PHYS 617 - Statistical Mechanics

A Modern Course in Statistical Physics by *Linda E. Reichl*
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Homework 9

Problem 1
Consider a monatomic non-relativistic gas of bosons.

(a) First, show the number density and energy density are described as follows:

$$n(\mu) = \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1},$$

$$\mathcal{E}(\mu) = E/V = \int \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \frac{\epsilon_p}{e^{\beta(\epsilon_p - \mu)} - 1},$$

with $\epsilon_p = p^2/2m$, where m is the boson mass.

(b) Show in the limiting case $\mu \to -\infty$, you recover the classical result for a monatomic ideal gas:

$$\mathcal{E}_{\text{classical}} = \frac{3}{2} nkT.$$

(c) In the limit $\mu \to 0, n(\mu) = n_{\text{crit}}(T)$. Compute this explicitly. You might have to look up an integral to do this problem; feel free to either leave your result in terms of a Riemann zeta function, or compute the decimal coefficient explicitly.

Solution. (a) The grand partition function is given by

$$\mathbb{Q} = \prod_q \left(\frac{1}{1 - \mathrm{e}^{-\beta(\epsilon_q - \mu)}} \right).$$

Taking the logarithm, we have

$$\ln(\mathbb{Q}) = -\sum_q \ln\left(1 - \mathrm{e}^{-\beta(\epsilon_q - \mu)}\right).$$

Now, we can find the number density and the energy density as follows

- **Number density:**

$$\begin{aligned}
\beta N &= \partial_\mu \ln(\mathbb{Q}) \\
&= \frac{\partial}{\partial \mu} \left[-\sum_q \ln\left(1 - \mathrm{e}^{-\beta(\epsilon_q - \mu)}\right) \right] \\
&= \sum_q \frac{\beta \mathrm{e}^{-\beta(\epsilon_q - \mu)}}{1 - \mathrm{e}^{-\beta(\epsilon_q - \mu)}} \\
&= \sum_q \frac{\beta}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \\
\implies N &= \sum_q \frac{1}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1}.
\end{aligned}$$

Making the jump from discrete to continuous, as we are assuming we have a gas of bosons, we have

$$\begin{aligned}
N &= \int \frac{1}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \frac{\mathrm{d}^3 x \, \mathrm{d}^3 p}{(2\pi\hbar)^3} \\
&= V \int \frac{1}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \\
\implies n(\mu) = \frac{N}{V} &= \frac{1}{(2\pi\hbar)^3} \int \frac{1}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \, \mathrm{d}^3 p
\end{aligned}$$

- **Energy density:**

$$\begin{aligned}
E - \mu N &= -\partial_\beta \ln(\mathbb{Q}) \\
&= -\frac{\partial}{\partial \beta} \left[-\sum_q \ln\left(1 - \mathrm{e}^{-\beta(\epsilon_q - \mu)}\right) \right] \\
&= \sum_q \frac{(\epsilon_q - \mu) \, \mathrm{e}^{-\beta(\epsilon_q - \mu)}}{1 - \mathrm{e}^{-\beta(\epsilon_q - \mu)}} \\
&= \sum_q \frac{\epsilon_q - \mu}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \\
\implies E &= \sum_q \frac{\epsilon_q - \mu}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} + \mu N \\
&= \sum_q \frac{\epsilon_q - \mu}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} + \mu \sum_q \frac{1}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \\
&= \sum_q \frac{\epsilon_q}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1}.
\end{aligned}$$

Making the jump from discrete to continuous, as we are assuming we have a gas of bosons, we have

$$\begin{aligned}
E &= \int \frac{\epsilon_q}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \frac{\mathrm{d}^3 x \, \mathrm{d}^3 p}{(2\pi\hbar)^3} \\
&= V \int \frac{\epsilon_q}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \frac{\mathrm{d}^3 p}{(2\pi\hbar)^3} \\
\implies \mathcal{E}(\mu) = \frac{E}{V} &= \frac{1}{(2\pi\hbar)^3} \int \frac{\epsilon_q}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \, \mathrm{d}^3 p.
\end{aligned}$$

