PHYS 603 - Methods of Theoretical Physics III

Lie Algebras in Particle Physics by *H. Georgi*

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Homework 1

Problem 1

The group S_3 discussed in the textbook can be viewed as a group of rotations, in the 3-dimensional space, that are symmetries of an equilateral triangle. Denote by r the $2\pi/3$ rotation about the 3rd-order axis perpendicular to the plane of the triangle and by s the π rotation about one of the in-plane 2nd-order axes.

(a) Verify that any of the 6 group elements (including $e =$ no rotation) can be written as either r^k or sr^k, where $k = 0$, 1, 2.

(b) The result from (a) can be used to compute the conjugacy classes, i.e., the sets formed by $g^{-1}r^kg$ and $g^{-1}sr^kg$ with g running over the entire group. Find these classes (there should be 3 of them).

(c) Verify that assigning value 1 to r and either 1 or -1 to s produces two one-dimensional irreducible representations of the group.

Solution. (a) The group elements are given by

$$S_3 = \{e, r, r^2, s, sr, sr^2\}.$$

Since $r^3 = e$, the group elements can be written as

$$S_3 = \{r^0, r^1, r^2, sr^0, sr^1, sr^2\}.$$

This covers all six elements uniquely.

(b) The conjugacy class of an element x in a group G is the set of all elements of the form $g^{-1}xg$, where g runs over the entire group. Let us compute the following conjugacy classes.

- **Conjugacy Class of** e: The conjugacy class of e is $\{e\}$ since

$$g^{-1}eg = e,$$

for any $g \in G$.

- **Conjugacy Class of** r: To find the conjugacy class of r, we have to search, for all elements $g \in G$, what the outcome of $g^{-1}rg$ is. We have

$$\begin{aligned} e^{-1}re &= r \\ r^{-1}r(r) &= r \\ (r^2)^{-1}r(r^2) &= r, \end{aligned}$$

and

$$\begin{aligned} s^{-1}r(s) &= s^{-1}sr^2 = r^2 \\ (sr)^{-1}r(sr) &= (sr)^{-1}sr^2r = r^{-1}s^{-1}s = r^{-1} = r^2 \\ (sr^2)^{-1}r(sr^2) &= (sr^2)^{-1}sr^2r^2 = (r^2)^{-1}s^{-1}sr = r^{-2}r = r^2. \end{aligned}$$

Thus, the conjugacy class of r is $\{r, r^2\}$.

- **Conjugacy Class of** s: To find the conjugacy class of s, we have to search, for all elements $g \in G$, what the outcome of $g^{-1}sg$ is. We have

$$\begin{aligned} e^{-1}s(e) &= s \\ r^{-1}s(r) = r^{-1}rsr^{-1} &= sr^{-1} = sr^2 \\ (r^2)^{-1}s(r^2) = r^{-2}rs &= r^2 s = sr, \end{aligned}$$

and

$$\begin{aligned} s^{-1}s(s) &= s \\ (sr)^{-1}s(sr) = (r^{-1}s^{-1})s(sr) = r^{-1}sr &= r^{-1}rsr^{-1} = sr^{-1} = sr^2 \\ (sr^2)^{-1}s(sr^2) = (r^{-2}s^{-1})s(sr^2) = r^{-2}sr^2 &= (rs)r^2(sr^2)r^2 = sr^4 = sr. \end{aligned}$$

Thus, the conjugacy class of s is $\{s, sr, sr^2\}$.

Therefore the three conjugacy classes are

$$\begin{aligned} e &\quad \text{(identity)} \\ r, r^2 &\quad \text{(3rd-order rotations)} \\ s, sr, sr^2 &\quad \text{(2nd-order rotations).} \end{aligned}$$

(c) We want to show that setting $r = 1$ and $s = \pm 1$ produces two one-dimensional irreducible representations of the group S_3. Let's verify this for both cases.

- **Case 1:** Let $r \mapsto 1$, $s \mapsto 1$. This maps every group element to 1, preserving multiplication.
- **Case 2:** Let $r \mapsto 1$, $s \mapsto -1$, then

$$\begin{aligned} r &\mapsto 1 \\ r^2 &\mapsto 1 \\ s &\mapsto -1 \\ sr &\mapsto -1 \\ sr^2 &\mapsto -1 \end{aligned}$$

We can verify these preserve multiplication. For example

$$\begin{aligned} (sr)(sr) &\mapsto (-1)(-1) = 1 \\ s(sr) &\mapsto (-1)(-1) = 1, \end{aligned}$$

and similarly for all other products.

Therefore, both assignments define valid 1-dimensional irreducible representations.

