Graded String Diagrams

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joint work with Fabio Zanasi

March 28th, 2025

Abstract

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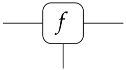

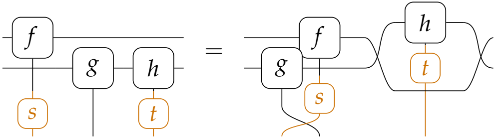

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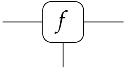
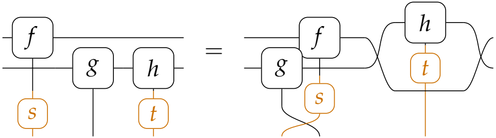




- Application to imprecise probability.

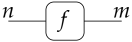
Outline

Graded String Diagrams

Imprecise Probability

Conclusion

Monoidal Theories and String Diagrams

- The signature Σ contains operations with (co)arities drawn as 

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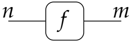
- The signature Σ contains operations with (co)arities drawn as 
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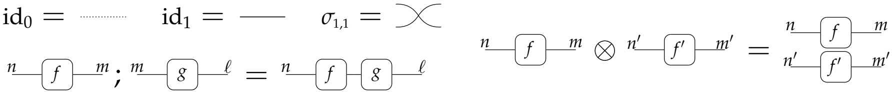
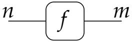
- Monoidal terms are constructed inductively:



- The signature Σ contains operations with (co)arities drawn as 

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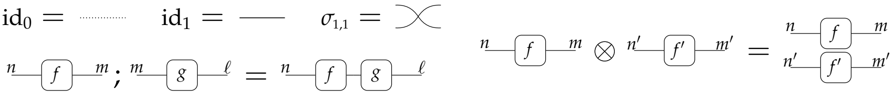
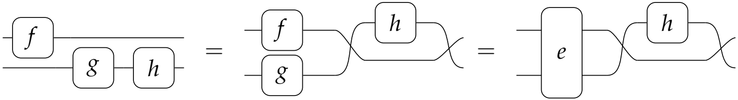


- From Σ and a set E of equations between terms, we generate a syntactic category $\mathcal{M}_{\Sigma,E}$ of string diagrams.



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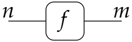
- The signature Σ contains operations with (co)arities drawn as 

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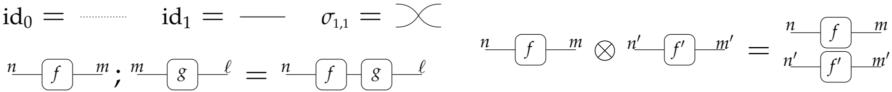

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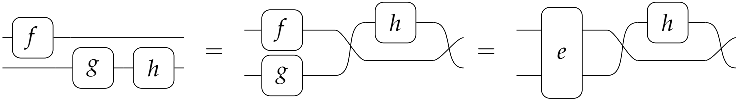
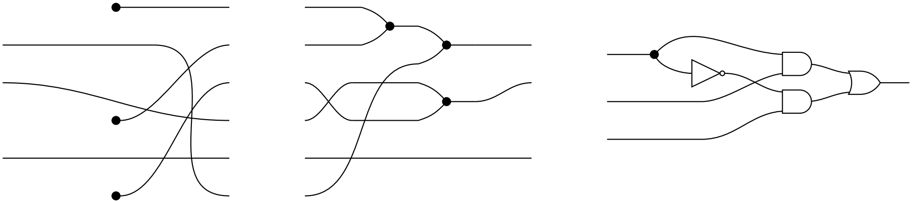


- Models of the theory (Σ, E) are monoidal functors $\mathcal{M}_{\Sigma,E} \to \mathbf{C}$.

Examples

1. With a single generator and no equations we get **FinInj**.
2. With a single generator that is *commutative and associative* we get **FinSurj**.
3. With generators , , 1 , 0 , , and , and equations from Boolean algebra, we get the subcategory of **FinSet** spanned by $\{\mathtt{1},\mathtt{0}\}^n$.



(All from Lafont [17].)

Graded Categories

Let $\mathbb{G}$ be a semicartesian monoidal category.