(b) If $\mu \to -\infty$, then $e^{\beta(\epsilon_q - \mu)} \gg 1$, and replacing the energy by $\frac{p^2}{2m}$, we have

$$\begin{aligned}
n(-\infty) &= \frac{1}{(2\pi\hbar)^3} \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} \, d^3p \\
&= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{e^{\beta(\epsilon_q - \mu)}} 4\pi p^2 \, dp \\
&= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} 4\pi p^2 \, dp \\
&= \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} \, dp \\
&= \frac{1}{2\pi^2\hbar^3} \int_0^\infty \frac{p^2}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} \, dp \\
&= \frac{1}{2\pi^2\hbar^3} \left(\frac{\sqrt{\pi} m^{\frac{3}{2}} e^{\beta\mu}}{\sqrt{2}\beta^{\frac{3}{2}}} \right) \\
&= \frac{m^{\frac{3}{2}}}{2\sqrt{2}\pi^{\frac{3}{2}}\hbar^3\beta^{\frac{3}{2}}} e^{\beta\mu} \\
&= \left(\frac{m}{2\pi\hbar^2\beta} \right)^{\frac{3}{2}} e^{\beta\mu}.
\end{aligned}$$

The energy density is then

$$\begin{aligned}
\mathcal{E}(-\infty) &= \frac{1}{(2\pi\hbar)^3} \int \frac{\epsilon_q}{e^{\beta(\epsilon_q - \mu)} - 1} \, d^3p \\
&= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{\epsilon_q}{e^{\beta(\epsilon_q - \mu)}} 4\pi p^2 \, dp \\
&= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{\frac{p^2}{2m}}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} 4\pi p^2 \, dp \\
&= \frac{4\pi}{2m(2\pi\hbar)^3} \int_0^\infty \frac{p^4}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} \, dp \\
&= \frac{1}{4m\pi^2\hbar^3} \int_0^\infty \frac{p^4}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} \, dp \\
&= \frac{1}{4m\pi^2\hbar^3} \left(\frac{3\sqrt{\pi} m^{\frac{5}{2}} e^{\beta\mu}}{\sqrt{2}\beta^{\frac{5}{2}}} \right) \\
&= \frac{3m^{\frac{3}{2}}}{4\sqrt{2}\pi^{\frac{3}{2}}\hbar^3\beta^{\frac{5}{2}}} e^{\beta\mu} \\
&= \frac{3}{2} k_B T \left(\frac{m}{2\pi\hbar^2\beta} \right)^{\frac{3}{2}} e^{\beta\mu} \\
&= \frac{3}{2} n k_B T,
\end{aligned}$$

yielding the classical result.

(c) If $\mu \to 0$, we have

$$\begin{aligned}
n(0) &= \frac{1}{(2\pi\hbar)^3} \int \frac{1}{\mathrm{e}^{\beta(\epsilon_q - \mu)} - 1} \,\mathrm{d}^3 p \\
&= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{\mathrm{e}^{\beta\epsilon_q} - 1} \,\mathrm{d}^3 p \\
&= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{\mathrm{e}^{\beta\epsilon_q} - 1} 4\pi p^2 \,\mathrm{d}p \\
&= \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2}{\mathrm{e}^{\beta\frac{p^2}{2m}} - 1} \,\mathrm{d}p \\
&= \frac{1}{2\pi^2\hbar^3} \int_0^\infty \frac{p^2}{\mathrm{e}^{\beta\frac{p^2}{2m}} - 1} \,\mathrm{d}p.
\end{aligned}$$

Now, let $x = \beta\frac{p^2}{2m} \implies \mathrm{d}x = \beta\frac{p}{m}\,\mathrm{d}p$. Then

$$\begin{aligned}
n(0) \equiv n_{\text{crit}}(T) &= \frac{1}{2\pi^2\hbar^3} \frac{m}{\beta} \int_0^\infty \frac{\frac{2m}{\beta} x}{\mathrm{e}^x - 1} \sqrt{\frac{\beta}{2mx}} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2}\pi^2\hbar^3} \left(\frac{m}{\beta}\right)^{\frac{3}{2}} \int_0^\infty \frac{x^{\frac{1}{2}}}{\mathrm{e}^x - 1} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2}\pi^2\hbar^3} \left(\frac{m}{\beta}\right)^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right) \\
&= \frac{1}{\sqrt{2}\pi^2\hbar^3} \left(\frac{m}{\beta}\right)^{\frac{3}{2}} \left(\frac{\sqrt{\pi}}{2}\right) (2.612).
\end{aligned}$$

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Problem 2
For μ in-between these limits, there is an analytic expression in terms of special functions, but we might as well do the integral numerically at that point. You are free to do the integrals either way (analytically or numerically). The goal will be to eliminate μ from the equations and make a plot of $\mathcal{E}(n, T)$.

