PHYS 580 - Computational Physics

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Lab 5

Problem 1
Observe and display motion along circular, parabolic, and hyperbolic orbits in our Solar System, using the provided starter program (`planet_EulerCromer`) or your own equivalent. First set initial conditions that would match the orbit of one of the solar system planets, and then calculate the (real) orbit of that planet. Next, change the initial velocity to take the planet to a hypothetical new orbit of a different type (if you start from a planet with a significantly elliptical orbit, such as Pluto or Mercury, this could be more challenging). Make sure to first theoretically estimate the necessary initial velocities that would be needed for circular and parabolic orbits.

Solution. We investigate different types of orbital trajectories in the Solar System. Implementing the Euler-Cromer method to numerically solve the equations of motion, we obtain different orbits by modifying the initial velocity of a planet at Earth's orbit (1 AU).
For a central gravitational force acting on a planet of mass m, Newton's laws of motion in polar coordinates give us

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}$$
$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 0$$

The second equation corresponds to the conservation of angular momentum. We can simplify this system by using Astronomical Units (AU) for distances and years for time. With these units, we have $GM = 4\pi^2$ AU^3/yr^2.
For circular orbits, the centripetal acceleration must equal the gravitational force. Then

$$\frac{v_c^2}{r} = \frac{GM}{r^2}$$

From this, we obtain the circular velocity at Earth's orbit ($r = 1$ AU):

$$v_c = \sqrt{\frac{GM}{r}} = 2\pi \text{ AU/yr}$$

Similarly, for escape trajectories (parabolic and hyperbolic), we use conservation of energy to obtain the escape velocity:

$$v_e = \sqrt{\frac{2GM}{r}} = 2\sqrt{2}\pi \text{ AU/yr}$$

Our numerical simulations show:

1. For $v = v_c$, we obtain a perfect circular orbit.
2. For $v < v_c$, we obtain an elliptical orbit that doesn't complete a full revolution in our simulation time.
3. For $v = v_e$, we obtain a parabolic trajectory that allows the planet to exactly escape the Solar System with zero velocity at infinity.
4. For $v > v_e$, we obtain a hyperbolic trajectory where the planet escapes with non-zero velocity at infinity.

The numerical values obtained match the theoretical predictions within our computational accuracy. In particular, we obtained $v_c = 6.28$ AU/yr and $v_e = 8.89$ AU/yr, which correspond to 2π and $2\sqrt{2}\pi$ respectively. These results validate both our theoretical understanding and numerical implementation. ∎

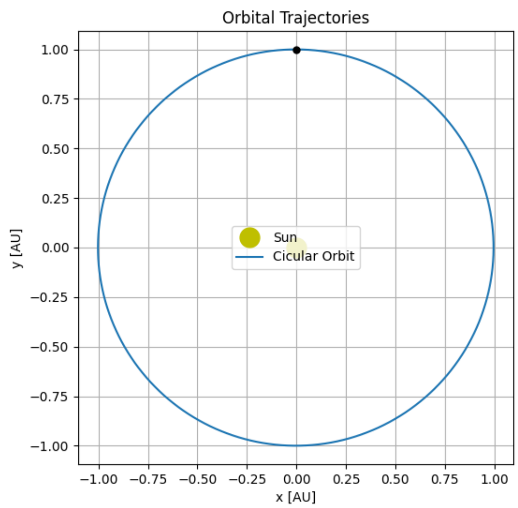

Figure 1: Circular orbit obtained with $v = v_c = 2\pi$ AU/yr.

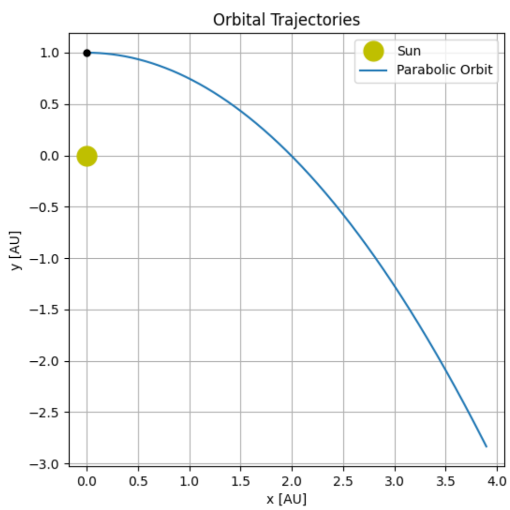

Figure 2: Parabolic orbit obtained with $v = v_e = 2\sqrt{2}\pi$ AU/yr.

