PHYS 630 - Advanced Electricity and Magnetism

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Homework 10

Problem 1
A relativistic electron with initial Lorentz factor $\gamma_0 \gg 1$ is moving in magnetic field of value B_0 (no parallel velocity). Estimate how the peak frequency of the synchrotron emission evolves with time (as long as $\gamma \geq 1$). You may use approximate values, neglecting factors of $\sim$ few.

Solution. Recall that, for a relativistic electron in a magnetic field, the synchrotron peak frequency is given by

$$\omega_{\text{peak}} = \frac{3}{2}\gamma^2 \omega_B,$$

where $\omega_B = \frac{eB_0}{m_e c}$ is the cyclotron frequency. Taking the time derivative of the synchroton peak frequency, we get

$$\begin{aligned}
\dot{\omega}_{\text{peak}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}\frac{eB_0}{m_e c}\gamma^2\right) \\
&= \frac{3}{2}\frac{eB_0}{m_e c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1-\frac{v^2}{c^2}}\right) \\
&= \frac{3}{2}\frac{eB_0}{m_e c}\frac{2v}{\left(1-\frac{v^2}{c^2}\right)^2} \\
&= \frac{3eB_0}{m_e c} v\gamma^4 \\
&= \frac{3eB_0}{m_e c}\frac{1}{\gamma}\sqrt{\gamma^2-1}\,\gamma^4 \\
&\sim \gamma^3.
\end{aligned}$$

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Problem 2
A circularly polarized electromagnetic wave propagating along $+z$ has an electric field

$$\mathbf{E}_w = \big(\sin(\omega t - kz),\, \cos(\omega t - kz),\, 0\big)E_0.$$

A conducting plate is located at $z = 0$ (so that, for $z > 0$, the total electric field is zero). Find the reflected wave, and the total electromagnetic field in the region $z < 0$.

Solution. The incident wave is

$$\mathbf{E}_i = E_0\big(\sin(\omega t - kz),\, \cos(\omega t - kz)\big).$$

For the reflected wave, we expect a similar form but with kz replaced by $-kz$ (propagating in $-z$ direction) and with possibly different amplitudes and phases, given by

$$\mathbf{E}_r = E_0\big(A\sin(\omega t + kz + \phi_1),\, B\cos(\omega t + kz + \phi_2)\big).$$

At the conducting plate ($z = 0$), the total tangential electric field must be zero. Thus,

$$\mathbf{E}_{\text{total}}(z=0) = \mathbf{E}_i(z=0) + \mathbf{E}_r(z=0) = 0.$$

This gives us the following two equations (for the x and y components)

$$\sin(\omega t) + A\sin(\omega t + \phi_1) = 0,$$
$$\cos(\omega t) + B\cos(\omega t + \phi_2) = 0.$$

These equations must be satisfied for all t, which requires that

$$A = B = 1, \quad \phi_1 = \phi_2 = \pi.$$

Therefore, the reflected wave is

$$\mathbf{E}_r = -E_0\big(\sin(\omega t + kz),\, \cos(\omega t + kz)\big).$$

The total field in the region $z < 0$ is

$$\begin{aligned}
\mathbf{E}_{\text{total}} &= \mathbf{E}_i + \mathbf{E}_r \\
&= E_0\big(\sin(\omega t - kz),\, \cos(\omega t - kz)\big) + E_0\big(-\sin(\omega t + kz),\, -\cos(\omega t + kz)\big) \\
&= E_0\big(\sin(\omega t - kz) - \sin(\omega t + kz),\, \cos(\omega t - kz) - \cos(\omega t + kz)\big) \\
&= E_0\big(2\cos(\omega t)\sin(kz),\, 2\sin(\omega t)\sin(kz)\big) \\
&= 2E_0\big(\cos(\omega t)\sin(kz),\, -\sin(\omega t)\sin(kz)\big).
\end{aligned}$$

The magnetic field can be found using Maxwell's equations or noting that $\mathbf{B} = \frac{1}{c}\hat{k} \times \mathbf{E}$ for each wave. The total magnetic field is then

$$\mathbf{B}_{\text{total}} = -\frac{2E_0}{c}\big(\sin(\omega t)\cos(kz),\, \cos(\omega t)\cos(kz)\big).$$

Therefore, the reflected wave has the same amplitude but opposite polarization, the total field is a standing wave with amplitude modulated by $\sin(kz)$, and at the conducting plate ($z = 0$), the total electric field is zero as required. The field maintains circular polarization at each point in space, but with an amplitude that varies with position. ■