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- ENMG 616 - Dr. Maher Nouiehed ;
- Advanced Optimization Techniques & Algorithms -

Assignment # 1 -

Prob 1:

$f(x;y) = (x^2-4)^2 + y^2$
$= x^4 - 8x^2 + 16 + y^2$
$= (x^4 - 8x^2 + y^2) + 16.$

We want to show that f has two global minima and one stationary pt, which is neither local max / min.

$\nabla f(x,y) = \begin{bmatrix} 4x^3 - 16x \\ 2y \end{bmatrix}$. We want to set $\nabla f(x,y) = 0$.

$4x^{*3} - 16x^* = 0.$
$x^*(4x^{*2} - 16) = 0 \iff x_1^* = 0$ or $4x^{*2} = 16.$
$x^{*2} = 4 \iff x_2^* = 2$
$x_3^* = -2.$

$2y^* = 0 \Rightarrow y^* = 0.$

Three critical points:
$(0;0); (-2;0); (2;0)$.

$\nabla^2 f(x,y) = \begin{bmatrix} 12x^2 - 16 & 0 \\ 0 & 2 \end{bmatrix}$

$\nabla^2 f(0;0) = \begin{pmatrix} -16 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow$ Stationary point.

$\nabla^2 f(-2;0) = \begin{pmatrix} 32 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_i > 0 \Rightarrow$ Positive Definite Matrix.
Convex fct° and $(-2;0)$ global Minima.

$\nabla^2 f(2;0) = \begin{pmatrix} 32 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_i > 0$; PD Matrix. (strictly >).
Convex; $(2;0)$ global minima.

Prob 2:

$f(x;y) = x^2 + y^2 + \beta xy + x + 2y.$ We want all stationary points:

$\nabla f(x;y) = \begin{pmatrix} \frac{\partial f(x;y)}{\partial x} \\ \frac{\partial f(x;y)}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x + \beta y + 1 \\ 2y + \beta x + 2 \end{pmatrix}.$

$\nabla f(x;y) = 0.$

$\begin{cases} 2x + \beta y + 1 = 0. & x = \frac{-1-\beta y}{2}. \\ 2y + \beta x + 2 = 0. & \Rightarrow 2y + \beta * \left(\frac{-1-\beta y}{2}\right) + 2 = 0. \end{cases}$

$2y + \frac{-\beta - \beta^2 y}{2} + 2 = 0.$

$4y - \beta - \beta^2 y + 4 = 0.$

$y(4 - \beta^2) = -4 + \beta$

$y = \frac{-4+\beta}{(2-\beta)(2+\beta)}$

$y = \frac{\beta - 4}{(2-\beta)(2+\beta)}$ $\quad x = \frac{-1 - \beta y}{2}$

$= \dfrac{-1 - \beta \times \frac{\beta - 4}{(2-\beta)(2+\beta)}}{2}$

$\Rightarrow \left\{ \left(\frac{-2(1-\beta)}{(2-\beta)(2+\beta)} ; \frac{\beta - 4}{(2-\beta)(2+\beta)} \right) \right\}$ $= \dfrac{-1 - \beta^2 - 4\beta / (2-\beta)(2+\beta)}{2}$

$= \dfrac{\cancel{1} - 4 + \cancel{\beta^2} - \cancel{\beta^2} + 4\beta}{2(4-\beta^2)}$

$\beta \neq -2$ and $\beta \neq +2$.

$= \dfrac{-4(-\beta+1)}{2(4-\beta^2)} = \dfrac{-2(1-\beta)}{(2-\beta)(2+\beta)}$

Global: $\nabla^2 f(x;y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 2 & \beta \\ \beta & 2 \end{pmatrix}$

if $\beta \notin [-2;2]$: we want $\nabla^2 f$ to be > 0 (Positive Definite).

if $\beta < -2$: $\lambda_1 > 0$, $\lambda_2 \leq 0$. $\beta > 2$: $\lambda_1 \leq 0$, $\lambda_2 > 0$. Saddle Points

only $\Rightarrow$ $\lambda i > 0$. let's find eigenvalues:

$\det(A - \lambda i) = 0$. $\Leftrightarrow \det \left\{ \begin{pmatrix} 2-\lambda & \beta \\ \beta & 2-\lambda \end{pmatrix} \right\} = 0 \Rightarrow (2-\lambda)^2 - \beta^2 = 0$.

