PHYS 660 - Quantum Mechanics I

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Homework 4

Problem 1
An electron is subject to a uniform, time-independent magnetic field $\vec{B} = B\hat{z}$ in the z direction. At $t = 0$, the electron is in an eigenstate of $\vec{S} \cdot \hat{n}$ with eigenvalue $\frac{\hbar}{2}$. Here $\hat{n}$ is an arbitrary unit vector with polar angles (θ, ϕ). (Similar to previous homework)

(a) Find the state of the electron at any subsequent time t.

(b) At a given time t, what are the possible results of measuring S_x, and their probabilities? Repeat the same for S_z and S_y.

(c) Compute the mean value of S_x, S_y, and S_z as a function of time.

Proof. (a) We have that $\hat{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$. Then

$$S_{\hat{n}} = \vec{S} \cdot \hat{n} = \frac{\hbar}{2}\begin{pmatrix} \cos(\theta) & \sin(\theta)\mathrm{e}^{-i\phi} \\ \sin(\theta)\mathrm{e}^{i\phi} & -\cos(\theta) \end{pmatrix}.$$

The state with eigenvalue $\frac{\hbar}{2}$ is

$$\frac{\hbar}{2}\begin{pmatrix} \cos(\theta) - 1 & \sin(\theta)\mathrm{e}^{-i\phi} \\ \sin(\theta)\mathrm{e}^{i\phi} & -\cos(\theta) - 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{cases} (\cos(\theta) - 1)\alpha + \sin(\theta)\mathrm{e}^{-i\phi}\beta = 0 \\ \sin(\theta)\mathrm{e}^{i\phi}\alpha - (\cos(\theta) + 1)\beta = 0 \end{cases}$$
$$\implies \alpha = \frac{\sin(\theta)\mathrm{e}^{-i\phi}}{1 - \cos(\theta)}\beta.$$

Let $\beta = 1 - \cos(\theta)$, then $\alpha = \sin(\theta)\mathrm{e}^{-i\phi}$. The initial state of $S_{\hat{n}}$ corresponding to the eigenvalue $\frac{\hbar}{2}$ at $t = 0$ is

$$|\alpha, t_0 = 0, 0\rangle = \sin(\theta)\mathrm{e}^{-i\phi}|\uparrow\rangle + (1 - \cos(\theta))|\downarrow\rangle.$$

Normalizing, we get

$$\begin{aligned} \left|\sin(\theta)\mathrm{e}^{-i\phi}\right|^2 + |1 - \cos(\theta)|^2 &= \sin^2(\theta) + (1 - \cos(\theta))^2 \\ &= \sin^2(\theta) + 1 - 2\cos(\theta) + \cos^2(\theta) \\ &= 2 - 2\cos(\theta) \\ &= 4\sin^2\left(\frac{\theta}{2}\right). \end{aligned}$$

Then,

$$\begin{aligned} |\alpha, t_0 = 0, 0\rangle &= \frac{1}{2\sin\left(\frac{\theta}{2}\right)}\left[\sin(\theta)\mathrm{e}^{-i\phi}|\uparrow\rangle + (1 - \cos(\theta))|\downarrow\rangle\right] \\ &= \frac{1}{2\sin\left(\frac{\theta}{2}\right)}\left[2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\phi}|\uparrow\rangle + 2\sin^2\left(\frac{\theta}{2}\right)|\downarrow\rangle\right] \\ &= \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\phi}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)|\downarrow\rangle \\ &= \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}|\downarrow\rangle. \end{aligned}$$

The magnetic field is given as $\vec{B} = B\hat{z}$, which yields the unitary operator $U(t,0) = \mathrm{e}^{\frac{-i\omega\hat{S}_z t}{\hbar}}$, where $\omega = \frac{eB}{mc}$.

So, at any time t, the state of our system is

$$\begin{aligned}
|\alpha, t_0 = 0, t\rangle &= U(t,0)\,|\alpha, t_0 = 0, 0\rangle \\
&= \mathrm{e}^{\frac{-i\omega\hat{S}_z t}{\hbar}}\,|\alpha, t_0 = 0, 0\rangle \\
&= \mathrm{e}^{\frac{-i\omega\hat{S}_z t}{\hbar}}\left[\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\,|{\uparrow}\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\,|{\downarrow}\rangle\right] \\
&= \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}\,|{\uparrow}\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\,|{\downarrow}\rangle\,,
\end{aligned}$$

where the last step is due to $\hat{S}_z\,|{\uparrow}\rangle = \frac{\hbar}{2}$ and $\hat{S}_z\,|{\downarrow}\rangle = -\frac{\hbar}{2}$. Additionally, the state for any time t is normalized since

$$\left|\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}\right|^2 + \left|\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\right|^2 = 1.$$

(b) We have

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

- **For S_x:** Finding the eigenvalues, we have

$$\begin{aligned}
|S_x - \lambda\mathbb{I}| &= 0 \\
\begin{vmatrix}-\lambda & \frac{\hbar}{2}\\ \frac{\hbar}{2} & -\lambda\end{vmatrix} &= 0 \\
\lambda^2 - \frac{\hbar^2}{4} &= 0 \\
\lambda &= \pm\frac{\hbar}{2}.
\end{aligned}$$

  The eigenvectors are given by

$$\begin{pmatrix}-\lambda & \frac{\hbar}{2}\\ \frac{\hbar}{2} & -\lambda\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \implies -\lambda\alpha + \frac{\hbar}{2}\beta = 0 \implies \beta = \frac{2\lambda}{\hbar}\alpha.$$

  - **For $\lambda = \frac{\hbar}{2}$:** We have $\beta = \alpha$. Since α is arbitrary, we can set $\alpha = 1 \implies \beta = 1$. Then, we have

$$|S_{x\uparrow}\rangle = \begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} = |{\uparrow}\rangle + |{\downarrow}\rangle\,.$$

    Normalizing the state, we get

$$|S_{x\uparrow}\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle + |{\downarrow}\rangle\right).$$

