Category Theory - Important Results

Ralph Sarkis

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This is a record of the important results we cover during the lectures we will have in the summer 2018. We will try to go over two sets of lecture notes by Mariusz Wodzicki. Our goal is to introduce the concept of categories and build enough familiarity with them to be able to see other mathematical concepts we know in a more categorical point of view.

1 Review

In this section, we will review some concepts that will be helpful in the study of category theory.

1.1 Operations on sets

We give formal definitions of common set operators, giving a bit of a taste of the language we will use.

Definition 1 (Union of sets)**.** Let X be a set, we can define the **union** operator like so:

$$\bigcup = A \mapsto \{x \in X \mid \exists S \in A, x \in S\} : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$$

Definition 2 (Intersection of sets)**.** Let X be a set, we can define the **intersection** operator like so:

$$\bigcap = A \mapsto \{x \in X \mid \forall S \in A, x \in S\} : \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(X)$$

Definition 3 (Difference of sets)**.** Let X be a set, we an define the **difference** operator like so:

$$\setminus = (S, T) \mapsto \{x \in X \mid x \in S \land x \notin T\} : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$$

Definition 4 (Cartesian product)**.** Let $(X_i)_{i \in I}$, where I is some index set, be a family of sets, the Cartesian product of these sets is

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} \mid \forall i \in I, x_i \in X_i\}.$$

We can also see each element as a function $f : I \to \cup_{i \in I} X_i$ such that $f(i) \in X_i$ for all $i \in I$.

If a family of set is closed under the three first operations, we call it a ring of sets.

Definition 5 (Ring of sets)**.** A non-empty family of sets R is called a **ring** of sets if for any two elements r and r', we have $r \cup r', r \cap r', r \setminus r' \in R$.

1.2 Classes vs. Sets

Several times in our coverage of category theory, we will need to use the concept of a class. It is very similar to that of a set and has one simple difference. While a set can contain another set, classes cannot contain other classes. This difference is necessary because some collections of objects can simply not form a set. Famous examples include the class of ordinal numbers which, by the Burali-Forti paradox, cannot be a set and the class of all sets that do not contain themselves which, by the Russel paradox, cannot be a set.

2 Introduction to categories

2.1 Basic definitions

Definition 6 (Oriented graph)**.** An **oriented graph** G consists of a class of nodes G_0, a class of arrows G_1 along with two functions $s, t : G_1 \to G_0$, so that each arrow $f \in G_1$ has a source $s(f)$ and a target $t(f)$.

Remark 7. The nodes can also be called vertices or objects while arrows are also known as morphisms in the context of categories.

Definition 8 (Paths)**.** A **path** in an oriented graph G is a sequence of arrows $(f_1, \dots, f_k)$ that are **composable** in the sense that $t(f_i) = s(f_{i-1})$ for $i = 2, \dots, k$. We will denote G_k to be the class of paths of length k and we often refer to G_2 simply as the class of composable arrows.

Remark 9. Note that the notation indicating the direction of the path does not translate well to what we usually think of as a path in a graph. The reason is that the arrows are more linked to the composition of functions than paths in graphs.

Definition 10 (Category)**.** An oriented graph C along with a map $\circ : C_2 \to C_1$ is a **category** if for any $(f, g, h) \in C_3$, we have $f \circ (g \circ h) = (f \circ g) \circ h$, namely, composition is associative.

Definition 11 (Unital category)**.** A category C is called **unital** if it is equipped with a map $u : C_0 \to C_1$ (for $A \in C_0$, we denote $u(A) = \mathrm{id}_A$) such that for any arrow $f : A \to B$, we have $f \circ \mathrm{id}_A = \mathrm{id}_B \circ f = f$.

Definition 12 (Hom sets)**.** Let C be a category and $A, B \in C_0$, we denote

$$\mathrm{Hom}_C(A, B) = \{f \in C_1 \mid s(f) = A \wedge t(f) = B\}.$$

Definition 13 (Small and discrete)**.** A category C is called **small** if the class of objects and morphisms is not proper (it is a set). It is called **discrete** if there are no morphisms and **discrete unital** if there are no morphisms other than the identity morphisms.

Definition 14 (Subcategory)**.** Let C be a category, a category C' is a **subcategory** of C if:

1. The objects and morphisms of C' are objects and morphisms of C (i.e.: $C'_0 \subseteq C_0$ and $C'_1 \subseteq C_1$).
2. For every morphism $f \in C'_1$, $s(f), t(f) \in C'_0$.
3. For every pair of composable arrows $(f, g) \in C'_2$, $f \circ_{C'} g = f \circ_C g \in C'_1$.

If we are working with unital categories we have the additional requirement that for any $A \in C'_0$, $u_{C'}(A) \in C'_1$. One can show that since composition is the same as in C, the identity must be the same.

Definition 15 (Full and wide)**.** A subcategory C' of C is called **full** if for any objects $A, B \in C'_0$, we have $\mathrm{Hom}_{C'}(A, B) = \mathrm{Hom}_C(A, B)$. It is called **wide** if $C'_0 = C_0$.

Definition 16 (Covariant functor)**.** Let C and D be categories, a **covariant functor** $F : C \rightsquigarrow D$ is a pair of maps $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ that are defined such that the following diagrams commute (where F_2 is induced by the definition of F_1 with $(f, g) \mapsto (F_1(f), F_1(g))$).

$$\begin{tikzcd} C_0 \arrow[d, "F_0"'] & C_1 \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "F_1"'] & C_0 \arrow[d, "F_0"'] \\ D_0 & D_1 \arrow[l, "s"] \arrow[r, "t"'] & D_0 \end{tikzcd} \qquad \begin{tikzcd} C_2 \arrow[r, "F_2"] \arrow[d, "\circ_C"'] & D_2 \arrow[d, "\circ_D"'] \\ C_1 \arrow[r, "F_1"'] & D_1 \end{tikzcd}$$

If we are working with unital categories, we may want to talk about a **unital** functor which requires this additional diagram to commute.

