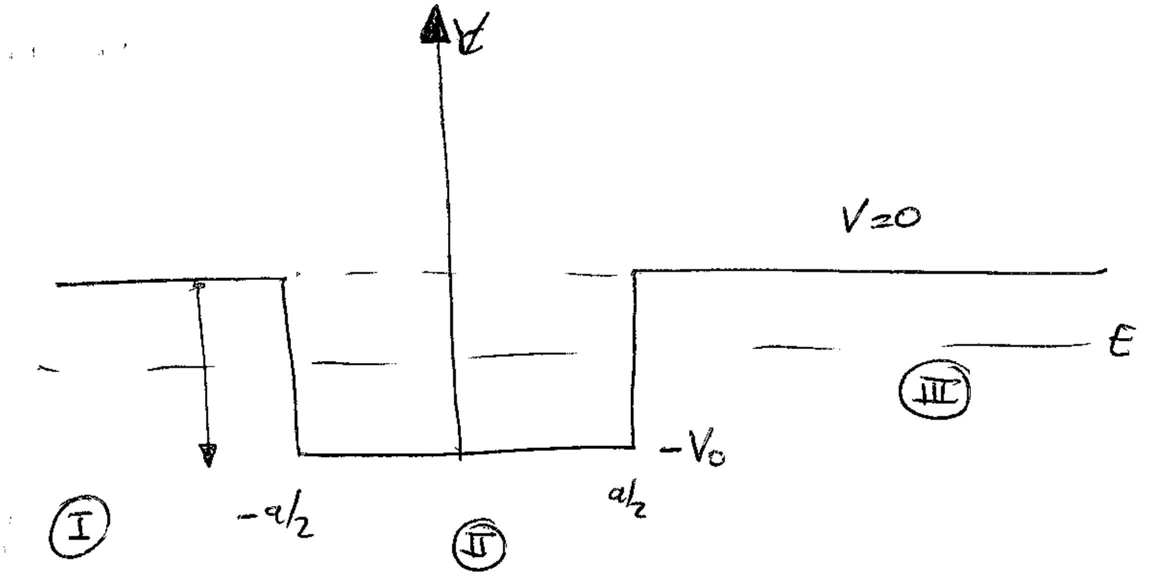


$$\psi_{I} = A e^{kx} + B e^{-kx}$$

$$\psi_{II} = C e^{iKx} + D e^{-iKx}$$

$$\psi_{III} = F e^{kx} + G e^{-kx}$$

$$k = \frac{\sqrt{-2mE}}{\hbar}$$

$$K = \frac{\sqrt{2m(V_0+E)}}{\hbar}$$

$$A e^{-ka/2} + B e^{ka/2} = C e^{-iK\frac{a}{2}} + D e^{iK\frac{a}{2}}$$

$$kA e^{-ka/2} - kB e^{ka/2} = iKC e^{-iK\frac{a}{2}} - iK D e^{iK\frac{a}{2}}$$

$$\begin{pmatrix} e^{-ka/2} & e^{ka/2} \\ ke^{-ka/2} & -ke^{ka/2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} e^{-iK\frac{a}{2}} & e^{iK\frac{a}{2}} \\ iK e^{-iK\frac{a}{2}} & -iK e^{iK\frac{a}{2}} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$C e^{iK\frac{a}{2}} + D e^{-iK\frac{a}{2}} = \cancel{ik} F e^{\frac{ka}{2}} + G e^{-\frac{ka}{2}}$$

$$iKC e^{iK\frac{a}{2}} - iKD e^{-iK\frac{a}{2}} = kF e^{\frac{ka}{2}} - kG e^{-\frac{ka}{2}}$$

$$\begin{pmatrix} e^{iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \\ iK e^{iK\frac{a}{2}} & -iK e^{-iK\frac{a}{2}} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} e^{\frac{ka}{2}} & e^{-\frac{ka}{2}} \\ ke^{\frac{ka}{2}} & -ke^{-\frac{ka}{2}} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc}$$

