PHYS 617 - Statistical Mechanics

A Modern Course in Statistical Physics by *Linda E. Reichl*

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Homework 8

$$n(\mu) = 2\int \frac{\mathrm{d}^3p}{(2\pi\hbar)^3}\frac{1}{1+\mathrm{e}^{\beta(\epsilon_p-\mu)}}$$

$$P(\mu) = \frac{2}{\beta}\int \frac{\mathrm{d}^3p}{(2\pi\hbar)^3}\ln\left(1+\mathrm{e}^{-\beta(\epsilon_p-\mu)}\right)$$

Problem 1

(a) Take the above two integrals derived in class for $P(\mu)$ and $n(\mu)$ and compute them both (as functions of μ) in the limit of low temperature (if you can't remember how to do this, try plotting the integrands at low temperature for inspiration!). Recall $\epsilon_p = \frac{p^2}{2m_e}$.

(b) What is a good estimate for μ for a white dwarf of mass of $M_\odot$ and radius of 10^4 km? Write your answer in eV.

(c) Plot this parametrically on a plot of log $P(\mu)$ vs. log $n(\mu)$. Allow μ to take on values in this vicinity of your estimate above; $0 < \mu < 10\mu_0$, where μ_0 is your estimate from part (b).

Solution. We have that $\epsilon_p = \frac{p^2}{2m_e}$.

(a) • Calculating n, we have

$$\begin{aligned}
n(\mu) &= 2\int \frac{\mathrm{d}^3p}{(2\pi\hbar)^3}\frac{1}{1+\mathrm{e}^{\beta(\epsilon_p-\mu)}} \\
&= \frac{2}{(2\pi\hbar)^3}\int \frac{1}{1+\mathrm{e}^{\beta(\epsilon_p-\mu)}}\,\mathrm{d}^3p \\
&= \frac{2}{(2\pi\hbar)^3}\int \frac{1}{1+\mathrm{e}^{\beta(\epsilon_p-\mu)}}4\pi p^2\,\mathrm{d}p \\
&= \frac{1}{\pi^2\hbar^3}\int_0^\infty \frac{p^2}{1+\mathrm{e}^{\beta(\epsilon_p-\mu)}}\,\mathrm{d}p.
\end{aligned}$$

For $T\to 0$, $\beta\to\infty$, which gives us

- When $\epsilon_p > \mu$, then $\mathrm{e}^{\beta(\epsilon_p-\mu)}\to\infty$.
- When $\epsilon_p < \mu$, then $\mathrm{e}^{\beta(\epsilon_p-\mu)}\to 0$.

Additionally, $\frac{p_0^2}{2m_e} = \mu \implies p_0 = \sqrt{2m_e\mu}$. Then

$$\begin{aligned}
n(\mu) &= \frac{1}{\pi^2\hbar^3}\int_0^{p_0} p^2\,\mathrm{d}p \\
&= \frac{p_0^3}{3\pi^2\hbar^3} \\
&= \frac{(2m_e\mu)^{\frac{3}{2}}}{3\pi^2\hbar^3}.
\end{aligned}$$

- Calculating P, we have

$$\begin{aligned} P(\mu) &= \frac{2}{\beta}\int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3}\ln\left(1+\mathrm{e}^{-\beta(\epsilon_p-\mu)}\right) \\ &= \frac{2}{(2\pi\hbar)^3\beta}\int \ln\left(1+\mathrm{e}^{-\beta(\epsilon_p-\mu)}\right)4\pi p^2\,\mathrm{d}p \\ &= \frac{1}{\pi^2\hbar^3\beta}\int \ln\left(1+\mathrm{e}^{-\beta(\epsilon_p-\mu)}\right)p^2\,\mathrm{d}p. \end{aligned}$$

  For $T\to 0$, $\beta\to\infty$, which gives us
  - When $\epsilon_p > \mu$, then $\ln\left(1+\mathrm{e}^{-\beta(\epsilon_p-\mu)}\right)\to 0$.
  - When $\epsilon_p < \mu$, then $\ln\left(1+\mathrm{e}^{-\beta(\epsilon_p-\mu)}\right)\to \beta(\mu-\epsilon_p)$.

