PHYS 662 - Quantum Field Theory I

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Homework 2

Problem 1 - Action Principle for a Relativistic Particle
Consider a free particle moving along $x^\mu(t)$ in Minkowski spacetime

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu.$$

Take as the action

$$S = -m\int \mathrm{d}s = -m\int \mathrm{d}t\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right).$$

Vary it and find the equations of motion.

Solution. We start by dividing by $(\mathrm{d}t)^2$ on both sides to get a time derivative of the infinitesimal path, and taking $\eta_{\mu,\nu}$ to be the Minkowski metric, we have

$$\begin{aligned}
\frac{(\mathrm{d}s)^2}{(\mathrm{d}t)^2} &= \frac{1}{(\mathrm{d}t)^2}\left(\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right)\\
\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= \frac{1}{(\mathrm{d}t)^2}\left[(\mathrm{d}t)^2-\left((\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2\right)\right]\\
\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= 1-\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\\
\frac{\mathrm{d}s}{\mathrm{d}t} &= \sqrt{1-\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2-\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\\
\dot{s} &= \sqrt{1-(\dot{x}+\dot{y}+\dot{z})}\\
\dot{s} &= \sqrt{\dot{x}_\mu\dot{x}^\mu} = \sqrt{\partial_0 x_\mu\partial_0 x^\mu}.
\end{aligned}$$

Our action becomes

$$S = -m\int_{t_i}^{t_f}\sqrt{\dot{x}_\mu\dot{x}^\mu}\,\mathrm{d}t,$$

where $\mathcal{L}(x^\mu,\dot{x}^\mu,t) = -m\sqrt{\dot{x}_\mu\dot{x}^\mu}$ is the Lagrangian. The Euler-Lagrange equations are given by

$$\frac{\partial\mathcal{L}}{\partial x^\mu}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = 0.$$

- **Varying** x^μ:
$$\frac{\partial\mathcal{L}}{\partial x^\mu} = 0.$$
- **Varying** $\dot{x}^\mu$:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = -m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\dot{x}^\mu}{2\sqrt{\dot{x}_\mu\dot{x}^\mu}}\right) = -m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}_\mu\dot{x}^\mu}}\right).$$

Then, plugging these into the Euler-Lagrange equation, we have

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}_\mu\dot{x}^\mu}}\right) = 0 \qquad\Longrightarrow\qquad \dot{x}^\mu = \text{constant}.$$

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Problem 2 - Coulomb Gauge
Consider classical Maxwell fields.

(a) Expand the Lagrangian density in the presence of a current in terms of $A^\mu = (\varphi, \vec{A})$, where φ is the electrostatic potential and $\vec{A}$ is the vector potential satisfying the Coulomb gauge $\vec{\nabla}\cdot\vec{A}(x) = 0$.

(b) Vary the action with respect to φ and $\vec{A}$ to find the equations of motion.

(c) Find the solution to the equations of motion using Fourier transform.

Solution. (a) Considering classical Maxwell fields, the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu.$$

