# Modelling Nondeterminism, Probability and Termination

The story of Ralph's internship with his amazing supervisors Matteo and Valeria.

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ENS de Lyon

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# Introduction

Overview of the presentation:

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- **Set** monads and equational theories modelling

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  - Nondeterministic choices

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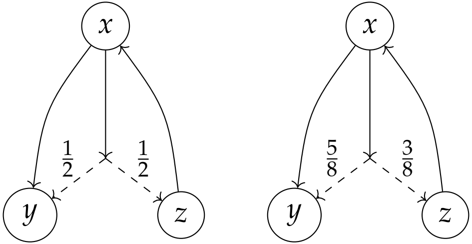
  - Nondeterminism and probability
- Combining nondeterminism, probability and termination NEW!

# Motivation



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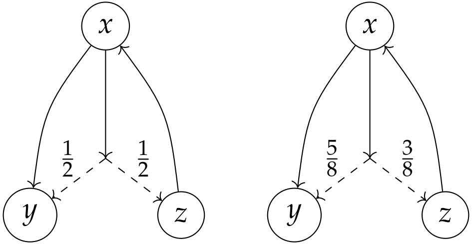


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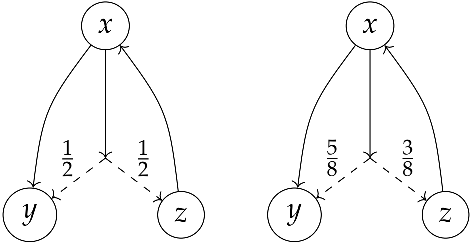


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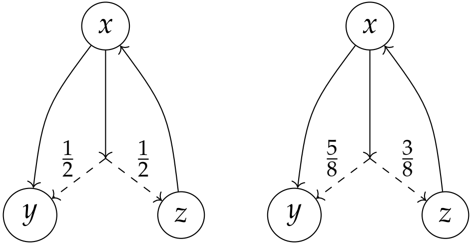


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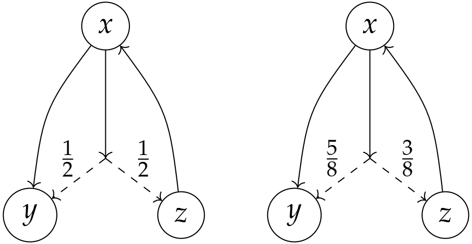


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# Algebraic Effects

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A link between these two pictures: **algebraic presentations of monads**.

# Nondeterministic Choices (in **Set**)

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$$\mathcal{P}(X) = \{\text{non-empty finite subsets of } X\}$$
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- Theory of (sup-)semilattices:
  A binary operation $\oplus$ satisfying

$$\begin{aligned} x \oplus x &= x & I \\ x \oplus y &= y \oplus x & C \\ (x \oplus y) \oplus z &= x \oplus (y \oplus z). & A \end{aligned}$$

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- Operational semantics

$$\frac{q \xrightarrow{\mathtt{a}} r \qquad q \xrightarrow{\mathtt{a}} s}{q \xrightarrow{\mathtt{a}} r \oplus s} \qquad \frac{q \xrightarrow{\mathtt{a}} r \qquad q' \xrightarrow{\mathtt{a}} r'}{q \oplus q' \xrightarrow{\mathtt{a}} r \oplus r'}$$

$$\frac{q \downarrow \mathsf{accept}}{q \oplus r \downarrow \mathsf{accept}}$$

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$$\begin{aligned}\mathcal{D}(X) = \{&\text{finitely supported}\\ &\text{probability distributions on } X\}\\ \eta_X &= x \mapsto \delta_x \text{ (Dirac)}\\ \mu_X &= \Phi \mapsto \sum_{\varphi \in \mathrm{supp}(\Phi)} \Phi(\varphi) \cdot \varphi\end{aligned}$$

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$$\begin{aligned} x +_p x &= x && I_p \\ x +_p y &= y +_{1-p} x && C_p \\ (x +_q y) +_p z &= x +_{pq} (y +_{\frac{p\bar{q}}{\overline{pq}}} z) && A_p \end{aligned}$$

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$$\frac{q \xrightarrow{\mathtt{a},p} r \qquad q \xrightarrow{\mathtt{a},1-p} s}{q \xrightarrow{\mathtt{a}} r +_p s} \qquad \frac{q \xrightarrow{\mathtt{a}} r \qquad q' \xrightarrow{\mathtt{a}} r'}{q +_p q' \xrightarrow{\mathtt{a}} r +_p r'}$$

$$\frac{q \downarrow o(q) \qquad r \downarrow o(r)}{q +_p r \downarrow po(q) + (1-p)o(r)}$$

# Termination (in **Set**)

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- Maybe monad $\cdot + \mathbf{1}$:

$$X + \mathbf{1} = X \sqcup \{\star\}$$
$$\eta_X = \mathsf{inl}^{X+\mathbf{1}}$$
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$$\frac{q \not\to}{q \to \star}$$

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- Operational semantics: Combine all previous rules.