  Additionally, $\frac{p_0^2}{2m_e}=\mu \implies p_0=\sqrt{2m_e\mu}$. Then

$$\begin{aligned} P(\mu) &= \frac{1}{\pi^2\hbar^3\beta}\int_0^{p_0} p^2\beta(\mu-\epsilon_p)\,\mathrm{d}p \\ &= \frac{1}{\pi^2\hbar^3}\int_0^{p_0} p^2\left(\mu-\frac{p^2}{2m_e}\right)\mathrm{d}p \\ &= \frac{1}{\pi^2\hbar^3}\left(\frac{\mu p_0^3}{3}-\frac{p_0^5}{10m_e}\right) \\ &= \frac{1}{\pi^2\hbar^3}\left(\frac{p_0^5}{6m_e}-\frac{p_0^5}{10m_e}\right) \\ &= \frac{p_0^5}{15m_e\pi^2\hbar^3}. \end{aligned}$$

(b) We have that

$$n=\frac{N}{V}\quad\text{and}\quad N=\frac{M_\odot}{m_p}=\frac{V(2m_e\mu)^{\frac{3}{2}}}{3\pi^2\hbar^3}$$

$$\implies \mu=\frac{1}{2m_e}\left(\frac{3\pi^2\hbar^3}{V}\frac{M_\odot}{m_p}\right)^{\frac{2}{3}}\sim 9\times 10^4\,\mathrm{eV}.$$

(c) The code and graph we obtain are listed below.

```python
import numpy as np
import matplotlib.pyplot as plt

##### CONSTANTS #####
hbar = 1.05 * 10**(-34)
k_B = 1.38 * 10**(-23)
m_e = 9.11 * 10**(-31)
mu_0 = 9 * 10**4
e = 1.6 * 10**(-19)

################## PROBLEM 1 ##################
mu = np.linspace(0, 10*mu_0, 1000)

def n(mu):
    return (2 * m_e * mu)**(3/2) / (3 * np.pi**2 * hbar**3)

def p_0(mu):
    return np.sqrt(2 * m_e * mu)

def P(mu):
    return p_0(mu)**5 / (15 * m_e * np.pi**2 * hbar**3)


plt.plot(np.log10(n(mu * e)), np.log10(P(mu * e)), 'r')
plt.xlabel('$\ln(n(\mu))$')
plt.ylabel('$\ln(P(\mu))$')
plt.show()
```

Listing 1: Code for Problem 1 (c)

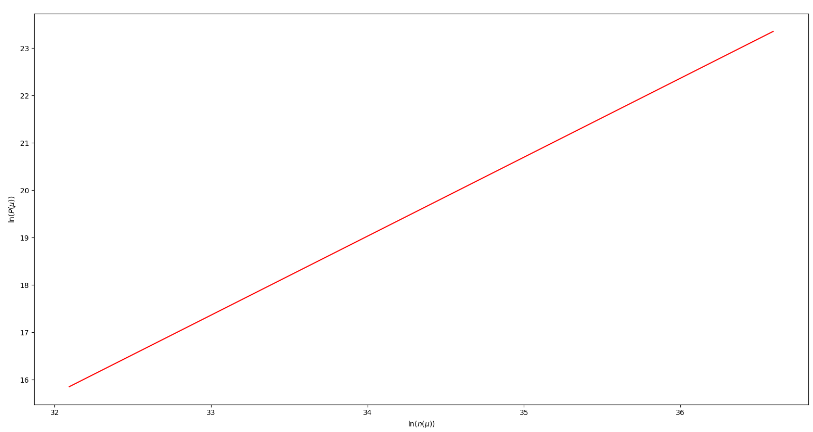

Figure 1: Plot of $\log P(\mu)$ vs. $\log n(\mu)$.

Problem 2
For non-zero temperature, you can just perform these integrals on the computer! Do so, and generate additional parametric curves of $(n(\mu), P(\mu))$ in the same plot. Be careful, when $T \neq 0$, μ is no longer the Fermi energy; in fact, you might need μ to take on negative values to reach sufficiently low densities. For values of the temperature, try $T = 10^5$ K, 10^6 K, 10^7 K, and maybe a few others. Thus, your plot should have several different curves of (n, P) at different temperatures. Congratulations! You've computed the equation of state of a white dwarf $P(n, T)$. Can you separate this into a "degeneracy pressure" component and a "gas pressure" component?

