

□ **Proposition 88.** *Let  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  and  $G : \mathbf{D} \rightsquigarrow \mathbf{E}$  be **functors** and  $G \circ F : \mathbf{C} \rightsquigarrow \mathbf{E}$  be their **composition** defined by  $G_0 \circ F_0$  on **objects** and  $G_1 \circ F_1$  on **morphisms**. Then,  $G \circ F$  is a **functor**.*

*Proof.* One could proceed with a really hands-on proof and show that  $G \circ F$  satisfies the three necessary properties in a straightforward manner. While this should not be too hard, the proof will end up involving **objects**, **morphisms** and the **composition** from all three different **categories**. This can easily lead to confusion or worse, boredom!

Instead, we will use the diagrams we introduced in the first definition of a **functor**. From the **functoriality** of  $F$  and  $G$ , we get two sets of three diagrams and combining them yields the diagrams for  $G \circ F$ .<sup>65</sup>

$$\begin{array}{ccccc}
 \mathbf{C}_0 & \xleftarrow{s} & \mathbf{C}_1 & \xrightarrow{t} & \mathbf{C}_0 \\
 F_0 \downarrow & & \downarrow F_1 & & \downarrow F_0 \\
 \mathbf{D}_0 & \xleftarrow{s} & \mathbf{D}_1 & \xrightarrow{t} & \mathbf{D}_0 \\
 G_0 \downarrow & & \downarrow G_1 & & \downarrow G_0 \\
 \mathbf{E}_0 & \xleftarrow{s} & \mathbf{E}_1 & \xrightarrow{t} & \mathbf{E}_0
 \end{array} \quad (12)$$

$$\begin{array}{ccccc}
 \mathbf{C}_2 & \xrightarrow{F_2} & \mathbf{D}_2 & \xrightarrow{G_2} & \mathbf{E}_2 \\
 \circ_{\mathbf{C}} \downarrow & & \downarrow \circ_{\mathbf{D}} & & \downarrow \circ_{\mathbf{E}} \\
 \mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1 & \xrightarrow{G_1} & \mathbf{E}_1
 \end{array} \quad (13)$$

$$\begin{array}{ccccc}
 \mathbf{C}_0 & \xrightarrow{F_0} & \mathbf{D}_0 & \xrightarrow{G_0} & \mathbf{E}_0 \\
 u_{\mathbf{C}} \downarrow & & \downarrow u_{\mathbf{D}} & & \downarrow u_{\mathbf{E}} \\
 \mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1 & \xrightarrow{G_1} & \mathbf{E}_1
 \end{array} \quad (14)$$

To finish the proof, you need to convince yourself that combining **commutative** diagrams in this way yields **commutative** diagrams. We proceed with a proof by example. Take diagram (14), we know the left and right square are **commutative** because  $F$  and  $G$  are functors. To show that the rectangle also **commutes**, we need to show the top path and bottom path from  $\mathbf{C}_0$  to  $\mathbf{E}_1$  compose to the same function. Here is the derivation:<sup>66</sup>

$$G_1 \circ F_1 \circ u_{\mathbf{C}} = G_1 \circ u_{\mathbf{D}} \circ F_0 \quad \text{left square commutes}$$

<sup>64</sup> In fact, we defined **functors** so that they preserve the structure of **categories**.

<sup>65</sup>

<sup>66</sup> In this case, both the diagram and the derivation are fairly simple. This will not stay true in the rest of the book, but the complexity of diagrams will grow way slower than the complexity of derivations and we will mostly omit the latter for this reason.

$$= u_E \circ G_0 \circ F_0 \quad \text{right square commutes}$$

□

Since **functors** are also a new structure, one might expect that there are transformations between **functors** that preserve it. It is indeed the case, they are called **natural transformations** and they are the main subject of Chapter ?? . Moreover, although we will not cover it, there is a whole tower of abstraction that one could build in this way and it is the subject of study of higher category theory.

## Products

There is one last thing we want to mention to end this chapter. We have defined two new mathematical objects, **categories** and **functors** and presented several examples of each. By defining products, we give you access to an unlimited amount of new **categories** and **functors** you can construct from known ones.<sup>67</sup>

▮ **Definition 89** (Product categories). Let  $\mathbf{C}$  and  $\mathbf{D}$  be two **categories**, the **product** of  $\mathbf{C}$  and  $\mathbf{D}$ , denoted  $\mathbf{C} \times \mathbf{D}$ , is the **category** whose **objects** are pairs of **objects** in  $\mathbf{C}_0 \times \mathbf{D}_0$  and for any two pairs  $(X, Y), (X', Y') \in (\mathbf{C} \times \mathbf{D})_0$ ,<sup>68</sup>

$$\text{Hom}_{\mathbf{C} \times \mathbf{D}}((X, Y), (X', Y')) := \text{Hom}_{\mathbf{C}}(X, X') \times \text{Hom}_{\mathbf{D}}(Y, Y').$$

The **identity morphisms** and the **composition** are defined componentwise, i.e.:  $\text{id}_{(X, Y)} = (\text{id}_X, \text{id}_Y)$  and if  $(f, f') \in \mathbf{C}_2$  and  $(g, g') \in \mathbf{D}_2$  are two **composable** pairs, then  $(f, g) \circ (f', g') = (f \circ f', g \circ g')$ .<sup>69</sup>

**Exercise 90.** Show that the assignment  $\Delta_{\mathbf{C}} : \mathbf{C} \rightsquigarrow \mathbf{C} \times \mathbf{C} = X \mapsto (X, X)$  is **functorial**, i.e.: give its action on **morphisms** and show it satisfies the relevant axioms. We call  $\Delta_{\mathbf{C}}$  the **diagonal functor**.

▮ **Definition 91** (Product functor). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$  and  $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$  be two **functors**, the **product** of  $F$  and  $G$ , denoted  $F \times G$ , is defined componentwise on **objects** and **morphisms**, i.e.: for any  $(X, Y) \in (\mathbf{C} \times \mathbf{D})_0$  and  $(f, g) \in (\mathbf{C} \times \mathbf{D})_1$

$$(F \times G)(X, Y) = (FX, GY) \text{ and } (F \times G)(f, g) = (Ff, Gg).$$

Let us check this defines a **functor**.

- i. By definition of  $\mathbf{C}' \times \mathbf{D}'$ ,  $(Ff, Gg)$  is a **morphism** from  $(FX, GY)$  to  $(FX', GY')$ .
- ii. For  $(f, f') \in \mathbf{C}_2$  and  $(g, g') \in \mathbf{D}_2$ , we have

$$\begin{aligned} (F \times G)((f, g) \circ (f', g')) &= (F \times G)(f \circ f', g \circ g') \\ &= (F(f \circ f'), G(g \circ g')) \\ &= (Ff \circ Ff', Gg \circ Gg') \\ &= (Ff, Gg) \circ (Ff', Gg') \\ &= (F \times G)(f, g) \circ (F \times G)(f', g'). \end{aligned}$$

<sup>67</sup> This is akin to product of **groups**, direct sums of **vector spaces**, etc.

<sup>68</sup> Explicitly, a **morphism**  $(X, Y) \rightarrow (X', Y')$  is a pair of **morphisms**  $X \rightarrow X'$  and  $Y \rightarrow Y'$ .

<sup>69</sup> We leave you to check that this defines the **composition** of all **morphisms** in  $\mathbf{C} \times \mathbf{D}$ . Namely, if  $(f, g)$  and  $(f', g')$  are **composable**, then  $f, f'$  and  $g, g'$  are **composable**. See solution.



iii. Since  $F$  and  $G$  preserve **identity morphisms**, we have

$$(F \times G)(\text{id}_{(X,Y)}) = (F \times G)(\text{id}_X, \text{id}_Y) = (F\text{id}_X, G\text{id}_Y) = (\text{id}_{FX}, \text{id}_{GY}) = \text{id}_{(FX,GY)}.$$

**Exercise 92 (NOW!).** Let  $F : \mathbf{C} \times \mathbf{C}' \rightarrow \mathbf{D}$  be a **functor**. For  $X \in \mathbf{C}_0$ , we define  $F(X, -) : \mathbf{C}' \rightsquigarrow \mathbf{D}$  on **objects** by  $Y \mapsto F(X, Y)$  and on **morphisms** by  $g \mapsto F(\text{id}_X, g)$ . Show that  $F(X, -)$  is a **functor**. Define  $F(-, Y)$  similarly.

**Exercise 93.** Let  $F : \mathbf{C} \times \mathbf{C}' \rightarrow \mathbf{D}$  be an action defined on **objects** and **morphisms** which is not necessarily a **functor**. Show that if for any  $X \in \mathbf{C}_0$  and  $Y \in \mathbf{C}'_0$ ,  $F(X, -)$  and  $F(-, Y)$  as defined above are **functors**, then  $F$  is a **functor**. In other words, the **functoriality** of  $F$  can be proven componentwise.

In the next chapters, we will present other interesting constructions, but we will stop here for now.

See solution.

▮ We will often use  $-$  as a **placeholder** for an input so that the latter remains nameless. For instance,  $f(-, -)$  means  $f$  takes two inputs. The type of the inputs and outputs will be made clear in the context.  
See solution.



# Duality

The concept of duality is ubiquitous throughout mathematics. It can relate two perspectives of the same object as for [dual vector spaces](#), two complementary problems such as a maximization and a minimization linear program and even two seemingly unrelated fields like topology and logic (cf. Stone dualities). While this vague principle of duality is the foundation of many groundbreaking results, the duality in question here is categorical [duality](#) and it is a bit more precise.

Informally, there is nothing more to say than “Take all the diagrams in a definition/theorem, reverse the arrows and reap the benefits of the [dual](#) concept/result.” The more formal version will follow after we first exhibit the principle in action.

Recall that, intuitively, a [functor](#) is a structure preserving transformation between [categories](#). A simple example we have seen was [functors](#) between [posets](#) that were [order-preserving](#) functions. However, as a consequence, one might conclude that [order-reversing](#) functions impair the structure of a [poset](#), which feels arbitrary. The same happens between [deloopings](#) of [groups](#) because [anti-homomorphisms](#)<sup>70</sup> cannot arise as [functors](#) between such [categories](#).

There are two options to remedy this discrepancy between intuition and formalism; both have [duality](#) as a guiding principle.

<sup>70</sup> An [anti-homomorphism](#)  $f : G \rightarrow H$  is a function satisfying  $f(gg') = f(g')f(g)$  and  $f(1_G) = f(1_H)$ .

## Contravariant Functors

By modifying Definition 79 to require that  $F(f)$  goes in the opposite direction, we obtain a [contravariant functor](#). Incidentally, what we defined as a [functor](#) then is also called a [covariant functor](#).

**Definition 94** (Contravariant functor). Let  $\mathbf{C}$  and  $\mathbf{D}$  be [categories](#), a [contravariant functor](#)  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  is a pair of maps  $F_0 : \mathbf{C}_0 \rightarrow \mathbf{D}_0$  and  $F_1 : \mathbf{C}_1 \rightarrow \mathbf{D}_1$  making diagrams (15), (16) and (17) commute.<sup>71</sup>

<sup>71</sup> Where  $F'_2$  is now induced by the definition of  $F_1$  with  $(f, g) \mapsto (F_1(g), F_1(f))$ .

$$\begin{array}{ccccc} \mathbf{C}_0 & \xleftarrow{s} & \mathbf{C}_1 & \xrightarrow{t} & \mathbf{C}_0 \\ F_0 \downarrow & & \downarrow F_1 & & \downarrow F_0 \\ \mathbf{D}_0 & \xleftarrow{t} & \mathbf{D}_1 & \xrightarrow{s} & \mathbf{D}_0 \end{array} \quad (15)$$

$$\begin{array}{ccc}
\mathbf{C}_2 & \xrightarrow{F'_2} & \mathbf{D}_2 \\
\circ_{\mathbf{C}} \downarrow & & \downarrow \circ_{\mathbf{D}} \\
\mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1
\end{array} \quad (16)$$

$$\begin{array}{ccc}
\mathbf{C}_0 & \xrightarrow{F_0} & \mathbf{D}_0 \\
u_{\mathbf{C}} \downarrow & & \downarrow u_{\mathbf{D}} \\
\mathbf{C}_1 & \xrightarrow{F_1} & \mathbf{D}_1
\end{array} \quad (17)$$

In words,  $F$  must satisfy the following properties.

- i. For any  $A, B \in \mathbf{C}_0$ , if  $f \in \text{Hom}_{\mathbf{C}}(A, B)$  then  $F(f) \in \text{Hom}_{\mathbf{D}}(F(B), F(A))$ .
- ii. If  $f, g \in \mathbf{C}_1$  are **composable**, then  $F(f \circ g) = F(g) \circ F(f)$ .
- iii. If  $A \in \mathbf{C}_0$ , then  $u_{\mathbf{D}}(F(A)) = F(u_{\mathbf{C}}(A))$ .

**Examples 95.** Just like their **covariant** counterparts, **contravariant functors** are quite numerous. Here are a few simple ones, we leave you to check that they satisfy the diagrams above.

1. **Contravariant functors**  $F : (X, \leq) \rightsquigarrow (Y, \sqsubseteq)$  correspond to **order-reversing** functions between the posets  $X$  and  $Y$  while contravariant functors  $F : \mathbf{BG} \rightsquigarrow \mathbf{BH}$  correspond to **anti-homomorphisms** between the **groups**  $G$  and  $H$ .
2. The **contravariant powerset functor**  $\widehat{\mathcal{P}} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$  sends a set  $X$  to its **powerset**  $\mathcal{P}(X)$  and a function  $f : X \rightarrow Y$  to the pre-image map  $\widehat{\mathcal{P}}(f) : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ , the latter sends a subset  $S \subseteq Y$  to

$$\widehat{\mathcal{P}}(f)(S) = f^{-1}(S) := \{x \in X \mid f(x) \in S\} \subseteq X.$$

Next, there is a couple of **functors** that are key to understand the philosophy put forward by category theory.<sup>72</sup>

**Example 96 (Hom functors).** Let  $\mathbf{C}$  be a **locally small category** and  $A \in \mathbf{C}_0$  one of its **objects**.<sup>73</sup> We define the **covariant** and **contravariant Hom functors** from  $\mathbf{C}$  to **Set**.

1. The **covariant functor**  $\text{Hom}_{\mathbf{C}}(A, -) : \mathbf{C} \rightsquigarrow \mathbf{Set}$  sends an **object**  $B \in \mathbf{C}_0$  to the **hom-set**  $\text{Hom}_{\mathbf{C}}(A, B)$  and a **morphism**  $f : B \rightarrow B'$  to the function

$$\text{Hom}_{\mathbf{C}}(A, f) : \text{Hom}_{\mathbf{C}}(A, B) \rightarrow \text{Hom}_{\mathbf{C}}(A, B') = g \mapsto f \circ g.$$

⌈ This function is called **post-composition by**  $f$  and is denoted  $f \circ (-)$ .<sup>74</sup> Let us show  $\text{Hom}_{\mathbf{C}}(A, -)$  is a **covariant functor**.

- i. For any  $f \in \mathbf{C}_1$ , it is clear from the definitions that

$$\text{Hom}_{\mathbf{C}}(A, s(f)) = s(\text{Hom}_{\mathbf{C}}(A, f)) \text{ and } \text{Hom}_{\mathbf{C}}(A, t(f)) = t(\text{Hom}_{\mathbf{C}}(A, f)).$$

- ii. For any  $(f_1, f_2) \in \mathbf{C}_2$ , we claim that

$$\text{Hom}_{\mathbf{C}}(A, f_1 \circ f_2) = \text{Hom}_{\mathbf{C}}(A, f_1) \circ \text{Hom}_{\mathbf{C}}(A, f_2).$$

In the L.H.S., an element  $g \in \text{Hom}_{\mathbf{C}}(A, s(f_1 \circ f_2))$  is mapped to  $(f_1 \circ f_2) \circ g$  and in the R.H.S., an element  $g \in \text{Hom}_{\mathbf{C}}(A, s(f_2))$  is mapped to  $f_1 \circ (f_2 \circ g)$ . Since  $s(f_1 \circ f_2) = s(f_2)$  and **composition** is **associative**, we conclude that the two maps are the same.

<sup>72</sup> We will talk more about it when covering the **Yoneda lemma** in Chapter ??.

<sup>73</sup> We need **local smallness** to have **functors** into **Set**.

<sup>74</sup> Some authors denote  $f \circ (-)$  as  $f^*$ , we prefer to keep this notation for later (see **pullbacks**).

iii. For any  $B \in \mathbf{C}_0$ , the **post-composition** by  $u_{\mathbf{C}}(B)$  is defined to be the identity,<sup>75</sup> hence (11) also commutes.

<sup>75</sup> Namely, for any  $f : A \rightarrow B$ ,  $u_{\mathbf{C}}(B) \circ f = f$ .

2. The **contravariant functor**  $\text{Hom}_{\mathbf{C}}(-, A) : \mathbf{C} \rightsquigarrow \mathbf{Set}$  sends an **object**  $B \in \mathbf{C}_0$  to the **hom-set**  $\text{Hom}_{\mathbf{C}}(B, A)$  and a **morphism**  $f : B \rightarrow B'$  to the function

$$\text{Hom}_{\mathbf{C}}(f, A) : \text{Hom}_{\mathbf{C}}(B', A) \rightarrow \text{Hom}_{\mathbf{C}}(B, A) = g \mapsto g \circ f.$$

□ This function is called **pre-composition by**  $f$  and is denoted  $(-) \circ f$ .<sup>76</sup> Let us show  $\text{Hom}_{\mathbf{C}}(-, A)$  is a **contravariant functor**.

<sup>76</sup> Some authors denote  $(-) \circ f$  as  $f_*$ , we prefer to keep this notation for later (see **pushouts**).

i. For any  $f \in \mathbf{C}_1$ , it is clear from the definitions that

$$\text{Hom}_{\mathbf{C}}(s(f), A) = t(\text{Hom}_{\mathbf{C}}(f, A)) \text{ and } \text{Hom}_{\mathbf{C}}(t(f), A) = s(\text{Hom}_{\mathbf{C}}(f, A)).$$

ii. For any  $(f_1, f_2) \in \mathbf{C}_2$ , we claim that

$$\text{Hom}_{\mathbf{C}}(f_1 \circ f_2, A) = \text{Hom}_{\mathbf{C}}(f_2, A) \circ \text{Hom}_{\mathbf{C}}(f_1, A).$$

In the L.H.S., an element  $g \in \text{Hom}_{\mathbf{C}}(t(f_1 \circ f_2), A)$  is mapped to  $g \circ (f_1 \circ f_2)$  and in the R.H.S., an element  $g \in \text{Hom}_{\mathbf{C}}(t(f_1), A)$  is mapped to  $(g \circ f_1) \circ f_2$ . Since  $t(f_1 \circ f_2) = t(f_1)$  and **composition** is **associative**, we conclude that the two maps are the same.

iii. For any  $B \in \mathbf{C}_0$ , **pre-composition** by  $u_{\mathbf{C}}(B)$  is defined to be the identity,<sup>77</sup> hence (17) also commutes.

<sup>77</sup> Namely, for any  $f : B \rightarrow A$ ,  $f \circ u_{\mathbf{C}}(B) = f$ .

We will not dwell too long on **contravariant functors** as we will see right away how they can be avoided.

## Opposite Category

Another way to deal with **order-reversing** maps  $(X, \leq) \rightarrow (Y, \subseteq)$  is to consider the reverse order on  $X$  and a **covariant functor**  $(X, \geq) \rightsquigarrow (Y, \subseteq)$ . This also works for **anti-homomorphisms** by constructing the opposite **group**  $G^{\text{op}}$  in which the operation is reversed, namely  $g \cdot^{\text{op}} h = hg$ . The **opposite category** is a generalization of these constructions.

□ **Definition 97** (Opposite category). Let  $\mathbf{C}$  be a **category**, we denote the **opposite category** with  $\mathbf{C}^{\text{op}}$  and define it by<sup>78</sup>

$$\mathbf{C}_0^{\text{op}} = \mathbf{C}_0, \mathbf{C}_1^{\text{op}} = \mathbf{C}_1, s^{\text{op}} = t, t^{\text{op}} = s, u_{\mathbf{C}^{\text{op}}} = u_{\mathbf{C}}$$

with the **composition** defined by  $f^{\text{op}} \circ^{\text{op}} g^{\text{op}} = (g \circ f)^{\text{op}}$ .<sup>79</sup> This naturally leads to the following **contravariant functor**  $(-)_{\mathbf{C}}^{\text{op}} : \mathbf{C} \rightsquigarrow \mathbf{C}^{\text{op}}$  which sends an **object**  $A$  to

□  $A^{\text{op}}$  and a **morphism**  $f$  to  $f^{\text{op}}$ . It is called the **opposite functor**.

<sup>78</sup> Intuitively, we reverse the direction of all **morphisms** in  $\mathbf{C}$  and reverse the order of **composition** as well.

<sup>79</sup> Note that the  $-^{\text{op}}$  notation here is just used to distinguish elements in  $\mathbf{C}$  and  $\mathbf{C}^{\text{op}}$  but the class of **objects** and **morphisms** are the same.

With this definition, one can see **contravariant functors** as **covariant functors**. Formally, let  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  be a **contravariant functor**, we can view  $F$  as **covariant**

functor from  $\mathbf{C}^{\text{op}}$  to  $\mathbf{D}$  or from  $\mathbf{C}$  to  $\mathbf{D}^{\text{op}}$  via the compositions  $F \circ (-)_{\mathbf{C}^{\text{op}}}^{\text{op}}$  and  $(-)_{\mathbf{D}}^{\text{op}} \circ F$  respectively.

In the rest of this book, we choose to work with functors of type  $\mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$  instead of contravariant functors.<sup>80</sup>

**Examples 98.** 1. As hinted at before, the category corresponding to  $(X, \geq)$  is the opposite category of  $(X, \leq)$  and  $(\mathbf{B}G)^{\text{op}}$  is the category corresponding to the opposite group of  $G$ , i.e.:  $\mathbf{B}G^{\text{op}} = \mathbf{B}G^{\text{op}}$ .

2. We have seen that functors  $\mathbf{B}G \rightsquigarrow \mathbf{Set}$  correspond to left actions of a group  $G$ . You can check that functors  $\mathbf{B}G^{\text{op}} \rightsquigarrow \mathbf{Set}$  correspond to right actions of  $G$ .

3. The two Hom functors defined in Example 96 are now written

$$\text{Hom}_{\mathbf{C}}(A, -) : \mathbf{C} \rightsquigarrow \mathbf{Set} \text{ and } \text{Hom}_{\mathbf{C}}(-, A) : \mathbf{C}^{\text{op}} \rightsquigarrow \mathbf{Set}.$$

By Exercise 93, they can be combined into a functor  $\text{Hom}_{\mathbf{C}}(-, -)$  acting on objects as  $(A, B) \mapsto \text{Hom}_{\mathbf{C}}(A, B)$  and on morphisms as  $(f, g) \mapsto (g \circ - \circ f)$ . This will be called the **Hom bifunctor**.

**Exercise 99 (NOW!).** Let  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  be a functor, show that its dual  $F^{\text{op}}$  defined by  $A^{\text{op}} \mapsto (FA)^{\text{op}}$  on objects and  $f^{\text{op}} \mapsto (Ff)^{\text{op}}$  on morphisms is a functor  $\mathbf{C}^{\text{op}} \rightsquigarrow \mathbf{D}^{\text{op}}$ .

## Duality in Action

Let us start illustrating how duality can be useful with some simple definitions and results.

**Definition 101** (Monomorphism). Let  $\mathbf{C}$  be a category, a morphism  $f \in \mathbf{C}_1$  is said to be **monic** (or a **monomorphism**) if for any  $(f, g), (f, h) \in \mathbf{C}_2$  where  $g$  and  $h$  have the same source,  $f \circ g = f \circ h$  implies  $g = h$ . Equivalently,  $f$  is **monic** if  $g = h$  whenever the following diagram commutes.

$$\begin{array}{ccc} \bullet & \xrightarrow{g} & \bullet \\ & \searrow h & \nearrow f \\ & \bullet & \xrightarrow{\quad} \bullet \end{array} \quad (18)$$

Standard notation for a monomorphism is  $\bullet \hookrightarrow \bullet$  ( $\hookrightarrow$  is `\hookrightarrow`).

**Proposition 102.** Let  $\mathbf{C}$  be a category and  $f : A \rightarrow B$  a morphism, if there exists  $f' : B \rightarrow A$  such that  $f' \circ f = \text{id}_A$ ,<sup>81</sup> then  $f$  is a monomorphism.

*Proof.* If  $f \circ g = f \circ h$ , then  $f' \circ f \circ g = f' \circ f \circ h$  implying  $g = h$ .  $\square$

**Definition 103.** A monomorphism with a left inverse is called a **split monomorphism**.

**Proposition 103.** Let  $\mathbf{C}$  be a category and  $(f_1, f_2) \in \mathbf{C}_2$ , if  $f_1 \circ f_2$  is a monomorphism, then  $f_2$  is a monomorphism.

*Proof.* Let  $g, h \in \mathbf{C}_1$  be such that  $f_2 \circ g = f_2 \circ h$ , we readily get that  $(f_1 \circ f_2) \circ g = (f_1 \circ f_2) \circ h$ . Since  $f_1 \circ f_2$  is a monomorphism, this implies  $g = h$ .  $\square$

<sup>80</sup> We still had to introduce the notion because you might see contravariant functors in the wild.

**Remark 100.** It is sometimes useful to compose the Hom bifunctor with other functors as follows. Given two functors  $F, G : \mathbf{C} \rightsquigarrow \mathbf{D}$ , there is a functor  $\text{Hom}_{\mathbf{D}}(F-, G-) : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightsquigarrow \mathbf{D}$  action on objects by  $(X, Y) \mapsto \text{Hom}_{\mathbf{D}}(FX, GY)$  and on morphisms by  $(f, g) \mapsto Gg \circ (-) \circ Ff$ . One can check functoriality by showing

$$\text{Hom}_{\mathbf{D}}(F-, G-) = \text{Hom}_{\mathbf{D}}(-, -) \circ (F^{\text{op}} \times G).$$

<sup>81</sup> We say that  $f'$  is a **left inverse** of  $f$ .

The last two results make it obvious that **monomorphisms** are analogous to injective functions and we will see that they are exactly the same in the **category Set**, but first let us introduce the **dual** concept.

**Definition 104** (Duality). Given a definition or statement in an arbitrary category  $\mathbf{C}$ , one could view this concept inside the category  $\mathbf{C}^{\text{op}}$  and obtain a similar definition or statement where all **morphisms** and the order of **composition** are reversed, this is called the **dual** concept. For a definition or result where multiple **categories** are involved, the **dual** version is obtained by taking the **opposite** of all **categories**.

Dualizing the definition of a **monomorphism** yields an **epimorphism**.

**Definition 105** (Epimorphism). Let  $\mathbf{C}$  be a **category**, a **morphism**  $f \in \mathbf{C}_1$  is said to be **epic** (or an **epimorphism**) if for any two **morphisms**  $(g, f), (h, f) \in \mathbf{C}_2$  where  $g$  and  $h$  have the same **target**,  $g \circ f = h \circ f$  implies  $g = h$ . Equivalently,  $f$  is **epic** if  $g = h$  whenever the following diagram commutes.<sup>82</sup>

$$\bullet \xrightarrow{f} \bullet \begin{array}{c} \xrightarrow{g} \bullet \\ \xleftarrow{h} \bullet \end{array} \quad (19)$$

Standard notation for an **epimorphism** is  $\bullet \twoheadrightarrow \bullet$  (`\twoheadrightarrow`).

The **dual** versions of Propositions 102 and 103 also hold. Although translating our previous proofs to the **dual** case is straightforward, we will do the two next proofs relying on **duality** to convey the general sketch that works anytime a **dual** result needs to be proven.

**Proposition 106.** Let  $\mathbf{C}$  be a **category** and  $f : A \rightarrow B$  a **morphism**, if there exists  $f' : B \rightarrow A$  such that  $f \circ f' = \text{id}_B$ , then  $f$  is **epic**.<sup>83</sup>

*Proof.* Observe that  $f$  is **epic** in  $\mathbf{C}$  if and only if  $f^{\text{op}}$  is **monic** in  $\mathbf{C}^{\text{op}}$  (reverse the arrows in the definition).<sup>84</sup> Moreover, by definition,

$$f'^{\text{op}} \circ f^{\text{op}} = (f \circ f')^{\text{op}} = \text{id}_B^{\text{op}} = \text{id}_{B^{\text{op}}},$$

so by the result for **monomorphisms**,  $f^{\text{op}}$  is **monic** and hence  $f$  is **epic**.  $\square$

□ An **epimorphism** with a **right inverse** is called a **split epimorphism**.

**Proposition 107.** Let  $\mathbf{C}$  be a **category** and  $(f_1, f_2) \in \mathbf{C}_2$ , if  $f_1 \circ f_2$  is **epic**, then  $f_2$  is **epic**.

*Proof.* Since  $f_2^{\text{op}} \circ f_1^{\text{op}} = (f_1 \circ f_2)^{\text{op}}$  is **monic**, the result for **monomorphisms** implies  $f_2^{\text{op}}$  is **monic** and hence  $f_2$  is **epic**.  $\square$

**Example 108 (Set).**

- A function  $f : A \rightarrow B$  is a **monomorphism** in **Set** if and only if it is injective:<sup>85</sup>  
 $(\Leftarrow)$  Since  $f$  is injective, it has a left inverse, so it is monic by Proposition 102.  
 $(\Rightarrow)$  Given  $a \in A$ , let  $g_a : \mathbf{1} := \{*\} \rightarrow A$  be the function sending  $*$  to  $a$ . For any  $a_1 \neq a_2 \in A$ , the functions  $g_{a_1}$  and  $g_{a_2}$  are different, hence  $f \circ g_{a_1} \neq f \circ g_{a_2}$ . Therefore,  $f(a_1) \neq f(a_2)$  and since  $a_1$  and  $a_2$  were arbitrary,  $f$  is injective.

<sup>82</sup> Seeing the diagrams make it clearer that the concepts are **dual**.

□ <sup>83</sup> We say that  $f'$  is a **right inverse** of  $f$ .

<sup>84</sup> This is one other way to see that two concepts are dual.

<sup>85</sup> As a consequence, since all injective functions have a left inverse, all the **monomorphisms** in **Set** are **split monic**.

- A function  $f : A \rightarrow B$  is an **epimorphism** if and only if it is surjective:<sup>86</sup>  
 $(\Leftarrow)$  Since  $f$  is surjective, it has a right inverse, so it is **epic** by Proposition 106.  
 $(\Rightarrow)$  Let  $h : B \rightarrow \{0, 1\} =: \mathbf{2}$  be the constant function at 1 and  $g : B \rightarrow \mathbf{2}$  be the indicator function of  $\text{Im}(f) \subseteq B$ , namely,

$$g(x) = \begin{cases} 1 & \exists a \in A, x = f(a) \\ 0 & \text{o/w} \end{cases}.$$

It is clear that  $g \circ f = h \circ f \equiv 1$  and since  $f$  is **epic**, it implies  $g = h$ . Thus, any element of  $B$  is in the image of  $f$ , that is  $f$  is surjective.

**Example 109 (Mon).** Inside the category **Mon**, the **monomorphisms** correspond exactly to injective **homomorphisms**.

$(\Rightarrow)$  Let  $f : M \rightarrow M'$  be an injective **homomorphism** and  $g_1, g_2 : N \rightarrow M$  be two **parallel homomorphisms**. Suppose that  $f \circ g_1 = f \circ g_2$ , then for all  $x \in N$ ,  $f(g_1(x)) = f(g_2(x))$ , so by injectivity of  $f$ ,  $g_1(x) = g_2(x)$ . Therefore  $g_1 = g_2$  and since  $g_1$  and  $g_2$  were arbitrary,  $f$  is a **monomorphism**.

$(\Leftarrow)$  Let  $f : M \rightarrow M'$  be a **monomorphism**. Let  $x, y \in M$  and define  $p_x : \mathbb{N} \rightarrow M$  by  $k \mapsto x^k$  and similarly for  $p_y$ . It is easy to show that  $p_x$  and  $p_y$  are **homomorphisms**.<sup>87</sup> If  $f(x) = f(y)$ , then, by the **homomorphism** property, for all  $k \in \mathbb{N}$

$$f(p_x(k)) = f(x^k) = f(x)^k = f(y)^k = f(y^k) = f(p_y(k)).$$

In other words, we get  $f \circ p_x = f \circ p_y$ , so  $p_x = p_y$  and  $x = y$ . This direction follows.

Conversely, an **epimorphism** is not necessarily surjective. For example, the inclusion **homomorphism**  $i : \mathbb{N} \rightarrow \mathbb{Z}$  is clearly not surjective but it is an **epimorphism**. Indeed, let  $g, h : \mathbb{Z} \rightarrow M$  be two **monoid homomorphisms** satisfying  $g \circ i = h \circ i$ . In particular,  $g(n) = h(n)$  for any  $n \in \mathbb{N} \subset \mathbb{Z}$ . It remains to show that also  $g(-n) = h(-n)$ : we have

$$h(n)g(-n) = g(n)g(-n) = g(n-n) = g(0) = 1_M = h(0) = h(-n+n) = h(-n)h(n),$$

$$\text{but then } g(-n) = h(-n)h(n)g(-n) = h(-n).$$

▮ **Definition 110 (Isomorphism).** Let **C** be a **category**, a **morphism**  $f : A \rightarrow B$  is said to be an **isomorphism** if there exists a **morphism**  $f^{-1} : B \rightarrow A$  such that  $f \circ f^{-1} = \text{id}_B$  and  $f^{-1} \circ f = \text{id}_A$ .<sup>88</sup>

As you might expect from the terminology, in general, we will not distinguish between **isomorphic objects** in a **category** because all the properties we care about will hold for one if and only if it holds for the other.

**Exercise 111.** Show that **composing monic/epic/isomorphisms** yields **monic/epic/isomorphisms**.

**Remark 112.** The results shown about **monic** and **epic morphisms**<sup>89</sup> imply that any **isomorphism** is **monic** and **epic**. However, the converse is not true as witnessed by the inclusion **morphism**  $i$  described in Example 109.<sup>90</sup> If there exists an **isomorphism** between two objects  $A$  and  $B$ , then they are **isomorphic**, denoted  $A \cong B$ . **Isomorphic** objects are also **isomorphic** in the **opposite category**,<sup>91</sup> that is, the concept of **isomorphism** is **self-dual**.

<sup>86</sup> If you assume the axiom of choice, all surjective functions have a right inverse and thus all **epimorphisms** in **Set** are **split epic**.

<sup>87</sup> It follows from the definition of  $x^k$  which is  $x \cdot \dots \cdot x$ .

▮ <sup>88</sup> Then  $f^{-1}$  is called the **inverse** of  $f$ . One can check that if  $f'$  is the **left inverse** of  $f$  and  $f''$  is its **right inverse**, then  $f' = f'' = f^{-1}$ .

See solution.

<sup>89</sup> Proposition 102 and 106.

<sup>90</sup> This is not akin to the situation in **Set** because, there, all monomorphisms and epimorphisms are **split** (assuming the axiom of choice).

<sup>91</sup> Because the **left inverse** becomes the **right inverse** and vice-versa.

**Example 113 (Set).** A function  $f : X \rightarrow Y$  in **Set**<sub>1</sub> has an *inverse*  $f^{-1}$  if and only if  $f$  is bijective, thus *isomorphisms* in **Set** are bijections. As a consequence, we have  $A \cong B$  if and only if  $|A| = |B|$ .<sup>92</sup>

**Example 114 (Cat).** An *isomorphism* in **Cat** is a *functor*  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  with an *inverse*  $F^{-1} : \mathbf{D} \rightsquigarrow \mathbf{C}$ . This implies that  $F_0$  and  $F_1$  are bijections<sup>93</sup> because  $F_0^{-1}$  is the inverse of  $F_0$  and  $F_1^{-1}$  is the inverse of  $F_1$ .

Conversely, if  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  is a *functor* whose components on *objects* and *morphisms* are bijective, we can check that defining  $F^{-1} : \mathbf{D} \rightsquigarrow \mathbf{C}$  with  $F_0^{-1} := (F_0)^{-1}$  and  $F_1^{-1} = (F_1)^{-1}$  yields a *functor*. Therefore, *isomorphisms* are precisely the *fully faithful* functors which are bijective on *objects*.

**Examples 115 (Concrete categories).** 1. It is a simple exercise in an algebra class to show that *isomorphisms* in the *categories* **Mon**, **Grp**, **Ring**, **Field** and **Vect**<sub>k</sub> are the *isomorphisms* in their respective theory.

2. In **Poset**, *isomorphisms* are bijective *order-preserving* functions.

3. In **Top**, it is not enough to have a bijective *continuous* function, we need to require that it has a *continuous inverse*. Such functions are called *homeomorphisms*.

**Definition 116 (Initial object).** Let **C** be a *category*, an object  $A \in \mathbf{C}_0$  is said to be *initial* if for any  $B \in \mathbf{C}_0$ ,  $|\text{Hom}_{\mathbf{C}}(A, B)| = 1$ , namely there are no two *parallel morphisms* with *source*  $A$  and every *object* has a *morphism* coming from  $A$ . The<sup>94</sup> *initial object* of a *category*, if it exists, is denoted  $\emptyset$  and the *unique morphism* from  $\emptyset$  to  $X \in \mathbf{C}_0$  is denoted  $() : \emptyset \rightarrow X$ .

**Definition 117 (Terminal object).** Let **C** be a *category*, an object  $A \in \mathbf{C}_0$  is said to be *terminal* (or *final*) if for any  $B \in \mathbf{C}_0$ ,  $|\text{Hom}_{\mathbf{C}}(B, A)| = 1$ , namely there are no two *parallel morphisms* with *target*  $A$  and every *object* has a *morphism* going to  $A$ . The *terminal object* of a *category*, if it exists, is denoted **1** and the *unique morphism* from  $X \in \mathbf{C}_0$  into **1** is denoted  $[] : X \rightarrow \mathbf{1}$ .

*Remark 118 (Notation).* The motivation behind the notations  $\emptyset$  and **1** is given shortly, but the notations for the *morphisms* will be explained in Chapter .

An *object* is *initial* in a *category* **C** if and only if it is *terminal* in **C**<sup>op</sup>. Also, if an *object* is *initial* and *terminal*, we say it is a *zero object* and usually denote it **0**.

**Example 119 (Set).** Let  $X$  be a set, there is a unique function from the empty set into  $X$ , it is the empty function.<sup>95</sup> We infer that the emptyset is the *initial object* in **Set**, hence the notation  $\emptyset$ . For the *terminal object*, we observe that there is a unique function  $X \rightarrow \{*\}$  sending all elements of  $X$  to  $*$ , thus  $\{*\}$  is *terminal* in **Set**.

In this example, we could have chosen any singleton to show it is *terminal*. However, that choice is irrelevant to a good category theorician since, as any two singletons are *isomorphic* (because they have the same cardinality), any two *terminal objects* are *isomorphic*.

**Proposition 120.** Let **C** be a *category* and  $A, B \in \mathbf{C}_0$  be *initial*, then  $A \cong B$ .

<sup>92</sup> This is in fact the definition of cardinality.

<sup>93</sup> Note that  $F_1$  being a bijection implies that  $F$  is *fully faithful*.

<sup>94</sup> We will soon see why we can use *the* instead of *an*.

<sup>95</sup> Recall (or learn here) that a function  $f : A \rightarrow B$  is defined via subset of  $f \subseteq A \times B$  that satisfies  $\forall a \in A, \exists ! b \in B, (a, b) \in f$ . When  $A$  is empty,  $A \times B$  is empty and the unique subset of  $\emptyset \subseteq A \times B$  satisfies the condition vacuously. In passing, when  $B$  is empty but  $A$  is not, the unique subset of  $A \times B$  does not satisfy the condition.



*Proof.* Let  $f$  be the single element in  $\text{Hom}_{\mathbf{C}}(A, B)$  and  $f'$  be the single element in  $\text{Hom}_{\mathbf{C}}(B, A)$ . Since the **identity morphisms** are the only elements of  $\text{Hom}_{\mathbf{C}}(A, A)$  and  $\text{Hom}_{\mathbf{C}}(B, B)$ ,  $f' \circ f$  and  $f \circ f'$ , belonging to these sets, must be the **identities**. In other words  $f$  and  $f'$  are **inverses**, thus  $A \cong B$ .  $\square$

The **dual** result follows.

**Proposition 121.** Let  $\mathbf{C}$  be a **category** and  $A, B \in \mathbf{C}_0$  be **terminal**, then  $A \cong B$ .

Moreover, **initial** (resp. **terminal**) **objects** are unique up to *unique isomorphisms*.

**Exercise 122.** Show that in **Cat**, the **initial object** is the empty **category** (no **objects** and no **morphisms**) and the **terminal object** is **1** (hence the notation).<sup>96</sup>

**Example 123 (Grp).** Similarly to **Set**, the **trivial group** with one element is **terminal** in **Grp**. Moreover, note that there are no empty **group** (because there is no identity element), but any **group homomorphism** from  $\{1\}$  into a **group**  $G$  must send 1 to  $1_G$ , which completely determines the **homomorphism**. Therefore, the **trivial group** is also **initial** in **Grp**, it is the **zero object**.

**Examples 124.** Here are more examples of **categories** where **initial** and **terminal objects** may or may not exist.

1.  $\exists$  **terminal**,  $\nexists$  **initial**: Consider the **poset**  $(\mathbb{N}, \geq)$  represented by diagram (20). It is clear that 0 is **terminal** and no element can be **initial** because  $0 \geq x$  implies  $x = 0$ .
2.  $\nexists$  **terminal**,  $\exists$  **initial**:<sup>97</sup> The **category** **FinGrpInj** where the **objects** are finite **groups** and the **morphisms** are injective **homomorphisms** only contains an **initial object**  $\{1\}$ . Indeed, an injective **homomorphism**  $G \hookrightarrow H$  can be seen as **subgroup** of  $H$  **isomorphic** to  $G$ . The **trivial group**  $\{1\}$  can only be **isomorphic** to the **subgroup**  $\{1_H\}$  as any other element has degree more than 1, so  $\{1\}$  is **initial**. Moreover, a **group**  $G$  cannot be **terminal** as  $G \times (\mathbb{Z}/2\mathbb{Z})$  cannot be **isomorphic** to any **subgroup** of  $G$ .
3.  $\nexists$  **terminal**,  $\nexists$  **initial**: Let  $G$  be a non **trivial group**, the **delooping** of  $G$  has no **terminal** and no **initial objects**. The **category** **BG** has a single **object**  $*$  with  $\text{Hom}_{\mathbf{BG}}(*, *) = G$ , so  $*$  cannot be **initial** nor **terminal** when  $|G| > 1$ .

For a more interesting example, consider the **category** **Field**. Its underlying **oriented graph** is disconnected<sup>98</sup> because there are no **field homomorphisms** between **fields** of different **characteristic**. Therefore, **Field** has no **initial** nor **terminal objects**.

4.  $\exists$  **terminal**,  $\exists$  **initial**: Let  $X$  be a non-empty **topological space** where  $\tau$  is the collection of **open sets**.<sup>99</sup> The **category** of **open sets**  $\mathcal{O}(X)$  satisfies

$$\text{Hom}_{\mathcal{O}(X)}(U, V) = \begin{cases} \{i_{U,V}\} & U \subseteq V \\ \emptyset & U \not\subseteq V \end{cases}$$

<sup>96</sup> **Hint:** the unique functor  $\square : \mathbf{C} \rightarrow \mathbf{1}$  is the **constant functor** at the **object**  $\bullet \in \mathbf{1}_0$ .

$$\overset{0}{\bullet} \longleftarrow \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \dots \quad (20)$$

<sup>97</sup> Of course, you could take the opposite of  $(\mathbb{N}, \geq)$ , that is  $(\mathbb{N}, \leq)$ , but that is not fun.

<sup>98</sup> There are **objects** with no **morphisms** between them.

<sup>99</sup> Recall that it must contain  $\emptyset$  and  $X$ .



Since the empty set is contained in every **open set**, it is an **initial object**. Since the full set  $X$  contains every **open set**, it is a **terminal object**. No other set can be **initial** as it cannot be contained in  $\emptyset$  nor be **terminal** as it cannot contain  $X$ . Moreover, note that the two objects are not **isomorphic** because  $X \not\subseteq \emptyset$ .

**Exercise 125.** Let  $\mathbf{C}$  be a **category** with a **terminal object**  $1$ . Show any **morphism**  $f : 1 \rightarrow X$  is **monic**. State and prove the **dual** statement.

**Example 126.** For our last application of **duality** in this chapter,<sup>100</sup> let  $X$  be a set and consider the **posetal category**  $(\mathcal{P}(X), \subseteq)$ . We would like to define the union of two subsets of  $X$  in this **category**. The usual definition  $A \cup B = \{x \in X \mid x \in A \text{ or } x \in B\}$  is not suitable because the data in the **posetal category**  $\mathcal{P}(X)$  never refers to elements of  $X$ . In particular, the subsets  $A, B \subseteq X$  are simply **objects** in the **category** and it is not clear to us how we can determine what elements are in  $A$  and  $B$  with our categorical tools (**objects** and **morphisms**).

We propose another characterization of the union of  $A$  and  $B$ . First, what is obvious,  $A \cup B$  contains  $A$  and it contains  $B$ . Second,  $A \cup B$  is the smallest subset of  $X$  containing  $A$  and  $B$ . Indeed, if  $Y \subseteq X$  contains all element in  $A$  and  $B$ , then it also contains  $A \cup B$ . Using the order  $\subseteq$  (or equivalently, the **morphisms** in the **category**  $\mathcal{P}(X)$ ), we have  $A, B \subseteq A \cup B$  and  $\forall Y$  s.t.  $A, B \subseteq Y$  then  $A \cup B \subseteq Y$ .<sup>101</sup> This yields a definition of  $\cup$  within the category  $\mathcal{P}(X)$ , which means we can **dualize** it.

The **dual** of this property (reversing all inclusions) is as follows.<sup>102</sup>

$$A \sqcap B \subseteq A, B \text{ and } \forall Y \text{ s.t. } Y \subseteq A, B \text{ then } Y \subseteq A \sqcap B$$

Putting this in words,  $A \sqcap B$  is the largest subset of  $X$  which is contained in  $A$  and  $B$ . That is, of course, the intersection  $A \cap B$ . In this way, union and intersection are **dual** operations. If you search your memory for properties about union and intersection that you proved when you first learned about sets, you will find that they usually come in pairs; the first property being the **dual** of the second.

## More Vocabulary

In the next chapter, we will start heavily using diagrams, so before going further, we need to define the formal notion that we will use.<sup>103</sup>

**Definition 127 (Diagram).** A **diagram** in  $\mathbf{C}$  is a **functor**  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  where  $\mathbf{J}$  is usually a **small** or even finite **category**.

**Remark 128.** **Diagrams** are usually represented by (partially) drawing the image of  $F$ . All the **diagrams** drawn up to this point define the domain of the **functor** implicitly. For instance, when considering a **commutative** square in  $\mathbf{C}$ , what is actually considered is the image from a **functor** with codomain  $\mathbf{C}$  and domain the **category**  $2 \times 2$  represented in (??).<sup>104</sup>

**Definition 129.** Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$  be a **functor** and  $P$  a property<sup>105</sup> of **diagrams** in  $\mathbf{C}$ .

See solution.

<sup>100</sup> Don't worry, we will have plenty of opportunities to use **duality** later.

<sup>101</sup> We leave it as an exercise to show that  $A \cup B$  is the only subset of  $X$  satisfying this property.

<sup>102</sup> The symbol  $\sqcap$  is a placeholder for the operation which we will find to be **dual** to union.

<sup>103</sup> In fact, we will now refrain from referring to every picture on the page as a diagram and keep this terminology for the formal use (without necessarily making it explicitly formal).

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array} \quad (21)$$

<sup>104</sup> If  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  is a **diagram** of shape  $\mathbf{J}$  in  $\mathbf{C}$  and  $G : \mathbf{C} \rightsquigarrow \mathbf{D}$  is a **functor**, then  $G \circ F$  is a **diagram** of shape  $\mathbf{J}$  in  $\mathbf{D}$ .

<sup>105</sup> This is intentionally a vague term.

- We say that  $F$  **preserves diagrams** with property  $P$  if for any **diagram**  $D : \mathbf{J} \rightsquigarrow \mathbf{C}$ . If  $D$  has property  $P$ , then  $F \circ D$  has property  $P$ .
- We say that  $F$  **reflects diagrams** with property  $P$  if for any **diagram**  $D : \mathbf{J} \rightsquigarrow \mathbf{C}$ . If  $F \circ D$  has property  $P$ , then  $D$  has property  $P$ .

*Warning 130.* **Preserving** and **reflecting** a property  $P$  are not **dual** notions.

It follows easily from this definition that **functors preserve commutative diagrams**. The following two exercises are a quick investigation in **preservation** and **reflection** of simple properties we have seen in this chapter.

**Exercise 131.** 1. Find an example of **functor** which does not **preserve monomorphisms**.<sup>106</sup>

2. Show that if  $f \in \mathbf{C}_1$  is a **split monomorphism**, then  $F(f)$  is also a **split monomorphism**, i.e.: any **functor preserves split monomorphisms**.

3. State and prove the **dual** statement.

4. Infer that all **functors preserve isomorphisms**, in particular **functors send isomorphic objects to isomorphic objects**.

**Exercise 132.** 1. Find an example of **functor** which does not **reflect monomorphisms**.<sup>107</sup>

2. Show that if  $F$  is **faithful**, then  $F$  **reflects monomorphisms**.

3. State and prove the **dual** statement.

**Exercise 133.** Let  $\mathbf{C}$  be a **category** and  $X \in \mathbf{C}_0$ , we define the relation  $\sim$  on **monomorphisms** with **target**  $X$  by

$$m \sim m' \Leftrightarrow \exists \text{ isomorphism } i, m = m' \circ i.$$

Show that  $\sim$  is an equivalence relation.

□ **Definition 134** (Subobject). Let  $\mathbf{C}$  be a **category**, a **subobject** of  $X \in \mathbf{C}_0$  is an equivalence class of the relation  $\sim$  defined above. We will often abusively refer to a **subobject** simply with a **monomorphism**  $Y \hookrightarrow X$ . The **collection** of **subobjects** of  $X$  is denoted  $\text{Sub}_{\mathbf{C}}(X)$ . If for any  $X \in \mathbf{C}_0$ ,  $\text{Sub}_{\mathbf{C}}(X)$  is a set, we say that  $\mathbf{C}$  is **well-powered**.

**Example 135 (Set).** Let  $X \in \text{Set}_0$ , **subobjects** of  $X$  correspond to subsets of  $X$ .<sup>108</sup> Indeed, any subset  $I \subseteq X$  has an inclusion function  $i : I \hookrightarrow X$  which is injective, hence **monic**. For the other direction, note that if  $j : J \hookrightarrow X$  is in the same equivalence class as  $i$  in  $\text{Sub}_{\text{Set}}(X)$ , there is a bijection  $f : I \rightarrow J$  such that  $i = j \circ f$ , hence the image of  $j$  is  $I \subseteq X$ . We conclude that the correspondence between  $\text{Sub}_{\text{Set}}(X)$  and  $\mathcal{P}(X)$  sends  $[i]$  to the image of  $i$  and  $I \subseteq X$  to the inclusion  $i : I \hookrightarrow X$ .

