PHYS 660 - Quantum Mechanics I

Modern Quantum Mechanics by *J. J. Sakurai*

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Homework 1

Problem 1

Prove that

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB$$

Proof. Left hand-side:

$$[AB, CD] = ABCD - CDAB \tag{1}$$

Right hand-side:

$$\begin{aligned}
-AC\{D, B\} &+ A\{C, B\}D - C\{D, A\}B + \{C, A\}DB \\
&= -AC(DB + BD) + A(CB + BC)D - C(DA + AD)B + (CA + AC)DB \\
&= -ACDB - ACBD + ACBD + ABCD - CDAB - CADB + CADB + ACDB \\
&= ABCD - CDAB
\end{aligned}$$

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Problem 2

Consider a Hermitian operator A, (i.e. $A = A^\dagger$). Let $\{|a_i\rangle, i = 1, \ldots, N\}$ be a basis of eigenstates $|a_i\rangle$ of A, with eigenvalues a_i. Assume for simplicity that there is no degeneracy, namely all the a_i are different.

(a) Prove that

$$\prod_{i=1}^{N}(A - a_i) = 0$$

(b) For a given value of i, consider the operator

$$P_i = \prod_{j=1, j\neq i}^{N}\left(\frac{A - a_j}{a_i - a_j}\right)$$

What does P_i do when applied to an arbitrary state?

(c) Illustrate points **(a)** and **(b)** by using the operator S_z of a spin 1/2 system.

(d) Discuss how to modify the formulas if there is a degeneracy in the spectrum of A.

Proof. (a) Since the product is over a complete set, the operator $\prod_{i=1}^{N}(A - a_i) = 0$ will always encounter an element $|a_j\rangle$ such that $a_i = a_j$ in which case the result is zero. Thus, for any state $|\psi\rangle$, we have

$$\begin{aligned}
\prod_{i=1}^{N}(A - a_i)|\psi\rangle &= \prod_{i=1}^{N}(A - a_i)\sum_{j=1}^{N}|a_j\rangle\langle a_j|\psi\rangle \\
&= \sum_{j=1}^{N}\prod_{i=1}^{N}(a_j - a_i)|a_j\rangle\langle a_j|\psi\rangle \\
&= \sum_{j=1}^{N} 0 \\
&= 0
\end{aligned}$$

(b) If the product instead is over all $a_i \neq a_j$, then the only surviving term in the sum is

$$\prod_{i=1}^{N}(a_j - a_i)\left|a_j\right\rangle\left\langle a_j|\psi\right\rangle$$

and dividing by the factors $(a_j - a_i)$ just gives the projection of $|\psi\rangle$ on the direction $|a_i\rangle$. Therefore, it is like a projection operator which projects the $|a_i\rangle$ component of $|\psi\rangle$.

(c) For the operator $A = S_z$ and $|a_i\rangle = \{|+\rangle, |-\rangle\}$, we have

$$\prod_{i=1}^{N}(A - a_i) = (S_z - |+\rangle)(S_z - |-\rangle) = \left(S_z - \frac{\hbar}{2}\right)\left(S_z + \frac{\hbar}{2}\right)$$

$$\prod_{j=1, j\neq i}^{N}\left(\frac{A - a_j}{a_i - a_j}\right) = \begin{cases} \left(\frac{S_z - |+\rangle}{|-\rangle - |+\rangle}\right) = \left(\frac{S_z - \hbar/2}{-\hbar}\right), & \text{for } a_j = |+\rangle \\ \left(\frac{S_z - |-\rangle}{|+\rangle - |-\rangle}\right) = \left(\frac{S_z + \hbar/2}{\hbar}\right), & \text{for } a_j = |-\rangle\,. \end{cases}$$

For the first equation, it is easy to see we get $S_z^2 - \frac{\hbar^2}{4} = 0$. For the second equation, we can work them out explicitly

$$\left(\frac{S_z - \hbar/2}{-\hbar}\right) = -\frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \mathbb{I}\right] = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \text{ projection on } |-\rangle$$

$$\left(\frac{S_z + \hbar/2}{\hbar}\right) = \frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \mathbb{I}\right] = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \text{ projection on } |+\rangle$$

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Problem 3
Consider the following Hamiltonian of a two-state system

$$H = E(|1\rangle\langle 1| - |2\rangle\langle 2|) + \Delta(|1\rangle\langle 2| + |2\rangle\langle 1|)$$

where E, Δ have dimension of energy. Find the energy eigenvalues and the corresponding eigenstates as linear combinations of $|1\rangle, |2\rangle$.