The probability of measuring $\frac{\hbar}{2}$ is

$$\begin{aligned}
P_{x\uparrow} &= \left|\langle S_{x\uparrow}|\alpha, t_0 = 0, t\rangle\right|^2 \\
&= \left|\left(\frac{1}{\sqrt{2}}\langle\uparrow| + \frac{1}{\sqrt{2}}\langle\downarrow|\right)\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}\,|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\,|\downarrow\rangle\right)\right|^2 \\
&= \frac{1}{2}\left|\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}} + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\right|^2 \\
&= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}} + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\right)\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}} + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}\right) \\
&= \frac{1}{2}\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\phi}\mathrm{e}^{i\omega t} + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\phi}\mathrm{e}^{-i\omega t}\right) \\
&= \frac{1}{2}\left(1 + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(\mathrm{e}^{i\phi}\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\phi}\mathrm{e}^{-i\omega t}\right)\right) \\
&= \frac{1}{2}\left(1 + 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos(\phi + \omega t)\right) \\
&= \frac{1}{2}\left(1 + \sin(\theta)\cos(\phi + \omega t)\right).
\end{aligned}$$

– **For $\lambda = -\frac{\hbar}{2}$:** We have $\beta = -\alpha$. Since α is arbitrary, we can set $\alpha = 1 \implies \beta = -1$. Then, we have

$$|S_{x\downarrow}\rangle = \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} = |\uparrow\rangle - |\downarrow\rangle.$$

Normalizing the state, we get

$$|S_{x\downarrow}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right).$$

The probability of measuring $-\frac{\hbar}{2}$ is

$$\begin{aligned}
P_{x\downarrow} &= \left|\langle S_{x\downarrow}|\alpha, t_0 = 0, t\rangle\right|^2 \\
&= \left|\left(\frac{1}{\sqrt{2}}\langle\uparrow| - \frac{1}{\sqrt{2}}\langle\downarrow|\right)\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}\,|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\,|\downarrow\rangle\right)\right|^2 \\
&= \frac{1}{2}\left|\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}} - \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\right|^2 \\
&= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}} - \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\right)\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}} - \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}\right) \\
&= \frac{1}{2}\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\phi}\mathrm{e}^{i\omega t} - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\phi}\mathrm{e}^{-i\omega t}\right) \\
&= \frac{1}{2}\left(1 - \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(\mathrm{e}^{i\phi}\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\phi}\mathrm{e}^{-i\omega t}\right)\right) \\
&= \frac{1}{2}\left(1 - 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\cos(\phi + \omega t)\right) \\
&= \frac{1}{2}\left(1 - \sin(\theta)\cos(\phi + \omega t)\right).
\end{aligned}$$

- **For S_y:** Finding the eigenvalues, we have

$$
\begin{aligned}
|S_y - \lambda \mathbb{I}| &= 0 \\
\begin{vmatrix} -\lambda & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & -\lambda \end{vmatrix} &= 0 \\
\lambda^2 - \frac{\hbar^2}{4} &= 0 \\
\lambda &= \pm\frac{\hbar}{2}.
\end{aligned}
$$

The eigenvectors are given by

$$
\begin{pmatrix} -\lambda & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & -\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -\lambda\alpha - \frac{i\hbar}{2}\beta = 0 \implies \beta = \frac{2i\lambda}{\hbar}\alpha.
$$

  - **For $\lambda = \frac{\hbar}{2}$:** We have $\beta = i\alpha$. Since α is arbitrary, we can set $\alpha = 1 \implies \beta = i$. Then, we have

$$
|S_{y\uparrow}\rangle = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |{\uparrow}\rangle + i\,|{\downarrow}\rangle .
$$

Normalizing the state, we get

$$
|S_{y\uparrow}\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle + i\,|{\downarrow}\rangle\right).
$$

The probability of measuring $\frac{\hbar}{2}$ is

$$
\begin{aligned}
P_{y\uparrow} &= \left|\langle S_{y\uparrow}|\alpha, t_0 = 0, t\rangle\right|^2 \\
&= \left|\left(\frac{1}{\sqrt{2}}\langle{\uparrow}| - \frac{i}{\sqrt{2}}\langle{\downarrow}|\right)\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}\,|{\uparrow}\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\,|{\downarrow}\rangle\right)\right|^2 \\
&= \frac{1}{2}\left|\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}} - i\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\right|^2 \\
&= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}} - i\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\right)\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}} + i\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}\right) \\
&= \frac{1}{2}\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) - i\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\phi}\mathrm{e}^{i\omega t} + i\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\phi}\mathrm{e}^{-i\omega t}\right) \\
&= \frac{1}{2}\left(1 - i\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(\mathrm{e}^{i\phi}\mathrm{e}^{i\omega t} - \mathrm{e}^{-i\phi}\mathrm{e}^{-i\omega t}\right)\right) \\
&= \frac{1}{2}\left(1 + 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\sin(\phi + \omega t)\right) \\
&= \frac{1}{2}\left(1 + \sin(\theta)\sin(\phi + \omega t)\right).
\end{aligned}
$$

  - **For $\lambda = -\frac{\hbar}{2}$:** We have $\beta = -i\alpha$. Since α is arbitrary, we can set $\alpha = 1 \implies \beta = -i$. Then, we have

$$
|S_{y\downarrow}\rangle = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |{\uparrow}\rangle - i\,|{\downarrow}\rangle .
$$

Normalizing the state, we get

$$
|S_{y\downarrow}\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow}\rangle - i\,|{\downarrow}\rangle\right).
$$