$\beta = |2 - \lambda|$.

$\Rightarrow 2 - \lambda = \beta$ or $2 - \lambda = -\beta$.

$\lambda_1 = 2 - \beta$ or $\lambda_2 = 2 + \beta$.

we want $\lambda_i > 0$. $\Leftrightarrow$ $2 - \beta > 0$ and $2 + \beta > 0$.

$\boxed{\beta < 2}$ and $\boxed{\beta > -2}$.

Prob 3: $\frac{1}{x} + x \geq 2$. $x > 0$

let $f(x) = 1/x + x$. $f'(x) = -1/x^2 + 1 = 0$.

$-1/x^2 = -1$

$f''(x) = 2/x^3$. $\underbrace{x^* = -1}_{\text{rejected}}$ or $\boxed{x^* = +1}$.

$f''(x^*) = f''(1) = 2 > 0$.

$\Rightarrow$ The pt at $x^* = 1$ is global min.

$f(x^* = 1) = 1 + 1 = 2$. 2 is min of the fct. $(1,2)$ is global min.

$\Rightarrow f(X) \geq 2$

$\boxed{1/x + x \geq 2}$ $x > 0$

Prob 4:

$\min_x e^{x_1} + e^{x_2} + \dots + e^{x_n}$

s.t: $x_1 + x_2 + \dots + x_n = S$

1) $f(x) = e^x$; $f'(x) = e^x$; $f''(x) > 0 \Rightarrow e^x$ convex.

The sum of convex fct^s is convex.

$\Rightarrow e^{x_1} + e^{x_2} + \dots + e^{x_n}$ convex.

$x_1 + x_2 + \dots + x_n = S$. $\quad Ax = b$. $\quad A = (1\ 1 \dots 1)$. $x = (x_1\ x_2 \dots)$ $b = (S)$.

$\Downarrow$ $\quad$ convex set.

CONVEX SET.

This optim. probl is convex.

let $u, v \in x$: $0 \leq \alpha \leq 1$. $u = (u_1, u_2 \dots)$ $v = (v_1, v_2 \dots)$

$\alpha u + (1-\alpha) v$

$\alpha S + (1-\alpha) S = S \Rightarrow$ CONVEX SET.

2) Change of variable: $x_n = s - x_1 - x_2 - (\dots) - x_{n-1}$

$f(x_1, x_2, \dots, x_n) = e^{x_1} + e^{x_2} + \dots + e^{x_n}$ with the change of var:

$f(x_1, x_2, \dots, x_{n-1}) = e^{x_1} + e^{x_2} + \dots + e^{x_{n-1}} + e^{s - x_1 - x_2 - (\dots) - x_{n-1}}$.

the cst is now satisfied in the obj fct°.

→ Optim. Prob: $\min\limits_{x \in \mathbb{R}^{n-1}} e^{x_1} + e^{x_2} + \dots + e^{x_{n-1}} + e^{s - x_1 - x_2 - (\dots) - x_{n-1}}$.
(unconstrained)

3) $\nabla f(x_1, x_2, \dots, x_{n-1}) = \begin{pmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \\ \vdots \\ \partial f / \partial x_{n-1} \end{pmatrix} = \begin{pmatrix} e^{x_1} - e^{s - x_1 - x_2 - (\dots) - x_{n-1}} \\ e^{x_2} - e^{s - x_1 - x_2 - (\dots) - x_{n-1}} \\ \vdots \\ e^{x_{n-1}} - e^{s - x_1 - x_2 - (\dots) - x_{n-1}} \end{pmatrix}$.

$\nabla f(x_1, \dots) = 0$.

$\Rightarrow \left.\begin{aligned} e^{x_1^*} &= e^{s - x_1^* - x_2^* - (\dots) - x_{n-1}^*} \\ e^{x_2^*} &= e^{s - x_1^* - x_2^* - (\dots) - x_{n-1}^*} \\ e^{x_{n-1}^*} &= e^{s - x_1^* - x_2^* - (\dots) - x_{n-1}^*} \end{aligned}\right\}$ $x_1^* = x_2^* = \dots = x_{n-1}^* = s - x_1^* - (\dots)$
Let them be $= x_i^*$

$\Rightarrow x_i^* = s - (n-1)x_i^* \iff \cancel{x_i^*} + n x_i^* - \cancel{x_i^*} = s$

