Category Theory - Important Results

Ralph Sarkis

August 19, 2018

This is a record of the important results we cover during the lectures we will have in the summer 2018. We will try to go over two sets of lecture notes by Mariusz Wodzicki. Our goal is to introduce the concept of categories and build enough familiarity with them to be able to see other mathematical concepts we know in a more categorical point of view.

1 Review

In this section, we will review some concepts that will be helpful in the study of category theory.

1.1 Operations on sets

We give formal definitions of common set operators, giving a bit of a taste of the language we will use.

Definition 1 (Union of sets). Let X be a set, we can define the **union** operator like so:

$$\bigcup = A \mapsto \{x \in X \mid \exists S \in A, x \in S\} : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$$

Definition 2 (Intersection of sets). Let X be a set, we can define the **intersection** operator like so:

$$\bigcap = A \mapsto \{x \in X \mid \forall S \in A, x \in S\} : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$$

Definition 3 (Difference of sets). Let X be a set, we an define the **difference** operator like so:

$$\setminus = (S,T) \mapsto \{x \in X \mid x \in S \wedge x \notin T\} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$$

Definition 4 (Cartesian product). Let $(X_i)_{i\in I}$, where I is some index set, be a family of sets, the Cartesian product of these sets is

$$\prod_{i \in I} X_i = \{(x_i)_{i\in I} \mid \forall i \in I, x_i \in X_i\}.$$

We can also see each element as a function $f : I \to \cup_{i\in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$.

If a family of set is closed under the three first operations, we call it a ring of sets.

Definition 5 (Ring of sets)**.** A non-empty family of sets R is called a **ring** of sets if for any two elements r and r', we have $r \cup r', r \cap r', r \setminus r' \in R$.

1.2 Classes vs. Sets

Several times in our coverage of category theory, we will need to use the concept of a class. It is very similar to that of a set and has one simple difference. While a set can contain another set, classes cannot contain other classes. This difference is necessary because some collections of objects can simply not form a set. Famous examples include the class of ordinal numbers which, by the Burali-Forti paradox, cannot be a set and the class of all sets that do not contain themselves which, by the Russel paradox, cannot be a set.

2 Introduction to categories

2.1 Basic definitions

Definition 6 (Oriented graph)**.** An **oriented graph** G consists of a class of nodes G_0, a class of arrows G_1 along with two functions $s, t : G_1 \to G_0$, so that each arrow $f \in G_1$ has a source $s(f)$ and a target $t(f)$.

Remark 7. The nodes can also be called vertices or objects while arrows are also known as morphisms in the context of categories.

Definition 8 (Paths)**.** A **path** in an oriented graph G is a sequence of arrows $(f_1, \ldots, f_k)$ that are **composable** in the sense that $t(f_i) = s(f_{i-1})$ for $i = 2, \ldots, k$. We will denote G_k to be the class of paths of length k and we often refer to G_2 simply as the class of composable arrows.

Remark 9. Note that the notation indicating the direction of the path does not translate well to what we usually think of as a path in a graph. The reason is that the arrows are more linked to the composition of functions than paths in graphs.

Definition 10 (Category)**.** An oriented graph C along with a map $\circ : C_2 \to C_1$ is a **category** if for any $(f, g, h) \in C_3$, we have $f \circ (g \circ h) = (f \circ g) \circ h$, namely, composition is associative.

Definition 11 (Unital category)**.** A category C is called **unital** if it is equipped with a map $u : C_0 \to C_1$ (for $A \in C_0$, we denote $u(A) = \mathrm{id}_A$) such that for any arrow $f : A \to B$, we have $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$.

Definition 12 (Hom sets)**.** Let C be a category and $A, B \in C_0$, we denote

$$\mathrm{Hom}_C(A, B) = \{f \in C_1 \mid s(f) = A \wedge t(f) = B\}.$$

Definition 13 (Small and discrete)**.** A category C is called **small** if the class of objects and morphisms is not proper (it is a set). It is called **discrete** if there are no morphisms and **discrete unital** if there are no morphisms other than the identity morphisms.

Definition 14 (Subcategory)**.** Let C be a category, a category C' is a **subcategory** of C if:

1. The objects and morphisms of C' are objects and morphisms of C (i.e.: $C'_0 \subseteq C_0$ and $C'_1 \subseteq C_1$).
2. For every morphism $f \in C'_1$, $s(f), t(f) \in C'_0$.
3. For every pair of composable arrows $(f, g) \in C'_2$, $f \circ_{C'} g = f \circ_C g \in C'_1$.

If we are working with unital categories we have the additional requirement that for any $A \in C'_0$, $u_{C'}(A) \in C'_1$. One can show that since composition is the same as in C, the identity must be the same.

Definition 15 (Full and wide)**.** A subcategory C' of C is called **full** if for any objects $A, B \in C'_0$, we have $\mathrm{Hom}_{C'}(A, B) = \mathrm{Hom}_C(A, B)$. It is called **wide** if $C'_0 = C_0$.