[BSV19]

# Metric Reasoning

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$$\forall x, x' \in X, d_Y(f(x), f(x')) \leq d_X(x, x').$$

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- Equations become quantitative

$$x =_\varepsilon y \qquad\qquad (\varepsilon \in [0, 1])$$

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- Quantitative equational logic

# Quantitative Equational Logic

| | |
|---|---|
| $\vdash t =_0 t$ | Refl |
| $\vdash t =_1 s$ | 1–bounded |
| $t =_\varepsilon s \vdash s =_\varepsilon t$ | Symm |
| $t =_\varepsilon s, s =_\delta u \vdash t =_{\varepsilon+\delta} u$ | Triang |
| $t =_\varepsilon s \vdash t =_{\varepsilon+\delta} s$ | Max (for $\delta > 0$) |
| $\forall \varepsilon > \delta, t =_\varepsilon s \vdash t =_\delta s$ | Arch (for $\delta \geq 0$) |
| $\forall 1 \leq i \leq n, t_1 =_\varepsilon s_n \vdash f(t_1, \ldots, t_n) =_\varepsilon f(s_1, \ldots, s_n)$ | Nexp (for $f$ an $n$–ary) |
| $\Gamma \vdash t =_\varepsilon s \implies \sigma(\Gamma) \vdash \sigma(t) =_\varepsilon \sigma(s)$ | Subst (for $\sigma$ a substitution) |
| $\phi \in \Gamma \implies \Gamma \vdash \phi$ | Assumption |

Introduced by Mardare, Panangaden and Plotkin [MPP16].

# Nondeterminism Choices (in **1Met**)

## Categorically

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## Categorically

- Hausdorff lifting of $d : X \times X \to [0,1]$ into $H(d) : \mathcal{P}(X) \times \mathcal{P}(X) \to [0,1]$:

$$H(d)(S,T) = \max \left\{ \sup_{s \in S} \inf_{t \in T} d(s,t), \sup_{t \in T} \inf_{s \in S} d(s,t) \right\}$$

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- Hausdorff lifting of $\mathcal{P}$: $\widehat{\mathcal{P}}(X,d) = (\mathcal{P}(X), H(d))$.

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- Theory of quantitative (sup-)semilattices: binary operation $\oplus$ satisfying

$$\begin{array}{rll}
& \vdash x \oplus x =_0 x & I \\
& \vdash x \oplus y =_0 y \oplus x & C \\
& \vdash (x \oplus y) \oplus z =_0 x \oplus (y \oplus z) & A \\
x_1 =_{\varepsilon_1} y_1, x_2 =_{\varepsilon_2} y_2 & \vdash x_1 \oplus x_2 =_{\max\{\varepsilon_1,\varepsilon_2\}} y_1 \oplus y_2 & H
\end{array}$$

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- Kantorovitch lifting of $d : X \times X \to [0,1]$ into $K(d) : \mathcal{D}(X) \times \mathcal{D}(X) \to [0,1]$:

$$K(d)(\varphi, \psi) = \inf_{\omega} \sum_{x_1, x_2 \in X} \omega(x_1, x_2) \cdot d(x_1, x_2),$$

where $\omega$ ranges over couplings of $\varphi$ and $\psi$.

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- Kantorovitch lifting of $\mathcal{D}$: $\widehat{\mathcal{D}}(X, d) = (\mathcal{D}(X), K(d))$.

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- Kantorovich lifting of $d : X \times X \to [0, 1]$ into $K(d) : \mathcal{D}(X) \times \mathcal{D}(X) \to [0, 1]$:

$$K(d)(\varphi, \psi) = \inf_{\omega} \sum_{x_1, x_2 \in X} \omega(x_1, x_2) \cdot d(x_1, x_2),$$

  where $\omega$ ranges over couplings of $\varphi$ and $\psi$.
- Kantorovich lifting of $\mathcal{D}$: $\widehat{\mathcal{D}}(X, d) = (\mathcal{D}(X), K(d))$.