The field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

By replacing that in the Lagrangian, we get

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + J^\mu A_\mu \\
&= -\frac{1}{4}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu\right] + J^\mu A_\mu \\
&= -\frac{1}{2}\left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right] + J^\mu A_\mu \\
&= -\frac{1}{2}\left[\left(\partial_t A_\nu\right)\left(\partial_t A^\nu\right) - \left(\partial_x A_\nu\right)\left(\partial_x A^\nu\right) - \left(\partial_y A_\nu\right)\left(\partial_y A^\nu\right) - \left(\partial_z A_\nu\right)\left(\partial_z A^\nu\right)\right. \\
&\quad \left. - \left(\partial_t A_\mu\right)\left(\partial^\mu A_t\right) + \left(\partial_x A_\mu\right)\left(\partial^\mu A_x\right) + \left(\partial_y A_\mu\right)\left(\partial^\mu A_y\right) + \left(\partial_z A_\mu\right)\left(\partial^\mu A_z\right)\right] - J^\mu A_\mu \\
&= -\frac{1}{2}\left[\cancel{\left(\partial_t A_t\right)\left(\partial_t A_t\right)} - \left(\partial_t A_x\right)\left(\partial_t A_x\right) - \left(\partial_t A_y\right)\left(\partial_t A_y\right) - \left(\partial_t A_z\right)\left(\partial_t A_z\right)\right. \\
&\quad - \left(\partial_x A_t\right)\left(\partial_x A_t\right) + \cancel{\left(\partial_x A_x\right)\left(\partial_x A_x\right)} + \left(\partial_x A_y\right)\left(\partial_x A_y\right) + \left(\partial_x A_z\right)\left(\partial_x A_z\right) \\
&\quad - \left(\partial_y A_t\right)\left(\partial_y A_t\right) + \left(\partial_y A_x\right)\left(\partial_y A_x\right) + \cancel{\left(\partial_y A_y\right)\left(\partial_y A_y\right)} + \left(\partial_y A_z\right)\left(\partial_y A_z\right) \\
&\quad - \left(\partial_z A_t\right)\left(\partial_z A_t\right) + \left(\partial_z A_x\right)\left(\partial_z A_x\right) + \left(\partial_z A_y\right)\left(\partial_z A_y\right) + \cancel{\left(\partial_z A_z\right)\left(\partial_z A_z\right)} \\
&\quad - \cancel{\left(\partial_t A_t\right)\left(\partial_t A_t\right)} + \left(\partial_t A_x\right)\left(\partial_x A_t\right) + \left(\partial_t A_y\right)\left(\partial_y A_t\right) + \left(\partial_t A_z\right)\left(\partial_z A_t\right) \\
&\quad + \left(\partial_x A_t\right)\left(\partial_t A_x\right) - \cancel{\left(\partial_x A_x\right)\left(\partial_x A_x\right)} - \left(\partial_x A_y\right)\left(\partial_y A_x\right) - \left(\partial_x A_z\right)\left(\partial_z A_x\right) \\
&\quad + \left(\partial_y A_t\right)\left(\partial_t A_y\right) - \left(\partial_y A_x\right)\left(\partial_x A_y\right) - \cancel{\left(\partial_y A_y\right)\left(\partial_y A_y\right)} - \left(\partial_y A_z\right)\left(\partial_z A_y\right) \\
&\quad \left. + \left(\partial_z A_t\right)\left(\partial_t A_z\right) - \left(\partial_z A_x\right)\left(\partial_x A_z\right) - \left(\partial_z A_y\right)\left(\partial_y A_z\right) - \cancel{\left(\partial_z A_z\right)\left(\partial_z A_z\right)}\right] - J^\mu A_M \\
&= -\frac{1}{2}\Big[- \left(\partial_t A_x - \partial_x A_t\right)^2 - \left(\partial_t A_y - \partial_y A_t\right)^2 - \left(\partial_t A_z - \partial_z A_t\right)^2 + \\
&\quad + \left(\partial_x A_y - \partial_y A_x\right)^2 + \left(\partial_x A_z - \partial_z A_x\right)^2 + \left(\partial_y A_z - \partial_z A_y\right)^2\Big] - J^\mu A_\mu \\
&= \frac{1}{2}\left[\left|\partial_t \mathbf{A} + \nabla\varphi\right|^2 - \left|\nabla\times\mathbf{A}\right|^2\right] - J^\mu A_\mu \\
&= \frac{1}{2}\left(\left|\partial_t \mathbf{A} + \nabla\varphi\right|^2 - \left|\nabla\times\mathbf{A}\right|^2\right) - \left(\rho\varphi - \mathbf{j}\cdot\mathbf{A}\right).
\end{aligned}$$

