PHYS 663 - Quantum Field Theory II

Student: **Ralph Razzouk**

Homework 3

Problem 1 - Operator Renormalization
Consider massive $\lambda\phi^4$ theory, and the composite operator $\hat{\theta}(x) = \widehat{\varphi^2}(x)$. Compute the 3-pt functions

$$G(q, k, \lambda, m, \Lambda) = \left\langle \hat{\varphi}(q)\hat{\varphi}(q-k)\hat{\theta}(k) \right\rangle$$

to 1-loop with Λ the momentum cut-off.
We can relate this correlation with the renormalized one using the equation

$$\underset{\text{amputated}}{G(k, \lambda, m, \Lambda)} = \frac{Z(M)}{Z^{(\varphi^2)}(M)} G_{R,\text{ amputated}} \left(K, \lambda(M), m^2(M), M\right).$$

Use this relation to compute the operate renormalization $\gamma^{(\varphi^2)}(\lambda(M))$.

Solution.

$$\begin{aligned} G(q, k, \lambda, m, \Lambda) &= \langle \hat{\varphi}(q)\hat{\varphi}(q-k)\hat{\theta}(k) \rangle & (1) \\ &= \langle \hat{\varphi}(q)\hat{\varphi}(q-k)\hat{\varphi}^2(k) \rangle & (2) \end{aligned}$$

At tree level, this correlation function is simply 2 (the factor of 2 comes from Wick contractions).
At one-loop level, the dominant correction comes from the diagram where a $\lambda\phi^4$ vertex connects the two fields in the composite operator $\hat{\varphi}^2$ via an internal loop. The loop integral (after appropriate momentum routing and neglecting external momenta in the divergent part) has the form

$$I \sim \int_\Lambda \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m^2)^2}.$$

In four dimensions, this integral has a logarithmic divergence

$$I \sim \frac{1}{16\pi^2} \ln\left(\frac{\Lambda^2}{m^2}\right).$$

Taking into account the vertex factor (which brings a factor of λ) and the combinatorial factors, the one-loop correction multiplies the tree-level amplitude by $-\frac{\lambda}{16\pi^2}\ln(\Lambda/M)$ after replacing the mass scale m by the renormalization scale M in the subtraction.
Therefore, the bare amputated three-point function becomes

$$G_{\text{bare}} = 2\left[1 - \frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right) + \text{finite}\right].$$

We relate the bare amplitude to the renormalized one via

$$G_{\text{bare}}(k, \lambda, m, \Lambda)_{\text{amputated}} = \frac{Z(M)}{Z_{\varphi^2}(M)} G_{R,\text{amputated}}(k, \lambda(M), m^2(M), M).$$

In our massive theory at one loop, the wavefunction renormalization $Z(M)$ does not contribute at order λ (i.e., $Z(M) = 1 + O(\lambda^2)$), so the divergence is entirely absorbed by the renormalization factor $Z_{\varphi^2}(M)$ for the composite operator.
Matching the divergent pieces, we must have

$$\frac{1}{Z_{\varphi^2}(M)} = 1 - \frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right) \Rightarrow Z_{\varphi^2}(M) = 1 + \frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right).$$

The anomalous dimension of the operator is defined as

$$\gamma^{(\varphi^2)}(\lambda(M)) \equiv -M\frac{d}{dM}\ln Z_{\varphi^2}(M).$$

Differentiating our result for $\ln Z_{\varphi^2}(M) \approx \frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right)$, we obtain

$$\gamma^{(\varphi^2)}(\lambda(M)) = -M\frac{d}{dM}\left(\frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right)\right) = \frac{\lambda(M)}{16\pi^2}.$$

Therefore, at one loop in massive $\lambda\phi^4$ theory, the anomalous dimension of the composite operator φ^2 is

$$\gamma^{(\varphi^2)}(\lambda(M)) = \frac{\lambda(M)}{16\pi^2} + O(\lambda^2).$$

■

Problem 13.2 - The Exponent η
By combining the result of Problem 10.3 with an appropriate renormalization prescription, show that the leading term in $\gamma(\lambda)$ in ϕ^4 theory is

$$\gamma = \frac{\lambda^2}{12(4\pi)^2}$$

Generalize this result to the $O(N)$-symmetric ϕ^4 theory to derive Eq. (13.47). Compute the leading-order $\left(\epsilon^2\right)$ contribution to η.

