Math 572, Spring 2025: Class Overview and Grades

This class is an introduction to the topic of algebraic topology. Although this is a relatively young mathematical area, we will only touch on topics near the infancy of the subject. I will break the class into 4 modules. Before going into the modules, let me setup what I view as one goal of this class from the perspective of a topologist.

Given a topological space X, our goal is to understand X in the context of all topological spaces. Given another topological space Y, one of the fundamental problems in topology then is how to decide if X and Y are homeomorphic. From this starting point, one can view algebraic topology as a mathematical framework for associating algebraic invariants (e.g. groups, rings, vector spaces) to a topological space. We also want certain functorial properties to be satisfied by these invariants. For instance, if $f: X \to Y$ is a continuous map, we would like maps between the invariants associated to X and Y. We will see that homology provides a family of vector spaces (modules) as the algebraic invariants and the maps are linear functions. Important here is that if some algebraic invariant of X is not the same as that of Y, then X and Y cannot be homeomorphic.

Given our goal is to produce algebraic invariants of topological spaces, we need to become good friends with some families of topological spaces in order to appreciate this topic. So the first module will be on constructing topological spaces. The second module will be on the fundamental group or π_1. The third module will be on covering space theory which we will see is just Galois theory. The fourth module is collectively the topics of homology, cohomology, and how they are related (e.g. duality theory).

Below is a preliminary list of some topics that I will try to cover.

(1) The Zoo of Topological Spaces

- Topological Words and Properties: (locally) compact, (path) connected, separability (e.g. Hausdorff), paracompact, homotopy, homotopy equivalence.
- Nice Spaces: Manifolds, homogenous spaces, Lie groups, topological groups, surfaces, 3-manifolds, smooth/analytic manifolds, CW and simplicial structures, geometric structures, real and complex affine spaces.
- Making New Spaces: Quotients, group actions and quotients, gluing things together, mapping cylinders, mapping tori, wedge product, cones, suspensions.

(2) The Fundamental Group.

- Paths and Loops.
- Fundamental Group and Functoriality.
- Seifert-Van Kampen and Amalgamated Free Products of Groups.
- $K(G,1)$ and a Detour Into Topological/Geometric Group Theory.

(3) Cover Space Theory

- Covering Spaces and Basics.
- The Universal Cover and the π_1–Action.
- The Galois Correspondence.
- Profinite Groups and Galois Theory
- Covering Space Theory After Grothendieck.
- A Detour Into Sheaf Theory.

(4) Homology, Cohomology, and Duality

- Singular Homology, Functoriality, and More.
- CW/Simplicial Homology.
- Singular Cohomology.
- De Rham Cohomology and Duality.
- Duality Theory.
- Cohomology With Non-Trivial Coefficients.
- More on $K(G,1)$.

1 Grades and Classwork

First, everyone will get an A. You should be motivated to learn topology. If you want to learn topology, you should work problems to make sure you understand what you are reading. I will produce a list of problems that you can try to solve. I think the class discussion forum on the brightspace will be a good place to discuss the problems. We have a grader for the class but I think that we should be able to decide if a solution is correct as a class without using the grader. I will likely add the grader to the brightspace and they can decide how much to help with questions posted on the discussion forum. We can also discuss the problems in class.