(b) The action of this system is given by

$$\begin{aligned}
S &= \int \mathcal{L}\, \mathrm{d}^4x \\
&= \int \left[\frac{1}{2}\left(\left|\partial_t \mathbf{A} + \nabla\varphi\right|^2 - \left|\nabla\times\mathbf{A}\right|^2\right) - \left(\rho\varphi - \mathbf{j}\cdot\mathbf{A}\right)\right] \mathrm{d}^4x.
\end{aligned}$$

Varying the action, we get

$$\begin{aligned}\delta S &= \int \left[\frac{1}{2}\left(2(\nabla\varphi)\cdot(\nabla(\delta\varphi)) + 2(\nabla\varphi)\cdot(\partial_t(\delta\mathbf{A})) + 2(\nabla(\delta\varphi))\cdot(\partial_t\mathbf{A})\right.\right.\\ &\quad \left.+2(\partial_t\mathbf{A})\cdot(\partial_t(\delta\mathbf{A})) - 2(\nabla\times\mathbf{A})\cdot(\nabla\times\delta\mathbf{A})\right) - \rho\delta\varphi + \mathbf{j}\cdot\delta\mathbf{A}\Big]\,\mathrm{d}^4x\\ &= \int\left[\left(-\nabla^2\varphi-\rho\right)\delta\varphi + \left(-\partial_t^2\mathbf{A} + \nabla^2\mathbf{A} - \partial_t(\nabla\varphi) + \mathbf{j}\right)\delta\mathbf{A}\right]\mathrm{d}^4x,\end{aligned}$$

which give us the following equations of motion

$$\begin{aligned}\nabla^2\varphi &= -\rho,\\ \left(-\partial_t^2+\nabla^2\right)\mathbf{A} &= \nabla(\partial_t\varphi) - \mathbf{j}.\end{aligned}$$

(c) Let $\nabla^2\varphi(\mathbf{x},t) = -\rho(\mathbf{x},t)$. Then

$$\begin{aligned}\nabla^2\int G(\mathbf{x},t,\mathbf{x}',t')\rho(\mathbf{x}',t')\,\mathrm{d}^3x'\,\mathrm{d}t' &= \nabla^2\int G(\mathbf{x}',\mathbf{x})\delta(t'-t)\rho(\mathbf{x}',t')\,\mathrm{d}^3x'\,\mathrm{d}t'\\ &= -\int\delta(\mathbf{x}'-\mathbf{x})\delta(t'-t)\rho(\mathbf{x}',t')\,\mathrm{d}^3x'\,\mathrm{d}t',\end{aligned}$$

where $G(\mathbf{x},t,\mathbf{x}',t')\rho(\mathbf{x}',t') = G(\mathbf{x}',\mathbf{x})\delta(t'-t)$ and $\nabla^2 G(\mathbf{x}',\mathbf{x}) = -\delta(\mathbf{x}'-\mathbf{x})$.
Consider the Fourier transform

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\nabla^2\int\tilde{G}(\mathbf{k})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\,\mathrm{d}^3k = -\frac{1}{(2\pi)^{\frac{3}{2}}}\int\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\,\mathrm{d}^3k,$$

then we can conclude that $\tilde{G}(\mathbf{k}) = \frac{1}{k^2}$.
Replacing, we have

$$G(\mathbf{x}',\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{1}{k^2}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\,\mathrm{d}^3k.$$

