

TD - Semantics and Verification

IV- Partial Orders and Lattices Friday 11th February 2022

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Partial Orders and Lattices

- A partial order is a pair (A, \leq) of a set A and binary relation \leq which is reflexive $(\forall a \in A, a \leq a)$, transitive $(\forall a, b, c \in A, a \leq b \text{ and } b \leq c \text{ implies } a \leq c)$, and antisymmetric $(\forall a, b \in A, a \leq b \text{ and } b \leq a \text{ implies } a = b)$.
- A join (or least upper bound or supremum) of $S \subseteq A$ is an upper bound of S smaller than every upper bound of S, it is denoted by $\bigvee S$.
- A meet (or greatest lower bound or infimum) of $S \subseteq A$ is a lower bound of S greater than every lower bound b of S, it is denoted by $\bigwedge S$.
- A complete lattice is a partial order such that every subset has both a join and a meet.
- Given a topological space (X, \mathcal{U}) , the *interior* of a set $A \subseteq X$ is $\mathring{A} = \bigcup \{U \in \mathcal{U} \mid U \subseteq A\}$.

Exercise 1.

Show that the following are equivalent for a partial order (L, \leq) :

- 1. (L, \leq) is a complete lattice,
- 2. every subset $S \subseteq L$ has a least upper bound $\bigvee S \in L$,
- 3. every subset $S \subseteq L$ has a greatest lower bound $\bigwedge S \in L$.

Exercise 2.

Consider the space A^{ω} with $A = \{a, b\}$. Show that $\bigcap_{n \in \mathbb{N}} \operatorname{ext}(a^n)$ is not open.

Closure Operators

A closure operator on a partial order (A, \leq) is a function $c: A \to A$ which is monotonote $(\forall a, b \in A, a \leq b \Rightarrow c(a) \leq c(b))$, expansive $(\forall a \in A, a \leq c(a))$, and idempotent $(\forall a \in A, c(c(a)) = c(a))$.

Exercise 3.

Consider a closure operator c on a complete lattice (L, \leq) . Show that $L^c = \{a \in L \mid c(a) = a\}$ is a complete lattice with greatest lower bounds $\prod S = \bigwedge S$ and least upper bounds $\coprod S = c(\bigvee S)$.

Exercise 4.

A Kuratowski closure operator is a closure operator $c: 2^X \to 2^X$ such that $c(\emptyset) = \emptyset$ and $c(A \cup B) = c(A) \cup c(B)$.

- 1. Consider a topological space (X,\mathcal{U}) . Show that $\overline{(-)}$ is a Kuratowski closure operator.
- 2. Given a Kuratowski closure operator $c: 2^X \to 2^X$, show that there is topology \mathcal{U} on X such that the closed sets for \mathcal{U} are exactly the closed sets for c, ie the sets such that A = c(A).

Galois Connection

• Given partial orders (A, \leq_A) and (B, \leq_B) , a Galois connection $g \dashv f : A \to B$ is given by a pair of functions $g : A \to B$ and $f : B \to A$ such that for all $a \in A$ and $b \in B$, we have

$$g(a) \leq_B b \text{ iff } a \leq_A f(b).$$

g (resp. f) is called the lower adjoint (resp. upper adjoint).

• Given a non-empty set A, we define

$$\operatorname{pref}: 2^{A^{\omega}} \mapsto 2^{A^*} \qquad \operatorname{cl}: 2^{A^*} \mapsto 2^{A^{\omega}}$$

$$P \mapsto \bigcup \{\operatorname{pref}(\sigma) \mid \sigma \in P\} \qquad W \mapsto \{\sigma \in A^{\omega} \mid \operatorname{pref}(\sigma) \subseteq W\}$$

Notice that the function cl defined in the second tutorial is actually cl(pref(P)) here.

Exercise 5.

Consider a Galois connection $g \dashv f : A \to B$.

- 1. Show that both f and g are monotone.
- 2. Show that $f \circ g$ is a closure operator.
- 3. Suppose $f': B \to A$ is such that $g \dashv f'$, show that f = f'.
- 4. Suppose $g': A \to B$ is such that $g' \dashv f$, show that g = g'.

In words, the last two points state that if a monotone function has a lower (or upper) adjoint, then it is unique.

Exercise 6.

Consider two complete lattices (A, \leq_A) and (B, \leq_B) .

- 1. Show that a function $f: B \to A$ preserves greatest lower bounds iff f has a lower adjoint.
- 2. Show that a function $g: A \to B$ preserves least upper bounds iff g has an upper adjoint.

Exercise 7.

Show that pref \dashv cl : $2^{A^{\omega}} \rightarrow 2^{A^*}$ form a Galois connection.

Exercise 8

Given $P \subseteq A^{\omega}$, show that $\overline{P} = \operatorname{cl}(\operatorname{pref}(P))$. Deduce that P is a safety property if and only if it is closed.

Exercise 9.

Show that $P \subseteq A^{\omega}$ is a liveness property if and only if it is dense. Deduce that any LTP is the interesection of a safety property and a liveness property.

Continous Functions

Consider topological spaces (X, \mathcal{U}_X) and (Y, \mathcal{U}_Y) . A function $f: X \to Y$ is continous if $f^{-1}(V)$ is open in X whenever V is open in Y.

Exercise 10.

Show that $f: A^{\omega} \to B^{\omega}$ is continuous iff

$$\forall n \in \mathbb{N}, \ \forall \alpha \in A^{\omega}, \ \exists k \in \mathbb{N}, \ \forall \beta \in A^{\omega} \Big(\beta(0) \cdots \beta(k) = \alpha(0) \cdots \alpha(k) \\$$
$$f(\beta)(0) \cdots f(\beta)(n) = f(\alpha)(0) \cdots f(\alpha)(n) \Big)$$