PHYS 630 - Advanced Electricity and Magnetism

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Homework 3

Problem 1
A sphere is charged with a surface charge $\sigma = \sigma_0 \cos(\theta)$. Find electric potential inside and outside of the sphere. Hint: since both inside and outside are vacuum, use an expansion in spherical harmonics plus a jump at the surface.

Solution. Let R be the radius of the charged sphere with surface charge $\sigma = \sigma_0 \cos(\theta)$. Given $\sigma(\theta)$, we could solve this using direct integration

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\theta)}{r} \, \mathrm{d}a,$$

but solving this using separation of variables is simpler.
The spherical harmonics expansion of the electric potential is given by

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos(\theta)).$$

$\boxed{\text{For } r \leq R}$

The term inversely proportional to $r^{\ell+1}$ diverges when $r = 0$, so we set $B_\ell = 0$. Then, we have

$$V_{\text{in}}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos(\theta)).$$

$\boxed{\text{For } r \geq R}$

The term proportional to r^ℓ diverges when $r \to \infty$, so we set $A_\ell = 0$. Then, we have

$$V_{\text{out}}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos(\theta)).$$

The piece-wise functions must be joined together using proper boundary conditions, which is that the functions V_{in} and V_{out} are continuous at the surface when $r = R$ and their radial derivatives are discontinuous at $r = R$.

- **Continuous**: We have that

$$\begin{aligned} V_{\text{in}}(R,\theta) &= V_{\text{out}}(R,\theta) \\ \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos(\theta)) &= \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos(\theta)). \end{aligned}$$

  We will now invoke the orthogonality property for the Legendre functions to simplify the summation and equate the coefficients, specifically, we have

$$\int_1^{-1} P_\ell(x) P_{\ell'}(x) \, \mathrm{d}x = \int_0^{\pi} P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta)) \sin(\theta) \, \mathrm{d}\theta = \begin{cases} 0, & \text{if } \ell \neq \ell', \\ \frac{2}{2\ell+1}, & \text{if } \ell = \ell'. \end{cases}$$

Thus, by multiplying both sides by $P_{\ell'}(\cos(\theta))\sin(\theta)$, and then integrating over θ from 0 to π, we have

$$\begin{aligned}
\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos(\theta)) \times P_{\ell'}(\cos(\theta))\sin(\theta) &= \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos(\theta)) \times P_{\ell'}(\cos(\theta))\sin(\theta) \\
\int_0^\pi \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos(\theta)) \times P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta &= \int_0^\pi \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos(\theta)) \times P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
\sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta &= \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} \int_0^\pi P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
A_\ell R^\ell \left(\frac{2}{2\ell+1}\right) &= \frac{B_\ell}{R^{\ell+1}} \left(\frac{2}{2\ell+1}\right) \\
A_\ell R^\ell &= \frac{B_\ell}{R^{\ell+1}} \\
\implies B_\ell &= A_\ell R^{2\ell+1}.
\end{aligned}$$

We used the orthogonality property for the Legendre functions to get rid of the summation and equate the coefficients.

- **Not Differentiable**: We have that

$$\begin{aligned}
\left.\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\right|_{r=R} &= -\frac{\sigma(\theta)}{\epsilon_0} \\
\left.\left(-\sum_{\ell=0}^{\infty}(\ell+1)\frac{B_\ell}{r^{\ell+2}} P_\ell(\cos(\theta)) - \sum_{\ell=0}^{\infty} \ell A_\ell r^{\ell-1} P_\ell(\cos(\theta))\right)\right|_{r=R} &= -\frac{\sigma(\theta)}{\epsilon_0} \\
-\sum_{\ell=0}^{\infty}(\ell+1)\frac{B_\ell}{R^{\ell+2}} P_\ell(\cos(\theta)) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= -\frac{\sigma(\theta)}{\epsilon_0} \\
-\sum_{\ell=0}^{\infty}(\ell+1)\frac{A_\ell R^{2\ell+1}}{R^{\ell+2}} P_\ell(\cos(\theta)) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= -\frac{\sigma(\theta)}{\epsilon_0} \\
-\sum_{\ell=0}^{\infty}(\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= -\frac{\sigma(\theta)}{\epsilon_0} \\
\sum_{\ell=0}^{\infty}(2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= \frac{\sigma(\theta)}{\epsilon_0}.
\end{aligned}$$