(a) For $T > T_{\text{crit}}$, first fix the temperature ($T =$ const.) and plot $\mathcal{E}/\mathcal{E}_{\text{classical}}$ as a function of n. If you're having trouble figuring out how to do this, first compute $\mathcal{E}(\mu)$ and $n(\mu)$ and you should be able to make a parametric plot that puts $n(\mu)$ on the x-axis and $\mathcal{E}(\mu)$ on the y-axis.

You should be able to show with this plot that $\mathcal{E}/\mathcal{E}_{\text{classical}}$ is well-approximated by a linear function of n. Describe this function, using the limits $n = 0$ and $n = n_{\text{crit}}$ from Problem 1; this gives us a decent analytical description of $\mathcal{E}(n, T)$ in this regime with no free parameters.

(b) For $T < T_{\text{crit}}$, the result is no longer dependent on n, and only on the temperature; explain why this is, and use this to compute $\mathcal{E}(T)$ for all $T < T_{\text{crit}}$.

(c) Make a plot of $\mathcal{E}(T)/\mathcal{E}_{\text{classical}}(T)$, as a function of all T, both above and below the critical temperature. Is your result continuous and differentiable everywhere? (If you want, you can fix $n = N_A/\text{cm}^3$).

(d) Finally, since you have an analytical expression everywhere, make a plot of the heat capacity, compared with the classical result $C_V/C_{V,\text{classical}}$. Is this function continuous and differentiable everywhere?

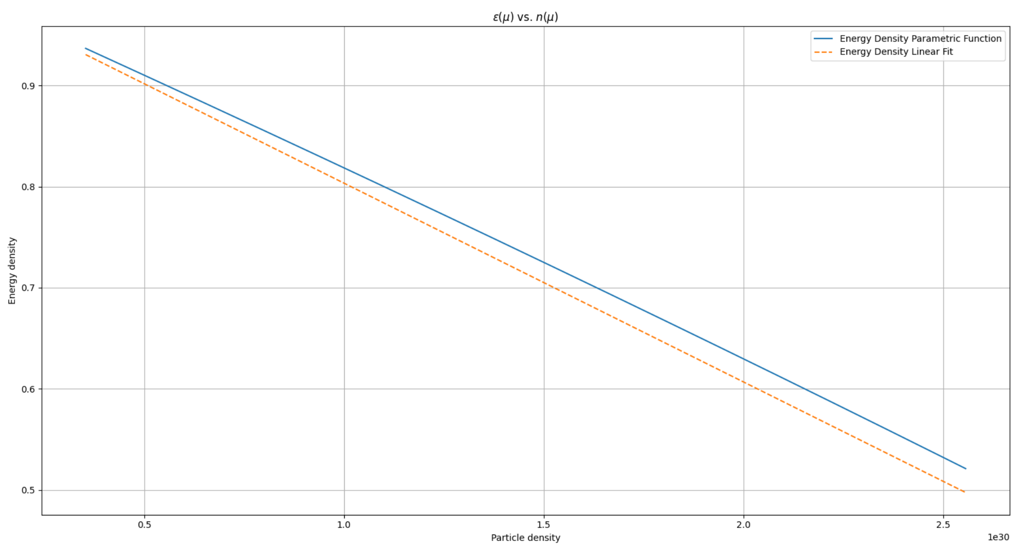

Figure 1: Plot of $\mathcal{E}(\mu)$ vs. $n(\mu)$.