$$\begin{tikzcd} C_0 \arrow[r, "F_0"] \arrow[d, "u_C"'] & D_0 \arrow[d, "u_D"'] \\ C_1 \arrow[r, "F_1"'] & D_1 \end{tikzcd}$$

Definition 17 (Contravariant functor)**.** Let C and D be categories, a **contravariant functor** $F : C \rightsquigarrow D$ is similar to a covariant functor except for the first diagram which changes a bit (see below) and the definition of F_2 which becomes: $(f, g) \mapsto (F_1(g), F_1(f))$.

$$
\begin{tikzcd}
C_0 & C_1 \arrow[l, "s"'] \arrow[r, "t"] & C_0 \\
D_0 \arrow[from=u, "F_0"'] & D_1 \arrow[from=u, "F_1"'] \arrow[l, "t"] \arrow[r, "s"'] & D_0 \arrow[from=u, "F_0"']
\end{tikzcd}
$$

Example 18 (Hom functors)**.** Let C be a category and $A \in C_0$ one of its object. We define the covariant and contravariant Hom functors from C to **Set**.

A. The functor $\mathrm{Hom}_C(A, -) : C \rightsquigarrow \textbf{Set}$ sends an object $B \in C_0$ to the hom set $\mathrm{Hom}_C(A, B)$ and a morphism $f : B \to B'$ to the function

$$\mathrm{Hom}_C(A, f) : \mathrm{Hom}_C(A, B) \to \mathrm{Hom}_C(A, B') = g \mapsto f \circ g.$$

Let us check that this is a covariant functor. We show the commutativity of the three squares in definition 16:

1. For $f \in C_1$, $\mathrm{Hom}_C(A, s(f)) = s(\mathrm{Hom}_C(A, f))$ follows from the definition.
2. For $f \in C_1$, $\mathrm{Hom}_C(A, t(f)) = t(\mathrm{Hom}_C(A, f))$ follows from the definition.
3. For $(f_1, f_2) \in C_2$, we claim that $\mathrm{Hom}_C(A, f_1 \circ f_2) = \mathrm{Hom}_C(A, f_1) \circ \mathrm{Hom}_C(A, f_2)$. In the L.H.S., an element $g \in \mathrm{Hom}_C(A, s(f_1 \circ f_2))$ is mapped to $(f_1 \circ f_2) \circ g$ and in the R.H.S., an element $g \in \mathrm{Hom}_C(A, s(f_2)$ is mapped to $f_1 \circ (f_2 \circ g)$. Since $s(f_1 \circ f_2) = s(f_2)$, we see that the two maps are the same.

B. The functor $\mathrm{Hom}_C(-, A) : C \rightsquigarrow \textbf{Set}$ sends an object $B \in C_0$ to the hom set $\mathrm{Hom}_C(B, A)$ and a morphism $f : B \to B'$ to the function

$$\mathrm{Hom}_C(f, A) : \mathrm{Hom}_C(B', A) \to \mathrm{Hom}_C(B, A) = g \mapsto g \circ f.$$

Let us check that this is a contravariant functor. We show the commutativity of the three squares in definition 17:

1. For $f \in C_1$, $\mathrm{Hom}_C(s(f), A) = s(\mathrm{Hom}_C(A, f))$ follows from the definition.
2. For $f \in C_1$, $\mathrm{Hom}_C(t(f), A) = t(\mathrm{Hom}_C(f, A))$ follows from the definition.
3. For $(f_1, f_2) \in C_2$, we claim that $\mathrm{Hom}_C(f_1 \circ f_2, A) = \mathrm{Hom}_C(f_2, A) \circ \mathrm{Hom}_C(f_1, A)$. In the L.H.S., an element $g \in \mathrm{Hom}_C(t(f_1 \circ f_2), A)$ is mapped to $g \circ (f_1 \circ f_2)$ and in the R.H.S., an element $g \in \mathrm{Hom}_C(t(f_1), A)$ is mapped to $(g \circ f_1) \circ f_2$. Since $t(f_1 \circ f_2) = t(f_1)$, we see that the two maps are the same.

Definition 19 (Full, faithfull and essentially surjective)**.** Let $F : C \rightsquigarrow D$ be a functor, then:

- If the restriction $F_{A,B} : \mathrm{Hom}_C(A, B) = \mathrm{Hom}_D(F(A), F(B))$ is injective for any $A, B \in C_0$, then we say F is faithfull.
- If $F_{A,B}$ is surjective for any $A, B \in C_0$, then F is full.

- If for any $X \in D$, there exists $Y \in C_0$ such that $D \cong F(Y)$, then F is essentially surjective.

Definition 20 (Natural transformation)**.** Let $F, G : C \rightsquigarrow D$ be two covariant functors, a **natural transformation** $\phi : F \Rightarrow G$ is a map $\phi : C_0 \to D_1$ that satisfies $\phi(A) \in \mathrm{Hom}_D(F(A), G(A))$ for all $A \in C_0$ and makes the following diagram commute for any $f \in \mathrm{Hom}_C(A, B)$:

$$\begin{array}{ccc} F(A) & \xrightarrow{\phi(A)} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\phi(B)]{} & G(B) \end{array}$$

For two contravariant functors, the vertical arrows are reversed.

Example 21. Let **CRing** denote the category of commutative rings, where objects are commutative rings, morphisms are ring homomorphisms, and composition is the usual composition of functions. Let **Grp** denote the category of groups, where objects are groups, morphisms are group homomorphisms, and composition is the usual composition of functions.

Fix some $n \in \mathbb{N}$, we define the functor $\mathrm{GL}_n : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$ by

$$\begin{aligned} R &\mapsto \mathrm{GL}_n(R) \text{ for any commutative ring } R \text{ and} \\ f &\mapsto \mathrm{GL}_n(f) \text{ for any ring homomorphism } f \end{aligned}$$

The map $\mathrm{GL}_n(f)$ is just the extension of f on $\mathrm{GL}_n(R)$ by applying f to every element of the matrices. The second functor is $(-)^\times : \mathbf{CRing} \rightsquigarrow \mathbf{Grp}$ which sends a commutative ring R to its group of units $R^\times$ under multiplication and a ring homomorphism f to $f^\times$, its restriction on $R^\times$. Checking these mappings define two covariant functors is left as an (simple) exercise, but one might expect these to be functors as they play nicely with the structure of the objects involved.