Definition

A $\mathbb{G}$-**graded category** consists of

- A collection of objects $\mathrm{Ob}(\mathbf{C})$ and sets of morphisms $\mathbf{C}_\gamma(X, Y)$ indexed by objects X, Y in $\mathbf{C}$ and γ in $\mathbb{G}$.
- Compositions and identities: $\fatsemi : \mathbf{C}_\gamma(X, Y) \times \mathbf{C}_\varepsilon(Y, Z) \to \mathbf{C}_{\gamma\varepsilon}(X, Z)$ and $\mathrm{id}_X \in \mathbf{C}_\mathbf{1}(X, X)$.
- **Regrading** operations $t \star - : \mathbf{C}_\varepsilon(X, Y) \to \mathbf{C}_\gamma(X, Y)$, for all $t : \gamma \to \varepsilon$.
- It is **monoidal** if it comes with a unit $I \in \mathrm{Ob}(\mathbf{C})$ and monoidal product $\otimes : \mathbf{C}_\gamma(X, Y) \times \mathbf{C}_\varepsilon(X', Y') \to \mathbf{C}_{\gamma\varepsilon}(X \otimes X', Y \otimes Y')$.

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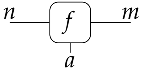
The $\mathbb{N}$-graded Kleisli category of the graded list monad has sets as objects, and a morphism $X \xrightarrow{n} Y$ is a function sending each $x \in X$ to a list of n elements in Y. Graded categories generalize enriched categories and actegories [19].

Graded Monoidal Theories

- A **graded signature** Σ contains operations with (co)arities and a grade, all in $\mathbb{N}$, drawn as  .

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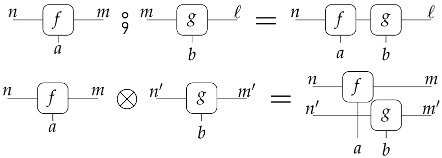

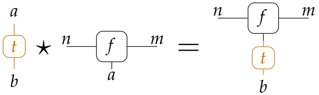

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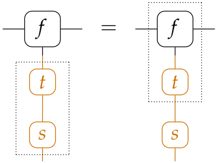

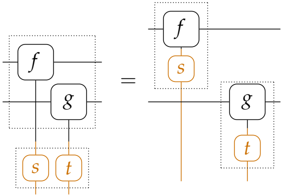

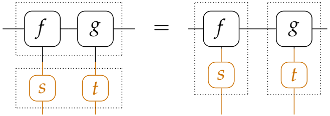

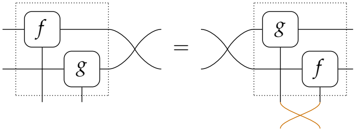

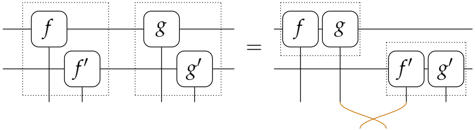

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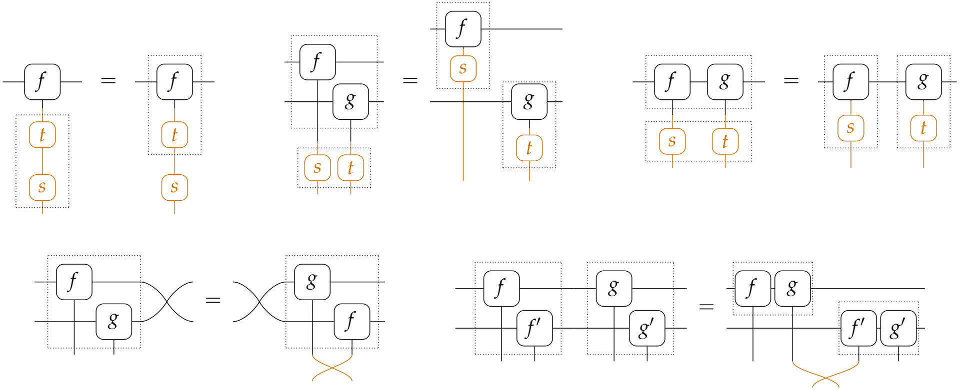

Models of the graded theory (Σ, E) are graded monoidal functors $\mathcal{S}_{\Sigma,E} \to \mathbf{C}$.