The probability of measuring $-\frac{\hbar}{2}$ is

$$\begin{aligned}
P_{y\downarrow} &= |\langle S_{y\downarrow}|\alpha, t_0 = 0, t\rangle|^2 \\
&= \left| \left(\frac{1}{\sqrt{2}} \langle\uparrow| + \frac{i}{\sqrt{2}} \langle\downarrow| \right) \left(\cos\left(\frac{\theta}{2}\right) \mathrm{e}^{-i\frac{\phi}{2}} \mathrm{e}^{\frac{-i\omega t}{2}} |\uparrow\rangle + \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{i\frac{\phi}{2}} \mathrm{e}^{\frac{i\omega t}{2}} |\downarrow\rangle \right) \right|^2 \\
&= \frac{1}{2} \left| \cos\left(\frac{\theta}{2}\right) \mathrm{e}^{-i\frac{\phi}{2}} \mathrm{e}^{\frac{-i\omega t}{2}} + i \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{i\frac{\phi}{2}} \mathrm{e}^{\frac{i\omega t}{2}} \right|^2 \\
&= \frac{1}{2} \left(\cos\left(\frac{\theta}{2}\right) \mathrm{e}^{-i\frac{\phi}{2}} \mathrm{e}^{\frac{-i\omega t}{2}} + i \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{i\frac{\phi}{2}} \mathrm{e}^{\frac{i\omega t}{2}} \right) \left(\cos\left(\frac{\theta}{2}\right) \mathrm{e}^{i\frac{\phi}{2}} \mathrm{e}^{\frac{i\omega t}{2}} - i \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{-i\frac{\phi}{2}} \mathrm{e}^{\frac{-i\omega t}{2}} \right) \\
&= \frac{1}{2} \left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + i \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{i\phi} \mathrm{e}^{i\omega t} - i \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{-i\phi} \mathrm{e}^{-i\omega t} \right) \\
&= \frac{1}{2} \left(1 + i \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \left(\mathrm{e}^{i\phi} \mathrm{e}^{i\omega t} - \mathrm{e}^{-i\phi} \mathrm{e}^{-i\omega t} \right) \right) \\
&= \frac{1}{2} \left(1 - 2 \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right) \sin(\phi + \omega t) \right) \\
&= \frac{1}{2} \left(1 - \sin(\theta) \sin(\phi + \omega t) \right).
\end{aligned}$$

- **For S_z:** Finding the eigenvalues, we have

$$\begin{aligned}
|S_z - \lambda \mathbb{I}| &= 0 \\
\begin{vmatrix} \frac{\hbar}{2} - \lambda & 0 \\ 0 & -\frac{\hbar}{2} - \lambda \end{vmatrix} &= 0 \\
-\left(\frac{\hbar}{2} - \lambda\right)\left(\frac{\hbar}{2} + \lambda\right) &= 0 \\
\lambda &= \pm\frac{\hbar}{2}.
\end{aligned}$$

The eigenvectors are given by

$$\begin{pmatrix} \frac{\hbar}{2} - \lambda & 0 \\ 0 & -\frac{\hbar}{2} - \lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \left(\frac{\hbar}{2} - \lambda\right) \alpha = 0 \text{ and } \left(\frac{\hbar}{2} + \lambda\right) \beta = 0.$$

  - **For $\lambda = \frac{\hbar}{2}$:** We have $0\alpha = 0$ and $\hbar\beta = 0$. Since α is arbitrary, we can set $\alpha = 1$ and then $\beta = 0$. Then, we have

$$|S_{z\uparrow}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\uparrow\rangle.$$

The state is already normalized.
The probability of measuring $\frac{\hbar}{2}$ is

$$\begin{aligned}
P_{z\uparrow} &= |\langle S_{z\uparrow}|\alpha, t_0 = 0, t\rangle|^2 \\
&= \left| \langle\uparrow| \left(\cos\left(\frac{\theta}{2}\right) \mathrm{e}^{-i\frac{\phi}{2}} \mathrm{e}^{\frac{-i\omega t}{2}} |\uparrow\rangle + \sin\left(\frac{\theta}{2}\right) \mathrm{e}^{i\frac{\phi}{2}} \mathrm{e}^{\frac{i\omega t}{2}} |\downarrow\rangle \right) \right|^2 \\
&= \left| \cos\left(\frac{\theta}{2}\right) \mathrm{e}^{-i\frac{\phi}{2}} \mathrm{e}^{\frac{-i\omega t}{2}} \right|^2 \\
&= \cos^2\left(\frac{\theta}{2}\right).
\end{aligned}$$

  - **For $\lambda = -\frac{\hbar}{2}$:** We have $\hbar\alpha = 0$ and $0\beta = 0$. Since β is arbitrary, we can set $\beta = 1$ and then $\alpha = 0$. Then, we have

$$|S_{x\downarrow}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |\downarrow\rangle.$$

The state is already normalized.
The probability of measuring $-\frac{\hbar}{2}$ is

$$\begin{aligned}
P_{z\downarrow} &= |\langle S_{z\downarrow}|\alpha, t_0 = 0, t\rangle|^2 \\
&= \left|\langle\downarrow|\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{\phi}{2}}\mathrm{e}^{\frac{-i\omega t}{2}}|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}|\downarrow\rangle\right)\right|^2 \\
&= \left|\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{\phi}{2}}\mathrm{e}^{\frac{i\omega t}{2}}\right|^2 \\
&= \sin^2\left(\frac{\theta}{2}\right).
\end{aligned}$$

(c) Formally, the expectation value is given by

$$\langle x\rangle = \sum_x xP(x).$$

- **For S_x:** We have

$$\begin{aligned}
\langle S_x\rangle &= \left(\frac{\hbar}{2}\right)P_{x\uparrow} + \left(-\frac{\hbar}{2}\right)P_{x\downarrow} \\
&= \frac{\hbar}{4}\left[(1+\sin(\theta)\cos(\phi+\omega t)) - (1-\sin(\theta)\cos(\phi+\omega t))\right] \\
&= \frac{\hbar}{2}\sin(\theta)\cos(\phi+\omega t).
\end{aligned}$$

- **For S_y:** We have

$$\begin{aligned}
\langle S_y\rangle &= \left(\frac{\hbar}{2}\right)P_{y\uparrow} + \left(-\frac{\hbar}{2}\right)P_{y\downarrow} \\
&= \frac{\hbar}{4}\left[(1+\sin(\theta)\sin(\phi+\omega t)) - (1-\sin(\theta)\sin(\phi+\omega t))\right] \\
&= \frac{\hbar}{2}\sin(\theta)\sin(\phi+\omega t).
\end{aligned}$$

- **For S_z:** We have

$$\begin{aligned}
\langle S_z\rangle &= \left(\frac{\hbar}{2}\right)P_{z\uparrow} + \left(-\frac{\hbar}{2}\right)P_{z\downarrow} \\
&= \frac{\hbar}{2}\left[\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right] \\
&= \frac{\hbar}{2}\cos(\theta).
\end{aligned}$$

■

Problem 2
Consider a one-dimensional harmonic oscillator with frequency ω. Using the algebraic method (*i.e.* operators a, $a^\dagger$) compute

(a) $\langle m|x|n\rangle$, $\langle m|p|n\rangle$, $\langle m|xp+px|n\rangle$, $\langle m|x^2|n\rangle$, $\langle m|p^2|n\rangle$, where $|n\rangle$ and $|m\rangle$ are eigenstates of energy.