The natural transformation between these two functors is $\det : \mathrm{GL}_n \Rightarrow (-)^\times$ which maps a commutative ring R to $\det_R$, the function calculating the determinant of a matrix in $\mathrm{GL}_n(R)$. The first thing to check is that $\det_R \in \mathrm{Hom}_{\mathbf{Grp}}(\mathrm{GL}_n(R), R^\times)$ which is clearly the case because the determinant of an invertible matrix is always a unit. The second thing is to verify that the following diagram commutes for any $f \in \mathrm{Hom}_{\mathbf{CRing}}(R, S)$:

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{\det_R} & R^\times \\ {\scriptstyle \mathrm{GL}_n(f)}\downarrow & & \downarrow{\scriptstyle f^\times = f|_{R^\times}} \\ \mathrm{GL}_n(S) & \xrightarrow[\det_S]{} & S^\times \end{array}$$

We will check the claim for $n = 2$, but the general proof should only involve more notation to write the bigger expressions. We can rewrite the diagram as $f^\times \circ \det_R =$

$\det_S \circ \mathrm{GL}_2(f)$ and show it holds as follows. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(R)$, we have

$$\begin{aligned}
(\det{}_S \circ \mathrm{GL}_2(f)) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) &= \det{}_S \left(\begin{bmatrix} f(a) & f(b) \\ f(c) & f(d) \end{bmatrix} \right) \\
&= f(a)f(d) - f(b)f(c) \\
&= f(ad - bc) \\
&= f^{\times}(ad - bc) \\
&= (f^{\times} \circ \det{}_R) \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).
\end{aligned}$$

We conclude that the diagram commutes and that det is indeed a natural transformation.

Definition 22 (Vertical composition)**.** Let $F, G, H : C \rightsquigarrow D$ be parallel functors and $\phi : F \Rightarrow G$ and $\psi : G \Rightarrow H$ be two natural transformations. Then the **vertical composition** of ϕ and ψ, denoted $\psi \cdot \phi : F \Rightarrow H$ is defined by $(\psi \cdot \phi)(A) = \psi(A) \circ \phi(A)$ for all $A \in C_0$. If $f : A \to B$ is a morphism in C, then we have the following diagram that commutes by naturality of ϕ and ψ:

$$\begin{tikzcd}
F(A) \arrow[r, "\phi(A)"] \arrow[d, "F(f)"'] & G(A) \arrow[r, "\psi(A)"] \arrow[d, "G(f)"'] & H(A) \arrow[d, "H(f)"'] \\
F(B) \arrow[r, "\phi(B)"'] & G(B) \arrow[r, "\psi(B)"'] & H(B)
\end{tikzcd}$$

This shows that $\psi \cdot \phi$ is a natural transformation from F to H. We call this vertical composition as opposed to horizontal composition that we introduce in definition 63.

Definition 23 (Opposite category)**.** Let C be a category, we denote the **opposite category** C^{op} and define it by

$$C_0^{\mathrm{op}} = C_0, C_1^{\mathrm{op}} = C_1, s^{\mathrm{op}} = t, t^{\mathrm{op}} = s,$$

with the correspondence defined by $f^{\mathrm{op}} \circ^{\mathrm{op}} g^{\mathrm{op}} = (g \circ f)^{\mathrm{op}}$. This canonically leads to the following contravariant functor $(-)^{\mathrm{op}}_C : C \rightsquigarrow C^{\mathrm{op}}$ which sends an object A to A^{op} and a morphism f to f^{op}. Note that the $^{\mathrm{op}}$ notation here is just used to distinguish elements in C and C^{op} although the class of objects and morphisms are the same.

Remark 24. The last definition helps us define the contravariant functors as covariant functors. Formally, let $F : C \rightsquigarrow D$ be a contravariant functor, we can see F as covariant functor from C^{op} to D or from C to D^{op} via the compositions $F \circ (-)^{\mathrm{op}}_{C^{\mathrm{op}}}$ and $(-)^{\mathrm{op}}_D \circ F$ respectively.

Definition 25 (Opposite of a functor)**.** Let $F : C \rightsquigarrow D$ be a covariant functor, then the **opposite** of this functor $F^{\mathrm{op}} : C^{\mathrm{op}} \rightsquigarrow D^{\mathrm{op}}$ is defined by $F^{\mathrm{op}} = (-)^{\mathrm{op}}_D \circ F \circ (-)^{\mathrm{op}}_{C^{\mathrm{op}}}$.

Definition 26 (Opposite functor)**.** The **opposite functor** $(-)^{\mathrm{op}}$: **Cat** $\rightsquigarrow$ **Cat** sends a category or a functor to its opposite. It is a covariant functor.

Definition 27 (Monomorphism)**.** Let C be a category, a morphism $f \in C_1$ is said to be a **monomorphism** if for any two morphisms $g, h \in C_1$ with $t(g) = t(h) = s(f)$, $f \circ g = f \circ h$ implies $g = h$.

Definition 28 (Epimorphism)**.** Let C be a category, a morphism $f \in C_1$ is said to be an **epimorphism** if for any two morphisms $g, h \in C_1$ with $s(g) = s(h) = t(g)$, $g \circ f = h \circ f$ implies $g = h$.

