

Lecture 2 - Duality and More Vocabulary

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Abstract

Through the exploration of duality and the presentation of more vocabulary, further familiarity with categories and functors will be built.

1 Duality

The concept of duality is ubiquitous throughout mathematics. It can relate two perspectives of the same object as for dual vector spaces, two complementary problems such as a maximization and a minimization linear program and even two seemingly unrelated fields like topology and logic (Stone dualities). While this vague principle of duality is the foundation of many groundbreaking results, the duality in question here is categorical duality and it is a bit more precise.

Informally, there is nothing more to say than “Take all the diagrams in a definition/theorem, reverse the arrows and reap the benefits of the dual concept/result.” The more formal version will follow after we first exhibit the principle in action.

Recall that, intuitively, a functor is a structure preserving transformation between categories. A simple example we gave last lecture was functors between posets that were order-preserving functions. However, as a consequence, one might conclude that order-reversing functions impair the structure of a poset, which feels arbitrary. The same happens between categories representing groups because anti-homomorphisms cannot arise as functors between such categories.

There are two options to remedy this discrepancy between intuition and formalism; both have duality as a guiding principle.

1.1 Contravariant functors

By modifying last week’s definition to require that $F(f)$ goes in the opposite direction, we obtain a **contravariant** functor. Incidentally, what we defined as a functor last week is more commonly called a **covariant** functor.

Definition 1 (Contravariant functor). Let C and D be categories, a **contravariant functor** $F : C \rightsquigarrow D$ is a pair of maps $F_0 : C_0 \rightarrow D_0$ and $F_1 : C_1 \rightarrow D_1$ such that the following diagrams commute (where F_2 is now induced by the definition of F_1 with $(f, g) \mapsto (F_1(g), F_1(f))$).

$$\begin{array}{ccccc}
C_0 & \xleftarrow{s} & C_1 & \xrightarrow{t} & C_0 \\
F_0 \downarrow & & \downarrow F_1 & & \downarrow F_0 \\
D_0 & \xleftarrow{t} & D_1 & \xrightarrow{s} & D_0
\end{array}
\quad
\begin{array}{ccc}
C_2 & \xrightarrow{F_2} & D_2 \\
\circ_C \downarrow & & \downarrow \circ_D \\
C_1 & \xrightarrow{F_1} & D_1
\end{array}
\quad
\begin{array}{ccc}
C_0 & \xrightarrow{F_0} & D_0 \\
u_C \downarrow & & \downarrow u_D \\
C_1 & \xrightarrow{F_1} & D_1
\end{array}$$

In words, F must satisfy the following properties.

- i. For any $A, B \in C_0$, if $f \in \text{Hom}_C(A, B)$ then $F(f) \in \text{Hom}_D(F(B), F(A))$.
- ii. If $f, g \in C_1$ are composable, then $F(f \circ g) = F(g) \circ F(f)$.
- iii. If $A \in C_0$, then $u_D(F(A)) = F(u_C(A))$.

Examples 2. Just like their covariant counterparts, contravariant functors are quite numerous. Here are a few simple ones, we leave you to check that they satisfy the diagrams above.

1. It is easy to verify that contravariant functors $F : (X, \leq) \rightsquigarrow (Y, \subseteq)$ correspond to order-reversing functions between the posets while contravariant functors $F : G* \rightsquigarrow H*$ correspond to anti-homomorphisms between the groups.
2. The contravariant powerset functor $\hat{\mathcal{P}} : \mathbf{Set} \rightsquigarrow \mathbf{Set}$ sends a set X to its power set $\mathcal{P}(X)$ and a function $f : X \rightarrow Y$ to the pre-image map $\mathcal{P}(f) : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$, the latter sends a subset $S \subseteq Y$ to

$$f^{-1}(S) = \{x \in X \mid f(x) \in S\} \subseteq X.$$

Next, there is a couple of functors that are key to understand the philosophy put forward by category theory (we will talk more about it when covering the Yoneda lemma).

