Category Theory - Limits and Colimits

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In this document we will look at the definitions of limits and colimits, as well as certain specific types of limits and colimits. It was written as notes for a lecture, so if there are any parts that can use more editing please let me know as some things may have been lost or forgotten while transcribing them.

For the examples, we will look at the following categories:

- **Set**, the familiar category of sets;
- $(\mathbf{P}, \leq)$ where $(P, \leq)$ is a poset;
- $(\mathbb{N}, |)$, the poset category where $|$ is the divisibility relation;
- **Top**, the topological spaces and continuous maps;
- **Grp**, the groups and group homomorphisms;
- **Rel**, the category whose objects are sets and whose arrows are relations (as opposed to functions).

1 Limits

In this section, we look at specific types of limits, before considering the general limit.

Definition 1.1 (Products)**.** Let $\mathcal{C}$ be a category, and let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be objects of $\mathcal{C}$. We say that the **product** of X and Y, if it exists, is an object (denoted $X \times Y$) in $\mathcal{C}$ along with maps (called projections) $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for any object A of $\mathcal{C}$ with arrows $f : A \to X$ and $g : A \to Y$, there is a unique $h : A \to X \times Y$ such that the following diagram commutes.

$$\begin{tikzcd}
& A \arrow[d, "h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\
& X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\
X & & Y
\end{tikzcd}$$

One often says that the maps f, g "factor through" the product $X \times Y$.

Example 1.2. Products in the category **Set** are the Cartesian product $X \times Y$ of sets, along with the regular projection maps. In this case, we would have $h(a) = (f(a), g(a))$. For this reason, the induced map h is often denoted (f, g).

Example 1.3. Products in $(\mathbb{N}, |)$ are greatest common divisors. To see this, take two objects $m, n \in \mathbb{N}$. Then there are arrows from $gcd(m, n)$ to m and n because it divides them both, and for any a dividing both m and n (ie having arrows to both m and n), a must divide their gcd and so there is a (necessarily unique) map from a to $gcd(m, n)$.

In a genera poset, one can check that the product of two objects X and Y (should it exist!) is their infimum/meet.

Example 1.4. In **Top**, the product of two spaces is simply the product of their sets endowed with the product topology (and this is why the product topology is the standard topology endowed on the Cartesian product of two spaces).

Similarly, in **Grp**, one takes the product of the two underlying sets endowed with the group structure $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$.

Remark 1.5. A good (and important) exercise is to check yourself that the above definitions are actually products in the sense defined above, in order to familiarize yourself with the definitions and become more comfortable with the constructions, which are common within category theory.

One can also verify that products are uniquely defined up to isomorphism (so that $A \times B \cong B \times A$), and that $(A \times B) \times C \cong A \times (B \times C)$.

Definition 1.6 (Pullbacks)**.** Let $\mathcal{C}$ be a category, $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ be objects of $\mathcal{C}$, and $p : X \to Z, q : Y \to Z$ be maps in $\mathcal{C}$. We say that the **pullback** of X and Y over Z, if it exists, is an object (denoted $X \times_Z Y$ or $X\,{}_p{\times}_q Y$) in $\mathcal{C}$ along with maps (called projections) $\pi_X : X \times_Z Y \to X$ and $\pi_Y : X \times_Z Y \to Y$ such that

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "q"] \\
X \arrow[r, "p"'] & Z
\end{tikzcd}$$

commutes, and for any object A of $\mathcal{C}$ with arrows $f : A \to X$ and $g : A \to Y$ such that

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "q"] \\
X \arrow[r, "p"'] & Z
\end{tikzcd}$$

commutes, there is a unique $h : A \to X \times_Z Y$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[dr, "h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "q"] \\
& X \arrow[r, "p"'] & Z
\end{tikzcd}$$

Example 1.7. In **Set**, the pullback $X \times_Z Y$ is the subset of $X \times Y$ consisting of pairs (x, y) such that $p(x) = q(y)$.

Given a map $f : X \to Z$, one calls $f^{-1}(z)$ the **fiber** of $z \in Z$. Thus, one can see the pullback $X \times_Z Y$ as the collection of products of fibers

$$X \times_Z Y = \bigcup_{z \in Z} p^{-1}(z) \times q^{-1}(z).$$

This is why the pullback is also often called the **fiber product** or **fibered product**.

Example 1.8. In poset categories, the pullback is the same as the product (if they exist).