**Exercise 136.** Let  $\mathbf{C}$  be a **category** and  $X \in \mathbf{C}_0$ , we define the relation  $\leq$  on  $\text{Sub}_{\mathbf{C}}(X)$  by

$$[m] \leq [m'] \Leftrightarrow \exists \text{ morphism } k, m = m' \circ k.$$

Show that  $\leq$  is a well-defined **partial order**.

See solution.

<sup>106</sup> We can see a **morphism** as a **diagram** of shape **2**. Indeed, a **functor**  $\mathbf{2} \rightsquigarrow \mathbf{C}$  amounts to a choice of a **morphism** in  $\mathbf{C}_1$ . Therefore, a **functor**  $F$  **preserves monomorphisms** if whenever  $f$  is **monic**,  $F(f)$  also is.

See solution.

<sup>107</sup> A **functor reflects monomorphisms** if whenever  $Ff$  is **monic**,  $f$  also is.

See solution.

<sup>108</sup> The notation  $\text{Sub}_{\text{Set}}(X)$  is perfect!

See solution.

We can use [duality](#) to obtain (for free) the notion of [quotient objects](#).

**Definition 137** (Quotients). Let  $\mathbf{C}$  be a [category](#) and  $X \in \mathbf{C}_0$ , there is an equivalence relation  $\sim$  on [epimorphisms](#) with [source](#)  $X$  defined by

$$q \sim q' \Leftrightarrow \exists \text{ isomorphism } i, q = i \circ q'.$$

⌈ A [quotient object](#) (or simply [quotient](#)) of  $X$  is an equivalence class of the relation  $\sim$  defined above.<sup>109</sup> The [collection](#) of [quotients](#) of  $X$  is denoted  $\text{Quot}_{\mathbf{C}}(X)$ . If for any  $X \in \mathbf{C}_0$ ,  $\text{Quot}_{\mathbf{C}}(X)$  is a set, we say that  $\mathbf{C}$  is [co-well-powered](#). There is a [partial order](#)  $\leq$  on  $\text{Quot}_{\mathbf{C}}(X)$  defined by

$$[q] \leq [q'] \Leftrightarrow \exists \text{ morphism } k, q = k \circ q'.$$

<sup>109</sup> We will often abusively refer to a [quotient](#) simply with an [epimorphism](#)  $X \rightarrow Y$ .



# Limits and Colimits

The unifying power of categorical abstraction is arguably its biggest benefit. Indeed, it is often the case that many mathematical objects or results from different fields fit under the same categorical definition or fact. In my opinion, category theory is at its peak of elegance when a complex idea becomes close to trivial when viewed categorically, and when this same view helps link together the intuitions behind many ideas throughout mathematics.

The next two chapters concern one particular instance of this power, that is, the use of [universal properties](#) to define mathematical constructions. This term is somewhat delicate to define, therefore, we postpone its definition to next chapter and for a while, we suggest the reader to try and recognize [universality](#) as the thing that all definitions of [\(co\)limits](#) have in common. This chapter will cover [limits](#) and [colimits](#) which are specific cases of [universal](#) constructions.

The first section presents several examples; each of its subsection is dedicated to one kind of [limit](#) or [colimit](#) of which a detailed example in [Set](#) is given along with a couple of interesting examples in other [categories](#). The second section gives a formal framework to talk about all the examples previously explored as well as a few general results. In the sequel,  $\mathbf{C}$  denotes a [category](#).

## Examples

### Product

Given two sets  $S$  and  $T$ , the most common construction of the Cartesian product  $S \times T$  is conceptually easy: you take all pairs of elements  $S$  and  $T$ , that is,

$$S \times T := \{(s, t) \mid s \in S, t \in T\}.$$

However, this does not have a nice categorical analog because it requires to pick out elements in  $S$  and  $T$ . If one hopes to generalize products to other [categories](#), the construction must only involve [objects](#) and [morphisms](#).

**Question 138.** What are significant functions ([morphisms](#) in [Set](#)) to consider when studying  $S \times T$ ?

*Answer.* Projection maps. They are functions  $\pi_1 : S \times T \rightarrow S$  and  $\pi_2 : S \times T \rightarrow T$ ,<sup>110</sup> but that is not enough to define the product. Indeed, there are projection maps

<sup>110</sup> The projections are defined by  $\pi_1(s, t) = s$  and  $\pi_2(s, t) = t$  for all  $(s, t) \in S \times T$ .

$\pi'_1 : S \times T \times S \rightarrow S$  and  $\pi'_2 : S \times T \times S \rightarrow T$ , but  $S \times T \times S$  is not always isomorphic to  $S \times T$ .  $\square$

**Question 139.** What is unique<sup>111</sup> about  $S \times T$  with the projections  $\pi_1$  and  $\pi_2$ ?

<sup>111</sup> Always up to isomorphism of course.

*Answer.* For one,  $\pi_1$  and  $\pi_2$  are surjective and while they are not injective, they have an invertible-like property. Namely, given  $s \in S$  and  $t \in T$ , the pair  $(s, t)$  is completely determined from  $\pi_1^{-1}(s) \cap \pi_2^{-1}(t)$ .  $\square$

Again, in order to discharge the references to specific elements, another point of view is needed. Let  $X$  be a set of *choices* of pairs, an element  $x \in X$  chooses elements in  $S$  and  $T$  via functions  $c_1 : X \rightarrow S$  and  $c_2 : X \rightarrow T$  (similar to the projections). Now, the *quasi-inverse* defined above yields a function

$$! : X \rightarrow S \times T = x \mapsto \pi^{-1}(c_1(x)) \cap \pi^{-1}(c_2(x)).$$

This function maps  $x \in X$  to an element in  $S \times T$  that makes the same choice as  $x$ , and it is the only one that does so. Categorically,  $!$  is the unique **morphism** in  $\text{Hom}_{\mathbf{C}}(X, S \times T)$  satisfying  $\pi_i \circ ! = c_i$  for  $i = 1, 2$ . Later, we will see that this property completely determines  $S \times T$ . For now, enjoy the power we gain from generalizing this idea.

**Definition 140** (Binary product). Let  $A, B \in \mathbf{C}_0$ . A (categorical) **binary product** of  $A$  and  $B$  is an **object**, denoted  $A \times B$ , along with two **morphisms**  $\pi_A : A \times B \rightarrow A$  and  $\pi_B : A \times B \rightarrow B$  called **projections** that satisfy the following **universal property**<sup>112</sup>: for every **object**  $X \in \mathbf{C}_0$  with **morphisms**  $f_A : X \rightarrow A$  and  $f_B : X \rightarrow B$ , there is a unique **morphism**  $! : X \rightarrow A \times B$  making diagram (22) **commute**.<sup>113</sup>

<sup>112</sup> Remember that the word **universal** is not yet defined, we are trying to give you an idea of what it means with these examples.

<sup>113</sup> It is common to denote  $! = (f_A, f_B)$ .

$$\begin{array}{ccccc} & & X & & \\ & f_A \swarrow & \downarrow ! & \searrow f_B & \\ A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \end{array} \quad (22)$$

**Example 141 (Set).** Cleaning up the argument above, we show that the Cartesian product  $A \times B$  with the usual projections is a **binary product** in **Set**. To show that it satisfies the **universal property**, let  $X, f_A$  and  $f_B$  be as in the definition. A function  $! : X \rightarrow A \times B$  that makes (22) **commute** must satisfy

$$\forall x \in X, \pi_A(!x) = f_A(x) \text{ and } \pi_B(!x) = f_B(x).$$

Equivalently,  $!(x) = (f_A(x), f_B(x))$ . Since this uniquely determines  $!$ ,  $A \times B$  is indeed the **binary product**.

**Examples 142.** Most of the constructions throughout mathematics with the name product can also be realized with a **binary product**. Examples include the **direct product** of **groups**, **rings** or **vector spaces**, the product of topologies, etc. The fact that all these constructions are based on the Cartesian product of the underlying sets is a corollary of a deeper result about the **forgetful functor** that all these **categories** have in common.<sup>114</sup>

<sup>114</sup> cf.

In another flavour, let  $X$  be a **topological space** and  $\mathcal{O}(X)$  be the **category** of **opens**. If  $A, B \subseteq X$  are **open**, what is their **product**? Following Definition 140, the existence of  $\pi_A$  and  $\pi_B$  imply that  $A \times B$ <sup>115</sup> is included in both sets, or equivalently  $A \times B \subseteq A \cap B$ .

Moreover, for any **open set**  $X$  included in  $A$  and  $B$  (via  $f_A$  and  $f_B$ ),  $X$  should be included in  $A \times B$  (via  $!$ ).<sup>116</sup> In particular,  $X$  can be  $A \cap B$  (it is **open** by definition of a **topology**), thus  $A \cap B \subseteq A \times B$ . In conclusion, the **product** of two **open sets** is their intersection. In an arbitrary **poset**, the same argument is used to show the **product** is the **greatest lower bound/infimum/meet**.

*Remark 143.* Given two **objects** in an arbitrary **category**, their **product** does not necessarily exist. Nevertheless, when it exists, one can (and we will) show that it is unique up to unique **isomorphism**.<sup>117</sup> Thus, in the sequel, we will speak of *the* **product** of two **objects** and similarly for other constructions presented in this chapter.

To generalize the categorical **product** to more than two **objects**, one can, for instance, define the **product** of a finite family of sets recursively with the **binary product**.<sup>118</sup> However, this implies having to show the associativity and commutativity of  $\times$  for it to be well-defined.<sup>119</sup> In contrast, generalizing the **universal property** illustrated in (22) yields a simpler definition that works even for arbitrary families.

□ **Definition 144** (Product). Let  $\{X_i\}_{i \in I}$  be an  $I$ -indexed family of **objects** of  $\mathbf{C}$ . The **product** of this family is an **object**, denoted  $\prod_{i \in I} X_i$  along with **projections**  $\pi_j : \prod_{i \in I} X_i \rightarrow X_j$  for all  $j \in I$  satisfying the following **universal property**: for any **object**  $X$  with **morphisms**  $\{f_j : X \rightarrow X_j\}_{j \in I}$ , there is a unique **morphism**  $! : X \rightarrow \prod_{i \in I} X_i$  making (23) **commute** for all  $j \in I$ .<sup>120</sup>

$$\begin{array}{ccc} X & & \\ \downarrow ! & \searrow f_j & \\ \prod_{i \in I} X_i & \xrightarrow{\pi_j} & X_j \end{array} \quad (23)$$

A family of **objects** in a **category** is also called a **discrete diagram**,<sup>121</sup> the **product** is then the **limit** of this **diagram**.

**Exercise 145.** Show that the **product** of an arbitrary family of sets is still the Cartesian product of this family.

**Exercise 146 (NOW!).** Let  $\{f_i : X_i \rightarrow Y_i\}_{i \in I}$  be a family of **morphisms** in  $\mathbf{C}$ , show that there is a unique **morphism**  $\prod_{i \in I} f_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i$  making the following square **commute** for all  $j \in I$ .

$$\begin{array}{ccc} \prod_{i \in I} X_i & \xrightarrow{\prod_{i \in I} f_i} & \prod_{i \in I} Y_i \\ \pi_j \downarrow & & \downarrow \pi_j \\ X_j & \xrightarrow{f_j} & Y_j \end{array} \quad (24)$$

□ In the finite case, we will write  $f_1 \times \cdots \times f_n$ .

<sup>115</sup> Recall that  $\times$  denotes the categorical **product**, not the Cartesian product of sets.

<sup>116</sup> Notice that uniqueness of  $!$  is already given in a **posetal category**.

<sup>117</sup> The uniqueness of the **isomorphism** is under the condition that it preserves the structure of the **product**. We will clear up this subtlety in Remark 178.

<sup>118</sup> For a family  $\{X_1, \dots, X_n\} \subseteq \mathbf{C}_0$ :

$$\prod_{i=1}^n X_i = \begin{cases} X_1 & n = 1 \\ \left( \prod_{i=1}^{n-1} X_i \right) \times X_n & n > 1 \end{cases}$$

<sup>119</sup> These proofs are not very involved, they heavily rely on uniqueness, cf. Exercises.

In the case of the **category** of **open** subsets of a **topological space**, the arbitrary **product** is not always the intersection. This is because arbitrary intersections of **open sets** are not necessarily **open**. To resolve this problem, it suffices to take the **interior** of the intersection which is **open** by definition.

<sup>120</sup> Analogously to the binary case, we may write  $! = (f_j)_{j \in I}$  or, in the finite case,  $! = (f_1, \dots, f_n)$ .

<sup>121</sup> The terminology comes from Definition 127.

See solution.

The big takeaway from last chapter is that each time we read a new definition, it is worth to **dualize**. Thus we ask, what is the **colimit** of a **discrete diagram**?

### Coproduct

▮ **Definition 147** (Coproduct). Let  $\{X\}_{i \in I}$  be an  $I$ -indexed family of **objects** in  $\mathbf{C}$ , its **coproduct** is an **object**, denoted  $\coprod_{i \in I} X_i$  (or  $X_1 + X_2$  in the binary case), along with **morphisms**  $\kappa_j : X_j \rightarrow \coprod_{i \in I} X_i$  for all  $j \in I$  called **coprojections** satisfying the following **universal property**: for any object  $X$  with **morphisms**  $\{f_j : X_j \rightarrow X\}_{j \in I}$ , there is a unique **morphism**  $! : \coprod_{i \in I} X_i \rightarrow X$  making (25) **commute** for all  $j \in I$ .<sup>122</sup>

$$\begin{array}{ccc} X_j & \xrightarrow{\kappa_j} & \coprod_{i \in I} X_i \\ & \searrow f_j & \downarrow ! \\ & & X \end{array} \quad (25)$$

<sup>122</sup> We may denote  $! = [f_j]_{j \in I}$  or, in the finite case,  $! = [f_1, \dots, f_n]$ .

Let us find out what **coproducts** of sets are.

**Example 148 (Set)**. Let  $\{X_i\}_{i \in I}$  be a family of sets, first note that if  $X_j = \emptyset$  for  $j \in I$ , then there is only one **morphism**  $X_j \rightarrow X$  for any  $X$ . In particular, (25) **commutes** no matter what  $\coprod_{i \in I} X_i$  and  $X$  are. Therefore, removing  $X_j$  from this family does not change how the **coproduct** behaves, hence no generality is loss from assuming all  $X_i$ s are non-empty.

Second, for any  $j \in I$ , let  $X = X_j$ ,  $f_j = \text{id}_{X_j}$  and for any  $j' \neq j$ , let  $f_{j'}$  be any **morphism** in  $\text{Hom}(X_{j'}, X_j)$ .<sup>123</sup> **Commutativity** of (25) implies  $\kappa_j$  has a **left inverse** because  $! \circ \kappa_j = f_j = \text{id}_{X_j}$ , so all **coprojections** are injective.

Third, we claim that for any  $j \neq j' \in I$ ,  $\text{Im}(\kappa_j) \cap \text{Im}(\kappa_{j'}) = \emptyset$ . Assume towards a contradiction that there exists  $j \neq j' \in I$ ,  $x \in X_j$  and  $x' \in X_{j'}$  such that  $\kappa_j(x) = \kappa_{j'}(x')$ . Then, let  $X = \{0, 1\}$ ,  $f_j \equiv 0$ ,  $f_{j'} \equiv 1$  and the other **morphisms** be chosen arbitrarily. The **universal property** implies that  $! \circ \kappa_j \equiv 0$  and  $! \circ \kappa_{j'} \equiv 1$ , but it contradicts  $!(\kappa_j(x)) = !(\kappa_{j'}(x'))$ .

Finally, the previous point says that  $\coprod_{i \in I} X_i$  contains distinct copies of the images of all **coprojections**. Furthermore, the  $\kappa_j$ s being injective, their image can be identified with the  $X_j$ s to obtain<sup>124</sup>

$$\bigsqcup_{i \in I} X_i \subseteq \coprod_{i \in I} X_i.$$

For the converse inclusion, in (25), let  $X$  be the disjoint union and the  $f_j$ s be the inclusions. Assume there exists  $x$  in the R.H.S. that is not in the L.H.S., then we can define  $! : \coprod_{i \in I} X_i \rightarrow \sqcup_{i \in I} X_i$  that only differs from  $!$  at  $x$ . Since  $x$  is not in the image of any of the  $\kappa_j$ , the diagrams still **commute** and this contradicts the uniqueness of  $!$ .

In conclusion, the **coproduct** in **Set** is the disjoint union and the **coprojections** are the inclusions.<sup>125</sup>

*Remark 149.* If this example looks more complicated than the **product** of sets, it is because we started knowing nothing concrete about **coproducts** of sets and gradually discovered what properties they had using specific **objects** and **morphisms** we

<sup>123</sup> One exists because  $X_j$  is non-empty.

<sup>124</sup> The symbol  $\sqcup$  denotes the disjoint union of sets.

<sup>125</sup> We recover the intuition for why empty sets can be ignored. This is a general fact proved in Exercise



know exist in **Set**. In contrast, we knew what **products** of sets were and we just had to show they satisfied the **universal property**.<sup>126</sup>

In general, the hard part is to find what construction satisfies a **universal property**, proving it does is easier.

**Examples 150.** In the **category of open sets** of  $(X, \tau)$ : let  $\{U_i\}_{i \in I}$  be a family of **open sets** and suppose  $\coprod_i U_i$  exists. The **coprojections** yield inclusions  $U_j \subseteq \coprod_i U_i$  for all  $j \in I$ , so  $\coprod_i U_i$  must contain all  $U_j$ s and thus  $\cup_i U_i$ . Moreover, in (25), letting  $f_j$  be the inclusion  $U_j \hookrightarrow \cup_i U_i$  for all  $j \in I$ ,<sup>127</sup> the existence of  $!$  yields an inclusion  $\coprod_i U_i \subseteq \cup_i U_i$ . We conclude that the **coproduct** in this **category** is the union. In an arbitrary **poset**, the same argument is used to show the **coproduct** is the **least upper bound/supremum/join**.

In **Vect**<sub>k</sub>: the **coproduct**, also called the direct sum, is defined by<sup>128</sup>

$$\coprod_{i \in I} V_i = \bigoplus_{i \in I} V_i := \left\{ v \in \prod_{i \in I} V_i \mid v(i) \neq 0 \text{ for finitely many } i\text{'s} \right\},$$

where  $\kappa_j : V_j \hookrightarrow \coprod_i V_i$  sends  $v$  to  $\bar{v} \in \prod_i V_i$  with  $\bar{v}_j = v$  and  $\bar{v}_{j'} = 0$  whenever  $j \neq j'$ . To verify this, let  $\{f_j : V_j \rightarrow X\}_{j \in I}$  be a family of **linear maps**. We can construct  $!$  by defining it on **basis** elements of the **direct sum**, which are just the **basis** elements of all  $V_j$ s seen as elements of the **sum** (via the **coprojections**).<sup>129</sup> Indeed, if  $b$  is in the **basis** of  $V_j$ , we let  $!(\bar{b}) = f_j(b)$ . Extending linearly yields a **linear map**  $! : \coprod_i V_i \rightarrow X$ . Uniqueness is clear because if  $h : \coprod_i V_i \rightarrow X$  differs from  $!$  on one of the basis elements, it does not make (25) commute.

**Exercise 151.** Show that **products** are **dual** to **coproducts**, namely, if a **product** of a family  $\{X_i\}_{i \in I}$  exists in **C**, then this **object** is the **coproduct** of this family in **C**<sup>op</sup> and vice-versa. Conclude that you can define the **coproduct morphisms** **dually** to Exercise 146, we denote them  $\coprod_{i \in I} f_i$  or  $f_1 + \cdots + f_n$  in the finite case.

**Exercise 152.** Let **C** be a **category** with a **terminal object** **1**. Show that the assignment  $X \mapsto X + \mathbf{1}$  is **functorial**, i.e.: define the action of  $(- + \mathbf{1})$  on **morphisms** and show it satisfies the axioms of a **functor**.<sup>130</sup>

In a very similar way to the **product** and **coproduct**, we will define various constructions in **Set** as **limits** or **colimits**.

### Equalizer

**Definition 153** (Fork). A **fork** in **C** is a **diagram** of shape (26) or (27) that **commutes**.<sup>131</sup>

$$O \xrightarrow{o} A \xrightleftharpoons[g]{f} B \quad (26) \quad A \xrightleftharpoons[g]{f} B \xrightarrow{o} O \quad (27)$$

Because these are **dual** notions, we will prefer to call (27) a **cofork**.

**Definition 154** (Equalizer). Let  $A, B \in \mathbf{C}_0$  and  $f, g : A \rightarrow B$  be **parallel morphisms**.

The **equalizer** of  $f$  and  $g$  is an **object**  $E$  and a **morphism**  $e : E \rightarrow A$  satisfying

<sup>126</sup> One might argue that coming up with this **universal property** was the hard part in that case.

<sup>127</sup> These **morphisms** are in  $\mathcal{O}(X)$  because  $\cup_i U_i$  is open.

<sup>128</sup> Here, the symbol  $\prod$  denotes the Cartesian product of the  $V_i$ s as sets. The categorical **product** of **vector spaces** is also the direct sum, where the **projections** are the usual ones.

<sup>129</sup> It is necessary to require finitely many non-zero entries, otherwise the **basis** of the **coproduct** would not be the union of all bases of the  $V_j$ s.

See solution.

See solution.

<sup>130</sup> We call  $(- + \mathbf{1})$  the **maybe functor**.

<sup>131</sup> Again, we make use of our convention that **commutativity** does not make **parallel morphisms** equal.

$f \circ e = g \circ e$  with the following **universal property**: for any **object**  $O$  with **morphism**  $o : O \rightarrow A$  satisfying  $f \circ o = g \circ o$ , there is a unique  $! : O \rightarrow E$  making (28) **commute**.

$$\begin{array}{ccc} O & & \\ \downarrow ! & \searrow o & \\ E & \xrightarrow{e} & A \xrightleftharpoons[f]{g} B \end{array} \quad (28)$$

**Example 155 (Set).** Let  $f, g : A \rightarrow B$  be two functions and suppose  $e : E \rightarrow A$  is their **equalizer**. By **associativity**, for any  $h : O \rightarrow E$ , the composite  $e \circ h$  is a candidate for  $o$  in diagram (28) because  $f \circ (e \circ h) = g \circ (e \circ h)$ . What is more, if  $h'$  is such that  $e \circ h = e \circ h'$ , then  $h = h'$  or it would contradict the uniqueness of  $!$ . In other words,  $e$  is **monic/injective**.<sup>132</sup>

This implies  $E$  can be identified with its image under  $e$ . Since  $e$  makes a **fork** with  $f$  and  $g$ , its image is contained in the subset  $\{a \in A \mid f(a) = g(a)\}$ . But, by the **universal property**, letting  $O$  be this set and  $o$  be the inclusion, there is an injection<sup>133</sup>  $! : \{a \in A \mid f(a) = g(a)\} \hookrightarrow E$ , thus both sets are equal. In conclusion, the **equalizer** of two parallel functions is the subset  $E$  in which they are equal and  $e : E \hookrightarrow A$  is the inclusion.

**Examples 156.** In a **posetal category**: **hom-sets** are singletons, so it must be the case that  $f = g$  whenever  $f$  and  $g$  are parallel. Therefore, any  $o : O \rightarrow A$  satisfies  $f \circ o = g \circ o$ . Written using the **order** notation, the **universal property** is then equivalent to the fact that  $O \leq A$  implies  $O \leq E$ . In particular, if  $O = A$ , then  $A \leq E$ , so  $A = E$  by **antisymmetry**.

In **Ab**, **Ring** or **Vect**: For the same reason that the Cartesian product of the underlying sets is the underlying set of the **product**,<sup>134</sup> the construction of **equalizers** is as in **Set**. Nevertheless, since each of these **categories** have a notion of additive inverse for **morphisms**, the **equalizer** of  $f$  and  $g$  has a cooler name, that is,  $\ker(f - g)$ .<sup>135</sup>

The **equalizer** of  $f$  and  $g$  is the **limit** of the **diagram** containing only the two **parallel morphisms**, we define its **colimit** in the next section.

### Coequalizer

**Definition 157 (Coequalizer).** Let  $A, B \in \mathbf{C}_0$  and  $f, g : A \rightarrow B$  be **parallel morphisms**. The **coequalizer** of  $f$  and  $g$  is an **object**  $D$  and a **morphism**  $d : B \rightarrow D$  satisfying  $d \circ f = d \circ g$  with the following **universal property**: for any **object**  $O$  with **morphism**  $o : B \rightarrow O$  satisfying  $o \circ f = o \circ g$ , there is a unique  $! : D \rightarrow O$  making (29) **commute**.

$$\begin{array}{ccc} A & \xrightleftharpoons[f]{g} & B \xrightarrow{d} D \\ & & \searrow o \downarrow ! \\ & & O \end{array} \quad (29)$$

**Example 158 (Set).** Let  $f, g : A \rightarrow B$  be two functions and suppose  $d : B \rightarrow D$  is their **coequalizer**. Similarly to the **dual** case, one can show that  $d$  is **epic/surjective**.

<sup>132</sup> This argument was independent of the **category**, hence we can conclude that an **equalizer** of **parallel morphisms** is always **monic**.

<sup>133</sup> The fact that  $!$  is an injection comes from the fact that the inclusion  $o$  is an injection and  $e \circ ! = o$ .

<sup>134</sup> We explain this later:

<sup>135</sup> The **equalizer** of  $f$  and  $g$  is the subset of  $A$  where  $f$  and  $g$  are equal, or equivalently, where  $f - g$  is 0 (when  $f - g$  and 0 are defined).

Since  $d \circ f = d \circ g$ , for any  $b, b' \in B$ ,

$$(\exists a \in A, f(a) = b \text{ and } g(a) = b') \implies d(b) = d(b'). \quad (*)$$

Denoting  $\sim$  to be the relation in the L.H.S. of  $(*)$ , the implication is  $b \sim b' \implies d(b) = d(b')$ . Note that  $\sim$  is not an **equivalence relation** while  $=$  is, thus, the converse implication does not always hold. For instance, when  $b \sim b' \sim b''$ ,  $d(b) = d(b'')$ , but it might not be the case that  $b \sim b''$ .

Consequently, it makes sense to consider the **equivalence relation** generated by  $\sim$ ,<sup>136</sup> denoted  $\simeq$ . As noted above, the forward implication  $b \simeq b' \implies d(b) = d(b')$  still holds. For the converse, in (29), let  $O := B/\simeq$  and  $o : B \rightarrow B/\simeq$  be the **quotient map**, by **post-composing** with  $!$ , we have

$$d(b) = d(b') \implies o(b) = o(b') \implies b \simeq b'.$$

In conclusion,  $D = B/\simeq$  and  $d : B \rightarrow D$  is the **quotient map**.

**Examples 159.** In a **posetal category**: an argument **dual** to the one for **equalizers** shows the **coequalizer** of  $f, g : A \rightarrow B$  is  $B$ .

In **Ab**, **Ring** or **Vect**<sub>k</sub>: Let  $f, g : A \rightarrow B$  be **homomorphisms** and suppose  $d : B \rightarrow D$  is their **coequalizers**. Consider the **homomorphism**  $f - g$ , since  $d$  makes a **cofork** with  $f$  and  $g$ ,  $d \circ (f - g) = d \circ f - d \circ g = 0$ , or equivalently,  $\text{Im}(f - g) \subseteq \ker(d)$ . Now, consider diagram (30) as a particular instance of (29), where  $q$  is the quotient map.<sup>137</sup>

$$\begin{array}{ccc} A & \xrightleftharpoons[g]{f} & B \\ & \searrow q & \downarrow ! \\ & & B/\text{Im}(f - g) \end{array} \quad \begin{array}{c} \xrightarrow{d} \\ \downarrow ! \end{array} \quad \begin{array}{c} D \\ \\ \end{array} \quad (30)$$

We claim that  $!$  has an inverse, implying that  $D \cong B/\text{Im}(f - g)$ . Indeed, for  $[x] \in B/\text{Im}(f - g)$ , we must have

$$!^{-1}([x]) = !^{-1}(q(x)) = !^{-1}(!d(x)) = d(x),$$

and it is only left to show  $!^{-1}$  is well-defined because the inverse of a **homomorphism** is a **homomorphism**. This follows because if  $[x] = [x']$ , then there exists  $y \in \text{Im}(f - g)$  such that  $x = x' + y$ , so

$$!^{-1}(x) = d(x) = d(x' + y) = d(x') + d(y) = d(x') + 0 = !^{-1}(x').$$

□ In the special case that  $g \equiv 0$ ,  $B/\text{Im}(f)$  is called the **cokernel** of  $f$ , denoted  $\text{coker}(f)$ .

### Pullback

□ **Definition 160** (Cospan). A **cospan** in **C** is comprised of three **objects**  $A, B, C$  and two **morphisms**  $f$  and  $g$  as in (31).

$$A \xrightarrow{f} C \xleftarrow{g} B \quad (31)$$

<sup>136</sup> In this case, it is simply the **transitive closure**.

<sup>137</sup> It is **commutative** because  $q \circ (f - g) = 0$  by definition of  $q$ .

□ **Definition 161** (Pullback). Let  $A \xrightarrow{f} C \xleftarrow{g} B$  be a **cospan** in  $\mathbf{C}$ . Its **pullback** is an **object**, denoted  $A \times_C B$ , along with **morphisms**  $p_A : A \times_C B \rightarrow A$  and  $p_B : A \times_C B \rightarrow B$  such that  $f \circ p_A = g \circ p_B$  and the following **universal property** holds: for any **object**  $X$  and **morphisms**  $s : X \rightarrow A$  and  $t : X \rightarrow B$  satisfying  $f \circ s = g \circ t$ , there is a unique **morphism**  $! : X \rightarrow A \times_C B$  making (32) **commute**.<sup>138</sup>

$$\begin{array}{ccccc}
 X & & & & \\
 \swarrow s & \searrow t & & & \\
 & A \times_C B & \xrightarrow{p_B} & B & \\
 & \downarrow p_A & \lrcorner & \downarrow g & \\
 & A & \xrightarrow{f} & C & 
 \end{array} \quad (32)$$

□ We will call  $p_A$  the **pullback** of  $g$  **along**  $f$  and sometimes denote it  $f^*(g)$ . Symmetrically,  $p_B$  is the **pullback** of  $f$  **along**  $g$ , denoted  $g^*(f)$ .

**Example 162 (Set)**. Let  $A \xrightarrow{f} C \xleftarrow{g} B$  be a **cospan** in **Set** and suppose that its **pullback** is  $A \xleftarrow{p_A} A \times_C B \xrightarrow{p_B} B$ . Observe that  $p_A$  and  $p_B$  look like **projections**, and in fact, by the **universality** of the **product**  $A \times B$ , there is a map  $h : A \times_C B \rightarrow A \times B$  such that  $h(x) = (p_A(x), p_B(x))$  ((33) **commutes**). Consider the image of  $h$ , if  $(a, b) \in \text{Im}(h)$ , then there exists  $x \in A \times_C B$  such that  $p_A(x) = a$  and  $p_B(x) = b$ . Moreover, the **commutativity** of the square in (33) implies  $f(a) = g(b)$ , hence

$$\text{Im}(h) \subseteq \{(a, b) \in A \times B \mid f(a) = g(b)\} =: E.$$

Now, letting  $X = E$ ,  $s = \pi_A$  and  $t = \pi_B$ , by definition,  $f \circ s = g \circ t$  hence, there is a unique  $! : E \rightarrow A \times_C B$  satisfying  $p_A \circ ! = \pi_A$  and  $p_B \circ ! = \pi_B$ . Viewing  $h$  as going in the opposite direction to  $!$ ,<sup>139</sup> it is easy to see that for any  $(a, b) \in E$ ,<sup>140</sup>

$$(h \circ !)(a, b) = (p_A(! (a, b)), p_B(a, b)) = (\pi_A(a, b), \pi_B(a, b)) = (a, b),$$

thus  $!$  has a **left inverse** and is injective. Assume towards a contradiction that it is not surjective, then let  $y \in A \times_C B$  not be in the image of  $!$  and denote  $x = !(p_A(y), p_B(y))$ . Define  $!'$  as acting exactly like  $!$  except on  $(p_A(y), p_B(y))$  where it goes to  $y$  instead of  $x$ . This ensure that  $!'$  still makes the diagram **commutes**, but this contradicts the uniqueness of  $!$ .

As a particular case, if a **cospan** is comprised of two inclusions  $A \hookrightarrow C \hookleftarrow B$ , then its **pullback** is the intersection  $A \cap B$  with  $p_A$  and  $p_B$  being the inclusions.

**Examples 163**. In a **posetal category**, the **commutativity** of the square in (32) does not depend on the **morphisms**, thus the **universal property** is equivalent to the property of being a **product**.

**Exercise 164**. Let  $f : X \rightarrow Y$  be a **morphism** in  $\mathbf{C}$ . Show  $f$  is **monic** if and only if the square in (34) is a **pullback**.<sup>141</sup>

<sup>138</sup> The  $\lrcorner$  symbol is a standard convention to specify that the square is not only **commutative**, but also a **pullback** square.

$$\begin{array}{ccccc}
 A \times_C B & \xrightarrow{p_B} & B & & \\
 \downarrow p_A & \searrow h & \swarrow \pi_B & & \\
 & A \times B & & & \\
 \swarrow \pi_A & \downarrow g & & & \\
 A & \xrightarrow{f} & C & & 
 \end{array} \quad (33)$$

<sup>139</sup> We just saw that the image of  $h$  is contained in  $E$ , so we can see  $h$  as a function  $h : A \times_C B \rightarrow A \times B$ .

<sup>140</sup> We use the fact that  $\pi_A \circ h \circ ! = p_A \circ !$  and similarly for  $B$ .

See solution.

<sup>141</sup> This result and its **dual** will sometimes be used to treat **monomorphisms** (resp. **epimorphisms**) as **limits** (resp. **colimits**). In most of these cases, it will be crucial that this **limit** (resp. **colimit**) only involves the **monomorphism** (resp. **epimorphism**) and the **identity morphism** which is **preserved** by any **functor**.

$$\begin{array}{ccc}
 X & \xrightarrow{\text{id}_X} & X \\
 \text{id}_X \downarrow & \lrcorner & \downarrow f \\
 X & \xrightarrow{f} & Y
 \end{array} \quad (34)$$

State and prove the **dual** statement.

### Pushout

▮ **Definition 165** (Span). A **span** in  $\mathbf{C}$  is comprised of three **objects**  $A, B, C$  and two **morphisms**  $f$  and  $g$  as in (35).

$$A \xleftarrow{f} C \xrightarrow{g} B \quad (35)$$

▮ **Definition 166** (Pushout). Let  $A \xleftarrow{f} C \xrightarrow{g} B$  form a **span** in  $\mathbf{C}$ . Its **pushout** is an **object**, denoted  $A +_C B$ , along with **morphisms**  $k_A : A \rightarrow A +_C B$  and  $k_B : B \rightarrow A +_C B$  such that  $k_A \circ f = k_B \circ g$  and the following **universal property** holds: for any **object**  $X$  and **morphisms**  $s : A \rightarrow X$  and  $t : B \rightarrow X$  satisfying  $s \circ f = t \circ g$ , there is a unique **morphism**  $! : A +_C B \rightarrow X$  making (36) **commute**.<sup>142</sup>

$$\begin{array}{ccc}
 C & \xrightarrow{g} & B \\
 f \downarrow & \lrcorner & \downarrow k_B \\
 A & \xrightarrow{k_A} & A +_C B \\
 & \searrow s & \nearrow t \\
 & & X
 \end{array} \quad (36)$$

<sup>142</sup> The  $\lrcorner$  symbol is a standard convention to specify that the square is not only **commutative**, but also a **pushout** square.

▮ We will call  $k_A$  the **pushout** of  $g$  **along**  $f$  and sometimes denote it  $f_*(g)$ . Symmetrically,  $k_B$  is the **pushout** of  $f$  **along**  $g$ , denoted  $g_*(f)$ .

**Example 167 (Set)**. Let  $A \xleftarrow{f} C \xrightarrow{g} B$  be a **span** in **Set** and suppose its **pushout** is  $A \xrightarrow{k_A} A +_C B \xleftarrow{k_B} B$ . Similarly to above, observe that  $k_A$  and  $k_B$  are like **coprojections**, so there is a unique map  $! : A + B \rightarrow A +_C B$  such that  $!(a) = k_A(a)$  and  $!(b) = k_B(b)$ . Furthermore, for any  $c \in C$ ,  $!(f(c)) = !(g(c))$ , thus

$$\exists c \in C, f(c) = a \text{ and } g(c) = b \implies !(a) = !(b).$$

This is very similar to what happened for **coequalizers** and after working everything out, we obtain that  $! : A + B \rightarrow A +_C B$  is the **coequalizer** of  $\kappa_A \circ f$  and  $\kappa_B \circ g$ . This is a general fact that does not only apply in **Set** but in every category with binary **coproducts** and **coequalizers**.

As a particular case, if  $C = A \cap B$  and  $f$  and  $g$  are simply inclusions, then  $A +_C B = A \cup B$  (the *non-disjoint union*).

## Generalization

In case you have not figured out the pattern, note that [products](#), [equalizers](#) and [pullbacks](#) are examples of [limits](#) while [coproducts](#), [coequalizers](#) and [pushouts](#) are examples of [colimits](#). These six examples give quite a good idea of what it is to be a [limit](#) or [colimit](#). Roughly, all of the definitions go as follows.

- Some shape is specified for a [diagram](#)  $D$  (i.e.: a [discrete diagram](#), two [parallel morphisms](#), a [span](#), a [cospan](#), etc.).
- The [limit](#) (resp. [colimit](#)) of  $D$  is an [object](#)  $L$  along with [morphisms](#) in  $\text{Hom}_{\mathbf{C}}(L, O)$  (resp.  $\text{Hom}_{\mathbf{C}}(O, L)$ ) for any [object](#)  $O$  in  $D$  such that combining  $D$  with these [morphisms](#) yields a [commutative](#) diagram.
- These [morphisms](#) satisfy a [universal property](#). More specifically, for any [object](#)  $L'$  with [morphisms](#) in  $\text{Hom}_{\mathbf{C}}(L', O)$  (resp.  $\text{Hom}_{\mathbf{C}}(O, L')$ ) [commuting](#) with  $D$ , there is a unique  $! : L' \rightarrow L$  (resp.  $L \rightarrow L'$ ) such that combining all the [morphisms](#) with  $D$  yields a [commutative](#) diagram.

The first step towards a formal generalization is to formally define a [diagram](#).

## Definitions

Next, notice that the [morphisms](#) given for  $L$  and  $L'$  have the same conditions, they form a [cone](#) or [cocone](#).

□ **Definition 168** (Cone). Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a [diagram](#). A [cone](#) from  $X$  to  $F$  is an [object](#)  $X \in \mathbf{C}_0$ , called the [tip](#), along with a family of [morphisms](#)  $\{\psi_Y : X \rightarrow F(Y)\}$  indexed by [objects](#)  $Y \in \mathbf{J}_0$  such that for any [morphism](#)  $f : Y \rightarrow Z$  in  $\mathbf{J}_1$ ,  $F(f) \circ \psi_Y = \psi_Z$ , i.e.: diagram (37) [commutes](#).

$$\begin{array}{ccc} & X & \\ \psi_Y \swarrow & & \searrow \psi_Z \\ F(Y) & \xrightarrow{F(f)} & F(Z) \end{array} \quad (37)$$

Often, the terminology [cone over](#)  $F$  is used.

Next, the fact that the [morphism](#)  $!$  keeps everything [commutative](#) can be generalized.

**Definition 169** (Morphism of cones). Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a [diagram](#) and  $\{\psi_Y : A \rightarrow F(Y)\}_{Y \in \mathbf{J}_0}$  and  $\{\phi_Y : B \rightarrow F(Y)\}_{Y \in \mathbf{J}_0}$  be two [cones](#) over  $F$ . A [morphism of cones](#) from  $A$  to  $B$  is a [morphism](#)  $g : A \rightarrow B$  in  $\mathbf{C}_1$  such that for any  $Y \in \mathbf{J}_0$ ,  $\phi_Y \circ g = \psi_Y$ , i.e.: (38) [commutes](#).

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \psi_Y \searrow & & \swarrow \phi_Y \\ & F(Y) & \end{array} \quad (38)$$

After verifying that **morphisms** can be composed, the last two definitions give rise to the **category** of **cones** over a **diagram**  $F$  which we denote  $\mathbf{Cone}(F)$ . Finally, the **universal property** can be stated in terms of **cones**, thus giving the general definition of a **limit**. Indeed, the **limit** of a **diagram**  $D$  is a **cone**  $L$  over  $D$  such that for every **cone**  $L'$  over  $D$ , there is a unique **cone morphism**  $! : L' \rightarrow L$ . Equivalently,  $L$  is the **terminal object** of  $\mathbf{Cone}(F)$ .

□ **Definition 170** (Limit). Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a **diagram**, the **limit** of  $F$  denoted  $\lim F$  (or  $\lim_{\mathbf{J}} F$ ), if it exists, is the **terminal object** of  $\mathbf{Cone}(F)$ .

*Remark 171.* Often,  $\lim F$  also designates the **tip** of the **cone** as an **object** in  $\mathbf{C}$  rather than the whole **cone**.

**Examples 172.** While you can play around with the three examples of **limits** we have already given and make them fit in this general definition, we add to this list a trivial example and a more complex one.

1. Consider an empty **diagram** in  $\mathbf{C}$ , that is, the only **functor**  $\emptyset$  from the empty **category** to  $\mathbf{C}$ . A **cone** from  $X$  to  $\emptyset$  is just an **object**  $X \in \mathbf{C}_0$  as there are no **objects** in the **diagram**. Consequently, a **morphism** in  $\mathbf{Cone}(\emptyset)$  is simply a **morphism** in  $\mathbf{C}$ , so  $\mathbf{Cone}(\emptyset)$  is the same as the original **category**  $\mathbf{C}$  and  $\lim \emptyset$  is the **terminal object** of  $\mathbf{C}$  if it exists.<sup>143</sup>
2. Let  $X = \{x_1, \dots, x_n\}$  be a set of indeterminates (also called variables) and  $k$  be a **field**,  $k[X]$  denotes the **ring** of polynomials over  $X$ .<sup>144</sup> We will construct  $k[[X]]$ , the **ring** of formal power series over  $X$ , using **limits**.

Let  $I = \langle X \rangle$  be the **ideal generated** by  $X$ , the following three key properties are satisfied.

- a) For any  $n < m \in \mathbb{N}$  and  $p \in k[X]/I^m$ , forgetting about all terms in  $p$  of degree at least  $n$  yields a **ring homomorphism**  $\pi_{m,n} : k[X]/I^m \rightarrow k[X]/I^n$ .
- b) For any  $n \in \mathbb{N}$ , we can do the same thing for power series to obtain a **homomorphism**  $\pi_{\infty,n} : k[[X]] \rightarrow k[X]/I^n$ .
- c) Any composition of the **homomorphisms** above can be seen as a single **homomorphism**. Namely,  $\forall n < m < l \in \mathbb{N} \cup \infty$ ,

$$\pi_{m,n} \circ \pi_{l,m} = \pi_{l,n}.$$

Consider the **posetal category**  $(\mathbb{N}, \geq)$ , a) and c) imply that  $F(n) := k[X]/I^n$  and  $F(m > n) := \pi_{m,n}$  defines a **functor**  $F : (\mathbb{N}, \geq) \rightarrow \mathbf{Ring}$ . This is represented in (39).

$$\dots \longrightarrow k[X]/I^n \xrightarrow{\pi_{n,n-1}} \dots \longrightarrow k[X]/I^2 \xrightarrow{\pi_{2,1}} k[X]/I \xrightarrow{\pi_{1,0}} k[X] \quad (39)$$

Now, using b) and c), we see that  $k[[X]]$  along with  $\{\pi_{\infty,n}\}_{n \in \mathbb{N}}$  is a **cone** over the **diagram**  $F$ . It is in fact the **terminal cone**. Let  $\{p_n : R \rightarrow k[X]/I^n\}$  be another **cone** over  $F$  and  $! : R \rightarrow k[[X]]$  a **morphism** of **cones**. By **commutativity**, the

<sup>143</sup> Dually,  $\text{colim} \emptyset$  is the **initial object** of  $\mathbf{C}$  if it exists (**colim** is defined in the next section).

<sup>144</sup> While, we will describe a nice categorical definition of  $k[[X]]$  in Chapter , let us assume we know what it is.

coefficients of  $!(r)$  must agree with  $p_n(r)$  on all monomials of degree at most  $n$ , thus,

$$!(r) = p_0(r) + \sum_{n>0} p_n(r) - p_{n-1}(r).$$

This completely determines  $!$ , so it is unique.<sup>145</sup>

<sup>145</sup> Existence follows from the same equation.

The construction of this **diagram** from quotienting different powers of the same **ideal** is used in different contexts, it is called the **completion** of  $k[X]$  with respect to  $I$ . For instance, one can define the  $p$ -adic integers with base ring  $\mathbb{Z}$  and the **ideal generated** by  $p$  for any prime  $p$ .

### Codefinitions

Put simply, a **colimit** in  $\mathbf{C}$  is a **limit** in  $\mathbf{C}^{\text{op}}$ . We suggests you spend a bit of time trying to **dualize** all of the previous section on your own, but we have done it for completeness.

□ **Definition 173** (Cocone). Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a diagram. A **cocone** from  $F$  to  $X$  is an **object**  $X \in \mathbf{C}_0$  along with a family of **morphisms**  $\{\psi_Y : F(Y) \rightarrow X\}$  indexed by **objects** of  $\mathbf{Y} \in \mathbf{J}_0$  such that for any **morphism**  $f : Y \rightarrow Z$  in  $\mathbf{J}$ ,  $\psi_Z \circ F(f) = \psi_Y$ , i.e.: (40) **commutes**.

$$\begin{array}{ccc} F(Y) & \xrightarrow{F(f)} & F(Z) \\ & \searrow \psi_Y & \swarrow \psi_Z \\ & X & \end{array} \quad (40)$$

**Definition 174** (Morphism of cocones). Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a **diagram** and  $\{\psi_Y : F(Y) \rightarrow A\}_{Y \in \mathbf{J}_0}$  and  $\{\phi_Y : F(Y) \rightarrow B\}_{Y \in \mathbf{J}_0}$  be two **cocones**. A **morphism of cocones** from  $A$  to  $B$  is a **morphism**  $g : A \rightarrow B$  in  $\mathbf{C}$  such that for any  $Y \in \mathbf{J}_0$ ,  $g \circ \psi_Y = \phi_Y$ , i.e.: (41) **commutes**.

$$\begin{array}{ccc} & F(Y) & \\ \psi_Y \swarrow & & \searrow \phi_Y \\ A & \xrightarrow{g} & B \end{array} \quad (41)$$

The **category** of **cocones** from  $F$ , sometimes called **cones** under  $F$ , is denoted  $\mathbf{Cocone}(F)$ .

□ **Definition 175** (Colimit). Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a **diagram**, the **colimit** of  $F$  denoted  $\text{colim} F$ , if it exists, is the **initial object** of  $\mathbf{Cocone}(F)$ .

**Example 176**. The **colimit** of the empty **diagram** is the **initial object** if it exists.

### Result

**Proposition 177** (Uniqueness). Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a **diagram**, the **limit** (resp. **colimit**) of  $F$ , if it exists, is unique up to unique **isomorphism**.



*Proof.* This follows from the uniqueness of **terminal** (resp. **initial**) **objects**.  $\square$

*Remark 178.* The **isomorphism** between two **limits** (also **colimits**) is unique when viewed as a **morphism** of **cone**. There might exist an **isomorphism** between the **tips** that is not a **morphism** of **cone**. For instance, let  $A, B$  and  $C$  be finite sets. One can check that both  $A \times (B \times C)$  and  $(A \times B) \times C$  are **products** of  $\{A, B, C\}$  (with the usual **projection** maps). Thus, there is an **isomorphism** between them. One can check that, for it to be a **morphism** of **cones**, it must send  $(a, (b, c))$  to  $((a, b), c)$ , but any other bijection between them is an **isomorphism** in **Set**.

For this reason, the **limit** really consists of the whole **cone**, and not just of the **object** at the **tip**! Unfortunately, this subtlety is not well cared for in the literature and it can and has led to errors.

### Diagram chasing

We show four results in increasing order of complexity to demonstrate **diagram chasing** through examples.

**Theorem 179.** Consider the **pullback** square in (42).

$$\begin{array}{ccc} A \times_C B & \xrightarrow{p_B} & B \\ p_A \downarrow & \lrcorner & \downarrow g \\ A & \xrightarrow{f} & C \end{array} \quad (42)$$

If  $g$  is **monic**, then  $p_A$  also is. Symmetrically, if  $f$  is **monic**, then  $p_B$  also is.<sup>146</sup>

*Proof.* Let  $h_1, h_2 : X \rightarrow A \times_C B$  be such that  $p_A \circ h_1 = p_A \circ h_2$ , we need to show that  $h_1 = h_2$ . First, observe that  $h_1$  and  $h_2$  yield two **cones** over the **cospan**  $A \xrightarrow{f} C \xleftarrow{g} B$  as depicted in (43).

$$\begin{array}{c} \begin{array}{ccccc} & & p_B \circ h_2 & & \\ & \searrow & \curvearrowright & \searrow & \\ X & & & & B \\ & \searrow h_2 & \searrow p_B \circ h_1 & \searrow & \\ & & A \times_C B & \xrightarrow{p_B} & \\ & \searrow h_1 & \downarrow p_A & \lrcorner & \downarrow g \\ & & A & \xrightarrow{f} & C \end{array} \\ p_A \circ h_1 = p_A \circ h_2 \end{array} \quad (43)$$

Furthermore,  $h_1$  and  $h_2$  are **cone morphisms** between  $X$  and  $A \times_C B$  and since the **pullback** is the **terminal cone** over this **cospan**, they are unique. Now, we already have that the **projections** onto  $A$  is the same for both new **cones**, but we claim this is also true for the **projections** onto  $B$ . Indeed, because  $g$  is **monic** and the square **commutes**, we have the following implications.

$$p_A \circ h_1 = p_A \circ h_2 \implies f \circ p_A \circ h_1 = f \circ p_A \circ h_2$$

<sup>146</sup> This is commonly stated simply as: “The **pullback** of a **monomorphism** is a **monomorphism**.”

The two **cones** are

$$\begin{array}{ccc} X & \xrightarrow{p_B \circ h_1} & B \\ p_A \circ h_1 \downarrow & & \downarrow \\ A & & \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{p_B \circ h_2} & B \\ p_A \circ h_2 \downarrow & & \downarrow \\ A & & \end{array}$$

They make the squares **commute** because the original **pullback** square **commutes**.

$$\begin{aligned}
&\implies g \circ p_B \circ h_1 = g \circ p_B \circ h_2 \\
&\implies p_B \circ h_1 = p_B \circ h_2
\end{aligned}$$

In other words, the two new **cones** are in fact the same **cones**, hence  $h_1$  and  $h_2$  are the same **morphisms** by uniqueness, which concludes our proof.  $\square$

**Corollary 180.** *The **pushout** of an **epimorphism** is an **epimorphism**.*

**Theorem 181** (Pasting Lemma). *Consider diagram (44), where the right square is a **pull-back**. This result is called the **pasting lemma**.*

$$\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\alpha \downarrow & & \beta \downarrow & \lrcorner & \downarrow \gamma \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C'
\end{array} \quad (44)$$

If (44) **commutes**, the left square is a **pullback** if and only if the rectangle is.