(b) $\langle\alpha|x|\alpha\rangle$, $\langle\alpha|p|\alpha\rangle$, $\langle\alpha|x^2|\alpha\rangle$, $\langle\alpha|p^2|\alpha\rangle$, where $|\alpha\rangle$ is a coherent state.

(c) Use the previous result to show that coherent states have minimum uncertainty.

Proof. We know that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right), \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right)$$

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n|0\rangle, \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle, \qquad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle,$$

(a) • Computing $\langle m|\hat{x}|n\rangle$, we have

$$\begin{aligned}
\langle m|\hat{x}|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left\langle m\middle|\hat{a}^\dagger + \hat{a}\middle|n\right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\left\langle m\middle|\hat{a}^\dagger\middle|n\right\rangle + \langle m|\hat{a}|n\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\langle m|n+1\rangle + \sqrt{n}\,\langle m|n-1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right).
\end{aligned}$$

• Computing $\langle m|\hat{p}|n\rangle$, we have

$$\begin{aligned}
\langle m|\hat{p}|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\left\langle m\middle|\hat{a}^\dagger - \hat{a}\middle|n\right\rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left(\left\langle m\middle|\hat{a}^\dagger\middle|n\right\rangle - \langle m|\hat{a}|n\rangle\right) \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\,\langle m|n+1\rangle - \sqrt{n}\,\langle m|n-1\rangle\right) \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right).
\end{aligned}$$

- Computing $\langle m|\hat{x}\hat{p}+\hat{p}\hat{x}|n\rangle$, we have

$$
\begin{aligned}
\langle m|\hat{x}\hat{p}+\hat{p}\hat{x}|n\rangle &= \langle m|\hat{x}\hat{p}|n\rangle + \langle m|\hat{p}\hat{x}|n\rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left\langle m\middle|\hat{x}\left(\hat{a}^\dagger - \hat{a}\right)\middle|n\right\rangle + \sqrt{\frac{\hbar}{2m\omega}}\left\langle m\middle|\hat{p}\left(\hat{a}^\dagger + \hat{a}\right)\middle|n\right\rangle \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\left\langle m\middle|\hat{x}\hat{a}^\dagger\middle|n\right\rangle - \langle m|\hat{x}\hat{a}|n\rangle\right] + \sqrt{\frac{\hbar}{2m\omega}}\left[\left\langle m\middle|\hat{p}\hat{a}^\dagger\middle|n\right\rangle + \langle m|\hat{p}\hat{a}|n\rangle\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\,\langle m|\hat{x}|n+1\rangle - \sqrt{n}\,\langle m|\hat{x}|n-1\rangle\right] \\
&\quad + \sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\,\langle m|\hat{p}|n+1\rangle + \sqrt{n}\,\langle m|\hat{p}|n-1\rangle\right] \\
&= i\sqrt{\frac{\hbar m\omega}{2}}\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\left\langle m\middle|\hat{a}^\dagger+\hat{a}\middle|n+1\right\rangle - \sqrt{n}\left\langle m\middle|\hat{a}^\dagger+\hat{a}\middle|n-1\right\rangle\right] \\
&\quad + \sqrt{\frac{\hbar}{2m\omega}}\,i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\left\langle m\middle|\hat{a}^\dagger-\hat{a}\middle|n+1\right\rangle + \sqrt{n}\left\langle m\middle|\hat{a}^\dagger-\hat{a}\middle|n-1\right\rangle\right] \\
&= \frac{i\hbar}{2}\left[\sqrt{n+1}\left(\left\langle m\middle|\hat{a}^\dagger\middle|n+1\right\rangle + \langle m|\hat{a}|n+1\rangle\right) - \sqrt{n}\left(\left\langle m\middle|\hat{a}^\dagger\middle|n-1\right\rangle + \langle m|\hat{a}|n-1\rangle\right)\right] \\
&\quad + \frac{i\hbar}{2}\left[\sqrt{n+1}\left(\left\langle m\middle|\hat{a}^\dagger\middle|n+1\right\rangle - \langle m|\hat{a}|n+1\rangle\right) + \sqrt{n}\left(\left\langle m\middle|\hat{a}^\dagger\middle|n-1\right\rangle - \langle m|\hat{a}|n-1\rangle\right)\right] \\
&= \frac{i\hbar}{2}\left[\sqrt{n+1}\left(\sqrt{n+2}\,\langle m|n+2\rangle + \sqrt{n+1}\,\langle m|n\rangle\right) - \sqrt{n}\left(\sqrt{n}\,\langle m|n\rangle + \sqrt{n-1}\,\langle m|n-2\rangle\right)\right] \\
&\quad + \frac{i\hbar}{2}\left[\sqrt{n+1}\left(\sqrt{n+2}\,\langle m|n+2\rangle - \sqrt{n+1}\,\langle m|n\rangle\right) + \sqrt{n}\left(\sqrt{n}\,\langle m|n\rangle - \sqrt{n-1}\,\langle m|n-2\rangle\right)\right] \\
&= \frac{i\hbar}{2}\left[\sqrt{(n+1)(n+2)}\delta_{m,n+2} + (n+1)\delta_{m,n} - n\delta_{m,n} - \sqrt{n(n-1)}\delta_{m,n-2}\right] \\
&\quad + \frac{i\hbar}{2}\left[\sqrt{(n+1)(n+2)}\delta_{m,n+2} - (n+1)\delta_{m,n} + n\delta_{m,n} - \sqrt{n(n-1)}\delta_{m,n-2}\right] \\
&= i\hbar\left[\sqrt{(n+1)(n+2)}\delta_{m,n+2} - \sqrt{n(n-1)}\delta_{m,n-2}\right].
\end{aligned}
$$

We could have also used the property $\hat{x}\hat{p}+\hat{p}\hat{x} = 2\hat{x}\hat{p} - [\hat{x}\hat{p}] = 2\hat{x}\hat{p} - i\hbar$.