Definition 16 (Covariant functor)**.** Let C and D be categories, a **covariant functor** $F : C \rightsquigarrow D$ is a pair of maps $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ that are defined such that the following diagrams commute (where F_2 is induced by the definition of F_1 with $(f, g) \mapsto (F_1(f), F_1(g))$).

$$\begin{tikzcd}
C_0 \arrow[d, "F_0"'] & C_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"'] \\
D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_2 \arrow[r, "F_2"] \arrow[d, "\circ_C"'] & D_2 \arrow[d, "\circ_D"'] \\
C_1 \arrow[r, "F_1"'] & D_1
\end{tikzcd}$$

If we are working with unital categories, we may want to talk about a **unital** functor which requires this additional diagram to commute.

$$\begin{tikzcd}
C_0 \arrow[r, "F_0"] \arrow[d, "u_C"'] & D_0 \arrow[d, "u_D"'] \\
C_1 \arrow[r, "F_1"'] & D_1
\end{tikzcd}$$

Definition 17 (Contravariant functor)**.** Let C and D be categories, a **contravariant functor** $F : C \rightsquigarrow D$ is similar to a covariant functor except for the first diagram which changes a bit (see below) and the definition of F_2 which becomes: $(f, g) \mapsto (F_1(g), F_1(f))$.

$$\begin{array}{ccccc} C_0 & \xleftarrow{s} & C_1 & \xrightarrow{t} & C_0 \\ {\scriptstyle F_0}\downarrow & & {\scriptstyle F_1}\downarrow & & {\scriptstyle F_0}\downarrow \\ D_0 & \xleftarrow[t]{} & D_1 & \xrightarrow[s]{} & D_0 \end{array}$$

Example 18 (Hom functors)**.** Let C be a category and $A \in C_0$ one of its object. We define the covariant and contravariant Hom functors from C to **Set**.

A. The functor $\mathrm{Hom}_C(A, -) : C \rightsquigarrow$ **Set** sends an object $B \in C_0$ to the hom set $\mathrm{Hom}_C(A, B)$ and a morphism $f : B \to B'$ to the function $\mathrm{Hom}_C(A, f) = g \mapsto f \circ g : \mathrm{Hom}_C(A, B) \to \mathrm{Hom}_C(A, B')$. Let us check that this is a covariant functor. We show the commutativity of the three squares in definition 16:

1. For $f \in C_1$, $\mathrm{Hom}_C(A, s(f)) = s(\mathrm{Hom}_C(A, f))$ follows by the definition.
2. For $f \in C_1$, $\mathrm{Hom}_C(A, t(f)) = t(\mathrm{Hom}_C(A, f))$ also follows by the definition.
3. For $(f_1, f_2) \in C_2$, we claim that $\mathrm{Hom}_C(A, f_1 \circ f_2) = \mathrm{Hom}_C(A, f_1) \circ \mathrm{Hom}_C(A, f_2)$. In the L.H.S., an element $g \in \mathrm{Hom}_C(A, s(f_1 \circ f_2))$ is mapped to $(f_1 \circ f_2) \circ g$ and in the R.H.S., an element $g \in \mathrm{Hom}_C(A, s(f_2)$ is mapped to $f_1 \circ (f_2 \circ g)$. Since $s(f_1 \circ f_2) = s(f_2)$, we see that the two maps are the same.

B. The functor $\mathrm{Hom}_C(-, A) : C \rightsquigarrow$ **Set** sends an object $B \in C_0$ to the hom set $\mathrm{Hom}_C(B, A)$ and a morphism $f : B \to B'$ to the function $\mathrm{Hom}_C(f, A) = g \mapsto g \circ f : \mathrm{Hom}_C(B', A) \to \mathrm{Hom}_C(B, A)$. Let us check that this is a contravariant functor. We show the commutativity of the three squares in definition 17:

1. For $f \in C_1$, $\mathrm{Hom}_C(s(f), A) = s(\mathrm{Hom}_C(A, f))$ follows by the definition.
2. For $f \in C_1$, $\mathrm{Hom}_C(t(f), A) = t(\mathrm{Hom}_C(f, A))$ also follows by the definition.
3. For $(f_1, f_2) \in C_2$, we claim that $\mathrm{Hom}_C(f_1 \circ f_2, A) = \mathrm{Hom}_C(f_2, A) \circ \mathrm{Hom}_C(f_1, A)$. In the L.H.S., an element $g \in \mathrm{Hom}_C(t(f_1 \circ f_2), A)$ is mapped to $g \circ (f_1 \circ f_2)$ and in the R.H.S., an element $g \in \mathrm{Hom}_C(t(f_1), A)$ is mapped to $(g \circ f_1) \circ f_2$. Since $t(f_1 \circ f_2) = t(f_1)$, we see that the two maps are the same.