Example 1.9. In **Top**, the pullback $X \times_Z Y$ is the same subset of $X \times Y$ as in **Set**, given the subspace topology. Similarly, in **Grp**, the pullback is the same subset of $X \times Y$ as in **Set** (one can check that this subset is indeed a subgroup).

Remark 1.10. The pullback of a mono is a mono, in the sense that if the map q in the definition of a pullback is a mono, then so is the map π_X.

One also has the pasting lemma, which states that in the following commuting diagram, assuming the right square is a pullback, the left square is a pullback if and only if the large rectangle is a pullback.

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & D \arrow[d] \\
C \arrow[r] & B \arrow[r] & A
\end{tikzcd}$$

Both of these are good exercises in diagram chasing, and are great practice both for diagram chasing and to make sure you understand what a pullback is.

Definition 1.11 (Equalizers)**.** Let $\mathcal{C}$ be a category, $X, Y \in \mathrm{Ob}(\mathcal{C})$ be objects of $\mathcal{C}$, and $f, g : X \to Y$ be arrows in $\mathcal{C}$. We say that the **equalizer** of f, g, if it exists, is an object E of $\mathcal{C}$ along with a map $e : E \to X$ such that the diagram

$$\begin{tikzcd}
E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

commutes, and if $h : A \to X$ is a map such that $f \circ h = g \circ h$, then there is a unique map $h' : A \to E$ making the following diagram commute.

$$\begin{tikzcd} E \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ A \arrow[u, "h'"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Example 1.12. In **Set**, the equalizer of $f, g : X \to Y$ is the subset of X of all x such that $f(x) = g(x)$, and e is the inclusion. The same is true for **Top** and **Grp**, where the former is given the subspace topology, and the latter inherits the group structure (one can check this subset is a subgroup).

Remark 1.13. Equalizers are monic. Again, this is a good exercise to try out.

Definition 1.14 (Limits)**.** A **diagram** in $\mathcal{C}$ is a functor $A : \mathcal{J} \to \mathcal{C}$ for a (small) category $\mathcal{J}$. For $j \in \mathrm{Ob}(\mathcal{J})$, write A_j for $A(j)$.

A **cone** over a diagram A is an element $C \in \mathrm{Ob}(\mathcal{C})$ along with maps $f_j : C \to A_j$ for $j \in \mathrm{Ob}(\mathcal{J})$ such that for all maps $u : j \to k$ in $\mathcal{J}$, $A(u) \circ f_j = f_k$. The idea is that this cone forms a large commuting diagram over the image of A in $\mathcal{C}$.

A **limit** over a diagram A, written

$$\lim_{\leftarrow \mathcal{J}} A,$$

is a cone (L, π_j) over A in $\mathcal{C}$ such that every other cone over A factors through it uniquely, in the sense that if (C, f_j) is another cone over A, there is a unique map $h : C \to L$ such that the induced diagram commutes (ie $\pi_j \circ h = f_j$ for all $j \in \mathrm{Ob}(\mathcal{J})$).

One says that the limit is **finite** if the indexing category $\mathcal{J}$ is finite, meaning it has finitely many objects and arrows.

Example 1.15. All the previous definitions (products, pullbacks, equalizers) are examples of limits. For example, consider the discrete category 2 with two objects and no morphisms (other than the identities). Then a functor $A : 2 \to \mathcal{C}$ is simply a choice of two elements X, Y of $\mathcal{C}$, and the limit over this diagram (if it exists) is the product $X \times Y$ in $\mathcal{C}$. Try to find the categories $\mathcal{J}$ corresponding to pullbacks and equalizers.

Example 1.16. One can generalize products to n-ary (or infinitary) products by choosing a discrete category $\mathcal{J}$ above with n (or infinitely many) objects. In this way, one can see that the terminal object is the "empty product". One often denotes the terminal object as one of $\mathbb{1}, 1, 1_{\mathcal{C}}$.

Also, if a category has binary products, then by induction it has all n-ary products for finite $n \geq 1$.

Definition 1.17 (Complete categories)**.** A category is **complete** if it has all limits, meaning that for any diagram $A : \mathcal{J} \to \mathcal{C}$ the limit over A exists. A category is **finitely complete** if it has all finite limits.

We will now give a way to check if a category is (finitely) complete.

Proposition 1.18. *A category has finite products and equalizers if it has pullbacks and a terminal object.*

Remark 1.19. I make a mistake in the lecture when showing that we can get equalizers from pullbacks. The correct derivation is in the proof below.