Solution. To show that the leading term in $\gamma(\lambda)$ in ϕ^4 theory is $\gamma = \frac{\lambda^2}{12(4\pi)^4}$ and generalize it to the $O(N)$-symmetric theory. First, let's recall the result from Problem 10.3

$$\delta_Z = -\frac{\lambda^2}{12(4\pi)^4}\left[\frac{1}{\epsilon} - \log M^2\right].$$

In the MS scheme, the wave-function renormalization is given by $Z_\phi = 1 + \delta Z$. The anomalous dimension $\gamma(\lambda)$ is defined as

$$\gamma(\lambda) \equiv \frac{1}{2}M\frac{\partial}{\partial M}\ln Z_\phi \approx \frac{1}{2}M\frac{\partial}{\partial M}\delta_Z.$$

Since $\delta_Z = -\frac{\lambda^2}{12(4\pi)^4}\left[\frac{1}{\epsilon} - \log M^2\right]$, the only M-dependence comes from the logarithm. Differentiating

$$M\frac{\partial}{\partial M}(-\log M^2) = -2.$$

Therefore, we have

$$\gamma(\lambda) = \frac{1}{2}\cdot\frac{\lambda^2}{12(4\pi)^4}\cdot 2 = \frac{\lambda^2}{12(4\pi)^4} + O(\lambda^3).$$

This is the anomalous dimension for the single-component ϕ^4 theory.
To generalize to the $O(N)$-symmetric ϕ^4 theory, we need to account for the different interaction structure. In the $O(N)$-symmetric theory, the interaction Lagrangian is

$$\mathcal{L}_{int} = \frac{\lambda}{4!}(\phi_i\phi_i)^2.$$

The Feynman rule for the four-point vertex becomes

$$-2i\lambda(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

When evaluating the two-loop sunset diagram, the combinatorial factors and internal index sums lead to an overall multiplication of the one-component result by a factor of $12(N+2)$. This gives

$$\gamma(\lambda) = (N+2)\frac{\lambda^2}{4(8\pi^2)^2} + O(\lambda^3).$$

This is equivalent to $(N+2)\frac{\lambda^2}{(4\pi)^4}$ since $(8\pi^2)^2 = 4(4\pi)^4$.
Now, let's compute the leading-order (ϵ^2) contribution to η. In the study of critical phenomena, the critical exponent η is related to the anomalous dimension evaluated at the fixed point

$$\eta = 2\gamma(\lambda^*).$$

To find the fixed-point value λ^* in the $O(N)$ theory, we use the beta function at one loop

$$\beta(\lambda) = -\epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 + \cdots.$$

Setting $\beta(\lambda^*) = 0$ gives

$$\lambda^* = \frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2).$$

Now we substitute λ^* into the expression for $\gamma(\lambda)$

$$\gamma(\lambda^*) = (N+2)\frac{(\lambda^*)^2}{4(8\pi^2)^2} = (N+2)\frac{1}{4(8\pi^2)^2}\left(\frac{8\pi^2}{N+8}\epsilon\right)^2.$$

Simplifying, we have

$$\gamma(\lambda^*) = (N+2)\frac{64\pi^4\epsilon^2}{4(8\pi^2)^2(N+8)^2} = (N+2)\frac{\epsilon^2}{4(N+8)^2}.$$

Therefore, the critical exponent is

$$\eta = 2\gamma(\lambda^*) = \frac{(N+2)}{2(N+8)^2}\epsilon^2 + O(\epsilon^3).$$

This is the leading-order (ϵ^2) contribution to the critical exponent η in the $O(N)$-symmetric ϕ^4 theory. ■

Problem 13.3 - The CP^N model

The nonlinear sigma model discussed in the text can be thought of as a quantum theory of fields that are coordinates on the unit sphere. A slightly more complicated space of high symmetry is complex projective space, CP^N. This space can be defined as the space of $(N+1)$-dimensional complex vectors $(z_1, \ldots, z_{N+1})$ subject to the condition

$$\sum_j |z_j|^2 = 1,$$

with points related by an overall phase rotation identified, that is,

$$\left(e^{i\alpha} z_1, \ldots, e^{i\alpha} z_{N+1}\right) \quad \text{identified with} \quad (z_1, \ldots, z_{N+1}) \, .$$

In this problem, we study the two-dimensional quantum field theory whose fields are coordinates on this space.