Example 3 (Hom functors). Let C be a locally-small category and $A \in C_0$ one of its object. We define the covariant and contravariant Hom functors from C to \mathbf{Set} .

1. The functor $\text{Hom}_C(A, -) : C \rightsquigarrow \mathbf{Set}$ sends an object $B \in C_0$ to the Hom-set $\text{Hom}_C(A, B)$ and a morphism $f : B \rightarrow B'$ to the function

$$\text{Hom}_C(A, f) : \text{Hom}_C(A, B) \rightarrow \text{Hom}_C(A, B') = g \mapsto f \circ g.$$

This function is called post-composition by f and is sometimes denoted $f \circ (-)$. Let us show $\text{Hom}_C(A, -)$ is a covariant functor.

- i. For any $f \in C_1$, it is clear from the definitions that

$$\text{Hom}_C(A, s(f)) = s(\text{Hom}_C(A, f)) \text{ and } \text{Hom}_C(A, t(f)) = t(\text{Hom}_C(A, f)).$$

- ii. For any $(f_1, f_2) \in C_2$, we claim that

$$\text{Hom}_C(A, f_1 \circ f_2) = \text{Hom}_C(A, f_1) \circ \text{Hom}_C(A, f_2).$$

In the L.H.S., an element $g \in \text{Hom}_C(A, s(f_1 \circ f_2))$ is mapped to $(f_1 \circ f_2) \circ g$ and in the R.H.S., an element $g \in \text{Hom}_C(A, s(f_2))$ is mapped to $f_1 \circ (f_2 \circ g)$. Since $s(f_1 \circ f_2) = s(f_2)$ and composition is associative, we conclude that the two maps are the same.

- iii. For any $A \in C_0$, the post-composition by $u_C(A)$ is defined to be the identity, hence the third diagram also commutes.
- 2. The functor $\text{Hom}_C(-, A) : C \rightsquigarrow \mathbf{Set}$ sends an object $B \in C_0$ to the Hom-set $\text{Hom}_C(B, A)$ and a morphism $f : B \rightarrow B'$ to the function

$$\text{Hom}_C(f, A) : \text{Hom}_C(B', A) \rightarrow \text{Hom}_C(B, A) = g \mapsto g \circ f.$$

This function is called pre-composition by f and is sometimes denoted $(-) \circ f$. Let us show $\text{Hom}_C(-, A)$ is a contravariant functor.

- i. For any $f \in C_1$, it is clear from the definitions that

$$\text{Hom}_C(s(f), A) = t(\text{Hom}_C(f, A)) \text{ and } \text{Hom}_C(t(f), A) = s(\text{Hom}_C(f, A)).$$

- ii. For any $(f_1, f_2) \in C_2$, we claim that

$$\text{Hom}_C(f_1 \circ f_2, A) = \text{Hom}_C(f_2, A) \circ \text{Hom}_C(f_1, A).$$

In the L.H.S., an element $g \in \text{Hom}_C(t(f_1 \circ f_2), A)$ is mapped to $g \circ (f_1 \circ f_2)$ and in the R.H.S., an element $g \in \text{Hom}_C(t(f_1), A)$ is mapped to $(g \circ f_1) \circ f_2$. Since $t(f_1 \circ f_2) = t(f_1)$ and composition is associative, we conclude that the two maps are the same.

- iii. For any $A \in C_0$, the pre-composition by $u_C(A)$ is defined to be the identity, hence the third diagram also commutes.

1.2 Opposite Category

Another way to deal with order-reversing maps $(X, \leq) \rightarrow (Y, \subseteq)$ is to consider the reverse order on X and a covariant functor $(X, \geq) \rightsquigarrow (Y, \subseteq)$. This also works for anti-homomorphisms by constructing the opposite group G^{op} in which the operation is reversed, namely $g \cdot^{\text{op}} h = hg$. The opposite category is a generalization of these constructions.