*Proof.* ( $\Rightarrow$ ) Explicitly, we have to show that  $\alpha : A' \leftarrow A \rightarrow C : g \circ f$  is the **pullback** of  $g' \circ f' : A' \rightarrow C' \leftarrow C : \gamma$ . The **commutativity**  $g' \circ f' \circ \alpha = \gamma \circ g \circ f$  implies this is already a **cone** over the **cospan** we just described. Now, suppose there is another **cone** over this **cospan**, namely, there exist **morphisms**  $p_{A'} : X \rightarrow A'$  and  $p_C : X \rightarrow C$  satisfying  $g' \circ f' \circ p_{A'} = \gamma \circ p_C$  as depicted in (45).

$$\begin{array}{c}
\begin{array}{ccccc}
X & & & & \\
& \searrow^{!_B} & & \searrow^{p_C} & \\
& & A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
& \swarrow_{!_A} & \alpha \downarrow & \lrcorner & \beta \downarrow & \lrcorner & \downarrow \gamma \\
& & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \\
& \swarrow_{p_{A'}} & & & & & 
\end{array}
\end{array} \quad (45)$$

Notice that composing  $p_{A'}$  with  $f'$ , we obtain a **cone** over the **cospan** in the right square and by **universality** of  $B$ , this yields a unique **morphism**  $!_B : X \rightarrow B$  satisfying  $g \circ !_B = p_C$  and  $\beta \circ !_B = f' \circ p_{A'}$ . This second equality yields **cone** over the **cospan** in the left square, thus we get a unique **morphism**  $!_A : X \rightarrow A$  satisfying  $\alpha \circ !_A = p_{A'}$  and  $f \circ !_A = !_B$ . Composing the last equality with  $g$ , we get

$$g \circ f \circ !_A = g \circ !_B = p_C,$$

showing that  $!_A$  is a **morphism** of **cones** over the rectangular **cospan**.

What is more, any other **morphism**  $m : X \rightarrow A$  of **cones** over this **cospan** must satisfy

$$g \circ f \circ m = p_C \text{ and } \beta \circ f \circ m = f' \circ \alpha \circ m = f' \circ p_{A'},$$

and thus,  $f \circ m$  is a **morphism** of **cones** over the **cospan** in the right rectangle. By uniqueness,  $f \circ m = !_B$ , so  $m$  is also a **morphism** of **cones** over the **cospan** in the left square, and by **universality** of  $A$ ,  $m = !_A$ .

( $\Leftarrow$ ) Explicitly, we have to show that  $\alpha : A' \leftarrow A \rightarrow B : f$  is the **pullback** of  $f' : A' \rightarrow B \leftarrow B : \beta$ .

$$\begin{array}{ccccc}
 X & & \xrightarrow{p_B} & & B \\
 \downarrow p_{A'} & \searrow !_A & & \searrow f & \\
 A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \\
 \uparrow \alpha & & \uparrow \beta & & \uparrow \gamma \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C
 \end{array}
 \quad (46)$$

Let  $p_{A'} : A' \leftarrow X \rightarrow B : p_B$  be a **cone** over the **cospan** of the left square (i.e.:  $\beta \circ p_B = f' \circ p_{A'}$ ). The **commutativity** of (44) implies  $p_{A'} : A' \leftarrow X \rightarrow C : g \circ p_B$  is a **cone** over the rectangle **cospan**, then by **universality** of  $A$ , there exists a unique  $!_A : X \rightarrow A$  such that  $g \circ f \circ !_A = g \circ p_B$  and  $\alpha \circ !_A = p_{A'}$ . Moreover, with the **commutativity** of the left square, we find that  $f \circ !_A$  is a **morphism of cones** over the right **cospan** satisfying  $\beta \circ f \circ !_A = f' \circ \alpha \circ !_A = f' \circ p_{A'} = \beta \circ p_B$  and  $g \circ f \circ !_A = g \circ p_B$ . But since our hypothesis on  $p_{A'}$  and  $p_B$  implies  $p_B$  is a **morphism of cones** satisfying the same equations, by **universality** of  $B$ ,  $p_B = f \circ !_A$ . Therefore,  $!_A$  is a **morphism of cone** over the left **cospan**.

Finally, if  $m : X \rightarrow A$  also satisfies  $\alpha \circ m = p_{A'}$  and  $f \circ m = p_B$ . We find in particular that  $m$  is a **morphism of cones** over the rectangle **cospan**, hence by **universality** of  $A$ ,  $m = !_A$ .  $\square$

**Corollary 182.** In diagram (44) where the right square is not necessarily a **pullback** but the left square is a **pushout**, the right square is a **pushout** if and only if the rectangle is.

See solution.

**Exercise 183.** Show that (47) is a **pullback** square. Let  $i : A' \rightarrow A$  be an **isomorphism**, show that (48) is a **pullback** square.

$$\begin{array}{ccc}
 A & \xrightarrow{\text{id}_A} & A \\
 f \downarrow & \lrcorner & \downarrow f \\
 B & \xrightarrow{\text{id}_B} & B
 \end{array}
 \quad (47)$$

$$\begin{array}{ccc}
 A' & \xrightarrow{i} & A \\
 f \circ i \downarrow & \lrcorner & \downarrow f \\
 B & \xrightarrow{\text{id}_B} & B
 \end{array}
 \quad (48)$$

**Definition 184** ((Co)completeness). A **category** is said to be **(co)complete** (resp. **finitely (co)complete**) if any **small** (resp. finite) **diagram** has a **(co)limit**.

**Theorem 185.** Suppose that a **category**  $\mathbf{C}$  has all **products** and **equalizers** then  $\mathbf{C}$  has all **limits**, i.e.:  $\mathbf{C}$  is **complete**.

*Proof.* Let  $F : J \rightsquigarrow \mathbf{C}$  be a **diagram**, we will show that the **limit** of  $F$  is obtained from the **equalizer** of two **morphisms**<sup>147</sup>

$$u_1, u_2 : \prod_{X \in J_0} F(X) \rightarrow \prod_{a \in J_1} F(t(a)),$$

which are defined below. The **equalizer** and the **products** it involves exist by hypothesis.

<sup>147</sup> Recall that  $s$  and  $t$  denote the **sources** and **targets** of **morphisms**.

Recall that for any  $X \in J_0$  and  $a \in J_1$ , we have two canonical **projections**

$$\pi_X : \prod_{X \in J_0} F(X) \rightarrow F(X) \quad \text{and} \quad \pi_a : \prod_{a \in J_1} F(t(a)) \rightarrow F(t(a)).$$

The first family of **projections** makes  $\prod_{X \in J_0} F(X)$  into a **cone** over  $\{F(t(a)) \mid a \in J_1\}$  with **projections**  $\pi_{t(a)}$ . Hence, there is a unique **morphism**  $u_1 : \prod_{X \in J_0} F(X) \rightarrow \prod_{a \in J_1} F(t(a))$  that satisfies  $\pi_a \circ u_1 = \pi_{t(a)}$ . What is more, there is another way to **project** from  $\prod_{X \in J_0} F(X)$  to  $F(t(a))$ , namely, via  $F(a) \circ \pi_{s(a)}$ , thus we get a unique **morphism**  $u_2 : \prod_{X \in J_0} F(X) \rightarrow \prod_{a \in J_1} F(t(a))$  that satisfies  $\pi_a \circ u_2 = F(a) \circ \pi_{s(a)}$ . The situation is summarized in (49).

$$\begin{array}{ccc} \prod_{X \in J_0} F(X) & \xlongequal{\quad} & \prod_{X \in J_0} F(X) \\ & \searrow u_1 \quad \swarrow u_2 & \\ & \prod_{a \in J_1} F(t(a)) & \\ & \downarrow \pi_a & \\ \pi_{t(a)} \swarrow & F(t(a)) & \nwarrow F(a) \circ \pi_{s(a)} \end{array} \quad (49)$$

Let  $e : E \rightarrow \prod_{X \in J_0} F(X)$  be the **equalizer** of  $u_1$  and  $u_2$  and for any  $X \in J_0$ , let  $\psi_X = \pi_X \circ e$ . For any  $f : Y \rightarrow Z$  in  $J$ , we have

$$\begin{aligned} F(f) \circ \psi_Y &= F(f) \circ \pi_Y \circ e && \text{(def. of } \psi_Y) \\ &= \pi_f \circ u_2 \circ e && \text{(def. of } u_2) \\ &= \pi_f \circ u_1 \circ e && \text{(def. of } e) \\ &= \pi_Z \circ e = \psi_Z, && \text{(def. of } u_1 \text{ and } \psi_Z) \end{aligned}$$

so we indeed obtain a **cone** from  $E$  to  $F$ , depicted in (50).

$$\begin{array}{ccc} & E & \\ \pi_X \circ e \swarrow & & \searrow \pi_Y \circ e \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array} \quad (50)$$

Next, any other **cone**  $\{U_X : O \rightarrow F(X)\}_{X \in J_0}$  over  $F$  can also be viewed as a **cone** over the **discrete diagram**  $\{F(t(a))\}_{a \in J_1}$  with **projections**  $\{U_{t(a)}\}_{a \in J_1}$ . Moreover, the **universality** of the **product** yields a unique **morphism**  $p : O \rightarrow \prod_{X \in J_0} F(X)$  such that  $\pi_X \circ p = U_X$ . We claim that both  $u_1 \circ p$  and  $u_2 \circ p$  make (51) **commute** for all  $a \in J_1$ .

$$\begin{array}{ccc} O & \xrightarrow{p} \prod_{X \in J_0} F(X) & \xrightarrow{u_i} \prod_{a \in J_1} F(t(a)) \\ & \searrow U_{t(a)} & \downarrow \pi_a \\ & & F(t(a)) \end{array} \quad (51)$$

This follows from two simple derivations.

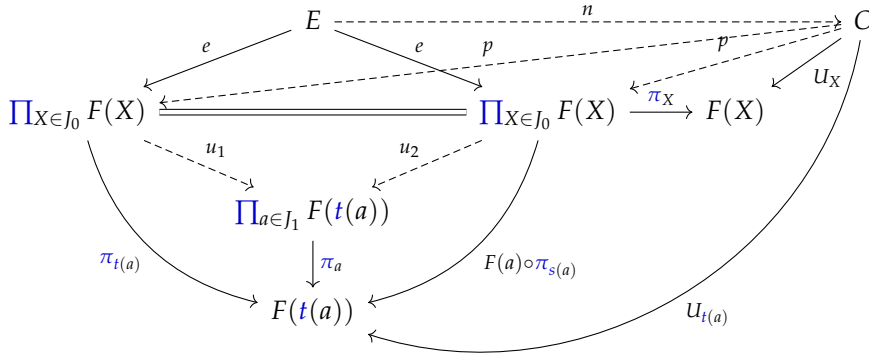
$$\begin{aligned} \pi_a \circ u_1 \circ p &= \pi_{t(a)} \circ p \\ &= U_{t(a)} \end{aligned} \qquad \begin{aligned} \pi_a \circ u_2 \circ p &= F(a) \circ \pi_{s(a)} \circ p \\ &= F(a) \circ U_{s(a)} \\ &= U_{t(a)} \end{aligned}$$

Hence,  $u_1 \circ p = u_2 \circ p$  as they are both **morphisms** of **cone** to the **terminal cone**  $\prod_{a \in J_1} F(t(a))$ . Now, by **universality** of the **equalizer**, we get a unique **morphism**  $n : O \rightarrow E$  such that  $e \circ n = p$ . Furthermore, for any  $X \in J_0$ , we have

$$\psi_X \circ n = \pi_X \circ e \circ n = \pi_X \circ p = U_X,$$

so  $n$  is also a **morphism** of **cones**  $(O, U_X) \rightarrow (E, \psi_X)$ . Since any other **morphism** of **cones**  $m$  needs to satisfy  $e \circ m = p$ , we see that  $n$  is unique and conclude that  $E$  is **lim** $F$ .

Just for fun, here is what the whole diagram would look like if it were drawn at once (on the board or on paper).



□

*Remark 186.* The same proof yields a more general statement: For any cardinal  $\kappa$ , if a **category**  $\mathbf{C}$  has all **products** of size less than  $\kappa$  and **equalizers**, then it has **limits** of any **diagram** with less than  $\kappa$  **objects** and **morphisms**.

▮ **Definition 187.** A **functor**  $\mathbf{C} \rightsquigarrow \mathbf{D}$  is said to be **(finitely) (co)continuous** if it **pre-serves** all (finite) **(co)limit**.

**Exercise 188.** Show that a **category** with all **pullbacks** and a **terminal object** is **finitely complete**.

See solution.



# Universal Properties

## Examples

### Free Monoid

The construction of a *free* object is common to different fields of mathematics and the example we will carry out in **Mon** can be carried out in many other *categories* like **Grp**, **Ab**, **Ring**, **Mod<sub>R</sub>** (we will do this one in the next section). In fact, one way to view this construction comes from the *forgetful functor* to **Set** that all these *categories* have in common. In Chapter , we will cover *adjoints* and recover the free constructions from  $U$ .

We choose **Mon** because the concrete characterization of a *free monoid* is the simplest.

□ **Definition 189** (Classical). A *monoid*  $M$  is said to be *free* if it can be *presented* by a set of *generators* without any *relations*, i.e.  $M = \langle A \mid \emptyset \rangle$ . In this case,  $M$  is called the *free monoid on  $A$*  and denoted  $A^*$ .

It is easy to check that  $A^*$  is the set of finite words with symbols in  $A$  with the operation being concatenation and identity being the empty word (denoted  $\epsilon$ ). In order to give a categorical characterization, we need to look at *homomorphisms* from or into the *free monoid*. Notice that any *homomorphism*  $h^* : A^* \rightarrow M$  is completely determined by where  $h^*$  sends elements of  $A$ . Indeed, in order to satisfy the *homomorphism* property, we must have for any  $a_1, a_2 \in A$ ,

$$h^*(a_1 a_2) = h^*(a_1) \cdot h^*(a_2) \text{ and } h^*(\epsilon) = 1_M.$$

In general, the unique *homomorphism* sending  $a \in A$  to  $h(a)$  can be defined recursively:

$$h^*(w) = \begin{cases} h(a) \cdot h^*(w') & a \in A, w \in A^*, w = aw' \\ 1_M & w = \epsilon \end{cases}.$$

Now, suppose that a *monoid*  $N$  contains  $A$  and satisfies the same property, that is for any (set-theoretic) function  $h : A \rightarrow M$ , there is a unique *homomorphism*  $h^* : N \rightarrow M$  with  $h^*(a) = h(a)$ .

If we take  $M = A^*$ , and  $h : A \rightarrow A^* = a \mapsto a$ , then we get a *homomorphism*  $h_N^* : N \rightarrow A^*$ . Moreover, taking  $M = N$  and  $i : A \hookrightarrow N$  be the inclusion, the

property of  $A^*$  means there is a unique **homomorphism**  $i^* : A^* \rightarrow N$ . Note that  $h_N^* \circ i^* : A^* \rightarrow A^*$  is a **homomorphism** satisfying  $a \mapsto a$ , so it must be the identity by uniqueness. We conclude that  $N$  and  $A^*$  are **isomorphic**.

□ **Definition 190** (Categorical). The **free monoid** of a set  $A$  is an object  $A^*$  in **Mon** along with a *canonical inclusion*  $i : A \rightarrow U(A^*)$  that satisfies the following **universal property**: for any **monoid**  $M$  and function  $h : A \rightarrow U(M)$ , there exists a unique **homomorphism**  $h^* : A^* \rightarrow M$  such that  $U(h^*) \circ i = h$ , namely,  $h^*(i(a)) = h(a)$ . This is summarized in (52), where we omit the  $U$  as the underlying set of a **monoid** is often denoted with the same symbol as the **monoid**.

$$\begin{array}{ccc}
 \text{in Set} & & \text{in Mon} \\
 A & \xrightarrow{i} & A^* \\
 & \searrow h & \downarrow h^* \\
 & & M
 \end{array}
 \quad
 \begin{array}{c}
 \text{forgetful} \\
 \longleftarrow
 \end{array}
 \quad
 \begin{array}{ccc}
 & & A^* \\
 & & \downarrow h^* \\
 & & M
 \end{array}
 \quad (52)$$

### Abelianization

□ **Definition 191** (Classical). Let  $G$  be a group, the **abelianization** of  $G$ , denoted  $G^{\text{ab}}$ , is the **quotient** of  $G$  with  $G' := \{xyx^{-1}y^{-1} \mid x, y \in G\} \leq G$ , called the **commutator subgroup**, that is  $G^{\text{ab}} := G/G'$ .

Let us get insight into this definition. The **abelianization** is supposed to be the **biggest abelian quotient** of  $G$ . To see why, note that if  $A$  is an **abelian group**, any **homomorphism**  $h : G \rightarrow A$  must satisfy  $h(xyx^{-1}y^{-1}) = 1_A$  for any  $x, y \in G$ . Hence,  $G'$  is contained in the **kernel** of  $h$ . This yields a factorization  $h = G \xrightarrow{\pi} G/G' \xrightarrow{h^*} A$  with  $h^*$  unique, where  $\pi$  is the canonical **quotient** map.

Moreover, since **Ab** is a **full subcategory** of **Grp**,  $h^*$  is also unique as a **morphism** in **Ab**. Using the fact that  $G/G'$  is **abelian**, we conclude the following categorical definition of  $G^{\text{ab}}$ .

**Definition 192** (Categorical). Let  $G$  be a group, the **abelianization** of  $G$  is an **abelian group**  $G^{\text{ab}}$  with a map  $\pi : G \rightarrow G^{\text{ab}}$  satisfying the following **universal property**: for any **homomorphism**  $h : G \rightarrow A$  where  $A$  is **abelian**, there is a unique **homomorphism**  $h^* : G^{\text{ab}} \rightarrow A$  such that  $h^* \circ \pi = h$ . This is summarized in (53).

$$\begin{array}{ccc}
 \text{in Grp} & & \text{in Ab} \\
 G & \xrightarrow{\pi} & G^{\text{ab}} \\
 & \searrow h & \downarrow h^* \\
 & & A
 \end{array}
 \quad
 \begin{array}{c}
 \text{forgetful} \\
 \longleftarrow
 \end{array}
 \quad
 \begin{array}{ccc}
 & & G^{\text{ab}} \\
 & & \downarrow h^* \\
 & & A
 \end{array}
 \quad (53)$$

### Vector Space Basis

□ **Definition 193** (Classical). Let  $V$  be a **vector space** over a **field**  $k$ , a **basis** for  $V$  is a subset  $S \subseteq V$  that is **linearly independent** and **generates**  $V$ , namely, any  $v \in V$  can be expressed as a **linear combination** of elements in  $S$  and any  $s \in S$  cannot be expressed as a **linear combination** of elements in  $S \setminus \{s\}$ .



Once again, we would like to get rid of the content of this definition talking about elements, so we focus on what this means for **linear maps** coming out of  $V$ . Let  $S$  be a **basis** of  $V$ ,  $W$  be another **vector space** over  $k$  and  $T : V \rightarrow W$  be a **linear map**. By **linearity**,  $T$  is completely determined by where it sends the elements of  $S$ . Indeed, for any  $v \in V$ , write  $v$  as a **linear combination**  $\sum_{s \in S} \lambda_s s$  with  $\lambda_s \in k$  (only finitely many of the coefficients are non-zero), then  $T(v) = \sum_{s \in S} \lambda_s T(s)$ . We conclude that any (set-theoretic) function  $t : S \rightarrow W$  extends to a unique **linear map**  $T : V \rightarrow W$ .

We claim that this property completely characterizes **bases** of  $V$ . Indeed, let  $S \subseteq V$  be such that for any  $t : S \rightarrow W$ , there is a unique **linear map**  $T : V \rightarrow W$  extending  $t$ . We will show that  $S$  is **generating** and **linearly independent**.

1. Assume towards a contradiction that  $S$  is not **generating**, that is, there exists  $v \in V$  that is not a **linear combination** of vectors in  $S$ . Equivalently, if  $U$  is the **subspace generated** by  $S$ , then  $V/U$  is not 0. Now, let  $t : S \rightarrow V/U$  be the 0 map, both the quotient map  $\pi : V \rightarrow V/U$  and the 0 map  $0 : V \rightarrow V/U$  extend  $t$ , and since  $V/U$  is not trivial, they are different maps.
2. Assume towards a contradiction that  $S$  is not **linearly dependent**, that is, there exists  $v \in S$  such that  $v = \sum_{s \in S-v} \lambda_s s$ . Consider the function

$$t : S \rightarrow V \oplus V = \begin{cases} (s, 0) & s \neq v \\ (0, v) & s = v \end{cases}.$$

There cannot exist a **linear map**  $T : V \rightarrow V \oplus V$  extending  $t$  because by **linearity**, we can show

$$(0, v) = t(v) = T(v) = T\left(\sum_{s \in S-v} \lambda_s s\right) = \sum_{s \in S-v} \lambda_s T(s) = \sum_{s \in S-v} \lambda_s (s, 0),$$

which is absurd.

In conclusion, we have the following alternate definition of a **vector space basis**.

**Definition 194** (Categorical). Let  $V$  be a **vector space**, a **basis** of  $V$  is a set  $S$  along with an inclusion  $i : S \rightarrow V$  satisfying the following **universal property**: for any function  $t : S \rightarrow W$  where  $W$  is a **vector space**, there is a unique **linear map**  $T : V \rightarrow W$  such that  $T \circ i = t$ . This is summarized in (54).

$$\begin{array}{ccc} \text{in } \mathbf{Set} & & \text{in } \mathbf{Vect}_k \\ S & \xrightarrow{i} & V \\ & \searrow t & \downarrow T \\ & & W \end{array} \quad \begin{array}{c} \xleftarrow{\text{forgetful}} \\ \downarrow T \\ W \end{array} \quad (54)$$

## Exponential Objects

This section and the following two are motivated by important constructions in **Set** that we want to define categorically. Going further in this direction amounts to doing topos theory, namely, studying **categories** which look a lot like **Set**.

**Exercise 195.** Let  $\mathbf{C}$  be a **category** and  $X \in \mathbf{C}_0$  be such that for any  $Y \in \mathbf{C}_0$ ,  $Y \times X$  exists. Show that  $- \times X$  is a **functor**  $\mathbf{C} \rightsquigarrow \mathbf{C}$ .

Let  $A$  and  $X$  be sets,  $A^X$  commonly denotes the set of functions  $X \rightarrow A$ . In hope to generalize this construction to other **categories**, let us study **morphisms** into  $A^X$ .

Given a set  $B$  and a **morphism**  $f : B \rightarrow A^X$ , there is a natural operation called **uncurrying** that takes  $f$  to  $\lambda^{-1}f : B \times X \rightarrow A$  which basically evaluates both  $f$  and its output at the same time. Namely,  $\lambda^{-1}f(b, x) = f(b)(x)$ .

As a particular case, we consider the identity function  $A^X \rightarrow A^X$ . **Uncurrying** yields the **evaluation** function  $\text{ev} : A^X \times X \rightarrow A$  that evaluates the function in the first coordinate at the second coordinate:  $\text{ev}(f, x) = f(x)$ .

Now, as the name suggests, **uncurrying** has an inverse operation called **currying** which takes  $g : B \times X \rightarrow A$  to  $\lambda g : B \rightarrow A^X$  defined by  $\lambda g(b) = x \mapsto g(b, x)$ . Morally,  $\lambda g$  delays the evaluation of  $g$  to later.<sup>148</sup> Moreover, notice that the **currying** of  $g$  satisfies  $\text{ev}(\lambda g(b), x) = g(b, x) \in A$  for any  $b \in B$  and  $x \in X$ . This along with the fact that **currying** and **uncurrying** are bijective operations<sup>149</sup> leads to a **universal property** that  $\text{ev}$  satisfies. It is summarized in (55).

$$\begin{array}{ccc}
 \text{in } \mathbf{Set} & & \text{in } \mathbf{Set} \\
 A & \xleftarrow{\text{ev}} & A^X \times X \\
 & \nwarrow g & \uparrow \lambda g \times \text{id}_X \\
 & & B \times X
 \end{array}
 \quad
 \begin{array}{ccc}
 & & A^X \\
 & & \uparrow \lambda g \\
 & & B
 \end{array}
 \xleftarrow{- \times X}
 \begin{array}{ccc}
 & & A^X \\
 & & \uparrow \lambda g \\
 & & B
 \end{array}
 \quad (55)$$

This is entirely categorical, so we can define **exponential objects** as follows.

**Definition 196** (Exponential). Let  $\mathbf{C}$  be a **category** and  $X \in \mathbf{C}_0$  be such that  $- \times X$  is a **functor**.<sup>150</sup> For  $A \in \mathbf{C}_0$ , the **exponential**  $A^X$  (if it exists) is an **object**  $A^X$  along with a **morphism**  $\text{ev} : A^X \times X \rightarrow A$  such that for all  $g : B \times X \rightarrow A$ , there is a unique  $\lambda g : B \rightarrow A^X$  making (55) **commute**.

### Subobject Classifier

**Exercise 197.** Let  $\mathbf{C}$  be a **well-powered category** with all **pullbacks**. We define  $\text{Sub}_{\mathbf{C}}$  on **morphisms**: it sends  $f : X \rightarrow Y$  to  $f^*(-) : \text{Sub}_{\mathbf{C}}(Y) \rightarrow \text{Sub}_{\mathbf{C}}(X)$  sending  $m : I \rightarrow Y$  to  $f^*(m)$  (the **pullback** of  $m$  **along**  $f$  as depicted in (56)). Show that this is well-defined and makes  $\text{Sub}_{\mathbf{C}}$  into a **functor**  $\mathbf{C}^{\text{op}} \rightsquigarrow \mathbf{Set}$ .

In **Set**, recall that **subobjects** are subsets. Hence, letting  $\Omega = \{\perp, \top\}$  there is a correspondence between  $\text{Sub}_{\mathbf{Set}}(X)$  and  $\text{Hom}_{\mathbf{Set}}(X, \Omega)$ , it sends  $I \subseteq X$  to the **characteristic** function  $\chi_I : X \rightarrow \Omega$ ,<sup>151</sup> and in the other direction  $f : X \rightarrow \Omega$  is sent to  $f^{-1}(\top) \subseteq X$ . Furthermore, recall that the preimage can be seen as a **pullback**, so we can define  $\chi_I$  as the unique function making (57) into a **pullback** square.

See solution.

<sup>148</sup> For computer scientists, this is also related to the concept of *continuations*.

<sup>149</sup> Check that  $\lambda \lambda^{-1}g = g$  and  $\lambda^{-1}\lambda g = g$ .

<sup>150</sup> i.e.: all **binary products** with  $X \in \mathbf{C}_0$  exist.

See solution.

$$\begin{array}{ccc}
 J & \longrightarrow & I \\
 f^*(m) \downarrow & \lrcorner & \downarrow m \\
 X & \xrightarrow{f} & Y
 \end{array}
 \quad (56)$$

<sup>151</sup> The **characteristic** function  $\chi_I$  is defined by

$$\chi_I(x) = \begin{cases} \top & x \in X \\ \perp & x \notin X \end{cases}$$

Uniqueness holds because this **pullback** implies  $I = \chi_I^{-1}(\top)$ .

$$\begin{array}{ccc} I & \longrightarrow & \mathbf{1} \\ \downarrow \lrcorner & & \downarrow \top \\ X & \xrightarrow{\chi_I} & \Omega \end{array} \quad (57)$$

The role played by the two element set  $\{\perp, \top\}$  can now be generalized to other **categories**.

▮ **Definition 198** (Subobject classifier). Let  $\mathbf{C}$  be a **category** with a **terminal** object  $\mathbf{1}$ . A **subobject classifier** is a **morphism**  $\top : \mathbf{1} \rightarrow \Omega \in \mathbf{C}_1$  such that for any **monomorphism**  $I \hookrightarrow X$  there is a unique **morphism**  $\chi_m : X \rightarrow \Omega$  such that (57) is a **pullback** square. We call  $\chi_I$  the **characteristic map** of  $I \hookrightarrow X$ .

Before drawing a diagram like those above to summarize the **universal property** of a **subobject classifier**, we need to make sure that the **characteristic maps** of two **monomorphisms** in the same equivalence class in  $\text{Sub}_{\mathbf{C}}(X)$  are equal. Looking at (58), the right square is a **pullback** by hypothesis and the left square is a **pullback** by Exercise 183. Therefore, the rectangle is a **pullback** by the **pasting lemma** and we see that  $\chi_{I'} = \chi_I \circ \text{id}_X$  by uniqueness of the **characteristic map**.

Now, in a **well-powered category**  $\mathbf{C}$  has a **terminal** object and all **pullbacks**,<sup>152</sup> a **subobject classifier**  $\top : \mathbf{1} \rightarrow \Omega$  is such that for any **subobject**  $m$  of  $X$ , which we identify as a **morphism**  $m : \mathbf{1} \rightarrow \text{Sub}_{\mathbf{C}}(X)$ , there is a unique **morphism**  $\chi_m : X \rightarrow \Omega$  such that  $\chi_m^*(\top) = m$ . This is summarized in (59).

$$\begin{array}{ccc} \text{in Set} & & \text{in } \mathbf{C} \\ \mathbf{1} & \xrightarrow{\top} & \text{Sub}_{\mathbf{C}}(\Omega) \\ & \searrow m & \downarrow \chi_m^*(-) \\ & & \text{Sub}_{\mathbf{C}}(X) \end{array} \quad \begin{array}{ccc} & \xleftarrow{\text{Sub}_{\mathbf{C}}} & \Omega \\ & \uparrow \chi_m & \\ & X & \end{array} \quad (59)$$

## Power Objects

Let  $X$  be a set, the **powerset** of  $X$ ,  $\mathcal{P}X$  is the set of all subsets of  $X$ .

$$\begin{array}{ccc} \text{in Set} & & \text{in } \mathbf{C} \\ \mathbf{1} & \xrightarrow{\exists_A} & \text{Sub}_{\mathbf{C}}(\mathcal{P}X \times X) \\ & \searrow m & \downarrow \chi_m^*(- \times \text{id}_X) \\ & & \text{Sub}_{\mathbf{C}}(Y \times X) \end{array} \quad \begin{array}{ccc} & \xleftarrow{\text{Sub}_{\mathbf{C}}(- \times X)} & \mathcal{P}X \\ & \uparrow \chi_m & \\ & Y & \end{array} \quad (60)$$

## Generalization

▮ **Definition 199** (Comma category). Given two **functors**  $\mathbf{D} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{E}$ , there is a **category**  $F \downarrow G$ ,<sup>153</sup> called the **comma category**, whose **objects** are triples  $(X, Y, \alpha)$

$$\begin{array}{ccccc} I' & \xleftarrow{\sim} & I & \longrightarrow & \mathbf{1} \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \top \\ X & \xrightarrow{\text{id}_X} & X & \xrightarrow{\chi_I} & \Omega \end{array} \quad \begin{array}{c} \searrow \chi_{I'} \nearrow \end{array} \quad (58)$$

<sup>152</sup> The definition of **subobject classifier** does not need the **well-poweredness** and the existence of all **pullbacks**, but they are necessary to have a **universal property** because it uses the **functor**  $\text{Sub}_{\mathbf{C}}$ . In any case, **subobject classifiers** are usually used when these conditions are satisfied.

Notice that the dashed arrow gets reversed because  $\text{Sub}_{\mathbf{C}}$  is **contravariant**. We could also write “in  $\mathbf{C}^{\text{op}}$ ” and not reverse the arrow.

A **finitely complete category** where every **object** has a **power object** is called an **(elementary) topos**. Topos theory is a vast subject concerned with properties and uses of toposes.

<sup>153</sup> Some authors denote this **category**  $F/G$ .

with  $X \in \mathbf{D}_0$ ,  $Y \in \mathbf{E}_0$  and  $\alpha : F(X) \rightarrow G(Y)$  (in  $\mathbf{C}_1$ ) and **morphisms** between  $(X_1, Y_1, \alpha)$  and  $(X_2, Y_2, \beta)$  are pairs of **morphisms**  $(f, g) \in \text{Hom}_{\mathbf{D}}(X_1, X_2) \times \text{Hom}_{\mathbf{E}}(Y_1, Y_2)$  yielding a **commutative** square as in (61).

$$\begin{array}{ccc} F(X_1) & \xrightarrow{F(f)} & F(X_2) \\ \alpha \downarrow & & \downarrow \beta \\ G(Y_1) & \xrightarrow{G(g)} & G(Y_2) \end{array} \quad (61)$$

**Definition 200** (Arrow category). In the setting of Definition 199, if  $F = G = \text{id}_{\mathbf{C}}$ , then  $\text{id}_{\mathbf{C}} \downarrow \text{id}_{\mathbf{C}}$  is called the **arrow category** of  $\mathbf{C}$  and denoted  $\mathbf{C}^{\rightarrow}$ . Its **objects** are **morphisms** in  $\mathbf{C}$  and its **morphisms** are **commutative** squares in  $\mathbf{C}$ .<sup>154</sup>

**Definition 201** (Slice category). In the setting of Definition 199, if  $F = \text{id}_{\mathbf{C}}$  and  $G = X : \mathbf{1} \rightsquigarrow \mathbf{C}$  is a **constant functor** selecting one **object**  $G(\bullet) = X \in \mathbf{C}_0$ , then  $\text{id}_{\mathbf{C}} \downarrow X$  is called the **slice category** over  $X$  and denoted  $\mathbf{C}/X$ .<sup>155</sup> Its **objects** are **morphisms** in  $\mathbf{C}$  with **target**  $X$  and its **morphisms** are **commutative** triangles with  $X$  as a tip as in (64).

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ & \searrow & \swarrow \\ & X & \end{array} \quad (63)$$

**Definition 202** (Coslice category). In the setting of Definition 199, if  $G = \text{id}_{\mathbf{C}}$  and  $F = X : \mathbf{1} \rightsquigarrow \mathbf{C}$  is a **constant functor** selecting one **object**  $F(\bullet) = X \in \mathbf{C}_0$ , then  $X \downarrow \text{id}_{\mathbf{C}}$  is called the **coslice category** under  $X$  and denoted  $X/\mathbf{C}$ .<sup>156</sup> Its **objects** are **morphisms** in  $\mathbf{C}$  with **source**  $X$  and its **morphisms** are **commutative** triangles with  $X$  as a tip as in (64).

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ A & \xrightarrow{\quad} & B \end{array} \quad (64)$$

**Exercise 203.** Show that for any **category**  $\mathbf{C}$  and **object**  $X \in \mathbf{C}_0$ , the **slice category**  $\mathbf{C}/X$  has a **terminal object**. State and prove the **dual** statement.

Back to **universal properties**.

**Definition 204** (Universal morphism). If  $F : \mathbf{D} \rightsquigarrow \mathbf{C}$  is a functor and  $X \in \mathbf{C}_0$ .  
A **universal morphism** from  $X$  to  $F$  is an **initial object** in  $X \downarrow F$ . Namely, it is a **morphism**  $a : X \rightarrow F(A)$  such that for any other **morphism**  $b : X \rightarrow F(B)$ , there is unique **commutative** triangle as in (65).

$$\begin{array}{ccc} & X & \\ \swarrow a & & \searrow b \\ F(A) & \xrightarrow{\quad F(f) \quad} & F(B) \end{array} \quad (65)$$

<sup>154</sup> Less concisely, a **morphism**  $\phi : f \rightarrow g$  between **morphisms**  $f : X \rightarrow Y$  and  $g : X' \rightarrow Y'$  is a pair of **morphisms**  $\phi_X : X \rightarrow X'$  and  $\phi_Y : Y \rightarrow Y'$  making (??) **commute**.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi_X \downarrow & & \downarrow \phi_Y \\ X' & \xrightarrow{g} & Y' \end{array} \quad (62)$$

<sup>155</sup> Some authors call this **category**  $\mathbf{C}$  over  $X$ .

<sup>156</sup> Some authors call this **category**  $\mathbf{C}$  under  $X$ .

See solution.

Notice that equivalently, one could say that for any  $b : X \rightarrow F(B)$ , there is a unique **morphism**  $f : A \rightarrow B$  in  $\mathbf{D}$  such that  $F(f) \circ a = b$ , which is summarized in (66).

$$\begin{array}{ccc}
 \text{in } \mathbf{C} & & \text{in } \mathbf{D} \\
 X & \xrightarrow{a} & FA \\
 & \searrow b & \downarrow Ff \\
 & & FB
 \end{array}
 \quad
 \begin{array}{ccc}
 & & A \\
 & \xleftarrow{F} & \downarrow f \\
 & & B
 \end{array}
 \quad (66)$$

The **dual** notion is a **universal morphism** from  $F$  to  $X$ , it is a **terminal object** in  $F\downarrow X$ . The **dual** of (66) is depicted below.

$$\begin{array}{ccc}
 \text{in } \mathbf{C} & & \text{in } \mathbf{D} \\
 A & \xleftarrow{a} & FA \\
 & \swarrow b & \uparrow Ff \\
 & & FB
 \end{array}
 \quad
 \begin{array}{ccc}
 & & A \\
 & \xleftarrow{F} & \uparrow f \\
 & & B
 \end{array}
 \quad (67)$$

□ **Definition 205** (Universal property). A **universal property** is the property of being a **universal morphism**.

**Examples 206.** Here we translate all the examples of this chapter into the general language.

1. The **free monoid** on a set  $A$  is the **universal morphism** from  $A$  to the **forgetful functor**  $\mathbf{Mon} \rightsquigarrow \mathbf{Set}$ .
2. The **abelianization** of a **group**  $G$  is the **universal morphism** from  $G$  to the **forgetful functor**  $\mathbf{Ab} \rightsquigarrow \mathbf{Grp}$ .
3. The set  $S \subseteq V$  is a **basis** for the **vector space**  $V$  when the inclusion  $S \hookrightarrow V$  is the **universal morphism** from  $S$  to the **forgetful functor**  $\mathbf{Vect}_k \rightsquigarrow \mathbf{Set}$ .
4. An **exponential object** is an **object**  $A^X$  along with a **universal morphism** from the **functor**  $- \times X$  to  $A$ .
5. A **subobject classifier** is a **morphism**  $\top : \mathbf{1} \hookrightarrow \Omega$  such that the corresponding function  $\top : \mathbf{1} \rightarrow \mathbf{Sub}_{\mathbf{C}}(\Omega)$  is a **universal morphism** from  $\mathbf{1}$  to the **functor**  $\mathbf{Sub}_{\mathbf{C}}$ .
6. A **power object** of  $X$  is an **object**  $\mathfrak{P}X$  along with a **universal morphism**  $\ni_X$  from  $\mathbf{1}$  to  $\mathbf{Sub}_{\mathbf{C}}(- \times X)$ .

We will not bother applying this general definition anymore because the formalism is not crucial to the study of **universal properties**.

We have to postpone to Chapter showing that, as we have claimed, any **(co)limit** satisfies a **universal property**. Still, you might have noticed that our definition of **universal property** also uses a special case of **(co)limits**, that is, **initial** and **terminal objects**. What is more, in the following chapters, we will introduce a couple more concepts which often coincide<sup>157</sup> with the concepts of **(co)limits** and **universal properties**.

<sup>157</sup> By *coincide*, we mean that one is a special case of the other or vice-versa or both directions.



# Natural Transformations

## Natural Transformations

**Natural transformations** are admittedly what made mathematicians want to study category theory in the first place. In short, they are **morphisms** between **functors**, i.e.: transformations that preserve the structure of **functors**.

The abstract structure of a **category** is very familiar because it resembles what is found in algebraic structures such as **groups**, **rings** or **vector spaces**. That is to say, it consists of the data of one or more sets with one or more operations satisfying one or more properties. In contrast, the definition of a functor is more opaque and by itself, the structure of a functor is not obvious. A functor is effectively a morphism between categories, hence a natural transformation will be a *morphism between morphisms*. Before moving on, one might find it enlightening to look for a satisfying definition of morphism between two group homomorphisms  $f, g : G \rightarrow H$  and then observe its meaning when  $f$  and  $g$  are seen as functors  $\mathbf{B}G \rightsquigarrow \mathbf{B}H$ .

For the general case, let  $F, G : \mathbf{C} \rightsquigarrow \mathbf{D}$  be **functors**. Morally, the structure of  $F$  and  $G$  is encapsulated in the following diagrams for every arrow,  $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$ .

$$\begin{array}{ccc} A & \xrightarrow{F_0} & F(A) \\ f \downarrow & & \downarrow F_1(f) \\ B & \xrightarrow{F_0} & F(B) \end{array} \quad (68)$$

$$\begin{array}{ccc} A & \xrightarrow{G_0} & G(A) \\ f \downarrow & & \downarrow G_1(f) \\ B & \xrightarrow{G_0} & G(B) \end{array} \quad (69)$$

Thus, a **morphism** between  $F$  and  $G$  should fit in this picture by sending diagram (68) to diagram (69) in a **commutative** way.

▮ **Definition 207** (Natural transformation). Let  $F, G : \mathbf{C} \rightsquigarrow \mathbf{D}$  be two (**covariant**) **functors**, a **natural transformation**  $\phi : F \Rightarrow G$  is a map  $\phi : \mathbf{C}_0 \rightarrow \mathbf{D}_1$  that satisfies  $\phi(A) \in \mathbf{Hom}_{\mathbf{D}}(F(A), G(A))$  for all  $A \in \mathbf{C}_0$  and makes (70) **commute** for any  $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$ :<sup>158</sup>

$$\begin{array}{ccc} F(A) & \xrightarrow{\phi(A)} & G(A) \\ F(f) \downarrow & & \downarrow G(f) \\ F(B) & \xrightarrow{\phi(B)} & G(B) \end{array} \quad (70)$$

▮ Each  $\phi(A)$  will be called a **component** of  $\phi$  and may also be denoted  $\phi_A$ .

<sup>158</sup> When doing proofs relying on **naturality** (i.e.: the property of being **natural**), we will use (70) where we instantiate  $\phi, F, G, A, B$  and  $f$  with the **natural transformation**, **functors**, **objects** and **morphism** that is needed in the proof. In order to make this instantiation less painful, we will use the shorthand  $\mathbf{NAT}(\phi, A, B, f)$  and instantiate the parameters (we can omit  $F$  and  $G$  because they should be known from the context).

As usual, there are trivial examples of **natural transformations** such as the **identity transformation**  $1_F : F \Rightarrow F$  that sends every **object**  $A$  to the **identity** map  $\text{id}_{F(A)}$ , but let us go back to the group case. Although very specific to **single object categories**, it is simple enough to quickly digest.

**Example 208.** Let  $f, g : \mathbf{BG} \rightsquigarrow \mathbf{BH}$  be **functors** (i.e.: **group homomorphisms**), both send the unique **object**  $*$  in  $\mathbf{BG}$  to  $*$  in  $\mathbf{BH}$ . Thus, a **natural transformation**  $\phi : f \Rightarrow g$  has a single **component**  $\phi(*) : * \rightarrow *$  in  $H$ , which is simply an element  $\phi \in H$ . The **commutativity** condition is then exhibited by diagram (71) (which lives in  $\mathbf{BH}$ ) for any  $x \in G$ .

$$\begin{array}{ccc} * & \xrightarrow{\phi} & * \\ f(x) \downarrow & & \downarrow g(x) \\ * & \xrightarrow{\phi} & * \end{array} \quad (71)$$

Recall that composition in  $\mathbf{BH}$  is just multiplication in  $H$ , so **naturality** of  $\phi$  says that for any  $x \in G$ ,  $\phi \cdot f(x) = g(x) \cdot \phi$ . Equivalently,  $\phi f(x) \phi^{-1} = g(x)$ . Therefore,  $g = c_\phi \circ f$  where  $c_\phi$  denotes **conjugation** by  $\phi$ .<sup>159</sup> In short, **natural transformations** between **group homomorphisms** correspond to factorizations through **conjugations**.

<sup>159</sup> In a **group**  $(H, \cdot)$ , **conjugation** by an element  $h \in H$  is the **homomorphism**  $c_h$  defined  $x \mapsto h x h^{-1}$ .

Next, an example closer to the general idea of a **natural transformation**.

**Example 209.** Fix some  $n \in \mathbb{N}$  and define the **functor**  $\text{GL}_n : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$  by<sup>160</sup>

$$\begin{aligned} R &\mapsto \text{GL}_n(R) \text{ for any commutative ring } R \text{ and} \\ f &\mapsto \text{GL}_n(f) \text{ for any ring homomorphism } f. \end{aligned}$$

The second **functor** is  $(-)^{\times} : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$  which sends a **commutative ring**  $R$  to its **group of units**  $R^{\times}$  and a **ring homomorphism**  $f$  to  $f^{\times}$ , its restriction on  $R^{\times}$ . Checking these mappings define two **covariant functors** is left as an (simple) exercise, but one might expect these to be **functors** as they play nicely with the structure of the **objects** involved.

A **natural transformation** between these two **functors** is  $\det : \text{GL}_n \Rightarrow (-)^{\times}$  which maps a **commutative ring**  $R$  to  $\det_R$ , the function calculating the **determinant** of a **matrix** in  $\text{GL}_n(R)$ . The first thing to check is that  $\det_R \in \text{Hom}_{\mathbf{Grp}}(\text{GL}_n(R), R^{\times})$  which is clear because the **determinant** of an **invertible matrix** is always a **unit**,  $\det_R(I_n) = 1$  and  $\det_R$  is a multiplicative map.<sup>161</sup> The second thing is to verify that diagram (72) **commutes** for any  $f \in \text{Hom}_{\mathbf{CRing}}(R, S)$ :

$$\begin{array}{ccc} \text{GL}_n(R) & \xrightarrow{\det_R} & R^{\times} \\ \text{GL}_n(f) \downarrow & & \downarrow f^{\times} = f|_{R^{\times}} \\ \text{GL}_n(S) & \xrightarrow{\det_S} & S^{\times} \end{array} \quad (72)$$

We will check the claim for  $n = 2$ , but the general proof should only involve more notation to write the bigger expressions, no novel idea. Let  $a, b, c, d \in R$ , we have

$$(\det_S \circ \text{GL}_2(f)) \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \det_S \left( \begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix} \right)$$

<sup>160</sup> The map  $\text{GL}_n(f)$  is just the extension of  $f$  on  $\text{GL}_n(R)$  by applying  $f$  to every element of the matrices.

<sup>161</sup> i.e.:  $\det_R(AB) = \det_R(A) \det_R(B)$ .



$$\begin{aligned}
&= f(a)f(d) - f(b)f(c) \\
&= f(ad - bc) \\
&= f^\times(ad - bc) \\
&= (f^\times \circ \det_R) \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).
\end{aligned}$$

We conclude that the diagram [commutes](#) and that  $\det$  is indeed a [natural transformation](#).<sup>162</sup>

<sup>162</sup> Modulo the cases  $n > 2$ .  
See solution.

**Exercise 210.** Let  $F, G : \mathbf{C} \times \mathbf{C}' \rightsquigarrow \mathbf{D}$  be two [functors](#). Show that a family

$$\{\phi_{X,Y} : F(X,Y) \rightarrow G(X,Y) \mid X \in \mathbf{C}_0, Y \in \mathbf{C}'_0\}$$

is a [natural transformation](#) if and only if for any  $X \in \mathbf{C}_0$  and  $Y \in \mathbf{C}'_0$ , both

$$\phi_{X,-} : F(X, -) \Rightarrow G(X, -) \text{ and } \phi_{-,Y} : F(-, Y) \Rightarrow G(-, Y)$$

are [natural](#).

Now, in order to talk about a [category](#) of [functors](#), it remains to describe the [composition](#) of [natural transformations](#).

**Definition 211** (Vertical composition). Let  $F, G, H : \mathbf{C} \rightsquigarrow \mathbf{D}$  be [parallel functors](#) and  $\phi : F \Rightarrow G$  and  $\eta : G \Rightarrow H$  be two [natural transformations](#). Then, the [vertical composition](#) of  $\phi$  and  $\eta$ , denoted  $\eta \cdot \phi : F \Rightarrow H$  is defined by  $(\eta \cdot \phi)(A) = \eta(A) \circ \phi(A)$  for all  $A \in \mathbf{C}_0$ . If  $f : A \rightarrow B$  is a [morphism](#) in  $\mathbf{C}$ , then diagram (73) [commutes](#) by [naturality](#) of  $\phi$  and  $\eta$ , showing that  $\eta \cdot \phi$  is a [natural transformation](#) from  $F$  to  $H$ .

$$\begin{array}{ccccc}
F(A) & \xrightarrow{\phi(A)} & G(A) & \xrightarrow{\eta(A)} & H(A) \\
F(f) \downarrow & & G(f) \downarrow & & H(f) \downarrow \\
F(B) & \xrightarrow{\phi(B)} & G(B) & \xrightarrow{\eta(B)} & H(B)
\end{array} \tag{73}$$

The meaning of *vertical* will come to light when [horizontal composition](#) is introduced in a bit.

**Definition 212** (Functor categories). For any two [categories](#)  $\mathbf{C}$  and  $\mathbf{D}$ , there is a [functor category](#) denoted  $[\mathbf{C}, \mathbf{D}]$ .<sup>163</sup> Its [objects](#) are [functors](#) from  $\mathbf{C}$  to  $\mathbf{D}$ , its [morphisms](#) are [natural transformations](#) between such [functors](#) and the [composition](#) is the [vertical composition](#) defined above. One can check that [associativity](#) of  $\cdot$  follows from [associativity](#) of [composition](#) in  $\mathbf{D}$  and that the [identity morphism](#) for a [functor](#)  $F$  is  $\mathbb{1}_F$ .

The notation  $\cdot$  is not widespread, most authors use  $\circ$  because [vertical composition](#) is the [composition](#) in a [functor category](#). We believe the distinction is helpful as you learn this material.

<sup>163</sup> Some authors denote it  $\mathbf{D}^{\mathbf{C}}$ , analogously to the exponential of sets.

**Example 213.** Recall that a [left action](#) of a [group](#)  $G$  on a set  $S$  is just a functor  $\mathbf{B}G \rightsquigarrow \mathbf{Set}$ . Now, between two such [functors](#)  $F, F' \in [\mathbf{B}G, \mathbf{Set}]$ , a [natural transformation](#) is a single map  $\sigma : F(*) \rightarrow F'(*)$  such that  $\sigma \circ F(g) = F'(g) \circ \sigma$  for any  $g \in G$ . In other words, denoting  $\cdot$  for both [group actions](#) on  $F(*)$  and on  $F'(*)$ ,  $\sigma$  satisfies

□  $\sigma(g \cdot x) = g \cdot (\sigma(x))$  for any  $g \in G$  and  $x \in F(*)$ . In group theory, such a map is called **G-equivariant**.

Therefore, the **category**  $[\mathbf{BG}, \mathbf{Set}]$  can be identified as the category of **G-sets** (sets equipped with an **action** of  $G$ ) with **A-equivariant** maps as the **morphisms**.

□ **Exercise 214 (NOW!).** **Isomorphisms** in a **functor category** are called **natural isomorphisms**. Show that they are precisely the **natural transformations** whose **components** are all **isomorphisms**.

**Examples 215.** We can recover constructions we have seen before by studying **categories** of **functors** with a simple domain.