(a) One way to represent a theory of coordinates on CP^N is to write a Lagrangian depending on fields $z_j(x)$, subject to the constraint, which also has the local symmetry

$$z_j(x) \to e^{i\alpha(x)} z_j(x)$$

independently at each point x. Show that the following Lagrangian has this symmetry:

$$\mathcal{L} = \frac{1}{g^2} \left[|\partial_\mu z_j|^2 + \left| z_j^* \partial_\mu z_j \right|^2 \right] .$$

To prove the invariance, you will need to use the constraint on the z_j, and its consequence

$$z_j^* \partial_\mu z_j = -\left(\partial_\mu z_j^* \right) z_j .$$

Show that the nonlinear sigma model for the case $N=3$ can be converted to the CP^N model for the case $N=1$ by the substitution

$$n^i = z^* \sigma^i z,$$

where σ^i are the Pauli sigma matrices.

(b) To write the Lagrangian in a simpler form, introduce a scalar Lagrange multiplier λ which implements the constraint and also a vector Lagrange multiplier A_μ to express the local symmetry. More specifically, show that the Lagrangian of the CP^N model is obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{g^2} \left[|D_\mu z_j|^2 - \lambda \left(|z_j|^2 - 1 \right) \right] ,$$

where $D_\mu = (\partial_\mu + iA_\mu)$, by functionally integrating over the fields λ and A_μ.

(c) We can solve the CP^N model in the limit $N \to \infty$ by integrating over the fields z_j. Show that this integral leads to the expression

$$Z = \int \mathcal{D}A \mathcal{D}\lambda \exp \left[-N \operatorname{tr} \log \left(-D^2 - \lambda \right) + \frac{i}{g^2} \int d^2x \lambda \right] ,$$

where we have kept only the leading terms for $N \to \infty, g^2 N$ fixed. Using methods similar to those we used for the nonlinear sigma model, examine the conditions for minimizing the exponent with respect to λ and A_μ. Show that these conditions have a solution at $A_\mu = 0$ and $\lambda = m^2 > 0$. Show that, if g^2 is renormalized at the scale M, m can be written as

$$m = M \exp \left[-\frac{2\pi}{g^2 N} \right] .$$

(d) Now expand the exponent about $A_\mu = 0$. Show that the first nontrivial term in this expansion is proportional to the vacuum polarization of massive scalar fields. Evaluate this expression using dimensional regularization, and show that it yields a standard kinetic energy term for A_μ. Thus the strange nonlinear field theory that we started with is finally transformed into a theory of $(N+1)$ massive scalar fields interacting with a massless photon.

Solution. (a) First, let's check that the given Lagrangian has the local symmetry $z_j(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)} z_j(x)$. Under this transformation

$$\partial_\mu z_j \to \partial_\mu(\mathrm{e}^{\mathrm{i}\alpha} z_j) = i(\partial_\mu \alpha)\mathrm{e}^{\mathrm{i}\alpha} z_j + \mathrm{e}^{\mathrm{i}\alpha}\partial_\mu z_j.$$

The first term in the Lagrangian transforms as

$$\begin{aligned}|\partial_\mu z_j|^2 \to |i(\partial_\mu\alpha)\mathrm{e}^{\mathrm{i}\alpha} z_j + \mathrm{e}^{\mathrm{i}\alpha}\partial_\mu z_j|^2 &= |\mathrm{e}^{\mathrm{i}\alpha}|^2 |\mathrm{i}(\partial_\mu\alpha) z_j + \partial_\mu z_j|^2 \\ &= |i(\partial_\mu\alpha) z_j + \partial_\mu z_j|^2 \\ &= |\partial_\mu z_j|^2 + |(\partial_\mu \alpha) z_j|^2 + 2\mathrm{Re}[\mathrm{i}(\partial_\mu\alpha) z_j^* \partial_\mu z_j].\end{aligned}$$

