PHYS 630 - Advanced Electricity and Magnetism

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Homework 5

Problem 1
At time $t = 0$, an electron has a velocity v_0 directed along the y-axis. There is an electric field E_0 along the z-axis. Find $y(t)$ and $z(t)$ (fully relativistic).

Solution. When our system is fully relativistic, we have that the momentum is given by $p = \gamma m v$, where γ is the Lorentz factor given by

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1-\beta^2}}.$$

The Lorentz force is given by

$$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right),$$

where we can also right, using Newton's second law, that

$$\dot{\mathbf{p}} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

assuming that the mass is constant.
At $t = 0$, the magnetic field has no effect, so $\dot{\mathbf{p}} = e\mathbf{E}$. Additionally, there is no force along the y-axis. From these, we have

$$\begin{cases} \dot{p}_x = F_x = 0 \\ \dot{p}_y = F_y = 0 \\ \dot{p}_z = F_z = eE_0 \end{cases} \implies \begin{cases} p_x = p_{0x} = 0 \\ p_y = p_{0y} = p_0 \\ p_z = eE_0 t \end{cases}$$

The kinetic energy of the particle is

$$\begin{aligned} \varepsilon &= \sqrt{m^2c^4 + (\mathbf{p}c)^2} \\ &= \sqrt{m^2c^4 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2} \\ &= \sqrt{m^2c^4 + p_0^2c^2 + (eE_0t)^2c^2} \\ &= \sqrt{\varepsilon_0^2 + (eE_0tc)^2}. \end{aligned}$$

Since this is a fully relativistic system, we know that $\varepsilon = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$, then by taking the ratio between them, we have

$$\frac{\mathbf{p}}{\varepsilon} = \frac{\mathbf{v}}{c^2} \implies \mathbf{v} = \frac{\mathbf{p}c^2}{\varepsilon}.$$

Solving for the components of $\mathbf{v}$ component-wise, we have

$$\begin{cases} v_y = \dot{y}(t) = \dfrac{p_y c^2}{\varepsilon} = \dfrac{p_0c^2}{\sqrt{\varepsilon_0^2 + (eE_0tc)^2}}, \\ v_z = \dot{z}(t) = \dfrac{p_z c^2}{\varepsilon} = \dfrac{eE_0tc^2}{\sqrt{\varepsilon_0^2 + (eE_0tc)^2}}, \end{cases} \implies \begin{cases} y(t) = p_0c^2 \displaystyle\int \frac{1}{\sqrt{\varepsilon_0^2 + (eE_0tc)^2}}\,\mathrm{d}t, \\ z(t) = eE_0c^2 \displaystyle\int \frac{t}{\sqrt{\varepsilon_0^2 + (eE_0tc)^2}}\,\mathrm{d}t, \end{cases}$$

$$\implies \begin{cases} y(t) = \dfrac{p_0c}{eE_0}\sinh^{-1}\left(\dfrac{eE_0tc}{\varepsilon_0}\right), \\ z(t) = \dfrac{1}{eE_0}\sqrt{\varepsilon_0^2 + (eE_0tc)^2}. \end{cases}$$

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Problem 2
Earth's magnetic field on the surface is approximately 1 Gauss. Earth magnetosphere extends approximately to 10 Earth radii, where it interacts with the solar wind, as shown in the figure below. The solar wind has a velocity $v_w \approx 500\,\mathrm{km/s}$. Find the Larmor radii of protons from the Solar wind as they penetrate in the outer parts of the Earth magnetosphere.

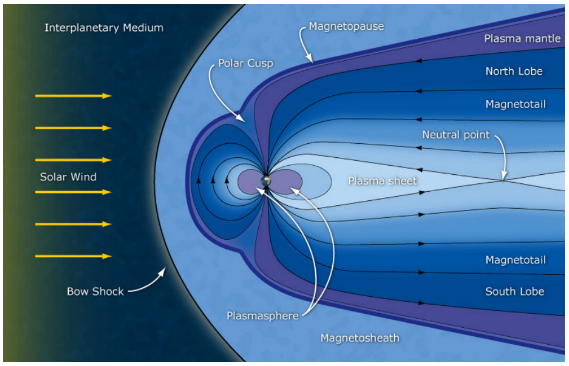

Figure 1: Magnetosphere of Earth

Solution. The Larmor radius is given by

$$r_L = \frac{\gamma v_\perp}{\omega_B},$$

where $v_\perp = v_0 \cos(\omega_c t)$, $\omega_B = \frac{eB}{mc}$, and $\omega_c = \frac{\omega_B}{\gamma}$ is the cyclotron frequency.
Since $v_w \ll c$, then $\gamma \approx 1$. Additionally, $\cos(\omega_c t) \approx 1$ by the small-angle formula since $\omega_c \approx \omega_B$ is very small. The Larmor radius is then

$$\begin{aligned} r_L &= \frac{\gamma\left(v_0 \cos(\omega_c t)\right)}{\left(\frac{eB}{mc}\right)} \\ &\approx \frac{mcv_0}{eB}\cos(\omega_c t) \quad (\text{take } c\cos(\omega_c t) \approx 1) \\ &= \frac{mv_w}{eB}. \end{aligned}$$

We are given that the magnetic field of Earth at the surface is $B_0 = 1\,\mathrm{Gs}$. The magnetic field of Earth at a distance of $10R_\oplus$ will be given by

$$\begin{aligned} \mathbf{B}(r) &= B_0\left(\frac{R^3}{r^3} - 1\right)\cos(\theta) \\ &\approx B_0\left(\frac{R_\oplus}{r}\right)^3 \\ &= B_0\left(\frac{R_\oplus}{10R_\oplus}\right)^3 \\ &= B_0\left(\frac{1}{10}\right)^3 \\ &= \frac{1}{1000B_0} \\ &= 10^{-3}\,\mathrm{Gs} \\ &= 10^{-7}\,\mathrm{T}. \end{aligned}$$

Thus, by replacing our values into the equation for the Larmor radius, we have that

$$\begin{aligned} r_L &= \frac{mv_w}{eB} \\ &= \frac{(1.67\times 10^{-27})(5\times 10^5)}{(1.602\times 10^{-19})(10^{-7})} \\ &= 52.122\,\text{km}. \end{aligned}$$

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Problem 3
Some laboratory device produces horizontal magnetic field of $10^3\,\text{G}$. A proton happens to be inside this device, as shown in the figure below. Find the drift velocity of a proton induced by the gravity of Earth.

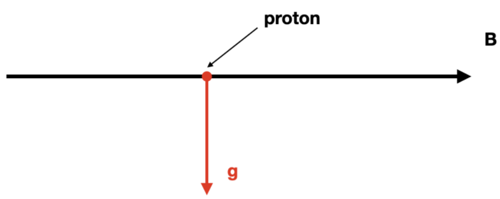

Figure 2: Free-body Diagram of a Proton

Solution. Assuming a slow drift, *i.e.* that the magnetic field $\mathbf{B}$ does not change fast enough on the Larmor scale, then the drift velocity can be approximated by

$$\begin{aligned} \frac{v_d}{c} &= \frac{\mathbf{F}\times\mathbf{B}}{qB^2} \\ &= \frac{m\mathbf{g}\times\mathbf{B}}{qB^2} \\ &= \frac{mgB_x}{eB_x^2} \\ &= \frac{(1.67\times 10^{-27})(9.81)}{(1.602\times 10^{-19})(10^{-1})} \\ &= 1.02\times 10^{-6}\,\text{m/s}, \end{aligned}$$

where v_d is in the $\hat{\mathbf{z}}$ direction (out of the page). ■