1. The **terminal category**  $\mathbf{1}$  has a single **object**  $\bullet$  and no **morphism** other than the **identity**. Notice that for any **category**  $\mathbf{C}$ , a **functor**  $F : \mathbf{1} \rightsquigarrow \mathbf{C}$  is a simply a choice of **object**  $F(\bullet) \in \mathbf{C}_0$  because  $F(\text{id}_\bullet) = \text{id}_{F(\bullet)}$ . If  $F, G \in [\mathbf{1}, \mathbf{C}]$ , then a **natural transformation**  $\phi : F \Rightarrow G$  is simply a choice of **morphism**  $\phi : F(\bullet) \rightsquigarrow G(\bullet)$  because **naturality** square (74) for the only **morphism**  $\text{id}_\bullet$  is trivially **commutative**. We conclude that  $[\mathbf{1}, \mathbf{C}]$  can be identified with the **category**  $\mathbf{C}$  itself.
2. Similarly, we can see a **functor**  $F : \mathbf{1} + \mathbf{1} \rightsquigarrow \mathbf{C}^{164}$  as a choice of two **objects**  $F(\bullet_1)$  and  $F(\bullet_2)$  and a **natural transformation**  $\phi : F \Rightarrow G$  between two such **functors** as a choice of two **morphisms**  $\phi_1 : F(\bullet_1) \rightarrow G(\bullet_1)$  and  $\phi_2 : F(\bullet_2) \rightarrow G(\bullet_2)$ . Therefore, we infer that  $[\mathbf{1} + \mathbf{1}, \mathbf{C}]$  can be identified with  $\mathbf{C} \times \mathbf{C}$ .
3. Let us go one level harder. A **functor**  $F : \mathbf{2} \rightsquigarrow \mathbf{C}^{165}$  is a choice of two **objects**  $FA$  and  $FB$  as well as a **morphism**  $Ff : FA \rightarrow FB$ . It can also be seen as a single choice of **morphism**  $Ff$  because  $FA$  and  $FB$  are determined to be the **source** and **target** of  $Ff$  respectively. A **natural transformation**  $\phi : F \Rightarrow G$  between two such **functors** is *not* simply a choice of two **morphisms**  $\phi_A : FA \rightarrow GA$  and  $\phi_B : FB \rightarrow GB$  because, while the **naturality** squares for  $\text{id}_A$  and  $\text{id}_B$  trivially **commute**, the **naturality** square (75) for  $f$  is an additional constraint on  $\phi$ . Namely, it says  $(\phi_A, \phi_B)$  makes a **commutative** square with  $Ff$  and  $Gf$ , hence we can identify  $[\mathbf{2}, \mathbf{C}]$  with the **arrow category**  $\mathbf{C}^\rightarrow$ .

It is now time to build intuition for the **horizontal composition** of **natural transformations** which will ultimately lead to the notion of a **2-category**.

**Definition 216** (The left action of functors). Let  $F, F' : \mathbf{C} \rightsquigarrow \mathbf{D}$ ,  $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$  be **functors** and  $\phi : F \Rightarrow F'$  a **natural transformation** as summarized in (76).<sup>166</sup>

$$\begin{array}{ccc} \mathbf{C} & \begin{array}{c} \xrightarrow{F} \\ \Downarrow \phi \\ \xrightarrow{F'} \end{array} & \mathbf{D} \xrightarrow{G} \mathbf{D}' \end{array} \quad (76)$$

The **functor**  $G$  acts on  $\phi$  by sending it to  $G\phi := A \mapsto G(\phi(A)) : \mathbf{C}_0 \rightarrow \mathbf{D}'_1$ . Showing that (77) **commutes** for any  $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$  will imply that  $G\phi$  is a **natural**

See solution.

**Functors** that are **naturally isomorphic** are essentially the same **functor**; they send the same **object** to **isomorphic objects** and the same **morphism** to **morphisms** that are well-behaved under **composition** with **isomorphisms** between the **source** and **targets**.

$$\begin{array}{ccc} F(\bullet) & \xrightarrow{F(\text{id}_\bullet)} & F(\bullet) \\ \phi \downarrow & & \downarrow \phi \\ G(\bullet) & \xrightarrow{G(\text{id}_\bullet)} & G(\bullet) \end{array} \quad (74)$$

<sup>164</sup> Recall  $\mathbf{1} + \mathbf{1}$  is the **category** depicted in (4).

<sup>165</sup> Recall  $\mathbf{2}$  is the **category** depicted in (5).

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \phi_A \downarrow & & \downarrow \phi_B \\ GA & \xrightarrow{Gf} & GB \end{array} \quad (75)$$

<sup>166</sup> Using squiggly arrows for **functors** in diagrams is very non-standard, but I believe it helps remember what kind of objects we are dealing with. Moreover, since these diagrams are not **commutative**, it makes a good contrast with the plain arrow notation which was mostly used for **commutative** diagrams.

transformation from  $G \circ F$  to  $G \circ F'$ .

$$\begin{array}{ccc} (G \circ F)(A) & \xrightarrow{G\phi(A)} & (G \circ F')(A) \\ (G \circ F)(f) \downarrow & & \downarrow (G \circ F')(f) \\ (G \circ F)(B) & \xrightarrow{G\phi(B)} & (G \circ F')(B) \end{array} \quad (77)$$

Consider this diagram after removing all applications of  $G$ , by **naturality** of  $\phi$ , it is **commutative**. Since **functors preserve commutativity**, the diagram still **commutes** after applying  $G$ , hence  $G\phi : G \circ F \Rightarrow G \circ F'$  is indeed **natural**.<sup>167</sup>

We leave you to check this constitutes a left action, namely, for any  $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$ ,  $G' : \mathbf{D}' \rightsquigarrow \mathbf{D}''$  and  $\phi : F \Rightarrow F'$ ,

$$\text{id}_{\mathbf{D}}\phi = \phi \text{ and } G'(G\phi) = (G' \circ G)\phi.$$

**Definition 217** (The right action of functors). Let  $F, F' : \mathbf{C} \rightsquigarrow \mathbf{D}$ ,  $H : \mathbf{C}' \rightsquigarrow \mathbf{C}$  be **functors** and  $\phi : F \Rightarrow F'$  a **natural transformation** as summarized in (78).

$$\begin{array}{ccc} \mathbf{C}' & \xrightarrow{H} & \mathbf{C} \\ & \searrow F & \downarrow \phi \\ & & \mathbf{D} \\ & \nearrow F' & \end{array} \quad (78)$$

The **functor**  $H$  acts on  $\phi$  by sending it to  $\phi H := A \mapsto \phi(H(A)) : \mathbf{C}'_0 \rightarrow \mathbf{D}_1$ . Showing that (79) **commutes** for any  $f \in \text{Hom}_{\mathbf{C}'}(A, B)$  will imply that  $\phi H$  is a **natural transformation** from  $F \circ H$  to  $F' \circ H$ .

$$\begin{array}{ccc} (F \circ H)(A) & \xrightarrow{\phi H(A)} & (F' \circ H)(A) \\ (F \circ H)(f) \downarrow & & \downarrow (F' \circ H)(f) \\ (F \circ H)(B) & \xrightarrow{\phi H(B)} & (F' \circ H)(B) \end{array} \quad (79)$$

**Commutativity** of (79) follows by **naturality** of  $\phi$ : change  $f$  in diagram (70) with the **morphism**  $H(f) : H(A) \rightarrow H(B)$ , i.e.: (79) is **NAT**( $\phi, HA, HB, Hf$ ).

We leave you to check this constitutes a right action, namely, for any  $H : \mathbf{C}' \rightsquigarrow \mathbf{C}$ ,  $H' : \mathbf{C}'' \rightsquigarrow \mathbf{C}'$  and  $\phi : F \Rightarrow F'$ ,

$$\phi \text{id}_{\mathbf{C}} = \phi \text{ and } (\phi H)H' = \phi(H \circ H').$$

**Proposition 218.** *The two actions commute, i.e.: in the setting of (80),  $G(\phi H) = (G\phi)H$ .*<sup>168</sup>

$$\begin{array}{ccccc} \mathbf{C}' & \xrightarrow{H} & \mathbf{C} & \begin{array}{c} \xrightarrow{F} \\ \downarrow \phi \\ \xrightarrow{F'} \end{array} & \mathbf{D} \xrightarrow{G} \mathbf{D}' \end{array} \quad (80)$$

*Proof.* In both the L.H.S. and the R.H.S., an object  $A \in \mathbf{C}_0$  is sent to  $G(\phi(H(A)))$ .  $\square$

<sup>167</sup> More concisely, we apply  $G$  to **NAT**( $\phi, A, B, f$ ) to obtain (77).

<sup>168</sup> For this reason and the associativity of the two actions, we will drop all the parentheses from such expressions. We will also drop the  $\circ$  for **composition of functors**. All in all, expect to find expressions like  $G'G\phi HH'$  and infer the **natural transformation**  $A \mapsto G'(G(\phi(H(H'(A)))))$ .

A very useful result following from the properties of these two actions is that for any **commutative diagram** in  $[\mathbf{C}, \mathbf{D}]$ , we can **pre-compose** and **post-compose** with any **functors** and still obtain a **commutative diagram**. For instance, if (81) **commutes** in  $[\mathbf{C}, \mathbf{D}]$ , then for any **functors**  $F : \mathbf{C}' \rightsquigarrow \mathbf{C}$  and  $G : \mathbf{D} \rightsquigarrow \mathbf{D}'$ , then (82) **commutes**.<sup>169</sup>

$$\begin{array}{ccc}
 X & \xrightarrow{\eta} & Y \\
 \phi \downarrow & & \downarrow \phi' \\
 X' & \xrightarrow{\eta'} & Y'
 \end{array} \quad (81) \qquad
 \begin{array}{ccc}
 F \circ X \circ G & \xrightarrow{F\eta G} & F \circ Y \circ G \\
 F\phi G \downarrow & & \downarrow F\phi' G \\
 F \circ X' \circ G & \xrightarrow{F\eta' G} & F \circ Y' \circ G
 \end{array} \quad (82)$$

<sup>169</sup> We will often use this property by writing things like “apply  $F(-)G$  to (81)” to use the **commutativity** of (82) in a proof.

□ We will refer to these two actions as the **biaction** of **functors** on **natural transformations** and they will motivate the definition of another way to **compose natural transformations**.

Let  $\mathbf{C}, \mathbf{D}$  and  $\mathbf{E}$  be **categories**,  $H, H' : \mathbf{C} \rightsquigarrow \mathbf{D}$  and  $G, G' : \mathbf{D} \rightsquigarrow \mathbf{E}$  be **functors** and  $\phi : H \Rightarrow H'$  and  $\eta : G \Rightarrow G'$  be **natural transformations**. This is summarized in (83).

$$\begin{array}{ccccc}
 \mathbf{C} & \xrightarrow{H} & & \xrightarrow{G} & \mathbf{E} \\
 & \Downarrow \phi & & \Downarrow \eta & \\
 \mathbf{C} & \xrightarrow{H'} & \mathbf{D} & \xrightarrow{G'} & \mathbf{E}
 \end{array} \quad (83)$$

The ultimate goal is to obtain a new **composition** of  $\phi$  and  $\eta$  that is a **natural transformation**  $G \circ H \Rightarrow G' \circ H'$ . Note that the **biaction** defined above yields four other **natural transformations**:

$$\begin{array}{ll}
 G\phi : G \circ H \Rightarrow G \circ H' & \eta H : G \circ H \Rightarrow G' \circ H \\
 G'\phi : G' \circ H \Rightarrow G' \circ H' & \eta H' : G \circ H' \Rightarrow G' \circ H'.
 \end{array}$$

All of the **functors** involved go from  $\mathbf{C}$  to  $\mathbf{E}$ , so all four **natural transformations** fit in diagram (84) that lives in the **functor category**  $[\mathbf{C}, \mathbf{E}]$ .

$$\begin{array}{ccc}
 G \circ H & \xrightarrow{G\phi} & G \circ H' \\
 \eta H \downarrow & & \downarrow \eta H' \\
 G' \circ H & \xrightarrow{G'\phi} & G' \circ H'
 \end{array} \quad (84)$$

At first glance, this suggests two different definitions for the **horizontal composition**, that is, the **composition** of the top **path** ( $\eta H' \cdot G\phi$ ) or the **composition** of the bottom **path** ( $G'\phi \cdot \eta H$ ). Surprisingly, both definitions coincide as shown in the next result.

**Lemma 219.** *Diagram (84) **commutes**, i.e.:  $\eta H' \cdot G\phi = G'\phi \cdot \eta H$ .*<sup>170</sup>

*Proof.* Fix an object  $A \in \mathbf{C}_0$ . Under  $\eta H' \cdot G\phi$ , it is sent to  $\eta(H'(A)) \circ G(\phi(A))$  and under  $G'\phi \cdot \eta H$ , it is sent to  $G'(\phi(A)) \circ \eta(H(A))$ . Thus, the proposition is equivalent

<sup>170</sup> Similarly to **NAT**, we will refer to the **commutativity** of (84) with **HOR**( $\phi, \eta$ ). We use **HOR** because this lemma is crucial in the definition of **horizontal composition**.

to saying diagram (85) is **commutative** (in  $\mathbf{E}$ ) for all  $A \in \mathbf{C}_0$ .

$$\begin{array}{ccc} (G \circ H)(A) & \xrightarrow{G(\phi(A))} & (G \circ H')(A) \\ \eta(H(A)) \downarrow & & \downarrow \eta(H'(A)) \\ (G' \circ H)(A) & \xrightarrow{G'(\phi(A))} & (G' \circ H')(A) \end{array} \quad (85)$$

This follows from  $\text{NAT}(\eta, HA, H'A, \phi(A))$ .  $\square$

Definition 220 (Horizontal composition). In the setting described in (83), we define the **horizontal composition** of  $\eta$  and  $\phi$  by  $\eta \diamond \phi = \eta H' \cdot G\phi = G'\phi \cdot \eta H$ .<sup>171</sup>

The most important part we expect from a notion of **composition** is **associativity**, so let us check  $\diamond$  is **associative**.

Proposition 221. In the setting of (86),  $\psi \diamond (\eta \diamond \phi) = (\psi \diamond \eta) \diamond \phi$ .

$$\begin{array}{ccccc} \mathbf{C} & \xrightarrow{H} & \mathbf{D} & \xrightarrow{G} & \mathbf{E} & \xrightarrow{K} & \mathbf{F} \\ & \Downarrow \phi & & \Downarrow \eta & & \Downarrow \psi & \\ \mathbf{C} & \xrightarrow{H'} & \mathbf{D} & \xrightarrow{G'} & \mathbf{E} & \xrightarrow{K'} & \mathbf{F} \end{array} \quad (86)$$

Proof. Similarly to how we constructed diagram (84) in  $[\mathbf{C}, \mathbf{E}]$  previously, we can use the **biaction** of **functors** and **composition** of **functors** to obtain the following diagram in  $[\mathbf{C}, \mathbf{E}]$ .<sup>172</sup>

$$\begin{array}{ccccc} & & K'GH & \xrightarrow{K'G\phi} & K'GH' \\ \psi GH \nearrow & & \downarrow K'\eta H & & \searrow \psi GH' \\ KGH & \xrightarrow{KG\phi} & KGH' & & \\ \downarrow K\eta H & & \downarrow K\eta H' & & \\ & & K'G'H & \xrightarrow{K'G'\phi} & K'G'H' \\ \psi G'H \nearrow & & \downarrow K\eta H' & & \searrow \psi G'H' \\ KG'H & \xrightarrow{KG'\phi} & KG'H' & & \end{array} \quad (87)$$

As detailed in the margin, this **commutes** because each face of the cube corresponds to a variant of diagram (84) (with some substitutions and application of a **functor**) and combining **commutative** diagrams yields **commutative** diagrams. Then, it follows that  $\diamond$  is associative because<sup>173</sup>  $\psi \diamond (\eta \diamond \phi)$  is the diagonal of the front face followed by the bottom right arrow and  $(\psi \diamond \eta) \diamond \phi$  is the top front arrow followed by the diagonal of the right face.  $\square$

There is one last thing to conclude that **Cat** is a **2-category**, namely, that the **vertical** and **horizontal compositions** interact nicely.

Proposition 222 (Interchange identity). In the setting of (89), the **interchange identity** holds:

$$(\eta' \cdot \eta) \diamond (\phi' \cdot \phi) = (\eta' \diamond \phi') \cdot (\eta \diamond \phi). \quad (88)$$

<sup>171</sup> The  $\diamond$  notation is not standard but there are no widespread symbol denoting **horizontal composition**. I have mostly seen  $*$  or plain juxtaposition. Hopefully, you will encounter papers/books clear enough that you can typecheck to find what **composition** is being used.

<sup>172</sup> All  $\circ$ 's are left out for simplicity.

Here is how each face **commutes**.

**Top:**  $\text{HOR}(\psi, G\eta)$

**Bottom:**  $\text{HOR}(\psi, G'\eta')$

**Left:**  $\text{HOR}(\psi, \eta H)$

**Right:**  $\text{HOR}(\psi, \eta H')$

**Front:**  $\text{HOR}(K\eta, \phi)$

**Back:**  $\text{HOR}(K'\eta', \phi)$

<sup>173</sup> We may have drawn only the front and right face, but the cube is cooler.

$$\begin{array}{ccc}
& H & G \\
& \Downarrow \phi & \Downarrow \eta \\
\mathbf{C} & \xrightarrow{H'} & \mathbf{D} \\
& \Downarrow \phi' & \Downarrow \eta' \\
& H'' & G''
\end{array}
\quad (89)$$

*Proof.* Akin to the other proofs, this is a matter of combining the right diagrams. After combining the diagrams in  $[\mathbf{C}, \mathbf{E}]$  corresponding to  $\eta \diamond \phi$  and  $\eta' \diamond \phi'$ , it is easy to see that the R.H.S. of (88) is the **morphism** going from  $G \circ H$  to  $G'' \circ H''$  (see (90)).

$$\begin{array}{ccccc}
G \circ H & \xrightarrow{G\phi} & G \circ H' & & \\
\eta H \downarrow & & \downarrow \eta H' & & \\
G' \circ H & \xrightarrow{G'\phi} & G' \circ H' & \xrightarrow{G'\phi'} & G' \circ H'' \\
& & \eta' H' \downarrow & & \downarrow \eta' H'' \\
& & G'' \circ H' & \xrightarrow{G''\phi'} & G'' \circ H''
\end{array}
\quad (90)$$

Moreover, observe that the diagram corresponding to the L.H.S. can be factored with the following equations (it also yields the factored diagram in (91)).

$$\begin{aligned}
(\eta' \cdot \eta)H &= \eta'H \cdot \eta H & (\eta' \cdot \eta)H'' &= \eta'H'' \cdot \eta H'' \\
G(\phi' \cdot \phi) &= G\phi' \cdot G\phi & G''(\phi' \cdot \phi) &= G''\phi' \cdot G''\phi
\end{aligned}$$

Combining the factored diagram with (90), we obtain (92) from which the **interchange identity** readily follows.<sup>174</sup>

$$\begin{array}{ccccc}
G \circ H & \xrightarrow{G\phi} & G \circ H' & \xrightarrow{G'\phi'} & G \circ H'' \\
\eta H \downarrow & & \downarrow \eta H' & & \downarrow \eta H'' \\
G' \circ H & \xrightarrow{G'\phi} & G' \circ H' & \xrightarrow{G'\phi'} & G' \circ H'' \\
\eta' H \downarrow & & \eta' H' \downarrow & & \downarrow \eta' H'' \\
G'' \circ H & \xrightarrow{G''\phi} & G'' \circ H' & \xrightarrow{G''\phi'} & G'' \circ H''
\end{array}
\quad (92)$$

It is in the drawing of (89) that the intuition behind the terms **vertical** and **horizontal** is taken.

$$\begin{array}{ccccc}
G \circ H & \xrightarrow{G\phi} & G \circ H' & \xrightarrow{G'\phi'} & G \circ H'' \\
\eta H \downarrow & & & & \downarrow \eta H'' \\
G' \circ H & & & & G' \circ H'' \\
\eta' H \downarrow & & & & \downarrow \eta' H'' \\
G'' \circ H & \xrightarrow{G''\phi} & G'' \circ H' & \xrightarrow{G''\phi'} & G'' \circ H''
\end{array}
\quad (91)$$

<sup>174</sup> The top right and bottom left square **commute** by  $\text{HOR}(\eta, \phi')$  and  $\text{HOR}(\eta', \phi)$  respectively. This implies all of (92) **commutes** and we have seen that the **path** from  $G \circ H$  to  $G'' \circ H''$  can be seen as the R.H.S. of (88) by looking at (90) or the L.H.S. by looking at (91). Thus, we infer the equality of (88).

□

□ **Definition 223** (Strict 2-category). A **strict 2-category** consists of

- a **category**  $\mathbf{C}$ ,
- for every  $A, B \in \mathbf{C}_0$  a **category**  $\mathbf{C}(A, B)$  with  $\text{Hom}_{\mathbf{C}}(A, B)$  as its **objects** (composition is denoted  $\cdot$  and **identities**  $\mathbb{1}$ ) and **morphisms** are called **2-morphisms**,
- a **category** with  $\mathbf{C}_0$  as its **objects**, where the **morphisms** are pairs of **parallel morphisms** of  $\mathbf{C}$  along with a 2-morphism between them<sup>175</sup> and the identity map sends  $A \in \mathbf{C}_0$  to the pair  $(\text{id}_A, \text{id}_A)$  and the 2-morphism  $\mathbb{1}_{\text{id}_A}$  (composition of 2-cells is denoted  $\diamond$ ),

<sup>175</sup> A **morphism** in this category is also called a **2-cell**.

such that the **interchange identity** (88) holds.

We will not cover it in this book, but there are notions of **morphisms** between **2-categories** (called 2-functors), between 3-categories as well as between  $n$ -categories for any  $n$  (even  $n = \infty$ !), these objects are more deeply studied in higher category theory.<sup>176</sup>

**Exercise 224 (NOW!).** Show that there is a **functor**  $[D, E] \times [C, D] \rightsquigarrow [C, E]$  whose action on **objects** is  $(F, G) \mapsto F \circ G$ .

## Equivalences

As is expected, an isomorphism of **categories** is an **isomorphism** in the **category Cat**, namely, a **functor**  $F : C \rightsquigarrow D$  with an inverse  $G : D \rightsquigarrow C$  such that  $F \circ G = \text{id}_D$  and  $G \circ F = \text{id}_C$ . As is typical in mathematics, one cannot distinguish between **isomorphic categories** as they only differ in notations and terminology.

### Examples 225.

1. It was already shown in Example 213 (the details were implicit) that for a group  $G$ , the category  $[\text{Set}, \mathbf{BG}]$  is **isomorphic** to the **category** of  $G$ -sets with  $G$ -equivariant maps as **morphisms**.
2. In Example 215, three other **isomorphisms** were implicitly given:

$$[1, C] \cong C \quad [1 + 1, C] \cong C \times C \quad [2, C] \cong C^{\rightarrow}.$$

3. The category **Rel** of sets with relations is **isomorphic** to **Rel**<sup>op</sup>.<sup>177</sup> The **functor**  $\text{Rel} \rightsquigarrow \text{Rel}^{\text{op}}$  is the identity on **objects** and sends a relation  $R \subseteq X \times Y$  to the opposite relation  $\mathfrak{R} \subseteq Y \times X$  (which is a **morphism**  $X \rightarrow Y$  in **Rel**<sup>op</sup>) defined by  $(y, x) \in \mathfrak{R} \Leftrightarrow (x, y) \in R$ . The inverse is defined similarly.
4. Given three **categories**  $C, D$  and  $E$ , there is an **isomorphism**<sup>178</sup>

$$[C \times D, E] \cong [C, [D, E]].$$

Let  $F : C \times D \rightsquigarrow E$ , the **currying** of  $F$  is  $\Lambda F : C \rightsquigarrow [D, E]$  defined as follows. For  $X \in C_0$ , the

Although there are other interesting instances of **isomorphic categories**, **natural transformations** lead to a more nuanced (and often more useful) equality between two **categories**, that is, **equivalence**.

- **Definition 226 (Equivalence).** A **functor**  $F : C \rightsquigarrow D$  is an **equivalence** of **categories** if there exists a **functor**  $G : D \rightsquigarrow C$  such that  $F \circ G \cong \text{id}_D$  and  $G \circ F \cong \text{id}_C$ .<sup>179</sup> This is clearly symmetric, so we say two **categories**  $C$  and  $D$  are **equivalent**, denoted  $C \simeq D$ , if there is an **equivalence** between them. Moreover, we say that  $G$  is a **quasi-inverse** of  $F$  and vice-versa.

<sup>176</sup> Most of higher category theory drops the *strict* part of our definition of 2-category because this condition is too strong. Very briefly, they allow the properties of **composition**, namely **associativity** and **identities**, to hold up to **natural isomorphisms**.

See solution.

Another example for readers who know a bit of advanced algebra. Let  $k$  be a **field** and  $G$  a finite **group**, the **categories** of  $k[G]$ -modules ( $k[G]$  is the group ring of  $k$  over  $G$ ) and of  $k$ -linear representations of  $G$  are **isomorphic**.

<sup>177</sup> An arbitrary **category**  $C$  is not always **isomorphic** to its **opposite**. While the **opposite functors**  $(-)^{\text{op}}_C : C \rightsquigarrow C^{\text{op}}$  and  $(-)^{\text{op}}_{C^{\text{op}}} : C^{\text{op}} \rightsquigarrow C$  are inverses of each other, they are **contravariant functors**.

<sup>178</sup> You might recognize a similarity with **exponentials** which rely on an **isomorphism**  $\text{Hom}_C(B \times X, A) \cong \text{Hom}_C(B, A^X)$ . The example here is more than an instance of **exponentials** of **categories** because the **isomorphism** is not only as sets but as **categories**.

<sup>179</sup> Recall that  $\cong$  between **functors** stands for **natural isomorphisms**.



In order to gain more intuition on how [equivalences](#) equate two [categories](#), let us observe what properties this forces on the [functor](#)  $F$ . For any [morphism](#)  $f \in \text{Hom}_{\mathbf{C}}(A, B)$ , the following square [commutes](#) where  $\phi(A)$  and  $\phi(B)$  are [isomorphisms](#).<sup>180</sup>

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \phi(A)^{-1} \uparrow \downarrow \phi(A) & & \phi(B) \uparrow \downarrow \phi(B)^{-1} \\ GF(A) & \xrightarrow{GF(f)} & GF(B) \end{array} \quad (93)$$

This implies that the map  $f \mapsto GF(f) : \text{Hom}_{\mathbf{C}}(A, B) \rightarrow \text{Hom}_{\mathbf{C}}(GF(A), GF(B))$  is a bijection. Indeed, [pre-composition](#) by  $\phi(A)^{-1}$  and [post-composition](#) by  $\phi(B)$  are both bijections,<sup>181</sup> so

$$f \mapsto \phi(B) \circ f \circ \phi(A)^{-1} = GF(f)$$

is a bijection. Since  $A$  and  $B$  are arbitrary,  $G \circ F$  is a [fully faithful functor](#) and a symmetric argument shows  $F \circ G$  is also [fully faithful](#). Then, it is easy to conclude that  $F$  and  $G$  must be [fully faithful](#) as well.

What is more, the existence of an [isomorphism](#)  $\eta(A) : A \rightarrow FG(A)$  for any object  $A$  implies  $F$  (symmetrically  $G$ ) has the following property.

□ **Definition 227** (Essentially surjective). A [functor](#)  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  is [essentially surjective](#) if for any  $X \in \mathbf{D}_0$ , there exists  $Y \in \mathbf{C}_0$  such that  $X \cong F(Y)$ .

We will show that these two properties ([full faithfulness](#) and [essential surjectivity](#)) are necessary and sufficient for  $F$  to be an [equivalence](#).

**Theorem 228.** A [functor](#)  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  is an [equivalence](#) of [categories](#) if and only if  $F$  is [fully faithful](#) and [essentially surjective](#).

*Proof.* ( $\Rightarrow$ ) Shown above.

( $\Leftarrow$ ) We construct a [functor](#)  $G : \mathbf{D} \rightsquigarrow \mathbf{C}$  such that  $G \circ F \cong \text{id}_{\mathbf{C}}$  and  $F \circ G \cong \text{id}_{\mathbf{D}}$ . Since  $F$  is [essentially surjective](#), for any  $A \in \mathbf{D}_0$ , there exists an object  $G(A) \in \mathbf{C}_0$  and an [isomorphism](#)  $\phi(A) : F(G(A)) \cong A$ . Hence,  $A \mapsto G(A)$  is a good candidate to describe the action of  $G$  on [objects](#).

Next, similarly to the converse direction, note that for any  $A, B \in \mathbf{D}_0$ , the map

$$f \mapsto \phi(B) \circ f \circ \phi(A)^{-1}$$

is a bijection from  $\text{Hom}_{\mathbf{D}}(A, B)$  to  $\text{Hom}_{\mathbf{D}}(FG(A), FG(B))$ . Moreover, since the functor  $F$  is [fully faithful](#), it induces a bijection

$$F_{A,B} : \text{Hom}_{\mathbf{C}}(G(A), G(B)) \rightarrow \text{Hom}_{\mathbf{D}}(FG(A), FG(B))$$

which in turns yields a bijection

$$G_{A,B} : \text{Hom}_{\mathbf{D}}(A, B) \rightarrow \text{Hom}_{\mathbf{C}}(G(A), G(B)) = f \mapsto F_1^{-1}(\phi(B) \circ f \circ \phi(A)^{-1}).$$

This is the action of  $G$  on [morphisms](#). Observe that the construction of  $G$  ensures that  $F \circ G \cong \text{id}_{\mathbf{D}}$  through the [natural transformation](#)  $\phi$ . It remains to show that  $G$  is indeed a [functor](#) and find a [natural isomorphism](#)  $\eta : G \circ F \cong \text{id}_{\mathbf{C}}$ .

<sup>180</sup> [Naturality](#) of  $\phi$  only gives us  $GF(f) \circ \phi(A) = \phi(B) \circ f$ , but by [composing](#) with  $\phi(A)^{-1}$  or  $\phi(B)^{-1}$ , we obtain the [commutativity](#) of all of (93). In particular, we have  $GF(f) = \phi(B) \circ f \circ \phi(A)^{-1}$ .

<sup>181</sup> Recall the definitions of [monomorphisms](#) and [epimorphisms](#) and the fact that [isomorphisms](#) are [monic](#) and [epic](#).



For any **composable morphisms**  $(f, g)$ , it is easy to verify that

$$F(G(f) \circ G(g)) = FG(f) \circ FG(g) = FG(f \circ g),$$

so **functoriality** of  $G$  follows after applying  $F_1^{-1}$ . To find  $\eta$ , recall that the definition of  $G$  yields **commutativity** of (94) for any  $f \in \mathbf{Hom}_{\mathbf{C}}(A, B)$ .

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \phi(F(A)) \uparrow & & \uparrow \phi(F(B)) \\ FGF(A) & \xrightarrow{FGF(f)} & FGF(B) \end{array} \quad (94)$$

Then, because  $F$  is **fully faithful**, the following square also **commutes** in  $\mathbf{C}$  where  $\eta = X \mapsto F_1^{-1}(\phi(F(X)))$  and we conclude that  $\eta$  is a **natural isomorphism**  $\text{id}_{\mathbf{C}} \cong G \circ F$ .

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \eta(A) \uparrow & & \uparrow \eta(B) \\ GF(A) & \xrightarrow{GF(f)} & GF(B) \end{array} \quad (95)$$

□

The insight to extract from this argument is that two categories are **equivalent** if they describe the same **objects** and **morphisms** with the only relaxation that **isomorphic objects** can appear any number of times in either **category**. In contrast, **categories** can only be **isomorphic** if they have exactly the same **objects** and **morphisms**.

*Remark 229.* We used the axiom of choice to construct the **quasi-inverse** of  $F$ .

We will detail a couple of *easy* examples of **equivalences** and briefly mention a few *harder* ones.

□ **Examples 230 (Easy).** 1. Consider the **full subcategory** of **FinSet** consisting only of the sets  $\emptyset, \{1\}, \{1, 2\}, \dots, \{1, \dots, n\}, \dots$ , denote it **FinOrd**.<sup>182</sup> The **inclusion functor** is **fully faithful** by definition and we claim it is **essentially surjective**. Indeed, any set  $X \in \mathbf{FinSet}_0$  has a finite cardinality  $n$ , so  $X \cong \{1, \dots, n\} \in \mathbf{FinOrd}_0$ .

2. In a very similar fashion, an early result in linear algebra says that any **finite dimensional vector space** over a **field**  $k$  is **isomorphic** to  $k^n$  for some  $n \in \mathbb{N}$ .

□ Thus, the **category** whose objects are  $k^n$  for all  $n \in \mathbb{N}$  and **morphisms** are  $m \times n$  **matrices** with entries in  $k$ ,<sup>183</sup> which we denote **Mat**( $k$ ), is **equivalent** to the **category** of **finite dimensional vector spaces**.

□ 3. A **partial function**  $f : X \rightarrow Y$  is a function that may not be defined on all of  $X$ .<sup>184</sup> There is **category Par** of sets and **partial functions** where **identity morphism** and **composition** are defined straightforwardly.<sup>185</sup> We can view a **partial function**  $f : X \rightarrow Y$  as a **total function**  $f' : X \rightarrow Y + \mathbf{1}$  which assigns to every  $x$  where  $f(x)$

<sup>182</sup> The name **FinOrd** is an abbreviation of finite ordinals, because we can also define **FinOrd** as the category of finite ordinals and functions between them.

<sup>183</sup> After making a choice of **basis** for all  $k^n$ , an  $m \times n$  matrix with entries in  $k$  corresponds to a **linear map**  $k^n \rightarrow k^m$ .

□ <sup>184</sup> In this context, a **normal function** defined on all of  $X$  is called **total**.

<sup>185</sup> You can view **Par** as the **subcategory** of **Rel** where you only take the relations  $R \subseteq X \times Y$  satisfying for any  $x \in X$  (cf. Remark 78),

$$|\{y \in Y \mid (x, y) \in R\}| \leq 1.$$

is undefined the value  $*$   $\in \mathbf{1}$ . Further extending  $f'$  to  $[f', \text{id}_1] : X + \mathbf{1} \rightarrow Y + \mathbf{1}$ , we can see any **partial** function as a function between **pointed** sets where the distinguished element corresponds to being undefined.

We claim that this yields a **fully faithful functor**  $\mathbf{Par} \rightsquigarrow \mathbf{Set}_*$  sending  $X$  to  $(X + \mathbf{1}, *)$  and  $f : X \rightarrow Y$  to  $[f', \text{id}_1]$ .

The first two examples and many other simple examples of **equivalences** are examples of **skeletons**. They are morally a **subcategory** where all the **isomorphic** copies are removed.

**Definition 231** (Skeleton). A **category** is called **skeletal** if there it contains no two **isomorphic objects**. A **skeleton** of a **category** is an **equivalent skeletal category**.

**Examples 232.** We have shown that  $\mathbf{FinOrd} \simeq \mathbf{FinSet}$  and  $\mathbf{Mat}(k) \simeq \mathbf{FDVect}_k$  and we leave to you the easy task to check that these are examples of **skeletons**.<sup>186</sup>

A **category** always has a **skeleton** if you assume the axiom of choice and the next result justifies say *the skeleton* of a **category**.

**Exercise 233.** Show that all **skeletons** of a **category** are **isomorphic**.

Here are other more interesting examples of **equivalent categories**.

**Example 234** (Medium). Let  $\mathbf{C}$  be a **category**, there is a **functor**  $F : \mathbf{C} \rightsquigarrow \mathbf{C}^\rightarrow$  sending  $X$  to  $\text{id}_X$  and  $f : X \rightarrow Y$  to the **commutative** square in (96). This **functor** is an **equivalence** if and only if all **morphisms** in  $\mathbf{C}$  are **isomorphisms**.<sup>187</sup> It is clearly **fully faithful**, so it is left to show  $F$  is **essentially surjective** if and only if  $\mathbf{C}$  is a **groupoid**.

( $\Rightarrow$ ) For any  $f : X \rightarrow Y \in \mathbf{C}_1$ , by hypothesis, there exists  $A \in \mathbf{C}_0$  such that  $\text{id}_A \cong f$  in  $\mathbf{C}^\rightarrow$ . Let  $(s : A \rightarrow X, t : A \rightarrow Y)$  be the **isomorphism**, its **inverse** must be  $(s^{-1}, t^{-1})$ . Looking at the chain of **commutative** squares in (97), we can infer that  $s \circ t^{-1}$  is the **inverse** of  $f$ .<sup>188</sup>

$$\begin{array}{ccccccc} Y & \xrightarrow{t^{-1}} & A & \xrightarrow{s} & X & \xrightarrow{s^{-1}} & A & \xrightarrow{s} & X \\ \text{id}_Y \downarrow & & \text{id}_A \downarrow & & f \downarrow & & \text{id}_A \downarrow & & \text{id}_X \downarrow \\ Y & \xrightarrow{t^{-1}} & A & \xrightarrow{t} & Y & \xrightarrow{t^{-1}} & A & \xrightarrow{s} & X \end{array} \quad (97)$$

( $\Leftarrow$ ) Let  $f : X \rightarrow Y$  be an **object** of  $\mathbf{C}^\rightarrow$ , the inverse of  $f$  satisfies  $f \circ f^{-1} = \text{id}_Y$  and  $f^{-1} \circ f = \text{id}_X$ , so the squares in (98) are **isomorphisms** in  $\mathbf{C}^\rightarrow$  (they are inverses of each other). Thus, we find that  $f$  is **isomorphic** to  $\text{id}_X$  which is in the image of  $F$ .

**Examples 235** (Hard). Examples of significant **equivalences** are all over the place in higher mathematics. However, they require a bit of work to describe them, thus let us only say a few words on them.

1. The **equivalence** between the **category** of affine schemes and the **opposite** of the **category** of **commutative rings** is a seminal result scheme theory, a huge part of modern algebraic geometry.

<sup>186</sup> Namely, you should show that no two sets in  $\mathbf{FinOrd}$  are **isomorphic** and no two spaces in  $\mathbf{Mat}(k)$  are **isomorphic**.

See solution.

<sup>187</sup> Such a **category** is called a **groupoid**.

$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{\text{id}_Y} & Y \end{array} \quad (96)$$

<sup>188</sup> The **composition**  $f \circ s \circ t^{-1}$  is the top path of the combined two leftmost squares, the bottom path is  $t \circ t^{-1} \circ \text{id}_Y = \text{id}_Y$ . The **composition**  $s \circ t^{-1} \circ f$  is the bottom path of the combined two rightmost squares, the top path is  $\text{id}_X \circ s \circ s^{-1} = \text{id}_X$ .

$$\begin{array}{ccc} X & \xrightarrow{\text{id}_X} & X \\ \text{id}_X \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{id}_X \downarrow & & \downarrow f^{-1} \\ X & \xrightarrow{\text{id}_X} & X \end{array} \quad (98)$$

2. The [equivalence](#) between Boolean lattices and Stone spaces is again seminal in the theory of Stone-type dualities. These can lead to deep connections between topology and logic. One application in particular is the study of the behavior of computer programs through formal semantics.

**Exercise 236.** Show that [equivalence](#) of [categories](#) is an equivalence relation.

See solution.

**Exercise 237.** Show that  $\mathbf{C} \simeq \mathbf{C}'$  and  $\mathbf{D} \simeq \mathbf{D}'$  implies  $[\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}', \mathbf{D}']$ .

See solution.



# Yoneda Lemma

## Representable Functors

Throughout, let  $\mathbf{C}$  be a **locally small category**. Recall that for an **object**  $A \in \mathbf{C}_0$ , there are two **Hom functors** from  $\mathbf{C}$  to **Set**. The **covariant** one,  $\text{Hom}_{\mathbf{C}}(A, -)$ , sends an **object**  $B \in \mathbf{C}_0$  to  $\text{Hom}_{\mathbf{C}}(A, B)$  and a **morphism**  $f : B \rightarrow B'$  to  $f \circ (-)$ . The **contravariant** one,  $\text{Hom}_{\mathbf{C}}(-, A)$ , sends an **object**  $B \in \mathbf{C}_0$  to  $\text{Hom}_{\mathbf{C}}(B, A)$  and a **morphism**  $f : B \rightarrow B'$  to  $(-) \circ f$ . In order to lighten the notation, we denote these functors  $H^A$  and  $H_A$  respectively.<sup>189</sup>

Although these **functors** are sometimes interesting on their own, their full power is unleashed when they are related to other **functors** through **natural transformations**. For instance, some of these **Hom functors** can be described in simpler terms.

### Examples 238.

1. Let  $\mathbf{1} = \{*\}$  be the **terminal object** in **Set**, then what is the action of  $H^{\mathbf{1}}$ ? For any **object**  $B$ ,

$$H^{\mathbf{1}}(B) = \text{Hom}_{\mathbf{Set}}(\mathbf{1}, B)$$

is easy to describe because for any element  $b \in B$ , there is a unique function  $f : \mathbf{1} \rightarrow B = * \mapsto b$ . Hence, there is an **isomorphism** from  $H^{\mathbf{1}}(B)$  to  $B$  for any  $B \in \mathbf{C}_0$ , it sends  $f$  to  $f(*)$  and its inverse sends  $b \in B$  to the map  $* \mapsto b$ . Moreover, these isomorphisms are natural in  $B$  because (99) clearly **commutes** for any  $f : B \rightarrow B'$ , yielding a **natural isomorphism**  $H^{\mathbf{1}} \cong \text{id}_{\mathbf{C}}$ .

$$\begin{array}{ccc} H^{\mathbf{1}}(B) & \xrightarrow{f \circ (-)} & H^{\mathbf{1}}(B') \\ \updownarrow & & \updownarrow \\ B & \xrightarrow{f} & B' \end{array} \quad (99)$$

2. Consider again the **terminal object** in the **category Grp**, namely, the **group 1** only containing an **identity**. Then, for any **group**  $G$ , the set  $H^{\mathbf{1}}(G)$  is a singleton because any **homomorphism**  $f : \mathbf{1} \rightarrow G$  must send the **identity** to the **identity** and no other choice can be made. Therefore, unlike in **Set**,  $H^{\mathbf{1}}$  is very uninteresting and acts like the **constant functor**  $\mathbf{1} : \mathbf{Grp} \rightsquigarrow \mathbf{Set}$ .
3. A better choice of **object** to mimic the behavior of  $\text{id}_{\mathbf{Grp}}$  is the additive **group**  $\mathbb{Z}$ . Indeed, for any  $g \in G$ , there is a unique **homomorphism**  $f : \mathbb{Z} \rightarrow G$  sending 0

<sup>189</sup> It might seem like this contradicts the notation used so far because  $H^A$  is **covariant** and  $H_A$  **contravariant**. However, this is not their *variance* in the parameter  $A$ , and we will show that in fact, the *variance* in  $A$  are opposites.

to the **identity** and 1 to  $g$ .<sup>190</sup> A very similar argument as above yields a **natural isomorphism**  $H^{\mathbb{Z}} \cong \text{id}_{\mathbf{Grp}}$ .

4. The **terminal object** in **Cat** is the **category 1** with a single **object**  $\bullet$  and no **morphism** other than the **identity**. Observe that for any **category**  $\mathbf{C}$ , a **functor**  $\mathbf{1} \rightsquigarrow \mathbf{C}$  is just a choice of **object**. Therefore, the same argument will show that  $H^{\mathbf{1}} \cong (-)_0$ , where  $(-)_0$  sends a **category** to its set<sup>191</sup> of **objects** and a **functor** to its action restricted on **objects**.

In order to obtain a similar way to extract **morphisms**, consider the category **2** with two **objects** and a single **morphism** between them. One obtains a **natural isomorphism**  $H^{\mathbf{2}} \cong (-)_1$ .

These examples suggest that **functors** that are **naturally isomorphic** to **Hom functors** have nice properties, they are said to be **representable**.

□ **Definition 239** (Representable functor). A **covariant** functor  $F : \mathbf{C} \rightsquigarrow \mathbf{Set}$  is **representable** if there is an **object**  $X \in \mathbf{C}_0$  such that  $F$  is **naturally isomorphic** to  $\text{Hom}_{\mathbf{C}}(X, -)$ . If  $F$  is **contravariant**, then it is **representable** if it is **naturally isomorphic** to  $\text{Hom}_{\mathbf{C}}(-, X)$ .

**Examples 240.** Let us give examples of the **contravariant** kind.

1. The **contravariant powerset** functor  $\widehat{\mathcal{P}} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$  sends a set  $X$  to its **powerset**  $\mathcal{P}(X)$  and a function  $f : X \rightarrow Y$  to the inverse image  $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ . It is common to identify subsets of a given set with functions from this set into  $2 = \{0, 1\}$ . Formally, this is an **isomorphism**  $\widehat{\mathcal{P}}(X) \cong H_2(X) = 2^X$  for any  $X$ , it maps  $S \subseteq X$  to the characteristic function  $\chi_S$ .<sup>192</sup> In the reverse direction, it sends a function  $g : X \rightarrow \{0, 1\}$  to  $g^{-1}(1)$ . It is easy to check that for any  $f : X \rightarrow Y$ , the **isomorphisms** make (100) **commute**, so  $\widehat{\mathcal{P}} \cong H_2$ .

$$\begin{array}{ccc} H_2(X) & \xrightarrow{f \circ (-)} & H_2(Y) \\ \updownarrow & & \updownarrow \\ \widehat{\mathcal{P}}(X) & \xrightarrow{f^{-1}} & \widehat{\mathcal{P}}(Y) \end{array} \quad (100)$$

2. In functional programming, it is often useful to transform a function taking multiple arguments so that it ends up taking a single argument but outputs another function. For instance, the multiplication function  $\text{mult} : \text{int} \times \text{int} \rightarrow \text{int}$  that takes two numbers as inputs and outputs their product can be rewritten as  $\text{multc} : \text{int} \rightarrow (\text{int} \rightarrow \text{int})$ . The function  $\text{multc}$  takes a number as input and outputs a function that outputs the product of its input and the initial input of  $\text{multc}$ . For example  $\text{multc}(3)$  is a function that outputs  $3 \cdot n$  when  $n$  is the input. This new function  $\text{multc}$  is said to be the **curried** version of  $\text{mult}$  in honor of Haskell Curry. This leads to a more general argument in **Set**.

Fix two sets  $A$  and  $B$ . The **functor**  $\text{Hom}(- \times A, B)$  maps a set  $X$  to  $\text{Hom}(X \times A, B)$  and a function  $f : X \rightarrow Y$  to the function  $(-) \circ (f \times \text{id}_A)$ .<sup>193</sup> As suggested

<sup>190</sup> Note that  $f$  is completely determined by  $f(1)$  because  $f(n) = f(1) + \dots + f(1)$  and  $f(0)$  must be the identity.

<sup>191</sup> Recall that **Cat** only contains **small categories**.

<sup>192</sup> It sends  $x \in X$  to 1 if  $x \in S$  and to 0 otherwise.

<sup>193</sup> You can see it as the composition  $H_B \circ (- \times A)$ .

by the [currying](#) process for mult, for any set  $X$ , there is a bijection  $\text{Hom}(X \times A, B) \cong \text{Hom}(X, B^A)$ . The image of  $f : X \times A \rightarrow B$  is denoted  $\lambda f$  and it satisfies  $f(x, a) = \lambda f(x)(a)$  for any  $x \in X$  and  $a \in A$ . It is easy to check that this is a bijection and also that it is [natural](#) in  $X$  because the (101) [commutes](#) for any  $f : X \rightarrow Y$ , so  $\text{Hom}(- \times A, B) \cong \text{Hom}(-, B^A)$ .

$$\begin{array}{ccc} \text{Hom}(X \times A, B) & \xrightarrow{(-) \circ (f \times \text{id}_A)} & \text{Hom}(Y \times A, B) \\ \updownarrow & & \updownarrow \\ \text{Hom}(X, B^A) & \xrightarrow{(-) \circ f} & \text{Hom}(Y, B^A) \end{array} \quad (101)$$

In the first item of Examples 238 and 240, we made an arbitrary choice of set. That is, we could have taken any singleton in the first case and any set with two elements in the second. More generally, one can show that if  $A \cong B$ , then  $H_A \cong H_B$  and  $H^A \cong H^B$ .

**Exercise 241.** Let  $A, B \in \mathbf{C}_0$  be [isomorphic objects](#). Show that  $H^A \cong H^B$ . [Dually](#), show that  $H_A \cong H_B$ .

Surprisingly, the converse is also true and it will follow from the [Yoneda lemma](#), but we prove it on its own first as a warm-up for the proof of the [lemma](#).

**Proposition 242.** Let  $A, B \in \mathbf{C}_0$  be such that  $H^A \cong H^B$ , then  $A \cong B$ .

*Proof.* The [natural isomorphism](#) gives two [natural transformations](#)  $\phi : H^A \Rightarrow H^B$  and  $\eta : H^B \Rightarrow H^A$  such that for any [object](#)  $X \in \mathbf{C}_0$ ,

$$\eta_X \circ \phi_X : H^A(X) \rightarrow H^A(X) \quad \text{and} \quad \phi_X \circ \eta_X : H^B(X) \rightarrow H^B(X)$$

are [identities](#). In order to show  $A \cong B$ , we will find two [morphisms](#)  $f : B \rightarrow A$  and  $g : A \rightarrow B$  such that  $f \circ g = \text{id}_A$  and  $g \circ f = \text{id}_B$ .

First, note that putting  $X$  equal to  $A$ , we get  $\eta_A(\phi_A(\text{id}_A)) = \text{id}_A$  and we claim that

$$\eta_A(\phi_A(\text{id}_A)) = \phi_A(\text{id}_A) \circ \eta_B(\text{id}_B).$$

Since  $\phi_A(\text{id}_A)$  is a [morphism](#) from  $B$  to  $A$ , (102) [commutes](#) by [naturality](#) of  $\eta$ . The equality then follows, starting with  $\text{id}_B \in H_B(B)$ .

$$\begin{array}{ccc} H_B(A) & \xrightarrow{\eta_A} & H_A(A) \\ \phi_A(\text{id}_A) \circ (-) \uparrow & & \uparrow \phi_A(\text{id}_A) \circ (-) \\ H_B(B) & \xrightarrow{\eta_B} & H_A(B) \end{array} \quad (102)$$

A [dual](#) argument shows that

$$\text{id}_B = \phi_B(\eta_B(\text{id}_B)) = \eta_B(\text{id}_B) \circ \phi_A(\text{id}_A),$$

so we can conclude, letting  $f = \phi_A(\text{id}_A)$  and  $g = \eta_B(\text{id}_B)$ , that  $A \cong B$ .  $\square$

For every  $A \in \mathbf{C}_0$ , there are two functors  $H^A$  and  $H_A$ , they are objects of  $[\mathbf{C}, \mathbf{Set}]$  and  $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$  respectively. It is then reasonable to expect that the assignments  $A \mapsto H^A$  and  $A \mapsto H_A$  are functorial.