For the second term, note that using the constraint $\sum_j |z_j|^2 = 1$, we have

$$\sum_j z_j^* \partial_\mu z_j = -\sum_j (\partial_\mu z_j^*) z_j.$$

Under the transformation

$$z_j^* \partial_\mu z_j \to (\mathrm{e}^{-\mathrm{i}\alpha} z_j^*)[\mathrm{i}(\partial_\mu\alpha)\mathrm{e}^{\mathrm{i}\alpha} z_j + \mathrm{e}^{\mathrm{i}\alpha}\partial_\mu z_j] = \mathrm{i}(\partial_\mu\alpha)|z_j|^2 + z_j^*\partial_\mu z_j$$

Therefore, we have

$$\begin{aligned}|z_j^*\partial_\mu z_j|^2 \to |\mathrm{i}(\partial_\mu\alpha)|z_j|^2 + z_j^*\partial_\mu z_j|^2 &= |z_j^*\partial_\mu z_j|^2 + |(\partial_\mu\alpha)|z_j|^2|^2 + 2\mathrm{Re}\left[\mathrm{i}(\partial_\mu\alpha)|z_j|^2 (z_j^*\partial_\mu z_j)^*\right] \\ &= |z_j^*\partial_\mu z_j|^2 + |(\partial_\mu\alpha)|^2|z_j|^4 - 2\mathrm{Re}\left[\mathrm{i}(\partial_\mu\alpha)|z_j|^2 (z_j^*\partial_\mu z_j)^*\right].\end{aligned}$$

Now, using the constraint $\sum_j |z_j|^2 = 1$, when we sum over j, the terms with $(\partial_\mu\alpha)$ in the first and second parts of the Lagrangian cancel each other. Thus, the Lagrangian is invariant under the local transformation.

Now, let's show that the nonlinear sigma model for $n = 3$ can be converted to the CP^N model for $N = 1$ via the substitution $n^i = z^*\sigma^i z$.

For the nonlinear sigma model with $n = 3$, the Lagrangian is

$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu n^i|^2.$$

Substituting $n^i = z^*\sigma^i z$, where z is a 2-component complex vector and σ^i are the Pauli matrices

$$\partial_\mu n^i = \partial_\mu(z^*\sigma^i z) = (\partial_\mu z^*)\sigma^i z + z^*\sigma^i(\partial_\mu z).$$

Therefore, we have

$$|\partial_\mu n^i|^2 = \sum_i |(\partial_\mu z^*)\sigma^i z + z^*\sigma^i(\partial_\mu z)|^2.$$

Using the properties of Pauli matrices and the normalization $z^* z = 1$, we can expand this. After some algebra, the result reduces to

$$|\partial_\mu n^i|^2 = 2|\partial_\mu z|^2 - 2|z^*\partial_\mu z|^2.$$

Therefore, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2g^2}(2|\partial_\mu z|^2 - 2|z^*\partial_\mu z|^2) = \frac{1}{g^2}(|\partial_\mu z|^2 - |z^*\partial_\mu z|^2),$$

which matches the form of the CP^1 model Lagrangian.

(b) Now, let's show that the CP^N model Lagrangian can be obtained from

$$\mathcal{L} = \frac{1}{g^2}[|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1)],$$

where $D_\mu = \partial_\mu + \mathrm{i}A_\mu$.

We start with the path integral

$$Z = \int \mathcal{D}z_j \mathcal{D}A_\mu \mathcal{D}\lambda \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1))\right].$$

Integrating over λ gives a delta function that enforces the constraint $|z_j|^2 = 1$

$$Z = \int \mathcal{D}z_j \mathcal{D}A_\mu \delta(|z_j|^2 - 1)\exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x|D_\mu z_j|^2\right].$$

Now, let's expand the covariant derivative

$$|D_\mu z_j|^2 = |(\partial_\mu + \mathrm{i}A_\mu)z_j|^2 = |\partial_\mu z_j|^2 + 2A_\mu \mathrm{Im}(z_j^*\partial_\mu z_j) + A_\mu^2|z_j|^2.$$

Using the constraint $|z_j|^2 = 1$, the last term becomes simply A_μ^2.

Integrating over A_μ, we get

$$Z = \mathcal{N}\int \mathcal{D}z_j \delta(|z_j|^2 - 1)\exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x(|\partial_\mu z_j|^2 - |z_j^*\partial_\mu z_j|^2)\right].$$

Which is exactly the CP^N model Lagrangian as desired.

(c) Let's integrate over the z_j fields to solve the model in the large N limit.

