Modelling Nondeterminism, Probability and Termination

The story of Ralph's internship with his amazing supervisors Matteo and Valeria.

Ralph Sarkis

ENS de Lyon

March 25th, 2021

Introduction

Overview of the presentation:

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- **Set** monads and equational theories modelling

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  - Nondeterministic choices

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- **1Met** monads and quantitative equational theories modelling

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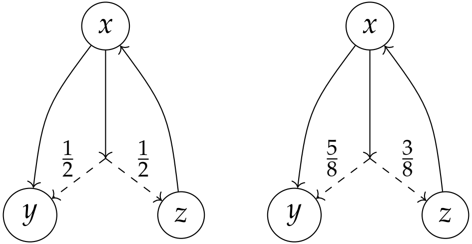
  - Nondeterminism and probability
- Combining nondeterminism, probability and termination NEW!

Motivation



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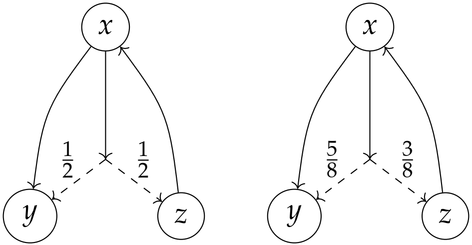

Questions

- Are the two systems equivalent?

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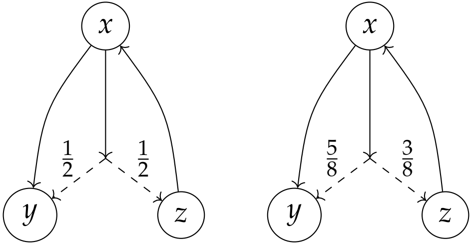

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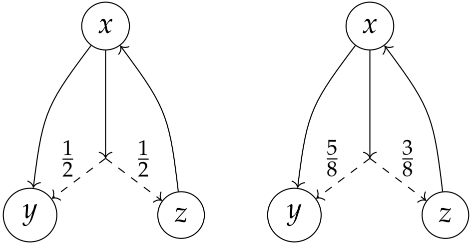

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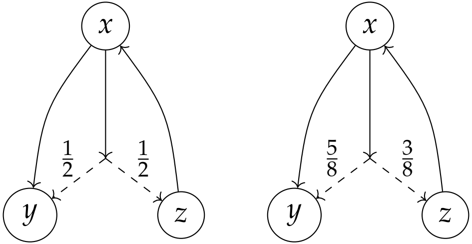

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Categorically

Algebraic Effects

Categorically

- We use **monads**: functors with additional structure that is closely related to computation

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A link between these two pictures: **algebraic presentations of monads**.

Nondeterministic Choices (in **Set**)

Categorically

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Categorically

- Powerset monad $\mathcal{P}$:

$$\mathcal{P}(X) = \{\text{non-empty finite subsets of } X\}$$
$$\eta_X = x \mapsto \{x\}$$
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- Theory of (sup-)semilattices:
  A binary operation $\oplus$ satisfying

$$x \oplus x = x \qquad I$$
$$x \oplus y = y \oplus x \qquad C$$
$$(x \oplus y) \oplus z = x \oplus (y \oplus z). \qquad A$$

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- Operational semantics

$$\frac{q \xrightarrow{\mathtt{a}} r \qquad q \xrightarrow{\mathtt{a}} s}{q \xrightarrow{\mathtt{a}} r \oplus s} \qquad \frac{q \xrightarrow{\mathtt{a}} r \qquad q' \xrightarrow{\mathtt{a}} r'}{q \oplus q' \xrightarrow{\mathtt{a}} r \oplus r'}$$

$$\frac{q \downarrow \mathsf{accept}}{q \oplus r \downarrow \mathsf{accept}}$$

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$$\begin{aligned}\mathcal{D}(X) = \{&\text{finitely supported}\\ &\text{probability distributions on } X\}\end{aligned}$$

$$\eta_X = x \mapsto \delta_x \text{ (Dirac)}$$

$$\mu_X = \Phi \mapsto \sum_{\varphi \in \mathrm{supp}(\Phi)} \Phi(\varphi) \cdot \varphi$$

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- Theory of convex algebras: For each $p \in (0,1)$, a binary operation $+_p$ satisfying

$$\begin{aligned} x +_p x &= x && I_p \\ x +_p y &= y +_{1-p} x && C_p \\ (x +_q y) +_p z &= x +_{pq} \left(y +_{\frac{p\bar{q}}{\overline{pq}}} z\right) && A_p \end{aligned}$$

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- Operational semantics

$$\frac{q \xrightarrow{\mathtt{a},p} r \qquad q \xrightarrow{\mathtt{a},1-p} s}{q \xrightarrow{\mathtt{a}} r +_p s} \qquad \frac{q \xrightarrow{\mathtt{a}} r \qquad q' \xrightarrow{\mathtt{a}} r'}{q +_p q' \xrightarrow{\mathtt{a}} r +_p r'}$$

$$\frac{q \downarrow o(q) \qquad r \downarrow o(r)}{q +_p r \downarrow po(q) + (1-p)o(r)}$$