Solution (a)

```
import numpy as np
import scipy.integrate as int
import matplotlib.pyplot as plt

##### CONSTANTS #####
c = 3e8                                         # Speed of light (m/s)
hbar = 6.582e-16                                # Planck's constant (eV/Hz)     |   1.05e-34 (J*s)
k_B = 8.617e-5                                  # Boltzmann constant (eV/K)     |   1.38e-23 (J/K)
# m_e = 9.11e-31                                # Electron mass (GeV/c^2)
# e = 1.6e-19                                   # Elementary charge (C)
N_A = 6.022e23                                  # Avogadro's number (cm^-3)

T = 1e5                                         # Temperature (K)
beta = 1 / (k_B*T)                              # Beta (1/eV)

m = 2829570 / c**2                              # Mass of Helium-4 Atom (eV/c)
zeta_1 = 2.612                                  # Riemann Zeta Function of 3/2
zeta_2 = 1.341                                  # Riemann Zeta Function of 5/2
C = 1/2 * np.pi**-2 * hbar**-3

################## PROBLEM 2 ##################
N = 10000
mu = np.linspace(-10, 0, N)                     # Chemical Potential (eV)
p = np.linspace(0, 1e5 / c, N)                  # Momentum (eV/c)
epsilon = p**2 / (2*m)                          # Energy of Particle (J)

n = np.zeros_like(mu)                           # Number Density
varepsilon = np.zeros_like(mu)                  # Energy density
varepsilon_classical = np.zeros_like(mu)        # Classical energy density


########## PART (a) ##########
for i, u in enumerate(mu):
    # Integrands for n(mu) and varepsilon(mu)
    n_i = 1/(np.exp(beta*(epsilon - u)) - 1)
    varepsilon_i = epsilon/(np.exp(beta*(epsilon - u)) - 1)

    # Integrating over momentum
    n[i] = C * np.trapz(p**2 * n_i, p)
    varepsilon[i] = C * np.trapz(p**2 * varepsilon_i, p)

varepsilon_classical = 3/2 * n * k_B * T

n_crit = (m / (2 * np.pi * hbar**2 * beta))**(3/2) * zeta_1
varepsilon_a = 1 - (0.51 / n_crit) * n
T_crit = (N_A * T**(3/2) / n_crit)**(2/3)


plt.plot(n, varepsilon/varepsilon_classical, label='Energy Density Parametric Function')
plt.plot(n, varepsilon_a, linestyle='--', label='Energy Density Linear Fit')

plt.title('$\\varepsilon(\mu)$ vs. $n(\mu)$')
plt.xlabel('Particle density')
plt.ylabel('Energy density')
plt.grid()
plt.legend()
plt.show()
```

Listing 1: Code for Problem 2 (a)

(b)

```
########## PART (b) ##########
T_n = np.linspace(0, T_crit, N) # Temperature (K)
beta_l = 1 / (k_B*T_n) # 1/eV
n_c = (m / (2 * np.pi * hbar**2 * beta_l))**(3/2) * zeta_1 # Critical density

varepsilon_classical = np.zeros_like(T) # Classical energy density
varepsilon_classical = 3/2 * N_A * k_B * T_n

varepsilon_l = 3/2 * n_c * beta_l**-1 * zeta_2 / zeta_1


plt.plot(T_n, varepsilon_l/varepsilon_classical, label='Energy Density Parametric Function')

plt.title('$\\varepsilon(T)$ vs. $T$')
plt.xlabel('Temperature')
plt.ylabel('Energy density')
plt.grid()
plt.legend()
plt.show()
```

Listing 2: Code for Problem 2 (b)