Proof. If 1 is the terminal object of $\mathcal{C}$, then the product of X and Y is the pullback of $X \to 1 \leftarrow Y$. Now 1 is the empty product, and by induction binary products give all n-ary products for $n \geq 1$, and so we have finite products. Also, the equalizer of $f, g : X \to Y$ is the pullback of the diagram

$$\begin{tikzcd} X \arrow[dr, "{(f,g)}"'] & & Y \arrow[dl, "\Delta"] \\ & Y \times Y & \end{tikzcd}$$

where $\Delta(y) = (y, y)$ is the diagonal map.

Proposition 1.20. *A category is (finitely) complete iff it has equalizers and (finite) products.*

We will prove this in the general case, but the finite case is identical.

Remark 1.21. The following proof is quite technical and messier than the ones done above. You may want to skip it for now, but I wanted to include it because it is an interesting and possibly useful result. I do think it's worth it to go through it yourself and make sure you can follow and understand it. The basic idea is simple, and the details are a lot easier to follow once you are more comfortable with limits, so if it feels hard to follow, come back to it later and try to follow along on a blackboard.

Proof. Clearly if a category is complete then it has equalizers and products, as these are limits.

Now suppose $\mathcal{C}$ has products and equalizers, and let $F : \mathcal{J} \to \mathcal{C}$ be a diagram. Define

$$X = \prod_{A \in \mathrm{Ob}(\mathcal{J})} FA.$$

For each arrow $f : A \to B$ in $\mathcal{J}$, we have the projection $\pi_B : X \to FB$. Thus, this forms a cone over the collection $FB = F(cod(f))$ for $f : A \to B$ in $\mathcal{J}$ (here $cod(f)$ means the codomain, or target object, of f). This in turn induces a unique map $\pi : X \to Y$, where

$$Y = \prod_{f : A \to B \text{ in } \mathcal{J}} FB$$

and $\pi_B = \pi_B \circ \pi$.

Similarly, for each arrow $f : A \to B$ in $\mathcal{J}$, we have the map $Ff \circ \pi_A : X \to FB$. This again forms a cone over the collection FB, and so we get a unique map $\tau : X \to Y$ such that $Ff \circ \pi_A = \pi_B \circ \tau$.

Now let E be the equalizer

$$E \xrightarrow{e} X \underset{\tau}{\overset{\pi}{\rightrightarrows}} Y \ .$$

We claim that E, along with the maps $\pi_A \circ e$ for $A \in \mathrm{Ob}(\mathcal{J})$, is cone over F, and is in fact the limit of F.

To see that this is a cone over F, we need to show that for $f : A \to B$ in $\mathcal{J}$, $Ff \circ (\pi_A \circ e) = \pi_B \circ e$. But this is clear because

$$\pi_B \circ e = \pi_B \circ \pi \circ e = \pi_B \circ \tau \circ e = Ff \circ \pi_A \circ e.$$

To see that this is the limit of F, let C, g_A be any other cone over F. These g_A induce a map $g : C \to X$. For any $f : A \to B$ in $\mathcal{J}$,

$$\pi_B \circ \pi \circ g = \pi_B \circ g = g_B = Ff \circ g_A = Ff \circ \pi_A \circ g = \pi_B \circ \tau \circ g,$$

and so there is a unique $h : C \to Y$ such that $\pi_B \circ h = \pi_B \circ \pi \circ g = \pi_B \circ \tau \circ g$ for all $f : A \to B$ in $\mathcal{J}$, and so $\pi \circ g = h = \tau \circ g$.

Because E is the equalizer of π, τ, this induces a unique map $h : C \to E$ such that $g = e \circ h$. For $A \in \mathrm{Ob}(\mathcal{J})$,

$$\pi_A \circ e \circ h = \pi_A \circ g = g_A,$$

and so this is a map of cones (which is unique as noted above). Thus E is the limit of F. 😸

Corollary 1.22. *A category is finitely complete iff it has finite products and equalizers iff it has pullbacks and a terminal object.* 😸

2 Colimits

Recall that for a category $\mathcal{C}$, we can form the category $\mathcal{C}^{\mathrm{op}}$ by reversing the arrows of $\mathcal{C}$. If a notion in $\mathcal{C}$ is exactly the same as q different notion in $\mathcal{C}^{\mathrm{op}}$, we call these notions **dual** to each other. For example, monomorphisms are a dual concept to epimorphisms, because a map is a monomorphism in $\mathcal{C}$ iff it is an epimorphism in $\mathcal{C}^{\mathrm{op}}$. In general, the correspondence of notions between $\mathcal{C}$ and $\mathcal{D}^{\mathrm{op}}$ for categories $\mathcal{C}$ and $\mathcal{D}$ is called **duality**, which is a concept we will (hopefully) explore later.

