304 Project 1 Final Report
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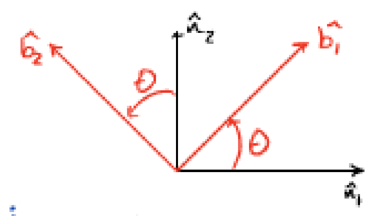
Summary:

This report investigates the circular restricted three body problem (CRTBP) using analytical and numerical methods. Using the conventions and graphics provided in the project prompt, Eq (2)-(8) were proven. Then using analytical methods along with the attached MATLAB script, the distances of points L4 and L5 in Earth-Moon, Earth-Sun, and Saturn-Titan systems were calculated. Lastly, eq(4) and (5) were linearized about equilibrium points, and both non-linearized and linearized equations of motion were integrated using the ode45 MATLAB function for the Earth-Moon system to plot the trajectories and departure motions.

Part A:

a)

$$r_1 = -\frac{m_2 R}{m_1+m_2} \; ; \; r_2 = \frac{m_1 R}{m_1+m_2}$$



$$V = -G\left(\frac{m_3 m_1}{\rho_1} + \frac{m_3 m_2}{\rho_2}\right)$$

$$\vec{r}_3 = x\hat{b}_1 + y\hat{b}_2 \; ; \; \dot{x}\hat{b}_1 + x\dot{\hat{b}}_1 + \dot{y}\hat{b}_2 + y\dot{\hat{b}}_2$$
$$= (\dot{x} - y\omega)\hat{b}_1 + (\dot{y} + x\omega)\hat{b}_2$$

$$\dot{\hat{b}}_n = \vec{\omega}_{B/N} \times \hat{b}_n$$
$$= \omega\hat{b}_3 \times \hat{b}_n$$
$$\therefore \; \dot{\hat{b}}_1 = \omega\hat{b}_2$$
$$\dot{\hat{b}}_2 = -\omega\hat{b}_1$$

$$T = \tfrac{1}{2} m_3(\vec{v}\cdot\vec{v})$$

$$\vec{v}\cdot\vec{v} = (\dot{x} - y\omega)^2 + (\dot{y} + x\omega)^2 = \dot{x}^2 - 2\dot{x}y\omega + (y\omega)^2$$
$$+ \dot{y}^2 + 2\dot{y}x\omega + (x\omega)^2$$
$$= \dot{x}^2 + \dot{y}^2 + \omega^2(x^2+y^2) + 2\omega(\dot{y}x - \dot{x}y)$$

$$\therefore \; T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \omega^2(x^2+y^2) + 2\omega(\dot{y}x - \dot{x}y))$$ *m_3 divides out

$$\therefore \; L = T - V = \tfrac{1}{2} m_3(\dot{x}^2 + \dot{y}^2 + \omega^2(x^2 - \;^2 + 2\omega(\dot{y}x - \dot{x}y))$$
$$+ G\left(\frac{m_3 m_1}{\rho_1} + \frac{m_3 m_2}{\rho_2}\right)$$

$$\left(\frac{\partial L}{\partial \dot{x}}\right) = \dot{x} - \omega y \rightarrow \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \ddot{x} - \omega\dot{y}$$

$$\frac{\partial L}{\partial x} = \omega^2 x + \omega\dot{y} \; \rightarrow \; \therefore \; \ddot{x} - 2\omega\dot{y} - \omega^2 x = V$$

$$\left(\frac{\partial L}{\partial \dot{y}}\right) = \dot{y} + \omega x \rightarrow \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) = \ddot{y} + \omega \dot{x}$$

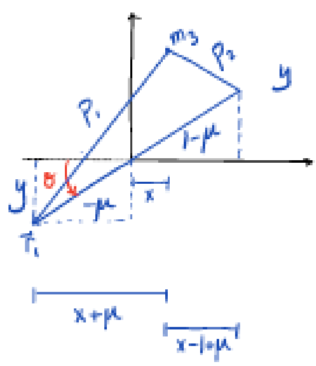
$$\frac{\partial L}{\partial y} = \omega^2 y + \omega \dot{x} \longrightarrow \therefore\ \ddot{y} + 2\omega\dot{x} - \omega^2 y = \nabla V$$

$$V = -\frac{G\cancel{m_3}m_1}{\rho_1} - \frac{G\cancel{m_3}m_2}{\rho_2} = \frac{-\omega^2(1-\mu)(x+\mu)}{\rho_1^3} - \frac{\omega^2\mu(x-1+\mu)}{\rho_2^3}$$

divides out

(in x dir.)