Graded Para Construction

An action of $\mathbb{G}$ on $\mathbf{C}$ given by a symmetric monoidal functor $\bullet : \mathbb{G} \times \mathbf{C} \to \mathbf{C}$ yields a $\mathbb{G}$-graded category through the **Para** construction. Objects of $\mathbf{Para}(\mathbb{G}, \mathbf{C})$ are objects of $\mathbf{C}$, and morphisms $X \xrightarrow{\gamma} Y$ are morphisms $\gamma \bullet X \to Y$ in $\mathbf{C}$.

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Theorem

If $\mathbf{C}$ is presented by $(\Sigma_{\mathbf{C}}, E_{\mathbf{C}})$ (i.e. $\mathcal{M}_{\Sigma_{\mathbf{C}}, E_{\mathbf{C}}} \cong \mathbf{C}$) and $\mathbf{1}$ is strictly terminal in $\mathbb{G}$, then $\mathbf{Para}(\mathbb{G}, \mathbf{C})$ is presented by $\mathsf{T} := (\Sigma, E_{\mathsf{h}})$ defined by

$$\Sigma := \Sigma_{\mathbf{C}} \sqcup \{ \; : 0 \xrightarrow{1} 1 \} \text{ and } E_{\mathsf{h}} := E_{\mathbf{C}} \cup \{ \boxed{g} = \boxed{g_0} \mid \boxed{g} \in \Sigma_{\mathsf{v}} \},$$

where g_0 is a term in the equivalence class of $g \bullet \mathrm{id}_0$.

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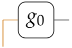
$\Sigma := \Sigma_{\mathbf{C}} \sqcup \{$  $: 0 \xrightarrow{1} 1\}$ *and* $E_{\mathsf{h}} := E_{\mathbf{C}} \cup \{$  $=$  $\mid$  $\in \Sigma_{\mathsf{v}}\}$,

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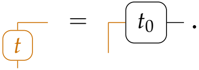
By construction, we have

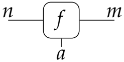
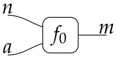
$$\mathcal{S}_{\mathsf{T}0}(n, m) \cong \mathcal{S}_{\Sigma_{\mathbf{C}}, E_{\mathbf{C}}}(n, m) \cong \mathbf{C}(n, m) \cong \mathbf{Para}(\mathbb{G}, \mathbf{C})_0(n, m)$$

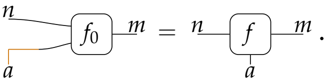
Modular Completeness

Lemma

1. *For any morphism*  *in* $\mathbb{G}$*, and any monoidal* $\Sigma_{\mathbf{C}}$*-term* t_0 *in the equivalence class of* $t \bullet \mathrm{id}_0 \in \mathbf{C}$*, the closure* $\overline{E_{\mathsf{h}}}$ *contains the equation*



2. *Given a graded* Σ*-term*  *, there is a term*  *such that*



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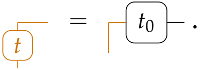
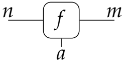
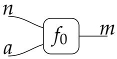
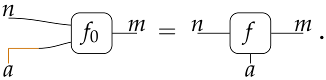


2. *Given a graded* Σ*-term* *, there is a term*  *such that*



Proof sketch for theorem.

Define the semantics of 0-graded generators as in $\mathbf{C}$. Define $[\![\ulcorner]\!] := \mathrm{id}_1 \bullet \mathrm{id}_0$. Soundness is easy to check. Completeness follows by decomposition above and isomorphism $\mathcal{M}_{\Sigma_{\mathbf{C}}, E_{\mathbf{C}}} \cong \mathbf{C}$. $\square$

Outline

BinStoch

The objects of **FinStoch** are natural numbers and its morphisms are

$$\{m \times n \text{ stochastic matrices}\} \Leftrightarrow \{\underline{n} \to \mathcal{D}\underline{m}\}\,.$$

The monoidal product is the Kronecker product:

$$A \otimes A' := \begin{bmatrix} a_{11}A' & \dots & a_{1n}A' \\ \vdots & \ddots & \vdots \\ a_{n'1}A' & \dots & a_{n'n}A' \end{bmatrix}.$$

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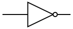
BinStoch is the full subcategory spanned by 2^n. It is a *prop* via **BinStoch**$(n, m) =$ **FinStoch**$(2^n, 2^m)$.