**Definition 243 (Yoneda embeddings).** The contravariant embedding  $H^{(-)} : \mathbf{C}^{\text{op}} \rightsquigarrow [\mathbf{C}, \mathbf{Set}]$  sends  $A \in \mathbf{C}_0$  to the Hom functor  $H^A$  and a morphism  $f : A' \rightarrow A$  to the natural transformation  $H^f : H^A \Rightarrow H^{A'}$  defined by  $H_B^f := \text{Hom}_{\mathbf{C}}(f, B) = (-) \circ f$  for every  $B \in \mathbf{C}_0$ . The naturality of  $H^f$  follows because (103) commutes (by associativity) for any  $g : B \rightarrow B'$ .

$$\begin{array}{ccc} H^A(B) & \xrightarrow{(-) \circ f} & H^{A'}(B) \\ g \circ (-) \downarrow & & \downarrow g \circ (-) \\ H^A(B') & \xrightarrow{(-) \circ f} & H^{A'}(B') \end{array} \quad (103)$$

The covariant embedding  $H_{(-)} : \mathbf{C} \rightsquigarrow [\mathbf{C}^{\text{op}}, \mathbf{Set}]$  sends  $B \in \mathbf{C}_0$  to the Hom functor  $H_B$  and a morphism  $f : B \rightarrow B'$  to the natural transformation  $H_f : H_B \rightarrow H_{B'}$  defined by  $H_f^A = \text{Hom}_{\mathbf{C}}(A, f) = f \circ (-)$  for any  $A \in \mathbf{C}_0$ . Naturality follows from a similar argument.

Functoriality is left for the reader to check. The embeddings are called like that because both functors are fully faithful as will follow from the Yoneda lemma.

### Yoneda Lemma

We have understood how an object  $A \in \mathbf{C}_0$  sees the category  $\mathbf{C}$  through representables, but since a representable is an object of another category, it is daring to study what representables see and how it relates to the object it represents. More formally, what is the functor  $\text{Hom}_{[\mathbf{C}, \mathbf{Set}]}(H^A, -)$  describing. For simplicity, we denote it  $\text{Nat}(H^A, -)$  because, for a functor  $F : \mathbf{C} \rightsquigarrow \mathbf{Set}$ ,  $\text{Nat}(H^A, F)$  is the collection<sup>194</sup> of natural transformations from  $H^A$  to  $F$ .

The surprising relation that the Yoneda lemma describes is that  $\text{Nat}(H^A, F)$  is isomorphic to  $F(A)$  naturally in  $F$  and  $A$ . We first show the isomorphism and then explain the naturality.

**Lemma 244 (Yoneda lemma I).** For any  $A \in \mathbf{C}_0$  and  $F : \mathbf{C} \rightsquigarrow \mathbf{Set}$ ,

$$\text{Nat}(H^A, F) \cong F(A).$$

*Proof.* Fix  $A$  and  $F$ , let  $\phi_{A,F} : \text{Nat}(H^A, F) \rightarrow F(A)$  be defined by  $\alpha \mapsto \alpha_A(\text{id}_A)$  (check that the types match). Let  $\eta_{A,F} : F(A) \rightarrow \text{Nat}(H^A, F)$  send an element  $a \in F(A)$  to the natural transformation that has components  $\eta_{A,F}(a)_B : f \mapsto F(f)(a) : \text{Hom}_{\mathbf{C}}(A, B) \rightarrow F(B)$  for any  $B \in \mathbf{C}_0$ . Checking (104) commutes for any  $g : B \rightarrow B'$

<sup>194</sup> Even if  $\mathbf{C}$  is locally small, there is no guarantee that  $[\mathbf{C}, \mathbf{Set}]$  is locally small. Nevertheless, one consequence of the Yoneda lemma is that  $\text{Nat}(F, G)$  is a set whenever  $F$  is representable.



shows that  $\eta_{A,F}(a)$  is a **natural transformation**.

$$\begin{array}{ccc} H^A(B) & \xrightarrow{F(-)(a)} & F(B) \\ g \circ (-) \downarrow & & \downarrow F(g) \\ H^A(B') & \xrightarrow{F(-)(a)} & F(B') \end{array} \quad (104)$$

We now check that  $\phi_{A,F}$  and  $\eta_{A,F}$  are inverses. First,  $(\eta \circ \phi)_{A,F}$  sends  $\alpha \in \mathbf{Nat}(H^A, F)$  to  $\eta_{A,F}(\alpha_A(\mathbf{id}_A))$ , and at any  $B \in \mathbf{C}_0$ , we have

$$\begin{aligned} \eta_{A,F}(\alpha_A(\mathbf{id}_A))_B(f) &= F(f)(\alpha_A(\mathbf{id}_A)) && \text{def of } \eta \\ &= \alpha_B(f \circ \mathbf{id}_A) && \text{naturality of } \alpha \\ &= \alpha_B(f), \end{aligned}$$

thus  $\alpha = (\eta \circ \phi)_{A,F}(\alpha)$ .

Conversely,  $(\phi \circ \eta)_{A,F}$  sends  $a \in F(A)$  to  $\eta_{A,F}(a)_A(\mathbf{id}_A) = F(\mathbf{id}_A)(a) = a$ .

We conclude that  $\eta_{A,F}$  and  $\phi_{A,F}$  are inverses.  $\square$

What this results first tells us is that  $\mathbf{Nat}(H^A, F)$  is a set (because it is **isomorphic** to  $F(A)$  which is a set). This lets us define two new **functors** to understand the second part of the **Yoneda lemma**.

The assignment  $(A, F) \mapsto \mathbf{Nat}(H^A, F)$  is a **functor**  $\mathbf{C} \times [\mathbf{C}, \mathbf{Set}] \rightsquigarrow \mathbf{Set}$ . We denote it  $\mathbf{Nat}(H^{(-)}, -)$ , it sends a **morphism**  $(g, \mu) : (A, F) \rightarrow (A', F')$  to  $\mu \cdot (-) \cdot H^g : \mathbf{Nat}(H^A, F) \rightarrow \mathbf{Nat}(H^{A'}, F')$ .<sup>195</sup>

The assignment  $(A, F) \mapsto F(A)$  is another **functor** of the same type. We denote it **Ev** (for evaluation), it sends a **morphism**  $(g, \mu) : (A, F) \rightarrow (A', F')$  to  $F'(g) \circ \mu_A : F(A) \rightarrow F'(A')$ .

**Lemma 245** (Yoneda lemma II). *There is a **natural isomorphism**  $\mathbf{Nat}(H^{(-)}, -) \cong \mathbf{Ev}$ .*

*Proof.* The **components** of this **isomorphism** are the ones described in the first part of the result. It remains to show that  $\phi$  is **natural** in  $(A, F)$ . For any  $(g, \mu) : (A, F) \rightarrow (A', F')$ , we need to show the following square **commutes**.

$$\begin{array}{ccc} \mathbf{Nat}(H^A, F) & \xrightarrow{\phi_{A,F}} & F(A) \\ \mu \cdot (-) \cdot H^g \downarrow & & \downarrow F'(g) \circ \mu_A \\ \mathbf{Nat}(H^{A'}, F') & \xrightarrow{\phi_{A',F'}} & F'(A') \end{array} \quad (105)$$

Starting with a **natural transformation**  $\alpha \in \mathbf{Nat}(H^A, F)$  the lower path sends it to  $(\mu \cdot \alpha \cdot H^g)_{A'}(\mathbf{id}_{A'})$  and the upper path sends it to  $(F'(g) \circ \mu_A)(\alpha_A(\mathbf{id}_A))$ . The following derivation shows they are equal.

$$\begin{aligned} (\mu \cdot \alpha \cdot H^g)_{A'}(\mathbf{id}_{A'}) &= (\mu_{A'} \circ \alpha_{A'}) (H_{A'}^g(\mathbf{id}_{A'})) && \text{def of composition} \\ &= (\mu_{A'} \circ \alpha_{A'})(g) && \text{def of } H_{A'}^g \\ &= (\mu_{A'} \circ \alpha_{A'}) (H_g^A(\mathbf{id}_A)) && \text{def of } H_g^A \end{aligned}$$

<sup>195</sup> As  $\mathbf{Nat}(-, -)$  is the **Hom bifunctor** of  $[\mathbf{C}, \mathbf{Set}]$ , we can see  $\mathbf{Nat}(H^{(-)}, -)$  as the **composition**

$$\mathbf{Nat}(-, -) \circ (H^{(-)} \times \mathbf{id}_{[\mathbf{C}, \mathbf{Set}]})$$

$$\begin{aligned}
&= (\mu_{A'} \circ \alpha_{A'} \circ H_g^A)(\text{id}_A) \\
&= (\mu_{A'} \circ F(g) \circ \alpha_A)(\text{id}_A) && \text{naturality of } \alpha \\
&= (F'(g) \circ \mu_A)(\alpha_A(\text{id}_A)) && \text{naturality of } \mu
\end{aligned}$$

□

**Corollary 246.** *The Yoneda embeddings  $H^{(-)}$  and  $H_{(-)}$  are fully faithful.*

*Proof.* Left as an exercise. □

**Example 247** (Cayley's theorem with the Yoneda lemma). Cayley's theorem states that any group is isomorphic to the subgroup of a permutation group. We will use the Yoneda lemma to show that.

Recall the first part of the Yoneda lemma which states that for a category  $\mathbf{C}$ , a functor  $F : \mathbf{C} \rightsquigarrow \mathbf{Set}$  and an object  $A \in \mathbf{C}_0$ , we have

$$\text{Nat}(\text{Hom}(A, -), F) \cong F(A).$$

Moreover, we know the explicit maps, namely, a natural transformation  $\phi$  in the L.H.S. is mapped to  $\phi_A(\text{id}_A)$  and an element  $u \in F(A)$  is mapped to the natural transformation  $\{\phi_B = f \mapsto F(f)(u) \mid B \in \mathbf{C}_0\}$ .

Let us apply this to  $\mathbf{C}$  being the delooping of  $G$ . Recall that any functor  $F : \mathbf{BG} \rightsquigarrow \mathbf{Set}$  sends  $*$  to a set  $S$  and any  $g \in G$  to a permutation of  $S$ , it corresponds to an action of  $G$  on  $S$ .

To use the Yoneda lemma, our only choice of object for  $A$  is  $*$  and we will choose for  $F$  the functor it represents, i.e.:  $F = \text{Hom}(*, -)$ . The Yoneda lemma yields

$$\text{Nat}(\text{Hom}(*, -), \text{Hom}(*, -)) \cong \text{Hom}(*, *).$$

We already know what the R.H.S. is  $G$ ,<sup>196</sup> but we have to do a bit of work to understand the L.H.S. First, observe that a natural transformation  $\phi : \text{Hom}(*, -) \Rightarrow \text{Hom}(*, -)$  is just one morphism  $\phi_* : \text{Hom}(*, *) \rightarrow \text{Hom}(*, *)$ . Namely, it is a map from  $G$  to  $G$ . Second, recalling that  $\text{Hom}(*, g) = g \circ (-)$  and that  $*$  is the only object in  $\mathbf{C}_0$ , we get that  $\phi_*$  must only make (106) commute.

<sup>196</sup> By definition of  $\mathbf{BG}$ .

$$\begin{array}{ccc}
G & \xrightarrow{\phi_*} & G \\
g \circ (-) \downarrow & & \downarrow g \circ (-) \\
G & \xrightarrow{\phi_*} & G
\end{array} \tag{106}$$

This is equivalent to  $\phi_*(g \cdot h) = g \cdot \phi_*(h)$ , and we get that each  $\phi_*$  is a  $G$ -equivariant map. Denote the set of  $G$ -equivariant maps  $\text{Hom}_G(G, G)$ . We obtain that, as sets,

$$\text{Hom}_G(G, G) \cong G.$$

Now, we can check that  $\text{Hom}_G(G, G)$  is a subgroup of  $\Sigma_G$  (the group of permutations of the set  $G$ ) and that the bijection is in fact an group isomorphism. Cayley's theorem follows.

To check that  $\text{Hom}_G(G, G) < \Sigma_G$ , we have to show that  $\text{id}_G$  is  $G$ -equivariant, that  $G$ -equivariant maps are bijective and that they are stable under composition and taking inverse. First, we have  $\text{id}_G(g \cdot h) = g \cdot h = g \cdot \text{id}_G(h)$ , so  $\text{id}_G \in \text{Hom}_G(G, G)$ . Second, let  $f$  be a  $G$ -equivariant map. For any  $g \in G$ , we have  $f(g) = f(g \cdot 1) = g \cdot f(1)$ . Thus,  $f$  is determined only by where it sends the identity. Additionally, since for any choice of  $f(1)$ ,  $g \cdot f(1)$  ranges over  $G$  when  $g$  ranges over  $G$ ,  $f$  is bijective. Therefore, if  $f$  and  $f'$  are both  $G$ -equivariant map, then

$$(f \circ f')(g \cdot h) = f(f'(g \cdot h)) = f(g \cdot f'(h)) = g \cdot (f \circ f')(h),$$

hence  $f \circ f'$  is  $G$ -equivariant. Finally,  $f^{-1}$  is the  $G$ -equivariant map sending 1 to  $f(1)^{-1}$  and we conclude that  $\text{Hom}_G(G, G)$  is a subgroup of  $\Sigma_G$ .

The final check is that the Yoneda bijection  $G \rightarrow \text{Hom}_G(G, G)$  sending  $g$  to  $(-) \cdot g$  is a group homomorphism.<sup>197</sup> It is clear that it sends the identity to the identity and for any  $g, h \in G$

$$(-) \cdot gh = ((-) \cdot g) \cdot h = ((-) \cdot h) \circ ((-) \cdot g),$$

so this is a group homomorphism.

### Universality as Representability

Representability is one of the two ways to describe universal constructions that we hinted at at the end of Chapter . In this section, we will explore how any universal property is equivalent to representability of some functor. Since (co)limits and universal morphisms are initial or terminal objects in some category, there is a first trivial way to express universality as representability.

**Exercise 248 (NOW!).** Let  $\mathbf{C}$  be a category,  $X \in \mathbf{C}_0$  and  $\mathbf{1} : \mathbf{C} \rightsquigarrow \mathbf{Set}$  be the constant functor at the singleton  $\mathbf{1} = \{\star\}$ . Show that  $\text{Hom}_{\mathbf{C}}(X, -) \cong \mathbf{1}$  if and only if  $X$  is initial. Dually,  $\text{Hom}_{\mathbf{C}}(-, X) \cong \mathbf{1}$  if and only if  $X$  is terminal.<sup>198</sup>

It turns out this result is not very useful.

**Proposition 249.** Let  $X, Y \in \mathbf{C}_0$ . The product of  $X$  and  $Y$  exists if and only if there exists  $P \in \mathbf{C}_0$  such that  $\text{Hom}_{\mathbf{C} \times \mathbf{C}}(\Delta_{\mathbf{C}}(-), (X, Y)) \cong \text{Hom}_{\mathbf{C}}(-, P)$ . The product is  $P$ .

*Proof.* ( $\Rightarrow$ ) Let  $P = X \times Y$ , for any  $A \in \mathbf{C}_0$ , there is an isomorphism

$$\text{Hom}_{\mathbf{C} \times \mathbf{C}}((A, A), (X, Y)) \cong \text{Hom}_{\mathbf{C}}(A, X \times Y)$$

which sends the pair  $(f : A \rightarrow X, g : A \rightarrow Y)$  to  $(f, g) : A \rightarrow X \times Y$ .<sup>199</sup> In the other direction,  $p : A \rightarrow X \times Y$  is sent to the pair  $(\pi_X \circ p, \pi_Y \circ p)$ . Let us show it is natural in  $A$ . For any  $m : A' \rightarrow A$ , (107) commutes because the top path sends the pair  $(f, g)$  to the morphism  $(f, g)$  then to  $(f, g) \circ m = (f \circ m, g \circ m)$  and the bottom path sends  $(f, g)$  to  $(f, g) \circ (m, m) = (f \circ m, g \circ m)$  which is then sent to  $(f \circ m, g \circ m)$ .

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C} \times \mathbf{C}}((A, A), (X, Y)) & \xrightarrow{\sim} & \text{Hom}_{\mathbf{C}}(A, X \times Y) \\ \downarrow - \circ (m, m) & & \downarrow - \circ m \\ \text{Hom}_{\mathbf{C} \times \mathbf{C}}((A', A'), (X, Y)) & \xrightarrow{\sim} & \text{Hom}_{\mathbf{C}}(A', X \times Y) \end{array} \quad (107)$$

<sup>197</sup> isomorphism follows because it is a bijection.

See solution.

<sup>198</sup> In the dual statement, the domain of  $\mathbf{1}$  is  $\mathbf{C}^{\text{op}}$ .

<sup>199</sup> Recall that  $(f, g)$  is the unique morphism satisfying  $\pi_X \circ (f, g) = f$  and  $\pi_Y \circ (f, g) = g$ . Be careful not to confuse it with a pair of morphisms.

( $\Leftarrow$ ) First, we define  $\pi_X$  and  $\pi_Y$  to be the pair of **morphisms** corresponding to  $\text{id}_P$  under the **isomorphism**  $\text{Hom}_{\mathbf{C} \times \mathbf{C}}((P, P), (X, Y)) \cong \text{Hom}_{\mathbf{C}}(P, P)$ . Given two **morphism**  $f : A \rightarrow X$  and  $g : A \rightarrow Y$ , the **isomorphism**

$$\text{Hom}_{\mathbf{C} \times \mathbf{C}}((A, A), (X, Y)) \cong \text{Hom}_{\mathbf{C}}(A, P)$$

yields a unique **morphism**  $! : A \rightarrow P$ . To see that  $\pi_X \circ ! = f$  and  $\pi_Y \circ ! = g$  we start with  $\text{id}_P$  in the top right of (108) which **commutes** by hypothesis.  $\square$

**Corollary 250 (Dual).** Let  $X, Y \in \mathbf{C}_0$ . The **coproduct** of  $X$  and  $Y$  exists if and only if there exists  $S \in \mathbf{C}_0$  such that  $\text{Hom}_{\mathbf{C} \times \mathbf{C}}((X, Y), \Delta_{\mathbf{C}}(-)) \cong \text{Hom}_{\mathbf{C}}(S, -)$ . The **coproduct** is  $S$ .<sup>200</sup>

In order to generalize these two results to arbitrary **(co)limits**, we defined the generalized version of  $\Delta_{\mathbf{C}}$ .

**Definition 251** (Generalized diagonal functor). Let  $\mathbf{J}$  and  $\mathbf{C}$  be **categories** the **generalized diagonal functor**  $\Delta_{\mathbf{C}}^{\mathbf{J}} : \mathbf{C} \rightsquigarrow [\mathbf{J}, \mathbf{C}]$  sends an **object**  $X \in \mathbf{C}_0$  to the **constant functor** at  $X$  and a **morphism**  $f : X \rightarrow Y \in \mathbf{C}_1$  to the **natural transformation** whose components are all  $f : X \rightarrow Y$ .

*Remark 252.* This is a generalization of the **diagonal functor**  $\Delta_{\mathbf{C}} : \mathbf{C} \rightsquigarrow \mathbf{C} \times \mathbf{C}$  because with the **isomorphism**  $[1 + 1, \mathbf{C}] \cong \mathbf{C} \times \mathbf{C}$  described in Example 215.2, we can identify  $\Delta_{\mathbf{C}}$  with  $\Delta_{\mathbf{C}}^{1+1}$ .

**Proposition 253.** Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a **diagram**. The **limit** of  $F$  exists if and only if there is an **object**  $L \in \mathbf{C}_0$  such that  $\text{Nat}(\Delta_{\mathbf{C}}^{\mathbf{J}}(-), F) \cong \text{Hom}_{\mathbf{C}}(-, L)$ .<sup>201</sup> The **tip** of the **limit cone** is  $L$ .

*Proof.* First, we note that for any  $X \in \mathbf{C}_0$ , a **natural transformation**  $\psi : \Delta_{\mathbf{C}}^{\mathbf{J}}(X) \Rightarrow F$  is a **cone over**  $F$  with **tip**  $X$ . Indeed, for any  $j : A \rightarrow B \in \mathbf{J}_1$ , the **naturality** square in (109) is **commutative**.

$$\begin{array}{ccc} X & \xrightarrow{X(j)=\text{id}_X} & X \\ \psi_A \downarrow & & \downarrow \psi_B \\ FA & \xrightarrow{F(j)} & FB \end{array} \quad (109)$$

This is equivalent to  $\{\psi_A : X \rightarrow FA\}_{A \in \mathbf{J}_0}$  being a **cone over**  $F$ . Furthermore, a **morphism** of **cones**  $\phi \rightarrow \psi$  is a **morphism**  $f$  between the **tips** such that  $\forall A \in \mathbf{J}_0, \phi_A = \psi_A \circ f$ . By looking at (110), we see this condition is equivalent to  $\phi = \psi \circ \Delta_{\mathbf{C}}^{\mathbf{J}}(f)$ .

( $\Rightarrow$ ) Let  $\{\psi_A : L \rightarrow FA\}_{A \in \mathbf{J}_0}$  be the **terminal cone over**  $F$  and see it as a **natural transformation**  $\psi : \Delta_{\mathbf{C}}^{\mathbf{J}}(L) \Rightarrow F$ . We need to define a **natural isomorphism**  $\text{Nat}(\Delta_{\mathbf{C}}^{\mathbf{J}}(-), F) \cong \text{Hom}_{\mathbf{C}}(-, L)$ . Similarly to the proofs of the previous section, we will see that we only need to see where  $\text{id}_L$  is sent to and the rest of the **natural transformation** will *construct itself*. Our only choice for the **cone** corresponding to  $\text{id}_L$  is  $\psi$  (it is the only **cone** we know exists).

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C} \times \mathbf{C}}((P, P), (X, Y)) & \xleftarrow{\sim} & \text{Hom}_{\mathbf{C}}(P, P) \\ \downarrow - \circ (!, !)^{\downarrow} & & \downarrow - \circ ! \\ \text{Hom}_{\mathbf{C} \times \mathbf{C}}((A, A), (X, Y)) & \xleftarrow{\sim} & \text{Hom}_{\mathbf{C}}(A, P) \end{array} \quad (108)$$

<sup>200</sup> We implicitly use the fact that  $(\mathbf{C} \times \mathbf{C})^{\text{op}} \cong \mathbf{C}^{\text{op}} \times \mathbf{C}^{\text{op}}$ .

We have  $\Delta_{\mathbf{C}}^{\mathbf{J}}(f) : X \Rightarrow Y$  because for any  $j \in \mathbf{J}_1$ , we have

$$\begin{array}{ccc} X & \xrightarrow{X(j)=\text{id}_X} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{Y(j)=\text{id}_Y} & Y \end{array}$$

<sup>201</sup> Recall that

$$\text{Nat}(\Delta_{\mathbf{C}}^{\mathbf{J}}(-), F) = \text{Nat}(-, F) \circ \Delta_{\mathbf{C}}^{\mathbf{J}}.$$

$$\begin{array}{ccccc} & Y & \xrightarrow{\text{id}_Y} & Y & \\ & \swarrow f & & \swarrow f & \\ X & \xrightarrow{\text{id}_X} & X & & \\ \swarrow \psi_A & \searrow \phi_A & & \searrow \psi_B & \downarrow \phi_B \\ & FA & \xrightarrow{F(j)} & FB & \end{array} \quad (110)$$

$$\begin{array}{ccc} \text{Nat}(\Delta_{\mathbf{C}}^{\mathbf{J}}(L), F) & \longleftrightarrow & \text{Hom}_{\mathbf{C}}(L, L) \\ \downarrow - \circ \Delta_{\mathbf{C}}^{\mathbf{J}}(f) & & \downarrow - \circ f \\ \text{Nat}(\Delta_{\mathbf{C}}^{\mathbf{J}}(X), F) & \longleftrightarrow & \text{Hom}_{\mathbf{C}}(X, L) \end{array} \quad (111)$$

Indeed, for any  $f : X \rightarrow L$  the **naturality** square in (111) means the **cone** corresponding to  $f : X \rightarrow L$  is  $\{\psi_A \circ f : X \rightarrow FA\}_{A \in \mathbf{J}_0}$  by starting with  $\text{id}_L$  in the top right. Now, since  $\psi$  is the **terminal cone**, for any **cone**  $\{\phi_A : X \rightarrow FA\}_{A \in \mathbf{J}_0}$ , there is a unique **morphism of cones**  $f : X \rightarrow L$  which satisfies  $\forall A \in \mathbf{J}_0, \psi_A \circ f = \phi_A$ . We conclude that  $f \mapsto \psi \circ \Delta_{\mathbf{C}}^{\mathbf{J}}(f)$  is a **natural isomorphism**.

( $\Leftarrow$ ) Let  $\psi : \Delta_{\mathbf{C}}^{\mathbf{J}}(L) \Rightarrow F$  be the **cone** corresponding to  $\text{id}_L \in \text{Hom}_{\mathbf{C}}(L, L)$  under the **natural isomorphism**, we will show it is **terminal**. By the **commutativity** of (111) and bijectivity of the horizontal arrows, for any **cone**  $\phi : \Delta_{\mathbf{C}}^{\mathbf{J}}(X) \Rightarrow F$ , there is a unique **morphism**  $f : X \rightarrow L$  such that  $\phi = \psi \circ \Delta_{\mathbf{C}}^{\mathbf{J}}(f)$ . By the first paragraph of the proof, this is the unique **morphism of cones** showing  $\psi$  is **terminal**.  $\square$

**Corollary 254 (Dual).** Let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a **diagram**. The **colimit** of  $F$  exists if and only if there is an object  $L \in \mathbf{C}_0$  such that  $\text{Nat}(F, \Delta_{\mathbf{C}}^{\mathbf{J}}(-)) \cong \text{Hom}_{\mathbf{C}}(L, -)$ . The **tip** of the **colimit cone** is  $L$ .

**Proposition 255.** Let  $U : \mathbf{Mon} \rightsquigarrow \mathbf{Set}$  be the **forgetful functor**,  $A$  be a set and  $A^*$  be the **free monoid** on  $A$ , we have  $\text{Hom}_{\mathbf{Set}}(A, U-) \cong \text{Hom}_{\mathbf{Mon}}(A^*, -)$ .

*Proof.* We have already shown before Definition 190 that sending  $h : A \rightarrow M$  to  $h^* : A^* \rightarrow M$  is a bijection.<sup>202</sup> Now, we need to show it is **natural** in  $M$ . For any **monoid homomorphism**  $f : M \rightarrow N$ , (112) **commutes** (we omitted applications of  $U$ ) because starting with  $h : A \rightarrow M$ , we have  $(f \circ h)^* = f \circ h^*$ .<sup>203</sup>

$$\begin{array}{ccc} \text{Hom}_{\mathbf{Set}}(A, M) & \xrightarrow{\sim} & \text{Hom}_{\mathbf{Mon}}(A^*, M) \\ f \circ - \downarrow & & \downarrow f \circ - \\ \text{Hom}_{\mathbf{Set}}(A, N) & \xrightarrow{\sim} & \text{Hom}_{\mathbf{Mon}}(A^*, N) \end{array} \quad (112)$$

<sup>202</sup> In the other direction,  $h : A^* \rightarrow M$  is sent to  $U(h) \circ i$  where  $i : A \hookrightarrow A^*$  is the inclusion.

<sup>203</sup> To check this, let  $w = a_1 \cdots a_n \in A^*$ , we have

$$\begin{aligned} (f \circ h)^*(w) &= fh(a_1) \cdots fh(a_n) \\ &= f(h(a_1) \cdots h(a_n)) \\ &= f(h(w)). \end{aligned}$$

$\square$

In the next Proposition, we will generalize this result to see how any **universal morphism** corresponds to some kind of **representability** and we will even give a converse direction. The generalizations of the proof is straightforward, so we suggest you try to get familiar with a specific case in the next exercise.

**Exercise 256.** Let  $\mathbf{C}$  be a **category** and  $X \in \mathbf{C}_0$  be such that  $- \times X$  is a **functor**. An **object**  $A \in \mathbf{C}_0$  has an **exponential**  $A^X \in \mathbf{C}_0$  if and only if  $\text{Hom}_{\mathbf{C}}(- \times X, A) \cong \text{Hom}_{\mathbf{C}}(-, A^X)$ .

**Proposition 257.** Let  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  be a **functor** and  $X \in \mathbf{D}_0$ . There is a **universal morphism** from  $X$  to  $F$  if and only if there exists  $A \in \mathbf{C}_0$  such that  $\text{Hom}_{\mathbf{D}}(X, F-) \cong \text{Hom}_{\mathbf{C}}(A, -)$ .

*Proof.* ( $\Rightarrow$ ) Let  $a : X \rightarrow FA$  be a **universal morphism**, by definition, for any  $b : X \rightarrow FB$ , there is a unique **morphism**  $\phi_B(b) : A \rightarrow B$  such that  $F(\phi_B(b)) \circ a = b$ . In the other direction,  $\phi_B^{-1}$  sending  $f : A \rightarrow B$  to  $Ff \circ a$  is the inverse of  $\phi_B$ .<sup>204</sup> Let us now check that  $\phi_B$  is natural. For any  $m : B \rightarrow B'$ , (113) **commutes** because when starting with  $f : A \rightarrow B$  in the top right, the top path sends it to  $Ff \circ a$  then to  $Fm \circ Ff \circ a$

See solution.

<sup>204</sup> We check they are inverses:

$$\begin{aligned} \phi_B^{-1}(\phi_B(b)) &= F(\phi_B(b)) \circ a = b \\ \phi_B(\phi_B^{-1}(f)) &= \phi_B(Ff \circ a) = f. \end{aligned}$$

and the bottom path sends it to  $m \circ f$  then to  $F(m \circ f) \circ a$ .

$$\begin{array}{ccc}
 \text{Hom}_{\mathbf{C}}(X, FB) & \xleftarrow{\sim} & \text{Hom}_{\mathbf{D}}(A, B) \\
 Fm \circ - \downarrow & & \downarrow m \circ - \\
 \text{Hom}_{\mathbf{C}}(X, FB') & \xleftarrow{\sim} & \text{Hom}_{\mathbf{D}}(A, B')
 \end{array} \quad (113)$$

( $\Leftarrow$ ) Let  $a : X \rightarrow FA$  be the image of  $\text{id}_A : A \rightarrow A$  under the isomorphism  $\text{Hom}_{\mathbf{C}}(X, FA) \cong \text{Hom}_{\mathbf{D}}(A, A)$ , we claim that  $a$  is a universal morphism from  $X$  to  $F$ . Given  $b : X \rightarrow FB$ , let  $\phi_B(b)$  be its image under the isomorphism  $\text{Hom}_{\mathbf{C}}(X, FB) \cong \text{Hom}_{\mathbf{D}}(A, B)$ , it satisfies  $F(\phi_B(b)) \circ a = b$  because (114) commutes (start with  $\text{id}_A$  in the top right corner). The morphism  $\phi_B(b)$  is unique with this property because any other  $f : A \rightarrow B$  is the image of some  $b' \neq b$  under  $\phi_B$  yielding  $Ff \circ a = b' \neq b$ .  $\square$

$$\begin{array}{ccc}
 \text{Hom}_{\mathbf{C}}(X, FA) & \xleftarrow{\sim} & \text{Hom}_{\mathbf{D}}(A, A) \\
 F(\phi_B(b)) \circ - \downarrow & & \downarrow \phi_B(b) \circ - \\
 \text{Hom}_{\mathbf{C}}(X, FB) & \xleftarrow{\sim} & \text{Hom}_{\mathbf{D}}(A, B)
 \end{array} \quad (114)$$

**Corollary 258 (Dual).** Let  $F : \mathbf{C} \rightsquigarrow \mathbf{D}$  be a functor and  $X \in \mathbf{D}_0$ . There is a universal morphism from  $F$  to  $X$  if and only if there exists  $A \in \mathbf{C}_0$  such that  $\text{Hom}_{\mathbf{D}}(F-, X) \cong \text{Hom}_{\mathbf{C}}(-, A)$ .

Comparing Propositions 253 and 258 and their duals, we infer that (co)limits satisfy universal properties.

**Theorem 259.** Let  $F \in [\mathbf{J}, \mathbf{C}]_0$  be a diagram.

- The limit of  $F$  exists if and only if there is a universal morphism from  $\Delta_{\mathbf{C}}^{\mathbf{J}}$  to  $F$ .
- The colimit of  $F$  exists if and only if there is a universal morphism from  $F$  to  $\Delta_{\mathbf{C}}^{\mathbf{J}}$ .

In the next chapter, we will lift these correspondence to a more global version. Namely, we will see how to assemble the universal morphisms for all diagrams of shape  $\mathbf{J}$  into a powerful object.

# Adjunctions

We start with a **universal morphism**  $\eta_X : X \rightarrow RLX$  for all  $X \in \mathbf{C}_0$  and develop a lot of things. First, we show that  $L$  is **functorial**. For any  $f : X \rightarrow Y$ , the **universality** of  $\eta_X$  yields a unique **morphism**  $Lf : LX \rightarrow LY$  satisfying  $RLf \circ \eta_X = \eta_Y \circ f$  as summarized in (115).

The **functoriality** follows from the following equalities showing that  $L(\text{id}_X) = \text{id}_{LX}$  and  $L(g \circ f) = Lg \circ Lf$  because these **morphisms** make the relevant diagrams **commute**:

$$\begin{aligned} R(\text{id}_{LX}) \circ \eta_X &= \text{id}_{RLX} \circ \eta_X = \eta_X = \eta_X \circ \text{id}_X \\ R(Lg \circ Lf) \circ \eta_X &= RLg \circ RLf \circ \eta_X = RLg \circ \eta_Y \circ f = \eta_Z \circ (g \circ f). \end{aligned}$$

Note that the definition of  $L$  on **morphisms** gives us that  $\eta$  is a **natural transformation**  $\text{id}_{\mathbf{C}} \Rightarrow RL$ . Next, we will define a **natural transformation**  $\varepsilon : LR \Rightarrow \text{id}_{\mathbf{D}}$ . For  $X \in \mathbf{D}_0$ , we let  $\varepsilon_X$  be the unique **morphism** given by the **universality** of  $\eta_{RX}$  such that  $R(\varepsilon_X) \circ \eta_{RX} = \text{id}_{RX}$  (see (116)).

Let us show that  $\varepsilon_X : LRX \rightarrow X$  is a **universal morphism** from  $L$  to  $X$ . For any  $f : LA \rightarrow X$ , if  $g : A \rightarrow RX \in \mathbf{C}_1$  is such that  $f = \varepsilon_X \circ Lg$ , then applying  $R$  and **pre-composing** with  $\eta_A$ , we obtain

$$\begin{aligned} Rf \circ \eta_A &= R\varepsilon_X \circ RLg \circ \eta_A \\ &= R\varepsilon_X \circ \eta_{RX} \circ g && \text{NAT}(\eta, A, RX, g) \\ &= \text{id}_{RX} \circ g && \text{definition of } \varepsilon_X \\ &= g. \end{aligned}$$

We conclude that  $g := Rf \circ \eta_A$  is the unique **morphism** such that  $f = \varepsilon_X \circ Lg$ , hence  $\varepsilon_X$  is **universal**. Next, we show that  $\varepsilon : LR \Rightarrow \text{id}_{\mathbf{D}}$  is **natural**. For any  $f : X \rightarrow Y \in \mathbf{D}_1$ , by **universality**, there is a unique **morphism**  $g : RX \rightarrow RY$  such that  $f \circ \varepsilon_X = \varepsilon_Y \circ Lg$  (see (117)) and by our derivation above,  $g = Rf \circ R\varepsilon_X \circ \eta_{RX} = Rf$ . Thus, we find that  $f \circ \varepsilon_X = \varepsilon_Y \circ LRf$ , namely  $\varepsilon$  is **natural**.

Second to last thing, we show that  $\eta$  and  $\varepsilon$  satisfy the **triangle identities** shown in (118) and (119) (they are **commutative** in  $[\mathbf{C}, \mathbf{D}]$  and  $[\mathbf{D}, \mathbf{C}]$  respectively).

$$\begin{array}{ccc} L & \xrightarrow{L\eta} & LRL \\ & \searrow \text{id}_L & \downarrow \varepsilon_L \\ & & L \end{array} \quad (118)$$

$$\begin{array}{ccc} RLRL & \xleftarrow{\eta^R} & R \\ R\varepsilon \downarrow & \swarrow \text{id}_R & \\ R & & \end{array} \quad (119)$$

$$\begin{array}{ccc} \text{in } \mathbf{C} & & \text{in } \mathbf{D} \\ X & \xrightarrow{\eta_X} & RLX \\ f \downarrow & \searrow \eta_Y \circ f & \downarrow RLf \\ Y & \xrightarrow{\eta_Y} & RLY \end{array} \xleftarrow{R} \begin{array}{ccc} \text{in } \mathbf{D} \\ LX \\ \downarrow Lf \\ LY \end{array} \quad (115)$$

$$\begin{array}{ccc} \text{in } \mathbf{C} & & \text{in } \mathbf{D} \\ RX & \xrightarrow{\eta_{RX}} & RLRLX \\ \text{id}_{RX} \searrow & \downarrow R\varepsilon_X & \downarrow \varepsilon_X \\ & RX & X \end{array} \xleftarrow{R} \begin{array}{ccc} \text{in } \mathbf{D} \\ LRX \\ \downarrow \varepsilon_X \\ X \end{array} \quad (116)$$

$$\begin{array}{ccc} \text{in } \mathbf{D} & & \text{in } \mathbf{C} \\ Y & \xleftarrow{\varepsilon_Y} & LRY \\ f \uparrow & \swarrow \varepsilon_X \circ f & \uparrow Lg \\ X & \xleftarrow{\varepsilon_X} & LRX \end{array} \xleftarrow{L} \begin{array}{ccc} \text{in } \mathbf{C} \\ RY \\ \uparrow g \\ RX \end{array} \quad (117)$$

The second one holds by definition of  $\varepsilon_X$  (for any  $X \in \mathbf{D}_0$ ,  $R\varepsilon_X \circ \eta_{RX} = \text{id}_{RX}$ ). For the first one, by **universality** there is a unique **morphism**  $g : X \rightarrow RLX$  such that  $\text{id}_{LX} = \varepsilon_{LX} \circ Lg$  (see (120)) and by our derivation above,  $g = R(\text{id}_{LX}) \circ \eta_X = \eta_X$ . We find that  $\varepsilon_{LX} \circ L\eta_X = \text{id}_{LX}$  as desired.

Finally, we now show that there is a **natural isomorphisms**

$$\Phi : \text{Hom}_{\mathbf{C}}(-, R-) \cong \text{Hom}_{\mathbf{D}}(L-, -) : \Phi^{-1}.$$

For  $g : X \rightarrow RY$ , we define  $\Phi_{X,Y}(g) = \varepsilon_Y \circ Lg$  and for  $f : LX \rightarrow Y$ , we define  $\Phi_{X,Y}^{-1}(f) = Rf \circ \eta_X$ .<sup>205</sup> The derivations below show these are inverses:

$$\begin{aligned} \Phi_{X,Y}^{-1}(\Phi_{X,Y}(g)) &= R\varepsilon_Y \circ RLg \circ \eta_X = R\varepsilon_Y \circ \eta_{RY} \circ g = g \\ \Phi_{X,Y}(\Phi_{X,Y}^{-1}(f)) &= \varepsilon_Y \circ LRf \circ L\eta_X = f \circ \varepsilon_{LX} \circ L\eta_X = f. \end{aligned}$$

To show that  $\Phi$  is **natural**, we need to show that (121) **commutes** for any  $x : X' \rightarrow X$  and  $y : Y \rightarrow Y'$ . Starting with  $g : X \rightarrow RY$  in the top left, the bottom path sends it to  $Ry \circ g \circ x$  then to  $\varepsilon_{Y'} \circ LRy \circ Lg \circ Lx$  and the top path sends  $g$  to  $\varepsilon_Y \circ Lg$  then to  $y \circ \varepsilon_Y \circ Lg \circ Lx$ . The end results are equal by **NAT**( $\varepsilon, Y, Y', y$ ).

**Definition 260** (Adjunction). An **adjunction** between a **functor**  $L : \mathbf{C} \rightsquigarrow \mathbf{D}$  and  $R : \mathbf{D} \rightsquigarrow \mathbf{C}$  is the following data:<sup>206</sup>

- ▮ A **natural transformation**  $\eta : \text{id}_{\mathbf{C}} \Rightarrow RL$  called the **unit** such that  $\eta_X$  is **initial** in  $X \downarrow R$  for each  $X \in \mathbf{C}_0$ .
- ▮ A **natural transformation**  $\varepsilon : LR \Rightarrow \text{id}_{\mathbf{D}}$  called the **counit** such that  $\varepsilon_X$  is **terminal** in  $L \downarrow X$  for each  $X \in \mathbf{D}_0$ .
- The **unit**  $\eta$  and **counit**  $\varepsilon$  satisfy the **triangle identities**.
- A **natural isomorphism**  $\Phi : \text{Hom}_{\mathbf{C}}(-, R-) \cong \text{Hom}_{\mathbf{D}}(L-, -) : \Phi^{-1}$  such that  $\Phi_{RX,X}(\text{id}_{RX}) = \varepsilon_X$  and  $\Phi_{X,LX}^{-1}(\text{id}_{LX}) = \eta_X$ .

▮ We denote  $L \dashv R$  when there is an **adjunction** between  $L$  and  $R$  and we call  $L$  the **left adjoint** and  $R$  the **right adjoint**.

**Theorem 261.** Two **functors**  $L : \mathbf{C} \rightsquigarrow \mathbf{D}$  and  $R : \mathbf{D} \rightsquigarrow \mathbf{C}$  are **adjoints** if at least one of the following holds.

- i. There is a **natural transformation**  $\eta : \text{id}_{\mathbf{C}} \Rightarrow RL$  such that  $\eta_X$  is **initial** in  $X \downarrow R$  for each  $X \in \mathbf{C}_0$ .
- ii. There is a **natural transformation**  $\varepsilon : LR \Rightarrow \text{id}_{\mathbf{D}}$  such that  $\varepsilon_X$  is **terminal** in  $L \downarrow X$  for each  $X \in \mathbf{D}_0$ .
- ▮ iii. There are two **natural transformations**  $\eta : \text{id}_{\mathbf{C}} \Rightarrow RL$  and  $\varepsilon : LR \Rightarrow \text{id}_{\mathbf{D}}$  that satisfy the **triangle identities**.<sup>207</sup>
- ▮ iv. There is a **natural isomorphism**  $\Phi : \text{Hom}_{\mathbf{C}}(-, R-) \cong \text{Hom}_{\mathbf{D}}(L-, -) : \Phi^{-1}$ .

$$\begin{array}{ccc} \text{in } \mathbf{D} & & \text{in } \mathbf{C} \\ LX & \xleftarrow{\varepsilon_{LX}} & LRLX \\ & \nwarrow \text{id}_{LX} & \uparrow Lg \\ & & LX \\ & & \uparrow g \\ & & X \end{array} \quad \xleftarrow{L} \quad \begin{array}{c} RLX \\ \uparrow g \\ X \end{array} \quad (120)$$

<sup>205</sup> Note because it will be useful that  $\Phi_{X,Y}(\text{id}_{RX}) = \varepsilon_X$  and  $\Phi_{X,Y}^{-1}(\text{id}_{LX}) = \eta_X$ .

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}}(X, RY) & \xleftarrow{\Phi_{X,Y}} & \text{Hom}_{\mathbf{D}}(LX, Y) \\ \downarrow Ry \circ - \circ x & & \downarrow y \circ - \circ Lx \\ \text{Hom}_{\mathbf{C}}(X', RY') & \xleftarrow{\Phi_{X',Y'}} & \text{Hom}_{\mathbf{D}}(LX', Y') \end{array} \quad (121)$$

<sup>206</sup> While this data is always part of an **adjunction**, we will prove in the next theorem that it is not necessary to specify all this data to obtain an **adjunction**. Moreover, this definition is not exhaustive in the sense that there is more things that you could construct and more properties you can derive from an **adjunction**. Still, we have to limit ourselves to a finite list and we mentioned the parts of an **adjunction** that are most commonly used. One notable omission is that of **adjunctions** as Kan extensions.

<sup>207</sup> They satisfy

$$\varepsilon L \cdot L\eta = \mathbb{1}_L \quad R\varepsilon \cdot \eta R = \mathbb{1}_R.$$



*Proof.* We have already shown that (i) gives rise to an **adjunction** at the start of the chapter.

For (ii), we can use **duality**. Indeed, taking the **dual** of Definition 260, we see that  $L \dashv R$  if and only if  $R^{\text{op}} \dashv L^{\text{op}}$  and  $\eta$  and  $\varepsilon$  swap their roles as **unit** and **counit**. Hence, from  $\varepsilon$ , we can derive an **adjunction**  $R^{\text{op}} \dashv L^{\text{op}}$  as we did at the start of the chapter and **duality** yields  $L \dashv R$ .

For (iii), it is enough to show  $\eta_X$  is **initial** in  $X \downarrow R$  and use (i).<sup>208</sup> Recall from our construction of  $\Phi$  and  $\Phi^{-1}$  above that for any  $g : X \rightarrow RY \in \mathbf{C}_1$ , there is a unique **morphism**  $\Phi_{X,Y}(g) = \varepsilon_Y \circ Lg$  such that  $R(\Phi_{X,Y}(g)) \circ \eta_X = \Phi_{X,Y}^{-1}(\Phi_{X,Y}(g)) = g$ . Thus,  $\eta_X$  is a **universal morphism** as required.

For (iv), we will construct a **unit** satisfying (i). Fix  $X \in \mathbf{C}_0$ , we have a **natural isomorphism**  $\Phi_{X,-} : \text{Hom}_{\mathbf{C}}(X, R-) \cong \text{Hom}_{\mathbf{D}}(LX, -)$ . By Proposition 257, there is a **universal morphism**  $\eta_X : X \rightarrow RLX$  from  $X$  to  $R$ .<sup>209</sup> This yields a **natural transformation**  $\eta : \text{id}_{\mathbf{C}} \Rightarrow RL$  because for any  $f : X \rightarrow Y$ , the **commutativity** of (122) implies (by starting with  $\text{id}_{LX}$  and  $\text{id}_{LY}$  in the top right and bottom right corners respectively)  $RLf \circ \eta_X = \Phi_{X,LY}^{-1}(Lf) = \eta_Y \circ f$ .  $\square$

Each points of Theorem 261 can be seen as a definition of **adjunctions**.<sup>210</sup> In the rest of this chapter, we will see many examples of **adjunctions** and results about **adjoint functors**, but we will rarely use a mix of the points of Theorem 261.

Our main point in the introduction to this chapter was that grouping **universal morphisms** together as we did into an **adjunction** yields a notion of *global universal construction*. In particular, we can characterize when a **category** has all **colimits** of shape  $\mathbf{J}$ .

**Theorem 262.** A **category**  $\mathbf{C}$  has all **limits** of shape  $\mathbf{J}$  if (and only if)<sup>211</sup> the **functor**  $\Delta_{\mathbf{C}}^{\mathbf{J}}$  has a **right adjoint**.

*Proof.* ( $\Rightarrow$ ) For each **diagram**  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$ , we pick (with the axiom of choice) a **limit**  $\lim_{\mathbf{J}} F$  given by **completeness** and a **universal morphism**  $\Delta_{\mathbf{C}}^{\mathbf{J}} \rightarrow F$  given by Theorem 259. By our argument at the start of the chapter, we get an **adjunction**  $\Delta_{\mathbf{C}}^{\mathbf{J}} \dashv \lim_{\mathbf{J}}$ .

( $\Leftarrow$ ) Suppose  $\mathbf{C} : \Delta_{\mathbf{C}}^{\mathbf{J}} \dashv L : [\mathbf{J}, \mathbf{C}]$  with **unit**  $\eta$  and let  $F : \mathbf{J} \rightsquigarrow \mathbf{C}$  be a **diagram**. By definition,  $\eta_F : \Delta_{\mathbf{C}}^{\mathbf{J}} L(F) \rightarrow F$  is a **universal morphism** from  $\Delta_{\mathbf{C}}^{\mathbf{J}}$  to  $F$ . Thus, by Theorem 259,  $L(F)$  is the **limit** of  $F$ .  $\square$

**Corollary 263 (Dual).** A **category**  $\mathbf{C}$  has all **colimits** of shape  $\mathbf{J}$  if and only if the **functor**  $\Delta_{\mathbf{C}}^{\mathbf{J}}$  has a **left adjoint**.

**Example 264 (Boring).** The **identity functor** on any **category** is self-adjoint, i.e.:  $\text{id}_{\mathbf{C}} \dashv \text{id}_{\mathbf{C}}$ . Both the **unit** and **counit** are  $\mathbb{1}_{\text{id}_{\mathbf{C}}}$ . This **adjunction** follows from the next result as  $\text{id}_{\mathbf{C}}$  is its own **inverse**.

**Example 265.** Recall from Exercise 152 the **maybe functor**  $- + \mathbf{1}$ . Denote  $\mathbf{1} = \{*\}$  for the **terminal object** of **Set**. We consider a very similar **functor**  $- + \mathbf{1} : \mathbf{Set} \rightsquigarrow \mathbf{Set}_*$  sending a set  $X$  to  $(X + \mathbf{1}, *)$  and  $f : X \rightarrow Y$  to  $f + \text{id}_{\mathbf{1}} : X + \mathbf{1} \rightarrow Y + \mathbf{1}$ . In the other direction, we have the **forgetful functor**  $U : \mathbf{Set}_* \rightsquigarrow \mathbf{Set}$  that forgets about the distinguished element of a **pointed set**. We claim that  $- + \mathbf{1} \dashv U$ .

<sup>208</sup> As before note that the **triangle identities** ensure that the **adjunction** constructed from (i) will have  $\varepsilon$  as a **counit**.

<sup>209</sup> From the proof of Proposition 257, we recover  $\eta_X = \Phi_{X,LX}^{-1}(\text{id}_{LX})$ .

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}}(X, RLX) & \longleftrightarrow & \text{Hom}_{\mathbf{D}}(LX, LX) \\ \downarrow RLf \circ - & & \downarrow Lf \circ - \\ \text{Hom}_{\mathbf{C}}(X, RLY) & \longleftrightarrow & \text{Hom}_{\mathbf{D}}(LX, LY) \\ - \circ f \uparrow & & \uparrow - \circ Lf \\ \text{Hom}_{\mathbf{C}}(Y, RLY) & \longleftrightarrow & \text{Hom}_{\mathbf{D}}(LY, LY) \end{array} \quad (122)$$

<sup>210</sup> In fact, that is how most textbooks present it.

<sup>211</sup>

First, for every set  $X$ , we need to define  $\eta_X : X \rightarrow U((X + \mathbf{1}, *)) = X + \mathbf{1}$ . The only obvious choice is to let  $\eta_X$  be the inclusion of  $X$  in  $X + \mathbf{1}$  and one can check it makes  $\eta$  into a **natural transformation**  $\text{id}_{\text{Set}} \Rightarrow U(- + \mathbf{1})$ .

Second, for every **pointed** set  $(X, x)$ , we need to define  $\varepsilon_{(X, x)} : (X + \mathbf{1}, *) \rightarrow (X, x)$ . Again, there is one clear choice, i.e.: acting like the identity on  $X$  and sending  $*$  to  $x$ , we will denote  $\varepsilon_{(X, x)} = [\text{id}_X, * \mapsto x]$ .

Finally, we need to check the **triangle identities** which we instantiate below.<sup>212</sup>

$$\begin{array}{ccc}
 (X + \mathbf{1}, *) & \xrightarrow{\eta_X + \text{id}_1} & ((X + \mathbf{1}) + \mathbf{1}, *) \\
 \searrow \text{id}_{X + \mathbf{1}} & & \downarrow [\text{id}_{X + \mathbf{1}}, * \mapsto *] \\
 & & (X + \mathbf{1}, *)
 \end{array}
 \quad
 \begin{array}{ccc}
 X & \xrightarrow{\eta_X} & X + \mathbf{1} \\
 \searrow \text{id}_X & & \downarrow [\text{id}_X, * \mapsto x] \\
 & & X
 \end{array}
 \quad (124)$$

(123)

We conclude that  $- + \mathbf{1} \dashv U$ . A good exercise in categorical thinking is to generalize this example to an arbitrary **category**  $\mathbf{C}$  with binary **coproducts** and a **terminal object**.<sup>213</sup>

**Proposition 266.** Let  $L : \mathbf{C} \rightsquigarrow \mathbf{D}$  and  $R : \mathbf{D} \rightsquigarrow \mathbf{C}$  be **quasi-inverses**, then  $L \dashv R$  and  $R \dashv L$ .