In this section we will define colimits, which are a dual concept to limits. This means that a colimit in $\mathcal{C}$ corresponds to a limit in $\mathcal{C}^{\mathrm{op}}$. In general, concepts that are dual to each other are prefixed by "co", and just as $\mathcal{C}^{\mathrm{op}\,\mathrm{op}} = \mathcal{C}$, if something would be prefixed by "co" twice we cancel them out. Thus we will look at coproducts and coequalizers in this section as well, before generalizing to colimits.

I recommend stopping at all definitions in this section, and trying to see how they are the "opposite" (dual) of the definitions in the previous section on limits.

Remark 2.1. The dual of pullback is not called co-pullback, but is instead called a pushout. If one looks at the diagram of the pullback and the pushout, the reason for these names is clear.

Remark 2.2. One may notice that I mentioned **Rel** at the start, and have yet to use it as an example. This is because $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$, and so limits and their corresponding colimits in **Rel** are exactly the same. We will talk about equivalence of categories later, but you can try to see why $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$.

Remark 2.3. By recognizing that limits and colimits are dual, we will see later that all the theorems we proved above for limits will carry over for colimits "for free".

Definition 2.4 (Coproducts)**.** Let $\mathcal{C}$ be a category, and let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be objects of $\mathcal{C}$. We say that the **coproduct** of X and Y, if it exists, is an object (denoted $X + Y$ or $X \coprod Y$) in $\mathcal{C}$ along with maps (called inclusions) $\iota_X : X \to X \coprod Y$ and $\iota_Y : Y \to X \coprod Y$ such that for any object A of $\mathcal{C}$ with arrows $f : X \to A$ and $g : Y \to A$, there is a unique $h : X \coprod Y \to A$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[dr, "\iota_X"] \arrow[ddr, bend right, "f"'] & & Y \arrow[dl, "\iota_Y"'] \arrow[ddl, bend left, "g"] \\
& X \coprod Y \arrow[d, "h"] & \\
& A &
\end{tikzcd}$$

Example 2.5. Coproducts in the category **Set** are the disjoint union of sets, along with the regular inclusions. In this case, for $z \in X + Y$, if $z \in X$ then $h(z) = f(z)$, and otherwise $z \in Y$ and $h(z) = g(z)$.

Example 2.6. Coproducts in $(\mathbb{N}, |)$ are least common multiples. In a general poset, a coproduct is a supremum/join (if it exists).

Example 2.7. In **Top**, the coproduct of two spaces X, Y is the disjoint union of their underlying sets, with open sets being disjoint unions of open sets in X and Y respectively.

Example 2.8. In **Grp**, the coproduct of two groups is the free product of the groups. This gives an example of a category whose objects are sets endowed with additional structure, but whose coproduct is not the coproduct of the underlying sets.

Example 2.9. In **Rel**, if X, Y are sets, their coproduct is their disjoint union, with the inclusion $\iota_X : X \to X + Y$ being $\iota_X = \{(x, (x, 0)) \mid x \in X\} \subseteq X \times (X \coprod Y)$.

As mentioned above, $\mathbf{Rel} \cong \mathbf{Rel}^{\mathrm{op}}$, and so this is actually the coproduct in $\mathbf{Rel}^{\mathrm{op}}$ as well. This means that this is the product in **Rel**! (You should verify this yourself directly). This gives an example of a category whose objects are sets, but whose product is not the product as sets.

Remark 2.10. Again, you should verify yourself that these are coproducts in order to familiarize yourself with the concept.

Also, as coproducts are dual to products, from the remark made for products one sees that coproducts are unique up to isomorphism, $A + B \cong B + A$, and $(A + B) + C \cong A + (B + C)$.

Definition 2.11 (Pushouts)**.** Let $\mathcal{C}$ be a category, $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ be objects of $\mathcal{C}$, and $p : Z \to X, q : Z \to Y$ be maps in $\mathcal{C}$. We say that the **pushout** of X and Y over Z, if it exists, is an object (denoted $X +_Z Y$ or $X\ {}_f{+}_g Y$) in $\mathcal{C}$ along with inclusion maps $\iota_X : X \to X +_Z Y$ and $\iota_Y : Y \to X +_Z Y$ such that

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "\iota_Y"] \\ X \arrow[r, "\iota_X"'] & X +_Z Y \end{tikzcd}$$

commutes, and for any object A of $\mathcal{C}$ with arrows $f : X \to A$ and $g : Y \to A$ such that

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & A \end{tikzcd}$$

commutes, there is a unique $h : X +_Z Y \to A$ such that the following diagram commutes.