$$\omega^2 = \frac{G(m_1+m_2)}{R^3}, \quad \therefore\ G = \frac{\omega^2 \cancel{R^3}\rightarrow 1}{m_1+m_2}$$



$$\therefore\ \frac{-\omega^2 m_1}{\rho_1(m_1+m_2)} - \frac{\omega^2 m_2}{\rho_2(m_1+m_2)}$$

$$\frac{-\omega^2 r_2}{\rho_1} - \frac{\omega^2 r_1}{\rho_2}$$

$$\frac{-\omega^2 r_2 (\rho_1)^2}{\rho_1^3} - \frac{\omega^2 r_1 (\rho_2)^2}{\rho_2^3}$$

by geometry, $\rho_1^2 = (x+\mu)^2 + (y)^2$
$\rho_2^2 = (x+\mu-1)^2 + y^2$

$$\therefore\ \ddot{x} - 2\omega\dot{y} - \omega^2 x = -\frac{\omega^2(1-\mu)(x+\mu)}{\rho_1^3} - \frac{\omega^2\mu(x-1+\mu)}{\rho_2^3}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = \frac{-\omega^2(1-\mu)y}{\rho_1^3} - \frac{\omega^2\mu y}{\rho_2^3}$$

$$\frac{dx}{dt}\left(\frac{dt}{d\tau}\right)=\frac{dx}{d\tau}\,; \quad \frac{dt}{d\tau}=\frac{1}{\omega} \quad \therefore$$

$$\therefore \quad \dot{x}=\omega x' \qquad \ddot{x}=\omega^2 x''$$
$$\dot{y}=\omega y' \qquad \ddot{y}=\omega^2 y''$$

$$\omega^2 x''-2\omega(\omega y')-\omega^2 x=-\frac{\omega^2(1-\mu)(x+\mu)}{P_1^3}-\frac{\omega^2\mu(x-1+\mu)}{P_2^3}$$

$$\omega^2 y''+2\omega(\omega x')-\omega^2 y=\frac{-\omega^2(1-\mu)y}{P_1^3}-\frac{\omega^2\mu y}{P_2^3}$$

$$x''-2y'-x=\frac{-(1-\mu)(x+\mu)}{P_1^3}-\frac{\mu(x-1+\mu)}{P_2^3}$$

$$y''+2x'-y=-\frac{(1-\mu)y}{P_1^3}-\frac{\mu y}{P_2^3}$$

$$\frac{\partial U}{\partial y}=-\frac{(1-\mu)y}{P_1^3}-\frac{\mu y}{P_2^3}+y$$

where $U=\frac{1}{2}(x^2+y^2)+\frac{1-\mu}{P_1}+\frac{\mu}{P_2}\,;\quad P_1^2=(x+\mu)^2+(y)^2$

$P_2^2=(x+\mu-1)^2+y^2$

$$=\frac{1}{2}(x^2+y^2)+\frac{1-\mu}{\sqrt{(x+\mu)^2+y^2}}+\frac{\mu}{\sqrt{(x+\mu-1)^2+y^2}}$$

$$\frac{\partial U}{\partial y}=y-\frac{y(1-\mu)}{((x+\mu)^2+y^2)^{3/2}}-\frac{\mu y}{((x+\mu-1)^2+y^2)^{3/2}}=y-\frac{y(1-\mu)}{P_1^3}-\frac{\mu y}{P_2^3}$$

repeating process in x:

$$\frac{\partial U}{\partial x}=x-\frac{(1-\mu)(x+\mu)}{P_1^3}-\frac{\mu(x-1+\mu)}{P_2^3}$$

where $U=\frac{1}{2}(x^2+y^2)+\frac{1-\mu}{P_1}+\frac{\mu}{P_2}\,;\quad P_1^2=(x+\mu)^2+(y)^2$