Proposition 29. *Let C be a category and $f : A \to B$ a morphism, if there exists $f' : B \to A$ such that $f' \circ f = id_A$, then f is a monomorphism.*

Proof. If $f \circ g = f \circ h$, then $f' \circ f \circ g = f' \circ f \circ h$ implying $g = h$. □

Proposition 30. *Let C be a category and $(f_1, f_2) \in C_2$, if $f_1 \circ f_2$ is a monomorphism, then f_2 is a monomorphism.*

Proof. Let $g, h \in C_1$ be such that $f_2 \circ g = f_2 \circ h$, we immediately get that $(f_1 \circ f_2) \circ g = (f_1 \circ f_2) \circ h$. Since $f_1 \circ f_2$ is a monomorphism, this implies $g = h$. □

Remark 31. The two dual propositions for epimorphisms also hold and are straightforward to prove.

Example 32 (Monomorphisms in the categories we know)**.**

1. Inside the category **Mon** where objects are monoids and morphims are monoid homomorphisms, the monomorphisms correspond exactly to injective homomorphims as shown below.
   - Let $f : M \to M'$ be an injective homomorphims and $g_1, g_2 : N \to M$ be two parallel homomorphisms. Suppose that $f \circ g_1 = f \circ g_2$, then for all $x \in N$, $f(g_1(x)) = f(g_2(x))$, so by injectivity of f, $g_1(x) = g_2(x)$. We conclude that $g_1 = g_2$ and since g_1 and g_2 were arbitrary, f is a monomorphism.
   - Let $f : M \to M'$ be a monomorphism. Let $x, y \in M$ and define $p_x : \mathbb{N} \to M$ by $k \mapsto x^k$ and similarly for p_y. It is trivial to show that p_x and p_y are homomorphism. If $f(x) = f(y)$, then by the homomorphism property, we get for all $k \in \mathbb{N}$:
   $$f(p_x(k)) = f(x^k) = f(x)^k = f(y)^k = f(y^k) = f(p_y(k)).$$
   In other words, we get $f \circ p_x = f \circ p_y$, so $p_x = p_y$ and $x = y$. We conclude that f is injective.

Example 33 (Epimorphisms in the categories we know)**.**

1. Inside the category **Mon** an epimorphism is not necessarily surjective. For example, the inclusion homomorphism $i : \mathbb{N} \to \mathbb{Z}$ is clearly not surjective but it is an epimorphism. Indeed, let $g, h : \mathbb{Z} \to M$ be two monoid homomorphisms satisfying $g \circ i = h \circ i$. In particular, we have $g(n) = h(n)$ for any $n \in \mathbb{N} \subset \mathbb{Z}$. It is left to show that also $g(-n) = h(-n)$, but if it were not the case for some n, $g(n)$ would have two left inverses $g(-n)$ and $h(-n)$ which is not possible. We conclude that $g = h$ and i is an epimorphism.

Definition 34 (Isomorphism)**.** Let C be a category, a morphism $f : A \to B$ is said to be an **isomorphism** if there exists a morphism $f^{-1} : B \to A$ such that $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$.

Proposition 35. *Let C be a category and $f \in C_1$ be an isomorphism, then f is a monomorphism and an epimorphism.*

Proof idea. If the compositions with f and two other morphisms are equal, compose with f^{-1} to obtain equality of the morphisms. □

Definition 36 (Natural isomorphism)**.** Let $\phi : F \to G$ be a natural transformation of functors $F, G : C \rightsquigarrow D$. If for every $A \in C_0$, $\phi(A)$ is an isomorphism in G, we say that ϕ is a **natural isomorphism** and we may write $\phi : F \cong G$.

Definition 37 (Equivalence of categories)**.**

Definition 38 (Subobject)**.**

Definition 39 (Quotient object)**.**

Definition 40 (Initial object)**.** Let C be a category, an object $A \in C_0$ is said to be **initial** if for any $B \in C_0$, $|\mathrm{Hom}_C(A, B)| = 1$, namely there are no two parallel morphisms with source A and every object has a morphism coming from A.

Definition 41 (Terminal object)**.** Let C be a category, an object $A \in C_0$ is said to be **terminal** if for any $B \in C_0$, $|\mathrm{Hom}_C(B, A)| = 1$, namely there are no two parallel morphisms with target A and every object has a morphism going to A.

Definition 42 (Zero object)**.** If an object is initial and terminal, we say it is a zero object and usually denote it 0.

Examples 43. We give examples of categories where initial and terminal objects may or may not exist.

1. $\exists$ terminal, $\nexists$ initial: Let **Sets'** denote the categories where objects are finite sets (excluding the empty set) and morphisms are surjective functions. Clearly, $\{1\}$ is final as any set can only map into $\{1\}$ by sending all their elements to 1. Suppose that a set S were initial, then it could be mapped surjectively to any other set T, implying that $|S| \geq |T|$ for any T. However, no finite number can be bigger than any other finite number, so we have a contradiction.

2. $\nexists$ terminal, $\exists$ initial: The category **GrpI** where the objects are groups and the morphisms are injective homomorphisms only contains an initial object $\{1\}$. Indeed, an injective homomorphism $G \to H$ can be seen as subgroup of H isomorphic to G. The identity group $\{1\}$ can only be isomorphic to the the identity subgroup as any other element has degree more than 1, so $\{1\}$ is initial. Moreover, a group G cannot be terminal as $G \times (\mathbb{Z}/2\mathbb{Z})$ cannot be isomorphic to any subgroup of G.

3. $\nexists$ terminal, $\nexists$ initial: Let G be a non trivial group. The category $G*$ has a single object $*$ with $\hom_{G*}(*,*) = G$ and the composition rule being the multiplication in G. The only object $*$ cannot be initial nor trivial as $|\hom_{G*}(*,*)| > 1$.

4. $\exists$ terminal, $\exists$ initial: Let X be a topological space where τ is the collection of open sets (recall that it must contain $\emptyset$ and X). We consider the category T_X where objects are the open sets and for any two open sets $U, V \in \tau$,

$$\hom_{T_X}(U,V) = \begin{cases} i_{U,V} & U \subseteq V \\ \emptyset & U \nsubseteq V \end{cases}$$

   Note that the composition rule can easily be inferred. Since the empty set is contained in every open set, it is an initial object. Since the full set X contains every open set, it is a terminal object. No other set can be initial as it cannot be contained in $\emptyset$ nor be terminal as it cannot contain X. Moreover, note that the two objects are not isomorphic as $\hom_{T_X}(X,\emptyset) = \emptyset$.