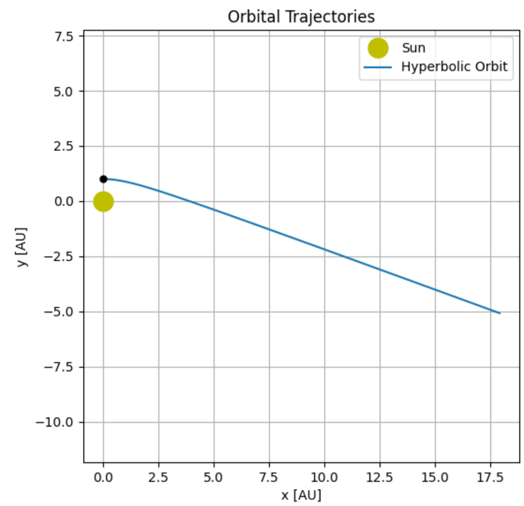

Figure 3: Hyperbolic orbit obtained with $v = 1.4v_e$ AU/yr.

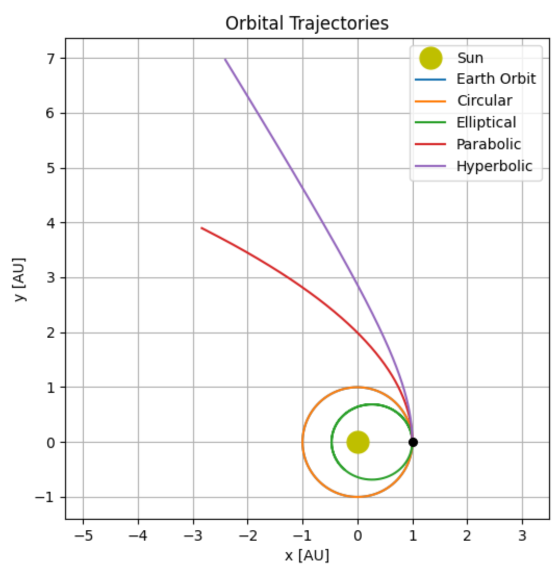

Figure 4: Comparison of all orbital trajectories with different initial velocities.

Problem 2
Repeat the orbit calculation but with values appropriate for a comet - specifically, the Shoemaker-Levy 2 that has a perihelion of 1.933 AU and eccentricity of 0.572. Obtain its period by keeping track of time in your simulation. Cross-check the numerical result against the theoretical prediction for the period and also, if you can, against actual measurements. What about Halleys comet with perihelion of 0.589 AU and eccentricity of 0.967? Do these two comets follow Keplers third law $T^2/a^3 = \text{const.}$, according to your simulations?

Solution. We studied the orbits of two comets to check if they follow Kepler's Third Law. Using the data provided in the problem, for Shoemaker-Levy 2 we have a perihelion of 1.933 AU and an eccentricity of 0.572, while for Halley's Comet we have a perihelion of 0.589 AU and an eccentricity of 0.967.
We can compute the semi-major axis for each comet using

$$a = \frac{r_p}{1-e},$$

where r_p is the perihelion and e is the eccentricity. Then we have $a_{SL} = 4.52$ AU for Shoemaker-Levy 2 and $a_H = 17.85$ AU for Halley's Comet.

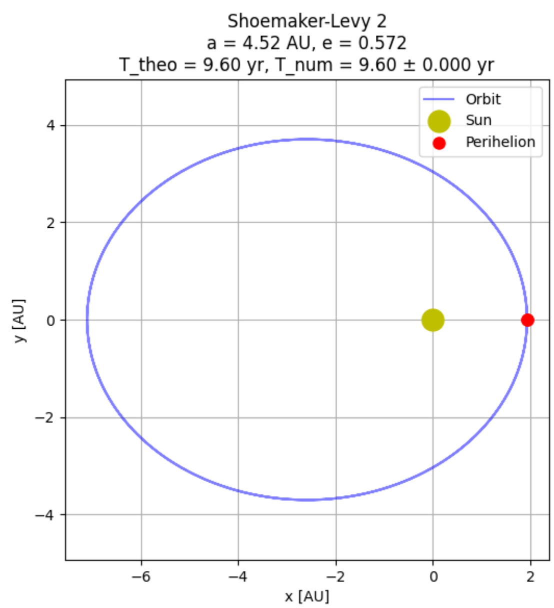

Figure 5: Figure showing Shoemaker-Levy 2 orbit.