Solution. The code and plot that we get are listed below.
The code and graph we obtain are listed below.

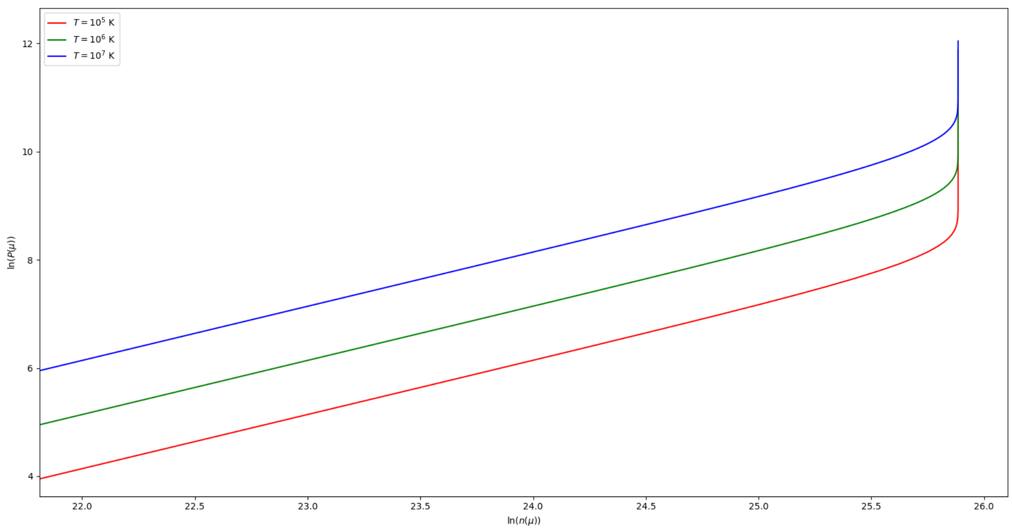

Figure 2: Plot of log $P(\mu)$ vs. log $n(\mu)$.

```python
import numpy as np
import matplotlib.pyplot as plt

##### CONSTANTS #####
hbar = 1.05 * 10**(-34)
k_B = 1.38 * 10**(-23)
m_e = 9.11 * 10**(-31)
mu_0 = 9 * 10**4
e = 1.6 * 10**(-19)

################## PROBLEM 2 ##################
T = np.array([10**5, 10**6, 10**7])
beta = 1 / (k_B * T)


def n(mu, beta):
    integrand = []
    for mu_var in mu:
        integrand.append(int.quad(
            lambda p: p**2 / (1 + np.exp(beta * (p**2/(2 * m_e) - mu_var))),
            0, 0.01*k_B)[0] / (np.pi**2 * hbar**3))
    return integrand

def P(mu, beta):
    integrand = []
    for mu_var in mu:
        integrand.append(int.quad(
            lambda p: p**2 * np.log(1 + np.exp(- beta * (p**2/(2 * m_e) - mu_var))),
            0, 0.01*k_B)[0] / (np.pi**2 * hbar**3 * beta))
    return integrand


mu = np.linspace(-mu_0, mu_0, 50000)
plt.plot(np.log10(n(mu*e, beta[0])), np.log10(P(mu*e, beta[0])), 'r', label="$T= {i}$ K")

mu = np.linspace(-mu_0, mu_0, 50000)
plt.plot(np.log10(n(mu*e, beta[1])), np.log10(P(mu*e, beta[1])), 'g', label="$T= {i}$ K")

mu = np.linspace(-mu_0, mu_0, 50000)
plt.plot(np.log10(n(mu*e, beta[2])), np.log10(P(mu*e, beta[2])), 'b', label="$T= {i}$ K")

plt.xlabel('$\ln(n(\mu))$')
plt.ylabel('$\ln(P(\mu))$')
plt.show()
```

Listing 2: Code for Problem 2

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Problem 3

(a) Repeat the simplified argument from class that gave us the estimate for the equation of state of a Fermi gas at low temperature:

$$P \cong \frac{\hbar^2}{m_e} n_e^{\frac{5}{3}}.$$

(b) At high enough densities, the Fermi energy is sufficiently large that $\epsilon_p \neq p^2/2m$ and instead it is given by the relativistic formula ($\epsilon_p = pc$). Estimate $P(\rho)$ in the relativistic regime.