Note: These are the only possible 1-dimensional representations because

- r must map to a cube root of unity that also equals 1 when squared.
- Once $r \mapsto 1$, s must map to either 1 or -1 to satisfy $s^2 = e$.

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Problem 2
For the same group as in Prob. 1, consider a (reducible) representation in the space V of complex vectors $v = (v_1,\, v_2,\, v_3)^T$, where the i-th component ($i = 1, 2, 3$) refers to the i-th vertex of the triangle. The action of the group in this representation is as follows: if a group element g replaces vertex i with vertex j, then the corresponding linear operator $D(g)$ replaces the i-th component of v with the j-th.

(a) Construct the 3×3 matrices $D(r)$ and $D(s)$ corresponding to r and s in this representation. Please use the basis formed by $e^{(1)} = (1, 0, 0)^T$, $e^{(2)} = (0, 1, 0)^T$, and $e^{(3)} = (0, 0, 1)^T$.

(b) The representation space V contains a one-dimensional invariant subspace W equivalent to one of the irreducible representations found in part (c) of Prob. 1. Find the general form of a vector $v \in W$.

(c) Consider the orthogonal complement $W^\perp$ to the subspace found in (b) (with the usual definition of inner product for complex vectors). Find an orthonormal basis in $W^\perp$ in which the 2×2 matrix corresponding to s is diagonal and obtain matrices for both s and r in that basis.

Solution. (a)
- **For the rotation r:** Vertex 1 goes to 2, vertex 2 goes to 3, and vertex 3 goes to 1. Thus,

$$D(r) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- **For the reflection s:** Vertex 1 goes to 2, vertex 2 goes to 1, and vertex 3 stays fixed. Thus,

$$D(s) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Looking for a one-dimensional invariant subspace equivalent to one of the irreducible representations from Problem 1(c), we need to consider both possibilities:

- **Trivial representation** (all elements map to 1): This would require $D(r)v = v$ and $D(s)v = v$, giving

$$v = c(1, 1, 1)^T, \quad c \in \mathbb{C}.$$

- **Sign representation** ($r \mapsto 1$, $s \mapsto -1$): This would require $D(r)v = v$ and $D(s)v = -v$.

For vector $v = (x, y, z)^T$, the condition $D(r)v = v$ means that

$$(z, x, y)^T = (x, y, z)^T$$

Thus, $x = y = z$.

The condition $D(s)v = -v$ means that

$$(y, x, z)^T = -(x, y, z)^T.$$

Thus, $y = -x$, $x = -y$, and $z = -z$, which has no non-zero solutions.

Therefore, the only one-dimensional invariant subspace corresponds to the trivial representation, spanned by $(1, 1, 1)^T$.

(c) We first need to find $W^\perp$. Since W is spanned by $(1, 1, 1)^T$, vectors in $W^\perp$ must satisfy

$$(x, y, z) \cdot (1, 1, 1) = x + y + z = 0.$$

We can find an orthonormal basis for this space as follows: take $(1,-1,0)$ as our first vector (after normalization) and then find a vector orthogonal to both $(1,1,1)$ and $(1,-1,0)$ (using Gram-Schmidt orthogonalization). This gives us

$$\begin{cases} u_1 = \dfrac{1}{\sqrt{2}}(1,-1,0)^T, \\ u_2 = \dfrac{1}{\sqrt{6}}(1,1,-2)^T. \end{cases}$$

Now, in this basis, how does s act? We know s swaps components 1 and 2, leaving 3 fixed.

- **For u_1:**
$$D(s)\frac{1}{\sqrt{2}}(1,-1,0)^T = \frac{1}{\sqrt{2}}(-1,1,0)^T = -u_1.$$
- **For u_2:**
$$D(s)\frac{1}{\sqrt{6}}(1,1,-2)^T = \frac{1}{\sqrt{6}}(1,1,-2)^T = u_2.$$

Thus, in this basis, we can write

$$D(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

For r, we can compute its action on u_1 and u_2 as

- **For u_1:**
$$D(r)u_1 = \frac{1}{\sqrt{2}}(0,1,-1)^T = -\frac{1}{2}u_1 - \frac{\sqrt{3}}{2}u_2.$$
- **For u_2:**
$$D(r)u_2 = \frac{1}{\sqrt{6}}(-2,1,1)^T = \frac{\sqrt{3}}{2}u_1 - \frac{1}{2}u_2.$$

Thus, in this basis, we can write

$$D(r) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

This is a 2-dimensional irreducible representation of S_3 because it cannot be further reduced (as s is already diagonal with different eigenvalues).

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