$$ad-bc = -iK - iK = -2iK$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{i}{2K} \begin{pmatrix} -iK e^{iK\frac{a}{2}} & -e^{iK\frac{a}{2}} \\ -iK e^{-iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \end{pmatrix} \begin{pmatrix} e^{-k\frac{a}{2}} & e^{k\frac{a}{2}} \\ k e^{-k\frac{a}{2}} & -k e^{k\frac{a}{2}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{1}{2k} \begin{pmatrix} -k e^{-k\frac{a}{2}} & -e^{-k\frac{a}{2}} \\ -k e^{k\frac{a}{2}} & e^{k\frac{a}{2}} \end{pmatrix} \begin{pmatrix} e^{iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \\ iK e^{iK\frac{a}{2}} & -iK e^{-iK\frac{a}{2}} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{4kK} \begin{pmatrix} -k e^{-ka/2} & -e^{-ka/2} \\ -k e^{ka/2} & e^{ka/2} \end{pmatrix} \underbrace{\begin{pmatrix} e^{iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \\ iK e^{iK\frac{a}{2}} & -iK e^{-iK\frac{a}{2}} \end{pmatrix} \begin{pmatrix} -iK e^{iK\frac{a}{2}} & -e^{iK\frac{a}{2}} \\ -iK e^{-iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} e^{-k\frac{a}{2}} & e^{k\frac{a}{2}} \\ k e^{-k\frac{a}{2}} & -k e^{k\frac{a}{2}} \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix}$$

$B = 0$, $F = 0$ $\Rightarrow$ eigenstate.

$$\begin{pmatrix} 0 \\ G \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix} = \begin{pmatrix} A t_{11} \\ A t_{21} \end{pmatrix} \Rightarrow \boxed{t_{11} = 0}$$

$$\mathbb{A} = \begin{pmatrix} -iK\left(e^{iKa} + e^{-iKa}\right) & -e^{iKa} + e^{-iKa} \\ K^2 e^{iKa} - K^2 e^{-iKa} & -iK e^{iKa} - iK e^{-iKa} \end{pmatrix} =$$

$$\mathbb{A} = \begin{pmatrix} -2iK \cos Ka & -2i \sin Ka \\ 2iK^2 \sin Ka & -2iK \cos Ka \end{pmatrix} = -2i \begin{pmatrix} K \cos Ka & \sin Ka \\ -K^2 \sin Ka & K \cos Ka \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{1}{2kK} \begin{pmatrix} -k e^{-ka/2} & -e^{-ka/2} \\ -k e^{ka/2} & e^{ka/2} \end{pmatrix} \begin{pmatrix} K cKa & sKa \\ -K^2 sKa & K cKa \end{pmatrix} \begin{pmatrix} e^{-ka/2} \\ k e^{-ka/2} \end{pmatrix} A$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{A e^{-ka/2}}{2kK} \begin{pmatrix} -k e^{-ka/2} & -e^{-ka/2} \\ -k e^{ka/2} & e^{ka/2} \end{pmatrix} \begin{pmatrix} k\,cKa + k\,sKa \\ -k^2 sKa + kK\,cKa \end{pmatrix} \qquad (3)$$

$$= -\frac{A e^{-ka/2}}{2kK} \begin{pmatrix} -kK\,cKa\, e^{-ka/2} - k^2 e^{-ka/2} sKa + K^2 sKa\, e^{-ka/2} - kK e^{-ka/2} cKa \\ e^{ka/2}\left(-kK\,cKa - k^2 sKa - \cancel{K}^2 sKa + kK\,cKa\right) \end{pmatrix}$$

$$-kK\,cKa - k^2 sKa + K^2 sKa - kK\,cKa = 0$$

$$(K^2 - k^2)\, sKa = 2kK\, cKa$$

$$\boxed{\tan Ka = \frac{2kK}{K^2 - k^2}}$$

$$G = -\frac{A}{2kK}\left(-(k^2+K^2)\right) sKa \Rightarrow \frac{G}{A} = \frac{k^2+K^2}{2kK}\, sKa$$