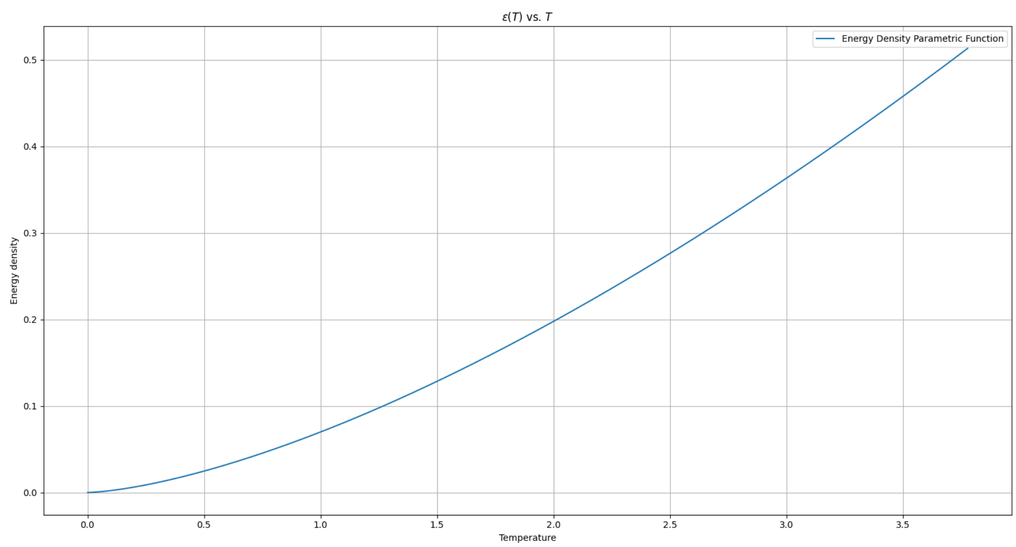

Figure 2: Plot of $\mathcal{E}(T)$ vs. T.

(c)
```
########## PART (c) ##########
T = np.linspace(T_crit, 5*T_crit, N) # K
beta = 1 / (k_B*T) # 1/eV

varepsilon_h = 1 - 0.51*(T_crit / T)**(3/2)


plt.plot(T_n, varepsilon_l/varepsilon_classical, label='Energy Density T < T_crit')
plt.plot(T, varepsilon_h, label='Energy Density T > T_crit')

plt.title('$\\varepsilon(T)$ vs $T$')
plt.xlabel('Temperature')
plt.ylabel('Energy density')
plt.grid()
plt.legend()
plt.show()
```

Listing 3: Code for Problem 2 (c)

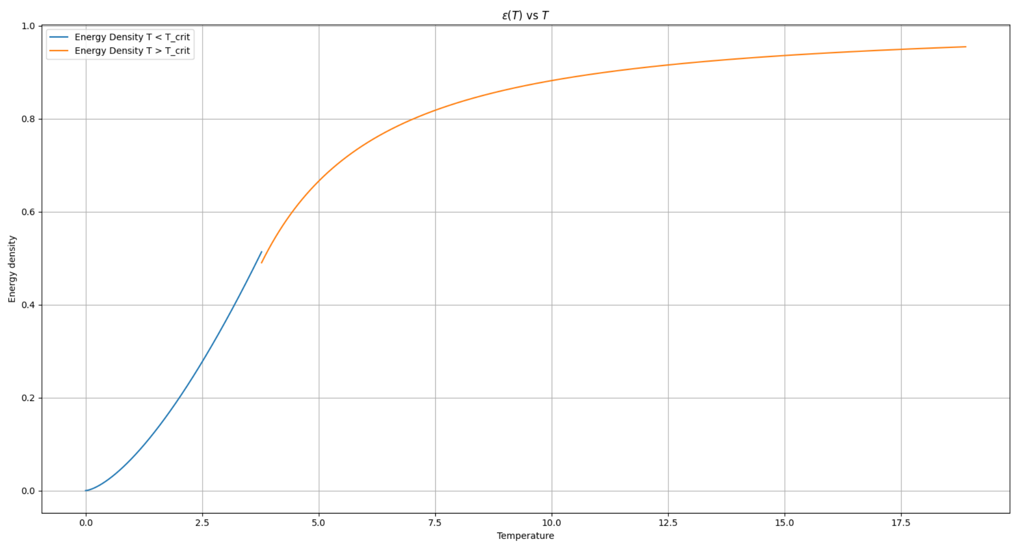

Figure 3: Plot of $\mathcal{E}(T)$ vs. T.