Similar to the case of continuity, we can determine the coefficients using the orthogonality of the Legendre polynomials. We have

$$\begin{aligned}
\sum_{\ell=0}^{\infty}(2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= \frac{\sigma(\theta)}{\epsilon_0} \\
\sum_{\ell=0}^{\infty}(2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta))\sin(\theta) &= \frac{\sigma(\theta)}{\epsilon_0} P_{\ell'}(\cos(\theta))\sin(\theta) \\
\int_0^\pi \sum_{\ell=0}^{\infty}(2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta &= \frac{1}{\epsilon_0}\int_0^\pi \sigma(\theta) P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
(2\ell+1)2A_\ell R^{\ell-1}\left(\frac{2}{2\ell+1}\right) &= \frac{1}{\epsilon_0}\int_0^\pi \sigma(\theta) P_\ell(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
A_\ell &= \frac{1}{2R^{\ell-1}\epsilon_0}\int_0^\pi \sigma(\theta) P_\ell(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta.
\end{aligned}$$

Now, considering the given conditions of our problem, we have $\sigma(\theta) = \sigma_0 \cos(\theta)$, for some constant σ_0, then notice that our surface charge density is proportional to $P_1(\cos(\theta))$, *i.e* $\sigma(\theta) = \sigma_0 P_1(\cos(\theta))$, which means that all A_ℓ's are null except for $\ell = 1$, which gives

$$
\begin{aligned}
A_1 &= \frac{1}{2R^0\epsilon_0} \int_0^\pi \sigma(\theta) P_1(\cos(\theta)) \sin(\theta)\,\mathrm{d}\theta \\
A_1 &= \frac{\sigma_0}{2\epsilon_0} \int_0^\pi \cos^2(\theta) \sin(\theta)\,\mathrm{d}\theta \\
A_1 &= \frac{\sigma_0}{2\epsilon_0} \left[-\frac{\cos^3(\theta)}{3} \right] \Bigg|_0^\pi \\
A_1 &= \frac{\sigma_0}{2\epsilon_0} \left(\frac{2}{3} \right) \\
A_1 &= \frac{\sigma_0}{3\epsilon_0}.
\end{aligned}
$$

From this, we have that

$$B_1 = A_1 R^3 = \frac{\sigma_0 R^3}{3\epsilon_0}.$$

The potential inside the sphere is

$$V_{\text{in}}(r, \theta) = A_1 r P_1(\cos(\theta)) = \frac{\sigma_0}{3\epsilon_0} r \cos(\theta).$$

The potential outside the sphere is

$$V_{\text{out}}(r, \theta) = \frac{B_1}{r^2} P_1(\cos(\theta)) = \frac{\sigma_0 R^3}{3\epsilon_0} \frac{1}{r^2} \cos(\theta).$$

Thus, the total electric potential is

$$V(r, \theta) = \begin{cases} \frac{\sigma_0}{3\epsilon_0} r \cos(\theta), & \text{if } r \leq R, \\ \frac{\sigma_0 R^3}{3\epsilon_0} \frac{1}{r^2} \cos(\theta), & \text{if } r \geq R. \end{cases}$$

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Problem 2
A straight current wire has current distribution $j_z = j_0 (1 - \varpi/\varpi_0)$ where $\varpi \leq \varpi_0$ is a cylindrical coordinate. Find the total current, the vector potential, and the magnetic field.

Solution. We have a current distribution of

$$j_x = j_y = 0, \quad j_z = j_0 \left(1 - \frac{\varpi}{\varpi_0}\right), \quad \varpi \leq \varpi_0.$$

The total current I is given by

$$\begin{aligned} I &= \int_{\mathcal{A}} \mathbf{j} \cdot \mathrm{d}\mathbf{A} \\ &= \int_0^{\varpi_0} j_0 \left(1 - \frac{\varpi}{\varpi_0}\right) 2\pi\varpi \, \mathrm{d}\varpi \\ &= 2\pi j_0 \int_0^{\varpi_0} \left(\varpi - \frac{\varpi^2}{\varpi_0}\right) \mathrm{d}\varpi \\ &= 2\pi j_0 \left(\frac{\varpi^2}{2} - \frac{\varpi^3}{3\varpi_0}\right)\Bigg|_0^{\varpi_0} \\ &= 2\pi j_0 \left(\frac{\varpi_0^2}{2} - \frac{\varpi_0^3}{3\varpi_0}\right) \\ &= \frac{\pi j_0 \varpi_0^2}{3}. \end{aligned}$$