Proof. We find the following inner products

$$\langle 1|H|1\rangle = \langle 1|E|1\rangle\langle 1|1\rangle = E$$

$$\langle 1|H|2\rangle = \langle 1|\Delta|1\rangle\langle 2|2\rangle = \Delta$$

$$\langle 2|H|1\rangle = \langle 2|\Delta|2\rangle\langle 1|1\rangle = \Delta$$

$$\langle 2|H|2\rangle = -\langle 2|E|2\rangle\langle 2|2\rangle = -E$$

H can be represented in matrix form as

$$H = \begin{pmatrix}E & \Delta\\ \Delta & -E\end{pmatrix}$$

The eigenvalues are then the solution to the following equation

$$\det(H - \lambda\mathbb{I}) = 0$$

$$\begin{vmatrix}E - \lambda & \Delta\\ \Delta & -E - \lambda\end{vmatrix} = (E - \lambda)(-E - \lambda) - \Delta^2 = \lambda^2 - E^2 - \Delta^2 = 0$$

$$\lambda_\pm = \pm\sqrt{E^2 + \Delta^2}$$

For the corresponding eigenvectors, we have

- For $\lambda_+ = \sqrt{E^2+\Delta^2}$:

$$\begin{cases} (E-\sqrt{E^2+\Delta^2})\alpha + \Delta\beta &= 0 \\ \Delta\alpha + (-E-\sqrt{E^2+\Delta^2})\beta &= 0 \end{cases}$$

$$\implies \beta = \frac{\sqrt{E^2+\Delta^2}-E}{\Delta}\alpha = \left(\sqrt{\left(\frac{E}{\Delta}\right)^2+1}-\frac{E}{\Delta}\right)\alpha$$

Thus, the normalized eigenvector $|+\rangle$ corresponding to λ_+ is

$$|+\rangle = \frac{1}{\sqrt{\left(\sqrt{\left(\frac{E}{\Delta}\right)^2+1}\right)^2 + \left(\frac{E}{\Delta}\right)^2 - 2\frac{E}{\Delta}\sqrt{\left(\frac{E}{\Delta}\right)^2+1}}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$= \frac{1}{\sqrt{2\left(\frac{E}{\Delta}\right)^2+1-\frac{2E}{\Delta}\sqrt{\left(\frac{E}{\Delta}\right)^2+1}}}\begin{pmatrix}1\\ \frac{\sqrt{E^2+\Delta^2}-E}{\Delta}\end{pmatrix}$$

- For $\lambda_- = -\sqrt{E^2+\Delta^2}$:

$$\begin{cases} (E+\sqrt{E^2+\Delta^2})\alpha + \Delta\beta &= 0 \\ \Delta\alpha + (-E+\sqrt{E^2+\Delta^2})\beta &= 0 \end{cases}$$

$$\implies \beta = \frac{-\sqrt{E^2+\Delta^2}-E}{\Delta}\alpha = \left(-\sqrt{\left(\frac{E}{\Delta}\right)^2+1}-\frac{E}{\Delta}\right)\alpha$$

Thus, the normalized eigenvector $|-\rangle$ corresponding to λ_- is

$$|-\rangle = \frac{1}{\sqrt{\left(\sqrt{\left(\frac{E}{\Delta}\right)^2+1}\right)^2 + \left(\frac{E}{\Delta}\right)^2 + 2\frac{E}{\Delta}\sqrt{\left(\frac{E}{\Delta}\right)^2+1}}}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$= \frac{1}{\sqrt{2\left(\frac{E}{\Delta}\right)^2+1+\frac{2E}{\Delta}\sqrt{\left(\frac{E}{\Delta}\right)^2+1}}}\begin{pmatrix}1\\ \frac{-\sqrt{E^2+\Delta^2}-E}{\Delta}\end{pmatrix}$$

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Problem 4
Consider the following Hamiltonian of a three-state system

$$H = \frac{\epsilon}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$$

where ϵ has dimension of energy. Find the energy eigenvalues and the corresponding eigenstates.