(d)

```
########## PART (d) ##########
C_l = np.diff(varepsilon_l)/np.diff(T_n) # heat capacity
C_h = np.diff(varepsilon_h)/np.diff(T) # heat capacity

varepsilon_classical = np.zeros_like(T_n) # Classical energy density
varepsilon_classical = 3/2 * N_A * k_B * T_n
C_classical = np.diff(varepsilon_classical)/np.diff(T_n) # Classical heat capacity


plt.plot(T_n[:-1], C_l/C_classical, label='Heat Capacity T < T_crit')
plt.plot(T[:-1], C_h + 1.02, label='Heat Capacity T > T_crit')

plt.axhline(y=1, color='r', linestyle='--', label='3/2 * N_A * k_B')
plt.axvline(x=T_crit, color='b', linestyle='--', label = 'T_crit')

plt.title('Heat Capacity vs Temperature')
plt.xlabel('Temperature')
plt.ylabel('Heat Capacity')
plt.grid()
plt.legend()
plt.show()
```

Listing 4: Code for Problem 2 (d)

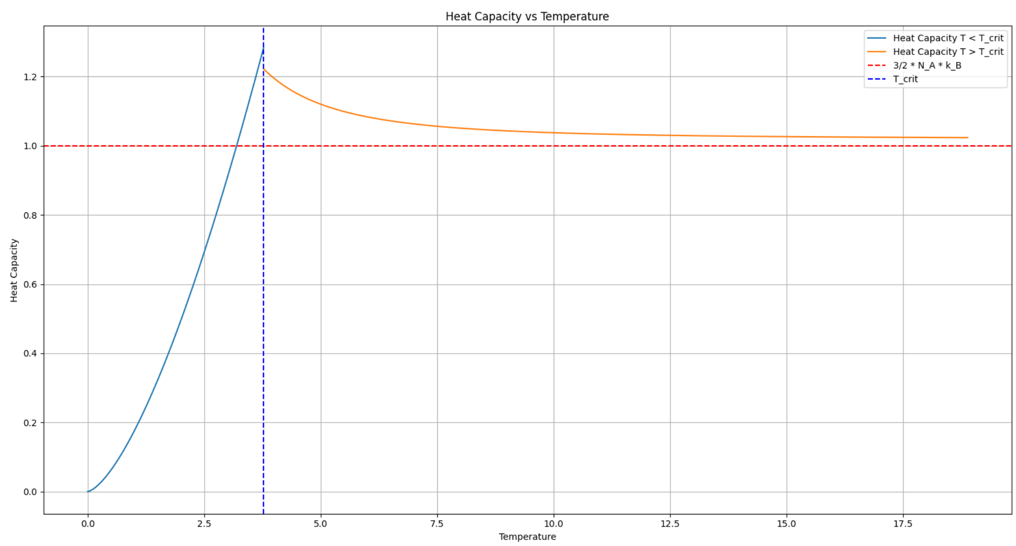

Figure 4: Plot of C_V vs. T.

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Problem 3
Estimate the critical condensation temperature T_c of a gas of massless particles where the number density n has been fixed (If you don't like the concept that we could fix the number of particles when they're massless, then just imagine they are ultra-relativistic particles with $E = pc$). Do photons have a critical condensation temperature T_c?

Solution. (a) From Problem 1, we have

$$\ln(\mathbb{Q}) = -\sum_q \ln\left(1 - \mathrm{e}^{-\beta(\epsilon_q-\mu)}\right),$$
$$n(\mu) = \frac{N}{V} = \frac{1}{(2\pi\hbar)^3}\int \frac{1}{\mathrm{e}^{\beta(\epsilon_q-\mu)}-1}\,\mathrm{d}^3p.$$

Here, we consider the number of particles $N(\mu)$, or the number density $n(\mu)$ equivalently, to be fixed. Since we are looking for a condensation temperature, then we are looking for the case when $\mu \to 0$. If $\mu \to 0$, then

$$\begin{aligned}
n(0) &= \frac{1}{(2\pi\hbar)^3}\int \frac{1}{\mathrm{e}^{\beta(\epsilon_q-\mu)}-1}\,\mathrm{d}^3p \\
&= \frac{1}{(2\pi\hbar)^3}\int_0^\infty \frac{1}{\mathrm{e}^{\beta\epsilon_q}-1}\,\mathrm{d}^3p \\
&= \frac{1}{(2\pi\hbar)^3}\int_0^\infty \frac{1}{\mathrm{e}^{\beta\epsilon_q}-1}4\pi p^2\,\mathrm{d}p \\
&= \frac{4\pi}{(2\pi\hbar)^3}\int_0^\infty \frac{p^2}{\mathrm{e}^{\beta pc}-1}\,\mathrm{d}p \\
&= \frac{1}{2\pi^2\hbar^3}\int_0^\infty \frac{p^2}{\mathrm{e}^{\beta pc}-1}\,\mathrm{d}p.
\end{aligned}$$

