PHYS 601 - Methods of Theoretical Physics II

Mathematical Methods for Physicists by *Arfken, Weber, Harris*

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Homework 7

Problem 1

In a spinning factory, a worker watches over several hundred spindles. As each spindle turns, the yarn breaks at chance due to irregularities in the tension, evenness of yarn, and etc. For purposes of quality control, it is important to know how frequently breaks occur. Assume that a given worker watches 800 spindles, and that the probability of a break during a given time interval τ is 0.005 for each spindle.

(a) Find the most probable number of breaks during the time interval τ.

(b) Find the probability that no more than 10 breaks will occur during τ. (Use Poisson distribution.)

Solution. (a) Let $n = 800$ and $p = 0.005$, then the most probable number of breaks during the time interval τ is $np = 4$.

(b) Since np is significantly smaller than n, then a Poisson distribution can be used. A Poisson distribution is given by

$$P(m) = \frac{\mu^m \mathrm{e}^{-\mu}}{m!},$$

where $\mu = np = 4$ is the mean.

The probability that no more than 10 breaks will occur is

$$\begin{aligned} P(m \leq 10) &= \sum_{m=0}^{10} P(m) \\ &= \sum_{m=0}^{10} \frac{\mu^m \mathrm{e}^{-\mu}}{m!} \\ &= \sum_{m=0}^{10} \frac{4^m \mathrm{e}^{-4}}{m!} \\ &= \mathrm{e}^{-4} \sum_{m=0}^{10} \frac{4^m}{m!} \\ &= \mathrm{e}^{-4}(54.4431) \\ &= 0.9971. \end{aligned}$$

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Problem 2
For a Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}\frac{(x-\bar{x})^2}{\sigma^2}}$$

centered at $\bar{x}$, compute

(a) the mean value $\langle x \rangle$ and

(b) the variance $\left\langle (x-\bar{x})^2 \right\rangle$.

(c) Show that $\sigma^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2$.

Solution. (a) The expectation value of a Gaussian distributed random variable is

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x p(x) \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (t+\bar{x})\, \mathrm{e}^{-\frac{t^2}{2\sigma^2}} \,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[\int_{-\infty}^{\infty} t \mathrm{e}^{-\frac{t^2}{2\sigma^2}} \,\mathrm{d}t + \bar{x} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{t^2}{2\sigma^2}} \,\mathrm{d}t \right] \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[0 + \bar{x}(\sqrt{2\pi}\sigma) \right] \\
&= \bar{x}.
\end{aligned}$$

(b) The variance of a Gaussian distributed random variable is

$$\begin{aligned}
\left\langle (x-\bar{x})^2 \right\rangle &= \int_{-\infty}^{\infty} (x-\bar{x})^2 p(x) \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\bar{x})^2 \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} t^2 \mathrm{e}^{-\frac{t^2}{2\sigma^2}} \,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[-\sigma^2 t \mathrm{e}^{-\frac{t^2}{2\sigma^2}} \Big|_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{t^2}{2\sigma^2}} \,\mathrm{d}t \right] \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[0 + \sigma^2(\sqrt{2\pi}\sigma) \right] \\
&= \sigma^2.
\end{aligned}$$

(c) We have that

$$\begin{aligned}
\sigma^2 &= \left\langle (x-\bar{x})^2 \right\rangle \\
&= \left\langle x^2 - 2x\bar{x} + \bar{x}^2 \right\rangle \\
&= \left\langle x^2 \right\rangle - 2 \langle x\bar{x} \rangle + \left\langle \bar{x}^2 \right\rangle \\
&= \left\langle x^2 \right\rangle - 2\bar{x} \langle x \rangle + \bar{x}^2 \langle 1 \rangle \\
&= \left\langle x^2 \right\rangle - 2\bar{x}\bar{x} + \bar{x}^2 \\
&= \left\langle x^2 \right\rangle - \bar{x}^2 \\
&= \left\langle x^2 \right\rangle - \langle x \rangle^2 .
\end{aligned}$$

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Problem 3
Starting with the binomial distribution, derive the Gaussian distribution. Make certain to clearly state all approximations that are made.