$P_2^2=(x+\mu-1)^2+y^2$

$$=\frac{1}{2}(x^2+y^2)+\frac{1-\mu}{\sqrt{(x+\mu)^2+y^2}}+\frac{\mu}{\sqrt{(x+\mu-1)^2+y^2}}$$

$$\frac{\partial U}{\partial x}=x-\frac{(1-\mu)(x+\mu)}{((x+\mu)^2+y^2)^{3/2}}-\frac{\mu(x-1+\mu)}{((x+\mu-1)^2+y^2)^{3/2}}$$

$$=x-\frac{(1-\mu)(x+\mu)}{(P_1)^3}-\frac{\mu(x-1+\mu)}{(P_2)^3}$$

Part B:

b)

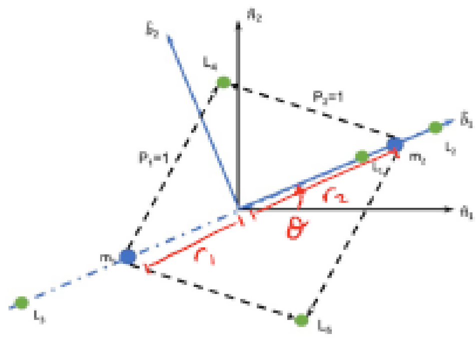
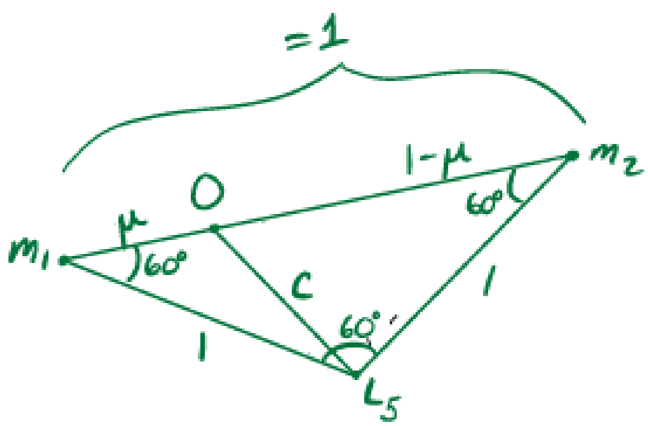


Figure 2: A Schematic of Lagrange Points.



Using law of cosines: $c=(a^2+b^2-2ab\cos(\gamma))$

$$= 1^2+\mu^2-2\mu\cos 60^\circ$$
$$= 1+\mu^2-\mu = \mu^2-\mu+1$$

Using data from table and attached MATLAB Script,

System	μ	L_4	L_5
Sun-Earth	3.0039×10^{-7}	0.999999	0.999999
Earth-Moon	1.2151×10^{-2}	0.987997	0.987997
Saturn-Titan	2.366×10^{-4}	0.999763	0.999763

Part C:

c)
$$\begin{bmatrix} \delta x' \\ \delta y' \\ \delta x'' \\ \delta y'' \end{bmatrix} = \begin{array}{c c} \begin{array}{c} \delta x' \\ \delta y' \\ \delta x' \\ \delta y'' \end{array} & \begin{bmatrix} \varnothing & \varnothing & 1 & \varnothing \\ \varnothing & \varnothing & \varnothing & 1 \\ U_{xx} & U_{xy} & \varnothing & 2 \\ U_{xy} & U_{yy} & -2 & \varnothing \end{bmatrix} \\ & \begin{array}{cccc} \delta x & \delta y & \delta x' & \delta y' \end{array} \end{array} \begin{bmatrix} \delta x \\ \delta y \\ \delta x' \\ \delta y' \end{bmatrix}$$

$\delta x' = \delta x'$ ✓
$\delta y' = \delta y'$ ✓
$\delta x'' = U_{xx}\delta x + U_{xy}\delta y + 2\delta y'$
$\delta y'' = U_{xy}\delta x + U_{yy}\delta y - 2\delta x'$

Since $U_x = f(x,y)$ the linearization of

$$\delta x'' = \left.\frac{\partial U_x}{\partial x}\right|_{x_0} + \left.\frac{\partial U_x}{\partial y}\right|_{y_0} + 2\delta y' = U_{xx}\delta x + U_{xy}\delta y + 2\delta y'$$ ✓

$$\delta y'' = \left.\frac{\partial U_y}{\partial x}\right|_{x_0} + \left.\frac{\partial U_y}{\partial y}\right|_{y_0} - 2\delta x' = U_{xy}\delta x + U_{yy}\delta y - 2\delta x'$$ ✓