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "\iota_Y"] \arrow[ddr, bend left, "g"] & \\ X \arrow[r, "\iota_X"'] \arrow[drr, bend right, "f"'] & X +_Z Y \arrow[dr, "h"] & \\ & & A \end{tikzcd}$$

Example 2.12. In **Set**, the pushout $X +_Z Y$ is a quotient of $X + Y$ (vs the subset taken in the pullback). Specifically, define a relation on $X + Y$ by $x \sim y$ iff there is some $z \in Z$ such that $p(z) = x$ and $q(z) = y$. Now this may not be an equivalence relation (try to come up with an example where it isn't), so we can't take the quotient of $X + Y$ by $\sim$. However, we can define $\sim^*$ to be the smallest equivalence relation containing $\sim$ (equivalently, $\sim^*$ is the equivalence relation generated by $\sim$), and take the quotient $X +_Z Y = (X + Y)/\sim^*$.

Given maps $f : X \to A$ and $g : Y \to A$ that commute with p, q, the induced map is $h([x]) = f(x)$ for $x \in X$ and $h([y]) = g(y)$ for $y \in Y$.

You should check yourself that this h is both a well-defined function and a unique one making the appropriate diagram commute (this is less straightforward than checking the construction given for the pullback defined above).

Remark 2.13. A fun example is that in **Set**, $A \cup B$ is the pushout of $A \leftarrow A \cap B \to B$, and vice versa, ie the following is both a pushout and pullback square, where all the arrows are inclusions.

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & A \cup B \end{tikzcd}$$

Example 2.14. As for pullbacks, the pushout in poset categories coincides with the coproduct.

Example 2.15. In **Top**, the pushout $X +_Z Y$ is the pushout of sets, given the quotient topology.

Similarly, the pushout in **Grp** is a quotient of the coproduct of groups. However, this coproduct is not the coproduct as sets, it's the free product. Also, we must take the normal subgroup generated by this equivalence relation in the quotient.

Remark 2.16. As we can see, pullbacks are often a subset of the product, while pushouts are often a quotient of the coproduct. Intuitively, one can think of the pullback $X\ {}_f{\times}_g Y$ to be asking "when is $f = g$?", while the pushout $X\ {}_f{+}_g Y$ is saying "lets force $f = g$". Not sure how helpful it is to think of things this way, but I like it.

Example 2.17. Pushouts do not in general exist in **Rel**. Thus, neither do pullbacks by duality. Also, since the existence of products and equalizers implies the existence of pullbacks, the existence of coproducts and coequalizers implies the existence of pushouts, and so coequalizers do not exist in general either.

Remark 2.18. Dual to the pullback, the pushout of an epi is an epi. One can also formulate the pasting lemma with respect to pushout squares instead of pullback squares. I'll leave this as an exercise.

Definition 2.19 (Coequalizers)**.** As I'm sure you can guess by now, in a category $\mathcal{C}$, given two objects X and Y and two arrows $f, g : X \to Y$, the **coequalizer** of f, g, if it exists, is an object E of $\mathcal{C}$ along with a map $e : Y \to E$ such that the diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "e"] & E \end{tikzcd}$$

commutes, and if $h : Y \to A$ is a map such that $h \circ f = h \circ g$, then there is a unique map $h' : E \to A$ making the following diagram commute.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "e"] \arrow[dr, "h"'] & E \arrow[d, "h'"] \\ & & A \end{tikzcd}$$

Example 2.20. In **Set**, the coequalizer of $f, g : X \to Y$ is the quotient of Y by the smallest equivalence relation such that $f(x) \sim g(x)$. Similarly, the coequalizer in **Top** and **Grp** are quotients of Y, in **Grp** by the normal subgroup generated by this relation.

One can see that for limits we take subsets and for colimits we take quotients, which lines up with the remark above that limits as "when is $f = g$" and colimits say "lets force $f = g$".

Remark 2.21. By duality again, coequalizers are epic.

Definition 2.22 (Colimits)**.** Recall that a **diagram** in $\mathcal{C}$ is a functor $A : \mathcal{J} \to \mathcal{C}$ for a (small) category $\mathcal{J}$. For $j \in \mathrm{Ob}(\mathcal{J})$, write A_j for $A(j)$.