The following gives alternate definitions for initial and terminal objects which have the advantage of being completely categorical, making no use of sets. They use the concept of representable functors which will be seen more in depth later.

Proposition 44. *Let C be a category and $\star : C \to$ **Set** be a functor sending objects to the singleton $\{1\}$ and morphisms to $id_{\{1\}}$. An object $A \in C_0$ is initial if and only if the functor* $\mathrm{Hom}_C(A,-)$ *is naturally isomorphic to $\star$.*

Proof. ($\Rightarrow$) Suppose that A is initial, then there is a natural transformation η from $\hom_C(A,-)$ to $\star$ that sends any object X to the only function between $\hom_C(A,X)$ and $\{1\}$. Since the $\hom_C(A,X)$ is also a singleton, this function is an isomorphism for all X and we conclude that η is a natural isomorphism.

($\Leftarrow$) Suppose that there is a natural isomorphism $\eta : \hom_C(A,-) \Rightarrow \star$, then there are isomorphisms between $\{1\}$ and $\hom_C(A,X)$ for all objects $X \in C_0$. This means that there is a unique morphism from A to X and that A is initial. ☐

Proposition 45. *Let C be a category and $\star$ be as above. An object $A \in C_0$ is terminal if and only if the functor* $\mathrm{Hom}_C(-,A)$ *is naturally isomorphic to $\star$.*

Proof. The proof is basically a copy of the last proof. ☐

Proposition 46. *Let C be a category, A and B are two initial (this also works for terminal) objects of C, then $A \cong B$.*

Proof. Let f be the single element in $\hom_C(A,B)$ and f' be the single element in $\hom_C(B,A)$. We claim that f and f' are inverses, thus that $A \cong B$. Since the identity morphisms are the only elements of $\hom_C(A,A)$ and $\hom_C(B,B)$, and $f' \circ f$ and $f \circ f'$, respectively, are elements of these sets, they must be the identities. ☐

Definition 47 (Product)**.** Let C be a category and $A, B \in C_0$. A **product** of A and B is an object denoted $A \times B$ along with two morphisms $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ (they are called projections) such that for any object V and morphisms $f : V \to A$ and $g : V \to B$, there exists a unique morphism $h : V \to A \times B$ such that this diagram commutes:

$$\begin{tikzcd} & V \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Example 48. Inside **Set**, the Cartesian products with the usual projection maps are products. Inside **Grp**, the direct products with the usual projection maps are products.

Definition 49 (Coproducts)**.** Let C be a category and $A, B \in C_0$. A **coproduct** of A and B is an object denoted $A \amalg B$ along with two morphisms $i_1 : A \to A \times B$ and $i_2 : B \to A \times B$ (they are called canonical injections) such that for any object V and morphisms $f : A \to V$ and $g : B \to V$, there exists a unique morphism $h : A \times B \to V$ such that this diagram commutes:

$$\begin{tikzcd} & V & \\ A \arrow[ur, "f"] \arrow[r, "i_1"'] & A \times B \arrow[u, dotted, "h"] & B \arrow[ul, "g"'] \arrow[l, "i_2"] \end{tikzcd}$$

Definition 50 (Pullback)**.** Let C be a category and $f : A \to C$ and $g : B \to C$ be in C_1. A **pullback** of f and g is an object denoted $A \times_C B$ along with two morphisms $p_1 : A \times_C B \to A$ and $p_2 : A \times_C B \to B$ such that this diagram commutes: and for any

$$\begin{tikzcd} A \times_C B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

object V and morphisms $s : V \to A$ and $t : V \to B$, there exists a unique morphism $h : V \to A \times_C B$ that makes this diagram commute:

Definition 51 (Pushout)**.**

Question 52. *Is the pullback object always a subobject of the product ? Is the pushout object always a subobject of the coproduct or quotient object of the product ? Why are these terms used ?*

$$\begin{tikzcd}
V \arrow[drr, bend left, "t"] \arrow[dr, dotted, "h"] \arrow[ddr, bend right, "s"'] & & \\
& A \times_C B \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Definition 53 (Commutative diagram)**.** Let C be a category. A **commutative diagram** in C is functor $F : D \to C$ where D is a small category. We usually draw diagrams by partially drawing the image of D as a graph where objects are vertices and morphisms are arrows. All the diagrams we have drawn up to this definition define the domain of the functor implicitly. For example, if we talk about a commutative square in C, the domain of this diagram can be drawn like so:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$

Remark 54. It follows trivially from this definition that functors preserve commutative diagrams.

Definition 55 (Equivalence)**.** A functor $F : C \rightsquigarrow D$ is an equivalence of categories if there exists a functor $G : D \rightsquigarrow C$ such that $FG \cong \mathrm{id}_C$ and $GF \cong \mathrm{id}_D$, where $\cong$ denote natural isomorphism.

Theorem 56. *A functor $F : C \rightsquigarrow D$ is an equivalence of categories if and only if F is fully faithfull and essaentially surjective.*

2.2 More on natural transformations

Definition 57 (The left action of functors)**.** Let $F, F' : C \rightsquigarrow D$, $G : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. The functor G acts on ϕ by sending it to $G\phi = A \mapsto G(\phi(A)) : C_0 \to E_1$. One can verify that this is a natural transformation from $G \circ F$ to $G \circ F'$ by verifying the diagram commutes for any $C_1 \ni f : A \to B$.

$$\begin{tikzcd}
(G \circ F)(A) \arrow[r, "G\phi(A)"] \arrow[d, "(G\circ F)(f)"'] & (G \circ F')(A) \arrow[d, "(G\circ F')(f)"] \\
(G \circ F)(B) \arrow[r, "G\phi(B)"] & (G \circ F')(B)
\end{tikzcd}$$

If we remove all applications of G, the diagram commutes by naturality of ϕ. Since functors preserve commuting diagrams, we get that $G\phi$ is a natural transformation.