- Computing $\left\langle m\middle|\hat{x}^2\middle|n\right\rangle$, we have

$$
\begin{aligned}
\left\langle m\middle|\hat{x}^2\middle|n\right\rangle &= \frac{\hbar}{2m\omega}\left\langle m\middle|\left(\hat{a}^\dagger+\hat{a}\right)\left(\hat{a}^\dagger+\hat{a}\right)\middle|n\right\rangle \\
&= \frac{\hbar}{2m\omega}\langle m|\left(\hat{a}^\dagger+\hat{a}\right)\left(\sqrt{n+1}\,|n+1\rangle + \sqrt{n}\,|n-1\rangle\right) \\
&= \frac{\hbar}{2m\omega}\langle m|\left(\sqrt{n+2}\sqrt{n+1}\,|n+2\rangle + \sqrt{n}\sqrt{n}\,|n\rangle + \sqrt{n+1}\sqrt{n+1}\,|n\rangle + \sqrt{n}\sqrt{n-1}\,|n-2\rangle\right) \\
&= \frac{\hbar}{2m\omega}\left[\sqrt{(n+1)(n+2)}\delta_{m,n+2} + n\delta_{m,n} + (n+1)\delta_{m,n} + \sqrt{n(n-1)}\delta_{m,n-2}\right] \\
&= \frac{\hbar}{2m\omega}\left[\sqrt{(n+1)(n+2)}\delta_{m,n+2} + (2n+1)\delta_{m,n} + \sqrt{n(n-1)}\delta_{m,n-2}\right].
\end{aligned}
$$

- Computing $\left\langle m\middle|\hat{p}^2\middle|n\right\rangle$, we have

$$\begin{aligned}
\left\langle m\middle|\hat{p}^2\middle|n\right\rangle &= -\frac{\hbar m\omega}{2}\left\langle m\middle|\left(\hat{a}^\dagger-\hat{a}\right)\left(\hat{a}^\dagger-\hat{a}\right)\middle|n\right\rangle \\
&= -\frac{\hbar m\omega}{2}\left\langle m\right|\left(\hat{a}^\dagger-\hat{a}\right)\left(\sqrt{n+1}\left|n+1\right\rangle-\sqrt{n}\left|n-1\right\rangle\right) \\
&= -\frac{\hbar m\omega}{2}\left\langle m\right|\left(\sqrt{n+2}\sqrt{n+1}\left|n+2\right\rangle-\sqrt{n}\sqrt{n}\left|n\right\rangle-\sqrt{n+1}\sqrt{n+1}\left|n\right\rangle+\sqrt{n}\sqrt{n-1}\left|n-2\right\rangle\right) \\
&= -\frac{\hbar m\omega}{2}\left[\sqrt{(n+1)(n+2)}\delta_{m,n+2}-n\delta_{m,n}-(n+1)\delta_{m,n}+\sqrt{n(n-1)}\delta_{m,n-2}\right] \\
&= -\frac{\hbar m\omega}{2}\left[\sqrt{(n+1)(n+2)}\delta_{m,n+2}-(2n+1)\delta_{m,n}+\sqrt{n(n-1)}\delta_{m,n-2}\right].
\end{aligned}$$

(b) Consider a normalized coherent state

$$\left|\alpha\right\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\mathrm{e}^{\alpha\hat{a}^\dagger}\left|0\right\rangle.$$

We have that $\left|\alpha\right\rangle$ is an eigenvector of $\hat{a}$: $\hat{a}\left|\alpha\right\rangle=\alpha\left|\alpha\right\rangle \iff \left\langle\alpha\right|\hat{a}^\dagger=\alpha^*\left\langle\alpha\right|$.
Additionally, we have

$$[\hat{a},\hat{a}^\dagger]=1, \qquad [\hat{a},\left(\hat{a}^\dagger\right)^n]=n\left(\hat{a}^\dagger\right)^{n-1}.$$

- Computing $\left\langle\alpha\middle|\hat{x}\middle|\alpha\right\rangle$, we have

$$\begin{aligned}
\left\langle\alpha\middle|\hat{x}\middle|\alpha\right\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left\langle\alpha\middle|\hat{a}^\dagger+\hat{a}\middle|\alpha\right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\left\langle\alpha\middle|\hat{a}^\dagger\middle|\alpha\right\rangle+\left\langle\alpha\middle|\hat{a}\middle|\alpha\right\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha^*+\alpha\right).
\end{aligned}$$

- Computing $\left\langle\alpha\middle|\hat{p}\middle|\alpha\right\rangle$, we have

$$\begin{aligned}
\left\langle\alpha\middle|\hat{p}\middle|\alpha\right\rangle &= i\sqrt{\frac{m\hbar\omega}{2}}\left\langle\alpha\middle|\hat{a}^\dagger-\hat{a}\middle|\alpha\right\rangle \\
&= i\sqrt{\frac{m\hbar\omega}{2}}\left(\left\langle\alpha\middle|\hat{a}^\dagger\middle|\alpha\right\rangle-\left\langle\alpha\middle|\hat{a}\middle|\alpha\right\rangle\right) \\
&= i\sqrt{\frac{m\hbar\omega}{2}}\left(\alpha^*-\alpha\right).
\end{aligned}$$