$$1 + \tan^2 Ka = \frac{1}{c^2 Ka} = 1 + \frac{4k^2K^2}{(K^2-k^2)^2} = \frac{(k^2+K^2)^2}{(k^2-K^2)^2} \Rightarrow c^2 Ka = \frac{(k^2-K^2)^2}{(k^2+K^2)^2}$$

$$s^2 Ka = \frac{4k^2K^2}{(k^2+K^2)^2} \qquad |sKa| = \frac{2kK}{k^2+K^2}$$

$$\boxed{\left|\frac{G}{A}\right| = 1}$$

sign(sKa) =
- $+$ even, symmetric

- $-$ odd, anti-symmetric.

$k - iK$; $k^2+K^2 = \frac{2m}{\hbar^2}\left(\cancel{-E}+V_0+\cancel{E}\right) = \frac{2mV_0}{\hbar^2} = k_0^2$

$\boxed{k-iK = k_0 e^{i\varphi}}$ $\quad k = k_0\cos\varphi \quad K = -k_0\sin\varphi$

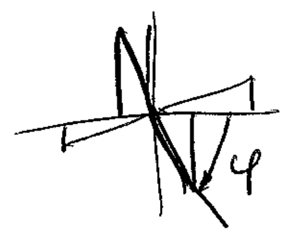
$$\frac{k-iK}{k+iK} = e^{2i\varphi} = \frac{(k-iK)^2}{k^2+K^2} = \frac{k^2-K^2}{k^2+K^2} - \frac{2ikK}{k^2+K^2}$$

$$\tan 2\varphi = -\frac{2kK}{k^2-K^2} = \tan ka$$

$\boxed{2\varphi = ka+n\pi}$ $\Rightarrow \varphi = \frac{ka}{2}+\frac{n\pi}{2}$

φ differs from $ka/2$ by $n\pi/2$ (for some n).

$K = -k_0\sin\left(\frac{ka}{2}+\frac{n\pi}{2}\right)$; $\left(\varphi - \frac{ka}{2}\right) = \frac{n\pi}{2}$