Proposition 58. *The previous definition constitutes a left action, namely,* $id_D\phi = \phi$ *and* $G_1(G_2\phi) = (G_1 \cdot G_2)\phi$.

Proof. $\square$

Definition 59 (The right action of functors)**.** Let $F, F' : C \rightsquigarrow D$, $G : E \rightsquigarrow C$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. The functor G acts on ϕ by sending it to $\phi G = A \mapsto \phi(G(A)) : E_0 \to D_1$. One can verify that this is a natural transformation from $F \circ G$ to $F' \circ G$ by verifying the diagram commutes for any $E_1 \ni f : A \to B$.

$$\begin{tikzcd}
(F \circ G)(A) \arrow[r, "\phi G(A)"] \arrow[d, "(F\circ G)(f)"'] & (F' \circ G)(A) \arrow[d, "(F'\circ G)(f)"] \\
(F \circ G)(B) \arrow[r, "\phi G(B)"] & (F' \circ G)(B)
\end{tikzcd}$$

It follows by naturality of ϕ; change f in the diagram of definition 20 with the morphism $G(f) : G(A) \to G(B)$.

Proposition 60. *The previous definition constitutes a right action, namely,* $\phi id_C = \phi$ *and* $(\phi G_1)G_2 = \phi(G_1 \cdot G_2)$.

Proof. $\square$

Proposition 61. *The two actions commute. Namely, if we let* $F, F' : C \rightsquigarrow D$, $G : D \rightsquigarrow E$, $H : E' \rightsquigarrow C$ *be functors and* $\phi : F \Rightarrow F'$ *be a natural transformation, then we have* $G(\phi E) = (G\phi)E$.

Proof. $\square$

We will refer to these two actions as the biaction of functors on natural transformations and they will motivate the definition of another way to compose natural transformations. Consider the following diagram to be the setting of this definition, with F, F', G and

$$\begin{tikzcd}
C \arrow[r, bend left=60, "F"{name=U}] \arrow[r, bend right=60, "F'"'{name=L}] \arrow[from=U, to=L, Rightarrow, "\phi"] & D \arrow[r, bend left=60, "G"{name=U2}] \arrow[r, bend right=60, "G'"'{name=L2}] \arrow[from=U2, to=L2, Rightarrow, "\psi"] & E
\end{tikzcd}$$

G' being functors and ϕ and ψ being natural transformations. With the two previous actions, we are able to construct four new transformations:

$$G\phi : G \circ F \Rightarrow G \circ F'$$
$$\psi F : G \circ F \Rightarrow G' \circ F$$
$$G'\phi : G' \circ F \Rightarrow G' \circ F'$$
$$\psi F' : G \circ F' \Rightarrow G' \circ F'$$

Observe that to go from $G \circ F$ to $G' \circ F'$, we have two paths yielding the following diagram:

$$\begin{tikzcd} & G \circ F' \arrow[dr, "\psi F'"] & \\ G \circ F \arrow[ur, "G\phi"] \arrow[dr, "\psi F"'] & & G' \circ F' \\ & G' \circ F \arrow[ur, "G'\phi"'] & \end{tikzcd}$$

Proposition 62. *The diagram above commutes.*

Proof. For a fixed element $c \in C_0$ we know that $a := F(c)$ and $b := F'(c)$ are two different elements of the category D and that we have an arrow $f := \phi(c)$ from a to b given by the natural transformation ϕ. But as ψ is a natural transformation $G \Rightarrow G'$, we know that the following diagram commutes:

$$\begin{tikzcd} & G(b) \arrow[dr, "\psi b"] & \\ G(a) \arrow[ur, "Gf"] \arrow[dr, "\psi a"'] & & G'(b) \\ & G'(a) \arrow[ur, "G'f"'] & \end{tikzcd}$$

Replacing a, b and f by their values we obtain what we wanted. $\square$

Definition 63 (Horizontal composition)**.** In the setting described above, we define the **horizontal composition** of ψ and ϕ by $\psi \diamond \phi = \psi F' \cdot G\phi = G'\phi \cdot \psi F$.

Proposition 64. *Horizontal composition is associative. Namely, if we let $F, F' : C_1 \rightsquigarrow C_2$, $G, G' : C_2 \rightsquigarrow C_3$ and $H, H' : C_3 \rightsquigarrow C_4$ be functors and $\phi : F \Rightarrow F'$, $\psi : G \Rightarrow G'$ and $\eta : H \Rightarrow H'$ be natural transformations, then we have $\eta \diamond (\psi \diamond \phi) = (\eta \diamond \psi) \diamond \phi$.*

Proof. $\square$

Proposition 65 (Interchange identity)**.** *Let $F, F', F'' : C \rightsquigarrow D$ and $G, G', G'' : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$, $\phi' : F' \Rightarrow F''$, $\psi : G \Rightarrow G'$ and $\psi' : G' \Rightarrow G''$ be natural transformations. Using $\cdot$ to denote vertical composition, the **interchange identity** holds:*

$$(\psi' \cdot \psi) \diamond (\phi' \cdot \phi) = (\psi' \diamond \phi') \cdot (\psi \diamond \phi)$$

Proof. The idea is to use the commutativity of $\psi' \circ \phi$ to switch from the LHS to the RHS of the equation. To make things clearer we first draw out the diagrams The LHS of the equation can be seen as the following diagram:

While the RHS would correspond to the following:

Joining the two diagrams, we obtain this huge one $\square$

$$\begin{tikzcd}
 & G' \circ F \arrow[dr, "G'\phi"] & & G'' \circ F' \arrow[dr, "G''\phi'"] & \\
G \circ F \arrow[ur, "\psi F"] \arrow[dr] & & G' \circ F' \arrow[ur, "\psi' F'"] \arrow[dr] & & G'' \circ F'' \\
 & G \circ F' \arrow[ur] & & G' \circ F'' \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & G'' \circ F \arrow[dr, "G''\phi"] & & \\
 & G' \circ F \arrow[ur, "\psi' F"] & & G'' \circ F' \arrow[dr, "G''\phi'"] & \\
G \circ F \arrow[ur, "\psi F"] \arrow[dr] & & & & G'' \circ F'' \\
 & G \circ F' \arrow[dr] & & G' \circ F'' \arrow[ur] & \\
 & & G \circ F'' \arrow[ur] & &
\end{tikzcd}$$

These definitions lead us to the first example of a 2-category.