## Equationally

# Probabilistic choices (in **1Met**)

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## Equationally

- Theory of quantitative convex algebras

$$\begin{array}{rll}
& \vdash x +_p x =_0 x & I_p \\
& \vdash x +_p y =_0 y +_p x & C_p \\
& \vdash (x +_q y) +_p z =_0 x +_{pq} (y +_{\frac{p\bar{q}}{\overline{pq}}} z) & A_p \\
x_1 =_{\varepsilon_1} y_1, x_2 =_{\varepsilon_2} y_2 & \vdash x_1 +_p x_2 =_{p\varepsilon_1 + (1-p)\varepsilon_2} y_1 +_p y_2 & K_p
\end{array}$$

# Termination (in **1Met**)

Categorically

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- The maybe monad $\cdot + \widehat{\mathbf{1}}$ puts $\star$ at distance 1 from every other point:

$$X + \widehat{\mathbf{1}} = X \sqcup \{\star\}$$

$$(d + d_{\widehat{\mathbf{1}}})(x, y) = \begin{cases} 1 & x =_0 \star \vee y = \star \\ d(x, y) & \text{otherwise} \end{cases}$$

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## Equationally

- Theory of pointed spaces:

$$\vdash x =_1 \star$$

# Nondeterminism and Probability (in **1Met**)

Categorically

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- Combine Kantorovitch and Hausdorff lifting of $d : X \times X \to [0,1]$ to obtain $\widehat{\mathcal{C}}(X,d) = (\mathcal{C}(X), HK(d))$.

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## Equationally

- Quantitative theory of convex semilattices: combine all previous axioms plus

$$\vdash (x \oplus y) +_p z =_0 (x +_p z) \oplus (y +_p z) \qquad D_p$$

[MV20]

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- On **Set**: $\mathcal{C}(\cdot + \mathbf{1})$ is presented by the theory of pointed convex semilattices.
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$$x \oplus \star = x \qquad (\bot) \qquad\qquad x +_p \star = \star \qquad (BH_p)$$

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## On **1Met**

- The black-hole equation trivializes the theory.
- There is (probably) no monad $\mathcal{C} + \mathbf{1}$ on **1Met**.

# Adding Termination (hard)

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- Remove the black-hole equation.
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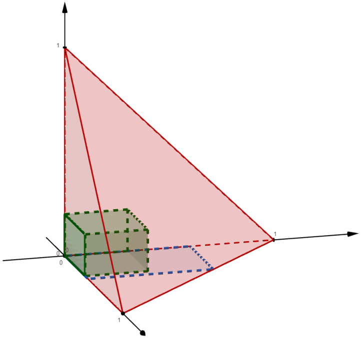

Figure 1: Examples of $\perp$–closed sets.

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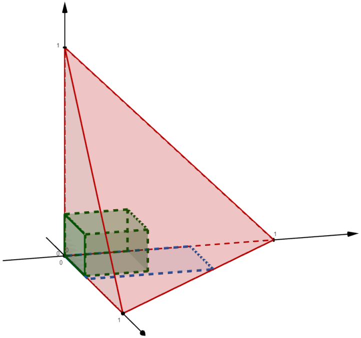

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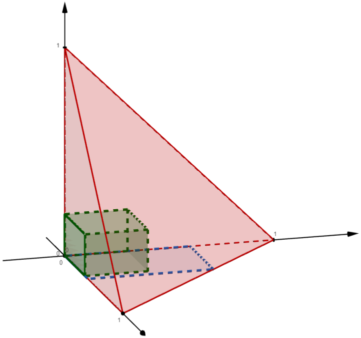


Figure 1: Examples of $\perp$–closed sets.

## On **1Met**

- The theory of quantitative convex semilattices with bottom corresponds to $\mathcal{C}^{\downarrow}$ with the Kantorovitch and Hausdorff liftings of the metrics.

# Summary

| On **Set** | | On **1Met** | |
|---|---|---|---|
| $\mathcal{P}$ | Semilattices | $\widehat{\mathcal{P}}$ | Quantitative semilattices |
| $\mathcal{D}$ | Convex algebras | $\widehat{\mathcal{D}}$ | Quantitative convex algebras |
| $\cdot + \mathbf{1}$ | Pointed sets | $\cdot + \widehat{\mathbf{1}}$ | Pointed spaces |
| $\mathcal{C}$ | Convex semilattices (CS) | $\widehat{\mathcal{C}}$ | Quantitative CS |
| $\mathcal{C}(\cdot + \mathbf{1})$ | Pointed CS | $\widehat{\mathcal{C}}(\cdot + \widehat{\mathbf{1}})$ | Quantitative pointed CS |
| $\mathcal{C} + \mathbf{1}$ | CS with $\perp$ and black-hole | Trivial | Quantitative CS with $\perp$ and black-hole |
| $\mathcal{C}^{\downarrow}$ | CS with $\perp$ | $\widehat{\mathcal{C}}^{\downarrow}$ | Quantitative CS with $\perp$ |

# Applications

See Section VI of preprint.

# Future Work

- Can our results be obtained *compositionally*?

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- Can our results be obtained *compositionally*? Close to answering yes with Daniela Petrişan.
- Where can quantitative algebraic reasoning be applied?

# References

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Merci !