*Proof.* It is enough to show  $L \dashv R$  as the definition of **quasi-inverses** is symmetric.  $\square$

**Example 267 (Top).** Let  $U : \mathbf{Top} \rightsquigarrow \mathbf{Set}$  be the **forgetful functor** sending a **topological space** to its underlying set. We will find a left and a right **adjoint** to  $U$ .

**Proposition 268.** Let  $\mathbf{C} : L \dashv R : \mathbf{D}$  be **adjoint functors** and  $g : X \rightarrow Y \in \mathbf{D}_1$  be an **monomorphism**, then  $R(g)$  is **monic**. In other words, **right adjoints preserve monomorphisms**.<sup>214</sup>

*Proof.* Let  $h_1, h_2 : Z \rightarrow R(X)$  be such that  $R(g) \circ h_1 = R(g) \circ h_2$ , we need to show that  $h_1 = h_2$ . Since  $L \dashv R$ , we have the following **commutative square**.

$$\begin{array}{ccc}
 \text{Hom}_{\mathbf{C}}(Z, RX) & \longleftrightarrow & \text{Hom}_{\mathbf{D}}(FZ, X) \\
 Rg \circ - \downarrow & & \downarrow g \circ - \\
 \text{Hom}_{\mathbf{C}}(Z, RY) & \longleftrightarrow & \text{Hom}_{\mathbf{D}}(FZ, Y)
 \end{array}
 \quad (125)$$

Starting with  $h_1$  and  $h_2$  in the top left corner, we find that<sup>215</sup>

$$g \circ h_1^t = Rg \circ h_1 = Rg \circ h_2 = g \circ h_2^t,$$

which, by **monicity** of  $g$  implies  $h_1^t = h_2^t$ . This in turn means that  $h_1 = h_2$  because  $(-)^t$  is a bijection.  $\square$

**Proposition 269.** Let  $\mathbf{C} : L \dashv R : \mathbf{D}$  be **adjoint functors** and  $X, Y \in \mathbf{D}_0$ , if  $X \times Y$  exists, then  $R(X \times Y)$  with the **projections**  $R(\pi_X)$  and  $R(\pi_Y)$  is the **product**  $R(X) \times R(Y)$ . In other words, **right adjoints preserve binary products**.<sup>216</sup>

Check  $\eta$  and  $\varepsilon$  are **natural**:

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_X} & X + \mathbf{1} & (X, x) & \xrightarrow{\varepsilon_{(X, x)}} & (X + \mathbf{1}, *) \\
 f \downarrow & & \downarrow f + \text{id}_1 & f \downarrow & & \downarrow f + \text{id}_1 \\
 Y & \xrightarrow{\eta_Y} & Y + \mathbf{1} & (Y, y) & \xrightarrow{\varepsilon_{(Y, y)}} & (Y + \mathbf{1}, *)
 \end{array}$$

<sup>212</sup> When dealing with a set  $(X + \mathbf{1}) + \mathbf{1}$ , we will denote  $*$  for the element of the inner  $\mathbf{1}$  and  $\star$  for the outer one.

In (124),  $X = U(X, x)$ .

<sup>213</sup> See ... for a solution.

<sup>214</sup> Dually, if  $f : A \rightarrow B \in \mathbf{C}_1$  is **epic**, then  $L(f)$  is **epic**. In other words, **left adjoints preserve epimorphisms**.

<sup>215</sup> The first and last equality follow from **commutativity** of (125) and the middle equality is a hypothesis.

<sup>216</sup> Dually, if  $A, B \in \mathbf{C}_0$  and  $A + B$  exists, then  $L(A + B)$  with the **coprojections**  $L(\kappa_A)$  and  $L(\kappa_B)$  is the **coproduct**  $L(A) \times L(B)$ . In other words, **left adjoints preserve binary coproducts**.

*Proof.* Let  $p_X : A \rightarrow RX$  and  $p_Y : A \rightarrow RY$  be such that (126) **commutes**.

$$\begin{array}{ccccc}
 & A & & & \\
 p_X \swarrow & & \searrow p_Y & & \\
 RX & \xleftarrow{R\pi_X} & R(X \times Y) & \xrightarrow{R\pi_Y} & RY
 \end{array} \quad (126)$$

We need to show there is a unique **mediating morphism**  $A \rightarrow R(X \times Y)$ . First, we will get rid of the applications of  $R$  at the bottom, in order to use the **universal property** of the **product**  $X \times Y$ . To do this, we apply  $L$  to (126) and use the **counit**  $\varepsilon : LR \Rightarrow \text{id}_D$  to obtain (127).

$$\begin{array}{ccccc}
 & LA & & & \\
 Lp_X \swarrow & & \searrow Lp_Y & & \\
 LRX & \xleftarrow{LR\pi_X} & LR(X \times Y) & \xrightarrow{LR\pi_Y} & LRY \\
 \varepsilon_X \downarrow & & \varepsilon_{X \times Y} \downarrow & & \varepsilon_Y \downarrow \\
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y
 \end{array} \quad (127)$$

The **universal property** of  $X \times Y$  tells us there is a unique  $! : LA \rightarrow X \times Y$  such that  $\pi_X \circ ! = \varepsilon_X \circ Lp_X$  and  $\pi_Y \circ ! = \varepsilon_Y \circ Lp_Y$ . We claim that  $!^t$  is the **mediating morphism** of (126), i.e.:  $R\pi_X \circ !^t = p_X$  and  $R\pi_Y \circ !^t = p_Y$ . Using the **adjunction**  $L \dashv R$ , we obtain the following **commutative square**.

$$\begin{array}{ccc}
 \text{Hom}_D(LA, X \times Y) & \longleftrightarrow & \text{Hom}_C(A, R(X \times Y)) \\
 \pi_X \circ - \downarrow & & \downarrow R\pi_X \circ - \\
 \text{Hom}_D(LA, X) & \longleftrightarrow & \text{Hom}_C(A, RX)
 \end{array} \quad (128)$$

Now, starting with  $!$  on the top left corner, we obtain the following derivation.

$$\begin{aligned}
 p_X &= p_X^{!^t} \\
 &= (\varepsilon_X \circ Lp_X)^t \\
 &= \pi_X \circ !^t && \text{definition of } ! \\
 &= R\pi_X \circ !^t && \text{commutativity of (128)}
 \end{aligned}$$

Replacing  $X$  with  $Y$  in the previous argument shows  $!^t$  makes (129) **commute**. For the uniqueness, note that if  $m : A \rightarrow R(X \times Y)$  can replace  $!^t$ , then (130) **commutes** which implies by uniqueness of  $!$  that  $m^t = \varepsilon_{X \times Y} \circ Lm = !$ . Transposing yields  $!^t = m$ .

$$\begin{array}{ccccc}
 & LA & & & \\
 Lp_X \swarrow & & \searrow Lp_Y & & \\
 LRX & \xleftarrow{LR\pi_X} & LR(X \times Y) & \xrightarrow{LR\pi_Y} & LRY \\
 \varepsilon_X \downarrow & & \varepsilon_{X \times Y} \downarrow & & \varepsilon_Y \downarrow \\
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y
 \end{array} \quad (130)$$

□

$$\begin{array}{ccccc}
 & LA & & & \\
 Lp_X \swarrow & & \searrow Lp_Y & & \\
 LRX & & & & LRY \\
 \varepsilon_X \downarrow & & ! \downarrow & & \varepsilon_Y \downarrow \\
 X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y
 \end{array}$$

$$\begin{array}{ccccc}
 & A & & & \\
 p_X \swarrow & & \searrow p_Y & & \\
 RX & \xleftarrow{R\pi_X} & R(X \times Y) & \xrightarrow{R\pi_Y} & RY
 \end{array} \quad (129)$$

**Theorem 270.** *Right adjoints are continuous.*

**Corollary 271 (Dual).** *Left adjoints are cocontinuous.*

**Theorem 272.** *If  $\mathbf{C} : L \dashv R : \mathbf{D}$  and  $\mathbf{D} : L' \dashv R' : \mathbf{E}$  are two adjunctions, then  $\mathbf{C} : L'L \dashv RR' : \mathbf{E}$  is an adjunction.*

*Proof.* Let  $\eta$  and  $\varepsilon$  be the unit and counit of the first adjunction and  $\eta'$  and  $\varepsilon'$  be the unit and counit of the second one. We define the following unit and counit for the composite adjunction:

$$\begin{aligned}\hat{\eta} &= R\eta'L \cdot \eta : \text{id}_{\mathbf{C}} \Rightarrow RR'L'L \\ \hat{\varepsilon} &= \varepsilon' \cdot L'\varepsilon R' : L'LRR' \Rightarrow \text{id}_{\mathbf{E}}.\end{aligned}$$

The following diagrams show the triangle identities.

$$\begin{array}{ccccc} & & L'L\hat{\eta} & & \\ & \nearrow & & \searrow & \\ L'L & \xrightarrow{L'L\eta} & L'LRL & \xrightarrow{L'LR\eta'L} & L'LRR'L'L \\ & \searrow \scriptstyle 1_{L'L} \text{ (a)} & \downarrow \scriptstyle L'\varepsilon L \text{ (b)} & & \downarrow \scriptstyle L'\varepsilon R' L'L \\ & & L'L & \xrightarrow{L'\eta'L} & L'R'L'L \\ & & \downarrow \scriptstyle 1_{L'L} \text{ (c)} & & \downarrow \scriptstyle \varepsilon' L'L \\ & & & & L'L \end{array} \quad \widehat{\varepsilon} L'L \quad (131)$$

Showing (131) commutes:

- (a) Apply  $L'(-)$  to the left triangle identity of  $\eta$  and  $\varepsilon$ .
- (b) Apply  $L'(-)L$  to  $\text{HOR}(\varepsilon, \eta')$ .
- (c) Apply  $(-)_L$  to the left triangle identity of  $\eta'$  and  $\varepsilon'$ .

$$\begin{array}{ccccc} & & \hat{\eta}RR' & & \\ & \nwarrow & & \swarrow & \\ RR'L'LRR' & \xleftarrow{R\eta'LRR'} & RLRR' & \xleftarrow{\eta RR'} & RR' \\ & \downarrow \scriptstyle RR'L'\varepsilon R' \text{ (a)} & \downarrow \scriptstyle R\varepsilon R' \text{ (b)} & & \downarrow \scriptstyle 1_{RR'} \\ RR'\hat{\varepsilon} & \xleftarrow{RR'L'R'} & RR' & & \\ & \downarrow \scriptstyle RR'\varepsilon' \text{ (c)} & & & \downarrow \scriptstyle 1_{RR'} \\ & & RR' & & \end{array} \quad (132)$$

Showing (132) commutes:

- (a) Apply  $R(-)R'$  to  $\text{HOR}(\eta', \varepsilon)$ .
- (b) Apply  $(-)_R$  to the right triangle identity of  $\eta$  and  $\varepsilon$ .
- (c) Apply  $R(-)$  to the right triangle identity of  $\eta'$  and  $\varepsilon'$ .

□

**Proposition 273.** *If  $\mathbf{D} : L \dashv R : \mathbf{E}$  is an adjunction, then there is an adjunction  $[\mathbf{C}, \mathbf{D}] : L- \dashv R- : [\mathbf{C}, \mathbf{E}]$ .*

*Proof.* First, we can see that  $L-$  and  $R-$  are functors by Exercise 224.<sup>217</sup> Composing them yields  $RL- : [\mathbf{C}, \mathbf{D}] \rightsquigarrow [\mathbf{C}, \mathbf{D}]$  and  $LR- : [\mathbf{C}, \mathbf{E}] \rightsquigarrow [\mathbf{C}, \mathbf{E}]$ . Let  $\eta : \text{id}_{\mathbf{D}} \Rightarrow RL$  and  $\varepsilon : LR \Rightarrow \text{id}_{\mathbf{E}}$  be the unit and counit of  $L \dashv R$ . We claim that  $\eta- = F \mapsto \eta F$  and  $\varepsilon- = G \mapsto \varepsilon G$  are the unit and counit of an adjunction  $L- \dashv R-$ .

<sup>217</sup> They are compositions:

$$\begin{aligned}L- &= (- \circ -) \circ (L \times \text{id}_{[\mathbf{C}, \mathbf{D}]}) \\ R- &= (- \circ -) \circ (R \times \text{id}_{[\mathbf{C}, \mathbf{E}]})\end{aligned}$$

Alternatively, we can use Example 225.4 where we described currying for functors. In that setting, we have

$$\begin{aligned}L- &= \Lambda(- \circ -)(L) \\ R- &= \Lambda(- \circ -)(R).\end{aligned}$$

These functors send a natural transformation  $\phi : F \Rightarrow G$  to  $L\phi$  and  $R\phi$  respectively.

To see that  $\eta-$  and  $\varepsilon-$  are **natural transformations** of the right type, we can recognize them in the image of  $\Lambda(- \circ -)$  (noting that  $\text{id}_{\mathbf{D}}- = \text{id}_{[\mathbf{C}, \mathbf{D}]}$  and  $\text{id}_{\mathbf{E}}- = \text{id}_{[\mathbf{C}, \mathbf{E}]}$ ):

$$\begin{aligned}\eta- &= \Lambda(- \circ -)(\eta) : \text{id}_{[\mathbf{C}, \mathbf{D}]} \Rightarrow RL- \\ \varepsilon- &= \Lambda(- \circ -)(\varepsilon) : LR- \Rightarrow \text{id}_{[\mathbf{C}, \mathbf{E}]}.\end{aligned}$$

It is left to show the **triangle identities** hold assuming they hold for  $\eta$  and  $\varepsilon$ . In the following derivations, we use three simple facts:<sup>218</sup>

- the **biaction** of  $F-$  and  $G-$  on  $\phi-$  yields  $(F\phi G)-$ ,
- $\phi- \cdot \phi'- = (\phi \cdot \phi')-$ , and
- $(\mathbb{1}_F)- = \mathbb{1}_{F-}$ .

Now, the **triangle identities** hold by:

$$\begin{aligned}(\varepsilon-)(L- \cdot (L-)(\eta-)) &= (\varepsilon L- \cdot (L\eta-)) = (\varepsilon L \cdot L\eta)- = (\mathbb{1}_L)- = \mathbb{1}_{L-} \\ (R-)(\varepsilon- \cdot (\eta-)(R-)) &= (R\varepsilon- \cdot (\eta R-)) = (R\varepsilon \cdot \eta R)- = (\mathbb{1}_R)- = \mathbb{1}_{R-}.\end{aligned}$$

□

**Corollary 274 (Dual).** *If  $\mathbf{D} : L \dashv R : \mathbf{E}$  is an **adjunction**, then there is an **adjunction**  $[\mathbf{C}, \mathbf{D}] : -L \dashv -R : [\mathbf{C}, \mathbf{E}]$ .*

**Theorem 275.** *Let  $\mathbf{D}$  be a **category** with all **limits** of shape  $\mathbf{J}$ . For any **category**  $\mathbf{C}$ , the **functor category**  $[\mathbf{C}, \mathbf{D}]$  has all **limits** of shape  $\mathbf{J}$  and the **limit** of any **diagram**  $F : \mathbf{J} \rightsquigarrow [\mathbf{C}, \mathbf{D}]$  satisfies for any  $X \in \mathbf{C}_0$ ,  $(\lim_{\mathbf{J}} F)(X) = \lim_{\mathbf{J}} (F(-)(X))$ .<sup>219</sup>*

*Proof.* From previous results, we have the following chain of **adjunctions**.

$$[\mathbf{C}, \mathbf{D}] \xrightleftharpoons[\lim_{\mathbf{J} \circ -}]{\Delta_{\mathbf{D}}^{\mathbf{J} \circ -}} [\mathbf{C}, [\mathbf{J}, \mathbf{D}]] \xrightleftharpoons[\Lambda]{\Lambda^{-1}} [\mathbf{C} \times \mathbf{J}, \mathbf{D}] \xrightleftharpoons[\text{--} \circ \text{swap}^{-1}]{\text{--} \circ \text{swap}} [\mathbf{J} \times \mathbf{C}, \mathbf{D}] \xrightleftharpoons[\Lambda^{-1}]{\Lambda} [\mathbf{J}, [\mathbf{C}, \mathbf{D}]] \quad (133)$$

From left to right. The first **adjunction** is induced by Proposition 273 and the **adjunction**  $\Delta_{\mathbf{D}}^{\mathbf{J}} \dashv \lim_{\mathbf{J}}$  given by **completeness** of  $\mathbf{D}$ . The second **adjunction** is obtained from Proposition 266 and the fact that  $\Lambda$  and  $\Lambda^{-1}$  are **inverses**. The third **adjunction** is induced by Corollary 274 and the canonical **isomorphism**  $\text{swap} : \mathbf{C} \times \mathbf{J} \rightsquigarrow \mathbf{J} \times \mathbf{C}$ . The fourth **adjunction** is similar to the second one.

Next, we show that the **composition** of the top **path** is  $\Delta_{[\mathbf{C}, \mathbf{D}]}^{\mathbf{J}}$ .

Finally, we obtain a **right adjoint** to  $\Delta_{[\mathbf{C}, \mathbf{D}]}^{\mathbf{J}}$  by **composing** all the **adjunctions** with Theorem 272 and we conclude by Theorem 262 that  $[\mathbf{C}, \mathbf{D}]$  is **complete**. □

**Corollary 276 (Dual).** *Let  $\mathbf{D}$  be a **category** with all **colimits** of shape  $\mathbf{J}$ . For any **category**  $\mathbf{C}$ , the **functor category**  $[\mathbf{C}, \mathbf{D}]$  has all **colimits** of shape  $\mathbf{J}$  and the **colimit** of any **diagram**  $F : \mathbf{J} \rightsquigarrow [\mathbf{C}, \mathbf{D}]$  satisfies for any  $X \in \mathbf{C}_0$ ,  $(\text{colim}_{\mathbf{J}} F)(X) = \text{colim}_{\mathbf{J}} (F(-)(X))$ .<sup>220</sup>*

<sup>220</sup> In other words, **colimits** are taken pointwise.

**Corollary 277.** *If a **category**  $\mathbf{D}$  is (finitely) **complete** or **cocomplete**, then so is  $[\mathbf{C}, \mathbf{D}]$  for any **category**  $\mathbf{C}$ .*

<sup>218</sup> They can be shown by proving the equality at each **component**.

<sup>219</sup> In other words (that you will often hear), **limits** in **functor categories** are taken pointwise.



# Monads and Algebras

## POV: Category Theory

We will start from the concept of an **adjunction** which, as we hope was made clear in the previous chapter, is ubiquitous and powerful throughout mathematics. However, we will start with a great oversimplification; we will assume the **categories** concerned are **posetal**.

An **adjunction** between **posets** is more commonly called a **Galois connection**.

□ **Definition 278** (Galois connection). Let  $(P, \leq)$  and  $(Q, \sqsubseteq)$  be **posets**. A **Galois connection** between them is a pair of **order-preserving** functions  $L : P \rightarrow Q$  and  $R : Q \rightarrow P$  satisfying for any  $p \in P$  and  $q \in Q$ ,  $L(p) \sqsubseteq q \iff p \leq R(q)$ .

We are now interested in the **composition**  $R \circ L$ . It is also a **monotonic** function but the results about **adjoints** we have seen yield a couple of interesting properties. First, the existence of the **unit**  $\eta : \text{id}_P \Rightarrow RL$  means that for any  $p \in P$ , there is  $\eta_p : p \rightarrow RL(p)$ , so  $RL$  is **extensive**.<sup>221</sup> Second, the existence of the **counit**  $\varepsilon : RL \Rightarrow \text{id}_P$  means that for any  $p \in P$ , there is  $R(\varepsilon_{L(p)}) : RLRL(p) \rightarrow RL(p)$  and  $RL(\eta_p) : RL(p) \rightarrow RLRL(p)$ , so  $RL$  is **idempotent** (i.e.:  $\forall p \in P, RL(p) = RLRL(p)$ ). We say that  $RL$  is a **closure operator**.

<sup>221</sup> i.e.:  $\forall p \in P, p \leq RL(p)$ .

□ **Definition 279** (Closure operator). Let  $(P, \leq)$  be a **poset**, a **closure operator** on  $P$  is a **monotone extensive** and **idempotent** function  $c : P \rightarrow P$ .

**Example 280.** We can give a very simple example hinting at the origins of the terminology. Consider the real numbers  $\mathbb{R}$  with the standard **topology**.<sup>222</sup> We know that  $(\mathcal{P}(\mathbb{R}), \subseteq)$  is a **poset** and we can define  $c : \mathcal{P}(\mathbb{R}) \rightarrow \mathcal{P}(\mathbb{R})$  sending  $U \subseteq \mathbb{R}$  to its **closure**  $c(U)$  in the topological sense as in  $c(U)$  is the set of **limit points** of  $U$ . Then, you probably have seen that for any  $U, V \subseteq \mathbb{R}$ ,  $U \subseteq V \implies c(U) \subseteq c(V)$ ,  $U \subseteq c(U)$  and  $c(U) = c(c(U))$ , thus operation of **closure** is a **closure operator**.

<sup>222</sup> This is the **topology** induced by the standard **metric**  $d(x, y) := |x - y|$ .

We will generalize this discussion to arbitrary categories now. Let  $L : \mathbf{C} \rightleftarrows \mathbf{D} : R$  be an **adjoint pair** of **functors**, we have two **natural transformations**  $\eta : \text{id}_{\mathbf{C}} \Rightarrow RL$  and  $R\varepsilon L : RLRL \Rightarrow RL$ . Recall these may interact well together via the **zig-zag identities** that we reformulate in (134) and (135) and we add to those the **commutativity** diagram (136) in the definition of  $\varepsilon \diamond \varepsilon$ .

$$\begin{array}{ccc}
L \xrightarrow{L\eta} LRL & & RLR \xleftarrow{\eta R} R \\
\searrow \scriptstyle 1_L \quad \downarrow \scriptstyle \varepsilon L & (134) & \downarrow \scriptstyle R\varepsilon \quad \swarrow \scriptstyle 1_L \\
L & & L
\end{array}
\quad
\begin{array}{ccc}
LRLR \xrightarrow{\varepsilon LR} LR & & \\
\downarrow \scriptstyle LR\varepsilon \quad \downarrow \scriptstyle \varepsilon & (135) & \downarrow \scriptstyle \varepsilon \\
LR \xrightarrow{\varepsilon} \text{id}_D & & LR \xrightarrow{\varepsilon} \text{id}_D
\end{array}
\quad
(136)$$

With a bit of tinkering, we can make these diagrams all about the **functor** we are interested in, namely  $RL$ . Indeed, if we denote  $M = RL$  and  $\mu = R\varepsilon L$  before acting by  $R$  on the right of (134),  $L$  on the left of (135) and  $R(\cdot)L$  on (136), we obtain the definition of a **monad**.

Definition 281 (Monad). A **monad** is a triple comprised of an **endofunctor**  $M : \mathbf{C} \rightsquigarrow \mathbf{C}$  and two **natural transformations**  $\eta : \text{id}_{\mathbf{C}} \Rightarrow M$  and  $\mu : M^2 \Rightarrow M$  called the **unit** and **multiplication** respectively that make (137) and (138) **commute** in  $[\mathbf{C}, \mathbf{C}]$ .

$$\begin{array}{ccc}
M \xrightarrow{M\eta} M^2 & \xleftarrow{\eta M} & M \\
\searrow \scriptstyle 1_M \quad \downarrow \scriptstyle \mu & & \swarrow \scriptstyle 1_M \\
M & & M
\end{array}
\quad (137)
\quad
\begin{array}{ccc}
M^3 & \xrightarrow{M\mu} & M^2 \\
\downarrow \scriptstyle \mu M & & \downarrow \scriptstyle \mu \\
M^2 & \xrightarrow{\mu} & M
\end{array}
\quad (138)$$

**Examples 282.** Our discussion above tells us that any **adjoint pair**  $L \dashv R$  corresponds to a **monad**  $(RL, \eta, R\varepsilon L)$ , so all the examples of **adjunctions** you have seen correspond to suitable examples of **monads**. For instance, all **closure operators** are **monads**. Here, we describe three simple yet very useful examples and let you ponder on the **adjunctions** they might or might not originate from.

1. Suppose  $\mathbf{C}$  has (binary) **coproducts** and a **terminal object**  $\mathbf{1}$ , then  $(\cdot + \mathbf{1}) : \mathbf{C} \rightsquigarrow \mathbf{C}$  is a **monad**. We write  $\text{inl}^{X+Y}$  (resp.  $\text{inr}^{X+Y}$ ) for the **coprojection** of  $X$  (resp.  $Y$ ) into  $X + Y$ . First, note that for a **morphism**  $f : X \rightarrow Y$ ,

$$f + \mathbf{1} = [\text{inl}^{Y+\mathbf{1}} \circ f, \text{inr}^{Y+\mathbf{1}}] : X + \mathbf{1} \rightarrow Y + \mathbf{1}.$$

The **components** of the **unit** are given by the **coprojections**, i.e.:  $\eta_X = \text{inl}^{X+\mathbf{1}} : X \rightarrow X + \mathbf{1}$ , and the **components** of the **multiplication** are

$$\mu_X = [\text{inl}^{X+\mathbf{1}}, \text{inr}^{X+\mathbf{1}}, \text{inr}^{X+\mathbf{1}}] : X + \mathbf{1} + \mathbf{1} \rightarrow X + \mathbf{1}.$$

Checking that (137) **commutes**, we have for any  $X \in \mathbf{C}$ :

$$\begin{aligned}
\mu_X \circ (\eta_X + \mathbf{1}) &= [\mu_X \circ \text{inl}^{(X+\mathbf{1})+\mathbf{1}} \circ \eta_X, \mu_X \circ \text{inr}^{(X+\mathbf{1})+\mathbf{1}}] \\
&= [[\text{inl}^{X+\mathbf{1}}, \text{inr}^{X+\mathbf{1}}] \circ \text{inl}^{X+\mathbf{1}}, \text{inr}^{X+\mathbf{1}}] \\
&= [\text{inl}^{X+\mathbf{1}}, \text{inr}^{X+\mathbf{1}}] \\
&= \text{id}_{X+\mathbf{1}} \\
&= [\text{inl}^{X+\mathbf{1}}, \text{inr}^{X+\mathbf{1}}] \\
&= \mu_X \circ \text{inl}^{(X+\mathbf{1})+\mathbf{1}} \\
&= \mu_X \circ \eta_{X+\mathbf{1}}
\end{aligned}$$

For (138), we have for any  $X \in \mathbf{C}$ :

$$\mu_X \circ (\mu_X + \mathbf{1}) = [\mu_X \circ \text{inl}^{(X+\mathbf{1})+\mathbf{1}} \circ \mu_X, \mu_X \circ \text{inr}^{(X+\mathbf{1})+\mathbf{1}}]$$



$$\begin{aligned}
&= [[\text{inl}^{X+1}, \text{inr}^{X+1}] \circ \mu_X, \text{inr}^{X+1}] \\
&= [[\text{inl}^{X+1}, \text{inr}^{X+1}, \text{inr}^{X+1}], \text{inr}^{X+1}] \\
&= [\mu_X, \text{inr}^{X+1}] \\
&= [[\text{inl}^{X+1}, \text{inr}^{X+1}], \text{inr}^{X+1}, \text{inr}^{X+1}] \\
&= [\mu_X \circ \text{inl}^{(X+1)+1}, \mu_X \circ \text{inr}^{(X+1)+1}, \mu_X \circ \text{inr}^{(X+1)+1}] \\
&= \mu_X \circ \mu_{X+1}
\end{aligned}$$

2. The **covariant powerset functor**  $\mathcal{P} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$  is a monad with the following unit and multiplication:

$$\eta_X : X \rightarrow \mathcal{P}(X) = x \mapsto \{x\} \text{ and } \mu_X : \mathcal{P}(\mathcal{P}(X)) \rightarrow \mathcal{P}(X) = F \mapsto \bigcup_{s \in F} s.$$

Checking that (137), we have for any  $S \subseteq \mathcal{P}(X)$ ,

$$\begin{aligned}
\mu_X(\mathcal{P}(\eta_X)(S)) &= \mu_X(\{\{x\} \mid x \in S\}) \\
&= \bigcup_{x \in S} \{x\} \\
&= S \\
&= \bigcup \{S\} \\
&= \mu_X(\{S\}) \\
&= \mu_X(\eta_{\mathcal{P}(X)}(S))
\end{aligned}$$

Checking that (138) commutes, we have for any  $\mathcal{F} \in \mathcal{P}(\mathcal{P}(X))$ ,

$$\begin{aligned}
\mu_X(\mu_{\mathcal{P}(X)}(\mathcal{F})) &= \mu_X\left(\bigcup_{F \in \mathcal{F}} F\right) \\
&= \bigcup_{\substack{s \in \mathcal{P}(X) \\ \exists F \in \mathcal{F}, s \in F}} s \\
&= \{x \in X \mid \exists s \in \mathcal{P}(X), x \in s \text{ and } \exists F \in \mathcal{F}, s \in F\} \\
&= \bigcup_{F \in \mathcal{F}} \bigcup_{s \in F} s \\
&= \mu_X\left(\left\{\bigcup_{s \in F} s \mid F \in \mathcal{F}\right\}\right) \\
&= \mu_X(\mathcal{P}(\mu_X)(\mathcal{F}))
\end{aligned}$$

3. The functor  $\mathcal{D} : \mathbf{Set} \rightarrow \mathbf{Set}$  sends a set  $X$  to the set of finitely supported distributions on  $X$ , i.e.:

$$\mathcal{D}(X) := \{\varphi \in [0, 1]^X \mid \sum_{x \in X} \varphi(x) = 1 \text{ and } \varphi(x) \neq 0 \text{ for finitely many } x's\}.$$

It sends a function  $f : X \rightarrow Y$  to the function between distributions

$$\lambda \varphi^{\mathcal{D}(X)}. \lambda y^Y. \varphi(f^{-1}(y)).$$

More verbosely, the weight of  $\mathcal{D}(f)(\varphi)$  at point  $y$  is equal to the total weight of  $\varphi$  on the preimage of  $y$  under  $f$ . It is a monad with unit  $\eta_X = x \mapsto \delta_x$ , where  $\delta_x$  is the Dirac distribution at  $x$  (all the weight is at  $x$ ), and multiplication

$$\mu_X = \Phi \mapsto \lambda x^X. \sum_{\phi \in \text{supp}(\Phi)} \Phi(\phi) \cdot \phi(x),$$

where  $\text{supp}(\Phi)$  is the support of  $\Phi$ , i.e.:  $\text{supp}(\Phi) := \{\varphi \mid \Phi(\varphi) \neq 0\}$ .

After looking long enough for adjunctions giving rise to the monads in Example 282, two questions dare to be asked. Does every monad arise from an adjunction? If yes, is that adjunction unique?

The second question might not be as natural to novices in category theory but it is almost as important as the first one. Indeed, uniqueness is a very strong property and if every monad had a unique corresponding adjunction, one might expect it to be fairly easy to find. This is part of the beauty of category theory. We are working with very little data  $M$ ,  $\eta$  and  $\mu$  so if it completely determined an adjunction  $L$ ,  $R$  and  $\text{Hom}(L(-), -) \cong \text{Hom}(-, R(-))$ , it could not do so in a very convoluted way merely because there is not that many ways to manipulated the original data.

In any case, we will respectively give a positive and negative answer to these questions. Fortunately, while we might not benefit from the power of uniqueness, there are two special adjunctions arising from a monad whose descriptions are fairly straightforward. In the order we present them, the first is due to Kleisli and the second to Eilenberg and Moore. In the rest of this section,  $(M, \eta, \mu)$  will be a monad on a category  $\mathbf{C}$ .

### Kleisli Category $\mathbf{C}_M$

An intuitive way to think about monads is through *generalized elements*. Given an object  $A \in \mathbf{C}_0$ , we can view  $MA$  as extending  $A$  with more general or structured elements built from  $A$ .

In this picture, the morphisms  $\eta_A : A \rightarrow MA$  give a way to understand anything inside  $A$  trivially as a general element of  $A$ . The morphisms  $\mu_A : M^2A \rightarrow MA$  imply that higher order structures can collapsed so that generalized elements over generalized elements of  $A$  are generalized elements of  $A$ . The functoriality of  $M$  implies that the new structures in  $MA$  are somewhat independent of  $A$ . Indeed, for every morphisms  $f : A \rightarrow B$ , there is a morphism  $Mf : MA \rightarrow MB$  which, by naturality of  $\eta$  ( $Mf(\eta_A(-)) = \eta_B(f(-))$ ), acts just like  $f$  on the trivial generalization of elements in  $A$ . Commutativity of (137) says that the trivial generalization (there are two ways to do it) of a generalized element is indeed trivial as after collapsing via  $\mu$ , we end up with what we started with. Finally, the associativity of  $\mu$  (i.e.: commutativity of (138)) corresponds to the fact that in higher order of generalizations, one can collapse the structure at every level in any order and end up with the same thing.

Now, we can also consider *generalized morphisms*. Let us say we were given an ill-defined morphism  $f : A \rightarrow B$  that sends some of the stuff in  $A$  outside of  $B$ . One

way to fix this might be to consider general elements of  $B$  and see  $f$  as a morphism  $A \rightarrow MB$ . We will call such morphisms **Kleisli morphisms** and write  $f : A \rightarrow MB$  or  $f : A \rightharpoonup B$ .

With an arbitrary functor  $F$ , you might have a hard time to come up with a way to compose them Kleisli morphisms  $A \rightarrow FB$  and  $B \rightarrow FC$  or define the identity Kleisli morphism  $A \rightarrow FA$ , but the data of a monad lets you do just that. We end up with the category  $\mathbf{C}_M$ .

**Definition 283** ( $\mathbf{C}_M$ ). Let  $\mathbf{C}$  be a category and  $(M, \eta, \mu)$  a monad on  $\mathbf{C}$ . The **Kleisli category** of  $M$ , denoted  $\mathbf{C}_M$  has objects  $\mathbf{C}_0$  and morphisms  $f : A \rightarrow MB \in \mathbf{C}_1$ . The identity for  $A$  is  $\eta_A : A \rightarrow MA$  and composition is given by

$$g \circ_M f = \mu_C \circ Mg \circ f : A \rightharpoonup C,$$

where  $f : A \rightharpoonup B$  and  $g : B \rightharpoonup C$ .

Checking that this is indeed a category is left as an exercise and you should remark that it requires using all the properties  $M$ ,  $\eta$  and  $\mu$  satisfy.

As we have constructed  $\mathbf{C}_M$  with objects and morphisms in  $\mathbf{C}$ , we would like to describe a forgetful functor  $U_M : \mathbf{C}_M \rightsquigarrow \mathbf{C}$ . A first guess to send objects  $A$  to themselves will fail when it is time to defining the action of  $U_M$  on morphisms for  $f : A \rightharpoonup B$  can send some stuff into generalized elements of  $B$ , and we have no way to get this back into  $B$ . Instead, we need to generalize everything before going into  $\mathbf{C}$ , namely,  $U_M(A) = MA$  and  $U_M f = \mu_B \circ Mf : MA \rightarrow MB$ .

We can check that for any  $A$ ,  $U_M(\eta_A) = \mu_A \circ M(\eta_A) \stackrel{(137)}{=} \text{id}_A$  and for any for any  $f : A \rightharpoonup B$  and  $g : B \rightharpoonup C$ ,

$$\begin{aligned} U_M(g \circ_M f) &= U_M(\mu_C \circ Mg \circ f) \\ &= \mu_C \circ M(\mu_C \circ Mg \circ f) \\ &= \mu_C \circ M(\mu_C) \circ MMg \circ Mf \\ &= \mu_C \circ \mu_{MC} \circ MMg \circ Mf && \text{by (138)} \\ &= \mu_C \circ Mg \circ \mu_B \circ Mf && \text{by naturality of } \mu \\ &= U_M(g) \circ U_M(f). \end{aligned}$$

We conclude that  $U_M$  is a functor.

In the opposite direction, we want a functor  $F_M : \mathbf{C} \rightsquigarrow \mathbf{C}_M$ . This time, we send  $A$  to itself because we can make any morphism  $f : A \rightarrow B$  generalized by post-composing with  $\eta_B$ , namely  $F_M(A) = A$  and  $F_M(f) = \eta_B \circ f$  for any  $f : A \rightarrow B$ . For functoriality,  $F_M(\text{id}_A)$  is trivial as  $\eta_A$  is the identity on  $A$  in  $\mathbf{C}_M$  and

$$\begin{aligned} F_M(g \circ f) &= \eta_C \circ g \circ f \\ &= Mg \circ \eta_B \circ f && \text{by naturality of } \eta \\ &= Mg \circ \mu_B \circ M(\eta_B) \circ \eta_B \circ f && \text{by (137)} \\ &= \mu_C \circ MMg \circ M(\eta_B) \circ \eta_B \circ f && \text{by naturality of } \mu \\ &= \mu_C \circ M(\eta_C) \circ Mg \circ \eta_B \circ f && \text{by naturality of } \eta \end{aligned}$$

$$= F_M(g) \circ_M F_M(f)$$

Before checking that this is indeed an adjunction, we need to make sure it composes to  $M$ , that is  $U_M F_M = M$ . On objects, it is clear. On a morphism  $f : A \rightarrow B$ , we have

$$U_M(F_M(f)) = U_M(\eta_B \circ f) = \mu_B \circ M(\eta_B) \circ Mf \stackrel{(137)}{=} Mf.$$

Let us now verify that  $F_M \dashv U_M$ . Let  $A, B \in \mathbf{C}_0$  (we view  $B$  as an object of  $\mathbf{C}_M$ ), we need to exhibit a natural isomorphism  $\text{Hom}_{\mathbf{C}_M}(F_M A, B) \cong \text{Hom}_{\mathbf{C}}(A, U_M B)$ . The isomorphism is clear as  $U_M B = MB$ ,  $F_M A = A$  and a morphism  $A \rightarrow MB$  is precisely a Kleisli morphism  $A \rightharpoonup B$ , we will denote it  $\text{id}$ . For the naturality, we need to show the following square commutes for any  $f : A' \rightarrow A$  and  $g : B \rightharpoonup B'$ .

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}_M}(A, B) & \xrightarrow{\text{id}} & \text{Hom}_{\mathbf{C}}(A, MB) \\ g \circ_M (-) \circ_M F_M f \downarrow & & \downarrow U_M g \circ (-) \circ f \\ \text{Hom}_{\mathbf{C}_M}(A', B') & \xrightarrow{\text{id}} & \text{Hom}_{\mathbf{C}}(A', MB') \end{array} \quad (139)$$

It follows from the following derivation starting with a morphism  $k : A \rightarrow MB$ .

$$\begin{aligned} g \circ_M k \circ_M F_M f &= \mu_{B'} \circ M(g) \circ \mu_B \circ M(k) \circ \eta_A \circ f \\ &= \mu_{B'} \circ M(g) \circ \mu_B \circ \eta_{MB} \circ k \circ f && \text{by naturality of } \eta \\ &= \mu_{B'} \circ M(g) \circ \text{id}_{MB} \circ k \circ f && \text{by (137)} \\ &= \mu_{B'} \circ M(g) \circ k \circ f \\ &= U_M g \circ k \circ f \end{aligned}$$

Recall that we claimed  $(F_M, U_M)$  was special in some way and that this was the (informal) reason why it was relatively easy to find, the next proposition will make this precise.

**Definition 284** ( $\text{Adj}_M$ ). Let  $\mathbf{C}$  be a category and  $(M, \eta, \mu)$  a monad on  $\mathbf{C}$ . The category of adjunctions inducing  $M$  is denoted  $\text{Adj}_M$ . Its objects are adjoint pairs  $(L, R)$  with  $R \circ L = M$  whose unit is  $\eta$  and whose counit  $\varepsilon$  satisfies  $R\varepsilon L = \mu$ . Its morphisms  $(L, R) \rightarrow (L', R')$  are functors  $K$  satisfying  $K \circ L = L'$  and  $R' \circ K = R$  as in (140).

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{K} & \mathbf{D}' \\ & \searrow R & \nearrow L' \\ & \swarrow L & \nwarrow R' \\ & & \mathbf{C} \end{array} \quad (140)$$

Let us first check that  $(F_M, U_M)$  induces  $(M, \eta, \mu)$ , we already know that  $U_M F_M = M$ , but it remains to find the unit and counit of this adjunction. We will need to use the correspondence between the definitions of adjoint pairs seen in the last lecture. Recall that the natural isomorphism  $\text{Hom}_{\mathbf{C}_M}(F_M -, -) \cong \text{Hom}_{\mathbf{C}}(-, U_M -)$  is simply the identity function between these two sets. Hence, the unit of the adjunction is  $\mathbf{C}_0 \ni A \mapsto \text{id}(u_{\mathbf{C}_M}(F_M A)) = \eta_A$  and the counit is  $(\mathbf{C}_M)_0 \ni A \mapsto \text{id}(u_{\mathbf{C}}(U_M A)) =: \varepsilon_A$ . Note that  $\varepsilon_A$  has type  $MA \rightharpoonup A$ . We check that  $U_M(\varepsilon_{F_M A}) = \mu_A$ :

$$U_M(\varepsilon_{F_M A}) = \mu_A \circ \varepsilon_{F_M A} = \mu_A \circ u_{\mathbf{C}}(MA) = \mu_A.$$

**Proposition 285.** *The adjunction  $(F_M, U_M)$  is initial in  $\text{Adj}_M$ .*

*Proof.* Let  $\mathbf{C} : L \dashv R : \mathbf{D} \in \text{Adj}_M$  with unit  $\eta$  and counit  $\varepsilon$ , we claim there is a unique functor  $K : \mathbf{C}_M \rightsquigarrow \mathbf{D}$  satisfying  $K \circ F_M = L$  and  $R \circ K = U_M$  as in (141).

$$\begin{array}{ccc}
 \mathbf{C}_M & \xrightarrow{\quad K \quad} & \mathbf{D} \\
 & \swarrow U_M \quad \searrow L & \\
 & \mathbf{C} & \\
 & \nwarrow F_M \quad \nearrow R &
 \end{array} \quad (141)$$

On objects,  $K$  is determined by  $KA = KF_MA = LA$ . To a morphism  $f : A \rightarrowtail B$ , we need to assign a morphism in  $Kf \in \text{Hom}_{\mathbf{D}}(LA, LB)$ . Denote  $\Phi_{A,B} : \text{Hom}_{\mathbf{C}_M}(F_MA, B) \cong \text{Hom}_{\mathbf{C}}(A, U_MB)$  and  $\Psi_{A,B} : \text{Hom}_{\mathbf{D}}(LA, B) \cong \text{Hom}_{\mathbf{C}}(A, RB)$  be given by the adjunctions  $F_M \dashv U_M$  and  $L \dashv R$  respectively. Using the fact that  $RK = U_M$ , we obtain the following commutative diagram.

$$\begin{array}{ccccccc}
 \text{Hom}_{\mathbf{C}_M}(F_MA, A) & \xrightarrow{\Phi_{A,A}} & \text{Hom}_{\mathbf{C}}(A, U_MA) & = & \text{Hom}_{\mathbf{C}}(A, RLA) & \xleftarrow{\Psi_{A,LA}} & \text{Hom}_{\mathbf{D}}(LA, LA) \\
 \downarrow f \circ_M (-) & & \downarrow U_M f \circ (-) & & \downarrow RK f \circ (-) & & \downarrow K f \circ (-) \\
 \text{Hom}_{\mathbf{C}_M}(F_MA, B) & \xrightarrow{\Phi_{A,B}} & \text{Hom}_{\mathbf{C}}(A, U_MB) & = & \text{Hom}_{\mathbf{C}}(A, RLB) & \xleftarrow{\Psi_{A,LB}} & \text{Hom}_{\mathbf{D}}(LA, LB)
 \end{array}$$

Because these adjunctions both have the same unit, we have  $\Phi_{A,A}(u_{\mathbf{C}_M}(F_MA)) = \eta_A = \Psi_{A,A}(u_{\mathbf{D}}(LA))$ . Therefore, if we start with  $\eta_A \in \text{Hom}_{\mathbf{C}_M}(F_MA, A)$  and follow the diagram, we infer that  $Kf = \Psi_{A,LB}^{-1}(\Phi_{A,B}(f))$ . Using the fact that  $\Phi_{A,B}(f) = f$  and  $\Psi_{A,LB}^{-1}(f) = \varepsilon_{LB} \circ Lf$  (follow the proof of the Yoneda lemma), we end up with  $Kf = \varepsilon_{LB} \circ Lf$  and we can verify it is indeed functorial.

$$K(u_{\mathbf{C}_M}(A)) = K(\eta_A) = \varepsilon_{LB} \circ L(\eta_A) \stackrel{\text{zig-zag}}{=} \text{id}_A$$

$$\begin{aligned}
 K(g \circ_M f) &= K(\mu_C \circ RLg \circ f) \\
 &= \varepsilon_{LC} \circ L(\mu_C) \circ LRLg \circ Lf \\
 &= \varepsilon_{LC} \circ LR\varepsilon_{LC} \circ LRLg \circ Lf && \text{by hypothesis on } \varepsilon \\
 &= \varepsilon_{LC} \circ \varepsilon_{LRLC} \circ LRLg \circ Lf && \text{by naturality of } \varepsilon \\
 &= \varepsilon_{LC} \circ Lg \circ \varepsilon_{LB} \circ Lf && \text{by naturality of } \varepsilon \\
 &= Kg \circ Kf
 \end{aligned}$$

□

### Eilenberg-Moore Category $\mathbf{C}^M$

For the second solution to the problem of finding an adjunction inducing a given monad, we look at the more structural side of monads.

**Definition 286** ( $M$ -algebra). Let  $(M, \eta, \mu)$  be a monad, an **Eilenberg-Moore algebra** for  $M$  or simply  $M$ -**algebra** is a pair  $(A, \alpha)$  consisting of an object  $A \in \mathbf{C}_0$  and morphism  $\alpha : MA \rightarrow A$  such that (142) and (143) commute.

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & MA \\ & \searrow \text{id}_A & \downarrow \alpha \\ & & A \end{array} \quad (142)$$

$$\begin{array}{ccc} M^2A & \xrightarrow{\mu_A} & MA \\ M(\alpha) \downarrow & & \downarrow \alpha \\ MA & \xrightarrow{\alpha} & A \end{array} \quad (143)$$

We will often denote an  $M$ -algebra using only its underlying set or morphism.

**Definition 287** ( $M$ -algebra homomorphism). Given two  $M$ -algebras  $(A, \alpha)$  and  $(B, \beta)$ , an  $M$ -algebra homomorphism  $f : (A, \alpha) \rightarrow (B, \beta)$  is a morphism  $h : A \rightarrow B$  making (156) commute.

$$\begin{array}{ccc} MA & \xrightarrow{Mh} & MB \\ \alpha \downarrow & & \downarrow \beta \\ A & \xrightarrow{h} & B \end{array} \quad (144)$$

For a monad  $M$ , the category of  $M$ -algebras is called the **Eilenberg-Moore category** of  $M$  and denoted  $\mathbf{C}^M$ , composition and identities are induced by the composition and identities in  $\mathbf{C}$ . Again, since  $\mathbf{C}^M$  was built from objects and morphisms in  $\mathbf{C}$ , there is an obvious candidate for a forgetful functor  $U^M : \mathbf{C}^M \rightsquigarrow \mathbf{C}$  sending an  $M$ -algebra  $(A, \alpha)$  to its underlying object  $A$  and a homomorphism to its underlying morphism. Next, we need a left adjoint to  $U^M$ ,  $F^M : \mathbf{C} \rightsquigarrow \mathbf{C}^M$ . Since we want  $U^M F^M = M$  to hold,  $F^M$  must send  $A \in \mathbf{C}_0$  to an  $M$ -algebra over  $MA$  and  $h \in \mathbf{C}_1$  to  $Mh$ . There is only one choice given to us by the data of  $M$ , that is,  $F^M A = (MA, \mu_A)$  and it turns out naturality of  $\mu_A$  yields

$$\begin{array}{ccc} M^2A & \xrightarrow{M^2h} & M^2B \\ \mu_A \downarrow & & \downarrow \mu_B \\ MA & \xrightarrow{Mh} & MB \end{array} \quad (145)$$

which implies  $Mh$  is indeed an  $M$ -algebra homomorphism. Let us now show that  $F^M \dashv U^M$  with unit  $\eta$  and counit  $\varepsilon$  satisfying  $U^M \varepsilon F^M = \mu$ .

We want to exhibit an isomorphism  $\text{Hom}_{\mathbf{C}^M}(\mu_A, \beta) \cong \text{Hom}_{\mathbf{C}}(A, B)$  natural in  $A$  and  $\beta : MB \rightarrow B$ . In the forward direction, we send  $h : MA \rightarrow B$  to  $h \circ \eta_A : A \rightarrow B$  which ensures the unit of this adjunction is  $\eta$ . In the backwards direction, we send  $h : A \rightarrow B$  to  $\beta \circ Mh$  which is a homomorphism by the following diagram where (a) commutes by naturality of  $\mu$  and (b) by  $\beta$  being an  $M$ -algebra.

$$\begin{array}{ccccc} M^2A & \xrightarrow{M^2h} & M^2B & \xrightarrow{M\beta} & MB \\ \mu_A \downarrow & \text{(a)} & \downarrow \mu_B & \text{(b)} & \downarrow \beta \\ MA & \xrightarrow{Mh} & MB & \xrightarrow{\beta} & B \end{array} \quad (146)$$

One can easily check that these operations are inverses of each other. Next, we turn to naturality. Let  $f : A' \rightarrow A \in \mathbf{C}_1$  and  $g : (B, \beta) \rightarrow (B', \beta') \in \mathbf{C}_1^M$ , we claim that

(147) commutes.

$$\begin{array}{ccc}
 \text{Hom}_{\mathbf{C}^M}(\mu_A, \beta) & \xrightarrow{(-) \circ \eta_A} & \text{Hom}_{\mathbf{C}}(A, B) \\
 g \circ (-) \circ Mf \downarrow & & \downarrow g \circ (-) \circ f \\
 \text{Hom}_{\mathbf{C}^M}(\mu_{A'}, \beta') & \xrightarrow{(-) \circ \eta_{A'}} & \text{Hom}_{\mathbf{C}}(A', B')
 \end{array} \quad (147)$$

Starting with a homomorphism  $h : (MA, \mu_A) \rightarrow (B, \beta)$  in the top-left object, we need to show  $g \circ h \circ Mf \circ \eta_{A'} = g \circ h \circ \eta_A \circ f$  which holds by naturality of  $\eta$ .

We find the counit of this adjunction is  $\varepsilon_\alpha = \alpha : (MA, \mu_A) \rightarrow (A, \alpha)$  which is a homomorphism because (143) commutes and we verify

$$U^M(\varepsilon_{FMA}) = U^M(\varepsilon_{\mu_A}) = U^M(\mu_A) = \mu_A.$$

Dually to Proposition 285, we show that this adjunction is special in a precise way.