Definition 66 (2-cateory)**.** A **2-category** consists of a class of objects C_0, a class of morphisms between objects C_1 and a class of 2-morphisms between parallel morphisms C_2 that satisfy the following conditions:

1. The objects and morphisms form a category under composition of morphisms.
2. For two objects $A, B \in C_0$, the morphisms from C to D and the 2-morphisms between them form a category under vertical composition.
3. If we consider 2-cells (two parallel morphisms with a 2-morphism between them) as morphisms, we get a category under horizontal composition.
4. The interchange identity hold for horizontal and vertical composition.

Example 67.

1. The 2-category of categories with functors and natural transformations as we just have proved.

Question 68. *Is the vertical composition of two natural isomorphisms also a natural isomorphism ? What about horizontal composition ?*

Definition 69 (Identity transformation)**.** Let $F : C \rightsquigarrow D$ be a functor, the identity natural transformation from F to itself is defined by $\mathrm{id}_F = A \mapsto \mathrm{id}_{F(A)} : C_0 \to D_1$ when the objects in the range of F all have an identity morphism.

$$
\begin{tikzcd}
 & & G'' \circ F \arrow[dr, "G''\phi"] & & \\
 & G' \circ F \arrow[ur, "\psi' F"] \arrow[dr, "G'\phi"] & 1 & G'' \circ F' \arrow[dr, "G''\phi'"] & \\
G \circ F \arrow[ur, "\psi F"] \arrow[dr] & & G' \circ F' \arrow[ur, "\psi' F'"] \arrow[dr] & & G'' \circ F'' \\
 & G \circ F' \arrow[ur] \arrow[dr] & & G' \circ F'' \arrow[ur] & \\
 & & G \circ F'' \arrow[ur] & &
\end{tikzcd}
$$

Proposition 70. *Let $F, F' : C \rightsquigarrow D$, $G : B \rightsquigarrow C$ and $H : D \rightsquigarrow E$ be functors and $\phi : F \Rightarrow F'$ be a natural transformation. Suppose that C and E are unital, then the following equations hold:*

1. $\phi G = \phi \diamond id_G$
2. $H\phi = id_H \diamond \phi$
3. $id_{id_D} \diamond \phi = \phi = \phi \diamond id_{id_C}$

Proof.

1. For any $x \in B_0$, we have the following:

$$
\begin{aligned}
(\phi \diamond \mathrm{id}_G)(x) &= \phi(G(x)) \circ F(\mathrm{id}_G(x)) && \text{(def of } \diamond\text{)} \\
&= \phi G(x) \circ F(\mathrm{id}_{G(x)}) && \text{(def of } \mathrm{id}_G\text{)} \\
&= \phi G(x) \circ \mathrm{id}_{F(G(x))} && \text{(functors preserve id morphisms)} \\
&= \phi G(x) && \text{(def of id morphisms)}
\end{aligned}
$$

   Thus, we conclude that $\phi \diamond \mathrm{id}_G = \phi G$.

2. For any $x \in C_0$, we have the following:

$$
\begin{aligned}
(\mathrm{id}_H \diamond \phi)(x) &= \mathrm{id}_H(F'(x)) \circ H(\phi(x)) && \text{(def of } \diamond\text{)} \\
&= \mathrm{id}_{H(F'(x))} \circ H\phi(x) && \text{(def of } \mathrm{id}_H\text{)} \\
&= H\phi(x) && \text{(def of id morphisms)}
\end{aligned}
$$

   Thus, we conclude that $\mathrm{id}_H \diamond \phi = H\phi$.

3. By swapping G for id_C and H for id_D in the two previous equations, we get the result we want.

$\square$

2.3 On our way to the Yoneda lemma

Definition 71 (Category of arrows)**.** Let C be a category, $\mathrm{Arr}(C)$ is the category of arrows of C. Its objects are morphisms in C and its morphisms are commutative squares ϕ. In other words, if f and g are morphisms in C and there exists maps ϕ_s and ϕ_t such that this diagram commutes

$$\begin{tikzcd} s(f) \arrow[r, "f"] \arrow[d, "\phi_s"'] & t(f) \arrow[d, "\phi_t"] \\ s(g) \arrow[r, "g"'] & t(g) \end{tikzcd},$$

then this square is a morphism from f to g. It is denoted by ϕ or (ϕ_s, ϕ_t).

Definition 72 (Source functor)**.** Let C be a category, the **source functor** is $S : \mathrm{Arr}\, C \rightsquigarrow C$ defined by:

$$\begin{aligned} S_0(f) &= s(f), \forall f \in C_1 = \mathrm{Arr}(C)_0 \\ S_1((\phi_s, \phi_t)) &= \phi_s \forall (\phi_s, \phi_t) \in \mathrm{Arr}(C)_1 \end{aligned}$$

Definition 73 (Target functor)**.** Let C be a category, the **target functor** is $T : \mathrm{Arr}\, C \rightsquigarrow C$ defined by:

$$\begin{aligned} T_0(f) &= t(f), \forall f \in C_1 = \mathrm{Arr}(C)_0 \\ T_1((\phi_s, \phi_t)) &= \phi_t \forall (\phi_s, \phi_t) \in \mathrm{Arr}(C)_1 \end{aligned}$$

Definition 74 (Tautological natural transformation)**.** Let C be a category, the **tautological natural transformation** is $\tau : S \Rightarrow T$ defined by $\tau(f) = f$ for all $f \in C_1 = \mathrm{Arr}(C)_0$. Note that we see the input as an object of $\mathrm{Arr}(C)$ and the output as a morphism of C.

