TD – Semantics and Verification

V– More Topology on Infinite Words

Friday 18th February 2022

TA: Ralph Sarkis
ralph.sarkis@ens-lyon.fr

We recall that

- Given sets X and Y and a function $f : X \to Y$, we define $f^{-1}(B) = \{x \mid f(x) \in B\}$ for $B \subseteq Y$.
- Given topological spaces X and Y, a function $X \to Y$ is continuous iff $f^{-1}(B)$ is open whenever B is open.
- A clopen is a set that is both open and closed.
- For a set A, a property $P \subseteq (2^A)^\omega$ is said obversable if P is a clopen.
- For a topological space $(X,\mathcal{U})$, a set $A \subseteq X$ is compact if every open cover of A has a finite subcover. If X is compact, $(X,\mathcal{U})$ is called a compact space.
- A topological space $(X,\mathcal{U})$ is Hausdorff if for any distinct points $x, y \in X$, there are disjoint opens U, V such that $x \in U$ and $y \in V$.

Exercise 1.
Show that $P \subseteq A^\omega$ is a liveness property if and only if it is dense.

Exercise 2.
For the following subsets $P_i \subseteq A^\omega$: (i) is P_i open? closed? dense?, (ii) compute the closure of P_i, (iii) compute the decomposition of P_i as the intersection of a closed and a dense subset of A^ω.

1. Let $A = \{\mathtt{a}, \mathtt{b}\}$.
   (a) $P_1 = \{\mathtt{b} \cdot \sigma \mid \sigma \in A^\omega\}$
   (b) $P_2 = \{\sigma \in A^\omega \mid \sigma(0) = \mathtt{a} \wedge \exists i, \sigma(i) = \mathtt{b}\}$
2. Let $A = \{\mathtt{a}, \mathtt{b}\}$.
   (a) $P_3 = \{\mathtt{a}^\omega\}$
   (b) $P_4 = \{\sigma \in A^\omega \mid \exists! i, \sigma(i) = \mathtt{b}\}$
3. Let $A = \{\mathtt{a}, \mathtt{b}\}$.
   (a) $P_5 = \{\sigma \in A^\omega \mid \exists^\infty i, \sigma(i) = \mathtt{b}\}$
   (b) $P_6 = \{\sigma \in A^\omega \mid \forall^\infty i, \sigma(i) = \mathtt{a}\}$
4. Let $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.
   (a) $P_7 = \{\sigma \in A^\omega \mid \exists^\infty i, \sigma(i) = \mathtt{a}$ and there are arbitrarily large consecutive sequences of $\mathtt{b}$s$\}$
   (b) $P_8 = \{\sigma \in A^\omega \mid \exists^\infty i, \sigma(i) = \mathtt{a} \wedge \exists k > i, \sigma(k) = \mathtt{b}\}$

Exercise 3.
Show that $f : A^\omega \to B^\omega$ is continuous iff

$$\forall n \in \mathbb{N},\ \forall \alpha \in A^\omega,\ \exists k \in \mathbb{N},\ \forall \beta \in A^\omega \Big(\beta(0) \cdots \beta(k) = \alpha(0) \cdots \alpha(k) \quad \Rightarrow \quad f(\beta)(0) \cdots f(\beta)(n) = f(\alpha)(0) \cdots f(\alpha)(n)\Big)$$

Exercise 4.

Let A be a set.

1. Show that $\mathrm{ext}(u)$ is a clopen for every $u \in A^*$.
2. Show that for every finite subset $U \subseteq A^*, \mathrm{ext}(U)$ is a clopen.
3. Let $A = \mathbb{N}$. Show that there is no finite $U \subseteq \mathbb{N}^*$ such that $\mathrm{ext}(U) = \mathrm{ext}(\mathbb{N}_{>0})$.

Exercise 5.

Show that a closed subset of a compact space is compact.

Exercise 6.

Show that if A is infinite, then A^ω is not compact.

Exercise 7.

Let AP be a finite set. Show that $P \subseteq (2^{\mathrm{AP}})^\omega$ is observable iff there is a finite $W \subseteq (2^{\mathrm{AP}})^*$ such that $P = \mathrm{ext}(W)$.

Exercise 8.

Let A be a set. Show that A^ω is Hausdorff.

Exercise 9.

Let AP be a finite set, recall that two MLM formulas φ and ψ are said to be logically equivalent, denoted $\varphi \equiv \psi$ if for any valuation ρ, $[\![\varphi]\!]_\rho = [\![\psi]\!]_\rho$. Show that the following equivalences hold.

$$\varphi \wedge \neg\varphi \equiv \bot \qquad \bigcirc(\varphi \vee \psi) = \bigcirc\varphi \vee \bigcirc\psi \qquad \bigcirc(\neg\varphi) = \neg \bigcirc \varphi$$

Exercise 10.

Let $\mathrm{AP} = \mathbb{N}$ and $2\mathbb{N} \subseteq \mathrm{AP}$ be the set of even numbers. Show that there is no closed LML-formula ϕ such that $[\![\phi]\!] = \mathrm{ext}(2\mathbb{N})$.