Let $\mathrm{d}^3k = k^2\sin(\theta)\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi$, which means $\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}) = k\,|\mathbf{x}'-\mathbf{x}|\cos(\theta)$. Thus,

$$\begin{aligned}G(\mathbf{x}',\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\lim_{\epsilon\to0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{k^2\sin(\theta)\mathrm{e}^{\mathrm{i}k\cdot|\mathbf{x}'-\mathbf{x}|\cos(\theta)}}{k^2+\epsilon^2}\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \frac{2\pi}{(2\pi)^{\frac{3}{2}}}\lim_{\epsilon\to0}\int_0^{\infty}\frac{k^2}{k^2+\epsilon^2}\int_0^{\pi}\sin(\theta)\mathrm{e}^{\mathrm{i}k\cdot|\mathbf{x}'-\mathbf{x}|\cos(\theta)}\,\mathrm{d}\theta\,\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\lim_{\epsilon\to0}\int_0^{\infty}\frac{k^2}{k^2+\epsilon^2}\left(\frac{\mathrm{e}^{\mathrm{i}k\cdot|\mathbf{x}'-\mathbf{x}|}-\mathrm{e}^{-\mathrm{i}k\cdot|\mathbf{x}'-\mathbf{x}|}}{\mathrm{i}k\,|\mathbf{x}'-\mathbf{x}|}\right)\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\mathrm{i}\,|\mathbf{x}'-\mathbf{x}|}\lim_{\epsilon\to0}\int_0^{\infty}\frac{k\left(\mathrm{e}^{\mathrm{i}k\cdot|\mathbf{x}'-\mathbf{x}|}-\mathrm{e}^{-\mathrm{i}k\cdot|\mathbf{x}'-\mathbf{x}|}\right)}{(k-\mathrm{i}\epsilon)(k+\mathrm{i}\epsilon)}\,\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{\mathrm{i}\,|\mathbf{x}'-\mathbf{x}|}\lim_{\epsilon\to0}\int_{-\infty}^{\infty}\frac{k\mathrm{e}^{\mathrm{i}k\cdot|\mathbf{x}'-\mathbf{x}|}}{(k-\mathrm{i}\epsilon)(k+\mathrm{i}\epsilon)}\,\mathrm{d}k\\ &= \frac{1}{|\mathbf{x}'-\mathbf{x}|}.\end{aligned}$$

Thus,

$$\varphi(\mathbf{x},t) = \int\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}'-\mathbf{x}|}\,\mathrm{d}^3x'.$$

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Problem 3 - Spontaneous Symmetry Breaking
Consider a complex relativistic scalar field with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \Phi^\dagger\right)\left(\partial^\mu \Phi\right) + \lambda\left(\Phi^\dagger \Phi - \phi_0\right)^2.$$

(a) Find the classical equations of motion.

(b) Consider constant field configurations and find the minima of action.

(c) Expand the field $\Phi(x^\mu)$ around this minimum to the second-order and find the mass of the excitations.

(d) Is there a massless mode?

Solution. (a) The classical equations of motion are given by the Euler-Lagrange equation for scalar fields given by

$$\frac{\partial \mathcal{L}}{\partial \Phi(x)} - D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi(x))} = 0.$$

Since we have two dynamical variables, Φ and $\Phi^\dagger$, we have to vary with respect to each of them to derive the equations of motion, *i.e.* we will have to take two Euler-Lagrange equations, one for Φ and another for $\Phi^\dagger$. When varying the first, the second must be considered as a constant, and vice versa.