Solving for U_{xx}, U_{yy}, and U_{xy} to be used in matrix A:

$$U_{xx} = \frac{-(1-\mu)}{((x+\mu)^2+y^2)^{3/2}} + \frac{3(1-\mu)(x-\mu)^2}{((x+\mu)^2+y^2)^{5/2}} - \frac{\mu}{((x+\mu-1)^2+y^2)^{3/2}} + \frac{3\mu(x+\mu-1)^2}{((x+\mu-1)^2+y^2)^{5/2}} + 1$$

$$U_{yy} = \frac{-(1-\mu)^2}{((x+\mu)^2+y^2)^{3/2}} + \frac{3(1-\mu)y^2}{((x+\mu)^2+y^2)^{5/2}} - \frac{\mu}{((x+\mu-1)^2+y^2)^{3/2}} + \frac{3\mu y^2}{((x+\mu-1)^2+y^2)^{5/2}} + 1$$

$$U_{xy} = \frac{-3y(\mu-1)(x+\mu)}{((x+\mu)^2+y^2)^{5/2}} + \frac{3y\mu(x+\mu-1)}{((x+\mu-1)^2+y^2)^{5/2}} = 0$$

Results for part C:

Earth Sun System Eigenvectors (points L1 thru L5)

L1	L2	L3	L4	L5
0.2034 + 0.0000i	-0.0000 + 0.2053i	0.0000 + 1.0000i	0.0000 - 0.0014i	0.0000 - 1.0000i
0.0000 + 1.0136i	-0.0000 - 0.2053i	0.0000 - 1.0000i	0.0000 + 0.0014i	0.0000 + 1.0000i
0.0000 - 1.0136i	-0.0000 - 0.9857i	-0.0000 + 0.0093i	-0.0000 + 1.0000i	-0.0000 + 0.0014i
-0.2034 + 0.0000i	-0.0000 + 0.9857i	-0.0000 - 0.0093i	-0.0000 - 1.0000i	-0.0000 - 0.0014i

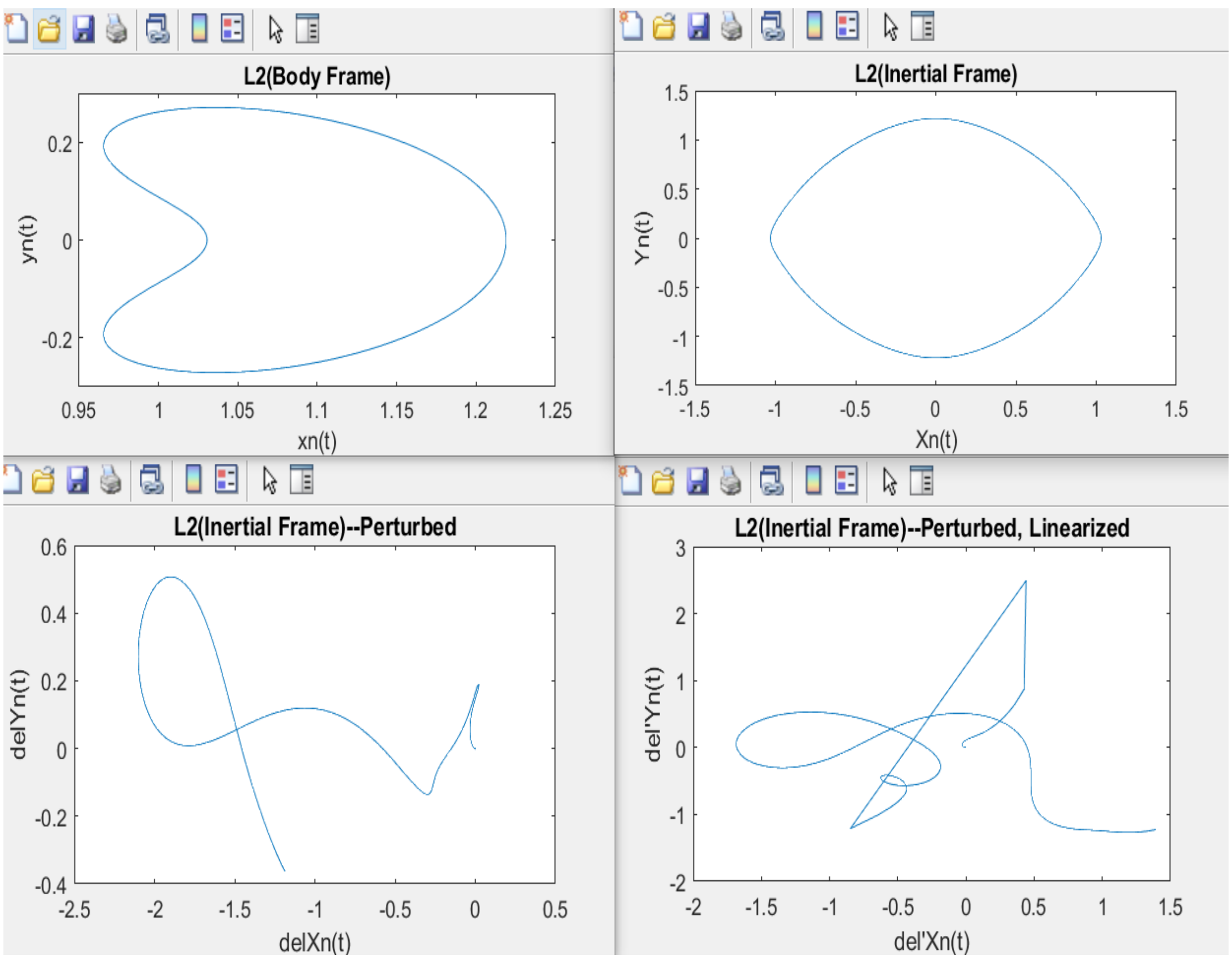
Earth Moon System Eigenvectors (points L1 thru L5)