Solution. The binomial distribution is given by

$$P(m) = \binom{n}{m} p^m (1-p)^{n-m} = \frac{n!}{m!(n-m)!} p^m (1-p)^{n-m}.$$

Considering $n, np \to \infty$, with $m \sim np$, we apply Stirling's approximation on the binomial coefficient, giving us

$$\begin{aligned}
\frac{n!}{m!(n-m)!} &= \frac{\sqrt{2\pi n} n^n e^{-n}}{\sqrt{2\pi m} m^m e^{-m} \sqrt{2\pi(n-m)}(n-m)^{(n-m)} e^{-(n-m)}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{n^{n+\frac{1}{2}}}{m^{m+\frac{1}{2}} (n-m)^{(n-m)+\frac{1}{2}}} \\
&= \frac{1}{\sqrt{2\pi n}} \frac{n^{n+1}}{m^{m+\frac{1}{2}} (n-m)^{(n-m)+\frac{1}{2}}} \\
&= \frac{1}{\sqrt{2\pi n}} \left(\frac{m}{n}\right)^{-m-\frac{1}{2}} \left(1 - \frac{m}{n}\right)^{-n+m-\frac{1}{2}}.
\end{aligned}$$

Plugging the coefficient back into the binomial distribution, we get

$$\begin{aligned}
P(m) &= \frac{1}{\sqrt{2\pi n}} \left(\frac{m}{n}\right)^{-m-\frac{1}{2}} \left(1 - \frac{m}{n}\right)^{-n+m-\frac{1}{2}} p^m (1-p)^{n-m} \\
&= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-\left(m+\frac{1}{2}\right)\ln\left(\frac{m}{n}\right) - \left(n-m+\frac{1}{2}\right)\ln\left(1-\frac{m}{n}\right) + m\ln(p) + (n-m)\ln(1-p)}.
\end{aligned}$$

Let $x = m - np$, then

$$\begin{aligned}
P(x) &= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-\left(x+np+\frac{1}{2}\right)\ln\left(\frac{x}{n}+p\right) - \left(n-x-np+\frac{1}{2}\right)\ln\left(1-\frac{x}{n}-p\right) + (x+np)\ln(p) + (n(1-p)-x)\ln(1-p)} \\
&= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(np+x)\ln\left(1+\frac{x}{np}\right) - \frac{1}{2}\ln\left(p+\frac{x}{n}\right) - (n-np-x)\ln\left(1-\frac{x}{n-np}\right) - \frac{1}{2}\ln\left(1-p-\frac{x}{n}\right)}.
\end{aligned}$$

Now, we have $\frac{x}{n} \ll p$, in addition to performing a Taylor expansion of the logarithmic functions of the form $\ln(1+x)$, where x is small, giving us

$$\begin{aligned}
P(x) &= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(np+x)\ln\left(1+\frac{x}{np}\right) - \frac{1}{2}\ln(p) - (n-np-x)\ln\left(1-\frac{x}{n-np}\right) - \frac{1}{2}\ln(1-p)} \\
&= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(np+x)\ln\left(1+\frac{x}{np}\right) - (n(1-p)-x)\ln\left(1-\frac{x}{n(1-p)}\right) - \frac{1}{2}\ln(p(1-p))} \\
&= \frac{1}{\sqrt{2\pi n}} \frac{1}{\sqrt{p(1-p)}} \mathrm{e}^{-(np+x)\left(\frac{x}{np} - \frac{1}{2}\left(\frac{x}{np}\right)^2 + \cdots\right) - (n(1-p)-x)\left(-\frac{x}{n(1-p)} - \frac{1}{2}\left(\frac{x}{n(1-p)}\right)^2 + \cdots\right)} \\
&= \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-x + \frac{1}{2}\frac{x^2}{np} - \frac{x^2}{np} + O(3) + x + \frac{1}{2}\frac{x^2}{n(1-p)} - \frac{x^2}{n(1-p)} + O(3)} \\
&= \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-\frac{1}{2}\frac{x^2}{np} - \frac{1}{2}\frac{x^2}{n(1-p)} + O(3)} \\
&= \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-\frac{x^2}{2n}\left[\frac{1}{p} + \frac{1}{1-p}\right]} \\
&= \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-\frac{1}{2}\frac{x^2}{np(1-p)}}.
\end{aligned}$$

Letting $\sigma = \sqrt{np(1-p)}$, we get

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{x^2}{2\sigma^2}},$$

which is the Gaussian distribution. ■