Termination (in **Set**)

Categorically

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Categorically

- Maybe monad $\cdot + \mathbf{1}$:

$$X + \mathbf{1} = X \sqcup \{\star\}$$
$$\eta_X = \mathsf{inl}^{X+\mathbf{1}}$$
$$\mu_X = [\mathsf{inl}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}, \mathsf{inr}^{X+\mathbf{1}}]$$

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$$t(x) = \begin{cases} y & x \xrightarrow{t} y \\ \star & x \not\to \end{cases}$$

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$$\frac{q \not\rightarrow}{q \rightarrow \star}$$

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- Transition system $t : X \to \mathcal{C}(X)$

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Equationally

- Theory of convex semilattices: A semilattice operation and convex algebra operations that distribute

$$(x \oplus y) +_p z = (x +_p z) \oplus (y +_p z) \qquad (D_p)$$

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- Operational semantics: Combine all previous rules.

[BSV19]

Metric Reasoning

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- Equations become quantitative

$$x =_\varepsilon y \qquad (\varepsilon \in [0, 1])$$

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Equationally

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- Quantitative equational logic

Quantitative Equational Logic

$$
\begin{array}{rl}
\vdash t =_0 t & \text{Refl} \\
\vdash t =_1 s & \text{1–bounded} \\
t =_\varepsilon s \vdash s =_\varepsilon t & \text{Symm} \\
t =_\varepsilon s, s =_\delta u \vdash t =_{\varepsilon+\delta} u & \text{Triang} \\
t =_\varepsilon s \vdash t =_{\varepsilon+\delta} s & \text{Max (for } \delta > 0\text{)} \\
\forall \varepsilon > \delta, t =_\varepsilon s \vdash t =_\delta s & \text{Arch (for } \delta \geq 0\text{)} \\
\forall 1 \leq i \leq n, t_1 =_\varepsilon s_n \vdash f(t_1, \ldots, t_n) =_\varepsilon f(s_1, \ldots, s_n) & \text{Nexp (for } f \text{ an } n\text{–ary)} \\
\Gamma \vdash t =_\varepsilon s \implies \sigma(\Gamma) \vdash \sigma(t) =_\varepsilon \sigma(s) & \text{Subst (for } \sigma \text{ a substitution)} \\
\phi \in \Gamma \implies \Gamma \vdash \phi & \text{Assumption}
\end{array}
$$

Introduced by Mardare, Panangaden and Plotkin [MPP16].

Nondeterminism Choices (in **1Met**)

Categorically

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Categorically

- Hausdorff lifting of $d : X \times X \to [0, 1]$ into $H(d) : \mathcal{P}(X) \times \mathcal{P}(X) \to [0, 1]$:

$$H(d)(S, T) = \max \left\{ \sup_{s \in S} \inf_{t \in T} d(s, t), \sup_{t \in T} \inf_{s \in S} d(s, t) \right\}$$

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- Hausdorff lifting of $\mathcal{P}$: $\widehat{\mathcal{P}}(X,d) = (\mathcal{P}(X), H(d))$.

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- Theory of quantitative (sup-)semilattices: binary operation $\oplus$ satisfying

$$\begin{array}{rll} & \vdash x \oplus x =_0 x & I \\ & \vdash x \oplus y =_0 y \oplus x & C \\ & \vdash (x \oplus y) \oplus z =_0 x \oplus (y \oplus z) & A \\ x_1 =_{\varepsilon_1} y_1, x_2 =_{\varepsilon_2} y_2 & \vdash x_1 \oplus x_2 =_{\max\{\varepsilon_1,\varepsilon_2\}} y_1 \oplus y_2 & H \end{array}$$

Probabilistic choices (in **1Met**)

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- Kantorovitch lifting of $d : X \times X \to [0,1]$ into $K(d) : \mathcal{D}(X) \times \mathcal{D}(X) \to [0,1]$:

$$K(d)(\varphi, \psi) = \inf_{\omega} \sum_{x_1, x_2 \in X} \omega(x_1, x_2) \cdot d(x_1, x_2),$$

where ω ranges over couplings of φ and ψ.

Probabilistic choices (in **1Met**)

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- Kantorovich lifting of $d : X \times X \to [0,1]$ into $K(d) : \mathcal{D}(X) \times \mathcal{D}(X) \to [0,1]$:

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Equationally

Probabilistic choices (in **1Met**)

Categorically

- Kantorovich lifting of $d : X \times X \to [0,1]$ into $K(d) : \mathcal{D}(X) \times \mathcal{D}(X) \to [0,1]$:

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Equationally

- Theory of quantitative convex algebras

$$\begin{array}{rlr} & \vdash x +_p x =_0 x & I_p \\ & \vdash x +_p y =_0 y +_p x & C_p \\ & \vdash (x +_q y) +_p z =_0 x +_{pq} (y +_{\frac{p\bar{q}}{\overline{pq}}} z) & A_p \\ x_1 =_{\varepsilon_1} y_1, x_2 =_{\varepsilon_2} y_2 & \vdash x_1 +_p x_2 =_{p\varepsilon_1 + (1-p)\varepsilon_2} y_1 +_p y_2 & K_p \end{array}$$

