ENMG 616 Advanced Optimization & Techniques

Homework Assignment 1

Due Date: September 24, 2023

*You need to show the steps followed to get the final answer (Do not just give the final result). The homework should be submitted to Moodle as **one** pdf file.*

1. Show that the 2-dimensional function $f(x, y) = (x^2 - 4)^2 + y^2$ has two global minima and one stationary point, which is neither a local maximum nor a local minimum.

2. For each value of the scalar β, find the set of all stationary points of the following function of the two variables x and y

$$f(x, y) = x^2 + y^2 + \beta x\, y + x + 2y.$$

   Which of these stationary points are global minima.

3. Use optimality conditions to show that for all $x > 0$ we have

$$\frac{1}{x} + x \geq 2.$$

4. Consider the following optimization problem

$$\begin{aligned} &\min_{\mathbf{x}} \; e^{x_1} + e^{x_2} + \ldots + e^{x_n} \\ &\text{s.t.} \quad x_1 + x_2 + \ldots + x_n = s \end{aligned} \tag{1}$$

   - Show that the above optimization problem is convex.
   - Rewrite problem (1) as an unconstrained optimization problem by applying the following change of variable $x_n = s - x_1 - x_2 - \ldots - x_{n-1}$.
   - Find the set of global optimal solutions of the unconstrained optimization problem.
   - Apply a change of variable $y_i = e^{x_i}$ to prove the Arithmetic-Geometric Mean Inequality

$$\frac{1}{n}\sum_{i=1}^{n} y_i \geq (y_1 y_2 \ldots y_n)^{1/n} \qquad \forall\, (y_1, \ldots, y_n) \in \mathbb{R}_+^n.$$

5. Determine whether the following functions are convex, concave or neither both:

   - $f(x,y) = \sqrt{x\,y}$

   - $f(x,y,z) = \sqrt{x\,y\,z}$

   - $f(\mathbf{x}) = \log\left(e^{\mathbf{a}_1\,\mathbf{x}+b_1} + e^{\mathbf{a}_2\,\mathbf{x}+b_2} + \ldots + e^{\mathbf{a}_m\,\mathbf{x}+b_m}\right)$

6. Consider the following optimization problem

$$\min_{x} f(x) \triangleq \frac{1}{2}\left(x^2 - 1\right)^2.$$

   - Is the optimization problem convex?
   - Find the set of stationary points, i.e. points $\bar{x}$ for which $\nabla f(\bar{x}) = 0$.
   - Is every local optimum of the problem global?

7. Consider the following optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}) \triangleq \frac{1}{2}\left\|\mathbf{x}\mathbf{x}^T - \mathbf{A}\right\|_F^2,$$

   where $\mathbf{A} = \mathbf{z}\mathbf{z}^T \neq 0$ is a constant rank 1 matrix that belongs to $\mathbb{R}^{n\times n}$ with $\mathbf{z}\in\mathbb{R}^n$.

   - Is the optimization problem convex?
   - Find the set of stationary points, i.e. points $\bar{\mathbf{x}}$ for which $\nabla f(\bar{\mathbf{x}}) = \mathbf{0}$.
   - Is every local optimum of the problem global?