A **cocone** over a diagram A is an element $C \in \mathrm{Ob}(\mathcal{C})$ along with maps $f_j : A_j \to C$ for $j \in \mathrm{Ob}(\mathcal{J})$ such that for all maps $u : j \to k$ in $\mathcal{J}$, $f_k \circ A(u) = f_j$. The idea is that this cocone forms a large commuting diagram under the image of A in $\mathcal{C}$.

A **colimit** over a diagram A, written

$$\lim_{\to \mathcal{J}} A,$$

is a cocone (L, π_j) over A in $\mathcal{C}$ such that every other cocone over A factors through it uniquely, in the sense that if (C, f_j) is another cocone over A, there is a unique map $h : L \to C$ such that the induced diagram commutes (ie $h \circ \pi_j = f_j$ for all $j \in \mathrm{Ob}(\mathcal{J})$).

One says that the colimit is **finite** if the indexing category $\mathcal{J}$ is finite, meaning it has finitely many objects and arrows.

The following comments and examples are dual to those in the previous section about limits (if you think they look familiar or copy-pasted ;)).

Example 2.23. All the previous definitions (coproducts, pushouts, coequalizers) are examples of colimits. For example, consider the discrete category 2 with two objects and no morphisms (other than the identities). Then a functor $A : 2 \to \mathcal{C}$ is simply a choice of two elements X, Y of $\mathcal{C}$, and the colimit over this diagram (if it exists) is the coproduct $X + Y$ in $\mathcal{C}$. Try to find the categories $\mathcal{J}$ corresponding to pushouts and coequalizers.

Example 2.24. One can generalize coproducts to n-ary (or infinitary) coproducts by choosing a discrete category $\mathcal{J}$ above with n (or infinitely many) objects. In this way, one can see that the initial object is the "empty coproduct". One often denotes the initial object as one of $0, 0_{\mathcal{C}}$.

Also, if a category has binary coproducts, then by induction it has all n-ary coproducts for finite $n \geq 1$.

Definition 2.25 (Cocomplete categories)**.** A category is **cocomplete** if it has all colimits, meaning that for any diagram $A : \mathcal{J} \to \mathcal{C}$ the colimit over A exists. A category is **finitely cocomplete** if it has all finite limits.

We will now give a way to check if a category is (finitely) cocomplete.

The proofs follow from the dual statements about limits by duality.

Proposition 2.26. *A category has finite coproducts and coequalizers if it has pushouts and an initial object.*

Proposition 2.27. *A category is (finitely) cocomplete iff it has coequalizers and (finite) coproducts.*

Corollary 2.28. *A category is finitely cocomplete iff it has finite coproducts and coequalizers iff it has pushouts and an initial object.* 😸

3 Universal Properties

Definition 3.1 (Universal properties)**.** An object or general construction is said to satisfy a certain **universal property** if this property essentially says that the object or construction is an initial object in an appropriate category.

Note that by duality, one could equivalently say that the object or construction is a terminal object in an appropriate category.

This is a very abstract notion, so to clear things up we will now present a bunch of examples.

Remark 3.2. This is a very deep idea, and it is equivalent or related to many other concepts we may see later. For example, since an initial object in a category has a unique map to any other object, one can argue that this means that a universal property is a property that defines an object by describing the maps that go out of it. This corresponds directly with the Yoneda lemma, which we will see later, which says that any object in a category is uniquely determined (defined) by the maps going into/out of it. We will see later how this relates objects defined by universal properties to the representable functors (those functors that are isomorphic to a hom functor).

Remark 3.3. It is easy to prove that initial objects in a category are all isomorphic (try this!), and so any object satisfying a universal property is unique up to isomorphism. Thus, defining things (should they exist) by the universal property they satisfy is a valid definition.

We will see now that all limits and colimits are examples of universal objects, and so once we show that the limit or colimit exists, we know it is unique (up to isomorphism), and we can take this as a proper definition of the object (which we did implicitly above).

Example 3.4. As mentioned above, limits and colimits are universal constructions. Specifically, limits correspond intuitively to terminal objects, and colimits to initial objects.