Definition 4 (Opposite category). Let C be a category, we denote the **opposite category** with C^{op} and define it by

$$C_0^{\text{op}} = C_0, C_1^{\text{op}} = C_1, s^{\text{op}} = t, t^{\text{op}} = s, u_{C^{\text{op}}} = u_C$$

with the composition defined by $f^{\text{op}} \circ^{\text{op}} g^{\text{op}} = (g \circ f)^{\text{op}}$. This canonically leads to the following contravariant functor $(-)^{\text{op}}_C : C \rightsquigarrow C^{\text{op}}$ which sends an object A to A^{op} and a morphism f to f^{op} . It is called the **opposite functor**.

Remark 5. This definition leads to seeing contravariant functors as covariant functors. Formally, let $F : C \rightsquigarrow D$ be a contravariant functor, we can view F as covariant functor from C^{op} to D or from C to D^{op} via the compositions $F \circ (-)^{\text{op}}_{C^{\text{op}}}$ and $(-)^{\text{op}}_D \circ F$ respectively. Note that the $^{\text{op}}$ notation here is just used to distinguish elements in C and C^{op} but the class of objects and morphisms are the same.

Examples 6. As hinted at before, the category corresponding to (X, \geq) is the opposite category of (X, \leq) and $(G^*)^{\text{op}}$ is the category corresponding to the opposite group of G . While there are other interesting examples, the opposite construction is usually used implicitly to avoid dealing with contravariant functors or to avoid proving the dual of an already proven result.

Let us continue to illustrate how duality can be useful with some simple definitions and results.

Definition 7 (Monomorphism). Let C be a category, a morphism $f \in C_1$ is said to be **monic** (or a **monomorphism**) if for any $(f, g), (f, h) \in C_2$ where g and h have the same source, $f \circ g = f \circ h$ implies $g = h$. Equivalently, f is monic if $g = h$ whenever the following diagram commutes.

$$\begin{array}{ccccc} & & g & & \\ & \nearrow & & \searrow & \\ \bullet & & & & \bullet \\ & \nwarrow & & \nearrow & \\ & & h & & \end{array} \quad \xrightarrow{f} \quad \bullet$$

Proposition 8. Let C be a category and $f : A \rightarrow B$ a morphism, if there exists $f' : B \rightarrow A$ such that $f' \circ f = \text{id}_A$, then f is a monomorphism.

Proof. If $f \circ g = f \circ h$, then $f' \circ f \circ g = f' \circ f \circ h$ implying $g = h$. \square

Proposition 9. Let C be a category and $(f_1, f_2) \in C_2$, if $f_1 \circ f_2$ is a monomorphism, then f_2 is a monomorphism.

Proof. Let $g, h \in C_1$ be such that $f_2 \circ g = f_2 \circ h$, we immediately get that $(f_1 \circ f_2) \circ g = (f_1 \circ f_2) \circ h$. Since $f_1 \circ f_2$ is a monomorphism, this implies $g = h$. \square

It is now obvious that monomorphisms are analogous to injective functions and we will see that they are exactly the same in the category **Set**, but first let us introduce the dual concept.

Definition 10 (Epimorphism). Let C be a category, a morphism $f \in C_1$ is said to be **epic** (or an **epimorphism**) if for any two morphisms $(g, f), (h, f) \in C_2$ where g and h have the same target, $g \circ f = h \circ f$ implies $g = h$. Equivalently, f is epic if $g = h$ whenever the following diagram commutes.

$$\begin{array}{ccccc} & & g & & \\ & \nearrow & & \searrow & \\ \bullet & & & & \bullet \\ & \nwarrow & & \nearrow & \\ & & h & & \end{array} \quad \xleftarrow{f} \quad \bullet$$

The duals to Propositions 8 and 9 also hold. Although doing the straightforward proofs is very easy, the two next proofs rely on duality and convey the general sketch that works anytime a dual result needs to be proven.

Proposition 11. Let C be a category and $f : A \rightarrow B$ a morphism, if there exists $f' : B \rightarrow A$ such that $f \circ f' = \text{id}_B$, then f is epic.

