ENS DE LYON

TD – Semantics and Verification

III– Topology

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In this set of exercises, we will discuss topological characterisations of safety and liveness properties.

Liveness Properties

Recall that

- a LT property is a liveness property if for all finite $\hat{\sigma} \in (2^{\mathrm{AP}})^*$, there is $\sigma \in P$ such that $\hat{\sigma} \subseteq \sigma$
- $\mathrm{pref}(P) = \{\hat{\sigma} \text{ finite } | \exists \sigma \in P, \hat{\sigma} \subseteq \sigma\}$
- $\mathrm{cl}(P) = \{\sigma | \mathrm{pref}(\sigma) \subseteq \mathrm{pref}(P)\}$

Exercise 1.

Show that

1. A LT property P is a liveness property if and only if $\mathrm{cl}(P) = (2^{\mathrm{AP}})^\omega$.
2. If P and Q are liveness properties, then so is $P \cup Q$.
3. There are two liveness properties P and Q such that $P \cap Q$ is not a liveness property.

Topological Spaces

- A *topological space* is a pair $(X, \mathcal{U})$ of a set X and a subset $\mathcal{U}$ of $\mathcal{P}(X)$, called the *open sets of* X, such that
  1. $\emptyset \in \mathcal{U}$ and $X \in \mathcal{U}$;
  2. for any set I and family $\{U_i \in \mathcal{U}\}_{i \in I}$, also $\bigcup_{i \in I} U_i \in \mathcal{U}$; and
  3. for all $U, V \in \mathcal{U}$, also $U \cap V \in \mathcal{U}$.

  A set $U \in \mathcal{U}$ is called *open* and elements $x \in X$ are called points. If $\mathcal{U}$ is clear from the context, we often refer to X is the topological space.
- Given a point $x \in X$, we say that N is a *neighbourhood* of x if there is an open set U, such that $x \in U$ and $U \subseteq N$. The collection of all neighbourhoods of x is denoted by $\mathcal{N}_x$.
- Given a topological X, we say that $F \in X$ is *closed*, if F^C is open.
- Give a topological space $(X, \mathcal{U})$, a set D is said dense if $D \cap U \neq \emptyset$ for all non-empty $U \in \mathcal{U}$.

Exercise 2.

Show that

1. $\emptyset$ and X are closed
2. for any set I and family $\{F_i \text{ closed}\}_{i \in I}$, also $\bigcap_{i \in I} F_i$ closed; and
3. for all closed F and G, also $F \cup G$ is closed.

For any set $A \subset X$, we define the *closure* $\overline{A}$ of A by

$$\overline{A} = \bigcap\{F \subseteq X \mid F \text{ closed and } A \subseteq F\},$$

which makes sense by the previous exercise.

Exercise 3.
Let $(X,\mathcal{U})$ be a topological space.

1. A set $A \subseteq X$ is open iff for every $x \in A$ there is an open set $U \in \mathcal{U}$ such that $x \in U$ and $U \subseteq A$.
2. A set $A \subseteq X$ is closed iff for every $x \notin A$ there is an open set $U \in \mathcal{U}$ such that $x \in U$ and $U \cap A = \emptyset$.

Exercise 4.
Let $(X,\mathcal{U})$ be a topological space and $A \subseteq X$.

1. Show that $\overline{A} = \{x \in X \mid \forall N \in \mathcal{N}_x.\, N \cap A \neq \emptyset\}$.
2. Show that A is closed iff $\overline{A} = A$.
3. Show A is dense iff $\overline{A} = X$.

Topology on Infinite Words

For A a non-empty set, we define

- $\mathrm{ext}(w) = \{\sigma \in A^\omega \mid w \subseteq \sigma\}$ for $w \in A^*$.
- $\mathrm{ext}(W) = \bigcup_{w \in W} \mathrm{ext}(w)$ for $W \subseteq A^*$.
- $\mathcal{U} = \{\mathrm{ext}(W) \mid W \subseteq A^*\}$.

Exercise 5.
Show that $(A^\omega,\mathcal{U})$ is a topological space.

Exercise 6.
Show that

1. A set $P \subseteq A^\omega$ is open iff for every $\sigma \in P$ there is a finite word $\hat{\sigma} \subseteq \sigma$ such that $\beta \in P$ for every $\beta \in A^\omega$ such that $\hat{\sigma} \subseteq \beta$.
2. A set $P \subseteq A^\omega$ is closed iff for every $\sigma \notin P$ there is a finite word $\hat{\sigma} \subseteq \sigma$ such that $\beta \notin P$ for every $\beta \in A^\omega$ such that $\hat{\sigma} \subseteq \beta$.

Exercise 7.
Show that

1. $P \subseteq (2^{\mathrm{AP}})^\omega$ is a safety property iff P is closed.
2. $P \subseteq (2^{\mathrm{AP}})^\omega$ is a liveness property iff P is dense.