- **For** Φ:
  - **First Term**:
  $$\frac{\partial \mathcal{L}}{\partial \Phi} = \lambda(2(\Phi^\dagger)^2\Phi - 2\phi_0\Phi^\dagger) = 2\lambda\Phi^\dagger(\Phi^\dagger\Phi - \phi_0).$$
  - **Second Term**:
  $$\begin{aligned} D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} &= -\frac{1}{2} D_\mu \left(\partial^\mu \Phi^\dagger\right) \\ &= -\frac{1}{2}\left(\partial_\mu \partial^\mu \Phi^\dagger + \delta^\mu_\nu \partial_\mu \partial^\nu \Phi^\dagger\right) \\ &= -\partial_\mu \partial^\mu \Phi^\dagger. \end{aligned}$$
  - **Euler-Lagrange**:
  $$\begin{gathered} \frac{\partial \mathcal{L}}{\partial \Phi} - D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0 \\ 2\lambda\Phi^\dagger(\Phi^\dagger\Phi - \phi_0) + \partial_\mu\partial^\mu\Phi^\dagger = 0. \end{gathered}$$
- **For** $\Phi^\dagger$:
  - **First Term**:
  $$\frac{\partial \mathcal{L}}{\partial \Phi^\dagger} = \lambda(2\Phi^\dagger\Phi^2 - 2\phi_0\Phi) = 2\lambda\Phi(\Phi^\dagger\Phi - \phi_0).$$
  - **Second Term**:
  $$\begin{aligned} D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\dagger)} &= -\frac{1}{2} D_\mu \left(\partial^\mu \Phi\right) \\ &= -\frac{1}{2}\left(\partial_\mu \partial^\mu \Phi + \delta^\mu_\nu \partial_\mu \partial^\nu \Phi\right) \\ &= -\partial_\mu \partial^\mu \Phi. \end{aligned}$$
  - **Euler-Lagrange**:
  $$\begin{gathered} \frac{\partial \mathcal{L}}{\partial \Phi^\dagger} - D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\dagger)} = 0 \\ 2\lambda\Phi(\Phi^\dagger\Phi - \phi_0) + \partial_\mu\partial^\mu\Phi = 0. \end{gathered}$$

Thus, the two equations of motions are given by

$$\begin{cases} 2\lambda\Phi^\dagger(\Phi^\dagger\Phi - \phi_0) + \partial_\mu\partial^\mu\Phi^\dagger & = 0, \\ 2\lambda\Phi(\Phi^\dagger\Phi - \phi_0) + \partial_\mu\partial^\mu\Phi & = 0. \end{cases}$$

(b) If $\Phi = \text{constant}$, then $\partial^\mu\Phi = 0$, and we have

$$2\lambda\Phi^\dagger(\Phi^\dagger\Phi - \phi_0) = 0,$$

$$\Longrightarrow \Phi = 0, \qquad \text{or} \qquad \Phi^\dagger\Phi = \phi_0$$
$$\Longrightarrow \Phi = \sqrt{\phi_0}\mathrm{e}^{\mathrm{i}\theta}.$$

The minimum of the action is then $S_{\min} = 0$.

(c) If

$$\Phi = \sqrt{\phi_0}\mathrm{e}^{\mathrm{i}\theta} + \Psi,$$
$$\Phi^\dagger = \sqrt{\phi_0}\mathrm{e}^{-\mathrm{i}\theta} + \Psi^\dagger,$$

where $|\Psi$ is small, then the Lagrangian becomes

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right) + \lambda\left[\left(\sqrt{\phi_0}\mathrm{e}^{-\mathrm{i}\theta} + \Psi^\dagger\right)\left(\sqrt{\phi_0}\mathrm{e}^{\mathrm{i}\theta} + \Psi\right) - \phi_0\right]^2 \\
&= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right) + \lambda\left[\left(\phi_0 + \sqrt{\phi_0}\mathrm{e}^{-\mathrm{i}\theta}\Psi + \sqrt{\phi_0}\mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger + \Psi^\dagger\Psi\right) - \phi_0\right]^2 \\
&= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right) + \lambda\left[\sqrt{\phi_0}\left(\mathrm{e}^{-\mathrm{i}\theta}\Psi + \mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger\right) + \Psi^\dagger\Psi\right]^2 \\
&= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right) + \lambda\left[\phi_0\left(\mathrm{e}^{-\mathrm{i}\theta}\Psi + \mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger\right)^2 + 2\sqrt{\phi_0}\left(\mathrm{e}^{-\mathrm{i}\theta}\Psi + \mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger\right)\Psi^\dagger\Psi + \left(\Psi^\dagger\Psi\right)^2\right] \\
&= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right) + \lambda\left[\phi_0\left(\mathrm{e}^{-2\mathrm{i}\theta}\Psi^2 + 2\Psi\Psi^\dagger + \mathrm{e}^{2\mathrm{i}\theta}\left(\Psi^\dagger\right)^2\right) + 2\sqrt{\phi_0}\left(\mathrm{e}^{-\mathrm{i}\theta}\Psi + \mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger\right)\Psi^\dagger\Psi + \left(\Psi^\dagger\Psi\right)^2\right].
\end{aligned}$$