Definition 19 (Natural transformation)**.** Let $F, G : C \rightsquigarrow D$ be two covariant functors, a **natural transformation** $\phi : F \Rightarrow G$ is a map $\phi : C_0 \to D_1$ that satisfies $\phi(A) \in \mathrm{Hom}_D(F(A), G(A))$ for all $A \in C_0$ and makes the following diagram commute for any $f \in \mathrm{Hom}_C(A, B)$: For two contravariant functors, the vertical arrows are reversed.

$$\begin{array}{ccc} F(A) & \xrightarrow{\phi(A)} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\phi(B)]{} & G(B) \end{array}$$

Example 20. Let **CRing** denote the category of commutative rings, where objects are commutative rings, morphisms are ring homomorphisms, and composition is the usual composition of functions. Let **Grp** denote the category of groups, where objects are groups, morphisms are group homomorphisms, and composition is the usual composition of functions.

Fix some $n \in \mathbb{N}$, we define the functor $\mathrm{GL}_n : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$ by

$$\begin{aligned} R &\mapsto \mathrm{GL}_n(R) \text{ for any commutative ring } R \text{ and} \\ f &\mapsto \mathrm{GL}_n(f) \text{ for any ring homomorphism } f \end{aligned}$$

The map $\mathrm{GL}_n(f)$ is just the extension of f on $\mathrm{GL}_n(R)$ by applying f to every element of the matrices. The second functor is $(-)^\times : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$ which sends a commutative ring R to its group of units $R^\times$ under multiplication and a ring homomorphism f to $f^\times$, its restriction on $R^\times$. Checking these mappings define two covariant functors is left as an (simple) exercise, but one might expect these to be functors as they play nicely with the structure of the objects involved.

The natural transformation between these two functors is $\det : \mathrm{GL}_n \Rightarrow (-)^\times$ which maps a commutative ring R to $\det_R$, the function calculating the determinant of a matrix in $\mathrm{GL}_n(R)$. The first thing to check is that $\det_R \in \mathrm{Hom}_{\mathbf{Grp}}(\mathrm{GL}_n(R), R^\times)$ which is clearly the case because the determinant of an invertible matrix is always a unit. The second thing is to verify that the following diagram commutes for any $f \in \mathrm{Hom}_{\mathbf{CRing}}(R, S)$:

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\det_R"] \arrow[d, "\mathrm{GL}_n(f)"'] & R^\times \arrow[d, "f^\times = f|_{R^\times}"] \\ \mathrm{GL}_n(S) \arrow[r, "\det_S"'] & S^\times \end{tikzcd}$$

We will check the claim for $n = 2$, but the general proof should only involve more notation to write the bigger expressions. We can rewrite the diagram as $f^\times \circ \det_R = \det_S \circ \mathrm{GL}_2(f)$ and show it holds as follows. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(R)$, we have

$$\begin{aligned} (\det{}_S \circ \mathrm{GL}_2(f)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) &= \det{}_S \left(\begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix} \right) \\ &= f(a)f(d) - f(b)f(c) \\ &= f(ad - bc) \\ &= f^\times(ad - bc) \\ &= (f^\times \circ \det{}_R) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right). \end{aligned}$$

We conclude that the diagram commutes and that det is indeed a natural transformation.

Definition 21 (Vertical composition)**.** Let $F, G, H : C \rightsquigarrow D$ be parallel functors and $\phi : F \Rightarrow G$ and $\psi : G \Rightarrow H$ be two natural transformations. Then the **vertical composition**

of ϕ and ψ, denoted $\psi \cdot \phi : F \Rightarrow H$ is defined by $(\psi \cdot \phi)(A) = \psi(A) \circ \phi(A)$ for all $A \in C_0$. If $f : A \to B$ is a morphism in C, then we have the following diagram that commutes by naturality of ϕ and ψ:

$$\begin{tikzcd} F(A) \arrow[r, "\phi(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\psi(A)"] \arrow[d, "G(f)"'] & H(A) \arrow[d, "H(f)"'] \\ F(B) \arrow[r, "\phi(B)"'] & G(B) \arrow[r, "\psi(B)"'] & H(B) \end{tikzcd}$$

This shows that $\psi \cdot \phi$ is a natural transformation from F to H. We call this vertical composition as opposed to horizontal composition that we introduce in definition 57.

Definition 22 (Opposite category)**.** Let C be a category, we denote the **opposite category** C^{op} and define it by

$$C_0^{\mathrm{op}} = C_0, C_1^{\mathrm{op}} = C_1, s^{\mathrm{op}} = t, t^{\mathrm{op}} = s,$$

with the correspondence defined by $f^{\mathrm{op}} \circ^{\mathrm{op}} g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$. This canonically leads to the following contravariant functor $(-)^{\mathrm{op}}_C : C \rightsquigarrow C^{\mathrm{op}}$ which sends an object A to A^{op} and a morphism f to f^{op}. Note that the $^{\mathrm{op}}$ notation here is just used to distinguish elements in C and C^{op} although the class of objects and morphisms are the same.