Proof. Observe that f is epic in C if and only if f^{op} is monic in C^{op} (reverse the arrows in the definition). Moreover, by definition,

$$f'^{\text{op}} \circ f^{\text{op}} = (f \circ f')^{\text{op}} = \text{id}_B^{\text{op}} = \text{id}_{B^{\text{op}}},$$

so by the result for monomorphisms, f^{op} is monic and hence f is epic. \square

Proposition 12. *Let C be a category and $(f_1, f_2) \in C_2$, if $f_1 \circ f_2$ is epic, then f_2 is epic.*

Proof. Since $f_2^{\text{op}} \circ f_1^{\text{op}} = (f_1 \circ f_2)^{\text{op}}$ is monic, the result for monomorphisms implies f_2^{op} is monic and hence f_2 is epic. \square

Example 13 (Set).

- A function $f : A \rightarrow B$ is a monomorphism if and only if it is injective:
 (\Leftarrow) Since f is injective, it has a left inverse, so it is epic by Proposition 8.
 (\Rightarrow) For any $a \in A$, let $g_a : \{1\} \rightarrow A$ be the function sending 1 to a . Fix $a_1, a_2 \in A$, the functions g_{a_1} and g_{a_2} are different, hence $f \circ g_{a_1} \neq f \circ g_{a_2}$. Therefore, $f(a_1) \neq f(a_2)$ and since a_1 and a_2 were arbitrary, f is injective.
- A function $f : A \rightarrow B$ is an epimorphism if and only if it is surjective:
 (\Leftarrow) Since f is surjective, it has a right inverse, so it is epic by Proposition 11.
 (\Rightarrow) Let h be the constant function at 1 and g be the indicator function of $\text{Im}(f) \subseteq B$, namely,

$$g(x) = \begin{cases} 1 & \exists a \in A, x = f(a) \\ 0 & \text{o/w} \end{cases}.$$

It is clear that $g \circ f = h \circ f$ and since f is epic, it implies $g = h$. Thus, any element of B is in the image of f , that is f is surjective.

Example 14 (Mon). Inside the category **Mon** where objects are monoids and morphisms are monoid homomorphisms, the monomorphisms correspond exactly to injective homomorphisms.

(\Rightarrow) Let $f : M \rightarrow M'$ be an injective homomorphism and $g_1, g_2 : N \rightarrow M$ be two parallel homomorphisms. Suppose that $f \circ g_1 = f \circ g_2$, then for all $x \in N$, $f(g_1(x)) = f(g_2(x))$, so by injectivity of f , $g_1(x) = g_2(x)$. Therefore $g_1 = g_2$ and since g_1 and g_2 were arbitrary, f is a monomorphism.

(\Leftarrow) Let $f : M \rightarrow M'$ be a monomorphism. Let $x, y \in M$ and define $p_x : \mathbb{N} \rightarrow M$ by $k \mapsto x^k$ and similarly for p_y . It is trivial to show that p_x and p_y are homomorphisms. If $f(x) = f(y)$, then, by the homomorphism property, for all $k \in \mathbb{N}$

$$f(p_x(k)) = f(x^k) = f(x)^k = f(y)^k = f(y^k) = f(p_y(k)).$$

In other words, we get $f \circ p_x = f \circ p_y$, so $p_x = p_y$ and $x = y$. We conclude that f is injective.

Conversely, an epimorphism is not necessarily surjective. For example, the inclusion homomorphism $i : \mathbb{N} \rightarrow \mathbb{Z}$ is clearly not surjective but it is an epimorphism. Indeed, let $g, h : \mathbb{Z} \rightarrow M$ be two monoid homomorphisms satisfying $g \circ i = h \circ i$. In particular, $g(n) = h(n)$ for any $n \in \mathbb{N} \subset \mathbb{Z}$. It remains to show that also $g(-n) = h(-n)$, but if it were not the case for some n , $g(n)$ would have two left inverses $g(-n)$ and $h(-n)$ which is not possible. Hence, $g = h$ and i is an epimorphism.

Definition 15 (Isomorphism). Let C be a category, a morphism $f : A \rightarrow B$ is said to be an **isomorphism** if there exists a morphism $f^{-1} : B \rightarrow A$ such that $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$.

