PHYS 662 - Quantum Field Theory I

Student: **Ralph Razzouk**

Homework 4

Problem 1 - Poincare Algebra
Consider the Poincare group $(\Lambda^\mu{}_\nu, a^\mu)$ in some representation $S(\Lambda, a)$. Expand the representation matrices around $\delta^\mu{}_\nu$ and derive the Poincare algebra by imposing the multiplication rule for the Poincare group.

Solution. Consider infinitesimal transformations around the identity

$$\begin{aligned}\Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \omega^\mu{}_\nu + O(\omega^2),\\ a^\mu &= \epsilon^\mu,\end{aligned}$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ (anti-symmetry from Lorentz condition).
The representation matrices can be expanded as

$$S(\Lambda, a) = 1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\epsilon_\mu P^\mu + O(\omega^2, \epsilon^2),$$

where $J^{\mu\nu}$ are generators of Lorentz transformations and P^μ are generators of translations.
For two transformations (Λ_1, a_1) and (Λ_2, a_2), we have

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2).$$

Expanding to first order, we get

$$\begin{aligned}\Lambda^\mu_{1\,\nu} &= \delta^\mu{}_\nu + \omega^\mu_{1\,\nu},\\ \Lambda^\mu_{2\,\nu} &= \delta^\mu{}_\nu + \omega^\mu_{2\,\nu},\\ a_1^\mu &= \epsilon_1^\mu,\\ a_2^\mu &= \epsilon_2^\mu.\end{aligned}$$

From representation multiplication, we get

$$S(\Lambda_1, a_1)S(\Lambda_2, a_2) = S((\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2))$$

The left hand-side expands to

$$(1 + \frac{i}{2}\omega_{1\mu\nu}J^{\mu\nu} + i\epsilon_{1\mu}P^\mu)(1 + \frac{i}{2}\omega_{2\mu\nu}J^{\mu\nu} + i\epsilon_{2\mu}P^\mu),$$

and the right hand-side expands to

$$1 + \frac{i}{2}(\omega_1 + \omega_2)_{\mu\nu}J^{\mu\nu} + i(\epsilon_1 + \epsilon_2 + \omega_1\epsilon_2)_\mu P^\mu.$$

Comparing terms and using anti-symmetry of $\omega^{\mu\nu}$, we have

- For $[J^{\mu\nu}, J^{\rho\sigma}]$:
$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}).$$
- For $[J^{\mu\nu}, P^\rho]$:
$$[J^{\mu\nu}, P^\rho] = i(\eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu).$$
- For $[P^\mu, P^\nu]$:
$$[P^\mu, P^\nu] = 0.$$

Therefore, the complete Poincar algebra is

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}),\\
[J^{\mu\nu}, P^{\rho}] &= i(\eta^{\nu\rho}P^{\mu} - \eta^{\mu\rho}P^{\nu}),\\
[P^{\mu}, P^{\nu}] &= 0.
\end{aligned}$$

These relations show that the Lorentz generators form a closed subalgebra, the translations commute with each other, and the Lorentz transformations act on translations as a vector representation.

■

Problem 2 - Symmetries of Euclidean Field Theory
Consider a Euclidean flat space in $\mathbb{R}^{d+1}$. Work out the commutators of the Lie algebra of $SO(d+1)$. Identify the generators of translations and rotations.

Solution. Let's first identify the generators of the Lie algebra $SO(d+1)$. In $\mathbb{R}^{d+1}$, we can denote the generators as $M_{\mu\nu}$ where $\mu, \nu = 0, 1, \ldots, d$. These generators are antisymmetric: $M_{\mu\nu} = -M_{\nu\mu}$.
For rotations in $\mathbb{R}^{d+1}$, the generators $M_{\mu\nu}$ act on coordinates x^{ρ} as

$$(M_{\mu\nu})^{\rho}_{\sigma} = \mathrm{i}(\delta^{\rho}_{\mu}\delta_{\nu\sigma} - \delta^{\rho}_{\nu}\delta_{\mu\sigma}).$$

The commutation relations can be worked out by considering the action on coordinates

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho}),$$

where $\eta_{\mu\nu}$ is the Euclidean metric ($\delta_{\mu\nu}$ in this case).
In a flat Euclidean space, translations are not part of $SO(d+1)$. The translation generators P_{μ} would be additional generators that satisfy $[P_{\mu}, P_{\nu}] = 0$ and rotations $[M_{\mu\nu}, P_{\rho}] = i(\eta_{\nu\rho}P_{\mu} - \eta_{\mu\rho}P_{\nu})$.
The rotation generators $M_{\mu\nu}$ can be interpreted geometrically. They generate rotations in the μ-ν plane and there are $\frac{d(d+1)}{2}$ independent generators, each corresponding to a plane of rotation in $\mathbb{R}^{d+1}$.
The algebra $SO(d+1)$ is completely characterized by the commutation relations mentioned above. These are the fundamental relationships that define the structure constants of the Lie algebra.