Remark 23. The last definition helps us define the contravariant functors as covariant functors. Formally, let $F : C \rightsquigarrow D$ be a contravariant functor, we can see F as covariant functor from C^{op} to D or from C to D^{op} via the compositions $F \circ (-)^{\mathrm{op}}_{C^{\mathrm{op}}}$ and $(-)^{\mathrm{op}}_D \circ F$ respectively.

Definition 24 (Opposite of a functor)**.** Let $F : C \rightsquigarrow D$ be a covariant functor, then the **opposite** of this functor $F^{\mathrm{op}} : C^{\mathrm{op}} \rightsquigarrow D^{\mathrm{op}}$ is defined by $F^{\mathrm{op}} = (-)^{\mathrm{op}}_D \circ F \circ (-)^{\mathrm{op}}_{C^{\mathrm{op}}}$.

Definition 25 (Opposite functor)**.** The **opposite functor** $(-)^{\mathrm{op}}$: **Cat** $\rightsquigarrow$ **Cat** sends a category or a functor to its opposite. It is a covariant functor.

Definition 26 (Monomorphism)**.** Let C be a category, a morphism $f \in C_1$ is said to be a **monomorphism** if for any two morphisms $g, h \in C_1$ with $t(g) = t(h) = s(f)$, $f \circ g = f \circ h$ implies $g = h$.

Definition 27 (Epimorphism)**.** Let C be a category, a morphism $f \in C_1$ is said to be an **epimorphism** if for any two morphisms $g, h \in C_1$ with $s(g) = s(h) = t(g)$, $g \circ f = h \circ f$ implies $g = h$.

Proposition 28. *Let C be a category and $f : A \to B$ a morphism, if there exists $f' : B \to A$ such that $f' \circ f = id_A$, then f is a monomorphism.*

Proof. If $f \circ g = f \circ h$, then $f' \circ f \circ g = f' \circ f \circ h$ implying $g = h$. $\square$

Proposition 29. *Let C be a category and $(f_1, f_2) \in C_2$, if $f_1 \circ f_2$ is a monomorphism, then f_2 is a monomorphism.*

Proof. Let $g, h \in C_1$ be such that $f_2 \circ g = f_2 \circ h$, we immediately get that $(f_1 \circ f_2) \circ g = (f_1 \circ f_2) \circ h$. Since $f_1 \circ f_2$ is a monomorphism, this implies $g = h$. $\square$

Remark 30. The two dual propositions for epimorphisms also hold and are straightforward to prove.

Example 31 (Monomorphisms in the categories we know)**.**

1. Inside the category **Mon** where objects are monoids and morphims are monoid homomorphisms, the monomorphisms correspond exactly to injective homomorphims as shown below.
   - Let $f : M \to M'$ be an injective homomorphims and $g_1, g_2 : N \to M$ be two parallel homomorphisms. Suppose that $f \circ g_1 = f \circ g_2$, then for all $x \in N$, $f(g_1(x)) = f(g_2(x))$, so by injectivity of f, $g_1(x) = g_2(x)$. We conclude that $g_1 = g_2$ and since g_1 and g_2 were arbitrary, f is a monomorphism.
   - Let $f : M \to M'$ be a monomorphism. Let $x, y \in M$ and define $p_x : \mathbb{N} \to M$ by $k \mapsto x^k$ and similarly for p_y. It is trivial to show that p_x and p_y are homomorphism. If $f(x) = f(y)$, then by the homomorphism property, we get for all $k \in \mathbb{N}$:

     $$f(p_x(k)) = f(x^k) = f(x)^k = f(y)^k = f(y^k) = f(p_y(k)).$$

     In other words, we get $f \circ p_x = f \circ p_y$, so $p_x = p_y$ and $x = y$. We conclude that f is injective.

Example 32 (Epimorphisms in the categories we know)**.**

1. Inside the category **Mon** an epimorphism is not necessarily surjective. For example, the inclusion homomorphism $i : \mathbb{N} \to \mathbb{Z}$ is clearly not surjective but it is an epimorphism. Indeed, let $g, h : \mathbb{Z} \to M$ be two monoid homomorphisms satisfying $g \circ i = h \circ i$. In particular, we have $g(n) = h(n)$ for any $n \in \mathbb{N} \subset \mathbb{Z}$. It is left to show that also $g(-n) = h(-n)$, but if it were not the case for some n, $g(n)$ would have two left inverses $g(-n)$ and $h(-n)$ which is not possible. We conclude that $g = h$ and i is an epimorphism.

