ENMG 616 Advanced Optimization & Techniques

Homework Assignment

Due Date: November 15, 2023

You need to show the steps followed to get the final answer (Do not just give the final result). The homework should be submitted to Moodle as ***one*** *pdf file. Please insert a copy of your Matlab code in the submitted file.*

1 Solving Quadratic Constrained Optimization

Consider the following quadratic optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} \frac{1}{2}\mathbf{x}^T\mathbf{Q}\,\mathbf{x} + \mathbf{q}^T\mathbf{x} \qquad \text{s.t.} \quad 1 \le \mathbf{x} \le 3$$

1. Randomly generate $\mathbf{Q}$ and $\mathbf{q}$ according to the following procedure:
   (a) Set $n = 100$;
   (b) Generate a random matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$ where each element is drawn from standard normal distribution. Matlab Function: $\mathbf{B} = \text{randn}(n, n)$.
   (c) Let $\mathbf{Q} = \mathbf{B} * \mathbf{B}'$.
   (d) For $\mathbf{Q}$ to be a PSD matrix, add its smallest eigenvalue plus 10. Matlab Function: $\mathbf{Q} = \mathbf{Q} + (\text{MinEig} + 10)\text{*eye}(n)$.
   (e) Generate a random vector $\mathbf{q}$ where each element is drawn from normal distribution $\mathcal{N}(0, 100)$. Matlab Function: $\mathbf{q} = 10 * \text{randn}(n, 1)$.
2. Compute the Lipschitz constant L of the objective function.
3. Implement the gradient projection method with constant step-size $\alpha_r = \dfrac{1}{L}$.
4. Implement the gradient projection method with diminishing step-size $\alpha_r = \dfrac{5\ln(r)}{rL}$.
5. Implement Frank-Wolfe method.
6. Run the above 3 algorithms from the point $\mathbf{x}^0 = 1$ for 2000 iterations, and plot the following error
$$\mathcal{E}(\mathbf{x}^r) = \left\|\mathbf{x}^r - [\mathbf{x}^r - \nabla f(\mathbf{x}^r)]_+\right\|_2^2$$
versus iteration number r.

1.1 Dual Problem

1. Write down the Lagrangian function (Hint you can split the constraints into $\mathbf{x} \leq 3$ and $\mathbf{x} \geq 1$).

2. Find the dual problem.

3. Write the K.K.T conditions.

4. Implement and run the gradient projection method with constant step-size for the primal and dual problem to reach convergence.

5. Conclude that strong duality holds?

1.2 Duality

Repeat the above experiment (only running the codes) when Q is generated as follows: $\mathbf{B} = \text{randn}(n, n)$; $\mathbf{Q} = \frac{1}{2}(\mathbf{B} + \mathbf{B}^T)$. Comment on the results.

2 Projection to a set of linear equality constraints

Consider the following optimization problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} \frac{1}{2}\|\mathbf{z} - \mathbf{x}\|_2^2 \qquad \text{s.t. } \mathbf{A}\mathbf{x} = \mathbf{b}.$$

This problem aims at finding the projection of $\mathbf{z}$ to the set $\{\mathbf{x} \,|\, \mathbf{A}\mathbf{x} = \mathbf{b}\}$. Assuming that $\mathbf{A}\mathbf{A}^T$ is invertible.

1. Write down the Lagrangian function.

2. Find the dual problem.

3. Write down the KKT conditions.

4. Using KKT conditions express the optimal solution $\mathbf{x}^*$ as a function of $\mathbf{z}$, $\mathbf{A}$, and $\mathbf{b}$.