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ENMG 616 - Advanced Optimization Techniques.

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Assignment 5:

1 - Hidden Partition Problem:

1.1 - Re-Formulating the optimization problem -

1) $x_i = \begin{cases} 1 & \text{if } i \in S_1 \\ -1 & \text{if } i \in S_2. \end{cases}$

we know that $(\hat{S_1}, \hat{S_2}) = \underset{|S_1|=|S_2|}{\text{argmax}} \; A_{ij}$
i,j same partition.

$x_i x_j = \begin{cases} 1 & \text{if both} \in \text{same partition.} \\ -1 & \text{otherwise.} \end{cases}$

$$(\hat{S_1}, \hat{S_2}) = \underset{|S_1|=|S_2|}{\text{argmax}} \sum_{\substack{i,j \\ \text{same partition}}} A_{ij}$$

$$= \underset{|S_1|=|S_2|}{\text{argmax}} \sum_{x_{i,j} \in S_1} A_{ij} \overbrace{(x_i x_j)}^{=+1} + \sum_{x_{i,j} \in S_2} A_{ij} \overbrace{(x_i x_j)}^{=+1}$$

$$+ \sum_{\substack{(x_i, x_j) \\ \neq \text{part.}}} A_{ij} \overbrace{(x_i x_j)}^{=-1}$$

$$= \underset{|S_1|=|S_2|}{\text{argmax}} \sum_{x_{i,j} \in S_1} A_{ij} + \sum_{x_{i,j} \in S_2} A_{ij} - \sum_{\substack{(x_i, x_j) \\ \neq}} A_{ij}$$

To obtain $1 + x_i x_j$:

$1 + x_i x_j = \begin{cases} 2 & \text{if same partit}^{\circ}. \\ 0 & \text{otherwise} \end{cases}$

$$= \text{argmax} \; \frac{1}{2} \sum_{\substack{x_i, x_j \\ \in S_1}} A_{ij}(1 + x_i x_j) + \frac{1}{2} \sum_{\substack{x_i, x_j \\ \in S_2}} A_{ij}(1 + x_i x_j) - \frac{1}{2} \sum_{\substack{(x_i, x_j) \\ \neq}} \cancel{A_{ij}(1 + x_i x_j)}$$

bcz 0 for $x_i, x_j \neq 0$

$$= \underset{x}{\text{argmax}} \sum A_{ij}(1 + x_i x_j).$$

SOLO

By saying $|S_1| = |S_2|$, $\Rightarrow$ nb of x_i's $= 1$ = nb of x_i's $= -1$.

$\Rightarrow$

$$\underset{x}{\text{argmax}} \sum_{i,j} A_{ij}(1 + x_i x_j) \quad \text{s.t:} \left\{ \sum_{i=1}^{m} x_i = 0 \right. \qquad x_i = \{-1, +1\}$$

2) Changes:

$$\underset{x}{\text{argmax}}\ x^T A x - \lambda \left(\sum_i x_i \right)^2 \quad \text{s.t } \{x_i = \{-1, +1\}\}.$$

$$x^T A x - \lambda \left(\sum_i x_i \right)^2 = x^T A x - \lambda x^T 1 1^T x$$
$$= Tr(A x x^T) - \lambda Tr(1 1^T x x^T).$$
$$= -\langle X, \hat{A} \rangle.$$

Show:

$$\underset{X}{\text{argmin}} \langle X, \hat{A} \rangle. ?? \quad X \succeq 0 \quad \text{rank}(X) \leq 1, \ X_{ii} = 1$$

$\Uparrow$ at most one eigenvalue positive

our problem is now to:

$$\underset{x}{\text{argmax}} - \langle X, \tilde{A} \rangle \equiv \min \langle X, \hat{A} \rangle$$

$X = x x^T$, $\tilde{A} = \lambda 1 1^T - A$. $\quad x x^T$. $\begin{pmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1} \begin{pmatrix} 1 & -1 & 1 & \cdots & 1 \end{pmatrix}_{1 \times n}$ $\Rightarrow x x^T = 1$ bcz $x_i * x_i$

The values of x are ... all $= 1$

$\Rightarrow$ at most one eigenvalue ≥ 0. $\Rightarrow$ rank$(X) \leq 1$.

can write our probl:

$$\underset{X}{\text{argmin}} \langle X, \hat{A} \rangle \quad \text{s.t: rank}(X) \leq 1; \ X \succeq 0, \ X_{ii} = 1.$$

3)

$$\min_{X,Z} \langle X, \hat{A} \rangle \quad \text{s.t: } X \succeq 0; \ Z_{ii} = 1; \ Z = X.$$

