MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*

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Homework 10

Problem 9-4

For any integer $n \geq 1$, define a flow on the odd-dimensional sphere $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ by $\theta(t, z) = e^{it}z$. Show that the infinitesimal generator of θ is a smooth non-vanishing vector field on $\mathbb{S}^{2n-1}$. [Remark: in the case $n = 2$, the integral curves of X are the curves γ_z of Problem 3-6, so this provides a simpler proof that each γ_z is smooth.]

Solution. Let's compute the infinitesimal generator X of the flow θ. Recall that for a flow, the infinitesimal generator at a point p is given by

$$X_p = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \theta(t, p).$$

For $z = (z_1, \ldots, z_n) \in \mathbb{S}^{2n-1}$, we have

$$\begin{aligned}\theta(t, z) &= \mathrm{e}^{\mathrm{i}t} z \\ &= \mathrm{e}^{\mathrm{i}t}(z_1, \ldots, z_n) \\ &= (\mathrm{e}^{\mathrm{i}t} z_1, \ldots, \mathrm{e}^{\mathrm{i}t} z_n).\end{aligned}$$

To compute X_z, we differentiate with respect to t at $t = 0$

$$\begin{aligned}X_z &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\mathrm{e}^{\mathrm{i}t} z_1, \ldots, \mathrm{e}^{\mathrm{i}t} z_n) \\ &= \left.\left(\mathrm{i}\mathrm{e}^{\mathrm{i}t} z_1, \ldots, \mathrm{i}\mathrm{e}^{\mathrm{i}t} z_n\right)\right|_{t=0} \\ &= (\mathrm{i}z_1, \ldots, \mathrm{i}z_n) \\ &= \mathrm{i}z.\end{aligned}$$

- **Smoothness:**

  The map $z \mapsto \mathrm{i}z$ is clearly $\mathbb{R}$-linear. All components are complex-valued multiplication, which is smooth.

  Thus, X is a smooth vector field on $\mathbb{S}^{2n-1}$.

- **Non-vanishing:**

  To show X is non-vanishing, let $z \in \mathbb{S}^{2n-1}$, we have

$$\begin{aligned}|X_z| &= |\mathrm{i}z| \\ &= |\mathrm{i}| \cdot |z| \\ &= |z| \\ &= 1.\end{aligned}$$

  The last equality follows since $z \in \mathbb{S}^{2n-1}$. Equivalently, we can show the latter by noticing that $X_z = 0$ only if $z = 0$, but $0 \notin \mathbb{S}^{2n-1}$.

  Thus, $X_z \neq 0$ for all $z \in \mathbb{S}^{2n-1}$ and has constant length 1 everywhere, so X is a non-vanishing vector field.

Therefore, the infinitesimal generator of θ is a smooth non-vanishing vector field on $\mathbb{S}^{2n-1}$.

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Problem 10-12
Let $\pi : E \to M$ and $\widetilde{\pi} : \widetilde{E} \to M$ be two smooth rank-k vector bundles over a smooth manifold M with or without boundary. Suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M such that both E and $\widetilde{E}$ admit smooth local trivializations over each U_α. Let $\{\tau_{\alpha\beta}\}$ and $\{\widetilde{\tau}_{\alpha\beta}\}$ denote the transition functions determined by the given local trivializations of E and $\widetilde{E}$, respectively. Show that E and $\widetilde{E}$ are smoothly isomorphic over M if and only if for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ such that

$$\tilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}, \quad p \in U_\alpha \cap U_\beta.$$

Solution. $\boxed{\Longrightarrow}$ Suppose that E and $\widetilde{E}$ are smoothly isomorphic. Then there exists a smooth bundle isomorphism $F : E \to \widetilde{E}$ covering the identity on M. Let $\{\phi_\alpha\}$ and $\{\widetilde{\phi}_\alpha\}$ be the local trivializations of E and $\widetilde{E}$ respectively over $\{U_\alpha\}$.
For each $\alpha \in A$, let $\mathrm{pr}_2 : U_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$ denote the projection onto the second factor, i.e., $\mathrm{pr}_2(p, v) = v$. Define $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ by

$$\sigma_\alpha(p) = \mathrm{pr}_2 \circ \widetilde{\phi}_\alpha \circ F \circ \phi_\alpha^{-1}(p, \cdot)$$

This map is well-defined because $\widetilde{\phi}_\alpha \circ F \circ \phi_\alpha^{-1}$ maps (p, v) to (p, w) for some $w \in \mathbb{R}^k$, and pr_2 extracts this w. Since F is a smooth bundle isomorphism, each σ_α is smooth and invertible.
For $p \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned}
\widetilde{\tau}_{\alpha\beta}(p) &= \widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}(p, \cdot) \\
&= \widetilde{\phi}_\alpha \circ F \circ F^{-1} \circ \widetilde{\phi}_\beta^{-1}(p, \cdot) \\
&= (\widetilde{\phi}_\alpha \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ F^{-1} \circ \widetilde{\phi}_\beta^{-1})(p, \cdot) \\
&= \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}.
\end{aligned}$$

Therefore, for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ such that

$$\tilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}, \quad p \in U_\alpha \cap U_\beta.$$

$\boxed{\Longleftarrow}$ Suppose we have smooth maps $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ satisfying the given relation. Define $F : E \to \widetilde{E}$ locally by

$$F|_{\pi^{-1}(U_\alpha)} = \widetilde{\phi}_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha} \times \sigma_\alpha) \circ \phi_\alpha$$

To show this defines a global bundle isomorphism, we need to verify that these local definitions agree on overlaps. For $p \in U_\alpha \cap U_\beta$:

$$\begin{aligned}
\widetilde{\phi}_\beta \circ F \circ \phi_\alpha^{-1}(p, v) &= (\mathrm{id}_{U_\beta} \times \sigma_\beta)(p,\, \tau_{\beta\alpha}(p)v) \\
&= (p,\, \sigma_\beta(p)\tau_{\beta\alpha}(p)v) \\
&= (p,\, \widetilde{\tau}_{\beta\alpha}(p)\sigma_\alpha(p)v) \\
&= \widetilde{\tau}_{\beta\alpha}(p)(p,\, \sigma_\alpha(p)v) \\
&= \widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}(p,\, \sigma_\alpha(p)v).
\end{aligned}$$

Thus, F is well-defined and each σ_α is smooth and invertible.
Therefore, F is a smooth bundle isomorphism covering the identity on M.

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Problem 11-5
For any smooth manifold M; show that T^*M is a trivial vector bundle if and only if TM is trivial.

Solution. We will proceed by proving both directions at the same time.

$$\begin{aligned}
T^*M \text{ is a trivial vector bundle} &\iff T^*M \cong M \times \mathbb{R}^n \\
&\iff \text{there exists a global frame of 1-forms } \{\omega_1, \omega_2, \ldots, \omega_n\} \\
&\iff \text{there exists a dual frame of vector fields } \{X_1, X_2, \ldots, X_n\} \text{ where } \omega_i(X_i) = \delta_{ij} \\
&\iff TM \cong M \times \mathbb{R}^n \\
&\iff TM \text{ is a trivial vector bundle.}
\end{aligned}$$

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Problem 11-16
Let M be a compact manifold of positive dimension. Show that every exact covector field on M vanishes at least at two points in each component of M.

Solution. Let M be a compact manifold of positive dimension and let ω be an exact covector field on M, then there exists a map $f \in C^\infty(M)$ such that $\omega = \mathrm{d}f$. The form $\mathrm{d}f$ vanishes at every global extremum of f, of which there exists at least two. ■