Definition 75 (Arr functor)**.** The **Arr functor** is a functor **Cat** $\rightsquigarrow$ **Cat** that sends a category C to its category of arrows and a functor $F : C \rightsquigarrow D$ to the functor $\mathrm{Arr}(F) : \mathrm{Arr}(C) \rightsquigarrow \mathrm{Arr}(D)$ defined by

$$\begin{aligned} &\mathrm{Arr}(F)_0 = f \mapsto F(f) \\ &\mathrm{Arr}(F)_1 = (\phi_s, \phi_t) \mapsto (F(\phi_s), F(\phi_t)) \end{aligned}$$

Proposition 76. *The correspondences $S = C \mapsto S_C$ and $T = C \mapsto T_C$ where S_C is the source functor and T_C is the target functor define natural transformations* $\mathrm{Arr} \mapsto id_{\mathbf{Cat}}$.

Proof. □

Definition 77 (Representable functors)**.** A covariant functor $F : C \to$ **Set** is said to be representable if there is an object $X \in C_0$ such that F is naturally isomorphic to $\hom_C(X, -)$. If F is contravariant, then we require it to be naturally isomorphic to $\hom_C(-, X)$.

Example 78. The functor $(-)^\times$: **Ring** $\rightsquigarrow$ **Set** is represented by $\mathbb{Z}[x, x^{-1}]$ because any unit of $R^\times$ corresponds to the unique homomorphism from $\mathbb{Z}[x, x^{-1}]$ to R sending x to that unit and every homomorphisms from $\mathbb{Z}[x, x^{-1}]$ to R must send x to a unit.

Example 79. The forgetful functor has a left adjoint implies it is representable. Look at what happens on a set with one element. Need to define forgetful functor and adjoint.

Example 80 (Cayley's theorem with the Yoneda Lemma)**.** Cayley's theorem states that any group is isomorphic to the subgroup of a permutation group. We will use the Yoneda lemma to show that.

Recall the first part of the Yoneda lemma which states that for a category C, a functor $F : C \rightsquigarrow$ **Sets** and an object A. We have

$$\mathrm{Nat}(\mathrm{Hom}(A, -), F) \cong F(A).$$

Moreover, we know the explicit maps, namely, a natural transformation φ in the L.H.S. is mapped to $\phi_A(\mathrm{id}_A)$ and an element $u \in F(A)$ is mapped to the natural transformation $\{\varphi_B = f \mapsto F(f)(u) \mid B \in C_0\}$.

Let us apply this to C being the category associated to a group G (i.e.: there is one object $\star$, $\mathrm{Hom}(\star, \star) = G$ and the composition law follows the group operation). Note that any functor $F : C \rightsquigarrow$ **Sets** sends $\star$ to a set S and any $g \in G$ to a permutation of S, otherwise $g \circ g^{-1} = 1$ cannot be satisfied.

To use the Yoneda lemma, our only choice for A is $\star$ and we will choose $F = \mathrm{Hom}(\star, -)$. The Yoneda correspondence becomes

$$\mathrm{Nat}(\mathrm{Hom}(\star, -), \mathrm{Hom}(\star, -)) \cong \mathrm{Hom}(\star, \star).$$

We already know what the R.H.S. is G, but we have to do a bit of work to understand the L.H.S. First, observe that a natural transformation $\varphi : \mathrm{Hom}(\star, -) \Rightarrow \mathrm{Hom}(\star, -)$ is just one morphism $\varphi_\star : \mathrm{Hom}(\star, \star) \to \mathrm{Hom}(\star, \star)$. Namely, it is a map from G to G. Second, recalling that $\mathrm{Hom}(\star, g) = g \circ (-)$ and that $\star$ is the only object in C_0, we get that $\varphi_\star$ must only satisfy one diagram.

$$\begin{tikzcd}
G \arrow[r, "\varphi_\star"] \arrow[d, "g\circ(-)"'] & G \arrow[d, "g\circ(-)"] \\
G \arrow[r, "\varphi_\star"'] & G
\end{tikzcd}$$

This is equivalent to $\varphi_\star(g \cdot h) = g \cdot \varphi_\star(h)$, and we get that each $\varphi_\star$ is just a G-equivariant map, denote these maps $\mathrm{Hom}_G(G, G)$. We obtain

$$\mathrm{Hom}_G(G, G) \cong G.$$

Now, it is easy to check that $\mathrm{Hom}_G(G, G)$ is a subgroup of Σ_G (the group of permutations of the set G) and that the correspondence is in fact an isomorphism of groups. Cayley's theorem follows.

Let us check that $\mathrm{Hom}_G(G, G) < \Sigma_G$. Let f be a G-equivariant map. For any $g \in G$, we have $f(g) = f(g \cdot 1) = g \cdot f(1)$. Thus, f is determined only by where it sends the identity. Additionally, since $g \cdot f(1)$ ranges over G when g ranges over G, for any choice of $f(1)$, f is bijective. Finally, if f and f' are both G-equivariant map, then

$$(f \circ f')(g \cdot h) = f(f'(g \cdot h)) = f(g \cdot f'(h)) = g \cdot (f \circ f')(h),$$

hence $f \circ g$ is G-equivariant. With the facts that f^{-1} is just the G-equivariant map sending 1 to $f(1)^{-1}$ and id is G-equivariant, it follows that $\mathrm{Hom}_G(G, G)$ is a subgroup of Σ_G.

The final check is that the Yoneda correspondence $G \to \mathrm{Hom}_G(G, G)$ sending g to $(-) \cdot g$ is a group homomorphism (isomorphism follows because it is a bijection). It is clear that is sends the identity to the identity and for any $g, h \in G$

$$(-) \cdot gh = ((-) \cdot g) \cdot h = ((-) \cdot h) \circ ((-) \cdot g),$$

so this is a group homomorphism.