The vector potential $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{c} \int_{\mathcal{V}} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V'.$$

Equivalently, we can solve the differential equation

$$\nabla^2 \mathbf{A} = \frac{1}{\rho} \partial_\rho \left(\rho \partial_\rho \mathbf{A}\right) = -\mu_0 \mathbf{j}(\rho) = -\frac{4\pi}{c} \mathbf{j}(\rho)$$

to get the vector potential, noting that there is no ϕ or z dependence since the vector potential does not rely on them. Additionally, since $j_x = j_y = 0$, then $A_x = A_y = 0$. We have

$$\begin{aligned} \nabla^2 A_z &= -\frac{4\pi}{c} j_z \\ \frac{1}{\varpi} \partial_\varpi \left(\varpi \partial_\varpi A_z\right) &= -\frac{4\pi j_0}{c} \left(1 - \frac{\varpi}{\varpi_0}\right) \\ \varpi \partial_\varpi A_z &= -\frac{4\pi j_0}{c} \left(\frac{\varpi^2}{2} - \frac{\varpi^3}{3\varpi_0}\right) + c_1 \\ \partial_\varpi A_z &= -\frac{4\pi j_0}{c} \left(\frac{\varpi}{2} - \frac{\varpi^2}{3\varpi_0}\right) + \frac{c_1}{\varpi} \\ A_z &= -\frac{4\pi j_0}{c} \left(\frac{\varpi^2}{4} - \frac{\varpi^3}{9\varpi_0}\right) + c_1 \ln(\varpi) + c_2. \end{aligned}$$

Thus

$$\mathbf{A} = (0, 0, A_z) = \left[-\frac{4\pi j_0}{c} \left(\frac{\varpi^2}{4} - \frac{\varpi^3}{9\varpi_0}\right) + c_1 \ln(\varpi) + c_2\right] \mathbf{z}.$$

The magnetic field $\mathbf{B}$ can be found by

$$
\begin{aligned}
\mathbf{B} = \nabla \times \mathbf{A} &= \frac{1}{c}\int_{\mathcal{V}} \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}V' \\
&= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}} \\
&= \frac{1}{\rho}\frac{\partial A_z}{\partial \varphi}\hat{\boldsymbol{\rho}} - \frac{\partial A_z}{\partial \rho}\hat{\boldsymbol{\varphi}} \\
&= -\frac{\partial A_z}{\partial \rho}\hat{\boldsymbol{\varphi}} \\
&\equiv -\frac{\partial A_z}{\partial \varpi}\hat{\boldsymbol{\varphi}} \\
&= \left[\frac{4\pi j_0}{c}\left(\frac{\varpi}{8} - \frac{\varpi^2}{27\varpi_0}\right) - \frac{c_1}{\varpi}\right]\hat{\boldsymbol{\varphi}}.
\end{aligned}
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Problem 3
Current density is given by

$$j_\phi = C_1 r^3 \mathrm{e}^{-\frac{2r}{3a}} \sin(\theta)\cos^2(\theta)$$

where C_1 is some constant and we are in spherical coordinates $r-\theta-\phi$. Find the magnetic moment.

Solution. The magnetic moment is given by

$$\boldsymbol{\mu} = \frac{1}{2}\int_{\mathcal{V}} \mathbf{r}\times\mathbf{j}\,\mathrm{d}V.$$

Using (r,θ,ϕ) coordinates, we have

$$\begin{aligned}\mathbf{r} &= (r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)),\\ \mathbf{j} &= (j_r, j_\theta, j_\phi),\end{aligned}$$

where

$$\begin{aligned}j_r = j_\theta &= 0,\\ j_\phi &= C_1 r^3 \mathrm{e}^{-\frac{2r}{3a}} \sin(\theta)\cos^2(\theta).\end{aligned}$$