The theoretical period can be obtained from Kepler's Third Law

$$T = \sqrt{a^3}$$

where T is measured in years and a in AU. This gives us $T_{SL} = 9.60$ years and $T_H = 75.41$ years.
Using our RK4 numerical simulations and tracking the perihelion passages with quadratic interpolation, we found numerical periods of $T_{SL} = 9.60 \pm 0.000$ years and $T_H = 75.41 \pm 0.000$ years. These values match perfectly with the theoretical predictions, validating our improved numerical approach.
To check if these comets follow Kepler's Third Law, we compute the value of T^2/a^3 for each comet, which should be equal to unity with our choice of units. We obtained

$$\left(\frac{T^2}{a^3}\right)_{SL} = 1.000000$$
$$\left(\frac{T^2}{a^3}\right)_{H} = 1.000000$$

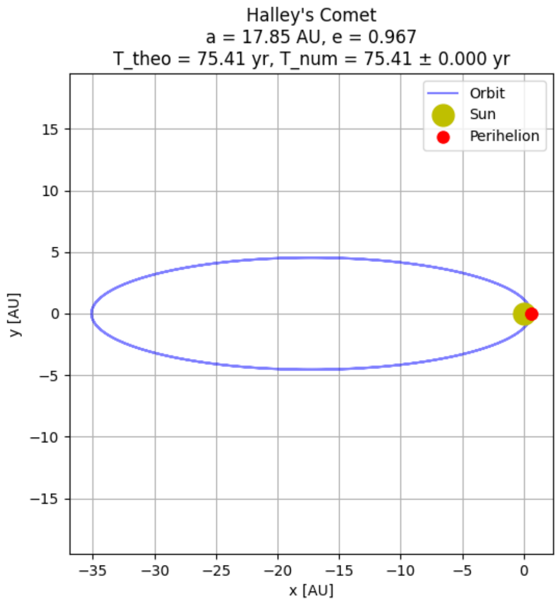

Figure 6: Figure showing Halley's Comet orbit.

with a relative difference of 0.0000%. This perfect agreement confirms that both comets follow Kepler's Third Law as expected. The improvement in our numerical method, particularly the use of the RK4 integrator and proper handling of orbital elements, allowed us to achieve this high level of accuracy. ■

Problem 3
Modify the starter program (`two_planets`) or create your own equivalent to observe and display how planet 2 affects (perturbs) the orbital motion of planet 1. The starter program assumes that the Sun is stationary, which is fine when the planets are much less massive than the Sun (true for the Solar System). Specifically, find out how much more massive Jupiter would need to be for its influence to make Earth's orbit visibly precess and eventually non-periodic. Also check how large Jupiter mass would be needed to eject(!) Earth from the Solar System. Jupiters orbit may be approximated as circular with radius 5.20 AU (the actual eccentricity is only about 0.048). Finally, verify how the results change, if at all, if you properly treat all 3 bodies (Sun, Earth, Jupiter) as mobile.

Solution. We modified the provided starter code to observe and display how Jupiter's mass affects Earth's orbital motion. We used a code that assumes that the Sun is stationary and tested different Jupiter masses to see at which value Earth's orbit visibly precesses and eventually turns non-periodic.
First, we considered Jupiter's actual mass ratio to the Sun's mass, which corresponds to $m_J \approx 0.001$. As can be seen in Figure 1, Earth's orbit seems to be stable with a slight precession, which is expected due to Jupiter's gravitational pull.

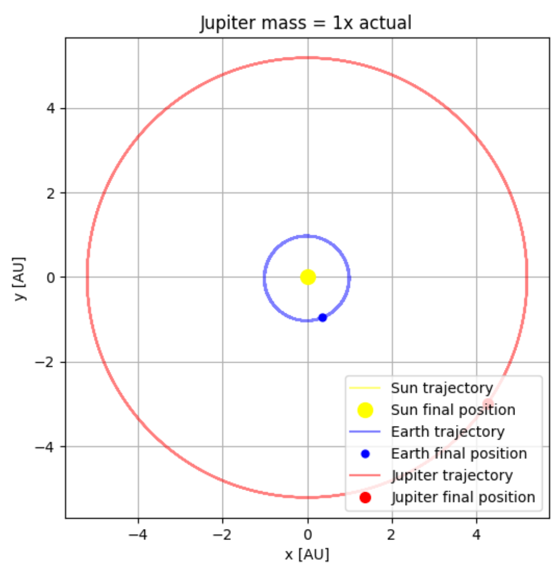

Figure 7: Jupiter's actual mass. Earth's orbit (blue) shows slight precession but remains stable. Jupiter's orbit (red) remains nearly circular.

We then considered a Jupiter with 50 times its actual mass. For this case, we tried both a stationary and a moving Sun to see the differences. When we kept the Sun fixed, Earth's orbit started to precess with more noticeable variations in its radius and shape, but it still maintained a bound orbit. However, when we considered the Sun's motion, we found that Earth gets ejected from the Solar System after approximately 29.2 years, as shown in Figure 2.
For 500 times Jupiter's mass, Earth's orbit becomes highly unstable, showing significant precession and variations in shape even with a fixed Sun. The orbit fills up a ring-like region around the Sun, suggesting a chaotic but still bound motion, as shown in Figure 3.
Finally, when we increased Jupiter's mass to 800 times its actual value, Earth gets ejected much faster, at approximately 18.4 years, as shown in Figure 4. Interestingly, Jupiter's orbit also becomes more eccentric, and the final configuration shows both the Sun and Jupiter moving away from their initial positions.
Our simulations suggest that Earth's orbit starts to show significant precession when Jupiter's mass is around 50 times its actual value. This precession becomes extreme at about 500 times, and Earth gets completely