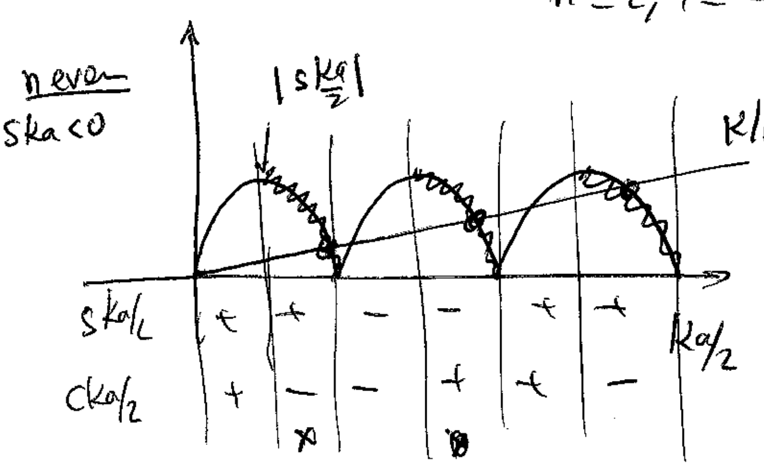


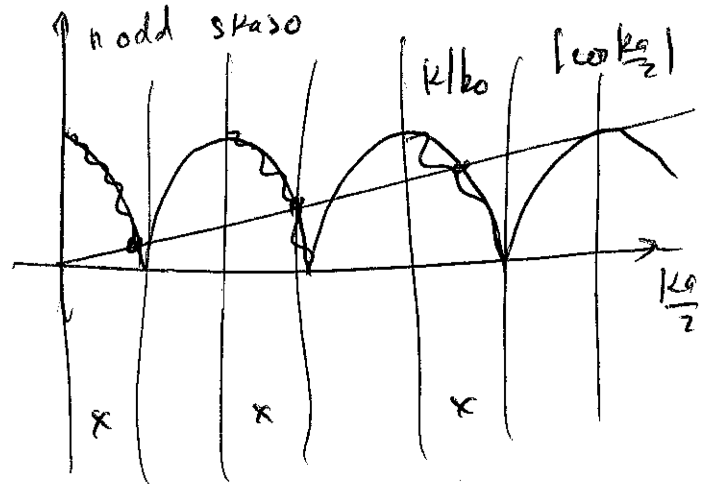
$\Rightarrow \sin\frac{ka}{2} = \pm\frac{K}{k_0}$ (n even) or $\cos\frac{ka}{2} = \pm\frac{K}{k_0}$ (n odd)

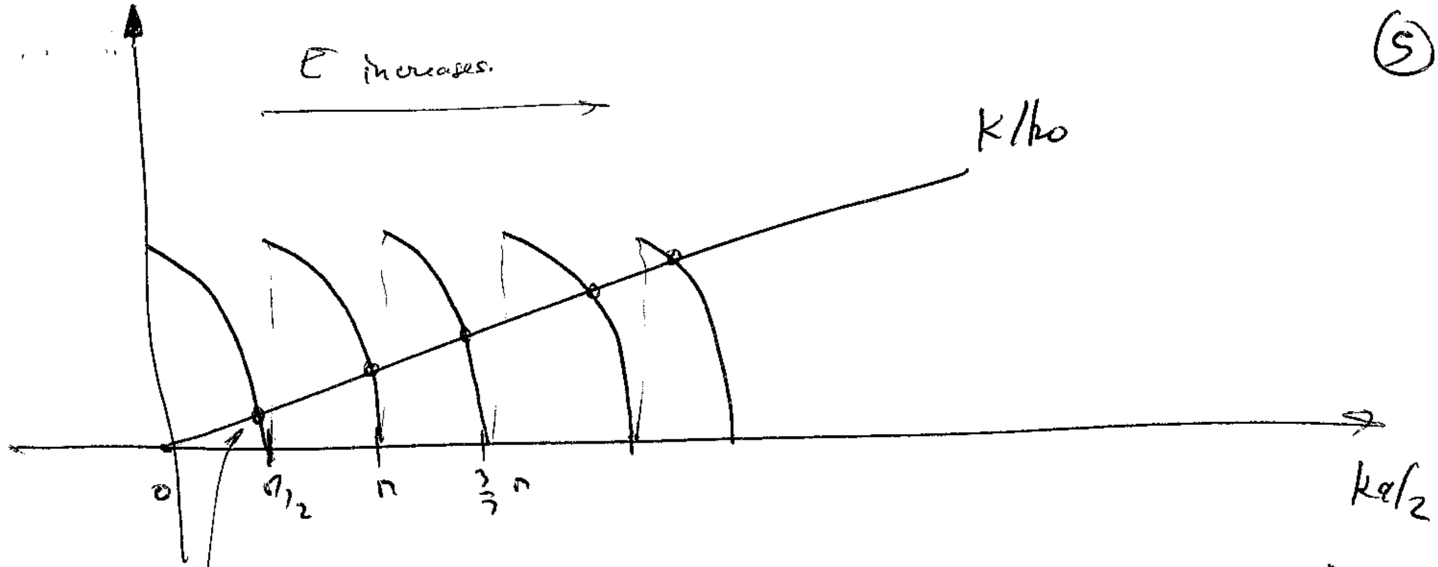
But $s\varphi<0$ $c\varphi>0$ $\quad s2\varphi<0 \Rightarrow s(ka+n\pi)<0 \Rightarrow (-)^n s(ka)<0$

sign$(ska) = (-)^{n+1}$ $\quad n=1,3,\ldots$ odd $\Rightarrow$ symmetric wave-function