$\boxed{x_i^* = s/n}$

we already proved that the fct° is convex:

$\nabla^2 f(x_1, x_2, \dots, x_{n-1}) = \begin{pmatrix} e^{x_1} + e^{s - x_1 - x_2 \dots} & e^{s - x_1 - x_2 (\dots)} & \dots & e^{s - x_1 - x_2 \dots} \\ e^{s - x_1 - x_2 \dots} & e^{x_2} + e^{s - x_1 - x_2 \dots} & e^{s - x_1 - x_2 \dots} \dots & e^{s - x_1 - x_2} \\ \vdots & & & \vdots \\ e^{s - x_1 - x_2 \dots} & e^{s - x_1 - x_2 \dots} & \dots & e^{x_{n-1}} - e^{s - x_1 - x_2 -} \end{pmatrix}$

$\nabla^2 f(s/n, s/n, \dots) = \begin{pmatrix} e^{s/n} + e^s & e^s & e^s & \dots & e^s \\ e^s & e^{s/n} + e^s & e^s & \dots & e^s \\ \vdots & & & & \vdots \\ e^s & e^s & e^s & \dots & e^{s/n} + e^s \end{pmatrix}$

if $x_i^* = s/n$ global min

$f(s/n, s/n, \dots, s/n) = e^{s/n} + e^{s/n} + \dots + e^{s - (n-1)\frac{s}{n}}$ $\qquad s - (n-1)\frac{s}{n} = \frac{ns - ns + s}{n} = s/n$

$= n e^{s/n}$.

4)

At $x_i^* = s/n$:

$f(x_i^*, \dots) = n e^{s/n}$ (min of the fction.

$\Rightarrow \sum\limits_{i=1}^{n} e^{x_i} \geq n e^{s/n}$

$\Rightarrow \frac{1}{n} \sum e^{x_i} \geq e^{s/n}$

But $s = x_1 + x_2 + \dots + x_n$

$$\frac{1}{n}\sum e^{x_i} \geq e^{\frac{x_1}{n}+\frac{x_2}{n}+\dots+\frac{x_n}{n}}$$

$$\frac{1}{n}\sum e^{x_i} \geq e^{x_1/n} e^{x_2/n} \dots e^{x_n/n}$$

change of variable:

$$\frac{1}{n}\sum e^{x_i} \geq y_1^{1/n}\, y_2^{1/n}\, y_3^{1/n} \dots y_n^{1/n}$$

$$\frac{1}{n}\sum_{i=1}^{n} y_i \geq (y_1 y_2 \dots y_n)^{1/n}$$

Prob 6:

$$\min_x \left[\frac{1}{2}(x^2-1)^2\right].$$

1) $f(x) = \frac{1}{2}(x^4 - 2x^2 + 1)$.

$f'(x) = \frac{1}{2}(4x^3 - 4x)$.

$f''(x) = \frac{1}{2}(12x^2 - 4) = 6x^2 - 2$.

$6x^2 - 2 = 0$. $x^2 = \frac{1}{3}$. $x = \pm\frac{\sqrt{3}}{3}$

x	$-\infty$		$-\sqrt{3}/3$		$+\sqrt{3}/3$		$+\infty$
$f''(x) = 6x^2 - 2$		+	0	−	0	+	

$\Rightarrow$ $f(x)$ is convex on the interval: $]-\infty ; -\frac{\sqrt{3}}{3}] \cup [+\frac{\sqrt{3}}{3} ; +\infty[$.

2) stationary Points:

$f'(x) = 0 \iff 2x^3 - 2x = 0$.

$x(2x^2 - 2) = 0$.

$x(x^2 - 1) = 0$.

$\Rightarrow x_1^* = 0 ;\ x_2^* = -1 ;\ x_3^* = +1$.

$f(-1) = 0$

$f(1) = 0$.

3) $f''(x) = 6x^2 - 2$;

$f''(0) = -2 < 0$. (local Max)

$\left.\begin{array}{l} f''(-1) = 4. \\ f''(1) = 4. \end{array}\right\}$ Global Min : $(-1, 0)$; $(+1, 0)$.