**Proposition 288.** *The adjunction  $(F^M, U^M)$  is terminal in  $\text{Adj}_M$ .*

*Proof.* Let  $\mathbf{C} : L \dashv R : \mathbf{D} \in \text{Adj}_M$  with unit  $\eta$  and counit  $\varepsilon$ , we claim there is a unique functor  $K : \mathbf{D} \rightsquigarrow \mathbf{C}^M$  satisfying  $K \circ L = F^M$  and  $U^M \circ K = R$  as in (148).

$$\begin{array}{ccc}
 \mathbf{D} & \xrightarrow{\quad K \quad} & \mathbf{C}^M \\
 \swarrow L & \begin{array}{c} \nearrow R \\ \nearrow F^M \end{array} & \searrow U^M \\
 & \mathbf{C} &
 \end{array} \quad (148)$$

As before, we can determine  $K$  by the equation  $U^M K = R$  which means it sends  $A \in \mathbf{D}_0$  to an algebra on  $RA$  and  $f : A \rightarrow B \in \mathbf{D}_1$  to an algebra homomorphism  $Rf : KA \rightarrow KB$ . The only missing piece of this puzzle is the algebra structure on  $KA$ . The only clue we have is the fact that  $Rf$  is a homomorphism so is  $KA = \alpha$  and  $KB = \beta$ , then (149) commutes.

$$\begin{array}{ccc}
 MRA & \xrightarrow{MRf} & MRB \\
 \alpha \downarrow & & \downarrow \beta \\
 RA & \xrightarrow{Rf} & RB
 \end{array} \quad (149)$$

Replacing  $M$  with  $RL$ , we recognize this as the naturality diagram for  $\varepsilon$  acted on by  $R(\cdot)$ . Hence, we find  $KA = (RA, R\varepsilon_A)$  is a suitable candidate after showing it is indeed an  $M$ -algebra. For the unit diagram, we have  $R\varepsilon_A \circ \eta_A = \text{id}_A$  by a zig-zag identity. For the multiplication diagram, we have

$$R\varepsilon_A \circ \mu_A = R\varepsilon_A \circ R\varepsilon_{LA} = R(\varepsilon_A \circ \varepsilon_{LA}) = R(\varepsilon_A \circ LR(\varepsilon_A)) = R\varepsilon_A \circ MR\varepsilon_A.$$

As  $K$  acts like  $R$  on morphisms, it is obviously functorial. We leave the proof that  $K$  is unique as an exercise as the argument should be somewhat similar to what we have done in the dual proposition.  $\square$

In summary, you can keep in mind the following diagram.

$$\begin{array}{ccccc}
 & & \mathbf{D} & & \\
 & \nearrow L & \uparrow \dashv \downarrow R & \nwarrow & \\
 \mathbf{C}_M & \xrightarrow{U_M} & \mathbf{C} & \xrightarrow{F^M} & \mathbf{C}^M \\
 & \xleftarrow{F_M} & & \xleftarrow{U^M} & \\
 & & \mathbf{C} & & 
 \end{array}
 \quad (150)$$

One last thing we want to mention is the embedding of the Kleisli category inside the Eilenberg-Moore category. After a bit more work, we can infer from the discussion above that the unique morphism of adjunctions  $K : \mathbf{C}_M \rightsquigarrow \mathbf{C}^M$  is a fully faithful functor sending an object  $A$  in  $\mathbf{C}_M$  to the algebra  $(MA, \mu_A)$ , called the **free algebra** on  $A$ , and a Kleisli morphism  $f : A \multimap B$  to the homomorphism  $\mu_B \circ Mf : (A, \mu_A) \rightarrow (B, \mu_B)$ .

### POV: Universal Algebra

In this section, we will highlight the link between algebraic structures as you have encountered them in other classes with the Eilenberg-Moore algebras discussed above. We will only work over the category **Set**. We start by developing an example.

**Example 289** ( $\mathcal{P}_{\text{ne}}$ ). Consider the non-empty finite powerset functor  $\mathcal{P}_{\text{ne}}$  sending  $X$  to  $\{S \in \mathcal{P}(X) \mid S \text{ is finite and non-empty}\}$ . The same unit and multiplication defined for  $\mathcal{P}$  make  $\mathcal{P}_{\text{ne}}$  into a monad. A  $\mathcal{P}_{\text{ne}}$ -algebra is a function  $\alpha : \mathcal{P}_{\text{ne}}(A) \rightarrow A$  satisfying the equations  $\alpha\{a\} = a$  and  $\alpha(\mathcal{P}_{\text{ne}}(\alpha)(S)) = \alpha(\bigcup S)$ . From this, we can extract a binary operation  $\oplus_\alpha : A \times A \rightarrow A$  by defining  $x \oplus_\alpha y = \alpha\{x, y\}$ . This operation is clearly commutative and idempotent, i.e.:  $x \oplus_\alpha y = y \oplus_\alpha x$  and  $x \oplus_\alpha x = x$ , but it is also associative by the following derivation.

$$\begin{aligned}
 (x \oplus_\alpha y) \oplus_\alpha z &= \alpha\{x, y\} \oplus_\alpha z \\
 &= \alpha\{\alpha\{x, y\}, z\} \\
 &= \alpha\{\alpha\{x, y\}, \alpha\{z\}\} \\
 &= \alpha\{\mathcal{P}_{\text{ne}}\alpha\{\{x, y\}, \{z\}\}\} \\
 &= \alpha\{\mu_A\{\{x, y\}, \{z\}\}\} \\
 &= \alpha\{x, y, z\}.
 \end{aligned}$$

Since a  $\mathcal{P}_{\text{ne}}$ -algebra homomorphism  $h : (A, \alpha) \rightarrow (B, \beta)$  commutes with  $\alpha$  and  $\beta$  it also commutes with  $\oplus_\alpha$  and  $\oplus_\beta$ .

Conversely, if  $(A, \oplus)$  is an idempotent, associative and commutative binary operation on  $A$ , we can define  $\alpha_\oplus$  on non-empty finite sets of  $A$  by iterating  $\oplus$ . Namely,

$$\alpha_\oplus\{x\} = x \oplus x \quad \text{and} \quad \alpha_\oplus\{x_1, \dots, x_n\} = x_1 \oplus x_2 \oplus \dots \oplus x_n.$$

It is well-defined by associativity and commutativity and we can check that it is the inverse of the operation described in the previous paragraph. That is to say, we can



check that  $\alpha_{\oplus_\alpha} = \alpha$  and  $\oplus_{\alpha_\oplus} = \oplus$ . For the former, it is clear for singleton sets and for any  $n > 1$ , we have the following derivation.

$$\begin{aligned}
 \alpha_{\oplus_\alpha} \{x_1, \dots, x_n\} &= x_1 \oplus_\alpha \dots \oplus_\alpha x_n \\
 &= \alpha \{x_1, x_2 \oplus_\alpha \dots \oplus_\alpha x_n\} \\
 &= \vdots \\
 &= \alpha \{x_1, \alpha \{x_2, \alpha \{\dots, \alpha \{x_n\}\}\}\} \\
 \text{using } \alpha \circ \mathcal{P}_{\text{ne}}(\alpha) &= \alpha \circ \mu_A = \alpha \{x_1, x_2, \alpha \{\dots, \alpha \{x_n\}\}\} \\
 &= \vdots \\
 &= \alpha \{x_1, \dots, x_n\}
 \end{aligned}$$

For the latter, we have

$$x \oplus_{\alpha_\oplus} y = \alpha_\oplus \{x, y\} = x \oplus y.$$

A set equipped with an idempotent, commutative and associative binary operation is called a **(join/meet)-semilattice** and we have shown above that  $\mathcal{P}_{\text{ne}}$ -algebras are in correspondence with semilattices. Through the introduction of basic notions in universal algebra, we will explain how this correspondence is functorial and generalize the core idea behind it.

**Definition 290** (Algebraic theory). An **algebraic signature** is a set  $\Sigma$  of operation symbols along with arities in  $\mathbb{N}$ , we denote  $f : n \in \Sigma$  for an  $n$ -ary operation  $f$  in  $\Sigma$ . Given a set  $X$ , one constructs the set of  $\Sigma$ -terms with variables in  $X$ , denoted  $T_\Sigma(X)$  by iterating operations symbols:

$$\begin{aligned}
 &\forall x \in X, x \in T_\Sigma(X) \\
 &\forall t_1, \dots, t_n \in T_\Sigma(X), f : n \in \Sigma, f(t_1, \dots, t_n) \in T_\Sigma(X).
 \end{aligned}$$

An **equation** (sometimes **axiom**)  $E$  over  $\Sigma$  is a pair of  $\Sigma$ -terms over a set of dummy variables. We will call the tuple  $(\Sigma, E)$  an **algebraic theory**.

**Definition 291** ( $(\Sigma, E)$ -algebras). Given a signature  $\Sigma$  and a set of equations  $E$  over this signature, a  $(\Sigma, E)$ -**algebra** is a set  $A$  along with operations  $f^A : A^n \rightarrow A$  for all  $f : n \in \Sigma$  such that the pair of terms in  $E$  are always equal when the operation symbols and dummy variables are instantiated in  $A$ . We usually denote  $\Sigma^A$  for the set operations  $f^A$ .

**Examples 292.** As is suggested by the terminology, the common algebraic structures can be define with simple algebraic theories.

1. We can define a monoid as an algebra for the signature  $\{\cdot : 2, 1 : 0\}$  and the equations  $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ ,  $1 \cdot x = x$ ,  $x \cdot 1 = x$ .
2. Adding the unary operation  $(-)^{-1}$  and the equations  $x \cdot x^{-1} = 1$  and  $x^{-1} \cdot x = 1$ , we obtain the theory of groups.
3. Adding the equation  $x \cdot y = y \cdot x$  yields the theory of abelian groups.

4. With the signature  $\{+ : 2, \cdot : 2, 1 : 0, 0 : 0\}$ , we can add the abelian group equations for the operation  $+$  (identity is 0), the monoid equations for  $\cdot$  (identity is 1) and the distributivity equation  $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$  and thus obtain the theory of rings.

We also have homomorphisms.

**Definition 293** ( $(\Sigma, E)$ -algebra homomorphisms). Given two  $(\Sigma, E)$ -algebras  $A$  and  $B$ , a **homomorphism** between them is a map  $h : A \rightarrow B$  commuting with all operations in  $\Sigma$ , that is  $\forall f : n \in \Sigma, h \circ f^A = f^B \circ h^n$ .

The category of  $(\Sigma, E)$ -algebras and their homomorphisms (with the obvious composition and identities) is denoted  $\text{Alg}(\Sigma, E)$ .

**Example 294**  $(\Sigma_S, E_S)$ . Recall in Example 289 that  $\mathcal{P}_{\text{ne}}$ -algebras correspond to semilattices. Formally, the theory of semilattices has a single binary operation  $\Sigma_S = \{\oplus : 2\}$  satisfying the following equations in  $E_S$ :

$$\begin{array}{ll} x \oplus x = x & I \\ x \oplus y = y \oplus x & C \\ (x \oplus y) \oplus z = x \oplus (y \oplus z). & A \end{array}$$

Up to a couple of missing functoriality arguments, we have shown that the categories  $\text{Set}^{\mathcal{P}_{\text{ne}}}$  and  $\text{Alg}(\Sigma_S, E_S)$  are isomorphic. We say that  $(\Sigma_S, E_S)$  is an **algebraic presentation** of the monad  $\mathcal{P}_{\text{ne}}$ .

It turns out all algebraic theories present at least one monad.

**Definition 295** (Term monad). Let  $(\Sigma, E)$  be an algebraic theory, one can assign to any set  $X$ , the set  $T_{\Sigma, E}(X)$  of terms in  $T_{\Sigma}(X)$  modulo the equations in  $E$ . This can be extended to functions  $f : X \rightarrow Y$ , by variable substitution, i.e.:  $T_{\Sigma}(f)$  acts on a term  $t$  by replacing all occurrences of  $x \in X$  by  $f(x) \in Y$  and  $T_{\Sigma, E}(f)$  acts on equivalence classes by  $[t] \mapsto [T_{\Sigma}(f)(t)]$ . We obtain a functor  $T_{\Sigma, E}$  on which we can put a monad structure.

The unit map is obvious because any element of  $X$  is a  $\Sigma$ -term, thus  $\eta_X : X \rightarrow T_{\Sigma, E}(X)$  maps  $x$  to the equivalence class containing the term  $x$ . The multiplication map is derived from the fact that applying operations in  $\Sigma$  to  $\Sigma$ -terms yields  $\Sigma$ -terms. More explicitly,  $\mu_X$  is a flattening operation defined recursively by

$$\begin{aligned} \forall t \in T_{\Sigma}(X), \mu_X([t]) &= [t] \\ \forall f : n \in \Sigma, t_1, \dots, t_n \in T_{\Sigma, E}(X), \mu_X([f(t_1, \dots, t_n)]) &= [f(\mu_X([t_1]), \dots, \mu_X([t_n]))] \end{aligned}$$

One can show that  $\text{Set}^{T_{\Sigma, E}}$  is the category of  $(\Sigma, E)$  algebras.

Unfortunately, the terms monads are not very simple to work with and it is often desirable to find other simpler monads which are presented by the same theory or conversely to find an algebraic presentation for a given monad.

**Examples 296.** 1. The algebraic theory for presenting  $\mathcal{D}$  is called the theory of **convex algebras** and is denoted  $(\Sigma_{CA}, E_{CA})$ , it consists of a binary operation  $+_p : 2$  for any  $p \in (0, 1)$  which is meant to represent a choice between the two terms in the operation, the left one being chosen with probability  $p$  and the second one with probability  $1 - p$ . There are three equations in the theory that morally ensure that terms representing the same choice are equal.

$$\begin{array}{ll} x +_p x = x & I_p: \text{ idempotence} \\ x +_p y = y +_{\bar{p}} x & C_p: \text{ skew-commutativity} \\ (x +_q y) +_p z = x +_{pq} (y +_{\frac{p\bar{q}}{pq}} z) & A_p: \text{ skew-associativity} \end{array}$$

These equations are necessary for every distribution in  $\mathcal{D}X$  to correspond uniquely to an equivalence class in  $T_{\Sigma_{CA}, E_{CA}}(X)$ .

2. The monad  $(\cdot + \mathbf{1})$  is particular because it is really simple and combines very well with other monads.

**Proposition 297.** *For any monad  $M$ , there is a monad structure on the composition  $M(\cdot + \mathbf{1})$ . Moreover, if  $M$  is presented by  $(\Sigma, E)$  the monad  $M(\cdot + \mathbf{1})$  is presented by  $(\Sigma \cup \{*: 0\}, E)$ , that is, the new theory only has an additional constant which is neutral with respect to the equations.*

We often qualify theories with an added constant as **pointed**. For instance, the theories presented by  $\mathcal{P}_{ne}(\cdot + \mathbf{1})$  and  $\mathcal{D}(\cdot + \mathbf{1})$  are those of **pointed semilattices** (PS) and **pointed convex algebras** (PCA) respectively.

*Remark 298* (Lawvere's way). There is another way to do universal algebra *more categorically* still very much linked to monads: Lawvere theories. Algebras over a Lawvere theory are defined more abstractly using the categorical language and, on this account, they enjoy straightforward generalization through enrichment or lifting to higher order categories.

## POV: Computer Programs

In this section, we will develop on an original idea by Eugenio Moggi that monads are suitable models for a general notion of *computation*. In the sequel, we will use the terms *type* and *set* interchangeably.

Moggi gave a justification for using monads in computer science (particularly in programming semantics) via the informal intuition of *computational types*. For a type  $A$ , the computational type of  $A$  should contain all computations which return a value of type  $A$ . It is intended for the interpretation of *computation* to be made explicit by an instance of a monad. In most cases, it can be thought of as a piece of code which returns some value, but for now, we start by building the intuition in an abstract sense.

Let  $MA$  denote the computational type of  $A$  and  $MMA$  the computational type of  $MA$ , that is computations returning values which are themselves computations of type  $A$ . The following items should coincide with our intuition of computation.

1. For any  $x \in A$ , there is a trivial computation  $\text{return } x \in MA$ .
2. For any  $C \in MMA$ , we can reduce  $C$  to  $\text{flatten}(C) \in MA$  which executes  $C$  and the computation returned by  $C$  to obtain a final return value of type  $A$ .
3. If  $C \in MA$ , then  $\text{flatten}(\text{return } C) = C$ .
4. If  $C \in MA$  and  $C' \in MMA$  does the same computation as  $C$  but instead of returning a value  $x$ , it returns the computation  $\text{return } x$ , then  $\text{flatten}(C') = C$ .
5. If  $MMMA$  is the computational type of  $MMA$  and  $C \in MMMA$ , then there are two ways to flatten  $C$ . First, there is the computation  $C_1$  which executes  $C$  and executes the returned computation (of type  $MMA$ ) to obtain a final value of type  $MA$ , hence  $C_1 \in MMA$  and  $\text{flatten}(C_1) \in MA$ . Second,  $C_2$  executes  $C$  and flattens the returned computation to obtain a final value of type  $MA$ ,  $C_2$  is also of type  $MMA$  and  $\text{flatten}(C_2) \in MA$ . These two operations should yield the same result.

Now, a monad  $M$  is a description of computational types that is general, namely, for any type  $A$ , the monad  $M$  gives a type  $MA$  behaving as expected. You can check that  $x \mapsto \text{return } x$  is the unit of this monad and  $\text{flatten}$  is the multiplication.

**Examples 299.** Here, we list more examples commonly used in computer science.

**List monad:** For any set  $X$ , let  $L(X)$  denote the set of all finite lists whose elements are chosen in  $X$ . This is a functor that sends a function  $f : X \rightarrow Y$  to its extension on lists  $L(f) : L(X) \rightarrow L(Y)$  which applies  $f$  to all elements on the list (in lots of programming languages, one writes  $L(f) := \text{map}(f, -)$ ). Then, we can put a monad structure on  $L$ . The unit maps send an element  $x \in X$  to the list containing only that element:  $\eta_X = x \mapsto [x]$ . The multiplication maps concatenate all the lists in a lists of lists:  $\mu_X = [\ell_1, \dots, \ell_n] \mapsto \ell_1 \ell_2 \dots \ell_n$ . It is easy to check diagrams (137) to (138) commute.

**Termination:** In order to model computations that might terminate with no output, the monad  $(\cdot + \mathbf{1})$  is often used. For any type  $X$ , the type  $X + \mathbf{1}$  has all the values of type  $X$  and an additional termination value denoted  $*$ . The behavior of the unit and multiplication of the monad can be interpreted as the fact that the stage of the computation that leads to a termination is irrelevant. This monad is also known as the Maybe monad.

**Non-deterministic choice:** The model for nondeterministic choice is given by the monad  $\mathcal{P}_{\text{ne}}$ . The elements of  $S \in \mathcal{P}_{\text{ne}}(X)$  are seen as the possible outcomes of a nondeterministic choice. The unit is basically viewing a deterministic choice as a nondeterministic choice. The multiplication reduces the number of choices without changing the behavior. For instance, consider a process that nondeterministically chooses between two boxes containing two coins each and then chooses a coin in the box. By simply observing the final choice, we would not be able to distinguish it from a process that nondeterministically chooses between the four coins from the start.

**Probabilistic choice:** In the same vein, probabilistic choice can be interpreted with the monad  $\mathcal{D}$  of finitely supported distributions.

**Exceptions:** As a generalization of termination, we can put a monad structure on the functor  $(\cdot + E)$  where  $E$  is a set of exceptions that the computation can raise.

This view sheds light on one important features of monads we have not yet explored. If  $M$  and  $\hat{M}$  are monads describing computational effects, it is natural to ask for a way to combine them. Indeed, it does not seem too ambitious to have a model for programs which, for instance, make nondeterministic choices and also might terminate with no output. It turns out there is a very useful tool to deal with this at the level of monads.

**Definition 300** (Monad distributive law). Let  $(M, \eta, \mu)$  and  $(\hat{M}, \hat{\eta}, \hat{\mu})$  be two monads on  $\mathbf{C}$ , a natural transformation  $\lambda : M\hat{M} \Rightarrow \hat{M}M$  is called a **monad distributive law of  $M$  over  $\hat{M}$**  if it makes (151), (152) commute.

$$\begin{array}{ccccc}
 M & \xrightarrow{M\hat{\eta}} & M\hat{M} & \xleftarrow{\hat{\eta}\hat{M}} & \hat{M} \\
 \searrow \hat{\eta}M & & \downarrow \lambda & & \swarrow \hat{M}\eta \\
 & & \hat{M}M & & 
 \end{array} \quad (151)$$

$$\begin{array}{ccccc}
 M\hat{M}\hat{M} & \xrightarrow{\mu\hat{M}} & M\hat{M} & \xleftarrow{M\hat{\mu}} & M\hat{M}\hat{M} \\
 M\lambda \downarrow & & \downarrow \lambda & & \downarrow \lambda\hat{M} \\
 M\hat{M}M & \xrightarrow{\lambda M} & \hat{M}MM & \xrightarrow{\hat{M}\mu} & \hat{M}M \\
 & & \downarrow \hat{\mu}M & & \downarrow \hat{M}\lambda \\
 & & \hat{M}M & \xleftarrow{\hat{\mu}M} & \hat{M}M \\
 & & \downarrow \hat{\mu}M & & \downarrow \hat{M}\lambda \\
 & & \hat{M}M & \xleftarrow{\hat{M}\lambda} & \hat{M}M
 \end{array} \quad (152)$$

**Proposition 301.** If  $\lambda : M\hat{M} \Rightarrow \hat{M}M$  is a monad distributive law, then the composite  $\bar{M} = \hat{M}M$  is a monad with unit  $\bar{\eta} = \hat{\eta} \diamond \eta$  and multiplication  $\bar{\mu} = (\hat{\mu} \diamond \mu) \cdot \hat{M}\lambda M$ .

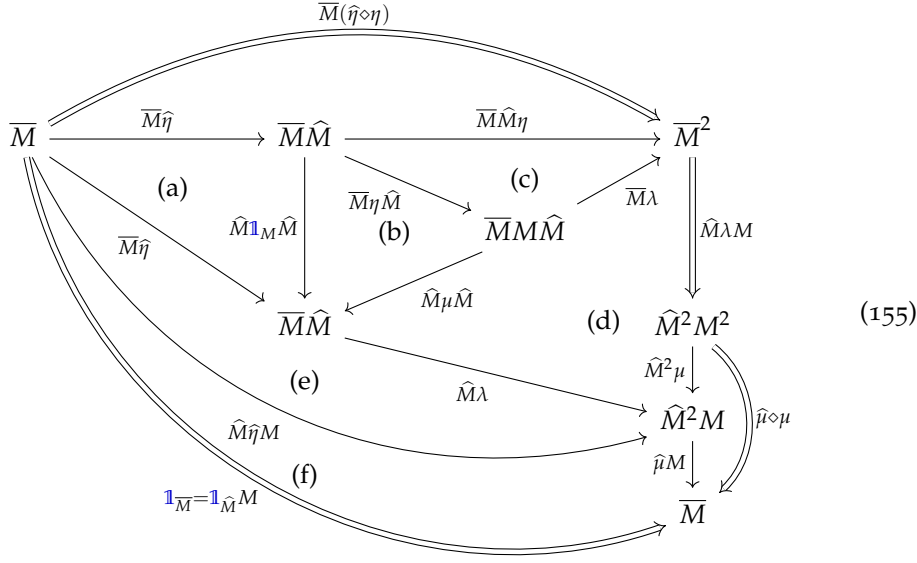
*Proof.* We have to show that the following instances of (137) and (138) commute.

$$\begin{array}{ccc}
 \bar{M} & \xrightarrow{\bar{M}(\hat{\eta} \diamond \eta)} & \bar{M}^2 \\
 \searrow \mathbb{1}_{\bar{M}} & & \downarrow \hat{M}\lambda M \\
 & & \hat{M}^2 M^2 \\
 & & \downarrow \hat{\mu} \diamond \mu \\
 & & \bar{M}
 \end{array} \quad (153)$$

$$\begin{array}{ccc}
 \bar{M}^3 & \xrightarrow{\bar{M}\hat{M}\lambda M} & \bar{M}\hat{M}^2 M^2 \\
 \hat{M}\lambda M \bar{M} \downarrow & & \downarrow \hat{M}\lambda M \\
 \hat{M}^2 M^2 \bar{M} & & \hat{M}^2 M^2 \\
 (\hat{\mu} \diamond \mu) \bar{M} \downarrow & & \downarrow \hat{\mu} \diamond \mu \\
 \bar{M}^2 & \xrightarrow{\hat{M}\lambda M} & \hat{M}^2 M^2 \\
 & & \downarrow \hat{\mu} \diamond \mu \\
 & & \bar{M}
 \end{array} \quad (154)$$

For the left part of (153), we have the following diagram, the justifications of each part is given below with what diagram has to be considered and what functors should be applied to it (recall that acting on the diagrams does not affect commutativity). The notation (137).L (resp. .R) means only the left (resp. right) part of the

diagram is considered.



(a)  $\hat{M}\mathbb{1}_M\hat{M}$  is the identity transformation.

(b) Act on (137).L with  $\hat{M}$  on the left and right.

(c) Act on (151).R with  $\bar{M}$  on the left.

(d) Act on (152).L with  $\hat{M}$  on the left.

(e) Act on (151).L with  $\hat{M}$  on the left.

(f) Act on (137) with  $M$  on the right.

Without a diagram, the derivation is this (we use ; to denote the opposite of  $\circ$ , i.e.: composition in the order read on the diagram):

$$\begin{aligned}
 \bar{M}(\hat{\eta} \diamond \eta); \hat{M}\lambda M; \hat{\mu} \diamond \mu &= \bar{M}\hat{\eta}; \bar{M}\hat{M}\eta; \hat{M}\lambda M; \hat{M}^2\mu; \hat{\mu}M && \text{def of } \diamond \\
 &= \bar{M}\hat{\eta}; \bar{M}\eta\hat{M}; \bar{M}\lambda; \hat{M}\lambda M; \hat{M}\hat{M}\mu; \hat{\mu}M && \bar{M}(151).R \\
 &= \bar{M}\hat{\eta}; \bar{M}\eta\hat{M}; \hat{M}\mu\hat{M}; \hat{M}\lambda; \hat{\mu}M && \hat{M}(152).L \\
 &= \bar{M}\hat{\eta}; \hat{M}\mathbb{1}_M\hat{M}; \hat{M}\lambda; \hat{\mu}M && \hat{M}(137).L\hat{M} \\
 &= \bar{M}\hat{\eta}; \hat{M}\lambda; \hat{\mu}M \\
 &= \hat{M}\hat{\eta}M; \hat{\mu}M && \hat{M}(151).R \\
 &= \mathbb{1}_{\hat{M}}M = \mathbb{1}_{\bar{M}} && (137).LM
 \end{aligned}$$

For the right part of (153), the derivation is very similar.

For (154), we do the same thing.

Then, by identifying products of sums (elements of  $T_{\text{Mon}}T_{\text{Ab}}X$ ) with sums of products (elements of  $T_{\text{Ab}}T_{\text{Mon}}X$ ) by *distributing* the product over the sum as we

are used to do with, say, real numbers, we obtain a monad distributive law of  $T_{\mathbf{Mon}}$  over  $T_{\mathbf{Ab}}$ . The resulting composite monad  $T_{\mathbf{Ab}}T_{\mathbf{Mon}}$  is the term monad for the theory of rings. The term distributive law comes from this example.

*Remark 304.* It is not always possible to combine monads in such a natural way. For instance, it was shown that no distributive law exist between  $\mathcal{P}_{\text{ne}}$  and  $\mathcal{D}$  and even that no monad structure can exist on  $\mathcal{P}_{\text{ne}}\mathcal{D}$  or  $\mathcal{D}\mathcal{P}_{\text{ne}}$ . Thus, modelling combined probabilistic and nondeterministic effects has been quite a hard endeavor and is still an active area of research I discovered in an internship with Matteo Mio and Valeria Vignudelli at ENS de Lyon last summer.

If you are looking for more applications of this perspective on monads and especially if you enjoyed the assignment on Brzozowski's algorithm, I suggest you look into the paper *Generalizing Determinization From Automata to Coalgebras* available at <https://arxiv.org/abs/1302.1046>.

### Exercises

1. Show that the triple  $(\mathcal{D}, \eta, \mu)$  described in Example 282.3 is a monad.
2. Show that the Kleisli category of the powerset monad is the category **Rel** of relations.
3. Show that  $\iota$  defined in the proof of Corollary 302 is a monad distributive law.
4. Show Proposition 297 with the monad structure on  $M(\cdot + \mathbf{1})$  given in Corollary 302.



## Bonus: (Co)algebras and (Co)induction

### (Co)algebras and (Co)induction

In this section, we will introduce (co)algebras and (co)induction and ask you to fill out some gaps. While the definitions are given for an arbitrary category  $\mathbf{C}$ , all of the examples and exercises will be done on  $\mathbf{Set}$ .

#### Algebras

The term *algebra* has a few different meanings and here we will more precisely consider  $F$ -algebras for some endofunctor  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$ . Nevertheless, all the objects that are referred to as algebras have a common motto: *algebras only care about structure*.

For instance, in a first year algebra course, groups are studied up to isomorphisms (maps that preserve the structure) because all the useful properties of a group are determined completely by how the operation acts on the underlying set. As a concrete example, the groups  $\mathbb{Z}_2 \times \mathbb{Z}_3$  and  $\mathbb{Z}_6$  are the same group even if their elements have different names. It is a similar situation for rings, vector spaces and a lot more mathematical objects.

Before giving the general definition of an  $F$ -algebra, we categorify the definition of a group.

**Example 305.** Usually, a group is defined as a set  $G$  along with an operation  $\cdot : G \times G \rightarrow G$  satisfying some conditions, namely, associativity, existence of an identity and existence of an inverse for each element. It is then a formal consequence that the identity and inverses are unique.

Therefore, it is equivalent to define a group as a set  $G$  with a binary operation  $\cdot$ , an identity  $1 \in G$  and an inverse  $g^{-1}$  for all  $g \in G$  that satisfy some properties. In order to abide to the categorical mindset, it is better to view the identity as a morphism  $1 : \mathbf{1} \rightarrow G$  ( $\mathbf{1}$  is the final object, i.e.: a singleton) and describe inverses with a morphism  $(-)^{-1} : G \rightarrow G$ . A few additional diagrams have to commute for  $G$  to satisfy all axioms of a group, but we leave their construction as an exercise. We conclude that a group can be seen as a morphism

$$[1, (-)^{-1}, \cdot] : \mathbf{1} + G + (G \times G) \rightarrow G.$$

This is our first example of an  $F$ -algebra, here  $F : \mathbf{Set} \rightsquigarrow \mathbf{Set}$  sends a set  $G$  to  $\mathbf{1} + G + (G \times G)$  and a morphism  $f$  to  $[\text{id}_{\mathbf{1}}, f, (f, f)]$ .

Note that since we have not used the fact that  $G$  is a set, this definition gives rise to groups in other categories than **Set** provided they have a final object, products and coproducts.

**Exercise 306** (1pt). Draw the additional diagrams that  $G$ ,  $\cdot$ ,  $1$  and  $(-)^{-1}$  should satisfy to obtain a group.

**Definition 307** ( $F$ -algebra). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor, an  $F$ -**algebra** is an object  $A \in \mathbf{C}_0$  along with a morphism  $\alpha : F(A) \rightarrow A \in \mathbf{C}_1$  called the **structure map**.

**Examples 308.**

1. Since a monoid  $M$  only has a binary operation and an identity, it can be represented as an algebra  $[1, \cdot] : \mathbf{1} + (M \times M) \rightarrow M$ . Similarly, one can construct algebras that represent rings and vector spaces, but not fields (why?).
2. We will see later that the induction principle we know comes from the algebra  $[0, \text{succ}] : \mathbf{1} + \mathbf{N} \rightarrow \mathbf{N}$ , where  $0(*) = 0$  and  $\text{succ}(n) = n + 1$ .
3. Although we will not use them often, there are algebras in different categories than **Set**. In computer science, we often use induction to reason about lists, we will see that this is because lists are algebras. More precisely, for a type  $A$ , the type  $A^*$  of lists with elements of type  $A$  has an algebra structure  $[\text{nil}, \text{cons}] : \mathbf{1} + (A \times A^*) \rightarrow A^*$  given by  $\text{nil}(*) = \varepsilon \in A^*$  (the empty list) and  $\text{cons}(a, w) = w \cdot a \in A^*$  (concatenation). The category in which this algebra lives depends on the programming language and types considered.

*Remark 309.* The components of the structure map (i.e.:  $0$  and  $\text{succ}$  in the second example) are often called the **constructors** because they define rules to construct elements of the algebra using other elements as building blocks.

As you might expect  $F$ -algebras form a category, denoted  $\text{Alg}(F)$ , with the following notion of morphism.

**Definition 310** ( $F$ -algebra homomorphism). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor and  $\alpha : F(A) \rightarrow A$  and  $\beta : F(B) \rightarrow B$  be  $F$ -algebras. An  $F$ -**algebra homomorphism** from the former to the latter is a morphism  $f : A \rightarrow B$  that makes this square commute.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \alpha \downarrow & & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array} \quad (156)$$

This definition also clarifies why we require  $F$  to be a functor.

**Example 311.** Let  $F = X \mapsto \mathbf{1} + X + (X \times X)$  be the functor discussed Example 305. An  $F$ -algebra homomorphism is represented by the following square.

$$\begin{array}{ccc} \mathbf{1} + G + (G \times G) & \xrightarrow{[\text{id}_1, f, (f, f)]} & \mathbf{1} + H + (H \times H) \\ [1_G, (-)^{-1}, \cdot] \downarrow & & \downarrow [1_H, (-)^{-1}, \cdot] \\ G & \xrightarrow{f} & H \end{array} \quad (157)$$

Unwrapped, this says that  $f(1_G) = 1_H$ ,  $f(g^{-1}) = f(g)^{-1}$  and  $f(g \cdot g') = f(g) \cdot f(g')$  for all  $g, g' \in G$ , i.e.: if both algebras represent groups as seen in Example 305, it is a group homomorphism.

### Coalgebras

Now that we have a categorical notion of algebra, we can look at its dual.

**Definition 312** ( $F$ -coalgebra). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor, an  $F$ -**coalgebra** is an object  $A \in \mathbf{C}_0$  called the **carrier** along with a morphism  $\omega : A \rightarrow F(A)$  called the **behavior map**. We will refer to a coalgebra with  $A, \omega$  or the pair  $(A, \omega)$ .

**Examples 313.** 1. If  $F$  is the identity on **Set**, then an  $F$ -coalgebra is just an endomorphism  $\omega : A \rightarrow A$  and it is sometimes called a **dynamical system**. You can think of the elements of  $A$  as states and  $\omega$  as the transition map for the system.

2. Let  $\text{Str}_{\mathbb{N}} : \mathbf{Set} \rightsquigarrow \mathbf{Set} = \mathbb{N} \times (-)$  be the functor sending a set  $X$  to  $\mathbb{N} \times X$  and a function  $f : X \rightarrow Y$  to  $\text{id}_{\mathbb{N}} \times f : \mathbb{N} \times X \rightarrow \mathbb{N} \times Y$ . An example of a  $\text{Str}_{\mathbb{N}}$ -coalgebra is the set  $\mathbb{N}^{\mathbb{N}}$  of all infinite sequences (also called **streams**) of natural numbers with the structure map  $(\text{head}, \text{tail}) : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$  given by

$$\text{head} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N} = \sigma \mapsto \sigma(0) \quad \text{and} \quad \text{tail} : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}} = \sigma \mapsto \sigma \circ \text{succ}.$$

Unsurprisingly, we call  $\text{head}(\sigma)$  and  $\text{tail}(\sigma)$  the **head** and **tail** of the stream  $\sigma$  respectively.

**Exercise 314** (1pt). Denoting  $2 = \{0, 1\}$ , let  $F = 2 \times (-)^A$  send a set  $X$  to  $2 \times X^A$  and a function  $f : X \rightarrow Y$  to  $\text{id}_2 \times (f \circ -) : 2 \times X^A \rightarrow 2 \times Y^A$ . When  $A$  is finite, show that there is a correspondence between  $F$ -coalgebras with a finite carrier and DFAs over the alphabet  $A$  without initial states.

*Remark 315.* The components of the behavior map (i.e.:  $h$  and  $t$  in the second example) are often called **destructors** or **observers** because they decompose elements of the coalgebra.

We define morphisms of  $F$ -coalgebras in order to obtain a category  $\text{Coalg}(F)$ .

**Definition 316** ( $F$ -coalgebra homomorphism). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor and  $\alpha : A \rightarrow F(A)$  and  $\beta : B \rightarrow F(B)$  be  $F$ -coalgebras. An  $F$ -**coalgebra homomorphism** from the former to the latter is a morphism  $f : A \rightarrow B \in \mathbf{C}_1$  that makes (158) commute.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \alpha \downarrow & & \downarrow \beta \\ F(A) & \xrightarrow{F(f)} & F(B) \end{array} \quad (158)$$

### Induction

Induction is a very well known and prevalent proof principle. In its most common form, it says that for any predicate  $P$  on  $\mathbb{N}$ , if  $P(0)$  is true and  $P(n) \implies P(n+1)$

is true for any  $n \in \mathbb{N}$ , then so is  $P(n)$  for any  $n \in \mathbb{N}$ . In this section, we use the power of algebras to generalize this proof principle and give a few examples.

**Definition 317** (Initial algebra). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor, an **initial algebra** is an initial object in the category of  $F$ -algebras. Namely, it is an algebra  $(A, \alpha)$  such that for any other algebra  $(B, \beta)$ , there is a unique  $f : A \rightarrow B$  making the following square commute.

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \alpha \downarrow & & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array} \quad (159)$$

**Example 318.** The algebra  $(\mathbb{N}, [0, \text{succ}])$  is initial for the functor  $\mathbf{1} + (-)$ . Indeed, let  $[z, s] : \mathbf{1} + X \rightarrow X$  be another algebra for this functor, then a map  $f : \mathbb{N} \rightarrow X$  that makes the following diagram commute must necessarily satisfy  $f(0) = z(*)$  and  $f(n) = s^n(z(*))$ .

$$\begin{array}{ccc} \mathbf{1} + \mathbb{N} & \xrightarrow{[\text{id}_1, f]} & \mathbf{1} + X \\ [0, \text{succ}] \downarrow & & \downarrow [z, s] \\ \mathbb{N} & \xrightarrow{f} & X \end{array} \quad (160)$$

This completely determines  $f$  and moreover, defining  $f$  like this for any  $(\mathbf{1} + (-))$ -algebra  $(X, [z, s])$  yields an algebra homomorphism.

**Exercise 319** (2pts). Show that the algebra for lists  $[\text{nil}, \text{cons}] : \mathbf{1} + A \times A^* \rightarrow A^*$  is initial for the functor  $\mathbf{1} + A \times (-) : \mathbf{Set} \rightsquigarrow \mathbf{Set}$  (you know its action on sets, on morphisms it sends  $f : X \rightarrow Y$  to  $[\text{id}_1, (\text{id}_A, f)]$ ).

We already know that initial objects are unique up to unique isomorphisms, but Lambek also showed furthermore that initial  $F$ -algebras are fixed points of  $F$ .

**Proposition 320** (Lambek). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$ , if  $(A, \alpha)$  is an initial  $F$ -algebra, then  $\alpha : F(A) \rightarrow A$  is an isomorphism.

**Exercise 321** (1.5pts). Prove Proposition 320. **Hint:** Consider the algebra  $F(\alpha) : F^2(A) \rightarrow F(A)$ .

Initial algebras generalize the inductive reasoning we use with the natural numbers to much more settings. We distinguish two cases where induction is used: inductive definitions and the induction proof principle.

For the former, the general idea is that, given an initial  $F$ -algebra  $(A, \alpha)$ , we can easily define a function  $f : A \rightarrow B$  by looking at how it acts on constructors. Indeed, with only this data, we can construct an  $F$ -algebra structure on  $B$  such that the unique homomorphism  $! : A \rightarrow B$  acts exactly like  $f$ .

**Example 322** (Inductive definition). Recall that  $(A^*, [\text{nil}, \text{cons}])$  is the initial  $(\mathbf{1} + A \times (-))$ -algebra. We would like to define the function  $\text{len} : A^* \rightarrow \mathbb{N}$  that computes the length of a list. Intuitively, it satisfies the equations

$$\text{len}(\text{nil}) = 0 \quad \text{len}(\text{cons}(a, l)) = 1 + \text{len}(l).$$

Then, if we construct the  $(1 + A \times (-))$ -algebra  $[z, s] : 1 + A \times \mathbb{N} \rightarrow \mathbb{N}$  defined by  $z(*) = 0$  and  $s(a, n) = 1 + n$ , we can verify that the unique algebra homomorphism  $! : A^* \rightarrow \mathbb{N}$  is the function `len` because both make the following diagram commute.

$$\begin{array}{ccc} 1 + A \times A^* & \xrightarrow{1 + \text{id}_A \times !} & 1 + A \times \mathbb{N} \\ \downarrow [\text{nil}, \text{cons}] & & \downarrow [z, s] \\ A^* & \xrightarrow{! = \text{len}} & \mathbb{N} \end{array} \quad (161)$$

**Exercise 323** (1pt). Use the initiality of  $\mathbb{N}$  for the functor  $1 + (-)$  to define the function  $n \mapsto 2^n$ .

Generalizing proofs by induction in this context is more involved and we will need the definition of  $F$ -congruences. While  $F$ -algebra homomorphisms are maps between algebras that preserve the structure, an  $F$ -congruence is a relation between two algebras that preserves the structure.

**Definition 324.** Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor and  $(A, \alpha)$  and  $(B, \beta)$  be  $F$ -algebras, a relation  $R \subseteq A \times B$  is an  **$F$ -congruence** if there is a structure map  $\gamma : F(R) \rightarrow R$  such that the projections  $\pi_1 : R \rightarrow A$  and  $\pi_2 : R \rightarrow B$  are algebra homomorphisms making this diagram commute.

$$\begin{array}{ccccc} F(A) & \xleftarrow{F\pi_1} & F(R) & \xrightarrow{F\pi_2} & F(B) \\ \alpha \downarrow & & \downarrow \gamma & & \downarrow \beta \\ A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \end{array} \quad (162)$$

**Example 325.** If  $F$  is the identity functor, then for any algebras  $(A, \alpha)$  and  $(B, \beta)$  and any relation  $R \subseteq A \times B$ ,  $\gamma = (\alpha \circ \pi_1, \beta \circ \pi_2)$  is a structure map making  $R$  into an  $F$ -congruence.

**Exercise 326** (2pts). Let  $F = 1 + (-)$ , we have already seen that  $\mathbb{N}$  is an initial  $F$ -algebra.

- (a) Give a necessary and sufficient condition for  $R \subseteq \mathbb{N} \times \mathbb{N}$  to be an  $F$ -congruence.
- (b) Conclude that for any  $F$ -congruence  $R \subseteq \mathbb{N} \times \mathbb{N}$ ,  $\forall n \in \mathbb{N}, (n, n) \in R$ .

The next theorem generalizes the previous exercise.

**Theorem 327** (General induction). *Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor and  $(A, \alpha)$  be an initial  $F$ -algebra, if  $R \subseteq A \times A$  is an  $F$ -congruence, then it is reflexive, that is  $(a, a) \in R$  for all  $a \in A$ .*

**Exercise 328** (1.5pts). Prove Theorem 327.

**Example 329** (Induction in  $\mathbb{N}$ ). We will see how the induction principle in Theorem 327 implies the usual induction principle. Let  $P$  be a predicate on  $\mathbb{N}$  that satisfies  $0 \in P$  and  $n \in P \implies n + 1 \in P$ . One can show that  $P \times P \subseteq \mathbb{N} \times \mathbb{N}$  is an  $F$ -congruence and by general induction,  $(n, n) \in P \times P$  for all  $n \in \mathbb{N}$ , i.e.:  $\forall n \in \mathbb{N}, n \in P$ .

Although going through all these abstractions and definitions seems like a really convoluted way to prove the induction principle, it lead us to two new concepts. First, we can now use inductive reasoning on all sorts of algebras even if they are in no way similar to  $\mathbb{N}$ . Second, we obtained an easy access to the dual of induction which we present in the following section.

### Coinduction

**Definition 330** (Final coalgebra). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor, a **final coalgebra** is a final object in the category of  $F$ -coalgebras. Namely, it is a coalgebra  $(A, \omega)$  such that for any other coalgebra  $(B, \psi)$ , there is a unique morphism  $f : B \rightarrow A$  making the following square commute.

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & A \\ \psi \downarrow & & \downarrow \omega \\ F(B) & \xrightarrow{\quad F(f) \quad} & F(A) \end{array} \quad (163)$$

Since final coalgebras are unique up to unique homomorphism, we will refer to **the** final coalgebra.

**Example 331.** The  $\text{Str}_{\mathbb{N}}$ -coalgebra  $(\text{head}, \text{tail}) : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$  is final. That is, for any  $\text{Str}_{\mathbb{N}}$ -coalgebra,  $(h, t) : X \rightarrow \mathbb{N} \times X$ , there is unique morphism  $! : X \rightarrow \mathbb{N}^{\mathbb{N}}$  making (164) commute.

$$\begin{array}{ccc} X & \xrightarrow{\quad ! \quad} & \mathbb{N}^{\mathbb{N}} \\ (h, t) \downarrow & & \downarrow (\text{head}, \text{tail}) \\ \mathbb{N} \times X & \xrightarrow{\quad \text{id}_{\mathbb{N}} \times ! \quad} & \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \end{array} \quad (164)$$

The equation corresponding to (164) is  $(\text{head}, \text{tail}) \circ ! = (\text{id}_{\mathbb{N}} \times !) \circ (h, t)$ . We can decompose it into  $\text{head} \circ ! = h$  and  $\text{tail} \circ ! = ! \circ t$ . The first equation tells us that  $!(x)$  starts with the number  $h(x)$  and the second equation tells us that the tail of  $!(x)$  is the stream corresponding to  $t(x)$  (via  $!$ ). In short, we have  $!(x) = h(x) \cdot !(t(x))$ . If we further decompose the tail, we obtain

$$!(x) = h(x) \cdot h(t(x)) \cdot h(t(t(x))) \cdots h(t^n(x)) \cdots$$

This should convince you that the only suitable choice for  $!$  is  $x \mapsto (n \mapsto h(t^n(x)))$ .

**Exercise 332** (3pts). Let  $F = 2 \times (-)^A$  with  $A$  finite and consider the  $F$ -coalgebra

$$(\varepsilon?, \omega) : 2^{A^*} \rightarrow 2 \times (2^{A^*})^A,$$

where for a language  $L \subseteq A^*$  ( $\varepsilon$  denotes the empty string),

$$\varepsilon?(L) = \begin{cases} 1 & \varepsilon \in L \\ 0 & \text{o/w} \end{cases}, \quad \omega(L) = a \mapsto L_a = \{w \in A^* \mid a \cdot w \in L\}.$$

The language  $L_a = \omega(L)(a)$  is sometimes called the left  $a$ -derivative of  $L$ . Given a DFA  $M$  corresponding to the coalgebra  $[f, \delta] : Q \rightarrow 2 \times Q^A$ , show that the function

$$o : Q \rightarrow 2^{A^*} = q \mapsto \{w \in A^* \mid M \text{ accepts } w \text{ when starting in state } q\}$$

is the only map making (165) commute.

$$\begin{array}{ccc} & & 2 \\ & \nearrow f & \uparrow \varepsilon? \\ X & \xrightarrow{o} & 2^{A^*} \\ \delta \downarrow & & \downarrow \omega \\ X^A & \xrightarrow{o^A} & (2^{A^*})^A \end{array} \quad (165)$$

*Remark 333.* Extending your proof to  $X$  not necessarily finite, you could obtain the fact that  $(2^{A^*}, (\varepsilon?, \omega))$  is the final  $2 \times (-)^A$  coalgebra.

**Proposition 334.** If  $\omega : A \rightarrow F(A)$  is a final  $F$ -coalgebra, then  $\omega$  is an isomorphism.

**Exercise 335** (0.5pts). Prove Proposition 334.

Final coalgebras lets us use **coinductive definitions**. You will see that they are quite similar to inductive definitions you are used to. In fact, as the name suggests, they are dual to each other, but we will not make this formal here.

**Examples 336** (Coinductive definitions). Fix some set  $A$  and consider the functor  $\text{Str}_A = A \times (-)$ , similarly to  $\text{Str}_{\mathbb{N}}$ , the set  $A^{\mathbb{N}}$  of streams in  $A$  is the final  $\text{Str}_A$ -coalgebra with the behavior map  $(\text{head}, \text{tail})$  as defined in Example 313. We will define three different maps using the finality of  $A^{\mathbb{N}}$ .

1. The function  $\text{even} : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$  takes a stream  $\sigma = (\sigma(0), \sigma(1), \dots)$  and maps it to the stream of elements of  $\sigma$  at even positions, namely  $\text{even}(\sigma) = (\sigma(0), \sigma(2), \dots)$ . To define it coinductively, we need to describe how destructors act on it. It is easy to verify that

$$\text{head}(\text{even}(\sigma)) = \text{head}(\sigma) \quad \text{and} \quad \text{tail}(\text{even}(\sigma)) = \text{even}(\text{tail}(\text{tail}(\sigma))).$$

Hence, if we define a new  $\text{Str}_A$ -coalgebra on  $A^{\mathbb{N}}$  by  $(h, t) = (\text{head}, \text{tail}^2 = \text{tail} \circ \text{tail})$ , then we conclude by finality and commutativity of the following diagram that  $! : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$  is the function  $\text{even}$ .

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{!} & A^{\mathbb{N}} \\ (\text{head}, \text{tail}^2) \downarrow & & \downarrow (\text{head}, \text{tail}) \\ A \times A^{\mathbb{N}} & \xrightarrow{\text{id}_A \times !} & A \times A^{\mathbb{N}} \end{array} \quad (166)$$

2. The operation of merging two streams is described by the function  $\text{merge} : A^{\mathbb{N}} \times A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$  mapping  $(\sigma, \tau)$  to  $(\sigma(0), \tau(0), \sigma(1), \tau(1), \dots)$ . Observe that destructors act as follows:

$$\text{head}(\text{merge}(\sigma, \tau)) = \text{head}(\sigma) \quad \text{and} \quad \text{tail}(\text{merge}(\sigma, \tau)) = \text{merge}(\tau, \text{tail}(\sigma)).$$

The existence of merge is then proven with finality of  $A^{\mathbb{N}}$  and the following coalgebra behavior map (where  $\pi_1$  and  $\pi_2$  are the projections):

$$(\text{head} \circ \pi_1, \pi_2, \text{tail} \circ \pi_1) : A^{\mathbb{N}} \times A^{\mathbb{N}} \rightarrow A \times A^{\mathbb{N}} \times A^{\mathbb{N}}.$$

**Exercise 337** (1pts). Similarly to the first item, show that the function  $\text{odd} : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$  mapping  $\sigma = (\sigma(0), \sigma(1), \dots)$  to  $\text{odd}(\sigma) = (\sigma(1), \sigma(3), \dots)$  can be defined coinductively.

There is also a dual to the induction proof principle for which we need to define bisimulation.

**Definition 338** ( $F$ -bisimulation). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor and  $(A, \omega)$  and  $(B, \psi)$  be  $F$ -coalgebras, a relation  $R \subseteq A \times B$  is an  $F$ -**bisimulation** if there is a behavior map  $\gamma : R \rightarrow F(R)$  such that the projections  $\pi_1 : R \rightarrow A$  and  $\pi_2 : R \rightarrow B$  are coalgebra homomorphisms making this diagram commute.