Proof. Finding the eigenvalues of the given matrix

$$\det(H-\lambda\mathbb{I}) = 0$$

$$\implies \begin{pmatrix}-\lambda&1&0\\1&-\lambda&1\\0&1&-\lambda\end{pmatrix} = -\lambda(\lambda^2-1)+\lambda = \lambda(2-\lambda^2) = 0$$

$$\implies \lambda_1 = -\sqrt{2},\ \lambda_2 = 0,\ \lambda_3 = \sqrt{2}.$$

Finding the eigenvectors

- For $\lambda_1 = -\sqrt{2}$:

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 1 \\ 0 & 1 & \sqrt{2} \end{pmatrix} |\lambda_1\rangle = 0 \implies |\lambda_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

- For $\lambda_2 = 0$:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} |\lambda_2\rangle = 0 \implies |\lambda_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

- For $\lambda_3 = \sqrt{2}$:

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix} |\lambda_3\rangle = 0 \implies |\lambda_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

Finally, the corresponding eigenstates of the Hamiltonian of our three-state system are

$$E_1 = -\epsilon,\ |E_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$E_2 = 0,\ |E_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$E_3 = \epsilon,\ |E_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}.$$

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Problem 5
Consider the following observables in a three-state system:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix},$$

where a, b are real numbers.

(a) The spectrum of A is degenerate. How about the spectrum of B?.

(b) Show that A and B commute.

(c) Find a new orthonormal basis where both A and B are diagonal. Do A and B form a complete set of observables for this system?

Proof. (a) The eigenvalues of A are obviously $\pm a$, with $-a$ twice. Finding the eigenvalues of B, we have

$$\det(B - \lambda\mathbb{I}) = 0$$

$$\implies \begin{pmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -ib \\ 0 & ib & -\lambda \end{pmatrix} = (\lambda - b)(\lambda^2 - b^2) = (\lambda - b)^2(\lambda + b) = 0$$

$$\implies \lambda_{1,2} = b,\ \lambda_3 = -b$$

Since we have degenerate eigenvalues, then the spectrum of B is also degenerate.

(b)

$$\begin{aligned}
[A,B] = AB - BA &= \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} - \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \\
&= \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} - \begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix} \\
&= 0
\end{aligned}$$

(c) Notice that A and B are Hermitian. Since A and B are Hermitian and $[A,B]=0$, then there exists a basis where both A and B are both diagonal. To find these, write the eigenvector components as $u_i,\ i=1,2,3$. Clearly, the basis states $|1\rangle$, $|2\rangle$, and $|3\rangle$ are eigenvectors of A with eigenvalues a, $-a$, and $-a$ respectively. We can notice that $|1\rangle$ is a common eigenvector for both A and B. We just need to work out the 2×2 block basis for $|2\rangle$ and $|3\rangle$. Indeed, both of these states have eigenvalues a for A, so one linear combinations should have eigenvalue b for B, and orthogonal combination with eigenvalue $-b$.

Let the eigenvector components be u_2 and u_3. Then, for eigenvalue b, we have

$$-ibu_3 = bu_2 \quad \text{and} \quad ibu_2 = bu_3$$

both of which imply $u_3 = iu_2$. For eigenvalue $-b$, we have

$$-ibu_3 = -bu_2 \quad \text{and} \quad ibu_2 = -bu_3$$

both of which imply $u_3 = -iu_2$.

Choosing u_2 to be real, then we have the set of simultaneous eigenstates

$$\lambda_{A1} = a,\ \lambda_{B1} = b:\ |1\rangle$$

$$\lambda_{A2} = -a,\ \lambda_{B2} = b:\ \frac{1}{\sqrt{2}}(|2\rangle + i\,|3\rangle)$$

$$\lambda_{A3} = -a,\ \lambda_{B3} = -b:\ \frac{1}{\sqrt{2}}(|2\rangle - i\,|3\rangle).$$

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