The cross product term is

$$\begin{aligned}\mathbf{r}\times\mathbf{j} &= \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z}\\ r\sin(\theta)\cos(\phi) & r\sin(\theta)\sin(\phi) & r\cos(\theta)\\ -j_\phi\sin(\phi) & j_\phi\cos(\phi) & 0\end{vmatrix}\\ &= -j_\phi r\cos(\theta)\cos(\phi)\mathbf{x} - j_\phi r\cos(\theta)\sin(\phi)\mathbf{y} + \left[j_\phi r\sin(\theta)\cos^2(\phi) + j_\phi\sin(\theta)\sin^2(\theta)\right]\mathbf{z}\\ &= -j_\phi r\cos(\theta)\cos(\phi)\mathbf{x} - j_\phi r\cos(\theta)\sin(\phi)\mathbf{y} + j_\phi r\sin(\theta)\mathbf{z}\\ &= j_\phi r\left[-\cos(\theta)\cos(\phi)\mathbf{x} - \cos(\theta)\sin(\phi)\mathbf{y} + \sin(\theta)\mathbf{z}\right].\end{aligned}$$

We will integrate coordinate-wise to solve for the magnetic moment $\boldsymbol{\mu}$.

- **Along x**:

$$\begin{aligned}\mu_x &= \frac{1}{2}\int_{\mathcal{V}} (\mathbf{r}\times\mathbf{j})_x\,\mathrm{d}V\\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(-j_\phi r\cos(\theta)\cos(\phi)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(-\left(C_1 r^3\mathrm{e}^{-\frac{2r}{3a}}\sin(\theta)\cos^2(\theta)\right) r\cos(\theta)\cos(\phi)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= -\frac{C_1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(r^6\mathrm{e}^{-\frac{2r}{3a}}\sin^2(\theta)\cos^3(\theta)\cos(\phi)\right)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= -\frac{C_1}{2}\left(\int_0^{\infty} r^6\mathrm{e}^{-\frac{2r}{3a}}\,\mathrm{d}r\right)\left(\int_0^{\pi}\sin^2(\theta)\cos^3(\theta)\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\cos(\phi)\,\mathrm{d}\phi\right)\\ &= 0,\end{aligned}$$

since $\int_0^{2\pi}\cos(\phi)\,\mathrm{d}\phi = 0$.

- **Along y**:

$$\begin{aligned}
\mu_y &= \frac{1}{2}\int_{\mathcal{V}} (\mathbf{r}\times\mathbf{j})_y \,\mathrm{d}V \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(-j_\phi r\cos(\theta)\sin(\phi)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(-\left(C_1 r^3 \mathrm{e}^{-\frac{2r}{3a}}\sin(\theta)\cos^2(\theta)\right) r\cos(\theta)\sin(\phi)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= -\frac{C_1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(r^6 \mathrm{e}^{-\frac{2r}{3a}}\sin^2(\theta)\cos^3(\theta)\sin(\phi)\right)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= -\frac{C_1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^6 \mathrm{e}^{-\frac{2r}{3a}}\sin^3(\theta)\cos^2(\theta)\cos(\phi)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= -\frac{C_1}{2}\left(\int_0^{\infty} r^6 \mathrm{e}^{-\frac{2r}{3a}}\,\mathrm{d}r\right)\left(\int_0^{\pi}\sin^2(\theta)\cos^3(\theta)\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\sin(\phi)\,\mathrm{d}\phi\right) \\
&= 0,
\end{aligned}$$

since $\int_0^{2\pi}\sin(\phi)\,\mathrm{d}\phi = 0$.

- **Along z**:

$$\begin{aligned}
\mu_z &= \frac{1}{2}\int_{\mathcal{V}} (\mathbf{r}\times\mathbf{j})_z \,\mathrm{d}V \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(j_\phi r\sin(\theta)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(\left(C_1 r^3 \mathrm{e}^{-\frac{2r}{3a}}\sin(\theta)\cos^2(\theta)\right) r\sin(\theta)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{C_1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(r^6 \mathrm{e}^{-\frac{2r}{3a}}\sin^3(\theta)\cos^2(\theta)\right)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{C_1}{2}\left(\int_0^{\infty} r^6 \mathrm{e}^{-\frac{2r}{3a}}\,\mathrm{d}r\right)\left(\int_0^{\pi}\sin^3(\theta)\cos^2(\theta)\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\mathrm{d}\phi\right) \\
&= \frac{C_1}{2}\left(\frac{98415a^7}{8}\right)\left(\frac{4}{15}\right)(2\pi) \\
&= \frac{6561\pi a^7}{2}C_1.
\end{aligned}$$

Thus, we have

$$\boldsymbol{\mu} = \left(0, 0, \frac{6561\pi a^7}{2}C_1\right) = \frac{6561\pi a^7}{2}C_1\mathbf{z}.$$

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