Monte Carlo Simulation for Option Pricing

1. Random Variates Generation

Monte Carlo simulations rely on generating paths using random variates from a standard normal distribution:

$$Z \sim \mathcal{N}(0, 1)$$

The asset price follows the geometric Brownian motion (GBM):

$$S_{t+1} = S_t \exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t} Z_t \right)$$

To reduce variance, **antithetic variates** are used. For each Z, we also simulate $-Z$.

2. European Call Option

The payoff of a European call option is:

$$\max(S_T - K, 0)$$

The Monte Carlo estimator is:

$$C = e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^{N} \max(S_T^{(i)} - K, 0)$$

Only the terminal price S_T is used. With antithetic variates:

$$S_T^{(i)} = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z_i \right), \quad Z_i \sim \mathcal{N}(0, 1)$$

3. Asian Call Option

Asian options depend on the average of the asset price over time. The arithmetic average Asian call has payoff:

$$\max\left(\frac{1}{M}\sum_{j=1}^{M} S_{t_j} - K, 0\right)$$

Simulate M steps per path and compute the average:

$$C = e^{-rT} \cdot \frac{1}{N}\sum_{i=1}^{N} \max\left(\bar{S}^{(i)} - K, 0\right)$$

where $\bar{S}^{(i)}$ is the average price for path i.

4. Bermudan Call Option

Bermudan options can be exercised at discrete time points. The price is computed by estimating the **continuation value**:

Least Squares Monte Carlo (LSM)

At each decision time t_k:

- For in-the-money paths, regress future discounted cash flows on S_{t_k}
- Compare immediate payoff $\max(S_{t_k} - K, 0)$ to the estimated continuation value

Let $C(S_{t_k})$ be the continuation value. Then the value at t_k is:

$$V_{t_k} = \begin{cases} \max(S_{t_k} - K, 0), & \text{if } \max(S_{t_k} - K, 0) > C(S_{t_k}) \\ \text{discounted value}, & \text{otherwise} \end{cases}$$

Nested Monte Carlo

At each decision point:

- Simulate multiple inner paths from S_{t_k}
- Estimate expected discounted future payoff (continuation value)
- Choose to exercise or continue

This is more accurate but computationally expensive.

5. Variance Reduction: Antithetic Variates

To reduce simulation variance:

- For every standard normal random draw Z, also simulate $-Z$
- Use the average of both paths in the Monte Carlo estimate

This technique improves estimator stability:

$$C = \frac{1}{2N} \sum_{i=1}^{N} [f(Z_i) + f(-Z_i)]$$

where f is the discounted payoff function.