Termination (in **1Met**)

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- The maybe monad $\cdot + \widehat{\mathbf{1}}$ puts $\star$ at distance 1 from every other point:

$$X + \widehat{\mathbf{1}} = X \sqcup \{\star\}$$

$$(d + d_{\widehat{\mathbf{1}}})(x, y) = \begin{cases} 1 & x =_0 \star \vee y = \star \\ d(x, y) & \text{otherwise} \end{cases}$$

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- Theory of pointed spaces:

$$\vdash x =_1 \star$$

Nondeterminism and Probability (in **1Met**)

Categorically

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- Combine Kantorovitch and Hausdorff lifting of $d : X \times X \to [0, 1]$ to obtain $\widehat{\mathcal{C}}(X, d) = (\mathcal{C}(X), HK(d))$.

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Equationally

- Quantitative theory of convex semilattices: combine all previous axioms plus

$$\vdash (x \oplus y) +_p z =_0 (x +_p z) \oplus (y +_p z) \qquad D_p$$

[MV20]

Adding Termination (easy)

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Equationally

- On **Set**: $\mathcal{C}(\cdot + \mathbf{1})$ is presented by the theory of pointed convex semilattices.
- On **1Met**: $\widehat{\mathcal{C}}(\cdot + \widehat{\mathbf{1}})$ is presented by the theory of quantitative pointed convex semilattices.

Adding Termination (medium)

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$$x \oplus \star = x \qquad (\bot) \qquad\qquad x +_p \star = \star \qquad (BH_p)$$

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- The black-hole equation trivializes the theory.

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On **1Met**

- The black-hole equation trivializes the theory.
- There is (probably) no monad $\mathcal{C} + \mathbf{1}$ on **1Met**.

Adding Termination (hard)

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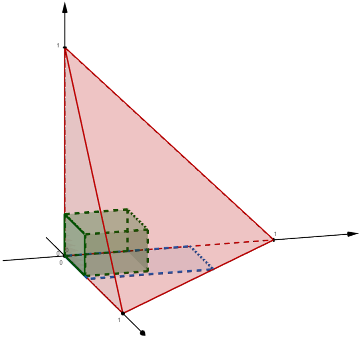

Figure 1: Examples of $\perp$–closed sets.

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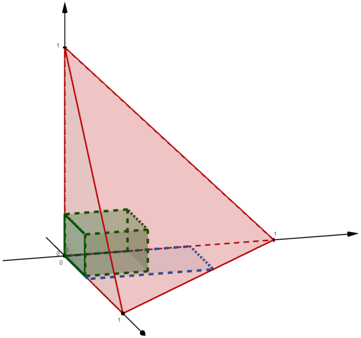

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On **1Met**

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On **1Met**

- The theory of quantitative convex semilattices with bottom corresponds to $\mathcal{C}^{\downarrow}$ with the Kantorovitch and Hausdorff liftings of the metrics.

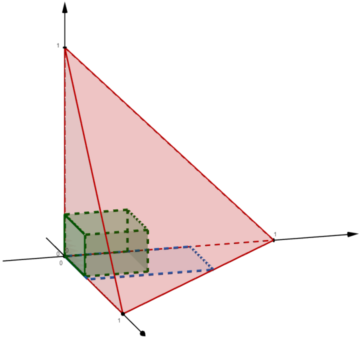

Figure 1: Examples of $\perp$–closed sets.

Summary

On **Set**		On **1Met**	
$\mathcal{P}$	Semilattices	$\widehat{\mathcal{P}}$	Quantitative semilattices
$\mathcal{D}$	Convex algebras	$\widehat{\mathcal{D}}$	Quantitative convex algebras
$\cdot + \mathbf{1}$	Pointed sets	$\cdot + \widehat{\mathbf{1}}$	Pointed spaces
$\mathcal{C}$	Convex semilattices (CS)	$\widehat{\mathcal{C}}$	Quantitative CS
$\mathcal{C}(\cdot + \mathbf{1})$	Pointed CS	$\widehat{\mathcal{C}}(\cdot + \widehat{\mathbf{1}})$	Quantitative pointed CS
$\mathcal{C} + \mathbf{1}$	CS with $\perp$ and black-hole	Trivial	Quantitative CS with $\perp$ and black-hole
$\mathcal{C}^{\downarrow}$	CS with $\perp$	$\widehat{\mathcal{C}}^{\downarrow}$	Quantitative CS with $\perp$

Applications

See Section VI of preprint.

Future Work

- Can our results be obtained *compositionally*?

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Future Work

- Can our results be obtained *compositionally*? Close to answering yes with Daniela Petrişan.
- Where can quantitative algebraic reasoning be applied?

References

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Merci !