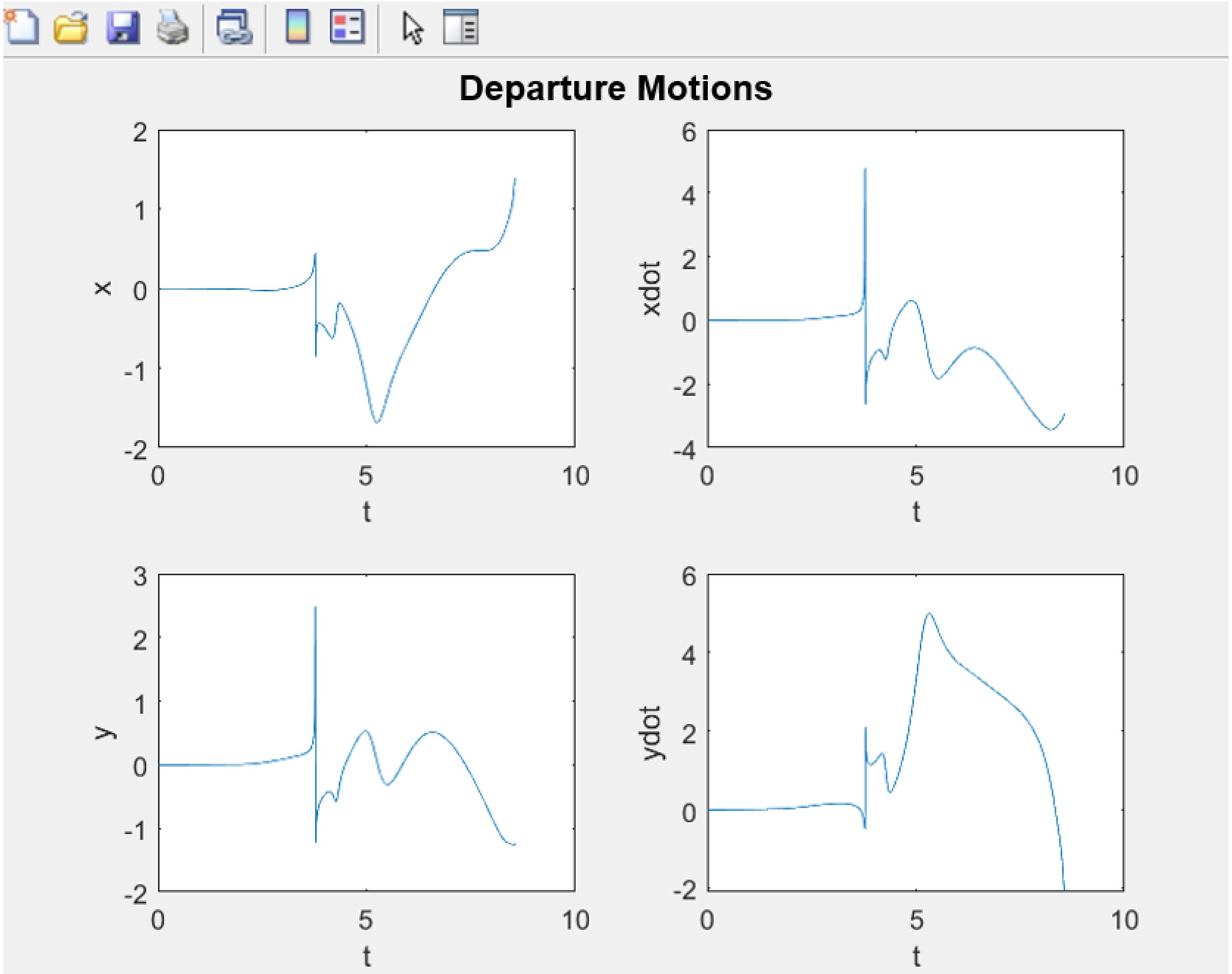
L1	L2	L3	L4	L5
1.1979 + 0.0000i	0.3025 + 0.8692i	0.2694 + 0.0000i	0.0000 + 0.3867i	0.0000 + 0.9324i
0.0000 + 1.3635i	0.3025 - 0.8692i	-0.0000 + 0.9686i	0.0000 - 0.3867i	0.0000 - 0.9324i
0.0000 - 1.3635i	-0.3025 - 0.8692i	-0.0000 - 0.9686i	0.0000 - 0.9324i	-0.0000 + 0.3867i
-1.1979 + 0.0000i	-0.3025 + 0.8692i	-0.2694 + 0.0000i	-0.0000 + 0.9324i	-0.0000 - 0.3867i

