ENS DE LYON

TD – Semantics and Verification

X– Duality and Ultrafilter Extension

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Duality

For a fixed set Act, recall that $\mathfrak{L}(\mathsf{HML})$ is the set of HML-formulas, it is a boolean algebra with the order $\varphi \leq \psi \Leftrightarrow \varphi \to \psi \equiv \top$ and the syntactically evident operations. Moreover, we let

- $\alpha \in \mathrm{Act}$, the functions $[\alpha] : \mathfrak{L}(\mathsf{HML}) \to \mathfrak{L}(\mathsf{HML})$ and $\langle\alpha\rangle : \mathfrak{L}(\mathsf{HML}) \to \mathfrak{L}(\mathsf{HML})$ are defined by
  - $[\alpha](\phi) = [\alpha]\phi$
  - $\langle\alpha\rangle(\phi) = \langle\alpha\rangle\phi$
- for $TS = (S, \mathrm{Act}, \to, I, \mathrm{AP}, L)$, the functions $[\![[\alpha]]\!]$ and $[\![\langle\alpha\rangle]\!]$ are defined by
  - $[\![[\alpha]]\!](A) = \{s \in S \mid \forall s' \in \mathrm{Succ}^{\alpha}(s), s' \in A\}$
  - $[\![\langle\alpha\rangle]\!](A) = \{s \in S \mid \exists s' \in \mathrm{Succ}^{\alpha}(s), s' \in A\}$

For two Boolean algebras B and B' and a function $f : B \to B'$, the **dual** of f is the function $f^{\partial} : B \to B'$ defined by $f^{\partial}(b) = \neg' f(\neg b)$.

Exercise 1.

1. Consider a transition system $TS = (S, \mathrm{Act}, \to, I, \mathrm{AP}, L)$ and $\alpha \in \mathrm{Act}$. Show that:
   - $[\![[\alpha]]\!] = [\![\langle\alpha\rangle]\!]^{\partial}$
   - $[\![\langle\alpha\rangle]\!] = [\![[\alpha]]\!]^{\partial}$
2. Let $\alpha \in \mathrm{Act}$. Show that:
   - $[\alpha] = \langle\alpha\rangle^{\partial}$
   - $\langle\alpha\rangle = [\alpha]^{\partial}$

Exercise 2.

Let B and B' be two Boolean algebras and $f : B \to B'$. Show that:

1. $f^{\partial^{\partial}} = f$
2. If f is map of join (resp. meet) semilattices, then f^{∂} is map of meet (resp. join) semilattices.
3. If f is a map of lattices, then $f^{\partial} = f$.

Ultrafilter Extension

Recall that:

- for a BAO $B^+ = (B, (f_\alpha)_{\alpha \in \mathrm{Act}})$, the ultrafilter frame $\mathfrak{Uf}(B^+)$ is defined as
  - the states are the ultrafilter over B, and
  - given $\mathcal{F}, \mathcal{H}$ two ultrafilters, $\mathcal{F} \xrightarrow{\alpha} \mathcal{H}$ iff $\forall b \in B, b \in \mathcal{H} \Rightarrow f_\alpha(b) \in \mathcal{F}$.
- for a set X, we define the function $\pi : X \to \mathfrak{Uf}(X)$ sending x to the principal filter at $\{x\}$ i.e.: $\pi(x) = \{A \in \mathcal{P}(X) \mid x \in A\}$.
- for a $TS = (S, \mathrm{Act}, \to, I, \mathrm{AP}, L)$, the ultrafilter extension $\mathfrak{Uf}(TS)$ is the transition system where
  - the states are the ultrafilters $\mathfrak{Uf}(S)$ on S
  - $\mathcal{F} \xrightarrow{\alpha} \mathcal{H}$ iff $[\![\langle\alpha\rangle]\!](A) \in \mathcal{F}$ whenever $A \in \mathcal{H}$
  - $a \in L_{\mathfrak{Uf}}(\mathcal{F})$ iff $\{s \in S \mid a \in L(s)\} \in \mathcal{F}$
  - the inital states are $\{\pi(s) \mid s \in I\}$

Exercise 3.

1. Let X be a finite set, show that π is a bijection. In other words, all ultrafilters are principal at some $\{x\}$ and principal filters $\pi(x)$ are ultrafilters.
2. Let X be an inifinite set, a subset of $S \subseteq X$ is called cofinite if its complement $X \setminus S$ is finite. Show that the family $\{S \subseteq X \mid S \text{ is cofinite}\}$ is an ultrafilter.

Exercise 4.

Consider a BAO $B^+ = (B, (f_\alpha)_{\alpha \in \mathrm{Act}})$. Show that in the ultrafilter frame $\mathfrak{Uf}(B^+)$, we have

$$\mathcal{F} \xrightarrow{\alpha} \mathcal{H} \text{ iff } \forall b \in B, f_\alpha^\partial(b) \in \mathcal{F} \Rightarrow b \in \mathcal{H}$$

Exercise 5.

Consider a $TS = (S, \mathrm{Act}, \to, I, \mathrm{AP}, L)$. Show that:

1. Given $s \in S$ and $a \in \mathrm{AP}$, $a \in L(s)$ iff $a \in L_{\mathfrak{Uf}}(\pi(s))$.
2. Show that
$$\mathcal{F} \xrightarrow{\alpha} \mathcal{H} \text{ iff } [\![[\alpha]]\!](A) \in \mathcal{F} \Rightarrow A \in \mathcal{H}$$
3. Given $s, s' \in S$ and $\alpha \in \mathrm{Act}$, $s \xrightarrow{\alpha} s'$ in TS iff $\pi(s) \xrightarrow{\alpha} \pi(s')$ in $\mathfrak{Uf}(TS)$.
4. Conclude that when TS is a finite transition system, then TS and $\mathfrak{Uf}(TS)$ are "the same" system.