■

Problem 3 - Proper Length
In $3+1$ dim, a Lorentz vector is the irreducible representation $\left(\frac{1}{2}, \frac{1}{2}\right)$. Write down its proper length in term of the spinor degrees of freedom.

Solution. Consider a Lorentz vector V^{μ} in $(3+1)$ dimensions. As a $(\frac{1}{2}, \frac{1}{2})$ representation, we can express it using the Pauli matrices $\sigma^{\mu} = (1, \vec{\sigma})$ as

$$V^{\alpha\dot{\beta}} = V^{\mu}(\sigma_{\mu})^{\alpha\dot{\beta}},$$

where α is the left-handed spinor index and $\dot{\beta}$ is the right-handed spinor index.
The proper length squared of the vector can be written as

$$V_{\mu}V^{\mu} = V_0^2 - \vec{V}^2.$$

In terms of spinor components, this becomes

$$V_{\mu}V^{\mu} = -\frac{1}{2}V_{\alpha\dot{\beta}}V^{\alpha\dot{\beta}},$$

where the indices are raised and lowered with the ϵ tensor

$$V^{\alpha\dot{\beta}} = \epsilon^{\alpha\gamma}\epsilon^{\dot{\beta}\dot{\delta}}V_{\gamma\dot{\delta}}.$$

The factor of $-\frac{1}{2}$ appears due to the normalization of the Pauli matrices

$$\mathrm{Tr}(\sigma^{\mu}\sigma^{\nu}) = 2\eta^{\mu\nu}.$$

This demonstrates how the Lorentz-invariant length of a vector can be expressed purely in terms of its spinor components, making manifest the $SL(2, \mathbb{C})$ structure of the Lorentz group. ■

Problem 4
Show that $SU(2)$ is a double cover of $SO(3)$.

Solution. To show that $SU(2)$ is a double cover of $SO(3)$, we need to demonstrate that there exists a surjective homomorphism from $SU(2)$ to $SO(3)$ where each element in $SO(3)$ is associated with exactly two elements in $SU(2)$ (meaning the kernel of the homomorphism is $\{\pm I\}$), effectively creating a "two-to-one" mapping between the groups.
Let's establish the map $\phi : SU(2) \to SO(3)$. For any $U \in SU(2)$ and vector $\vec{v} \in \mathbb{R}^3$, we can form:

$$V = \vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$$

where σ_i are the Pauli matrices. The map ϕ is defined by,

$$\phi(U) : \vec{v} \mapsto \vec{v}' \quad \text{where} \quad V' = UVU^\dagger.$$

is a homomorphism. We will show that

(i) ϕ maps to $SO(3)$: V' is Hermitian since $(UVU^\dagger)^\dagger = UVU^\dagger$. Additionally, $\det(V') = \det(V)$ since $\det(U) = 1$. The trace is preserved $\mathrm{Tr}(V') = \mathrm{Tr}(V)$. Therefore, $\phi(U)$ preserves length and orientation.

(ii) ϕ is a homomorphism:

$$\begin{aligned}\phi(U_1)\phi(U_2)V &= U_1(U_2VU_2^\dagger)U_1^\dagger \\ &= (U_1U_2)V(U_1U_2)^\dagger \\ &= \phi(U_1U_2)V\end{aligned}$$

(iii) $\ker(\phi) = \{\pm I\}$: If $UVU^\dagger = V$ for all V, then U must commute with all Pauli matrices, but only scalar matrices commute with all Pauli matrices, and since $\det(U) = 1$, then $U = \pm I$.

(iv) ϕ is surjective: Any rotation in $SO(3)$ can be parameterized by axis $\hat{n}$ and angle θ. The corresponding $SU(2)$ element is $U = \exp(-\mathrm{i}\frac{\theta}{2}\hat{n} \cdot \vec{\sigma})$. This covers all of $SO(3)$, but with U and $-U$ giving the same rotation.

We have shown that $SU(2)$ maps onto $SO(3)$ via ϕ and the map is 2-to-1 since $\ker(\phi) = \{\pm I\}$. Therefore, $SU(2)$ is a double cover of $SO(3)$.
This relationship explains why spinors must rotate by 4π to return to their original state, while vectors only need 2π. ∎

Problem 5 - Clifford Algebra
Show that $\sigma^\mu_{\alpha\dot{\alpha}} \quad \bar{\sigma}^\mu_{\alpha i}$ satisfy the Clifford algebra

$$\begin{aligned}(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta &= 2\eta^{\mu\nu}\delta_\alpha{}^\beta, \\ (\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} &= 2\eta^{\mu\prime}\delta^{\dot{\alpha}}{}_{\dot{\beta}}.\end{aligned}$$

Solution. Recall the explicit forms

$$\begin{aligned}\sigma^\mu &= (1,\, \vec{\sigma})_{\alpha\dot{\alpha}}, \\ \bar{\sigma}^\mu &= (1,\, -\vec{\sigma})^{\dot{\alpha}\alpha},\end{aligned}$$

where $\vec{\sigma}$ are the Pauli matrices satisfying $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.
The properties we will apply are

- The anti-commutation relations of Pauli matrices: $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.
- The Minkowski metric: $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

- The identity $(\sigma^i)^2 = 1$ for each Pauli matrix.