Saturn Titan System Eigenvectors (points L1 thru L5)

L1	L2	L3	L4	L5
0.6068 + 0.0000i	0.0765 + 0.7518i	0.0351 + 0.0000i	0.0000 + 0.9986i	0.0000 + 0.0516i
-0.0000 + 1.1094i	0.0765 - 0.7518i	0.0000 + 0.9993i	0.0000 - 0.9986i	0.0000 - 0.0516i
-0.0000 - 1.1094i	-0.0765 + 0.7518i	-0.0000 - 0.9993i	-0.0000 + 0.0516i	-0.0000 - 0.9988i
-0.6068 + 0.0000i	-0.0765 - 0.7518i	-0.0351 + 0.0000i	-0.0000 - 0.0516i	-0.0000 + 0.9988i

Part D Results:

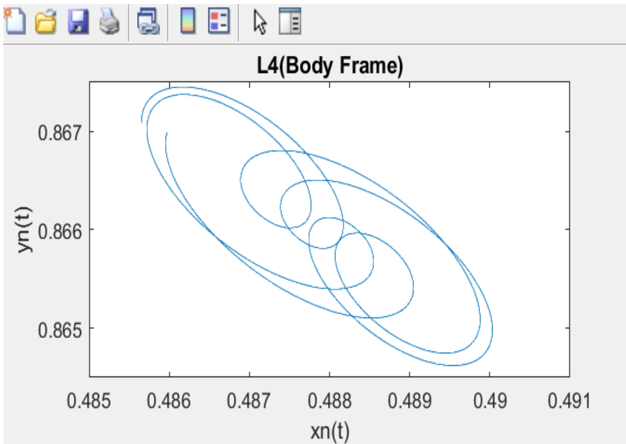


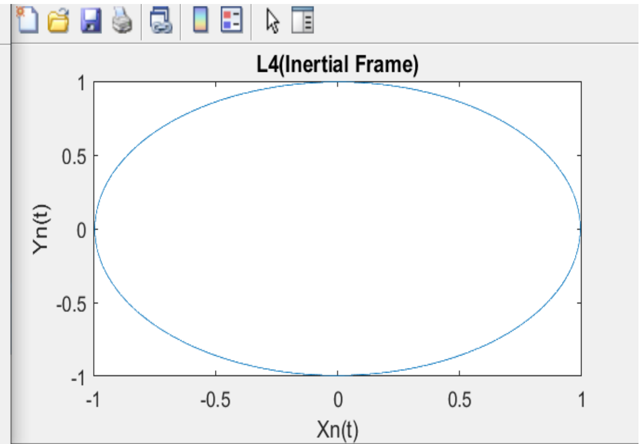


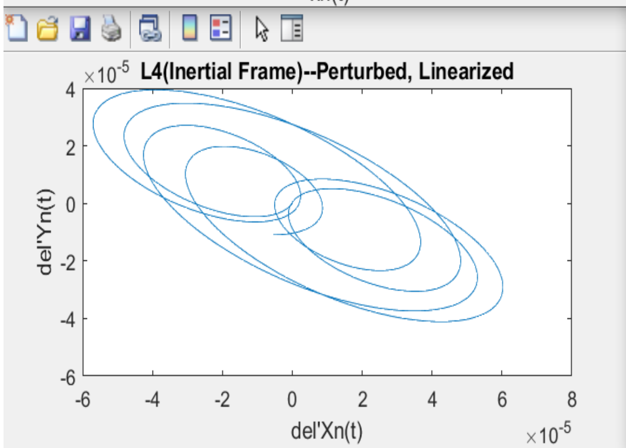
Part D Discussion:

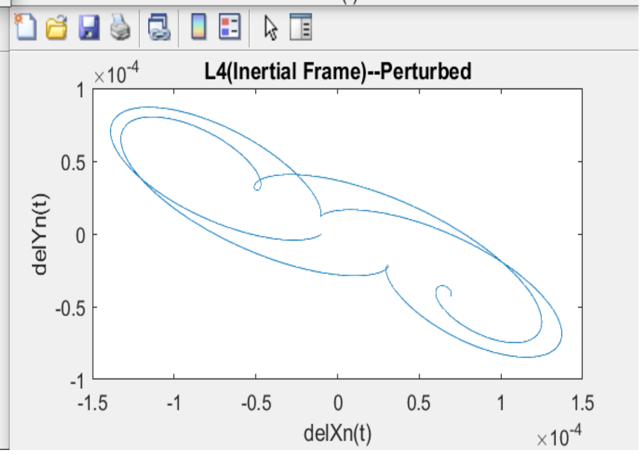
The behavior demonstrated by the simulation of the L2 Lyapunov orbit in an Earth-Moon system programmed in *project1partsbc.m* is consistent with traditional understanding of Euler point behavior in two main regards. Firstly, the shape of the orbit plotted in the state space (x vs y) is closed and appropriate for the equations of motion. Secondly, the system is unstable— demonstrated by its inability to return to the equilibrium state after a small perturbation—as described in the project prompt.

