PHYS 603 - Methods of Theoretical Physics III

Lie Algebras in Particle Physics by *H. Georgi*

Student: **Ralph Razzouk**

Homework 7

Problem 1

Decompose each of the following into a sum of tensors transforming as irreducible representations of $SU(3)$:

(a) The product in Problem 10.A of the textbook (u^i transforms as a 3). Problem 10.A: Decompose the product of tensor components $u^i v^{jk}$ where $v^{jk} = v^{kj}$ transforms like a 6 of $SU(3)$.

(b) $u^{ij} v_k$, where u^{ij} is a 6, and v_k is a $\overline{3}$.

Solution. (a) To decompose the product $u^i v^{jk}$ where u^i transforms as a 3 and $v^{jk} = v^{kj}$ transforms as a 6 of $SU(3)$, I need to find the irreducible components of this tensor product.

First, I note that the tensor product 3 6 has 18 components total. To decompose this, I'll look for ways to construct tensors with definite transformation properties.

I can construct a completely symmetric tensor by symmetrizing over all three indices

$$S^{ijk} = \frac{1}{3}(u^i v^{jk} + u^j v^{ik} + u^k v^{ij}).$$

Since v^{jk} is already symmetric in j and k, this simplifies to

$$S^{ijk} = \frac{1}{3}(u^i v^{jk} + u^j v^{ik} + u^k v^{ij}).$$

The space of completely symmetric tensors with three indices in $SU(3)$ transforms as the 10-dimensional representation. We can verify this using the dimension formula (10.43) from the textbook:

$$D(3,0) = \frac{(3+1)(0+1)(3+0+2)}{2} = \frac{4 \cdot 1 \cdot 5}{2} = 10.$$

Now, to find the remaining piece, we define

$$A^{ijk} = u^i v^{jk} - S^{ijk}.$$

This tensor A^{ijk} must transform as an 8-dimensional representation of $SU(3)$, since the total dimension is 18 and we've already identified a 10-dimensional subspace.

We can verify this is orthogonal to S^{ijk} in the sense of irreducible subspaces. Additionally, we can construct an object with two indices that transforms as the adjoint representation (8) of $SU(3)$ by contracting with the ϵ tensor:=

$$O^i_l = \epsilon_{jkl} A^{ijk}.$$

Therefore, the decomposition is

$$u^i v^{jk} = S^{ijk} + A^{ijk},$$

and the tensor product can be written as:

$$3 \otimes 6 = 10 \oplus 8.$$

(b) For the tensor product $T^{ij}_k = u^{ij} v_k$, where u^{ij} is a 6 and v_k is a $\overline{3}$, Matt first notes that the total tensor has $6 \times 3 = 18$ components.

A natural decomposition approach is to contract an upstairs index with the downstairs index. Since u^{ij} is symmetric, there's essentially one independent contraction

$$w^i = u^{ij} v_j.$$

This w^i transforms as a 3, the fundamental representation of $SU(3)$, accounting for 3 of the 18 components. The remaining $18 - 3 = 15$ components must form an irreducible subspace. To isolate this piece, Matt defines a tensor R^{ij}_k that satisfies

$$R^{ij}_k = R^{ji}_k,$$
$$R^{ij}_i = 0.$$

The most general form that satisfies these conditions is

$$R^{ij}_k = u^{ij} v_k - \alpha(\delta^i_k u^{jl} v_l + \delta^j_k u^{il} v_l).$$

The value of α is determined by imposing the tracelessness condition

$$R^{ij}_i = w^j - \alpha(3w^j + w^j) = 0,$$
$$w^j - 4\alpha w^j = 0.$$

This gives $\alpha = \frac{1}{4}$, so:

$$R^{ij}_k = u^{ij} v_k - \frac{1}{4}(\delta^i_k u^{jl} v_l + \delta^j_k u^{il} v_l).$$

This 15-dimensional irreducible piece corresponds to the (2,1) representation of SU(3).

Therefore, the decomposition is

$$6 \otimes \overline{3} = 15 \oplus 3.$$

■

Problem 10.B
Find the matrix elements $\langle u|T_a|v\rangle$ where T_a are the $SU(3)$ generators and $|u\rangle$ and $|v\rangle$ are tensors in the adjoint representation of $SU(3)$ with components u^i_j and v^i_j. Write the result in terms of the tensor components and the λ_a matrices of (7.4).

Solution. In this problem, we need to find the matrix elements of the SU(3) generators T_a between states in the adjoint representation. We know that $|u\rangle$ and $|v\rangle$ are tensors in the adjoint representation with components u^i_j and v^i_j.
First, recall that the action of the generators on a tensor operator is given by the commutator

$$[T_a, O] = T_a O - O T_a.$$

For a tensor in the adjoint representation, we have

$$\begin{aligned}\left[T_a, v^i_j\right] &= (T_a)_{ik} v^k_j - v^i_k (T_a)_{kj} \\ &= \frac{1}{2}[(\lambda_a)_{ik} v^k_j - v^i_k (\lambda_a)_{kj}],\end{aligned}$$

where we've used the relation $T_a = \frac{1}{2}\lambda_a$.
Now, the matrix element we're looking for can be written as

$$\begin{aligned}\langle u|T_a|v\rangle &= \sum_{i,j} (u^i_j)^* \langle i,j|T_a|v\rangle \\ &= \sum_{i,j} (u^i_j)^* [T_a v^i_j].\end{aligned}$$

Since the action of T_a on the tensor components is given by the commutator, we have

$$\begin{aligned}\langle u|T_a|v\rangle &= \sum_{i,j} (u^i_j)^* \frac{1}{2}[(\lambda_a)_{ik} v^k_j - v^i_k (\lambda_a)_{kj}] \\ &= \frac{1}{2}\sum_{i,j,k} (u^i_j)^* (\lambda_a)_{ik} v^k_j - \frac{1}{2}\sum_{i,j,k} (u^i_j)^* v^i_k (\lambda_a)_{kj}.\end{aligned}$$

Using the properties of tensor transformations and the fact that u^i_j and v^i_j are traceless, we can rewrite this in terms of traces

$$\begin{aligned}\langle u|T_a|v\rangle &= \frac{1}{2}\mathrm{Tr}(u^\dagger \lambda_a v) - \frac{1}{2}\mathrm{Tr}(u^\dagger v \lambda_a) \\ &= \frac{1}{2}\left[\mathrm{Tr}(u^\dagger \lambda_a v) - \mathrm{Tr}(u^\dagger v \lambda_a)\right].\end{aligned}$$

Therefore, the matrix elements are

$$\langle u|T_a|v\rangle = \frac{1}{2}\left[\mathrm{Tr}(u^\dagger \lambda_a v) - \mathrm{Tr}(u^\dagger v \lambda_a)\right].$$

We can also express this in terms of the tensor components

$$\langle u|T_a|v\rangle = \frac{1}{2}\sum_{i,j,k}\left[(u^i_j)^* (\lambda_a)_{ik} v^k_j - (u^i_j)^* v^i_k (\lambda_a)_{kj}\right].$$

This is the expression for the matrix elements of the SU(3) generators between adjoint representation states in terms of the tensor components and the λ_a matrices. ■