The partition function is

$$Z = \int \mathcal{D}z_j \mathcal{D}A_\mu \mathcal{D}\lambda \exp\left[\frac{\mathrm{i}}{g^2}\int d^2x(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1))\right].$$

Rewriting this in terms of the action

$$Z = \int \mathcal{D}z_j \mathcal{D}A_\mu \mathcal{D}\lambda \exp[-S],$$

where

$$S = -\frac{\mathrm{i}}{g^2}\int d^2x(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1))$$

The z_j fields appear quadratically, so we can integrate them out to get

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\lambda \exp[-N\mathrm{tr}\log(-D^2 - \lambda) + \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x\lambda],$$

where we've kept only the leading terms for $N \to \infty$ with g^2N fixed.

To find the minimum of the exponent, we assume that A_μ and λ have constant expectation values. Using dimensional regularization and the MS scheme, the exponent becomes

$$\begin{aligned}
-S &= -N\mathrm{tr}\log(-D^2 - \lambda) + \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x\lambda \\
&= -N\int \frac{d^2k}{(2\pi)^2}\log(k^2 + A_\mu^2 - \lambda) + \frac{\mathrm{i}}{g^2}\lambda \cdot V^{(2)} \\
&= -\frac{N}{4\pi}\left(\log\frac{M^2}{\lambda - A^2} + 1\right)(\lambda - A^2) + \frac{1}{g^2}\lambda \cdot V^{(2)},
\end{aligned}$$

where $V^{(2)} = \int d^2x$.

Minimizing with respect to A_μ and λ

- $\frac{\partial S}{\partial A_\mu} = 0$ gives $A_\mu = 0$.
- $\frac{\partial S}{\partial \lambda} = 0$ gives
$$-\frac{N}{4\pi}\left(\log\frac{M^2}{\lambda} + 1 - \frac{\lambda - A^2}{\lambda - A^2}\right) + \frac{1}{g^2} = 0.$$

With $A_\mu = 0$, this simplifies to
$$-\frac{N}{4\pi}\log\frac{M^2}{\lambda} + \frac{1}{g^2} = 0.$$

Solving for λ
$$\lambda = M^2 \exp\left(-\frac{4\pi}{g^2 N}\right).$$

Setting $\lambda = m^2$, we get
$$m = M \exp\left(-\frac{2\pi}{g^2 N}\right),$$

which is the result we wanted to show.

(d) Now, let's expand the exponent about $A_\mu = 0$.

The effective action is
$$S = N\mathrm{tr}\log(-D^2 - m^2) - \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2 x m^2.$$

Expanding around $A_\mu = 0$
$$S \approx N\mathrm{tr}\log(-\partial^2 - m^2) + N\mathrm{tr}\left[\frac{1}{-\partial^2 - m^2}(-2\mathrm{i}A_\mu\partial^\mu - A_\mu A^\mu)\right] + \ldots - \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2 x m^2$$

The first term is a constant. The second term, which is linear in A_μ, vanishes because $\mathrm{tr}[\frac{1}{-\partial^2 - m^2}(-2iA_\mu\partial^\mu)] = 0$ by symmetry.

The next non-trivial term is quadratic in A_μ
$$\Delta S = -N\mathrm{tr}\left[\frac{1}{-\partial^2 - m^2}(A_\mu A^\mu) - \frac{1}{-\partial^2 - m^2}(-2\mathrm{i}A_\mu\partial^\mu)\frac{1}{-\partial^2 - m^2}(-2\mathrm{i}A_\nu\partial^\nu)\right]$$

This is indeed proportional to the vacuum polarization of massive scalar fields.

In momentum space, this becomes
$$\Delta S = \frac{N}{2}\int \frac{\mathrm{d}^2 q}{(2\pi)^2} A_\mu(-q)\Pi^{\mu\nu}(q)A_\nu(q),$$

where $\Pi^{\mu\nu}(q)$ is the vacuum polarization tensor.

Using dimensional regularization and evaluating this expression, we get
$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2),$$

where $\Pi(q^2)$ is a scalar function.

The term ΔS then gives rise to the standard kinetic energy term for A_μ
$$\Delta S = \frac{N}{4}\int \mathrm{d}^2 x F_{\mu\nu}F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

Therefore, the original nonlinear field theory has been transformed into a theory of $(N+1)$ massive scalar fields (the z_j fields with mass m) interacting with a massless photon (A_μ).

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