Prob 7:

$$x^T x = \|x\|_2^2$$

1) $\min_x \frac{1}{2}\|xx^T - A\|_F^2$; we'll use the following methd to see if this is convex: $f(\alpha u + (1-\alpha)v) \leq \alpha f(u) + (1-\alpha) f(v)$. $\alpha \in [0;1]$

Let's take $f(x) = \frac{1}{2}\|A - xx^T\|_F^2$.

u, v are 2 vectors, s.t.; $u = -v$

$$1/2\,\|A - (\alpha u + (1-\alpha)v)(\alpha u + (1-\alpha)v)^T\|_F^2$$

$$(\alpha u + (1-\alpha)v)(\alpha u^T + (1-\alpha)v^T)$$

$$-(\alpha^2 uu^T + \alpha(1-\alpha)uv^T + \alpha(1-\alpha)vu^T + (1-\alpha)^2 vv^T$$

$\frac{1}{2} \| A - \alpha^2 uu^T - \alpha(1-\alpha) uv^T - \alpha(1-\alpha) vu^T - (1-\alpha)^2 vv^T \|_F^2$

Let's find α s.t this is not less than $\alpha f(u) + (1-\alpha) f(v)$.

Let's take $u = -v$: $\| A - \alpha^2 uu^T + \alpha(1-\alpha) uu^T + \alpha(1-\alpha) uu^T - (1-\alpha)^2 uu^T \|_F^2$

$\frac{1}{2} \| A - (\alpha^2 - 2\alpha(1-\alpha) + (1-\alpha)^2) uu^T \|_2$

$A = ZZ^T$; let $Z = u$.

$\frac{1}{2} \| (1 - \alpha^2 + 2\alpha(1-\alpha) - (1-\alpha)^2) uu^T \|_2^2$.

Let $\alpha = 1/2$:

$\frac{1}{2} \| (1 - 0.25 + 0.5 - (0.5)^2) uu^T \|_2^2$

$\frac{1}{2} \| 0.75 + 0.5 - 0.25 \; uu^T \|_2^2$

$\frac{1}{2} \| uu^T \|^2 > 0$.

$\alpha f(u) + (1-\alpha) f(v)$ where $u = -v$.
and $A = uu^T$.

$\alpha f(u) + (1-\alpha) f(-u)$.

$= \alpha \cdot \frac{1}{2} \| uu^T - uu^T \|_F^2 + (1-\alpha) \| uu^T - uu^T \|_F^2$

$= 0$.

$\Rightarrow f(\alpha u + (1-\alpha) v) > \alpha f(u) + (1-\alpha) f(v)$.

$\therefore$ This optimization problem is not convex.

2) $\frac{1}{2} \| A - xx^T \|_F^2$:

$\nabla f = -(A - xx^T) x$.

$\nabla f = 0 \Rightarrow$ i) $x^* = 0$ (if $A - xx^T$ is full rank).

ii) $A = xx^T$
$ZZ^T = xx^T$ $\Big\} \to$ $Z = x$, $x = \pm Z$

iii) $A - xx^T$ not full rank $\therefore x \in$ Null space $A - xx^T$.

$S = \{ x, 0 \vee \pm Z \mid Z \in \mathbb{R}^n \}$ is the set of stationary pts.

3) $\nabla^2 f = -A - \|x\|_2^2 - 2xx^T$.

At $x^* = 0$:

$\nabla^2 f = -A$. let $v \in \mathbb{R}^n$.

$-v^T A v = -v^T ZZ^T v$
$= -\| v^T Z \|_2^2 \leq 0$. negative semi definite. Local maxima. $\to$ GLOBAL.

$\Rightarrow \pm Z$ is my local minima. at $x = \pm Z$: $f(x) = 0$. GLOBAL.

Prob 5:

1) $\alpha u + (1-\alpha) v$

$\alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$= \begin{bmatrix} \alpha x_1 + (1-\alpha) x_2 \\ \alpha y_1 + (1-\alpha) y_2 \end{bmatrix}$.

$f(\alpha u + (1-\alpha) v) = f(\alpha x_1 + (1-\alpha) x_2 ; \alpha y_1 + (1-\alpha) y_2)$

$= \sqrt{(\alpha x_1 + (1-\alpha) x_2) * (\alpha y_1 + (1-\alpha) y_2)}$

For $\alpha = 1/2$: $= \sqrt{(1/2 x_1 + 1/2 x_2) * (\frac{1}{2} y_1 + \frac{1}{2} y_2)}$

$\alpha f(u) + (1-\alpha) f(v) = \alpha f(x_1, y_1) + (1-\alpha) f(x_2, y_2)$.