Now, let $x = \beta pc \implies \mathrm{d}x = \beta c\,\mathrm{d}p$. Then

$$\begin{aligned}
n(0) &= \frac{1}{2\pi^2\hbar^3}\frac{1}{(\beta c)^3}\int_0^\infty \frac{x^2}{\mathrm{e}^x-1}\,\mathrm{d}x \\
&= \frac{1}{2\pi^2\hbar^3}\frac{1}{(\beta c)^3}\Gamma(3)\zeta(3) \\
&= \frac{1}{\pi^2\hbar^3}\frac{1}{(\beta c)^3}\zeta(3) \\
&= \frac{1}{\pi^2\hbar^3}\frac{(k_BT)^3}{c^3}\zeta(3) \\
\implies T_c &= \left(\frac{\pi^2\hbar^3c^3n(0)}{k_B^3\zeta(3)}\right)^{\frac{1}{3}} \\
&= \frac{\hbar c}{k_B}\left(\frac{\pi^2 n(0)}{\zeta(3)}\right)^{\frac{1}{3}}.
\end{aligned}$$

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Problem 4
We showed in class that below T_{crit} , the fraction of bosons in the ground state goes like

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{crit}}}\right)^{\frac{3}{2}}.$$

How does this behavior depend on the number of dimensions? What happens when we consider massless bosons (or equivalently, ultra-relativistic bosons)?

Solution. In n dimensions, we will now have to integrate over $\mathrm{d}^n x$ and $\mathrm{d}^n p$. Let S_{n-1} be the surface area of the $(n-1)$-sphere embedded in n-dimensional Euclidean space. The integration over $\mathrm{d}^n x$ will give us V and we will convert the integral over $\mathrm{d}^n p$ by using the surface area of a $(n-1)$-sphere with radius p. Then our number of particles $N(\mu)$ will be

$$\begin{aligned}
N(\mu) &= \frac{1}{(2\pi\hbar)^n}\int \frac{1}{\mathrm{e}^{\beta(\epsilon_q-\mu)}-1}\,\mathrm{d}^n x\,\mathrm{d}^n p \\
&= \frac{V}{(2\pi\hbar)^n}\int \frac{1}{\mathrm{e}^{\beta(\epsilon_q-\mu)}-1}\,\mathrm{d}^n p \\
&= \frac{V}{(2\pi\hbar)^n}\int_0^\infty \frac{1}{\mathrm{e}^{\beta(\epsilon_q-\mu)}-1}\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}p^{n-1}\,\mathrm{d}p \\
&= \frac{V}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{p^{n-1}}{\mathrm{e}^{\beta(\epsilon_q-\mu)}-1}\,\mathrm{d}p.
\end{aligned}$$

Taking the $\mu \to 0$ limit, we have

$$\begin{aligned}
n(0) \equiv \frac{N(\mu)}{V} &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{p^{n-1}}{\mathrm{e}^{\beta(\epsilon_q-\mu)}-1}\,\mathrm{d}p \\
&= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{p^{n-1}}{\mathrm{e}^{\beta\epsilon_q}-1}\,\mathrm{d}p \\
&= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{p^{n-1}}{\mathrm{e}^{\beta\frac{p^2}{2m}}-1}\,\mathrm{d}p.
\end{aligned}$$

Letting $x = \beta \frac{p^2}{2m} \implies \mathrm{d}x = \beta \frac{p}{m}\,\mathrm{d}p$, we have

