PHYS 630 - Advanced Electricity and Magnetism

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Homework 6

Problem 1
A particle moves in the dipolar Earth magnetosphere. It is reflected at a polar angle $\theta_r = \frac{\pi}{4}$. Find its pitch angle α at the equator, as shown in the figure below.
Hint: The particle moves along a given dipolar field line parameterized by $r = R\sin^2(\theta)$. Using the expression for the strength of the dipolar magnetic field at a given radius r and angle θ, use $r \to R\sin^2(\theta)$. This will give the magnetic field along a given field line. Use conservation of the adiabatic invariant to find $\alpha(\theta)$ (reflection point corresponds to $\alpha = \frac{\pi}{2}$).

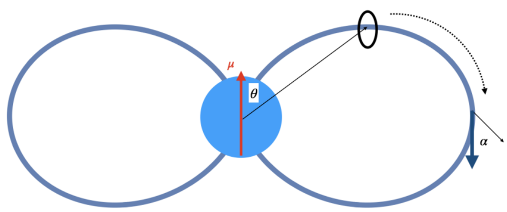

Figure 1: Bouncing particle in the dipolar magnetosphere. At the reflection point (a ring), the pitch angle is $\alpha = \frac{\pi}{2}$.

Solution. The magnetic field of a dipole is given by

$$\mathbf{B} = \{2\cos(\theta), \sin(\theta), 0\}\left(\frac{R}{r}\right)^3 B_0,$$

where $r = R\sin^2(\theta)$, giving us

$$\mathbf{B} = \{2\cos(\theta), \sin(\theta), 0\}\frac{B_0}{\sin^6(\theta)}.$$

Finding the magnitude of $\mathbf{B}$, we have

$$\begin{aligned}
|\mathbf{B}| &= \sqrt{\mathbf{B}_r^2 + \mathbf{B}_\theta^2 + \mathbf{B}_\varphi^2} \\
&= \sqrt{\left(\frac{2\cos(\theta)B_0}{\sin^6(\theta)}\right)^2 + \left(\frac{\sin(\theta)B_0}{\sin^6(\theta)}\right)^2 + 0} \\
&= \frac{B_0}{\sin^6(\theta)}\sqrt{4\cos^2(\theta) + \sin^2(\theta)}.
\end{aligned}$$

From the conservation of adiabatic invariant, we know that

$$\frac{\sin^2(\alpha)}{B_\alpha} = \frac{\sin^2(\alpha_r)}{B_r},$$

where α is the pitch angle at the equator, $B_\alpha = B_0$ is the magnetic field at the equator, and the subscript r refers to the reflection point, *i.e.* when $\alpha_r = \frac{\pi}{2}$. We know that $\theta_r = \frac{\pi}{4}$, then

$$\begin{aligned}
\frac{\sin^2(\alpha)}{B_\alpha} &= \frac{1}{B_r} \\
\sin^2(\alpha) &= \frac{B_\alpha}{B_r} \\
\sin(\alpha) &= \sqrt{\frac{B_\alpha}{B_r}} \\
\alpha &= \sin^{-1}\left(\sqrt{\frac{B_\alpha}{B_r}}\right) \\
\alpha &= \sin^{-1}\left(\sqrt{\frac{B_0}{\frac{B_0}{\sin^6(\theta_r)}\sqrt{4\cos^2(\theta_r)+\sin^2(\theta_r)}}}\right) \\
\alpha &= \sin^{-1}\left(\sqrt{\frac{\sin^6(\theta_r)}{\sqrt{4\cos^2(\theta_r)+\sin^2(\theta_r)}}}\right) \\
\alpha &= \sin^{-1}\left(\frac{\sin^3(\theta_r)}{\left(4\cos^2(\theta_r)+\sin^2(\theta_r)\right)^{\frac{1}{4}}}\right) \\
\alpha &= \sin^{-1}\left(\frac{\sin^3\left(\frac{\pi}{4}\right)}{\left(4\cos^2\left(\frac{\pi}{4}\right)+\sin^2\left(\frac{\pi}{4}\right)\right)^{\frac{1}{4}}}\right) \\
\alpha &= 0.285 \text{ rad.}
\end{aligned}$$

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Problem 2 - Lorentz Transformations for a Particle in a Magnetic Field
Consider a particle moving in a magnetic field along a helical trajectory with a velocity β and a pitch angle α, so that

$$\begin{aligned}
\mu &= \cos(\alpha), \\
\beta_\parallel &= \beta\mu, \\
\beta_\perp &= \beta\sqrt{1-\mu^2}.
\end{aligned}$$

Make a boost with $\beta_\parallel$ to the frame K', where the parallel velocity is zero, $\beta'_\parallel = 0$. (Prime denotes that frame K'). In the frame K', find the velocity $\beta'_\perp$, momentum $p'_\perp$, and Lorentz factor $\gamma'_\perp$.