Definition 33 (Isomorphism)**.** Let C be a category, a morphism $f : A \to B$ is said to be an **isomorphism** if there exists a morphism $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

Proposition 34. *Let C be a category and $f \in C_1$ be an isomorphism, then f is a monomorphism and an epimorphism.*

Proof idea. If the compositions with f and two other morphisms are equal, compose with f^{-1} to obtain equality of the morphisms. $\square$

Definition 35 (Natural isomorphism)**.** Let $\phi : F \to G$ be a natural transformation of functors $F, G : C \rightsquigarrow D$. If for every $A \in C_0$, $\phi(A)$ is an isomorphism in G, we say that ϕ is a **natural isomorphism** and we may write $\phi : F \cong G$.

Definition 36 (Subobject)**.**

Definition 37 (Quotient object)**.**

Definition 38 (Initial object)**.** Let C be a category, an object $A \in C_0$ is said to be **initial** if for any $B \in C_0$, $|\operatorname{Hom}_C(A, B)| = 1$, namely there are no two parallel morphisms with source A and every object has a morphism coming from A.

Definition 39 (Terminal object)**.** Let C be a category, an object $A \in C_0$ is said to be **terminal** if for any $B \in C_0$, $|\operatorname{Hom}_C(B, A)| = 1$, namely there are no two parallel morphisms with target A and every object has a morphism going to A.

Definition 40 (Zero object)**.** If an object is initial and terminal, we say it is a zero object and usually denote it 0.

The following are alternate definitions for initial and terminal objects which have the advantage of being completely categorical, making no use of sets. They use the concept of representable functors which will be seen more in depth later.

Proposition 41. *Let C be a category and $\star : C \to$ **Set** be a functor sending objects to the singleton $\{1\}$ and morphisms to $id_{\{1\}}$. An object $A \in C_0$ is initial if and only if the functor* $\operatorname{Hom}_C(A, -)$ *is naturally isomorphic to $\star$.*

Proof. ($\Rightarrow$) Suppose that A is initial, then there is a natural transformation η from $\hom_C(A, -)$ to $\star$ that sends any object X to the only function between $\hom_C(A, X)$ and $\{1\}$. Since the $\hom_C(A, X)$ is also a singleton, this function is an isomorphism for all X and we conclude that η is a natural isomorphism. ($\Leftarrow$) Suppose that there is a natural isomorphism $\eta : \hom_C(A, -) \Rightarrow \star$, then there are isomorphisms between $\{1\}$ and $\hom_C(A, X)$ for all objects $X \in C_0$. This means that there is a unique morphism from A to X and that A is initial. $\square$

Proposition 42. *Let C be a category and $\star$ be as above. An object $A \in C_0$ is terminal if and only if the functor* $\operatorname{Hom}_C(-, A)$ *is naturally isomorphic to $\star$.*

Proof. The proof is basically a copy of the last proof. $\square$

Definition 43 (Product). Let C be a category and $A, B \in C_0$. A **product** of A and B is an object denoted $A \times B$ along with two morphisms $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ (they are called projections) such that for any object V and morphisms $f : V \to A$ and $g : V \to B$, there exists a unique morphism $h : V \to A \times B$ such that this diagram commutes:

$$\begin{tikzcd} & V \arrow[dl, "f"'] \arrow[d, dashed, "h"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Example 44. Inside **Set**, the Cartesian products with the usual projection maps are products. Inside **Grp**, the direct products with the usual projection maps are products.

Definition 45 (Coproducts)**.** Let C be a category and $A, B \in C_0$. A **coproduct** of A and B is an object denoted $A \amalg B$ along with two morphisms $i_1 : A \to A \times B$ and $i_2 : B \to A \times B$ (they are called canonical injections) such that for any object V and morphisms $f : A \to V$ and $g : B \to V$, there exists a unique morphism $h : A \times B \to V$ such that this diagram commutes:

$$\begin{tikzcd} & V & \\ A \arrow[ur, "f"] \arrow[r, "i_1"'] & A \times B \arrow[u, dotted, "h"] & B \arrow[ul, "g"'] \arrow[l, "i_2"] \end{tikzcd}$$

Definition 46 (Pullback)**.** Let C be a category and $f : A \to C$ and $g : B \to C$ be in C_1. A **pullback** of f and g is an object denoted $A \times_C B$ along with two morphisms $p_1 : A \times_C B \to A$ and $p_2 : A \times_C B \to B$ such that this diagram commutes: and for any

$$\begin{tikzcd} A \times_C B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

object V and morphisms $s : V \to A$ and $t : V \to B$, there exists a unique morphism $h : V \to A \times_C B$ that makes this diagram commute:

$$\begin{tikzcd} V \arrow[dr, dotted, "h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & A \times_C B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

Definition 47 (Pushout)**.**

Question 48. *Is the pullback object always a subobject of the product ? Is the pushout object always a subobject of the coproduct or quotient object of the product ? Why are these terms used ?*

Definition 49 (Commutative diagram)**.** Let C be a category. A **commutative diagram** in C is functor $F : D \to C$ where D is a small category. We usually draw diagrams by partially drawing the image of D as a graph where objects are vertices and morphisms are arrows. All the diagrams we have drawn up to this definition define the domain of the functor implicitly. For example, if we talk about a commutative square in C, the domain of this diagram can be drawn like so:

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

Remark 50. It follows trivially from this definition that functors preserve commutative diagrams.