(i) **The first relation**: Let us consider cases.

- **Case 1:** $\mu = \nu = 0$

$$\begin{aligned}(\sigma^0\bar{\sigma}^0 + \sigma^0\bar{\sigma}^0)_\alpha{}^\beta &= (1\cdot 1 + 1\cdot 1)_\alpha{}^\beta \\ &= 2\delta_\alpha{}^\beta \\ &= 2\eta^{00}\delta_\alpha{}^\beta.\end{aligned}$$

- **Case 2:** $\mu = 0, \nu = i$ (or vice versa)

$$\begin{aligned}(\sigma^0\bar{\sigma}^i + \sigma^i\bar{\sigma}^0)_\alpha{}^\beta &= (1\cdot(-\sigma^i) + \sigma^i\cdot 1)_\alpha{}^\beta \\ &= (-\sigma^i + \sigma^i)_\alpha{}^\beta \\ &= 0 \\ &= 2\eta^{0i}\delta_\alpha{}^\beta.\end{aligned}$$

- **Case 3:** $\mu = i, \nu = j$

$$\begin{aligned}(\sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i)_\alpha{}^\beta &= (\sigma^i(-\sigma^j) + \sigma^j(-\sigma^i))_\alpha{}^\beta \\ &= -(\sigma^i\sigma^j + \sigma^j\sigma^i)_\alpha{}^\beta \\ &= -2\delta^{ij}\delta_\alpha{}^\beta \\ &= 2\eta^{ij}\delta_\alpha{}^\beta.\end{aligned}$$

(ii) **The second relation**: The proof follows analogously, with $\bar{\sigma}^\mu$ appearing first in each product. The key difference is the arrangement of dotted indices, but the algebraic structure remains the same due to the properties of the Pauli matrices.

Therefore, both Clifford algebra relations are satisfied for all values of the indices. ∎

Problem 6
P^2 is a Lorentz-invariant quantity and commutes with all Poincare generators (it is a quadratic Casimir). The $3+1$ dim Poincare algebra has another quadratic casimir:

$$W^2 = W^\mu W_\mu,$$

where

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}$$

Show that

$$W^2 = \frac{1}{2}P^2 J^{\mu\nu}J_{\nu\mu} + P_\mu J^{\mu\nu}P_\lambda J^\lambda_\nu.$$

Apply this to massless particle states and write it in terms of A, B, and J (the generators we encountered in the little group of massless particles).

Solution. Starting with $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}$, we have

$$\begin{aligned}W^2 &= W^\mu W_\mu \\ &= \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\rho\sigma\lambda}P_\nu J_{\alpha\beta}P^\rho J^{\sigma\lambda}.\end{aligned}$$

Using the identity for product of two Levi-Civita symbols, we get

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\rho\sigma\lambda} = -\delta^\nu_\rho\delta^\alpha_\sigma\delta^\beta_\lambda + \delta^\nu_\rho\delta^\alpha_\lambda\delta^\beta_\sigma + \delta^\nu_\sigma\delta^\alpha_\lambda\delta^\beta_\rho - \delta^\nu_\sigma\delta^\alpha_\rho\delta^\beta_\lambda + \delta^\nu_\lambda\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\nu_\lambda\delta^\alpha_\sigma\delta^\beta_\rho,$$

which gives

$$\begin{aligned}
W^2 &= \frac{1}{4}[-(P \cdot P)(J^{\alpha\beta} J_{\beta\alpha}) + 2P_\mu J^{\mu\nu} P_\lambda J_\nu^\lambda] \\
&= \frac{1}{2} P^2 J^{\mu\nu} J_{\nu\mu} + P_\mu J^{\mu\nu} P_\lambda J_\nu^\lambda .
\end{aligned}$$

For massless states ($P^2 = 0$), we choose the frame where

$$P^\mu = (E, 0, 0, E).$$

In terms of the little group generators, we have

$$\begin{aligned}
J_{12} &= J \\
J_{1-} &= A \\
J_{2-} &= B
\end{aligned}$$

where "$-$" refers to the light-cone combination $x^0 - x^3$.
Thus

$$\begin{aligned}
W^2 &= P_\mu J^{\mu\nu} P_\lambda J_\nu^\lambda \\
&= E^2(J^2 + A^2 + B^2).
\end{aligned}$$

For massless states, the first term vanishes due to $P^2 = 0$ and the remaining term gives the square of the helicity operator (J^2) plus contributions from the translational generators ($A^2 + B^2$). For physical states, we typically require $A = B = 0$, leaving only the helicity contribution.
Thus, for physical massless states with $A = B = 0$, we have

$$W^2 = E^2 J^2.$$

This is proportional to the square of the helicity, which is a key quantum number for massless particles. ■