ENS DE LYON

TD – Semantics and Verification

VII– Büchi Automata and ω-Regular Properties

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In the rest of the exercises, we will discuss properties of ω-regular expressions and Büchi automata. Given $U \subseteq \Sigma^*$ and $A \subseteq \Sigma^\omega$, recall that

- A is ω-regular iff there are regular languages $E_1, ..., E_n, F_1, ..., F_n \subseteq \Sigma^*$ such that for all i, $\epsilon \notin F_i$ and
$$A = E_1 \cdot F_1^\omega + ... + E_n \cdot F_n^\omega$$
- $U \cdot A = \{\hat{\sigma} \cdot \sigma \in \Sigma^\omega \mid \hat{\sigma} \in U \text{ and } \sigma \in A\}$
- If $\epsilon \notin U$, $U^\omega = \{\sigma \in \Sigma^\omega \mid \exists (u_k \in U)_{k\in\mathbb{N}}. \sigma = u_0 \cdot u_1 \cdot u_2 \cdots\}$

Exercise 1.
Let φ, ψ with parameters ρ and let $\theta(X) := \psi \vee (\varphi \wedge \bigcirc X)$. Let further $P := [\![(\varphi \mathsf{U} \psi) \vee \Box\varphi]\!]_\rho$. Show that P is a fixpoint of $[\![\theta]\!]_\rho(X)$.

Exercise 2.
Let $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$.For the following subsets $P_i \subseteq A^\omega$: is P_i open? closed? dense?, ω-regular?

1. $P_1 = \{\sigma \in A^\omega \mid \exists^\infty i, \sigma(i) = \mathtt{a}$ and there are arbitrarily large consecutive sequences of $\mathtt{b}$s$\}$
2. $P_2 = \{\sigma \in A^\omega \mid \exists^\infty i, \sigma(i) = \mathtt{a} \wedge \exists k > i, \sigma(k) = \mathtt{b}\}$

Büchi Automata

Exercise 3.

1. Let AP $= \{a, b\}$. Give an non-deterministic Büchi automaton (NBA) that accepts "b holds for a finite time until a holds forever and b never holds again". You may use propositional formulas as labels.
2. Depict an NBA for the language described by the ω-regular expression $(AB+C)^*((AA+B)C)^\omega + (A^*C)^\omega$.

Constructions on Büchi Automata

Exercise 4.
Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Büchi automata, and $\mathcal{A}$ an NFA.

1. Show that there is a Büchi automaton $\mathcal{A}_1 + \mathcal{A}_2$ with $\mathcal{L}_\omega(\mathcal{A}_1 + \mathcal{A}_2) = \mathcal{L}_\omega(\mathcal{A}_1) \cup \mathcal{L}_\omega(\mathcal{A}_2)$.
2. Show that there is a Büchi automaton $\mathcal{A} \odot \mathcal{A}_1$ with $\mathcal{L}_\omega(\mathcal{A} \odot \mathcal{A}_1) = \mathcal{L}(\mathcal{A}) \cdot \mathcal{L}(\mathcal{A}_1)$.
3. Show that if $\epsilon \notin \mathcal{L}(\mathcal{A})$, there is a Büchi automaton $\mathcal{A}_\omega$ such that $\mathcal{L}_\omega(\mathcal{A}_\omega) = \mathcal{L}(\mathcal{A})^\omega$.
4. Show that there is a Büchi automaton $\mathcal{A}_1 \sqcap \mathcal{A}_2$ with $\mathcal{L}_\omega(\mathcal{A}_1 \sqcap \mathcal{A}_2) = \mathcal{L}_\omega(\mathcal{A}_1) \cap \mathcal{L}_\omega(\mathcal{A}_2)$.

Decomposition of ω-regular Linear Time Properties

Exercise 5.

Let $U \subseteq \Sigma^*$ and $A, B \subseteq \Sigma^\omega$.

1. Show that $\mathrm{pref}(A \cup B) = \mathrm{pref}(A) \cup \mathrm{pref}(B)$.
2. Show that $\mathrm{pref}(U \cdot A) = \mathrm{pref}(U) \ \cup \ U \cdot \mathrm{pref}(A)$.
3. Show that $\mathrm{pref}(U^\omega) = \mathrm{pref}(U^*)$.

Exercise 6.

We want to show the decomposition theorem for ω-regular properties.

1. Show that if $A \subseteq \Sigma^\omega$ is ω-regular, then $\mathrm{pref}(A) \subseteq \Sigma^*$ is regular. (You may use that $\mathrm{pref}(U)$ is regular if $U \subseteq \Sigma^*$ is regular.)
2. Show that $\mathrm{cl}(U)$ is a safety property induced by $P_{bad} = U^c$.
3. Deduce that if $P \subseteq \mathcal{P}(\mathrm{AP})^\omega$ is an ω-regular safety property, then P is a *regular* safety property.
4. Let P_{bad} be a set of bad prefix of $\mathrm{cl}(U)$ and $\mathcal{A}$ be a complete deterministic automaton recognizing P_{bad} such that for all final state q_F of $\mathcal{A}$ and all $a \in \Sigma, \delta(q_F, a) = q_F$. Show that $\mathcal{L}_\omega(\mathcal{A}) = (2^{\mathrm{AP}})^\omega \backslash \mathrm{cl}(U)$.
5. Show that if U is regular then $\mathrm{cl}(U)$ is ω-regular *(we recall that if* $\mathrm{cl}(U)$ *is a regular safety property, we can always find* P_{bad} *and* $\mathcal{A}$ *satisfying the properties of the previous question).*
6. Show that for every ω-regular linear time property P, there is a ω-regular safety property P_{safe} and a ω-regular liveness property P_{live} such that $P = P_{safe} \cap P_{live}$.