PHYS 603 - Methods of Theoretical Physics III

Lie Algebras in Particle Physics by *H. Georgi*

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Homework 2

Problem 1

The group S_3 discussed in Homework 1 has a subgroup, called C_3 (also Z_3), which consists of three elements: e, r, and r^2. The group is Abelian, so each element is a separate conjugacy class. The character table for the irreps of C_3 is as follows

	e	r	r^2
A	1	1	1
C	1	ϵ	ϵ^2
C^*	1	ϵ^2	ϵ

where $\epsilon = e^{2\pi i/3}$.

(a) Consider the 3-dimensional representation (call it $\mathcal{D}$) described in Homework 1 Prob. 2, now as a representation of C_3. The matrix corresponding to r in it is

$$D(r) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Compute the characters of this representation.

(b) Decompose $\mathcal{D}$ into a direct sum of the irreducible representations of C_3. (You do not have to construct the projectors on the corresponding subspaces.)

Solution. (a) Let's find the characters of representation $\mathcal{D}$ for each element of C_3.

- **For e (identity):** We have

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\chi_{\mathcal{D}}(e) = \text{Tr}[D(e)] = 3$.

- **For r:** We have

$$D(r) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $\chi_{\mathcal{D}}(r) = \text{Tr}[D(r)] = 0$.

- **For r^2:** We square the matrix $D(r)$, and we get

$$D(r^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then $\chi_{\mathcal{D}}(r^2) = \text{Tr}[D(r^2)] = 0$.

(b) To decompose $\mathcal{D}$, we use the fact that the number of times an irreducible representation appears is given by

$$n_\alpha = \frac{1}{|G|} \sum_{g \in G} \chi_\alpha^*(g) \chi_{\mathcal{D}}(g).$$

- **For the irreducible representation A:**

$$n_A = \frac{1}{3}(1 \cdot 3 + 1 \cdot 0 + 1 \cdot 0) = 1$$

- **For the irreducible representation C:**

$$n_C = \frac{1}{3}(1 \cdot 3 + \epsilon^* \cdot 0 + (\epsilon^2)^* \cdot 0) = 1.$$

- **For the irreducible representation C^*:**

$$n_{C^*} = \frac{1}{3}(1 \cdot 3 + \epsilon^2 \cdot 0 + \epsilon \cdot 0) = 1.$$

Therefore, we can write

$$\mathcal{D} = A \oplus C \oplus C^*.$$

This decomposition makes sense because the dimensions add up correctly: $1 + 1 + 1 = 3$.

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Problem 2.A
Find all components of the matrix $\mathrm{e}^{\mathrm{i}\alpha A}$ where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Solution. Note that to find $\mathrm{e}^{\mathrm{i}\alpha A}$, we need to calculate the power series

$$\mathrm{e}^{\mathrm{i}\alpha A} = I + \mathrm{i}\alpha A + \frac{(\mathrm{i}\alpha A)^2}{2!} + \frac{(\mathrm{i}\alpha A)^3}{3!} + \cdots$$

Let's calculate powers of A. We have

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$A^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A.$$

We can see that $A^3 = A$, which means $A^4 = A^2$, $A^5 = A^3 = A$, and so on. Now we can regroup the series

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\alpha A} &= I + \mathrm{i}\alpha A + \frac{(\mathrm{i}\alpha)^2}{2!}A^2 + \frac{(\mathrm{i}\alpha)^3}{3!}A^3 + \frac{(\mathrm{i}\alpha)^4}{4!}A^2 + \cdots \\ &= I + \left(\frac{(\mathrm{i}\alpha)^2}{2!} + \frac{(\mathrm{i}\alpha)^4}{4!} + \cdots\right)A^2 + \mathrm{i}\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right)A. \end{aligned}$$

The series in parentheses are familiar, given by

$$\begin{cases} 1 - \dfrac{\alpha^2}{2!} + \dfrac{\alpha^4}{4!} - \cdots = \cos(\alpha) \\ \alpha - \dfrac{\alpha^3}{3!} + \dfrac{\alpha^5}{5!} - \cdots = \sin(\alpha) \end{cases}$$

Therefore, we have

$$\begin{aligned}
e^{i\alpha A} &= I + \sin(\alpha)A + (1-\cos(\alpha))A^2 \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + i\sin(\alpha)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + (1-\cos(\alpha))\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}$$

The final result is

$$e^{i\alpha A} = \begin{pmatrix} \cos(\alpha) & 0 & i\sin(\alpha) \\ 0 & 1 & 0 \\ i\sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix}.$$

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Problem 3

(a) Prove the repeated commutator formula

$$e^{\tau A} B e^{-\tau A} = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} [A,[A,\dots[A,B]],$$

where A and B are square matrices of the same size and τ is a parameter; in the n-th term on the right-hand side, A occurs n times. *Hint:* Consider the n-the derivative with respect to τ.

(b) Use the formula from (a) to solve Problem 2.B from the textbook, stating: If $[A,B]=B$, calculate

$$e^{i\alpha A} B e^{-i\alpha A}.$$

Solution. (a) Let's define

$$f(\tau) = e^{\tau A} B e^{-\tau A}.$$

Finding the derivative, we get

$$\begin{aligned}
\frac{df}{d\tau} &= A e^{\tau A} B e^{-\tau A} - e^{\tau A} B e^{-\tau A} A \\
&= A f(\tau) - f(\tau) A \\
&= [A, f(\tau)].
\end{aligned}$$

For higher derivatives, we have

$$\begin{aligned}
\frac{d^2 f}{d\tau^2} &= [A,[A,f(\tau)]] \\
\frac{d^3 f}{d\tau^3} &= [A,[A,[A,f(\tau)]]] \\
&\vdots
\end{aligned}$$

We see that the n-th derivative is the n-fold nested commutator of A with $f(\tau)$.

Now, at $\tau = 0$, we have

$$\begin{aligned}
f(0) &= B \\
\left.\frac{df}{d\tau}\right|_{\tau=0} &= [A,B] \\
\left.\frac{d^2 f}{d\tau^2}\right|_{\tau=0} &= [A,[A,B]] \\
&\vdots
\end{aligned}$$

Using the Taylor series expansion around $\tau = 0$, we get

$$f(\tau) = f(0) + \tau f'(0) + \frac{\tau^2}{2!} f''(0) + \frac{\tau^3}{3!} f'''(0) + \cdots$$

We get the desired formula

$$\mathrm{e}^{\tau A} B \mathrm{e}^{-\tau A} = B + \tau[A, B] + \frac{\tau^2}{2!}[A, [A, B]] + \frac{\tau^3}{3!}[A, [A, [A, B]]] + \cdots .$$

(b) From Problem 2.B in the textbook, if $[A, B] = B$, we need to calculate

$$\mathrm{e}^{A} B \mathrm{e}^{-A}.$$

Using our formula with $\tau = 1$, we get

$$\begin{aligned} \mathrm{e}^{A} B \mathrm{e}^{-A} &= B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots \\ &= B + B + \frac{1}{2!} B + \frac{1}{3!} B + \cdots . \end{aligned}$$

This is because $[A, B] = B$ implies each nested commutator just gives B again. Therefore

$$\begin{aligned} \mathrm{e}^{A} B \mathrm{e}^{-A} &= B(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots) \\ &= B\mathrm{e}. \end{aligned}$$

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