- Computing $\left\langle\alpha\middle|\hat{x}^2\middle|\alpha\right\rangle$, we have

$$\begin{aligned}
\left\langle\alpha\middle|\hat{x}^2\middle|\alpha\right\rangle &= \frac{\hbar}{2m\omega}\left\langle\alpha\middle|\left(\hat{a}^\dagger+\hat{a}\right)\left(\hat{a}^\dagger+\hat{a}\right)\middle|\alpha\right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle\alpha\middle|\left(\hat{a}^\dagger\right)^2+\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger+\hat{a}^2\middle|\alpha\right\rangle \\
&= \frac{\hbar}{2m\omega}\left[\left\langle\alpha\middle|\left(\hat{a}^\dagger\right)^2\middle|\alpha\right\rangle+\left\langle\alpha\middle|\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger\middle|\alpha\right\rangle+\left\langle\alpha\middle|\hat{a}^2\middle|\alpha\right\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\left\langle\alpha\middle|\left(\hat{a}^\dagger\right)^2\middle|\alpha\right\rangle+\left\langle\alpha\middle|2\hat{a}^\dagger\hat{a}-[\hat{a}^\dagger,\hat{a}]\middle|\alpha\right\rangle+\left\langle\alpha\middle|\hat{a}^2\middle|\alpha\right\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\left\langle\alpha\middle|\left(\hat{a}^\dagger\right)^2\middle|\alpha\right\rangle+\left\langle\alpha\middle|2\hat{a}^\dagger\hat{a}\middle|\alpha\right\rangle-\left\langle\alpha\middle|[\hat{a}^\dagger,\hat{a}]\middle|\alpha\right\rangle+\left\langle\alpha\middle|\hat{a}^2\middle|\alpha\right\rangle\right] \\
&= \frac{\hbar}{2m\omega}\left[\left(\alpha^*\right)^2+2\alpha^*\alpha+1+\alpha^2\right] \\
&= \frac{\hbar}{2m\omega}\left[\left(\alpha^*+\alpha\right)^2+1\right].
\end{aligned}$$

- Computing $\langle \alpha | \hat{p} | \alpha \rangle$, we have

$$\begin{aligned}
\left\langle \alpha \middle| \hat{p}^2 \middle| \alpha \right\rangle &= -\frac{m\hbar\omega}{2} \left\langle \alpha \middle| \left(\hat{a}^\dagger - \hat{a}\right) \left(\hat{a}^\dagger - \hat{a}\right) \middle| \alpha \right\rangle \\
&= -\frac{m\hbar\omega}{2} \left\langle \alpha \middle| \left(\hat{a}^\dagger\right)^2 - \hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2 \middle| \alpha \right\rangle \\
&= -\frac{m\hbar\omega}{2} \left[\left\langle \alpha \middle| \left(\hat{a}^\dagger\right)^2 \middle| \alpha \right\rangle - \langle \alpha | \hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger | \alpha \rangle + \left\langle \alpha \middle| \hat{a}^2 \middle| \alpha \right\rangle \right] \\
&= -\frac{m\hbar\omega}{2} \left[\left\langle \alpha \middle| \left(\hat{a}^\dagger\right)^2 \middle| \alpha \right\rangle - \langle \alpha | 2\hat{a}^\dagger \hat{a} - [\hat{a}^\dagger, \hat{a}] | \alpha \rangle + \left\langle \alpha \middle| \hat{a}^2 \middle| \alpha \right\rangle \right] \\
&= -\frac{m\hbar\omega}{2} \left[\left\langle \alpha \middle| \left(\hat{a}^\dagger\right)^2 \middle| \alpha \right\rangle - \langle \alpha | 2\hat{a}^\dagger \hat{a} | \alpha \rangle + \left\langle \alpha \middle| [\hat{a}^\dagger, \hat{a}] \middle| \alpha \right\rangle + \left\langle \alpha \middle| \hat{a}^2 \middle| \alpha \right\rangle \right] \\
&= -\frac{m\hbar\omega}{2} \left[(\alpha^*)^2 - 2\alpha^*\alpha - 1 + \alpha^2 \right] \\
&= -\frac{m\hbar\omega}{2} \left[(\alpha^* - \alpha)^2 - 1 \right].
\end{aligned}$$

(c) We need to show that coherent states have minimum uncertainty. In other words, we need to prove that

$$\Delta x \Delta p = \frac{\hbar}{2}.$$

- Calculating the uncertainty Δx, we have

$$\begin{aligned}
(\Delta x)^2 &= \left\langle \alpha \middle| \hat{x}^2 \middle| \alpha \right\rangle - \langle \alpha | \hat{x} | \alpha \rangle^2 \\
&= \frac{\hbar}{2m\omega} \left[(\alpha^* + \alpha)^2 + 1 \right] - \left(\sqrt{\frac{\hbar}{2m\omega}} \left(\alpha^* + \alpha\right) \right)^2 \\
&= \frac{\hbar}{2m\omega} \left[(\alpha^* + \alpha)^2 + 1 \right] - \frac{\hbar}{2m\omega} \left(\alpha^* + \alpha\right)^2 \\
&= \frac{\hbar}{2m\omega} \\
\implies \Delta x &= \sqrt{\frac{\hbar}{2m\omega}}.
\end{aligned}$$

- Calculating the uncertainty Δp, we have

$$\begin{aligned}
(\Delta p)^2 &= \left\langle \alpha \middle| \hat{p}^2 \middle| \alpha \right\rangle - \langle \alpha | \hat{p} | \alpha \rangle^2 \\
&= -\frac{m\hbar\omega}{2} \left[(\alpha^* - \alpha)^2 - 1 \right] - \left(i\sqrt{\frac{m\hbar\omega}{2}} \left(\alpha^* - \alpha\right) \right)^2 \\
&= -\frac{m\hbar\omega}{2} \left[(\alpha^* - \alpha)^2 - 1 \right] + \frac{m\hbar\omega}{2} \left(\alpha^* - \alpha\right)^2 \\
&= \frac{m\hbar\omega}{2} \\
\implies \Delta p &= \sqrt{\frac{m\hbar\omega}{2}}.
\end{aligned}$$

Thus,

$$\Delta x \Delta p = \left(\sqrt{\frac{\hbar}{2m\omega}} \right) \left(\sqrt{\frac{m\hbar\omega}{2}} \right) = \frac{\hbar}{2},$$

which is what we needed to prove.