A morphism $f : n \to m$ in **BinStoch** is determined by its action on the basis generated by $|1\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ and $|0\rangle = \begin{bmatrix}0\\1\end{bmatrix}$: $\{|\vec{x}\rangle \mid \vec{x} \in \{1, 0\}^n\}$.

Presentation of **BinStoch**

Piedeleu et al. [24] add probabilistic choices to Lafont's presentation,

$\Sigma :=$ {  ,  ,  ,  ,  } , $\qquad p \in [0, 1]$

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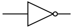
$[\![$  $]\!]\, |x\rangle|y\rangle = |x \texttt{ and } y\rangle \quad [\![$  $]\!]\, |x\rangle = |\texttt{not } x\rangle \quad [\![$  $]\!]\, |x\rangle = |\rangle$

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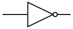
$\Sigma := \{$  ,  ,  ,  ,  $\}, \qquad p \in [0,1]$

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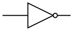
$\llbracket$  $\rrbracket\, |x\rangle|y\rangle = |x \texttt{ and } y\rangle$ $\quad\llbracket$  $\rrbracket\, |x\rangle = |\texttt{not } x\rangle$ $\quad\llbracket$  $\rrbracket\, |x\rangle = |\rangle$

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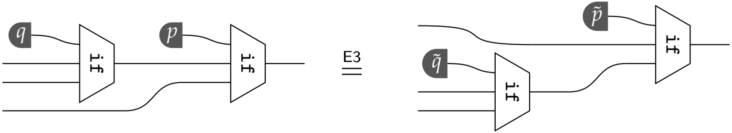
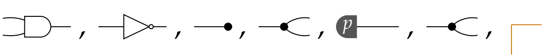
[24, Corollary 3.18] shows that **BinStoch** is presented by these generators and a cleverly chosen set of equations containing, e.g.,



$$\text{where } \tilde{p} = pq \quad \text{and } \tilde{q} = \frac{p(1-q)}{1-pq} \text{ for } pq \neq 1$$

Presentation of **BImP**

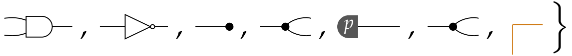
We add nondeterministic choices to Piedeleu et al.'s presentation.

$$\Sigma := \Big\{$$  $$\Big\}, \qquad p \in [0,1]$$

The generator $: 0 \xrightarrow{1} 1$ is interpreted as providing nondeterministic bits:

$$[\![\ ]\!]\,|\rangle = |1\rangle \oplus |0\rangle \qquad [\![\ ]\!] = \left[\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right] \qquad [\![\ ]\!]\,|x] = |x\rangle$$

We add nondeterministic choices to Piedeleu et al.'s presentation.

$\Sigma := \Big\{$  $\Big\}, \qquad\qquad p \in [0,1]$

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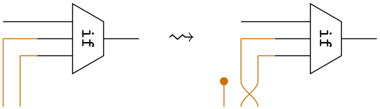
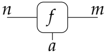
The grade of a circuit in $\mathcal{S}_\Sigma$ is the number of nondeterministic choices at its disposal. We can regrade it with a morphism in $\mathbf{FinInj}^{\mathrm{op}}$ that discards or swaps nondeterministic bits:



Presentation of **BImP**

We add two equations

to obtain a graded presentation of **BImP**. A morphism 

in **BImP** is a list of $2^m \times 2^n$ stochastic matrices of length 2^a.

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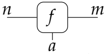
to obtain a graded presentation of **BImP**. A morphism 

in **BImP** is a list of $2^m \times 2^n$ stochastic matrices of length 2^a. Both composition and monoidal product are computed submatrix-wise.

$$\left[\begin{array}{cc|cc|cc|cc} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right] \left[\begin{array}{c|c} 1 & 0.5 \\ 0 & 0.5 \end{array}\right] = \left[\begin{array}{c|c|c|c|c|c|c|c} 1 & 1 & 0 & 0 & 1 & 0.5 & 0.5 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0.5 & 0.5 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \otimes \left[\begin{array}{c|c} .5 & 0 \\ .5 & 1 \end{array}\right] = \left[\begin{array}{cc|cc|cc|cc} .5 & 0 & 0 & 0 & 0 & .5 & 0 & 0 \\ .5 & 0 & 1 & 0 & 0 & .5 & 0 & 1 \\ 0 & .5 & 0 & 0 & .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & 1 & .5 & 0 & 1 & 0 \end{array}\right]$$

Regrading swaps or copies matrices.