ADMM algorithm: let's see how the algo. works:

$$\min f_1(x_1) + f_2(x_2) + \cdots f_m(x_m) \quad \text{s.t: } A_1 x_1 + A_2 x_2 + \cdots + A_m x_m = b.$$

one fixing variable. $\leftarrow$
$$\left\{ \begin{array}{l} L_\rho(x_1, \ldots, x_m, \lambda) = \sum f_i(x_i) + \lambda^T \left(\sum_{i=1}^{m} A_i x_i - b \right) + \rho \| \sum A_i x_i - b \|^2. \\ \Rightarrow x_i^{k+1} = \text{argmin}\ L_\rho(x_1^{k+1}, x_2^{k+1}, \ldots, x_i, x_{i+1}^k, \ldots, x_m^k, \lambda^k). \\ \lambda^{k+1} = \lambda^k + \rho \left(\sum A_i x_i^{k+1} - b \right). \end{array} \right.$$

Our opt. probl $\underset{X}{\text{argmin}} \langle X, \hat{A} \rangle$ s.t rank ≤ 1; $X \succeq 0$ - $X_{ii} = 1$.

we can write it as a summ. over all x.

$\min_X \sum_i (X_i, \hat{A})$. For $Z = X$, ADMM will select "i" from X with others fixed (in Z), and min:

SOLO

$$\min \mathcal{L} = \sum_i \langle X_i, \hat{A} \rangle + \langle \lambda, \sum_i X_i - Z_i \rangle + \rho \| \sum_i X_i - Z_i \|^2.$$

$$\Rightarrow \min_{X, Z} \langle X, \hat{A} \rangle \text{ s.t: } X \geq 0 \; ; \; Z_{ii} = 1, Z = X.$$

4) **ADMM Update Rules:**

$$X^{k+1} = \operatorname{argmin} \mathcal{L}(X, Z^k, \lambda^k)$$
$$Z^{k+1} = \operatorname{argmin} \mathcal{L}(X^{k+1}, Z, \lambda^k).$$
$$\lambda^{k+1} = \lambda^k + \rho\,(X^{k+1} - Z^{k+1}).$$

1.2: Min Max Problem:

$$\min_x \max_y x^T A y. \rightarrow \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} = Ay \\ \frac{\partial f}{\partial y} = x^T A \end{pmatrix}$$

Has many applications such as GANS --

Please Refer to the PDF file for this part. (MATLAB Code).

EG Method: (Extra Gradient):

Update Rule:

$$x^{k+1/2} = x^k - \alpha \nabla f_x(x^k, y^k)$$
$$y^{k+1/2} = y^k + \alpha \nabla f_{(y)}(x^k, y^k).$$

then:

$$x^{k+1} = x^k - \alpha \nabla f_x(x^{k+\frac{1}{2}}, y^{k+1/2})$$
$$y^{k+1} = y^k + \alpha \nabla f_y(x^{k+\frac{1}{2}}, y^{k+\frac{1}{2}}).$$

Gradient computed above.

OGDA (Optimistic Gradient descent ascent).

$$x^{k+1} = x^k - \alpha \nabla f_{(x)}(x^k, y^k) - \alpha\left(\nabla f_{(x)} f(x^{k-1}, y^{k-1})\right).$$
$$y^{k+1} = y^k + \alpha \nabla f_{(y)}(x^k, y^k) - \alpha\left(\nabla f_{(y)} f(x^{k-1}, y^{k-1})\right)$$

$$\Rightarrow x^{k+1} = x^k - 2\alpha A y^k + \alpha A y^{k-1}$$
$$y^{k+1} = y^k + 2\alpha A^T x^k - \alpha A^T x^{k-1}.$$

Check Matlab code.

PP method:

need to find x^{k+1} & y^{k+1} s.t: $(x^{k+1}, y^{k+1}) = \min_x \max_y \left(x^T A y + \frac{1}{2\alpha}\|x - x^k\|^2 - \frac{1}{2\alpha}\|y - y^k\|^2\right)$

$$x^{k+1} = x^k - \alpha \nabla_{(x)} f(x^{k+1}, y^{k+1}).$$

$$y^{k+1} = x^k - \alpha_{(y)} f(x^{k+1}, y^{k+1}).$$

$$\nabla_{(x)} f = Ay \rightarrow x^{k+1} = x - \alpha A y^{k+1}.$$

$$\nabla_{(y)} f = x^T A \rightarrow x^{k+1} = x - \alpha x^{k+1^T} A$$

$$\Rightarrow x^{k+1} = x^k - \alpha A (y^k - \alpha x^{k+1^T}) A$$

$$= x^k - \alpha A y^k - \alpha A \alpha x^{k+1^T} A$$

$$= x^k - \alpha A y^k - \alpha^2 A x^{k+1^T} A.$$

$$\Rightarrow x^{k+1} + \alpha^2 A x^{k+1^T} A = x^k - \alpha A y^k$$

$$x^{k+1}(I + \alpha^2 A A^T) = x^k - \alpha A y^k$$

$$x^{k+1} = (x^k - \alpha A y^k)(I + \alpha^2 A A^T)^{-1}.$$

check MATLAB code.

For other qsts, please check the MATLAB codes with their interpretat° on PDF.