Invoking the fact that Ψ is small, we will only consider terms up to the second-order in Ψ and ignore those of higher orders. Then, we have

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right) + \lambda\phi_0\left(\mathrm{e}^{-2\mathrm{i}\theta}\Psi^2 + 2\Psi\Psi^\dagger + \mathrm{e}^{2\mathrm{i}\theta}\left(\Psi^\dagger\right)^2\right).$$

We define the following

$$\begin{cases} \varphi_1 = \mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger + \mathrm{e}^{-\mathrm{i}\theta}\Psi, \\ \varphi_2 = \mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger - \mathrm{e}^{-\mathrm{i}\theta}\Psi, \end{cases} \Longleftrightarrow \begin{cases} \Psi = \mathrm{e}^{\mathrm{i}\theta}\left(\varphi_1 - \varphi_2\right), \\ \Psi^\dagger = \mathrm{e}^{-\mathrm{i}\theta}\left(\varphi_1 + \varphi_2\right). \end{cases}$$

Replacing in the Lagrangian, we have

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\left(\partial_\mu\left(\mathrm{e}^{-\mathrm{i}\theta}\left(\varphi_1 + \varphi_2\right)\right)\right)\left(\partial^\mu\left(\mathrm{e}^{\mathrm{i}\theta}\left(\varphi_1 - \varphi_2\right)\right)\right) \\
&+ \lambda\phi_0\left[\mathrm{e}^{-2\mathrm{i}\theta}\left(\mathrm{e}^{\mathrm{i}\theta}\left(\varphi_1 - \varphi_2\right)\right)^2 + 2\left(\mathrm{e}^{\mathrm{i}\theta}\left(\varphi_1 - \varphi_2\right)\right)\left(\mathrm{e}^{-\mathrm{i}\theta}\left(\varphi_1 + \varphi_2\right)\right) + \mathrm{e}^{2\mathrm{i}\theta}\left(\mathrm{e}^{-\mathrm{i}\theta}\left(\varphi_1 + \varphi_2\right)\right)^2\right] \\
= &-\frac{1}{2}\left(\partial_\mu\left(\varphi_1 + \varphi_2\right)\right)\left(\partial^\mu\left(\varphi_1 - \varphi_2\right)\right) + \lambda\phi_0\left[\left(\varphi_1 - \varphi_2\right)^2 + 2\left(\varphi_1 - \varphi_2\right)\left(\varphi_1 + \varphi_2\right) + \left(\varphi_1 + \varphi_2\right)^2\right] \\
= &-\frac{1}{2}\left(\partial_\mu\varphi_1 + \partial_\mu\varphi_2\right)\left(\partial^\mu\varphi_1 - \partial^\mu\varphi_2\right) + \lambda\phi_0\left[\left(\varphi_1 - \varphi_2\right) + \left(\varphi_1 + \varphi_2\right)\right]^2 \\
= &-\frac{1}{2}\left(\partial_\mu\varphi_1\partial^\mu\varphi_1 - \partial_\mu\varphi_2\partial^\mu\varphi_2\right) + 4\lambda\phi_0\varphi_1^2.
\end{aligned}$$

$$\frac{m_1^2}{2} = 4\lambda\phi_0, \quad \text{and} \quad \frac{m_2^2}{2} = 0,$$
$$m_1 = \sqrt{8\lambda\phi_0}, \quad \text{and} \quad m_2 = 0.$$

(d) From part (c), we can see the massless mode is m_2.

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