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\ \omega \downarrow & & \downarrow \gamma & & \downarrow \psi \\ F(A) & \xleftarrow{F\pi_1} & F(R) & \xrightarrow{F\pi_2} & F(B) \end{array} \quad (167)$$

**Theorem 339** (Coinductive proof principle). Let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}$  be a functor and  $(A, \omega)$  be the final  $F$ -coalgebra, if  $R \subseteq A \times A$  is an  $F$ -bisimulation, then it is contained in the diagonal relation, that is  $(a, a') \in R$  implies  $a = a'$ .

**Exercise 340** (0.5pts). Prove Theorem 339.

**Example 341.** We will use coinduction to prove that  $\text{odd}(\text{merge}(\sigma, \tau)) = \tau$ . By the previous theorem, it is enough to show that  $\mathcal{R} = \{(\text{odd}(\text{merge}(\sigma, \tau)), \tau) \mid \sigma, \tau \in A^{\mathbb{N}}\}$  is an  $F$ -bisimulation. We claim that  $\gamma = (x, y) \mapsto (\text{head}(x), (\text{tail}(x), \text{tail}(y)))$  makes the following diagram commute.

$$\begin{array}{ccccc} A^{\mathbb{N}} & \xleftarrow{\pi_1} & \mathcal{R} & \xrightarrow{\pi_2} & A^{\mathbb{N}} \\ (\text{head}, \text{tail}) \downarrow & & \downarrow \gamma & & \downarrow (\text{head}, \text{tail}) \\ A \times A^{\mathbb{N}} & \xleftarrow{\text{id}_A \times \pi_1} & A \times \mathcal{R} & \xrightarrow{\text{id}_A \times \pi_2} & A \times A^{\mathbb{N}} \end{array} \quad (168)$$

To prove our claim, first note that

$$\begin{aligned} \text{head}(\text{tail}(\text{merge}(\sigma, \tau))) &= \text{head}(\text{merge}(\tau, \text{tail}(\sigma))) \\ &= \text{head}(\tau), \end{aligned}$$

so if we can show that  $(\text{tail}(x), \text{tail}(y)) \in \mathcal{R}$  for any  $(x, y) \in \mathcal{R}$ , then we would conclude that the diagram commutes. This last part follows from the derivation

$$\begin{aligned} \text{tail}(\text{odd}(\text{merge}(\sigma, \tau))) &= \text{odd}(\text{tail}(\text{tail}(\text{merge}(\sigma, \tau)))) \\ &= \text{odd}(\text{tail}(\text{merge}(\tau, \text{tail}(\sigma)))) \end{aligned}$$



$$= \text{odd}(\text{merge}(\text{tail}(\sigma), \text{tail}(\tau))).$$

Indeed, we obtain

$$(\text{tail}(\text{odd}(\text{merge}(\sigma, \tau), \text{tail}(\tau))) = (\text{odd}(\text{merge}(\text{tail}(\sigma), \text{tail}(\tau))), \text{tail}(\tau)) \in \mathcal{R}.$$

**Exercise 342** (3pts). (a) Given  $f : A \rightarrow A$  coinductively define  $\text{map}_f : A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$  such that  $\text{map}_f(\sigma) = (f(\sigma(0)), f(\sigma(1)), \dots)$ .

(b) Show by coinduction that for any  $\sigma \in A^{\mathbb{N}}$ ,  $\text{even}(\text{map}_f(\sigma)) = \text{map}_f(\text{even}(\sigma))$ .

### Preliminaries on Automata

There are no questions in this section, only stuff we believe you learned in a first course in theoretical CS. Give it a quick read in order to at least to identify my notation, the most important part is the definition of Brzozowski's algorithm in Section .

### Deterministic Finite Automata

**Definition 343** (DFA). A **deterministic finite automaton** (DFA)  $M$  is composed of a **finite** alphabet  $\Sigma$  (a set of symbols), a finite set of **states**  $Q$  with a starting state identified by  $q_0$ , a **transition function**  $\delta : Q \times \Sigma \rightarrow Q$  and a subset  $F \subseteq Q$  of **accepting** states.

An input for  $M$  is a finite word (concatenation of finitely many elements) of  $\Sigma$ , we will denote it  $w \in \Sigma^*$ . The automaton reads its input letter by letter, starting in state  $q_0$ , and changes its state according to  $\delta$ :  $\delta(q, a) = q'$  means that if  $M$  is in state  $q$  and reads the symbol  $a$ , then  $M$  ends up in state  $q'$ . After reading all of its input  $M$  is in some state  $q$  and outputs “Accept” if  $q \in F$  and “Reject” otherwise.

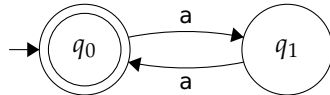
The global behavior of  $M$  is described by the subset  $L \subseteq \Sigma^*$  of words that  $M$  accepts. We denote this subset  $L(M)$  and say that  $L(M)$  is the **language recognized** by  $M$ .

We will denote  $\delta^*$  the extension of  $\delta$  to  $\Sigma^*$ , it is defined inductively by

$$\delta^*(x, \varepsilon) = x \quad \text{and} \quad \delta^*(x, a \cdot w) = \delta^*(\delta(x, a), w).$$

**Examples 344.** Typically, it is more readable to describe DFAs by drawing a graphical representation than by defining each component.

1. Consider the DFA described by  $\Sigma = \{a\}$ ,  $Q = \{q_0, q_1\}$ ,  $\delta = (q_0, a) \mapsto q_1, (q_1, a) \mapsto q_0$ ,  $F = \{q_0\}$ . It is represented by the following diagram.

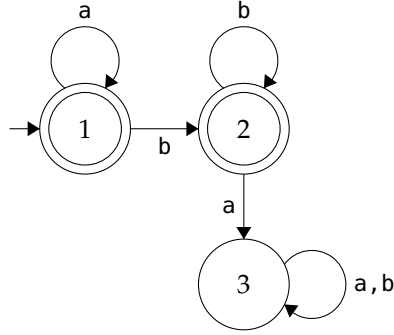


The circles represent the states and accepting states are denoted with a second inner circle. The arrows represent transitions and the labels are the symbols that

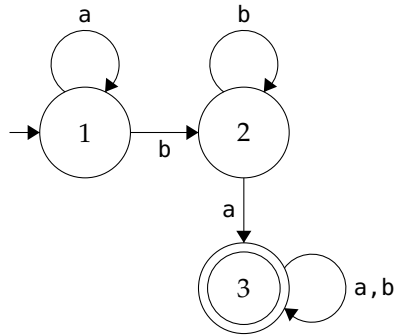
need to be read for this transition to occur. The smaller arrow with no label designates the starting state.

It is easy to see that the language recognized by this DFA consists of all the words with an even number of a's (i.e.:  $L(M) = \{a^{2n} : n \in \mathbb{N}\}$ ).

2. The DFA recognizing the language  $\{a^n b^m \mid n, m \in \mathbb{N}\}$  can be described as follows.



3. For any DFA  $M$  on an alphabet  $\Sigma$ , it is very easy to describe a DFA  $M'$  that recognizes the complement of  $L(M)$ , i.e.: such that  $L(M') = \Sigma^* \setminus L(M)$ . We just invert the roles of accepting and non-accepting states. Here is the DFA recognizing the complement of  $\{a^n b^m \mid n, m \in \mathbb{N}\}$ .



*Remark 345.* The term **deterministic** means that the transitions are completely determined by the input, that is, you can run the DFA on the same input many many times and it will always end in the same state and output the same thing. Mathematically, determinism comes from the fact that  $\delta(q, a)$  takes the value of only one state for any  $q \in Q$  and  $a \in \Sigma$ . If we allow  $\delta$  to be nondeterministic, we get the definition of an NFA.

### Nondeterministic Finite Automata

**Definition 346.** An **nondeterministic automaton** (NFA)  $M$  consists of a finite alphabet  $\Sigma$ , a finite set of states  $Q$  with a starting state identified by  $q_0$ , a transition function  $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$  and a subset  $F \subseteq Q$  of accepting states.

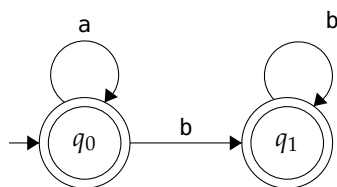
There are two main differences with a DFA. First, an NFA can sometimes make a transition without reading a symbol from the input but rather by reading  $\varepsilon$  (denoting an empty string). Second, the image of  $\delta$  is a set of possible states for the transition to end in. Let us see how this affects the behavior of  $M$ .

The automaton still reads its input letter by letter starting in state  $q_0$ , but now instead of making the only possible transition, it makes all of them at the same time and continues the computation on multiple branches. The output of  $M$  is “Accept” if in at least one branch, all the input was read and  $M$  is in an accepting state, otherwise  $M$  outputs “Reject”. The language recognized by  $M$  is defined as for DFAs.

Another way to view nondeterminism is to consider that  $M$  has access to an all-knowing oracle that will choose which transition to make (out of the possible ones). This oracle always make the choices that lead to accepting the input if possible, hence,  $M$  accepts  $w \in \Sigma^*$  if and only if there is a sequence of choices that lead to an accepting state after reading all the input.

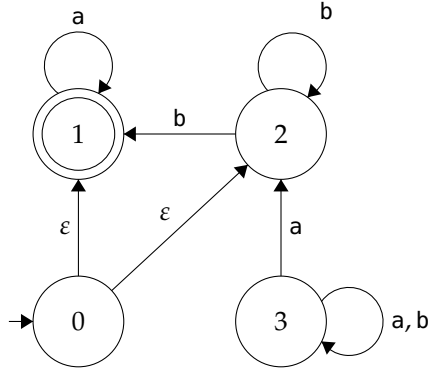
**Examples 347.** The representation of NFAs is really similar to that of DFAs but now, arrows can be labeled with an  $\varepsilon$ , multiple arrows coming out of the same state can have the same label, and states can have no arrows coming out with a specific label  $a \in \Sigma$  (corresponding to the fact that  $\delta(q, a) = \emptyset$ ).

1. The NFA recognizing  $\{a^n b^m \mid n, m \in \mathbb{N}\}$  is simpler than the DFA we drew above.

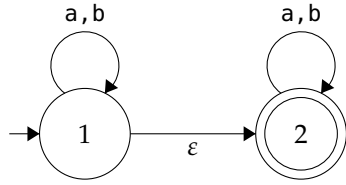


2. For any DFA  $M$ , we can easily construct an NFA  $M'$  that recognizes the reverse of  $L(M)$ , i.e.:  $L(M') = \{w^{\mathcal{R}} \mid w \in L(M)\}$ , where  $\mathcal{R}$  denotes the reverse of a word. Indeed, from the representation of a DFA, we can reverse all arrows, make the initial state an accepting state and add a pseudo initial state with  $\varepsilon$ -transitions to all the old accepting states as shown by the illustration below.

The reverse of  $\{a^n b^m \mid n, m \in \mathbb{N}\}$  is  $\{b^n a^m \mid n, m \in \mathbb{N}\}$  and it is recognized by the following NFA.



3. Finding the complement of an NFA is not as easy as for DFAs. For instance, consider the following NFA on the alphabet  $\{a, b\}$ .



It recognizes all words generated by the alphabet, but if you swap the accepting and non-accepting states, the new NFA will also accept all words.

*Remark 348.* Since an NFA can have  $\varepsilon$ -transitions, the extension of  $\delta$  to  $\Sigma^*$  is defined differently,  $\delta^*(x, \varepsilon)$  is the  $\varepsilon$ -closure of  $x$  and  $\delta^*(x, w)$  is defined inductively. More formally,

$$\delta^*(x, \varepsilon) = \{q \in Q : \exists \{x = x_0, x_2, \dots, x_n = q\}, \forall 1 \leq i \leq n, x_i \in \delta(x_{i-1}, \varepsilon)\}$$

$$\delta^*(x, a \cdot w) = \bigcup_{q \in \delta(x, a)} \bigcup_{q' \in \delta^*(q, w)} \delta^*(q', \varepsilon).$$

Intuitively,  $\delta^*(x, w)$  is the set of words that  $x$  can reach while reading the input  $w$ , taking into account that the machine can take any  $\varepsilon$ -transition.

One question that quickly arises is whether there are languages that can be recognized by an NFA, but not by any DFA. The converse is clearly false because any DFA can be written as an NFA where the image of the transition function only contains singletons, i.e.: there is only one possible state for each transition.

Surprisingly, the original question also has a negative answer. In other words, any NFA has a DFA recognizing the same language. To prove this, we describe the powerset construction transforming an NFA into an equivalent DFA.

*Proof sketch.* Let  $M = (\Sigma, Q, q_0, \delta, F)$  be an NFA, we construct an equivalent DFA  $M' = (\Sigma, Q', q'_0, \delta', F')$  as follows.

- The states of  $M'$  are sets of states of  $M$ , that is  $Q' = \mathcal{P}(Q)$ .

- The initial state  $q'_0$  is the set of states that can be reached by reading no symbol when running  $M$ . Formally, we have

$$q'_0 = \delta^*(q_0, \varepsilon).$$

- The transition function will simulate all the possible choices by transitioning between sets of states. For any  $S \subseteq Q$  and  $a \in \Sigma$ ,

$$\delta'(S, a) = \bigcup_{q \in S} \delta^*(q, a)$$

- A set of states is accepting if and only if it contains an accepting state.

It is left to show that  $L(M) = L(M')$ . □

The transformation we just described implies that NFAs are not necessary and we could always work with DFAs. However, observe that the size of the automaton increases exponentially in this procedure, so it is not practical to work with the resulting DFA. Consequently, one could ask whether we could further transform the DFA into a smaller (maybe smallest) but equivalent DFA. This is called DFA minimization and there are several polynomial time algorithms that solve this problem (Hopcroft's and Moore's). Another algorithm by Brzozowski is conceptually much simpler although it runs in exponential time in the worst case.

### Brzozowski's Algorithm

**Definition 349.** Let  $M$  be a DFA, **Brzozowski's algorithm** is the following procedure.

1. Reverse  $M$  to obtain an NFA  $M^{\mathcal{R}}$  which recognizes the reverse of  $L(M)$ .
2. Use the powerset construction to obtain a DFA  $D(M^{\mathcal{R}})$ , still recognizing the reverse of  $L(M)$ .
3. Discard all unreachable states of  $D(M^{\mathcal{R}})$ , denote the automaton obtained with  $N$ .
4. Apply the same procedure (steps 1 to 3) to  $N$ , i.e.: reverse  $N$ , determinize the result and discard unreachable states to obtain  $O$ .

**Proposition 350.** The final automata  $O$  is the automaton with the least number of states satisfying  $L(O) = L(M)$ . Another way to say this: if **rev** denotes the reversing operation, **det** denotes the determinization and **reach** denotes the operation of removing unreachable states, then

$$\text{reach}(\text{det}(\text{rev}(\text{reach}(\text{det}(\text{rev}(M))))))$$

is the minimal automaton equivalent to  $M$ .

*Remark 351.* This result should be very surprising. Brzozowski's algorithm simply reverses the automaton twice to obtain a minimal form. Moreover, during the procedure, there are two steps inducing an exponential blow-up of the number of states. Indeed, determinization via the powerset construction leads to an automaton with  $2^{|Q|}$  states. That means that in the worst cases,  $\text{det}(\text{rev}(\text{reach}(\text{det}(\text{rev}(M))))$  could have  $2^{2^{|Q|}}$  states. This is the reason why Brzozowski's algorithm can take exponential time. Nonetheless, in the end, you still end up with a minimal automaton. Another surprising thing, this algorithm often performs better than the worst case scenario.

This algorithm was first proven correct using complex combinatorial arguments in 1963, but in this assignment, we will give a much simpler proof using coalgebras. The next section is dedicated to introducing coalgebras and coinduction via their duals, algebras and induction.

### Brzozowski's Algorithm Coalgebraically

In this section we follow the proof given in this paper (answers are in there).

#### Reachability and Observability

Let  $A$  be an alphabet, and  $(X, i, \delta, F)$  be deterministic automaton, where  $X$  is the set of states,  $\delta : X \times A \rightarrow X$  is the transition function,  $i \in X$  is the initial state and  $F$  is the set of accepting states. For our purposes, we will view  $i$  as a function  $i : \mathbf{1} \rightarrow X$ ,  $F$  as a function  $f : X \rightarrow 2 = \{0, 1\}$  and  $\delta$  will also denote the curried version  $\delta : X \rightarrow X^A$ . This leads to the following simple representation of the automaton.

$$\begin{array}{ccc}
 \mathbf{1} & & 2 \\
 & \searrow i & \nearrow f \\
 & X & \\
 & \downarrow \delta & \\
 & X^A & 
 \end{array} \tag{169}$$

In the light of the previous sections, we decompose this automaton into an algebra and a coalgebra.

First, we have the algebra  $[i, \delta] : \mathbf{1} + (A \times X) \rightarrow X$  for the functor  $\mathbf{1} + A \times (-)$ . Recall from Exercise 319 that  $(A^*, [\text{nil} := \varepsilon, \text{cons}])$  is initial for the functor  $\mathbf{1} + (A \times -)$ , so after currying  $\text{cons}$ , we obtain a unique morphism  $r : A^* \rightarrow X$  that makes the following diagram commute.

$$\begin{array}{ccc}
 \mathbf{1} & & 2 \\
 \varepsilon \downarrow & \searrow i & \nearrow f \\
 A^* & \xrightarrow{\quad r \quad} & X \\
 \text{cons} \downarrow & & \downarrow \delta \\
 (A^*)^A & \xrightarrow{\quad r^A \quad} & X^A
 \end{array} \tag{170}$$

Explicitly, the commutativity of the left half of diagram 170 amounts to

$$r(\varepsilon) = i \quad \text{and} \quad r(\text{cons}(w, a)) = \delta(r(w), (a)), \forall a \in A$$

We infer that for any word  $w \in A^*$ ,  $r(w)$  is the state reached by the automaton after starting in state  $i$  and reading input  $w$ .

Second, we have the coalgebra  $(f, \delta) : X \rightarrow 2 \times X^A$ . Recall from diagram 165 that  $(2^{A^*}, (\varepsilon?, \omega))$  is final for the functor  $2 \times (-)^A$ . Thus, we obtain a unique morphism  $o : X \rightarrow 2^{A^*}$  that completes diagram 170.

$$\begin{array}{ccccc}
 \mathbf{1} & & & & \mathbf{2} \\
 \text{nil} \downarrow & \searrow i & & \nearrow f & \uparrow \varepsilon? \\
 A^* & \xrightarrow{\quad r \quad} & X & \xrightarrow{\quad o \quad} & 2^{A^*} \\
 \text{cons} \downarrow & & \downarrow \delta & & \downarrow \omega \\
 (A^*)^A & \xrightarrow{\quad r^A \quad} & X^A & \xrightarrow{\quad o^A \quad} & (2^{A^*})^A
 \end{array} \tag{171}$$

We recall from Exercise 332 that for any  $x \in X$ ,  $o(x)$  is the language in accepted by the automaton if started on state  $x$ .

**Definition 352.** In the setting above, the automaton  $(X, i, \delta, f)$  is said to be

1. **reachable** if  $r$  is surjective,
2. **observable** if  $o$  is injective, and
3. **minimal** if it is both reachable and observable.

**Exercise 353** (1pts). (a) Describe, in automata theoretic terms, what reachability and observability mean.

(b) Informally explain why an automaton has a minimal number of states if and only if it is both reachable and observable.

### Reversing an Automaton

We will use our new method of representing automata to give the construction of an automata which recognizes the reverse language. Morally, our procedure does the same thing as the original reversing algorithm and the powerset construction at the same time.

In the sequel, let  $2^{(-)}$  denote the contravariant powerset functor, namely, it sends  $X$  to  $2^X$  and  $f : X \rightarrow Y$  to  $2^f : 2^Y \rightarrow 2^X = S \mapsto \{x \in X : f(x) \in S\}$ .

The reversed powerset construction goes like this. Given a transition function  $\delta : X \rightarrow X^A$ , we can uncurry it to get  $\delta : X \times A \rightarrow X$ , then apply  $2^{(-)}$  to obtain  $2^\delta : 2^X \rightarrow 2^{X \times A}$  and finally curry the elements in the codomain to obtain  $2^\delta : 2^X \rightarrow (2^X)^A$ . This is now a transition function for an automata whose states are set of states of the original automata.

**Exercise 354** (3pts). (a) Describe the action of  $2^\delta$ .

- (b) Continue applying  $2^{(-)}$  to the L.H.S. of diagram (171) to obtain (172). Briefly describe what each depicted morphism does.

$$\begin{array}{ccc}
 & & 2 \\
 & \nearrow 2^i & \uparrow 2^{\text{nil}} \\
 2^X & \xrightarrow{2^r} & 2^{A^*} \\
 \downarrow 2^\delta & & \downarrow 2^{\text{cons}} \\
 (2^X)^A & \xrightarrow{2^{r^A}} & (2^{A^*})^A
 \end{array} \quad (172)$$

**Warning:** For cons, you will need to use the same trick as for  $\delta$ .

By currying  $f : X \rightarrow 2$  to  $f : 1 \rightarrow 2^X$ , we obtain an  $1 + A \times (-)$ -algebra structure on  $2^X$  which, by initiality of  $A^*$ , gives the following diagram.

$$\begin{array}{ccccc}
 1 & & & & 2 \\
 \text{nil} \downarrow & \searrow f & & \nearrow 2^i & \uparrow 2^{\text{nil}} \\
 A^* & \xrightarrow{\quad R \quad} & 2^X & \xrightarrow{2^r} & 2^{A^*} \\
 \text{cons} \downarrow & & \downarrow 2^\delta & & \downarrow 2^{\text{cons}} \\
 (A^*)^A & \xrightarrow{\quad R^A \quad} & (2^X)^A & \xrightarrow{2^{r^A}} & (2^{A^*})^A
 \end{array} \quad (173)$$

This is very close to what we are looking for:

- We have an automaton whose states are  $2^X$ ,
- its initial state is the set of states which contain at least one final state of the original automaton,
- its final states are the set of states which contain the initial state of the original automaton, and
- the morphism  $R$  tells us which states are reachable.

However, the R.H.S. of (173) is not exactly what we need to talk about observability of this automaton. You can see that  $2^{\text{nil}} = \varepsilon?$ , but  $2^{\text{cons}}$  is a kind of reversed version of  $\omega$ , the concatenation is done in the opposite way.

**Exercise 355** (1pts). Describe the unique way to complete (173) into (174).

$$\begin{array}{ccccc}
 1 & & & & 2 \\
 \text{nil} \downarrow & \searrow f & & \nearrow 2^i & \uparrow \varepsilon? \\
 A^* & \xrightarrow{\quad R \quad} & 2^X & \xrightarrow{\quad O \quad} & 2^{A^*} \\
 \text{cons} \downarrow & & \downarrow 2^\delta & & \downarrow \omega \\
 (A^*)^A & \xrightarrow{\quad R^A \quad} & (2^X)^A & \xrightarrow{\quad O^A \quad} & (2^{A^*})^A
 \end{array} \quad (174)$$



In conclusion, we have an automaton  $(2^X, f, 2^\delta, 2^i)$  which is reachable if  $R$  is surjective and observable if  $O$  is injective. Moreover, we can show two important properties.

**Exercise 356** (4pts). (a) If  $(X, i, \delta, f)$  recognizes the language  $L$ , then  $(2^X, f, 2^\delta, 2^i)$  recognizes the reverse of  $L$ .

(b) If  $(X, i, \delta, f)$  is reachable, then  $(2^X, f, 2^\delta, 2^i)$  is observable.

### *Correctness of the Algorithm*

**Exercise 357** (1pt). Show that Brzozowski's algorithm is correct. Namely, if  $(X, i, \delta, f)$  recognizes a language  $L$ , then

- applying the (new) reverse construction,
- keeping only the reachable states,
- applying the reverse construction again, and
- keeping only the reachable states

yields an automaton which is minimal and recognizes the language  $L$ .



# Solutions to Exercises

## Solutions to Chapter

*Solution to Exercise 73.* Take any **monoid**  $M$  with an idempotent element  $x \neq 1_M$  (it satisfies  $x \cdot x = x$ ). Letting  $\mathbf{C}$  be  $\mathbf{B}M$  and  $\mathbf{C}'$  contain the **object**  $*$  and only the **morphism**  $x$  yields a suitable example because the **identity** in  $\mathbf{C}'$  is  $x$ .  $\square$

*Solution to Exercise 90.* On **morphisms**, we define  $\Delta_{\mathbf{C}}(f) = (f, f)$ . The **functoriality** properties hold because everything in  $\mathbf{C} \times \mathbf{C}$  is done componentwise.

- i. For  $f : X \rightarrow Y$ , we have  $(f, f) : (X, X) \rightarrow (Y, Y)$ .
- ii. For  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , we have  $(g, g) \circ (f, f) = (g \circ f, g \circ f)$ .
- iii. For any  $X \in \mathbf{C}_0$ , we have  $\Delta_{\mathbf{C}}(\text{id}_X) = (\text{id}_X, \text{id}_X) = \text{id}_{(X, X)}$ .

$\square$

*Solution to Exercise 92.* A quick way to show  $F(X, -)$  is a **functor** is to recognize it as the **composition** of  $F$  with  $X \times \text{id}_{\mathbf{C}'}$ , where  $X$  is the **constant functor** at  $X$ . Similarly,  $F(-, Y) := F \circ (\text{id}_{\mathbf{C}} \times Y)$ .  $\square$

*Solution to Exercise 93.* Let us show the three properties of **functoriality**.

- i. For any  $(f, g) : (X, Y) \rightarrow (X', Y')$

$\square$

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*Solution to Exercise 111.* Let us have two **morphisms**  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ .

- Suppose  $f$  and  $g$  are **monic**. For any  $h_1, h_2 : Z \rightarrow Z'$  satisfying  $h_1 \circ g \circ f = h_2 \circ g \circ f$ , **monicity** of  $f$  implies  $h_1 \circ g = h_2 \circ g$  which in turn, by **monicity** of  $g$  imply  $h_1 = h_2$ . Thus,  $g \circ f$  is **monic**.
- We apply **duality**. Suppose  $f$  and  $g$  are **epic**, then  $f^{\text{op}}$  and  $g^{\text{op}}$  are **monic** so  $(g \circ f)^{\text{op}} = f^{\text{op}} \circ g^{\text{op}}$  is **monic**, thus  $g \circ f$  is **epic**.
- If  $f$  and  $g$  are **isomorphisms**, then it is easy to check that  $f^{-1} \circ g^{-1}$  is the **inverse** of  $g \circ f$ , implying  $g \circ f$  is an **isomorphism**.

□

*Solution to Exercise 131.* 1. Let  $f : A \rightarrow B$  be the only non-identity **morphism** in **2**, it is a **monomorphism** vacuously because there is only one **morphism** with **target**  $A$  ( $\text{id}_A$ ). Now, for any **morphism**  $m : X \rightarrow Y \in \mathbf{C}_1$ , we can define  $F : \mathbf{2} \rightsquigarrow \mathbf{C}$  by  $FA = X$ ,  $FB = Y$  and  $Ff = m$  and it will be a **functor**. Thus, choosing  $m$  that is not **monic** yields the required example.

2. If  $f$  is **split monic**, it has a **right inverse**  $f'$ . This implies  $Ff'$  is the **right inverse** of  $Ff$  because  $Ff \circ Ff' = F(f \circ f') = F(\text{id}) = \text{id}$ . We conclude that  $Ff$  is **split monic**.
3. We need to show that **functors preserve split epimorphisms**. By **duality**, if  $f$  is **split epic**, then  $f^{\text{op}}$  is **split monic**, thus it is **preserved** by the **functor**  $F^{\text{op}}$ . And  $Ff = (F^{\text{op}}(f^{\text{op}}))^{\text{op}}$  is **split epic**.
4. **Functors preserve isomorphisms** because a **morphism** is an **isomorphism** if and only if it is **split epic** and **split monic**.<sup>223</sup> If  $A \cong B$  and  $i : A \rightarrow B$  is an **isomorphism**, then  $Fi : FA \rightarrow FB$  is an **isomorphism**, so  $FA \cong FB$ .

□

<sup>223</sup> Because split epic is equivalent to having a left inverse and split monic is equivalent to having a right inverse.

*Solution to Exercise 132.* 1. Let  $\mathbf{C}$  be a **category** with at least one **morphism**  $f$  that is not **monic**, the only **functor**  $[] : \mathbf{C} \rightsquigarrow \mathbf{1}$  sends  $f$  to  $\text{id}_\bullet$  which is **monic**.

2. Suppose that  $F(f)$  is **monic** and let  $g$  and  $h$  be such that  $f \circ g = f \circ h$ . By **monicity** of  $F(f)$ ,  $F(f) \circ F(g) = F(f) \circ F(h)$  implies  $F(g) = F(h)$ . Since  $F$  is **faithful**,  $g = h$ .
3. We need to show **faithful functors reflect epimorphisms**.

□

*Solution to Exercise 133.* Let us have three **monomorphisms**  $m : Y \hookrightarrow X$ ,  $n : Z \hookrightarrow X$  and  $o : W \hookrightarrow X$ .

**Reflexivity:** We have  $m \circ \text{id}_Y = m$  thus  $m \sim m$ .

**Symmetry:** Suppose that  $m \sim n$ , namely, there is an **isomorphism**  $i : Y \rightarrow Z$  such that  $m = n \circ i$ . Then, **pre-composing** with the **isomorphism**  $i^{-1}$  yields  $m \circ i^{-1} = n$  which implies  $n \sim m$ .

**Transitivity:** If  $m \sim n$  and  $n \sim o$ , then there exist **isomorphisms**  $i : Y \rightarrow Z$  and  $i' : Z \rightarrow W$  satisfying  $m = n \circ i$  and  $n = o \circ i'$ . Therefore, we have  $m = o \circ i' \circ i$  which implies  $m \sim o$ .<sup>224</sup>

□

*Solution to Exercise 136.* Let us have five **monomorphisms**  $m : Y \hookrightarrow X$ ,  $m' : Y' \hookrightarrow X$ ,  $n : Z \hookrightarrow X$ ,  $n' : Z' \hookrightarrow X$  and  $o : W \hookrightarrow X$ .<sup>225</sup>

**Well-defined:** Suppose that  $m \leq n$ ,  $m' \sim m$  and  $n \sim n'$ , namely, there is a **morphism**  $k : Y \rightarrow Z$  and **isomorphisms**  $i : Y \rightarrow Y'$  and  $i' : Z' \rightarrow Z$  such that  $m = n \circ k$ ,  $m' = m \circ i$  and  $n = n' \circ i'$ . Combining these equalities yields  $m' = n' \circ i' \circ k \circ i$  which witnesses  $m' \leq n'$ .

**Reflexivity:** We have  $m \circ \text{id}_Y = m$  thus  $m \leq m$ .

<sup>224</sup> Recall that the **composition** of two **isomorphisms** is an **isomorphism**.

<sup>225</sup> Recall that we often use  $m$  to refer to  $[m]$ .

**Antisymmetry:** If  $m \leq n$  and  $n \leq m$ , then there exist **morphisms**  $k : Y \rightarrow Z$  and  $k' : Z \rightarrow Y$  satisfying  $m = n \circ k$  and  $n = m \circ k'$ . Combining these two equalities yield  $m = m \circ k' \circ k$  and  $n = n \circ k \circ k'$ . Therefore, since  $m$  and  $n$  are **monic**, we infer that  $k' \circ k = \text{id}_Y$  and  $k \circ k' = \text{id}_Z$ . This means  $k$  is an **isomorphism** and  $m \sim n$  (so  $[m] = [n]$ ).

**Transitivity:** If  $m \leq n$  and  $n \leq o$ , then there exist **morphisms**  $k : Y \rightarrow Z$  and  $k' : W \rightarrow Z$  satisfying  $m = n \circ k$  and  $n = o \circ k'$ . Therefore, we have  $m = o \circ k' \circ k$  which implies  $m \leq o$ .  $\square$

## Solutions to Chapter

*Solution to Exercise 146.* The existence and uniqueness of  $\prod_{i \in I} f_i$  is given by the **universal property** of the **product**  $\prod_{i \in I} Y_i$  with for each  $j \in I$ , the **morphism**  $f_j \circ \pi_j : \prod_{i \in I} X_i \rightarrow Y_j$ .  $\square$

*Solution to Exercise 164.*  $(\Rightarrow)$  Suppose  $f : X \rightarrow Y$  is **monic**, **commutativity** of (34) is trivial. For any  $X \xleftarrow{g} Z \xrightarrow{h} X$  satisfying  $f \circ g = f \circ h$ , we have  $g = h$ . Thus  $g = h$  is the **mediating morphism**  $!$  of (175), it is unique because  $\text{id}_X \circ m = g$  implies  $m = g$ .

$(\Leftarrow)$  For any  $g, h : Z \rightarrow X$  satisfying  $f \circ g = f \circ h$ , the **universal property** of the **pullback** tells us there is a unique  $! : Z \rightarrow X$  making (175) **commute**. Since  $!$  satisfies  $g = \text{id}_X \circ ! = h$ , we conclude  $g = ! = h$ , thus  $f$  is a **monomorphism**.

The **dual** statement is that  $f : X \rightarrow Y$  is **epic** if and only if (176) is a **pushout**. We leave the proof to you.  $\square$

*Solution to Exercise 183.* Let  $p_A : X \rightarrow A$  and  $p_B : X \rightarrow B$  be such that (177) **commutes**. A **mediating morphism**  $! : X \rightarrow A$  must satisfy  $\text{id}_A \circ ! = p_A$  and  $f \circ ! = p_B$ . The first equality ensures  $! = p_A$  is unique and satisfies the second equality because the outer square **commuting** yields  $f \circ p_A = p_B$ .

$$\begin{array}{ccc} X & \xrightarrow{p_A} & A \\ & \searrow p_B & \downarrow f \\ & B & \xrightarrow{\text{id}_B} B \end{array} \quad (177)$$

$$\begin{array}{ccc} X & \xrightarrow{p_A} & A \\ & \searrow p_B & \downarrow f \circ i \\ & B & \xrightarrow{\text{id}_B} B \end{array} \quad (178)$$

Let  $p_A : X \rightarrow A$  and  $p_B : X \rightarrow B$  be such that (178) **commutes**. A unique **mediating morphism**  $! : X \rightarrow A$  must satisfy  $i \circ ! = p_A$  and  $f \circ i \circ ! = p_B$ . **Post-composing** the first equality by  $i^{-1}$  implies  $! = i^{-1} \circ p_A$  is unique and satisfies the second equality because  $f \circ i \circ i^{-1} \circ p_A = f \circ p_A = p_B$ .  $\square$

*Solution to Exercise 188.* We will show that if  $\mathbf{C}$  has all **pullbacks** and a **terminal object**, then it has all finite **products** and **equalizers**. This implies, using Remark 186, that  $\mathbf{C}$  is **finitely complete**.

For finite **products**, recall that it is enough to show that  $\mathbf{C}$  has all **binary products** as it already has the empty **product** (the **terminal object**). We claim that the **pullback** of  $A \xrightarrow{\pi_A} \mathbf{1} \xleftarrow{\pi_B} B$  is the **binary product**  $A \times B$ .

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ & \searrow g & \downarrow \text{id}_X \\ & X & \xrightarrow{f} Y \end{array} \quad (175)$$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f \downarrow & \lrcorner & \downarrow \text{id}_Y \\ Y & \xrightarrow{\text{id}_Y} & Y \end{array} \quad (176)$$

$$\begin{array}{ccc} A \times B & \xrightarrow{\pi_B} & B \\ \pi_A \downarrow & \lrcorner & \downarrow \pi_B \\ A & \xrightarrow{\pi_A} & \mathbf{1} \end{array} \quad (179)$$

Indeed, for any  $A \xleftarrow{p_A} X \xrightarrow{p_B} B$ , we have  $\square \circ p_A = \square \circ p_B$ , thus, there is a unique **morphism**  $! : X \rightarrow A \times_1 B$  making (180) **commute**. Since the **commutativity** of the squares always hold, this is equivalent to the **universal property** of the **binary product**. Hence  $A \times B \cong A \times_1 B$ .

$$\begin{array}{ccccc}
 X & & & & \\
 \swarrow p_A & \dashrightarrow ! & \searrow p_B & & \\
 & A \times_1 B & \xrightarrow{\pi_B} & B & \\
 \searrow \pi_A & \downarrow \square & & \downarrow \square & \\
 & A & \xrightarrow{\quad} & 1 &
 \end{array}
 \quad (180)$$

□

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*Solution to Exercise 195.* We define  $- \times X$  on **morphisms** by sending  $f : Y \rightarrow Y' \in \mathbf{C}_1$  to  $f \times \text{id}_X : Y \times X \rightarrow Y' \times X$ . **Functoriality** follows from the definition of  $\times$  on **morphisms**. Indeed,  $\text{id}_Y \times \text{id}_X$  is the only **morphism** making (181) **commute** and  $(g \circ f) \times \text{id}_X$  is the only **morphism** making (182) **commute**.

$$\begin{array}{ccc}
 Y & \xleftarrow{\pi_Y} & Y \times X \\
 \text{id}_Y \downarrow & & \downarrow \text{id}_Y \times \text{id}_X \\
 Y & \xleftarrow{\pi_Y} & Y \times X
 \end{array}
 \quad (181)$$

$$\begin{array}{ccccc}
 Y & \xleftarrow{\pi_Y} & Y \times X & & \\
 \downarrow f & & \downarrow f \times \text{id}_X & & \\
 Y' & \xleftarrow{\pi_{Y'}} & Y' \times X & & \\
 \downarrow g & & \downarrow g \times \text{id}_X & & \\
 Y'' & \xleftarrow{\pi_{Y''}} & X & &
 \end{array}
 \quad (182)$$

□

*Solution to Exercise 197.* First, we know that the **pullback** of the **monomorphism**  $m$  **along**  $f$  is **monic** by Theorem 179. Next, for  $n : I' \hookrightarrow X \in \text{Sub}_{\mathbf{C}}(Y)$ , we need to show  $[m] = [n]$  implies  $[f^*(m)] = [f^*(n)]$ .<sup>226</sup> In (183), we need to show there is an **isomorphism**  $i' : J \rightarrow J'$  making everything **commute**.

$$\begin{array}{ccccc}
 J' & \xrightarrow{j'} & I' & & \\
 \uparrow f^*(n) & & \uparrow i & & \uparrow n \\
 J & \xrightarrow{j} & I & & \\
 \searrow f^*(m) & & \searrow m & & \downarrow \\
 X & \xrightarrow{f} & Y & &
 \end{array}
 \quad (183)$$

By the **pullback** property of  $J'$ , there is a unique **mediating morphism**  $i' : J \rightarrow J'$  **commuting** with (183).<sup>227</sup> Similarly, the **pullback** property of  $J$ , there is a unique **mediating morphism**  $i'^{-1} : J' \rightarrow J$  **commuting** with (183).<sup>228</sup> The fact that  $i'$  and

Recall that if  $f : A \rightarrow A'$  and  $g : B \rightarrow B'$ ,  $f \times g : A \times B \rightarrow A' \times B'$  is the unique **morphism** making the diagram below **commute**:

$$\begin{array}{ccccc}
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B \\
 f \downarrow & & \downarrow f \times g & & \downarrow g \\
 A' & \xleftarrow{\pi_{A'}} & A' \times B' & \xrightarrow{\pi_{B'}} & B'
 \end{array}$$

<sup>226</sup> Recall that  $[m] = [n]$  when there is an **isomorphism**  $i$  satisfying  $n = m \circ i$ .

<sup>227</sup> Use the fact that  $n \circ i^{-1} \circ j = m \circ j = f \circ f^*(m)$ .

<sup>228</sup> Use the fact that  $m \circ i \circ j' = n \circ j' = f \circ f^*(n)$ .

$i'^{-1}$  are inverses follows from viewing  $i'^{-1} \circ i'$  as a **mediating morphism** from the **pullback**  $J$  to itself which must be the identity by uniqueness. Similarly for  $i' \circ i'^{-1}$ .

For **functoriality** of  $\text{Sub}_{\mathbf{C}}$ , we need to show  $\text{id}^*(m) = m$  and  $g^*(f^*(m)) = f \circ g^*(m)$ . The first equality follows from Exercise 183 and the second from the **pasting lemma**.  $\square$

*Solution to Exercise 203.* The **terminal object** of  $\mathbf{C}/X$  is the **identity morphism**  $\text{id}_X : X \rightarrow X$ . For any **object** of the **slice category**  $f : A \rightarrow X$ , we have the **commutative triangle** (184) with  $! = f$ . Uniqueness of  $!$  follows from  $\text{id}_X \circ ! = f \implies ! = f$ .

The **dual** statement is that  $\text{id}_X$  is the **initial object** of  $X/\mathbf{C}$ .  $\square$

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & X \\ & \searrow f & \swarrow \text{id}_X \\ & X & \end{array} \quad (184)$$

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*Solution to Exercise 210.*  $(\implies)$  For any  $g : Y \rightarrow Y'$ , the **naturality** of  $\phi$  yields this **commutative square**.

$$\begin{array}{ccc} F(X, Y) & \xrightarrow{\phi_{X,Y}} & G(X, Y) \\ F(X, g) = F(\text{id}_X, g) \downarrow & & \downarrow G(\text{id}_X, g) = G(X, g) \\ F(X, Y') & \xrightarrow{\phi_{X,Y'}} & G(X, Y') \end{array} \quad (185)$$

We conclude that  $\phi_{X,-}$  is a **natural transformation**  $F(X, -)$ . A symmetric argument works for  $\phi_{-,Y}$  (see (186)).

$(\Leftarrow)$  For any  $(f, g) : (X, Y) \rightarrow (X', Y')$ , we note that, by **functoriality**,  $F(f, g) = F(f, \text{id}_{Y'}) \circ F(\text{id}_X, g)$  and similarly for  $G$ . Thus, we can combine the **naturality** of  $\phi_{X,-}$  and  $\phi_{-,Y}$  to obtain the **commutativity** of  $\phi_{X,Y}$  as shown in (187).

$$\begin{array}{ccc} F(X, Y) & \xrightarrow{\phi_{X,Y}} & G(X, Y) \\ F(f, \text{id}_{Y'}) \downarrow & & \downarrow G(f, \text{id}_{Y'}) \\ F(X', Y) & \xrightarrow{\phi_{X',Y}} & G(X', Y) \end{array} \quad (186)$$

$$\begin{array}{ccc} F(X, Y) & \xrightarrow{\phi_{X,Y}} & G(X, Y) \\ \left( \begin{array}{ccc} \downarrow F(\text{id}_X, g) & G(\text{id}_X, g) \downarrow \\ F(X, Y') & \xrightarrow{\phi_{X,Y'}} & G(X, Y') \\ \downarrow F(f, \text{id}_{Y'}) & G(f, \text{id}_{Y'}) \downarrow \\ F(X', Y') & \xrightarrow{\phi_{X',Y'}} & G(X', Y') \end{array} \right) & & G(f, g) \end{array} \quad (187)$$

$\square$

*Solution to Exercise 214.* Let  $F, G : \mathbf{C} \rightsquigarrow \mathbf{D}$  be **functors**.

$(\implies)$  If  $\phi : F \Rightarrow G$  is a **natural isomorphism**, then it has an **inverse**  $\phi^{-1} : G \Rightarrow F$  which satisfies  $\phi \cdot \phi^{-1} = \mathbb{1}_G$  and  $\phi^{-1} \cdot \phi = \mathbb{1}_F$ . Looking at each **components**, we find  $\phi_X \circ (\phi^{-1})_X = \text{id}_X$  and  $(\phi^{-1})_X \circ \phi_X = \text{id}_X$ , hence they are **isomorphisms**.

$(\Leftarrow)$  Let  $\phi : F \Rightarrow G$  be a **natural transformation** such that  $\phi_X$  is an **isomorphism** for each  $X \in \mathbf{C}_0$ . We claim that the family  $\phi_X^{-1}$  is the **inverse** of  $\phi$ . After we show that this family is a **natural transformation**  $G \Rightarrow F$ , the construction implies it is the **inverse** of  $\phi$ . For any  $f : X \rightarrow Y \in \mathbf{C}_1$ , the **naturality** of  $\phi$  implies  $\phi_Y \circ F(f) =$

$G(f) \circ \phi_X$ . **Pre-composing** with  $\phi_X^{-1}$ , we have  $G(f) = \phi_Y \circ F(f) \circ \phi_X^{-1}$  and therefore

$$\phi_Y^{-1} \circ G(f) = \phi_Y^{-1} \circ \phi_Y \circ F(f) \circ \phi_X^{-1} = F(f) \circ \phi_X^{-1}$$

yields the **naturality** of  $\phi^{-1}$ .  $\square$

*Solution to Exercise 224.* On **morphisms**, this **functor** must send a pair of **natural transformations**  $\eta : F \Rightarrow F'$  and  $\phi : G \Rightarrow G'$  to a **natural transformation**  $FG \Rightarrow F'G'$ . This is exactly what **horizontal composition** does.

To see that **horizontal composition** is **functorial**, first note that  $\mathbb{1}_F \diamond \mathbb{1}_G = \mathbb{1}_{FG}$ . Next, the fact that **horizontal composition** commutes with **composition** of **functors** is exactly the **interchange identity**.  $\square$

*Solution to Exercise 236.* We need to show that  $\simeq$  is reflexive, symmetric and transitive. Symmetry is trivial because the definition of  $\mathbf{C} \simeq \mathbf{D}$  is symmetric. Reflexivity follows from the fact that the **identity functor** on any **category** is **fully faithful** and **essentially surjective**.

For transitivity, given the **categories** and **functors** represented in (188) with **natural isomorphisms**  $\phi : FG \Rightarrow \text{id}_{\mathbf{D}}$ ,  $\psi : GF \Rightarrow \text{id}_{\mathbf{C}}$ ,  $\phi' : F'G' \Rightarrow \text{id}_{\mathbf{E}}$  and  $\psi' : G'F' \Rightarrow \text{id}_{\mathbf{D}}$ , we claim that the **composition**  $G \circ G'$  is the **quasi-inverse** of  $F' \circ F$ .

Since the **biaction** of **functors** preserves **natural isomorphisms**,<sup>229</sup> we have two **natural isomorphisms**

$$\phi' \cdot (F' \phi G') : F'FGG' \Rightarrow \text{id}_{\mathbf{E}} \text{ and } \psi \cdot (G \psi' F) : GG'F'F \Rightarrow \text{id}_{\mathbf{C}},$$

which shows  $\mathbf{C} \simeq \mathbf{E}$ .  $\square$

*Solution to Exercise 237.* We will show the following two implications

$$\mathbf{C} \simeq \mathbf{C}' \implies [\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}', \mathbf{D}]$$

$$\mathbf{D} \simeq \mathbf{D}' \implies [\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}, \mathbf{D}']$$

and infer that  $\mathbf{C} \simeq \mathbf{C}'$  and  $\mathbf{D} \simeq \mathbf{D}'$  implies

$$[\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}', \mathbf{D}] \simeq [\mathbf{C}', \mathbf{D}'].$$

For the first implication, let  $F : \mathbf{C} \rightsquigarrow \mathbf{C}'$  and  $G : \mathbf{C}' \rightsquigarrow \mathbf{C}$  be **quasi-inverses**. We define the **functor**  $(-)F : [\mathbf{C}', \mathbf{D}] \rightsquigarrow [\mathbf{C}, \mathbf{D}]$  that acts on **functors** by **pre-composition** and on **natural transformations** by the right action in Definition 217.<sup>230</sup> Similarly, we define the **functor**  $(-)G : [\mathbf{C}, \mathbf{D}] \rightsquigarrow [\mathbf{C}', \mathbf{D}]$ . We claim that  $(-)F$  and  $(-)G$  are **quasi-inverses**.

Let  $\Phi : GF \Rightarrow \text{id}_{\mathbf{C}}$  be a **natural isomorphism** witnessing  $F$  and  $G$  being **quasi-inverses**, then  $(-) \Phi$  is a **natural isomorphism** from  $(-)GF$  to  $\text{id}_{[\mathbf{C}, \mathbf{D}]}$ . Indeed, for any  $\phi : H \Rightarrow H' \in [\mathbf{C}, \mathbf{D}]_1$ , (189) **commutes** as the top path and bottom path are both equal to  $\phi \diamond \Phi$  and  $H\Phi$  is an **isomorphism** because  $\Phi$  is and **functors** preserve **isomorphisms**.

$$\begin{array}{ccc} HGF & \xrightarrow{H\Phi} & H \\ \phi GF \downarrow & & \downarrow \phi \\ H'GF & \xrightarrow{H'\Phi} & H' \end{array} \quad (189)$$

$$\begin{array}{ccccc} & F & & F' & \\ \mathbf{C} & \xrightarrow{\quad} & \mathbf{D} & \xrightarrow{\quad} & \mathbf{E} \\ & G & & G' & \end{array} \quad (188)$$

<sup>229</sup> This holds because acting on the left or right with a **functor** is a **functor**, part of this is shown in the next solution and it also follows from the previous exercise.

<sup>230</sup> i.e.:  $H : \mathbf{C} \rightsquigarrow \mathbf{D}$  is mapped to  $HF = H \circ F$  and  $\phi : H \Rightarrow H'$  is mapped to  $\phi F$ . **Functoriality** follows from the properties of the right action.

Another way to show **functoriality** is to recall that  $\phi F = \phi \diamond \mathbb{1}_F$  and hence  $(-)F$  is the **composition** of the **functor**

$$\text{id}_{[\mathbf{C}', \mathbf{D}]} \times F : [\mathbf{C}', \mathbf{D}] \times \mathbf{1} \rightsquigarrow [\mathbf{C}', \mathbf{D}] \times [\mathbf{C}, \mathbf{C}']$$

with the **horizontal composition functor** defined in Exercise 224.



We leave to you the symmetric argument showing  $(- )FG \cong \text{id}_{[C', D]}$  and the similar argument for the second implication.  $\square$

### Solutions to Chapter

*Solution to Exercise 248.*  $(\Rightarrow)$  Suppose there is a **natural isomorphism**  $\phi : \text{Hom}_{\mathbf{C}}(X, -) \Rightarrow \mathbf{1}$ , then for any **object**  $Y \in \mathbf{C}_0$ , there is a bijection  $\text{Hom}_{\mathbf{C}}(X, Y) \cong \{\star\}$ . Hence, there is a unique **morphism**  $X \rightarrow Y$ .

$(\Leftarrow)$  Suppose that  $X$  is **initial**, then for any  $Y \in \mathbf{C}_0$ , we have an **isomorphism**  $\phi_Y : \text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \mathbf{1}(Y)$  which sends the unique **morphism**  $X \rightarrow Y$  to  $\star$ . We need to show this family is **natural** in  $Y$ . Let  $f : Y \rightarrow Y' \in \mathbf{C}_1$ , (190) clearly **commutes** because all sets are singletons.  $\square$

$$\begin{array}{ccc} \text{Hom}_{\mathbf{C}}(X, Y) & \xrightarrow{\phi_Y} & \mathbf{1}(Y) \\ f \circ - \downarrow & & \downarrow \text{id}_{\mathbf{1}} \\ \text{Hom}_{\mathbf{C}}(X, Y') & \xrightarrow{\phi_{Y'}} & \mathbf{1}(Y') \end{array} \quad (190)$$

### Solutions to Chapter

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