

TD – Semantics and Verification  
**IV – Partial Orders and Lattices**  
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## Partial Orders and Lattices

- A *partial order* is a pair  $(A, \leq)$  of a set  $A$  and binary relation  $\leq$  which is
  1. reflexive:  $a \leq a$  for all  $a \in A$ ,
  2. transitive: if  $a \leq b$  and  $b \leq c$  then  $a \leq c$ ,
  3. antisymmetric: if  $a \leq b$  and  $b \leq a$  then  $a = b$ .
- A *join (or least upper bound)* of  $S \subseteq A$  is an upper bound  $\bigvee S$  such that  $\bigvee S \leq b$  for every upper bound  $b$  of  $S$ .
- A *meet (or greatest lower bound)* of  $S \subseteq A$  is a lower bound  $\bigwedge S$  such that  $b \leq \bigwedge S$  for every lower bound  $b$  of  $S$ .
- A *complete lattice* is a partial order  $(A, \leq)$  such that every subset  $S \subseteq L$  has both a join and a meet.
- Given a topological space  $(X, \mathcal{U})$ , the *interior* of a set  $A$  is  $\mathring{A} = \bigcup \{U \in \mathcal{U} \mid U \subseteq A\}$ .

### Exercise 1.

Show that the following are equivalent for a partial order  $(L, \leq)$ :

1.  $(L, \leq)$  is a complete lattice,
2. every subset  $S \subseteq L$  has a least upper bound  $\bigvee S \in L$ ,
3. every subset  $S \subseteq L$  has a greatest lower bound  $\bigwedge S \in L$ .

### Exercise 2.

Consider the space  $A^\omega$  with  $A = \{a, b\}$ . Show that  $\bigcap_{n \in \mathbb{N}} \text{ext}(a^n)$  is not open.

## Closure operators

A *closure operator* on a partial order  $(L, \leq)$  is a function  $c : L \rightarrow L$  which is:

- monotone:  $c(a) \leq c(b)$  if  $a \leq b$ ,
- expansive:  $a \leq c(a)$ ,
- idempotent:  $c(c(a)) = c(a)$ .

### Exercise 3.

Consider a closure operator  $c$  on a complete lattice  $(L, \leq)$ . Show that  $L^c = \{a \in L \mid c(a) = a\}$  is a complete lattice with greatest lower bounds  $\bigwedge S = \bigwedge S$  and least upper bounds  $\bigvee S = c(\bigvee S)$ .

**Exercise 4.**

A *Kuratowski closure operator* is a closure operator  $c : 2^X \rightarrow 2^X$  such that  $c(\emptyset) = \emptyset$  and  $c(A \cup B) = c(A) \cup c(B)$ .

1. Consider a topological space  $(X, \mathcal{U})$ . Show that  $\overline{(-)}$  is a Kuratowski closure operator.
2. Given a Kuratowski closure operator  $c : 2^X \rightarrow 2^X$ , show that there is topology  $\mathcal{U}$  on  $X$  such that the closed sets for  $\mathcal{U}$  are exactly the closed sets for  $c$ , ie the sets such that  $A = c(A)$ .

**Galois connexion**

- Given partial orders  $(A, \leq_A)$  and  $(B, \leq_B)$ , a *Galois connection*  $g \dashv f : A \rightarrow B$  is given by a pair of functions  $g : A \rightarrow B$  and  $f : B \rightarrow A$  such that for all  $a \in A$  and  $b \in B$ , we have

$$g(a) \leq_B b \text{ iff } a \leq_A f(b).$$

$g$  (resp.  $f$ ) is called the *lower adjoint* (resp. *upper adjoint*).

- Given a non-empty set  $A$ , we define

$$\begin{aligned} \text{pref} & : 2^{A^\omega} \rightarrow 2^{A^*} \\ P & \mapsto \bigcup \{\text{pref}(\sigma) \mid \sigma \in P\} \\ \\ \text{cl} & : 2^{A^*} \rightarrow 2^{A^\omega} \\ W & \mapsto \{\sigma \in A^\omega \mid \text{pref}(\sigma) \subseteq W\} \end{aligned}$$

Notice that the function  $\text{cl}$  defined in the second tutorial is actually  $\text{cl}(\text{Pref}(P))$  here.

**Exercise 5.**

Consider a Galois connection  $g \dashv f : A \rightarrow B$ .

1. Show that both  $f$  and  $g$  are monotone.
2. Show that  $f \circ g$  is a closure operator.

**Exercise 6.**

Consider two complete lattices  $(A, \leq_A)$  and  $(B, \leq_B)$ .

1. Show that a function  $f : B \rightarrow A$  preserves greatest lower bounds iff  $f$  has a lower adjoint  $g : A \rightarrow B$ .
2. Show that a function  $g : A \rightarrow B$  preserves least upper bounds iff  $g$  has an upper adjoint  $f : B \rightarrow A$ .

**Exercise 7.**

Show that  $\text{Pref} \dashv \text{cl} : 2^{A^\omega} \rightarrow 2^{A^*}$  form a Galois connection.

**Exercise 8.**

Given  $P \subseteq A^\omega$ , show that  $\overline{P} = \text{cl}(\text{pref}(P))$ . Deduce that  $P$  is a safety property if and only if it is closed.

**Exercise 9.**

Show that  $P \subseteq A^\omega$  is a liveness property if and only if it is dense. Deduce that any LTP is the intersection of a safety property and a liveness property.