ENS DE LYON

TD – Semantics and Verification

VI– LTL

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Recall that:

- $[\![\Diamond\varphi]\!]_\rho = \{\sigma \in (2^{\mathrm{AP}})^\omega \mid \exists i, \sigma\restriction i \in [\![\varphi]\!]_\rho\}$
- $[\![\Box\varphi]\!]_\rho = \{\sigma \in (2^{\mathrm{AP}})^\omega \mid \forall i, \sigma\restriction i \in [\![\varphi]\!]_\rho\}$
- $[\![\varphi \mathsf{\ U\ } \psi]\!]_\rho = \{\sigma \in (2^{\mathrm{AP}})^\omega \mid \exists i, \sigma\restriction i \in [\![\psi]\!]_\rho, \forall j < i, \sigma\restriction j \in [\![\varphi]\!]_\rho\}$
- $\varphi \mathsf{\ W\ } \psi := \neg(\neg\psi \mathsf{\ U\ } \neg(\varphi \vee \psi))$

Moreover, a state s of a transitition system satisfies a LTL formula φ with parameter ρ if and only if $\mathrm{Traces}(s) \subseteq [\![\varphi]\!]_\rho$.

Exercise 1.

Let L be a complete lattice and let $f : L \to L$ be a monotone function. Show that $\mu(f) = \bigwedge\{a \in L \mid f(a) \leq a\}$ (resp. $\nu(f) = \bigvee\{a \in L \mid a \leq f(a)\}$) is the least fixpoint (resp. greatest fixpoint) of f.

Exercise 2.

Show that:

1. $\neg(\varphi \mathsf{\ W\ } \psi) \equiv \neg\psi \mathsf{\ U\ } (\neg\varphi \wedge \neg\psi)$
2. $\neg(\varphi \mathsf{\ U\ } \psi) \equiv \neg\psi \mathsf{\ W\ } (\neg\varphi \wedge \neg\psi)$
3. $\bigcirc(\varphi \mathsf{\ U\ } \psi) \equiv \bigcirc\varphi \mathsf{\ U\ } \bigcirc\psi$
4. $\Diamond\phi \equiv \neg\Box\neg\phi$
5. $\Diamond\phi \equiv \phi \vee \bigcirc\Diamond\phi$

Exercise 3.

Let ϕ be a formula with parameters ρ. We define $\phi_\Box(X) = \phi \wedge \bigcirc X$. Show that for every valuation ρ, $[\![\Box\phi]\!]_\rho$ is the greatest fixpoint of $[\![\phi_\Box]\!]_\rho$.

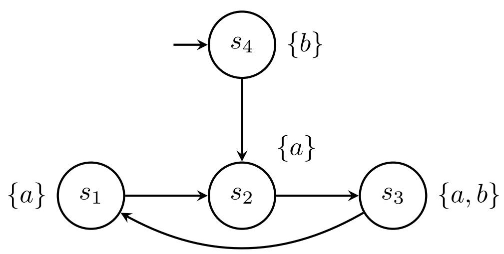
Exercise 4.

Show that:

1. $[\![\top \mathsf{\ U\ } \varphi]\!]_\rho = \{\sigma \in (2^{\mathrm{AP}})^\omega \mid \exists i, \sigma\restriction i \in [\![\varphi]\!]_\rho\}$
2. $[\![\varphi \mathsf{\ W\ } \bot]\!]_\rho = \{\sigma \in (2^{\mathrm{AP}})^\omega \mid \forall i, \sigma\restriction i \in [\![\varphi]\!]_\rho\}$

Exercise 5.

Consider the following transition system over the set of atomic propositions $\{a, b\}$:



Indicate for each of the following LTL formulae the set of states for which these formulae are fulfilled:

A. $\bigcirc a$

B. $\bigcirc\bigcirc\bigcirc b$

C. $\Box b$

D. $\Box\Diamond a$

E. $\Box(b \mathsf{U} a)$

F. $\Diamond(a \mathsf{U} b)$

Exercise 6.

A formula ϕ is said to be in positive normal form if the negations is only applied to atoms or variables (for instance, with $a, b \in \text{AP}$, $a \wedge \neg b$ is in positive normal form but $\neg(a \vee b)$ is not).

1. Show that every formula is equivalent to a formula in positive normal form.
2. We replace the operator W by the release operator R satisfying $\neg(\phi \mathsf{R} \psi) \equiv (\neg\phi) \mathsf{U} (\neg\psi)$.
   Show that, with this operator, every formula is still equivalent to a formula in positive normal form.
3. Compare the two methods regarding the size of the formulae.