ENS DE LYON

TD – Semantics and Verification

III– Topology

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TA: Ralph Sarkis
ralph.sarkis@ens-lyon.fr

In this set of exercises, we will discuss topological characterisations of safety and liveness properties.

Liveness Properties

Recall that an LTP is a liveness property if for all finite $\hat{\sigma} \in (2^{\mathrm{AP}})^*$, there is $\sigma \in P$ such that $\hat{\sigma} \subseteq \sigma$. Let

$$\mathrm{pref}(P) := \{\hat{\sigma} \text{ finite } \mid \exists \sigma \in P, \hat{\sigma} \subseteq \sigma\} \qquad \text{and} \qquad \mathrm{cl}(P) := \{\sigma \mid \mathrm{pref}(\sigma) \subseteq \mathrm{pref}(P)\}.$$

Exercise 1.

Show that

1. An LTP P is a liveness property if and only if $\mathrm{cl}(P) = (2^{\mathrm{AP}})^\omega$.
2. If P and Q are liveness properties, then so is $P \cup Q$.
3. There are two liveness properties P and Q such that $P \cap Q$ is not a liveness property.

Topological Spaces

- A *topological space* is a pair $(X, \mathcal{U})$ comprising a set X and a subset $\mathcal{U}$ of $\mathcal{P}(X)$, called the *open sets of* X, such that
  1. $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$;
  2. for any set I and family $\{U_i \in \mathcal{U}\}_{i \in I}$, also $\bigcup_{i \in I} U_i \in \mathcal{U}$; and
  3. for all $U, V \in \mathcal{U}$, also $U \cap V \in \mathcal{U}$.

  A set $U \in \mathcal{U}$ is called *open* and elements $x \in X$ are called points. If $\mathcal{U}$ is clear from the context, we often refer to X as the topological space.
- Given a point $x \in X$, we say that $N \subseteq X$ is a *neighbourhood* of x if there is an open set $U \in \mathcal{U}$, such that $x \in U$ and $U \subseteq N$. The collection of all neighbourhoods of x is denoted by $\mathcal{N}_x$.
- For any $F \subseteq X$, we denote by F^{c} the complement of F relative to X, i.e.

  $$F^{\mathsf{c}} = X \setminus F = \{x \in X \mid x \notin F\}.$$

  We say that $F \subseteq X$ is *closed*, if F^{c} is open.
- A subset $D \subseteq X$ is said to be dense if $D \cap U \neq \emptyset$ for all non-empty $U \in \mathcal{U}$.

Exercise 2.

For any topological space $(X, \mathcal{U})$, show that

1. $\emptyset$ and X are closed
2. for any set I and family $\{F_i \text{ closed}\}_{i \in I}$, also $\bigcap_{i \in I} F_i$ closed; and
3. for all closed F and G, also $F \cup G$ is closed.

For any set $A \subset X$, we define the *closure* $\overline{A}$ of A by

$$\overline{A} = \bigcap\{F \subseteq X \mid F \text{ closed and } A \subseteq F\}.$$

One can show that $\overline{A}$ is the smallest closed subset of X containing A.

Exercise 3.

For any topological space $(X,\mathcal{U})$, show that

1. A set $A \subseteq X$ is open iff for every $x \in A$ there is an open set $U \in \mathcal{U}$ such that $x \in U$ and $U \subseteq A$.
2. A set $A \subseteq X$ is closed iff for every $x \notin A$ there is an open set $U \in \mathcal{U}$ such that $x \in U$ and $U \cap A = \emptyset$.

Exercise 4.

Let $(X,\mathcal{U})$ be a topological space and $A \subseteq X$. An *adherent point* (or *point of closure*) of A is a point $x \in X$ such that for any neighborhood N of x, $N \in \mathcal{N}_x$.

1. Show that $\overline{A}$ is the set of adherent point of A, i.e. $\overline{A} = \{x \in X \mid \forall N \in \mathcal{N}_x.\, N \cap A \neq \emptyset\}$.
2. Show that A is closed iff $\overline{A} = A$.
3. Show A is dense iff $\overline{A} = X$.

Metric and Topology on Infinite Words

Exercise 5.

Let (X, d) be a metric space, the open ball of radius $\varepsilon \in [0, \infty)$ centered at $x \in X$ is denoted by $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$. The open ball topology associated to (X, d) is defined by

$$\mathcal{U} = \{U \subseteq X \mid \forall x \in U.\, \exists\varepsilon > 0.\, B_\varepsilon(x) \subseteq U\}.$$

1. Show that $(X,\mathcal{U})$ is indeed a topological space.
2. Show that for any $S \subseteq X$ that we have $\overline{S} = \{x \in X \mid \forall\varepsilon > 0.\, B_\varepsilon(x) \cap S \neq \emptyset\}$.

Exercise 6.

Let A be a non-empty set. The set of infinite sequences over A is denoted by A^ω as before. Let $d\colon A^\omega \times A^\omega \to \mathbb{R}_{\geq 0}$ be given by

$$d(\sigma, \tau) = \begin{cases} 0, & \sigma = \tau \\ 2^{-\min\{k \in \mathbb{N} \mid \sigma(k) \neq \tau(k)\}}, & \sigma \neq \tau \end{cases}$$

Let us also denote by $\sigma|_n$ the prefix of length n of σ. Show that (A^ω, d) is a metric space.

Exercise 7.

For A a non-empty set, we define

- $\mathrm{ext}(w) = \{\sigma \in A^\omega \mid w \subseteq \sigma\}$ for $w \in A^*$.
- $\mathrm{ext}(W) = \bigcup_{w \in W} \mathrm{ext}(w)$ for $W \subseteq A^*$.
- $\mathcal{U} = \{\mathrm{ext}(W) \mid W \subseteq A^*\}$.

Show that $(A^\omega,\mathcal{U})$ is a topological space.

Exercise 8.

1. Show that a set $P \subseteq A^\omega$ is open iff for every $\sigma \in P$ there is a finite word $\hat{\sigma} \subseteq \sigma$ such that $\mathrm{ext}(\hat{\sigma}) \subseteq P$.
2. Show that A set $P \subseteq A^\omega$ is closed iff for every $\sigma \notin P$ there is a finite word $\hat{\sigma} \subseteq \sigma$ such that $\mathrm{ext}(\hat{\sigma}) \cap P = \emptyset$.