$n=2,4,\ldots$ even $\Rightarrow$ antisymmetric wave-function.







at least one eigenstate (for any k_0, i.e. V_0, a)

$$E = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{\hbar^2}{2m}\left(k_0^2 - k^2\right) = -V_0 + \frac{\hbar^2 k^2}{2m}$$

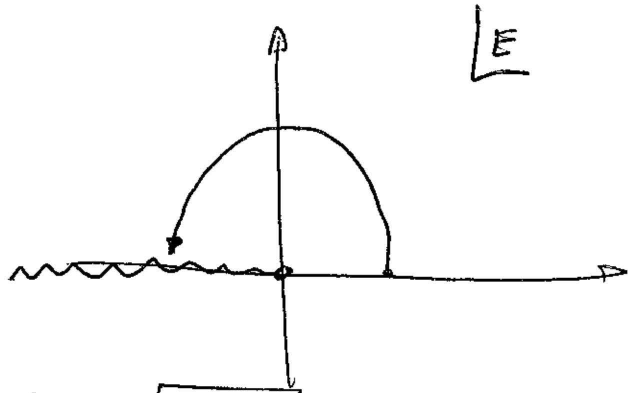
$k_0 \to \infty$ $\qquad \frac{ka}{2} = n\frac{\pi}{2}$ $\qquad \boxed{ka = n\pi}$ ∞ particle in a box.

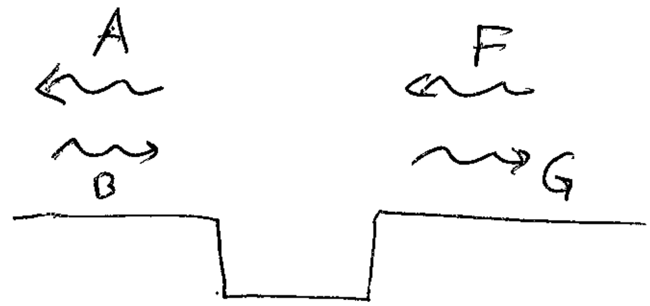
Continuum spectrum

$\tilde{k} = \frac{\sqrt{2mE}}{\hbar}$ $\quad K = \frac{\sqrt{2m(V_0+E)}}{\hbar}$ 6

$E>0$

$K^2 - \tilde{k}^2 = k_0^2$





$$\tilde{k} = \frac{\sqrt{2mE}}{\hbar} \longrightarrow ik$$

$$\tilde{k} = ik \qquad k = -i\tilde{k}$$

($B=0$, $F=0$, $A, G \neq 0$
bound state)

$$\psi_I = A e^{-i\tilde{k}x} + B e^{i\tilde{k}x}$$

$$\psi_{II} = C e^{iKx} + D e^{-iKx}$$

$$\psi_{III} = F e^{-i\tilde{k}x} + G e^{i\tilde{k}x}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = \frac{-2i}{4\tilde{k}K} \begin{pmatrix} i\tilde{k}\, e^{i\tilde{k}a/2} & -e^{i\tilde{k}a/2} \\ i\tilde{k}\, e^{-i\tilde{k}a/2} & e^{-i\tilde{k}a/2} \end{pmatrix} \begin{pmatrix} K cKa & sKa \\ -K^2 sKa & K cKa \end{pmatrix} \begin{pmatrix} e^{i\frac{\tilde{k}a}{2}} & e^{-i\frac{\tilde{k}a}{2}} \\ -i\tilde{k}e^{i\frac{\tilde{k}a}{2}} & i\tilde{k}e^{-i\frac{\tilde{k}a}{2}} \end{pmatrix} \times \begin{pmatrix} A \\ B \end{pmatrix}$$