2.2 More on natural transformations

Definition 51 (The left action of functors)**.** Let $F, F' : C \rightsquigarrow D$, $G : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. The functor G acts on ϕ by sending it to $G\phi = A \mapsto G(\phi(A)) : C_0 \to E_1$. One can verify that this is a natural transformation from $G \circ F$ to $G \circ F'$ by verifying the diagram commutes for any $C_1 \ni f : A \to B$.

$$\begin{tikzcd} (G \circ F)(A) \arrow[r, "G\phi(A)"] \arrow[d, "(G\circ F)(f)"'] & (G \circ F')(A) \arrow[d, "(G\circ F')(f)"] \\ (G \circ F)(B) \arrow[r, "G\phi(B)"] & (G \circ F')(B) \end{tikzcd}$$

If we remove all applications of G, the diagram commutes by naturality of ϕ. Since functors preserve commuting diagrams, we get that $G\phi$ is a natural transformation.

Proposition 52. *The previous definition constitutes a left action, namely,* $id_D\phi = \phi$ *and* $G_1(G_2\phi) = (G_1 \cdot G_2)\phi$*.*

Proof. □

Definition 53 (The right action of functors)**.** Let $F, F' : C \rightsquigarrow D$, $G : E \rightsquigarrow C$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. The functor G acts on ϕ by sending it to $\phi G = A \mapsto \phi(G(A)) : E_0 \to D_1$. One can verify that this is a natural transformation from $F \circ G$ to $F' \circ G$ by verifying the diagram commutes for any $E_1 \ni f : A \to B$.

$$\begin{tikzcd} (F \circ G)(A) \arrow[r, "\phi G(A)"] \arrow[d, "(F\circ G)(f)"'] & (F' \circ G)(A) \arrow[d, "(F'\circ G)(f)"] \\ (F \circ G)(B) \arrow[r, "\phi G(B)"] & (F' \circ G)(B) \end{tikzcd}$$

It follows by naturality of ϕ; change f in the diagram of definition 19 with the morphism $G(f) : G(A) \to G(B)$.

Proposition 54. *The previous definition constitutes a right action, namely, $\phi id_C = \phi$ and $(\phi G_1)G_2 = \phi(G_1 \cdot G_2)$.*

Proof. $\square$

Proposition 55. *The two actions commute. Namely, if we let $F, F' : C \rightsquigarrow D$, $G : D \rightsquigarrow E$, $H : E' \rightsquigarrow C$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation, then we have $G(\phi E) = (G\phi)E$.*

Proof. $\square$

We will refer to these two actions as the biaction of functors on natural transformations and they will motivate the definition of another way to compose natural transformations. Consider the following diagram to be the setting of this definition, with F, F', G and

$$\begin{tikzcd}
C \arrow[r, bend left=60, "F"{name=F}] \arrow[r, bend right=60, "F'"'{name=Fp}] & D \arrow[r, bend left=60, "G"{name=G}] \arrow[r, bend right=60, "G'"'{name=Gp}] & E
\arrow[from=F, to=Fp, "\phi"]
\arrow[from=G, to=Gp, "\psi"]
\end{tikzcd}$$

G' being functors and ϕ and ψ being natural transformations. With the two previous actions, we are able to construct four new transformations:

$$\begin{aligned}
G\phi &: G \circ F \Rightarrow G \circ F' \\
\psi F &: G \circ F \Rightarrow G' \circ F \\
G'\phi &: G' \circ F \Rightarrow G' \circ F' \\
\psi F' &: G \circ F' \Rightarrow G' \circ F'
\end{aligned}$$

Observe that to go from $G \circ F$ to $G' \circ F'$, we have two paths yielding the following diagram:

$$\begin{tikzcd}
 & G \circ F' \arrow[dr, "\psi F'"] & \\
G \circ F \arrow[ur, "G\phi"] \arrow[dr, "\psi F"'] & & G' \circ F' \\
 & G' \circ F \arrow[ur, "G'\phi"'] &
\end{tikzcd}$$

Proposition 56. *The diagram above commutes.*

Proof. For a fixed element $c \in C_0$ we know that $a := F(c)$ and $b := F'(c)$ are two different elements of the category D and that we have an arrow $f := \phi(c)$ from a to b given by

$$\begin{tikzcd}
& G(b) \arrow[dr, "\psi b"] & \\
G(a) \arrow[ur, "Gf"] \arrow[dr, "\psi a"'] & & G'(b) \\
& G'(a) \arrow[ur, "G'f"'] &
\end{tikzcd}$$

the natural transformation ϕ. But as ψ is a natural transformation $G \Rightarrow G'$, we know that the following diagram commutes:

Replacing a, b and f by their values we obtain what we wanted. $\square$

Definition 57 (Horizontal composition)**.** In the setting described above, we define the **horizontal composition** of ψ and ϕ by $\psi \diamond \phi = \psi F' \cdot G\phi = G'\phi \cdot \psi F$.