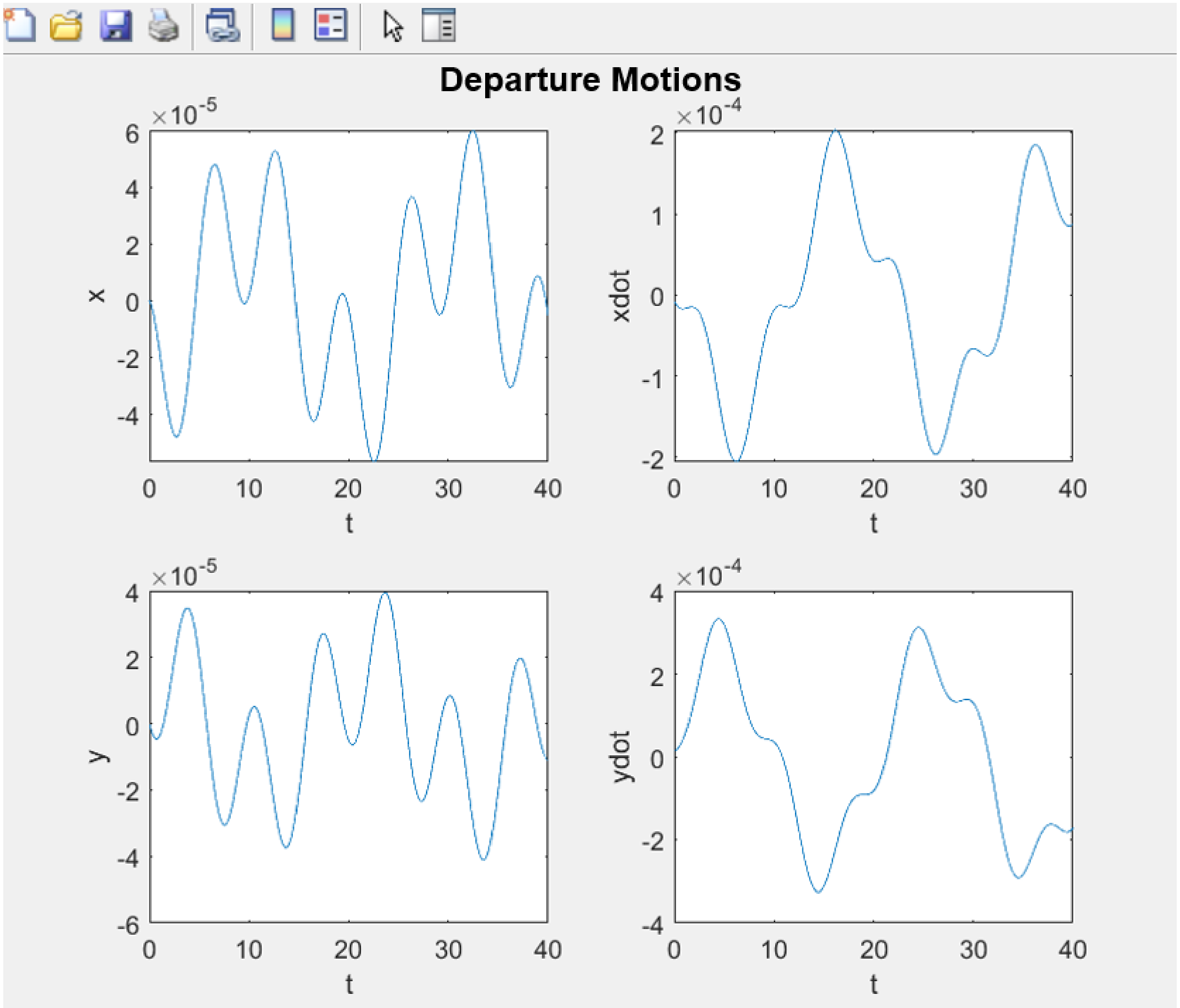
Part E Results:


L4(Body Frame)
yn(t)
0.867
0.866
0.865
0.485 0.486 0.487 0.488 0.489 0.49 0.491
xn(t)


L4(Inertial Frame)
Yn(t)
1
0.5
0
-0.5
-1
-1 -0.5 0 0.5 1
Xn(t)


L4(Inertial Frame)--Perturbed, Linearized
$\times 10^{-5}$
del'Yn(t)
4
2
0
-2
-4
-6
-6 -4 -2 0 2 4 6 8
del'Xn(t)
$\times 10^{-5}$


L4(Inertial Frame)--Perturbed
$\times 10^{-4}$
delYn(t)
1
0.5
0
-0.5
-1
-1.5 -1 -0.5 0 0.5 1 1.5
delXn(t)
$\times 10^{-4}$



Part E Discussion:

Similarly to results from Part D, the behavior demonstrated by the simulation of the L4 Lyapunov orbit (written in the same MATLAB script) are as expected. Firstly, when plotted in the inertial frame, a perfect closed circle is shown. This makes sense given the definition of L4 and L5— they are the points which form equilateral triangles between the two large masses of the CRTBP (in this case, the Earth and the Moon). Furthermore, these equilibria are stable, as demonstrated by the system's ability to resist perturbations.

Work Distribution

Name :	Psu Email:	Percentage:	Description:
Ralph Quartiano	rjq5038	60%	Computation, Coding, Debugging
Michael Da Rocha	mad6437	40%	Computation, Debugging, Reporting

Honor Pledge:

All members have neither given nor received assistance on this project.

- Michael Da Rocha, Ralph Quartiano