For example, consider the product $X \times Y$ in a category $\mathcal{C}$. Assuming this exists, in which category would this be a terminal object? Well, we know that the defining notion of the product says that given maps $f : A \to X$ and $g : A \to Y$, we get a unique map from A to $X \times Y$ such that the following diagram commutes.

$$\begin{tikzcd} & A \arrow[d, "h"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

We know that this map h to $X \times Y$ depends on f, g, but given a choice of f, g it exists and is unique (like what we require for a terminal object!). Thus, we are led to the following construction: let our category $\mathcal{D}$ be the category of objects A in $\mathcal{C}$ along with maps $A \to X$ and $A \to Y$, or equivalently the category of diagrams

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

Morphisms in this category between objects (A, f, g) and (B, f', g') will be morphisms $h : A \to B$ in $\mathcal{C}$ such that "pasting" these two diagrams along h will give a commuting diagram:

$$\begin{tikzcd} & A \arrow[dd, "h"] \arrow[dddl, bend right, "f"'] \arrow[dddr, bend left, "g"] & \\ & & \\ & B \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & & Y \end{tikzcd}$$

The definition of the product $X \times Y$ is exactly the statement that $X \times Y$ is the terminal object in this category.

Similarly, for the pullback $X \times_Z Y$, the category $\mathcal{D}$ is taken to be commuting diagrams of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

with morphisms being maps in $\mathcal{C}$ that make the appropriate diagrams commute. The coproduct $X \times_Z Y$, should it exist, is the terminal object in this category.

I leave as an exercise convincing yourself that these are proper categories and that the limits are indeed the terminal objects in them. You should also try to formulate the appropriate categories and initial objects for the colimits defined above. Finally, try to see if you can convince yourself that every limit is a terminal object, and every colimit an initial one, in a certain category that depends canonically on the diagram being used.

We now introduce other examples of universal properties.

Example 3.5. The natural numbers are the universal inductive set. Specifically, we say a category has a **natural numbers object** $\mathbb{N}$ if it has a terminal object $\mathbb{1}$, and maps

$$\mathbb{1} \xrightarrow{0} \mathbb{N} \xrightarrow{S} \mathbb{N}$$

such that every collection of maps $\mathbb{1} \to A$ and $f : A \to A$ induces a unique map $g : \mathbb{N} \to A$ such that the following diagram commutes.

$$\begin{tikzcd} \mathbb{1} \arrow[r, "0"] \arrow[dr] & \mathbb{N} \arrow[r, "S"] \arrow[d, "g"'] & \mathbb{N} \arrow[d, "g"] \\ & A \arrow[r, "f"'] & A \end{tikzcd}$$

In **Set**, the function 0 picks out the natural number 0, and the function S is the successor function $S(n) = n + 1$. Additionally, any maps to an object A as above pick out an element $a_0 \in A$ and induce the map $g : \mathbb{N} \to A$ defined as $g(0) = a_0$ and $g(n + 1) = f(g(n))$ (one can also see these maps to A as defining the sequence $a_{n+1} = f(a_n)$, and then g is just the function $n \mapsto a_n$).

It should be clear that the natural numbers object is the initial object in the category of diagrams of the form $\mathbb{1} \to A \to A$, and that in the category **Set** this exactly captures our intuition of $\mathbb{N}$.

Example 3.6. A poset is directed if for all x, y there is a z such that $x, y \leq z$ (ie such that every pair of elements has an upper bound). If P is a directed poset, the limit of a diagram $A : P \to \mathcal{C}$ is called an **inverse limit** or a **projective limit**.

The p-adic integers are an example of a projective limit, namely the limit of the diagram

$$\cdots \longrightarrow \mathbb{Z}/p^{n+1}\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}$$

from $\mathbb{N}^{\mathrm{op}}$ to **Grp** (here the maps are projection maps).

Also, the profinite groups are the groups that are isomorphic to a projective limit of finite groups, which we have shown exist (because all limits exist in **Grp**). These are exactly the Galois groups of field extensions of infinite degree.

For completeness, note that the colimit of a diagram $A : P \to \mathcal{C}$ for a directed poset P is called a **direct limit**.

Example 3.7. Recall that an integral domain is a commutative ring with no zero-divisors (where x, y are zero-divisors if $xy = 0$ but neither x nor y are equal to 0). The field of fractions of an integral domain is a universal construction. Specifically, after defining the field of fractions F of an integral domain R, one usually proves the universal property that for any field K and ring homomorphism $R \to K$, there is a unique map $F \to K$ making the following diagram commute.

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & F \arrow[d] \\ & K \end{tikzcd}$$

In which category is this an initial/terminal object?

Example 3.8. Given a group G, one can take its Abelianization G^{ab}, which is in some sense the "largest" Abelian group one can form from G. After defining it, one usually proves the universal property that for any other Abelian group H and group homomorphism $G \to H$, there is a unique group homomorphism $G^{ab} \to H$ such that the following diagram commutes.

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & G^{ab} \arrow[d] \\ & H \end{tikzcd}$$

What is the category in which this is an initial/terminal object?