(c) Estimate the density n and pressure P at which the equation of state transitions from non-relativistic to relativistic. Estimate the mass of a white dwarf at the transition point. Give your answer in solar masses.

(d) (optional, but maybe fun?) Extend your code from Problem 2 to include the full relativistic expression $\epsilon_p = \sqrt{p^2c^2 + m_e^2c^4} - m_ec^2$, to get curves $P(\rho)$ that transition from the non-relativistic to the relativistic regime, also at non-zero temperature. If you want you can compare your result to some industry standard EOS codes used in astrophysics: `https://cococubed.com/code_pages/eos.shtml`.

Solution. (a) From Heisenberg's uncertainty principle, we have $\Delta x \Delta p \sim \hbar$. Since $\Delta x \sim n_e^{-\frac{1}{3}}$, then $\Delta p \sim \hbar n_e^{\frac{1}{3}}$.

The pressure P can be written as energy per volume. Then

$$\begin{aligned} P &= \frac{(\Delta p)^2}{2m_e} n_e \\ &\sim \frac{\hbar^2 n_e^{\frac{2}{3}}}{2m_e} n_e \\ &\sim \frac{\hbar^2 n_e^{\frac{5}{3}}}{m_e}. \end{aligned}$$

(b) Similarly, we write

$$\begin{aligned} P &= (\Delta p)\, c n_e \\ &\sim \hbar c n_e^{\frac{1}{3}} n_e \\ &\sim \hbar c n_e^{\frac{4}{3}} \\ &\sim \hbar c \left(\frac{\rho}{m_e}\right)^{\frac{4}{3}}. \end{aligned}$$

(c) For a transition from non-relativistic to relativistic to occur, we need

$$\frac{(\Delta p)^2}{2m_e} \sim m_e c^2.$$

Working this out, we have

$$\Delta p \sim \sqrt{2} m_e c \sim m_e c.$$

Replacing in Heisenberg's uncertainty principle, we have

$$\Delta x \Delta p \sim \hbar$$

$$\Delta x \sim \frac{\hbar}{\sqrt{2} m_e c}.$$

Then

$$n = \frac{1}{(\Delta x)^3} = \frac{2^{\frac{3}{2}} m_e^3 c^3}{\hbar^3} \sim \left(\frac{m_e c}{\hbar}\right)^3.$$

Thus,

$$\begin{aligned} P &= (\Delta p)\, c n_e \\ &\sim m_e c^2 n_e \\ &\sim m_e c^2 \left(\frac{m_e c}{\hbar}\right)^3 \\ &\sim \frac{m_e^4 c^5}{\hbar^3}. \end{aligned}$$

Assuming hydrostatic equilibrium, we have

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}R} &\sim \frac{P}{R} \sim \rho g \\ \implies \frac{P}{R} &\sim \frac{M}{R^3} \frac{GM}{R^2} \\ \implies P &\sim \frac{GM^2}{R^4}. \end{aligned}$$

From $\rho \sim n m_p \sim \frac{M}{R^3}$, we get $R \sim \left(\frac{M}{n m_p}\right)^{\frac{1}{3}}$. Then

$$\begin{aligned} P &\sim GM^2 \left(\frac{M}{n m_p}\right)^{-\frac{4}{3}} \\ \implies \frac{P}{G} &\sim M^{\frac{2}{3}} (n m_p)^{\frac{4}{3}} \\ \implies M^{\frac{2}{3}} &\sim \frac{P}{G (n m_p)^{\frac{4}{3}}} \\ \implies M &\sim \left(\frac{P}{G}\right)^{\frac{3}{2}} \frac{1}{(n m_p)^2}. \end{aligned}$$

Plugging in $P \sim 1.4 \times 10^{24}\,\mathrm{Pa}$ and $n \sim 1.7 \times 10^{37}\,\mathrm{m}^{-3}$, we get

$$M \sim 4 \times 10^{30}\,\mathrm{kg} \sim 2 M_\odot.$$

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