■

Problem 3
Continuing from Problem 2, consider $x_H(t)$, the position operator in the Heisenberg picture. Evaluate the correlation function

$$C(t) = \langle 0|x_H(t)x_H(0)|0\rangle\,,$$

where $|0\rangle$ is the ground state.

Proof. In the Heisenberg picture, the position operator $x_H(t)$ is given by

$$x_H(t) = \hat{U}^\dagger(t)\hat{x}_H(0)\hat{U}(t),$$

where $x_H(0) = \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right)$, $\hat{U}(t) = \mathrm{e}^{-\frac{i\hat{H}t}{\hbar}}$, and $\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$ with $\hat{H} = \hat{H}^\dagger$.
Additionally, we know that $\hat{a}\,|0\rangle = 0 \iff \langle 0|\,\hat{a}^\dagger = 0$.
We have

$$\begin{aligned}
C(t) &= \langle 0|\hat{x}_H(t)\hat{x}_H(0)|0\rangle \\
&= \left\langle 0\middle|\hat{U}^\dagger(t)\hat{x}_H(0)\hat{U}(t)\hat{x}_H(0)\middle|0\right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle 0\middle|\mathrm{e}^{\frac{i\hat{H}t}{\hbar}}\left(\hat{a}^\dagger + \hat{a}\right)\mathrm{e}^{-\frac{i\hat{H}t}{\hbar}}\left(\hat{a}^\dagger + \hat{a}\right)\middle|0\right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle 0\middle|\mathrm{e}^{\frac{i\hat{H}t}{\hbar}}\left(\hat{a}^\dagger + \hat{a}\right)\mathrm{e}^{-\frac{i\hat{H}t}{\hbar}}\hat{a}^\dagger\middle|0\right\rangle \\
&= \frac{\hbar\sqrt{1}}{2m\omega}\left\langle 0\middle|\mathrm{e}^{\frac{i\hat{H}t}{\hbar}}\left(\hat{a}^\dagger + \hat{a}\right)\mathrm{e}^{-\frac{i\hat{H}t}{\hbar}}\middle|1\right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle 0\middle|\mathrm{e}^{i\omega t\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)}\left(\hat{a}^\dagger + \hat{a}\right)\mathrm{e}^{-i\omega t\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)}\middle|1\right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle 0\middle|\mathrm{e}^{i\omega t\hat{a}^\dagger\hat{a}}\left(\hat{a}^\dagger + \hat{a}\right)\mathrm{e}^{-i\omega t\hat{a}^\dagger\hat{a}}\middle|1\right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle 0\middle|\left(\hat{a}^\dagger + \hat{a}\right)\mathrm{e}^{-i\omega t\left(\hat{a}^\dagger\hat{a}-\hat{a}\hat{a}^\dagger\right)}\middle|1\right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle 0\middle|\left(\hat{a}^\dagger + \hat{a}\right)\mathrm{e}^{-i\omega t[\hat{a}^\dagger,\hat{a}]}\middle|1\right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle 0\middle|\left(\hat{a}^\dagger + \hat{a}\right)\mathrm{e}^{i\omega t}\middle|1\right\rangle \\
&= \frac{\hbar}{2m\omega}\mathrm{e}^{i\omega t}\,\langle 0|\hat{a}|1\rangle \\
&= \frac{\hbar}{2m\omega}\mathrm{e}^{i\omega t}\,\langle 0|0\rangle \\
&= \frac{\hbar}{2m\omega}\mathrm{e}^{i\omega t}.
\end{aligned}$$

■

Problem 4
A particle of mass m in one dimension is bound to a fix center by an attractive δ-function potential

$$V(x) = -\lambda\delta(x), \quad (\lambda > 0).$$

At $t = 0$, the potential is suddenly switched off, that is, $V = 0$ for $t > 0$. At $t = 0$, the wave function is the one of the bound state since it cannot change instantaneously ($\partial_x\psi$ is finite for finite Hamiltonian). Compute the wave function for $t > 0$.

Proof. Given that the potential at $t = 0$ is $V(x) = -\lambda\delta(x)$ for $\lambda > 0$.

- **At $x = 0$:**
  The Schrodinger equation is given by
  $$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) - \lambda\delta(x)\psi(x) = E\psi(x).$$
  Integrating over space right around $x = 0$, *i.e.* within a range $(-\epsilon, \epsilon)$ and then taking the limit as $\epsilon \to 0$, we have
  $$\begin{aligned} -\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\frac{\partial^2}{\partial x^2}\psi(x)\,\mathrm{d}x - \lambda\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,\mathrm{d}x &= E\int_{-\epsilon}^{\epsilon}\psi(x)\,\mathrm{d}x \\ -\frac{\hbar^2}{2m}\left.\frac{\partial\psi(x)}{\partial x}\right|_{-\epsilon}^{\epsilon} - \lambda\psi(0) &= 0 \\ -\frac{\hbar^2}{2m}\left(\frac{\partial\psi(0^+)}{\partial x} - \frac{\partial\psi(0^-)}{\partial x}\right) - \lambda\psi(0) &= 0 \\ \frac{\partial\psi(0^+)}{\partial x} - \frac{\partial\psi(0^-)}{\partial x} &= -\frac{2m\lambda}{\hbar^2}\psi(0). \end{aligned}$$

- **At $x \neq 0$:**
  The Schrodinger equation is given by
  $$\begin{aligned} -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) &= E\psi(x) \\ \frac{\partial^2\psi(x)}{\partial x^2} &= -\frac{2mE}{\hbar^2}\psi(x). \end{aligned}$$
  Let $k^2 \equiv -\frac{2mE}{\hbar^2}$, then
  $$\frac{\partial^2\psi(x)}{\partial x^2} = k^2\psi(x) \implies \psi(x) = \begin{cases} A\mathrm{e}^{-kx}, & \text{if } x > 0 \\ B\mathrm{e}^{kx}, & \text{if } x < 0 \end{cases}$$
  - **Continuity at $x = 0$:**
    We have that
    $$\begin{aligned} \psi(0^+) = A = \psi(0^-) = B = \psi(0), \\ \partial_x\psi(0^+) = -kA, \quad \partial_x\psi(0^+) = kB. \end{aligned}$$