Remark 16. The results shown about monic and epic morphisms imply that any isomorphism is monic and epic. However, the converse is not true as witnessed by the inclusion morphism i described in the example above. If there exists an isomorphism between two objects A and B , then they are **isomorphic**, denoted $A \cong B$. Isomorphic objects are also isomorphic in the opposite category as is easy to verify.

Definition 17 (Initial object). Let C be a category, an object $A \in C_0$ is said to be **initial** if for any $B \in C_0$, $|\text{Hom}_C(A, B)| = 1$, namely there are no two parallel morphisms with source A and every object has a morphism coming from A .

Definition 18 (Terminal object). Let C be a category, an object $A \in C_0$ is said to be **terminal** (or **final**) if for any $B \in C_0$, $|\text{Hom}_C(B, A)| = 1$, namely there are no two parallel morphisms with target A and every object has a morphism going to A .

It is clear that an object is initial in a category C if and only if it is terminal in C^{op} . Also, if an object is initial and terminal, we say it is a **zero** object and usually denote it 0.

Examples 19. Here are examples of categories where initial and terminal objects may or may not exist.

1. \nexists terminal, \nexists initial: Consider the poset (\mathbb{N}, \geq) represented by the diagram below. It is clear that 0 is terminal and no element can be initial because $0 \geq x$ implies $x = 0$.

$$\overset{0}{\bullet} \longleftarrow \overset{1}{\bullet} \longleftarrow \overset{2}{\bullet} \longleftarrow \dots$$

2. \nexists terminal, \exists initial: The category **GrpInj** where the objects are groups and the morphisms are injective homomorphisms only contains an initial object $\{1\}$. Indeed, an injective homomorphism $G \rightarrow H$ can be seen as subgroup of H isomorphic to G . The identity group $\{1\}$ can only be isomorphic to the identity subgroup as any other element has degree more than 1, so $\{1\}$ is initial. Moreover, any group G cannot be terminal as $G \times (\mathbb{Z}/2\mathbb{Z})$ cannot be isomorphic to any subgroup of G .
3. \nexists terminal, \nexists initial: Let G be a non trivial group. The category G_* has a single object $*$ with $\text{Hom}_{G_*}(*, *) = G$ and the composition rule being the multiplication in G . The only object $*$ cannot be initial nor terminal as $|\text{Hom}_{G_*}(*, *)| > 1$.

The category whose objects are fields and morphisms are field homomorphisms also has no initial nor terminal objects because there are no field homomorphisms between fields of different characteristics.

4. \exists terminal, \exists initial: Let X be a topological space where τ is the collection of open sets (recall that it must contain \emptyset and X). Let T_X be the category representing the

poset of open sets with inclusion as the relation. Namely, the objects are the open sets and for any two open sets $U, V \in \tau$,

$$\text{Hom}_{T_X}(U, V) = \begin{cases} i_{U,V} & U \subseteq V \\ \emptyset & U \not\subseteq V \end{cases}$$

Since the empty set is contained in every open set, it is an initial object. Since the full set X contains every open set, it is a terminal object. No other set can be initial as it cannot be contained in \emptyset nor be terminal as it cannot contain X . Moreover, note that the two objects are not isomorphic as $\text{Hom}_{T_X}(X, \emptyset) = \emptyset$.

Proposition 20. *Let C be a category and $A, B \in C_0$ be initial, then $A \cong B$.*

Proof. Let f be the single element in $\text{Hom}_C(A, B)$ and f' be the single element in $\text{Hom}_C(B, A)$. Since the identity morphisms are the only elements of $\text{Hom}_C(A, A)$ and $\text{Hom}_C(B, B)$, $f' \circ f$ and $f \circ f'$, belonging to these sets, must be the identities. In other words f and f' are inverses, thus $A \cong B$. \square

The dual result follows.

Proposition 21. *Let C be a category and $A, B \in C_0$ be terminal, then $A \cong B$.*