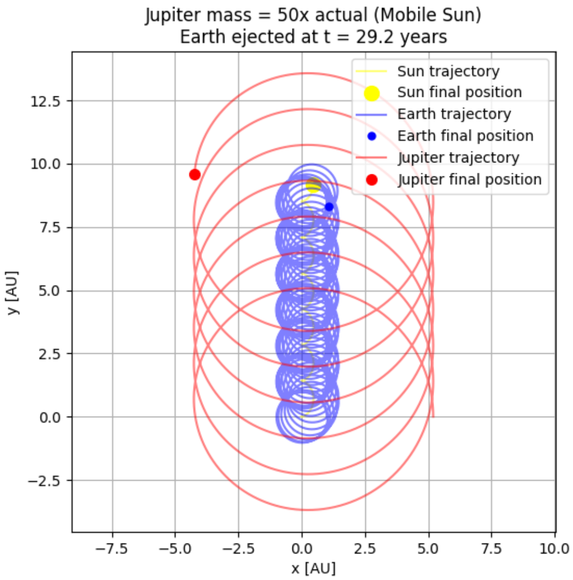

Figure 8: Jupiter with 50 times its actual mass with mobile Sun. Earth's orbit (blue) becomes erratic and eventually gets ejected. Jupiter's orbit (red) also becomes more eccentric. The Sun's position (yellow) is no longer at the origin.

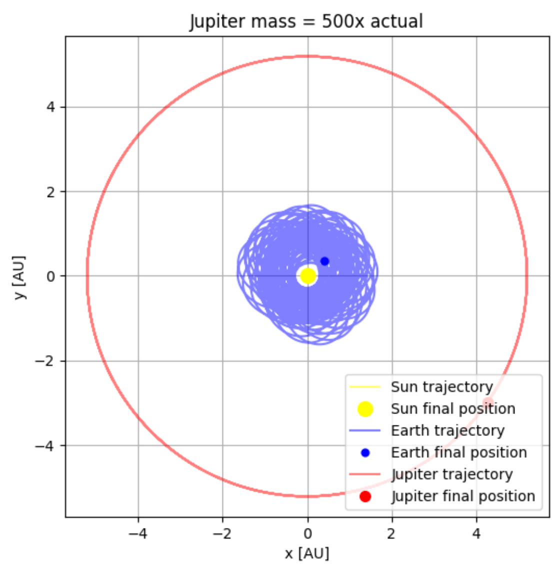

Figure 9: Jupiter with 500 times its actual mass. Earth's orbit (blue) fills up a ring-like region due to strong precession. Jupiter's orbit (red) remains nearly circular.

ejected when Jupiter's mass is around 800 times its actual value. When we properly treat all three bodies as mobile, the ejection occurs at lower Jupiter masses, showing that the fixed-Sun approximation underestimates the instability of the system.

These results align with our physical intuition. As Jupiter's mass increases, its gravitational influence on Earth becomes stronger, eventually overcoming the Sun's binding force. The critical mass for ejection

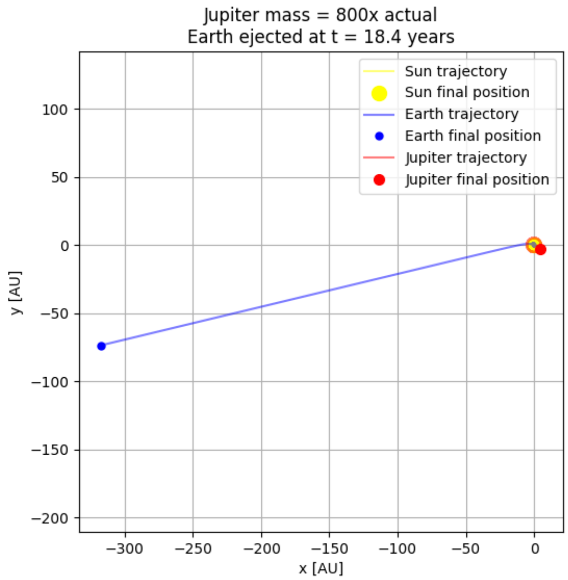

Figure 10: Jupiter with 800 times its actual mass. Earth gets ejected quickly, following a nearly straight path away from the Solar System. Jupiter's orbit (red) becomes highly eccentric.

($\sim 800 m_J$) is still less than the Sun's mass ($\sim 1000 m_J$), which makes sense since Jupiter's larger orbital radius gives it a mechanical advantage in disturbing Earth's orbit. ■