Replacing in the Schrodinger equation, we have

$$\begin{aligned} \frac{\partial\psi(0^+)}{\partial x} - \frac{\partial\psi(0^-)}{\partial x} &= -\frac{2m\lambda}{\hbar^2}\psi(0) \\ -kA - kB &= -\frac{2m\lambda}{\hbar^2}A \\ 2kA &= \frac{2m\lambda}{\hbar^2}A \\ k &= \frac{m\lambda}{\hbar^2}. \end{aligned}$$

Thus, the solution is

$$\psi(x,0) = A\mathrm{e}^{-\frac{m\lambda}{\hbar^2}|x|}.$$

To find A, we use the normalization condition, and hence

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi(x)|^2 \,\mathrm{d}x &= 1 \\
2\int_{0}^{\infty} |A|^2 \mathrm{e}^{-\frac{2m\lambda}{\hbar^2}x} \,\mathrm{d}x &= 1 \\
2|A|^2 \left(-\frac{\hbar^2}{2m\lambda}\right) \mathrm{e}^{-\frac{2m\lambda}{\hbar^2}x}\Big|_0^{\infty} &= 1 \\
2|A|^2 \left(\frac{\hbar^2}{2m\lambda}\right) &= 1 \\
|A|^2 \left(\frac{\hbar^2}{m\lambda}\right) &= 1 \\
|A|^2 &= \frac{m\lambda}{\hbar^2} \\
A &= \sqrt{\frac{m\lambda}{\hbar^2}}.
\end{aligned}$$

Thus,

$$\psi(x,0) = \sqrt{\frac{m\lambda}{\hbar^2}}\mathrm{e}^{-\frac{m\lambda}{\hbar^2}|x|}.$$

Let $\frac{m\lambda}{\hbar^2} \equiv \frac{1}{x_0}$, then

$$\psi(x,0) = \frac{1}{\sqrt{x_0}}\mathrm{e}^{-\frac{|x|}{x_0}}.$$

Now, we need to find $\psi(x,t)$ when $\hat{H} = \frac{\hat{p}^2}{2m}$. We will use the propagator, given by

$$K(x,t,x',t_0) = \sum_{a'} \langle x|a'\rangle \langle a'|x'\rangle \,\mathrm{e}^{-\frac{iE_{a'}(t-t_0)}{\hbar}} = \sqrt{\frac{m}{2i\pi\hbar t}}\mathrm{e}^{-\frac{1}{2}\frac{m(x-x')^2}{i\hbar t}}.$$

Then, we have

$$\begin{aligned}
\psi(x,t) &= \int_{-\infty}^{+\infty} K(x,t,x',0)\psi(x',0)\,\mathrm{d}x' \\
&= \sqrt{\frac{m}{2i\pi\hbar t}}\sqrt{\frac{m\lambda}{\hbar^2}}\int_{-\infty}^{\infty} \mathrm{e}^{\frac{im(x-x')^2}{2\hbar t}}\mathrm{e}^{-\frac{m\lambda}{\hbar^2}|x'|}\,\mathrm{d}x' \\
&= m\sqrt{\frac{2\lambda}{i\pi\hbar^2 t}}\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}\frac{m(x-x')^2}{i\hbar t}}\mathrm{e}^{-\frac{m\lambda}{\hbar^2}x'}\,\mathrm{d}x' \\
&= m\sqrt{\frac{2\lambda}{i\pi\hbar^2 t}}\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}\frac{m(x^2-2xx'+x'^2)}{i\hbar t}}\mathrm{e}^{-\frac{m\lambda}{\hbar^2}x'}\,\mathrm{d}x' \\
&= m\sqrt{\frac{2\lambda}{i\pi\hbar^2 t}}\int_0^{\infty} \mathrm{e}^{\frac{imx^2}{2\hbar t}}\mathrm{e}^{-\frac{imx}{\hbar t}x'}\mathrm{e}^{\frac{im}{2\hbar t}x'^2}\mathrm{e}^{-\frac{m\lambda}{\hbar^2}x'}\,\mathrm{d}x' \\
&= m\sqrt{\frac{2\lambda}{i\pi\hbar^2 t}}\mathrm{e}^{\frac{imx^2}{2\hbar t}}\int_0^{\infty} \mathrm{e}^{-\frac{m}{2i\hbar t}x'^2}\mathrm{e}^{\left(\frac{mx}{i\hbar t}-\frac{m\lambda}{\hbar^2}\right)x'}\,\mathrm{d}x'.
\end{aligned}$$

This is an integral solution of our equation for any time t. An analytic closed form solution in terms of error functions exists. We obtained this solution via a symbolic integral calculator, which gave

$$
\begin{aligned}
-\frac{\sqrt{\pi}\hbar t \mathrm{e}^{-\frac{mx\lambda}{\hbar^2}}}{4\sqrt{m}} \Bigg[& \operatorname{erf}\left(\frac{\sqrt{m}((i+1)t\lambda+(i-1)\hbar x)}{2\hbar^{\frac{3}{2}}\sqrt{t}}\right) \\
& + \mathrm{e}^{\frac{2mx\lambda}{\hbar^2}}\left(\operatorname{erf}\left(\frac{\sqrt{m}((i+1)t\lambda-(i-1)\hbar x)}{2\hbar^{\frac{3}{2}}\sqrt{t}}\right)+\operatorname{erf}\left(\frac{\sqrt{-i}\sqrt{m}(it\lambda+\hbar x)}{\sqrt{2t}\hbar^{\frac{3}{2}}}\right)-2\right) \\
& + \operatorname{erf}\left(\frac{\sqrt{-im}(it\lambda-\hbar x)}{\sqrt{2t}\hbar^{\frac{3}{2}}}\right)-2\Bigg]\left((i-1)\sin\left(\frac{mt\lambda^2}{2h^3}\right)+(i+1)\cos\left(\frac{mt\lambda^2}{2h^3}\right)\right).
\end{aligned}
$$

■