$$= -\frac{i}{2\tilde{k}K} \begin{pmatrix} i\tilde{k}K cKa\, e^{i\frac{\tilde{k}a}{2}} + K^2 sKa\, e^{i\frac{\tilde{k}a}{2}} & e^{i\frac{\tilde{k}a}{2}}(i\tilde{k} sKa - K cKa) \\ e^{-i\frac{\tilde{k}a}{2}}(i\tilde{k} K cKa - K^2 sKa) & e^{-i\frac{\tilde{k}a}{2}}(i\tilde{k} sKa + K cKa) \end{pmatrix} \begin{pmatrix} e^{i\frac{\tilde{k}a}{2}} & e^{-i\frac{\tilde{k}a}{2}} \\ -i\tilde{k}e^{i\frac{\tilde{k}a}{2}} & i\tilde{k}e^{-i\frac{\tilde{k}a}{2}} \end{pmatrix}$$

⑦

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{2\tilde{k}K}\begin{pmatrix} e^{i\tilde{k}a}\left(i\tilde{k}K\,cKa + K^2 sKa + \tilde{k}^2 sKa + iK\tilde{k}\,cKa\right) & i\tilde{k}K\,cKa + K^2 sKa - \tilde{k}^2 sKa - i\tilde{k}K\,cKa \\ i\tilde{k}K\,cKa - K^2 sKa + \tilde{k}^2 sKa - i\tilde{k}K\,cKa & e^{-i\tilde{k}a}\left(i\tilde{k}K\,cKa - K^2 sKa - \tilde{k}^2 sKa + i\tilde{k}K\,cKa\right) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{2\tilde{k}K}\begin{pmatrix} e^{i\tilde{k}a}\left((K^2+\tilde{k}^2)\,sKa + 2iK\tilde{k}\,cKa\right) & k_0^2\,sKa \\ -k_0^2\,sKa & e^{-i\tilde{k}a}\left(-(K^2+\tilde{k}^2)\,sKa + 2iK\tilde{k}\,cKa\right) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\det T = -\frac{1}{4\tilde{k}^2K^2}\Big[-4\tilde{k}^2K^2c^2Ka\underbrace{-\,(K^2+\tilde{k}^2)^2s^2Ka + \overset{(K^2-\tilde{k}^2)^2}{k_0^4}\,s^2Ka}_{-4K^2\tilde{k}^2s^2Ka}$$

$$= \cancel{-}\,\cancel{\frac{1}{4\tilde{k}^2K^2}}\,(\cancel{-4\tilde{k}^2K^2})\,(c^2Ka + s^2Ka) = 1.$$

$$F = At_{11} + Bt_{12}$$

$$G = At_{21} + Bt_{22}$$

Scattering matrix

$$\begin{pmatrix} G \\ A \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} B \\ F \end{pmatrix}$$

$$A = \frac{1}{t_{11}} F - \frac{t_{12}}{t_{11}} B \qquad ; \; G = \frac{t_{21}}{t_{11}} F - \frac{t_{21} t_{12}}{t_{11}} B + B t_{22}$$

$$G = \frac{\det T}{t_{11}} B + \frac{t_{21}}{t_{11}} F = \frac{1}{t_{11}} B + \frac{t_{21}}{t_{11}} F$$

$$\begin{pmatrix} G \\ A \end{pmatrix} = \frac{1}{t_{11}} \begin{pmatrix} 1 & t_{21} \\ -t_{12} & 1 \end{pmatrix} \begin{pmatrix} B \\ F \end{pmatrix}$$

$$S = \frac{i\, 2\tilde{k} K e^{-i\tilde{k}a}}{(k^2+\tilde{k}^2) ska + 2ik\tilde{k} cka} \begin{pmatrix} 1 & +\frac{i k_0^2\, ska}{2\tilde{k}K} \\ +\frac{i k_0^2\, ska}{2\tilde{k}K} & 1 \end{pmatrix}$$

$$S = \frac{2i\tilde{k}K\, e^{-i\tilde{k}a}}{(k^2+\tilde{k}^2) ska + 2i\tilde{k}Kcka} \begin{pmatrix} 1 & +\frac{i k_0^2\, ska}{2\tilde{k}K} \\ \frac{i k_0^2\, ska}{2\tilde{k}K} & 1 \end{pmatrix}$$

$$\boxed{S = \frac{e^{-i\tilde{k}a}}{(k^2+\tilde{k}^2) ska + 2i\tilde{k}Kcka} \begin{pmatrix} 2i\tilde{k}K & -k_0^2\, ska \\ -k_0^2\, ska & 2i\tilde{k}K \end{pmatrix}}$$

Spectrum vs double degeneracy.