$$\begin{aligned}
n(0) &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{\left(\frac{2mx}{\beta}\right)^{\frac{n-1}{2}}}{\mathrm{e}^x - 1} \frac{mx}{\beta}\sqrt{\frac{\beta}{2mx}}\,\mathrm{d}x \\
&= \frac{2^{\frac{n-1}{2}}}{2^{n-1}\sqrt{2}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{\left(\frac{mx}{\beta}\right)^{\frac{n-1}{2}}}{\mathrm{e}^x - 1}\sqrt{\frac{mx}{\beta}}\,\mathrm{d}x \\
&= \frac{2^{\frac{n-1}{2}}}{2^{n-\frac{1}{2}}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{\beta}\right)^{\frac{n}{2}} \int_0^\infty \frac{x^{\frac{n}{2}}}{\mathrm{e}^x - 1}\,\mathrm{d}x \\
&= \frac{1}{2^{\frac{n}{2}}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{\beta}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}+1\right)\zeta\left(\frac{n}{2}+1\right) \\
&= \frac{1}{2^{\frac{n}{2}}\pi^{\frac{n}{2}}\hbar^n} \left(\frac{m}{\beta}\right)^{\frac{n}{2}} \left(\frac{n}{2}\right)\zeta\left(\frac{n}{2}+1\right) \\
&= \frac{n}{2^{\frac{n}{2}+1}\pi^{\frac{n}{2}}\hbar^n} \left(\frac{m}{\beta}\right)^{\frac{n}{2}} \zeta\left(\frac{n}{2}+1\right) \\
&= \frac{n}{2^{\frac{n}{2}+1}\pi^{\frac{n}{2}}\hbar^n} \left(mk_BT\right)^{\frac{n}{2}} \zeta\left(\frac{n}{2}+1\right) \\
&= \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{\frac{n}{2}} \frac{n}{2}\zeta\left(\frac{n}{2}+1\right).
\end{aligned}$$

Then, we define

$$T_{\mathrm{crit}} \equiv \left(\frac{2\pi\hbar^2}{mk_B}\right)\left(\frac{2}{n\zeta\left(\frac{n}{2}+1\right)}\right)^{\frac{2}{n}}$$

Thus,

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\mathrm{crit}}}\right)^{\frac{n}{2}}.$$

When we consider massless bosons, *i.e.* ultra-relativistic bosons, then we have

$$\begin{aligned}
n(0) \equiv \frac{N(\mu)}{V} &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{p^{n-1}}{\mathrm{e}^{\beta\epsilon_q} - 1}\,\mathrm{d}p \\
&= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{p^{n-1}}{\mathrm{e}^{\beta pc} - 1}\,\mathrm{d}p.
\end{aligned}$$

Letting $x = \beta p c \implies \mathrm{d}x = \beta c\,\mathrm{d}p$, then

$$
\begin{aligned}
n(0) \equiv \frac{N(\mu)}{V} &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{\left(\frac{x}{\beta c}\right)^{n-1}}{\mathrm{e}^x - 1}\frac{1}{\beta c}\,\mathrm{d}x \\
&= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \left(\frac{1}{\beta c}\right)^n \int_0^\infty \frac{x^{n-1}}{\mathrm{e}^x - 1}\,\mathrm{d}x \\
&= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \left(\frac{1}{\beta c}\right)^n \Gamma(n)\zeta(n) \\
&= \frac{\Gamma(n)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)} \left(\frac{1}{\pi\hbar^2}\left(\frac{1}{\beta c}\right)^2\right)^{\frac{n}{2}} \zeta(n) \\
&= \frac{\Gamma(n)}{2^{n-1}\Gamma\left(\frac{n}{2}\right)} \left(\frac{(k_B T)^2}{\pi c^2\hbar^2}\right)^{\frac{n}{2}} \zeta(n) \\
&= \frac{2^{n-1}\Gamma\left(\frac{n+1}{2}\right)}{2^{n-1}\sqrt{\pi}} \left(\frac{(k_B T)^2}{\pi c^2\hbar^2}\right)^{\frac{n}{2}} \zeta(n) \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}} \left(\frac{(k_B T)^2}{\pi c^2\hbar^2}\right)^{\frac{n}{2}} \zeta(n) \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}} \left(\frac{k_B T}{\sqrt{\pi}c\hbar}\right)^{n} \zeta(n).
\end{aligned}
$$

Then, we define

$$
T_{\text{crit}} \equiv \left(\frac{\sqrt{\pi}c\hbar}{k_B}\right)\left(\frac{\sqrt{\pi}}{\Gamma\left(\frac{n+1}{2}\right)\zeta\left(n\right)}\right)^{\frac{1}{n}}.
$$

Thus,

$$
\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{crit}}}\right)^n.
$$

■