

TD – Semantics and Verification

III– Topology

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In this set of exercises, we will discuss topological characterisations of safety and liveness properties.

Liveness Properties

Recall that

- a LT property is a liveness property if for all finite $\hat{\sigma} \in (2^{AP})^*$, there is $\sigma \in P$ such that $\hat{\sigma} \subseteq \sigma$
- $\text{pref}(P) = \{\hat{\sigma} \text{ finite} \mid \exists \sigma \in P, \hat{\sigma} \subseteq \sigma\}$
- $\text{cl}(P) = \{\sigma \mid \text{pref}(\sigma) \subseteq \text{pref}(P)\}$

Exercise 1.

Show that

1. A LT property P is a liveness property if and only if $\text{cl}(P) = (2^{AP})^\omega$.
2. If P and Q are liveness properties, then so is $P \cup Q$.
3. There are two liveness properties P and Q such that $P \cap Q$ is not a liveness property.

Topological Spaces

- A *topological space* is a pair (X, \mathcal{U}) of a set X and a subset \mathcal{U} of $\mathcal{P}(X)$, called the *open sets of X* , such that
 1. $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$;
 2. for any set I and family $\{U_i \in \mathcal{U}\}_{i \in I}$, also $\bigcup_{i \in I} U_i \in \mathcal{U}$; and
 3. for all $U, V \in \mathcal{U}$, also $U \cap V \in \mathcal{U}$.

A set $U \in \mathcal{U}$ is called *open* and elements $x \in X$ are called points. If \mathcal{U} is clear from the context, we often refer to X as the topological space.

- Given a point $x \in X$, we say that $N \subseteq X$ is a *neighbourhood* of x if there is an open set $U \in \mathcal{U}$, such that $x \in U$ and $U \subseteq N$. The collection of all neighbourhoods of x is denoted by \mathcal{N}_x .
- For any $F \subseteq X$, we denote by F^c the complement of F relative to X , i.e.

$$F^c = X \setminus F = \{x \in X \mid x \notin F\}.$$

We say that $F \subseteq X$ is *closed*, if F^c is open.

- A subset $D \subseteq X$ is said to be *dense* if $D \cap U \neq \emptyset$ for all non-empty $U \in \mathcal{U}$.

Exercise 2.

For any topological space (X, \mathcal{U}) , show that

1. \emptyset and X are closed

2. for any set I and family $\{F_i \text{ closed}\}_{i \in I}$, also $\bigcap_{i \in I} F_i$ closed; and
3. for all closed F and G , also $F \cup G$ is closed.

For any set $A \subseteq X$, we define the *closure* \overline{A} of A by

$$\overline{A} = \bigcap \{F \subseteq X \mid F \text{ closed and } A \subseteq F\}.$$

One can show that \overline{A} is the smallest closed subset of X containing A .

Exercise 3.

For any topological space (X, \mathcal{U}) , show that

1. A set $A \subseteq X$ is open iff for every $x \in A$ there is an open set $U \in \mathcal{U}$ such that $x \in U$ and $U \subseteq A$.
2. A set $A \subseteq X$ is closed iff for every $x \notin A$ there is an open set $U \in \mathcal{U}$ such that $x \in U$ and $U \cap A = \emptyset$.

Exercise 4.

Let (X, \mathcal{U}) be a topological space and $A \subseteq X$. An *adherent point* (or *point of closure*) of A is a point $x \in X$ such that for any neighborhood N of x , $N \cap A \neq \emptyset$.

1. Show that \overline{A} is the set of adherent point of A , i.e. $\overline{A} = \{x \in X \mid \forall N \in \mathcal{N}_x. N \cap A \neq \emptyset\}$.
2. Show that A is closed iff $\overline{A} = A$.
3. Show A is dense iff $\overline{A} = X$.

Metric and Topology on Infinite Words

Exercise 5.

Let (X, d) be a metric space, the open ball of radius ε and centered at x is denoted by $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$. The open ball topology associated to (X, d) is defined by

$$\mathcal{U} = \{U \subseteq X \mid \forall x \in U. \exists \varepsilon > 0. B_\varepsilon(x) \subseteq U\}.$$

1. Show that (X, \mathcal{U}) is indeed a topological space.
2. Show that for any $S \subseteq X$ that we have $\overline{S} = \{x \in X \mid \forall \varepsilon > 0. B_\varepsilon(x) \cap S \neq \emptyset\}$.

Exercise 6.

Let A be a non-empty set. The set of infinite sequences over A is denoted by A^ω as before. Let $d: A^\omega \times A^\omega \rightarrow \mathbb{R}_{\geq 0}$ be given by

$$d(\sigma, \tau) = \begin{cases} 0, & \sigma = \tau \\ 2^{-\min\{k \in \mathbb{N} \mid \sigma(k) \neq \tau(k)\}}, & \sigma \neq \tau \end{cases}$$

Let us also denote by $\sigma|_n$ the prefix of length n of σ . Show that (A^ω, d) is a metric space.

Exercise 7.

For A a non-empty set, we define

- $\text{ext}(w) = \{\sigma \in A^\omega \mid w \subseteq \sigma\}$ for $w \in A^*$.
- $\text{ext}(W) = \bigcup_{w \in W} \text{ext}(w)$ for $W \subseteq A^*$.
- $\mathcal{U} = \{\text{ext}(W) \mid W \subseteq A^*\}$.

Show that (A^ω, \mathcal{U}) is a topological space.

Exercise 8.

Show that

1. A set $P \subseteq A^\omega$ is open iff for every $\sigma \in P$ there is a finite word $\hat{\sigma} \subseteq \sigma$ such that $\beta \in P$ for every $\beta \in A^\omega$ such that $\hat{\sigma} \subseteq \beta$.

2. A set $P \subseteq A^\omega$ is closed iff for every $\sigma \notin P$ there is a finite word $\hat{\sigma} \subseteq \sigma$ such that $\beta \notin P$ for every $\beta \in A^\omega$ such that $\hat{\sigma} \subseteq \beta$.

Exercise 9.

Show that

1. $P \subseteq (2^{\text{AP}})^\omega$ is a safety property iff P is closed.
2. $P \subseteq (2^{\text{AP}})^\omega$ is a liveness property iff P is dense.