$= \alpha \sqrt{x_1 y_1} + (1-\alpha) \sqrt{x_2 y_2}$.

$= \frac{1}{2} \sqrt{x_1 y_1} + \frac{1}{2} \sqrt{x_2 y_2}$.

At $(3,4) ; (2,7)$:

$\left. \begin{array}{l} f(\alpha u + (1-\alpha) v) = 3.71 \\ \alpha f(u) + (1-\alpha) f(v) = 3.6 \end{array} \right\}$ Here concave.

At $(1,1) ; (-1,-1)$:

$\left. \begin{array}{l} f(\alpha u + (1-\alpha) v) = 0. \\ \alpha f(u) + (1-\alpha) f(v) = 1. \end{array} \right\}$ Here convex.

It's Neither.

2) $u = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$; $v = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$.

$\alpha u + (1-\alpha) v$

$= \alpha \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + (1-\alpha) \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$

$= \begin{pmatrix} \alpha x_1 + (1-\alpha) x_2 \\ \alpha y_1 + (1-\alpha) y_2 \\ \alpha z_1 + (1-\alpha) z_2 \end{pmatrix}$

$f(\alpha u + (1-\alpha) v) = \sqrt{[\alpha x_1 + (1-\alpha) x_2] * [\alpha y_1 + (1-\alpha) y_2] * [\alpha z_1 + (1-\alpha) z_2]}$

$\alpha = 1/2$:

$= \sqrt{\left(\frac{1}{2} x_1 + \frac{1}{2} x_2\right) * \left(\frac{1}{2} y_1 + \frac{1}{2} y_2\right) * \left(\frac{1}{2} z_1 + \frac{1}{2} z_2\right)}$

$\alpha f(u) + (1-\alpha) f(v)$

$= \frac{1}{2} \sqrt{x_1 y_1 z_1} + \frac{1}{2} \sqrt{x_2 y_2 z_2}$

At $(1,1,1)$ & $(0,0,0)$: $f(\alpha u + (1-\alpha) v) = 0{,}354$.
$\alpha f(u) + (1-\alpha) f(v) = \frac{1}{2}$.
} Here, ~~concave~~ convex.

At $(2,1,9)$ & $(4,8,1)$:
$f(\alpha u + (1-\alpha) v) = 8{,}21$
$\alpha f(u) + (1-\alpha) f(v) = 4{,}94$
} Here it's concave.

This fct° is neither.

3) $f(x) = \log(e^{a_1x+b_1} + e^{a_2x+b_2} + \dots + e^{a_mx+b_m})$.

$$\nabla f(x) = \frac{a_1 e^{a_1x+b_1} + a_2 e^{a_2x+b_2} + \dots + a_m e^{a_mx+b_m}}{e^{a_1x+b_1} + e^{a_2x+b_2} + \dots + e^{a_mx+b_m}} . \quad \frac{u'v - v'u}{v^2}$$

$$\nabla^2 f(x) = \frac{(a_1^2 e^{a_1x+b_1} + a_2^2 e^{a_2x+b_2} + \dots + a_m^2 e^{a_mx+b_m})(e^{a_1x+b_1} + \dots e^{a_mx+b_m}) \underset{(-)}{} (a_1 e^{a_1x+b_1} + a_2 e^{a_2x+b_2} + (\dots) + a_m e^{a_mx+b_m})(a_1 e^{a_1x+b_1} + a_2 e^{a_2x+b_2} + \dots + a_m e^{a_mx+b_m}) \; (+)}{(e^{a_1x+b_1} + e^{a_2x+b_2} + \dots)^2} \to (+)$$

Let's see what the sign of numerator will be:

$$(a_1 e^{a_1x+b_1} + \dots + a_m e^{a_mx+b_m})^2 \leq (a_1^2 + \dots + a_m^2)(e^{2(a_1x+b_1)} + \dots + e^{2(a_mx+b_m)}).$$

$$(a_1^2 e^{a_1x+b_1} + \dots)(e^{a_1x+b_1} + \dots)$$

$\nabla^2 f(x) > 0$.

$\Rightarrow f(x)$ is a convex fct°. all 3 (+)