MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*
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Homework 8

Problem 9-1
Suppose M is a smooth manifold, $X \in \mathfrak{X}(M)$, and γ is a maximal integral curve of X.

(a) We say γ is periodic if there is a number $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Show that exactly one of the following holds:

- γ is constant.
- γ is injective.
- γ is periodic and non-constant.

(b) Show that if γ is periodic and non-constant, then there exists a unique positive number T (called the period of $\boldsymbol{\gamma}$) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$ for some $k \in \mathbb{Z}$.

(c) Show that the image of γ is an immersed submanifold of M, diffeomorphic to $\mathbb{R}, \mathbb{S}^1$, or $\mathbb{R}^0$.

Solution. (a) We will solve this by exhaustion.

- If the vector field X is zero everywhere, then, considering a specific point p, $X_p = 0 \implies \gamma'(t) = 0$. Thus, γ is constant.
- If the vector field X is non-zero, then γ is non-constant. We now have two cases:
  - If γ is injective, this is the second bullet point.
  - If γ is not injective, then at some time $t_1 > t_0$, we will have that $\gamma(t_1) = \gamma(t_0)$. Since a vector field is non-invariant with respect to its own flow, then γ is periodic.

(b) Let $\gamma : \mathbb{R} \to \mathbb{R}$ be periodic and non-constant. Since γ is periodic, there exists a unique positive number $T > 0$ such that $\gamma(t+T) = \gamma(t)$. Since γ is non-constant, then $\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \neq 0, \ \forall t \in \mathbb{R}$.

Let us consider all the periods T for which we will have $\gamma(t) = \gamma(t')$, from which we can form a set, say $\mathcal{T}$. This set is discrete since γ is non-constant. Additionally, this set is bounded from below since γ is periodic, otherwise, if a smallest positive number T does not exist, then there exists some infimum $\tilde{T}$ of those periods and hence a Cauchy sequence that converges to it. If T_1 and T_2 are in the set, then so is their absolute difference, and since the sequence is Cauchy, the infimum $\tilde{T}$ must be zero. Fix some time t_0 and choose a small enough period $\epsilon > 0$, then $\gamma(t_0)$ is in every neighborhood of $\gamma(t)$, $\forall t$. Pick a neighborhood of $\gamma(t)$ in M, then we can find a period p small enough such taht $t_0 + kp$, for some $k \in \mathbb{Z}$ lies in $(t - \epsilon, t + \epsilon)$. This contradicts the Hausdorff property of the manifold M. Thus, there must exists a smallest positive period T.

All the other periods will be multiples of T, otherwise we can subtract T enough times to find a yet another smaller period, which is not possible since T is the smallest. We know that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT \in \mathcal{T}$. To show uniqueness, consider another period that is an integer multiple of T, then it would not divide $(t+T) - t$, even though $\gamma(t) = \gamma(t+T)$.

(c)

- If γ is constant, then the image of γ is an immersed submanifold M diffeomorphic to $\mathbb{R}^0$.
- If γ is non-constant and injective, then the vector field is non-zero, which means that $\gamma'(t) \neq 0$. From that, we see that γ has full rank, which is one, the dimension of the domain. Thus, γ is a smooth immersion of the open interval J into M. Thus, the image γ is an immersed submanifold M diffeomorphic to $\mathbb{R}$ due to injectivity.

- If γ is non-constant and periodic, we can image t smoothly looping on a circle with period T. To make this rigorous, we consider the smooth covering map

$$\begin{aligned} \pi : \mathbb{R} &\to \mathbb{S}^1, \\ t &\mapsto \mathrm{e}^{2\pi \mathrm{i} \frac{t}{T}}. \end{aligned}$$

Since γ is constant on the fibers of π, then there must be a map $\tilde{\gamma} : \mathbb{S}^1 \to M$ such that $\tilde{\gamma} \circ \pi = \gamma$. From part (b), $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$, which means that $\tilde{\gamma}$ is injective. Since π is a local diffeomorphism, then $\tilde{\gamma}$ is a smooth immersion. Thus, the image of γ is an immersed submanifold of M diffeomorphic to $\mathbb{S}^1$.

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Problem 9-7
Let M be a connected smooth manifold. Show that the group of diffeomorphisms of M acts transitively on M: that is, for any $p, q \in M$, there is a diffeomorphism $F : M \to M$ such that $F(p) = q$. [Hint: first prove that if $p, q \in \mathbb{B}^n$ (the open unit ball in $\mathbb{R}^n$), there is a compactly supported smooth vector field on $\mathbb{B}^n$ whose flow θ satisfies $\theta_1(p) = q$.]

Solution. We start by proving the hint. Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$ and let $p, q \in \mathbb{B}$. Define

$$L = \{p + t(q - p) \mid 0 \le t \le 1\}$$

to be the line connecting p and q. Since $\mathbb{B}^n$ is convex, then $L \subset \mathbb{B}^n$. We now define a constant vector field X on the convex set L by $X_\ell = q - p$. It is clear that X is smooth. From that, there is a smooth vector field $\tilde{X}$ on $\mathbb{B}^n$ such that $\tilde{X}|_L = X$ and $\operatorname{supp}(\tilde{X}) \subset \mathbb{B}^n$. Thus, $\tilde{X}$ is a compactly supported smooth vector field on $\mathbb{B}^n$. We can define γ as

$$\gamma(t) = p + t(q - p)$$

to be the integral curve of $\tilde{X}$ since $\gamma'(t) = q - p = \tilde{X}_{\gamma(t_0)}$, $\forall t_0 \in [0, 1]$. Therefore, if θ is the flow of $\tilde{X}$, then the flow satisfies $\theta_1(p) = \gamma(1) = q$.
Now, let M be a connected smooth manifold and, for a fixed $p \in M$, define U_p to be the orbit of p under the action given, giving us

$$U_p = \{q \in M \mid \exists \text{ diffeomorphism } F : M \to M \text{ such that } F(p) = q\}.$$

- **Non-empty**: The identity on M is a diffeomorphism, so $p \in U_p$. Thus, U_p is non-empty.
- **Open**: Let $q \in U_p$. We need to show that there is a neighborhood of q contained in U_p. Since $q \in U_p$, there exists a diffeomorphism F such that $F(p) = q$. Since M is a smooth manifold, there exists a coordinate chart (V, φ) around q where V is open and $\varphi : V \to \mathbb{B}^n$ is a diffeomorphism onto its image. By the hint, we proved, for any point $r \in V$, there exists a compactly supported smooth vector field X on $\mathbb{B}^n$ whose flow θ_t takes $\varphi(q)$ to $\varphi(r)$ at time 1. Pull this vector field back to V via φ to get a compactly supported vector field $Y = (\varphi^{-1})^* X$ on V. The flow ψt of Y then takes q to r at time 1. The composition $\psi_1 \circ F$ is a diffeomorphism taking p to r. Then, $V \subseteq U_p$. Thus, U_p is open.
- **Closed**: Let q be a limit point of U_p. We need to show $q \in U_p$. Take a coordinate chart (V, φ) around q as before. Since q is a limit point, there exists some $r \in U_p \cap V$. By using the hint, we can construct a diffeomorphism taking r to q. If G is the diffeomorphism taking p to r, then composing these diffeomorphisms gives us one taking p to q. Then, $q \in U_p$. Thus, U_p is closed.

Thus, U_p is non-empty, open, and closed. Since M is connected, then $U_p = M$, and hence, this is only one orbit. Therefore, the action is transitive. ■