Proposition 58. *Horizontal composition is associative. Namely, if we let $F, F' : C_1 \rightsquigarrow C_2$, $G, G' : C_2 \rightsquigarrow C_3$ and $H, H' : C_3 \rightsquigarrow C_4$ be functors and $\phi : F \Rightarrow F'$, $\psi : G \Rightarrow G'$ and $\eta : H \Rightarrow H'$ be natural transformations, then we have $\eta \diamond (\psi \diamond \phi) = (\eta \diamond \psi) \diamond \phi$.*

Proof. $\square$

Proposition 59 (Interchange identity)**.** *Let $F, F', F'' : C \rightsquigarrow D$ and $G, G', G'' : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$, $\phi' : F' \Rightarrow F''$, $\psi : G \Rightarrow G'$ and $\psi' : G' \Rightarrow G''$ be natural transformations. Using $\cdot$ to denote vertical composition, the **interchange identity** holds:*

$$(\psi' \cdot \psi) \diamond (\phi' \cdot \phi) = (\psi' \diamond \phi') \cdot (\psi \diamond \phi)$$

Proof. The idea is to use the commutativity of $\psi' \circ \phi$ to switch from the LHS to the RHS of the equation. To make things clearer we first draw out the diagrams The LHS of the equation can be seen as the following diagram:

$$\begin{tikzcd}
& G' \circ F \arrow[dr, "G'\phi"] & & G'' \circ F' \arrow[dr, "G''\phi'"] & \\
G \circ F \arrow[ur, "\psi F"] \arrow[dr] & & G' \circ F' \arrow[ur, "\psi' F'"] \arrow[dr] & & G'' \circ F'' \\
& G \circ F' \arrow[ur] & & G' \circ F'' \arrow[ur] &
\end{tikzcd}$$

While the RHS would correspond to the following:

Joining the two diagrams, we obtain this huge one $\square$

These definitions lead us to the first example of a 2-category.

Definition 60 (2-cateory)**.** A **2-category** consists of a class of objects C_0, a class of morphisms between objects C_1 and a class of 2-morphisms between parallel morphisms C_2 that satisfy the following conditions:

$$
\begin{tikzcd}
& & G''\circ F \arrow[dr, "G''\phi"] & & \\
& G'\circ F \arrow[ur, "\psi' F"] & & G''\circ F' \arrow[dr, "G''\phi'"] & \\
G\circ F \arrow[ur, "\psi F"] \arrow[dr] & & & & G''\circ F'' \\
& G\circ F' \arrow[dr] & & G'\circ F'' \arrow[ur] & \\
& & G\circ F'' \arrow[ur] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& & G''\circ F \arrow[dr, "G''\phi"] \arrow[dd, phantom, "1"] & & \\
& G'\circ F \arrow[ur, "\psi' F"] \arrow[dr, "G'\phi"] & & G''\circ F' \arrow[dr, "G''\phi'"] & \\
G\circ F \arrow[ur, "\psi F"] \arrow[dr] & & G'\circ F' \arrow[ur, "\psi' F'"] \arrow[dr] & & G''\circ F'' \\
& G\circ F' \arrow[dr] \arrow[ur] & & G'\circ F'' \arrow[ur] & \\
& & G\circ F'' \arrow[ur] & &
\end{tikzcd}
$$

1. The objects and morphisms form a category under composition of morphisms.
2. For two objects $A, B \in C_0$, the morphisms from C to D and the 2-morphisms between them form a category under vertical composition.
3. If we consider 2-cells (two parallel morphisms with a 2-morphism between them) as morphisms, we get a category under horizontal composition.
4. The interchange identity hold for horizontal and vertical composition.

Example 61.

1. The 2-category of categories with functors and natural transformations as we just have proved.

Question 62. *Is the vertical composition of two natural isomorphisms also a natural isomorphism ? What about horizontal composition ?*

Definition 63 (Identity transformation)**.** Let $F : C \rightsquigarrow D$ be a functor, the identity natural transformation from F to itself is defined by $\mathrm{id}_F = A \mapsto \mathrm{id}_{F(A)} : C_0 \to D_1$ when the objects in the range of F all have an identity morphism.

Proposition 64. *Let $F, F' : C \rightsquigarrow D$, $G : B \rightsquigarrow C$ and $H : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. Suppose that C and E are unital, then the following equations hold:*

1. $\phi G = \phi \diamond id_G$
2. $H\phi = id_H \diamond \phi$
3. $id_{id_D} \diamond \phi = \phi = \phi \diamond id_{id_C}$

Proof.