Presentation of **BImP**

Lemma

The **FinInj**$^{\mathrm{op}}$*-graded category* **BImP** *is obtained by applying the* **Para** *construction to an action of* **FinInj**$^{\mathrm{op}}$ *on* **BinStoch***.*

Theorem

The graded category **BImP** *is presented by the ciruits of Piedeleu et al. [24] extended with the graded generator and the two equations above.*

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The graded category **BImP** *is presented by the ciruits of Piedeleu et al. [24] extended with the graded generator and the two equations above.*

Piedeleu et al. [24] also add conditioning to present **FinSubStoch**:

$$[\![\ \ ]\!]\,|x\rangle|y\rangle = \begin{cases} |x\rangle & x = y \\ 0 & x \neq y \end{cases}$$

Theorem

The graded category **BImSubP** *is presented by the theory above extended with and the equations from [24].*

Boy or Girl Paradox

Your neighbors have two kids and at least one of them is a girl. What is the probability that both kids are girls?

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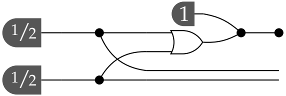

$$\llbracket - \rrbracket \mapsto \begin{bmatrix} 0.33 \\ 0.33 \\ 0.33 \\ 0 \end{bmatrix}$$

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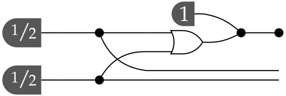

$$\xmapsto{[\![-]\!]} \begin{bmatrix} 0.33 \\ 0.33 \\ 0.33 \\ 0 \end{bmatrix}$$

Your neighbors have two kids: a toddler and a teen. You meet one of them, and she is a girl. What is the probability that both kids are girls?

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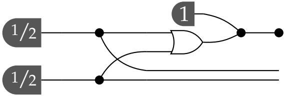

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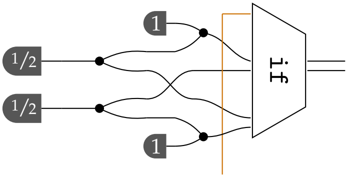


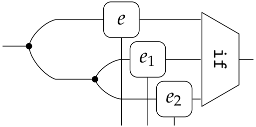
$$\xmapsto{[\![-]\!]} \left[\begin{array}{c|c} 0.25 & 0.25 \\ 0.25 & 0 \\ 0 & 0.25 \\ 0 & 0 \end{array}\right]$$

Imprecise Probabilistic Programming

There is a natural way to interpret a programming language with the following expressions in **BImP** (**BImSubP**).

$$e, e_1, e_2 ::= x \mid \texttt{flip}(p) \mid \texttt{knight} \mid \langle e_1, e_2 \rangle \mid \texttt{fst}\, e \mid \texttt{snd}\, e \mid \\ \ldots\ \texttt{if}\, e\, \texttt{then}\, e_1\, \texttt{else}\, e_2 \mid \texttt{let}\, x = e_1\, \texttt{in}\, e_2 \mid (\texttt{observe}\, x)$$

e.g.



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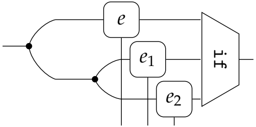
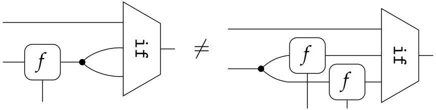


This language is commutative and affine (without conditioning), but it does not satisfy hoisting.



Outline

Graded String Diagrams

Imprecise Probability

Conclusion

Future Work

- Use our modular completeness 'hammer' to hit some nails, e.g. **Gauss**, optics, graded trace semantics,etc.
- Compare graded monoidal theories and graded algebraic theories.
- Graded tape/sheet diagrams to model the distributive fragment.
- Quantitative equations to model probabilistic refinement of nondeterminism.

Merci !

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