Solution. We choose the axes such that

$$\frac{v_x}{c} = \beta_\parallel, \quad \frac{v_y}{c} = \beta_\perp, \quad v_z = 0,$$

$$\implies v_x = c\beta_\parallel, \quad v_y = c\beta_\perp, \quad v_z = 0.$$

In the frame K', we have $\beta'_\parallel = 0$, then

$$v'_x = c\beta'_\parallel = 0, \quad v'_y = c\beta'_\perp, \quad v'_z = 0.$$

The Lorentz boosts are then given by

$$\begin{aligned} v_x &= \frac{v'_x + V}{1 + \frac{V}{c^2} v'_x}, \\ v_y &= \frac{v'_y \sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V}{c^2} v'_x}, \\ v_z &= \frac{v'_z \sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V}{c^2} v'_x}. \end{aligned}$$

We have that $v'_x = 0$, then

$$\begin{aligned} v_x &= V = c\beta_{\|} = c\beta\mu, \\ v_y &= v'_y \sqrt{1 - \frac{V^2}{c^2}} \implies v'_y = \frac{v_y}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{c\beta\sqrt{1-\mu^2}}{\sqrt{1-(\beta\mu)^2}} \equiv c\beta'_{\perp}, \end{aligned}$$

where

$$\begin{aligned} \beta'_{\perp} &\equiv \frac{\beta\sqrt{1-\mu^2}}{\sqrt{1-(\beta\mu)^2}} \\ &= \frac{\beta\sin(\alpha)}{\sqrt{1-(\beta\mu)^2}} \\ &= \gamma_{\|}\beta\sin(\alpha). \end{aligned}$$

The relativistic momentum in the frame K' is given by $\mathbf{p} = \gamma m \mathbf{v}$, divided in to parallel and perpendicular components

$$\begin{aligned} p'_{\|} &= \gamma' m \beta'_{\|}, \\ p'_{\perp} &= \gamma' m \beta'_{\perp}, \end{aligned}$$

where γ' is the Lorentz factor in the frame K'. Since we only have the perpendicular component of the velocity, we have that $\gamma'(\beta'_{\perp}) = \gamma'_{\perp}$, which gives

$$\begin{aligned} \gamma'_{\perp} &= \frac{1}{\sqrt{1 - \left(\frac{\beta'_{\perp}}{c}\right)^2}} \\ &= \frac{1}{\sqrt{1 - \left(\frac{\beta\sin(\alpha)}{\sqrt{1-(\beta\mu)^2}}\right)^2}} \\ &= \frac{1}{\sqrt{1 - \left(\frac{\beta^2\sin^2(\alpha)}{1-(\beta\mu)^2}\right)}} \\ &= \frac{\sqrt{1-(\beta\mu)^2}}{\sqrt{1-(\beta\mu)^2 - \beta^2\sin^2(\alpha)}} \\ &= \frac{\sqrt{1-(\beta\mu)^2}}{\sqrt{1-\beta^2\cos^2(\alpha) - \beta^2\sin^2(\alpha)}} \\ &= \frac{1}{\gamma_{\|}\sqrt{1-\beta^2}} \\ &= \frac{\gamma}{\gamma_{\|}}. \end{aligned}$$

Thus, the perpendicular momentum component in the frame K' is

$$\begin{aligned} p'_\perp &= \gamma'_\perp m \beta'_\perp \\ &= \gamma m \beta \sin(\alpha) \\ &= p_\perp . \end{aligned}$$

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Problem 3 - Lorentz Transformations for Fields
A magnetic dipole μ is aligned with z-axis and is moving with a velocity βc along the x direction. For small $\beta \ll 1$, find the electric field in the lab frame.

Solution. The electromagnetic field strength tensor, considering $c = 1$, is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

and a Lorentz transformation for a boost along the x direction with velocity βc can be written as

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^\mu{}_\nu \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}.$$

For $\beta \ll 1$, then $\gamma = 1$ and relativistic effects are ignored. Additionally, we will use the Minkowski metric given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} F^{\mu\nu} &= \Lambda^\mu{}_\lambda F_{\lambda\rho} \Lambda_\rho{}^\nu \\ &= \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\beta E_x & E_x & E_y & E_z \\ -E_x & \beta E_x & -B_z & B_y \\ -E_y - \beta B_z & \beta E_y + B_z & 0 & -B_x \\ -E_z - \beta B_y & \beta E_z - B_y & B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E_x(1-\beta^2) & E_y + \beta B_z & E_z - \beta B_y \\ E_x(\beta^2 - 1) & \beta E_x & -\beta E_y - B_z & -\beta E_z + B_y \\ -E_y - \beta B_z & \beta E_y + B_z & 0 & -B_x \\ -E_z + \beta B_y & \beta E_z - B_y & B_x & 0 \end{pmatrix}. \end{aligned}$$

Thus, the electric field in the lab frame is

$$\mathbf{E} = E_x(1-\beta^2)\hat{\mathbf{x}} + (E_y + \beta B_z)\hat{\mathbf{y}} + (E_z - \beta B_y)\hat{\mathbf{z}}.$$

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