1. For any $x \in B_0$, we have the following:

$$\begin{aligned}
(\phi \diamond \mathrm{id}_G)(x) &= \phi(G(x)) \circ F(\mathrm{id}_G(x)) && \text{(def of } \diamond) \\
&= \phi G(x) \circ F(\mathrm{id}_{G(x)}) && \text{(def of id}_G) \\
&= \phi G(x) \circ \mathrm{id}_{F(G(x))} && \text{(functors preserve id morphisms)} \\
&= \phi G(x) && \text{(def of id morphisms)}
\end{aligned}$$

   Thus, we conclude that $\phi \diamond \mathrm{id}_G = \phi G$.

2. For any $x \in C_0$, we have the following:

$$\begin{aligned}
(\mathrm{id}_H \diamond \phi)(x) &= \mathrm{id}_H(F'(x)) \circ H(\phi(x)) && \text{(def of } \diamond) \\
&= \mathrm{id}_{H(F'(x))} \circ H\phi(x) && \text{(def of id}_H) \\
&= H\phi(x) && \text{(def of id morphisms)}
\end{aligned}$$

   Thus, we conclude that $\mathrm{id}_H \diamond \phi = H\phi$.

3. By swapping G for id_C and H for id_D in the two previous equations, we get the result we want.

□

2.3 On our way to the Yoneda lemma

Definition 65 (Category of arrows)**.** Let C be a category, $\mathrm{Arr}(C)$ is the category of arrows of C. Its objects are morphisms in C and its morphisms are commutative squares ϕ. In other words, if f and g are morphisms in C and there exists maps ϕ_s and ϕ_t such that this diagram commutes

$$\begin{tikzcd}
s(f) \arrow[r, "f"] \arrow[d, "\phi_s"'] & t(f) \arrow[d, "\phi_t"] \\
s(g) \arrow[r, "g"'] & t(g)
\end{tikzcd},$$

then this square is a morphism from f to g. It is denoted by ϕ or (ϕ_s, ϕ_t).

Definition 66 (Source functor)**.** Let C be a category, the **source functor** is $S : \mathrm{Arr}\, C \rightsquigarrow C$ defined by:

$$\begin{aligned} S_0(f) &= s(f), \forall f \in C_1 = \mathrm{Arr}(C)_0 \\ S_1((\phi_s, \phi_t)) &= \phi_s \forall (\phi_s, \phi_t) \in \mathrm{Arr}(C)_1 \end{aligned}$$

Definition 67 (Target functor)**.** Let C be a category, the **target functor** is $T : \mathrm{Arr}\, C \rightsquigarrow C$ defined by:

$$\begin{aligned} T_0(f) &= t(f), \forall f \in C_1 = \mathrm{Arr}(C)_0 \\ T_1((\phi_s, \phi_t)) &= \phi_t \forall (\phi_s, \phi_t) \in \mathrm{Arr}(C)_1 \end{aligned}$$

Definition 68 (Tautological natural transformation)**.** Let C be a category, the **tautological natural transformation** is $\tau : S \Rightarrow T$ defined by $\tau(f) = f$ for all $f \in C_1 = \mathrm{Arr}(C)_0$. Note that we see the input as an object of $\mathrm{Arr}(C)$ and the output as a morphism of C.

Definition 69 (Arr functor)**.** The **Arr functor** is a functor **Cat** $\rightsquigarrow$ **Cat** that sends a category C to its category of arrows and a functor $F : C \rightsquigarrow D$ to the functor $\mathrm{Arr}(F) : \mathrm{Arr}(C) \rightsquigarrow \mathrm{Arr}(D)$ defined by

$$\begin{aligned} &\mathrm{Arr}(F)_0 = f \mapsto F(f) \\ &\mathrm{Arr}(F)_1 = (\phi_s, \phi_t) \mapsto (F(\phi_s), F(\phi_t)) \end{aligned}$$

Proposition 70. *The correspondences $S = C \mapsto S_C$ and $T = C \mapsto T_C$ where S_C is the source functor and T_C is the target functor define natural transformations* $\mathrm{Arr} \mapsto id_{\mathbf{Cat}}$.

Proof. $\square$

Definition 71 (Representable functors)**.** A covariant functor $F : C \to$ **Set** is said to be representable if there is an object $X \in C_0$ such that F is naturally isomorphic to $\hom_C(X, -)$. If F is contravariant, then we require it to be naturally isomorphic to $\hom_C(-, X)$.

Example 72. The functor $(-)^\times$: **Ring** $\rightsquigarrow$ **Set** is represented by $\mathbb{Z}[x, x^{-1}]$ because any unit of $R^\times$ corresponds to the unique homomorphism from $\mathbb{Z}[x, x^{-1}]$ to R sending x to that unit and every homomorphisms from $\mathbb{Z}[x, x^{-1}]$ to R must send x to a unit.