Remark 3.9. In the previous two examples, I showed the universal property for the field of fractions of an integral domain and the Abelianization of a group without defining the field of fractions or the Abelianization explicitly. By the remarks above that any universal object is unique up to isomorphism, this shows that defining these objects by their universal properties is a valid definition (and is in fact often the true definition we care about).

However, when defining something by a universal property, one must be careful to show that this universal property is satisfied by some object (ie that the object they define actually exists). This is why these objects are often defined explicitly and constructively, and then proven to satisfy the universal property that defines them, rather than defined by the universal property directly.

4 Other Examples

Here are a few other examples of limits and colimits, as well as a theorem about them that will be useful later.

Example 4.1. A lattice is a poset with products and coproducts. A bounded lattice is a lattice with an initial and terminal object. Thus, one can equivalently define a bounded lattice to be a finitely complete and finitely cocomplete category.

A complete lattice is a lattice with arbitrary (not just finite) meets and joins, and so a complete lattice is a poset with arbitrary products and coproducts, and so it is a complete and cocomplete poset.

The following two examples are things we will hopefully return to later over the summer.

Example 4.2. Given a typed programming language, one can define the category whose objects are types and whose arrows are terms in the language. Specifically, if t is a term in the language that describes a function that takes as input type A and has as output type B, then it will be a map $A \to B$ in our category.

In this category, products are the product type, and coproducts the disjoint union type (often characterized by a case or switch statement being used in the code). Other limits and colimits, should they exist, correspond to more general type constructors in the language.

Example 4.3. In a similar vein to the previous example, one can specify a formal language $\mathscr{L}$ in a certain proof system, and then take the category corresponding to this language to be the category whose objects are statements in $\mathscr{L}$, and whose arrows $\phi \to \psi$ are formal proofs that ϕ logically entails ψ.

In this category, products will be the conjunction of statements ($\phi \wedge \psi$), and coproducts will be disjunction ($\phi \vee \psi$).

Lemma 4.4. *Hom functors preserve limits. That is,* $\mathrm{Hom}(\lim_{\to \mathcal{J}} A, C) = \lim_{\leftarrow \mathcal{J}} Hom(A, C)$ *and* $\mathrm{Hom}(C, \lim_{\leftarrow \mathcal{J}} A) = \lim_{\leftarrow \mathcal{J}} Hom(C, A)$.

This lemma requires understanding the composition of functors, which we have covered previously. Note that in one case colimits become limits, because the Hom functors is contravariant in the first coordinate.

Proof. Suppose $A : \mathcal{J} \to \mathcal{C}$ is a diagram and $L = \lim_{\leftarrow \mathcal{J}} A$. Note that in general, if $F : \mathcal{C} \to \mathcal{D}$ is a functor, then $F \circ A$ is a diagram in $\mathcal{D}$. Also, if $C \in \mathrm{Ob}(\mathcal{C})$ forms a cone over A, then FC forms a cone over $F \circ A$. Thus, for any object $C \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}(C, L)$ is a cone over $\mathrm{Hom}(C, A) = \mathrm{Hom}(C, -) \circ A$. It remains to show that it is a universal cone.

To see this, suppose M forms a cone (in **Set**) over $\mathrm{Hom}(C, A)$, with maps $h_j : M \to \mathrm{Hom}(C, A_j)$. For all $m \in M$, we get maps $h_j(m) \in \mathrm{Hom}(C, A_j)$ such that $A(u) \circ h_j(m) = h_k(m)$ for $u : j \to k$ in $\mathcal{J}$ (this follows because M is a cone over $\mathrm{Hom}(C, A)$). Thus, C along with these maps forms a cone in $\mathcal{C}$ over A, and by the universal property of limits this induces a unique map $\xi_m : C \to L$ that commutes with this cone.

By construction, the map $m \mapsto \xi_m$ forms a commuting map of cones $M \to \mathrm{Hom}(C, L)$. Since each ξ_m was uniquely defined to make this map commute, this map is unique. Thus $\mathrm{Hom}(C, L)$ is the universal cone in **Set** of $\mathrm{Hom}(C, A)$, which is what we set out to show.

That $\mathrm{Hom}(L, C)$ is the limit of $\mathrm{Hom}(A, C)$ where $L = \lim_{\to \mathcal{J}} A$ follows similarly. Note that the colimit in $\mathcal{C}$ turns into a limit in **Set** due to the contravariant nature of $\mathrm{Hom}(-, C)$. 😸