2 states $\boxed{\begin{matrix} B=1, & F=0 \\ G=S_{11} & A=S_{21} \end{matrix}}$; $\boxed{\begin{matrix} B=0; & F=1 \\ G=S_{12}, & A=S_{22} \end{matrix}}$

$$SS^{+} = \frac{1}{(k^2+\tilde{k}^2)^2 s^2 ka + 4\tilde{k}^2 k^2 \underbrace{c^2 ka}_{1-s^2 ka}} \begin{pmatrix} 2i\tilde{k}k & -k_0^2 ska \\ -k_0^2 ska & 2i\tilde{k}k \end{pmatrix} \begin{pmatrix} -2i\tilde{k}k & -k_0^2 ska \\ -k_0^2 ska & -2i\tilde{k}k \end{pmatrix}$$

$$\begin{pmatrix} 4\tilde{k}^2 k^2 + k_0^2 s^2 ka & \cancel{-2i\tilde{k}k k_0^2 ska + 2i\tilde{k}k k_0^2 ska} \\ 0 & k_0^4 s^2 ka + 4\tilde{k}^2 k^2 \end{pmatrix}$$

$$SS^{+} = \frac{\cancel{4\tilde{k}^2 k^2 + k_0^2 s^2 ka}}{\cancel{4\tilde{k}^2 k^2 + [(k^2+\tilde{k}^2)^2 - 4\tilde{k}^2 k^2]} s^2 ka} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(k^2 - \tilde{k}^2)^2 = k_0^2$$

$$\boxed{SS^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{unitary!}}$$

Other properties :

$k_0 \to 0$ (no potential) $\quad K^2 = \tilde{k}^2 \quad K = \tilde{k}$

$$S = \frac{e^{-i\tilde{k}a}}{(K^2+k^2)\, sKa + 2ik^2 cKa} \begin{pmatrix} 2ik^2 & 0 \\ 0 & 2ik^2 \end{pmatrix}$$

$$= \frac{\cancel{2}ik^2\, e^{-i\tilde{k}a}}{\cancel{2}k^2(sKa + icKa)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \cancel{\frac{i\, e^{-ika}}{i\, e^{-ika}}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$S \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ no potential.

Bound states. analytic continuation $\tilde{k} \to k$

$B, F = 0 \quad , \quad A, G \neq 0 \qquad S^{-1}\begin{pmatrix} G \\ A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$S \to \infty \quad S$ has poles.

$(K^2 + \tilde{k}^2)\, sKa + 2i\tilde{k}K\, cKa = 0$ not possible but

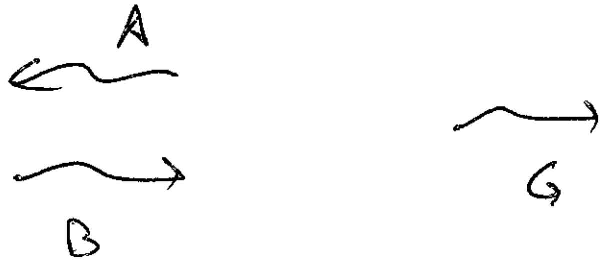
after $\tilde{k} \to ik$

$$(K^2 - k^2)\, sKa - 2kK\, cKa = 0 \Rightarrow \boxed{\tan Ka = \frac{2kK}{K^2 - k^2}}$$

✓

Transmission & reflection coefficients

F = 0



$$T = \frac{|G|^2}{|B|^2} \qquad R = \frac{|A|^2}{|B|^2}$$

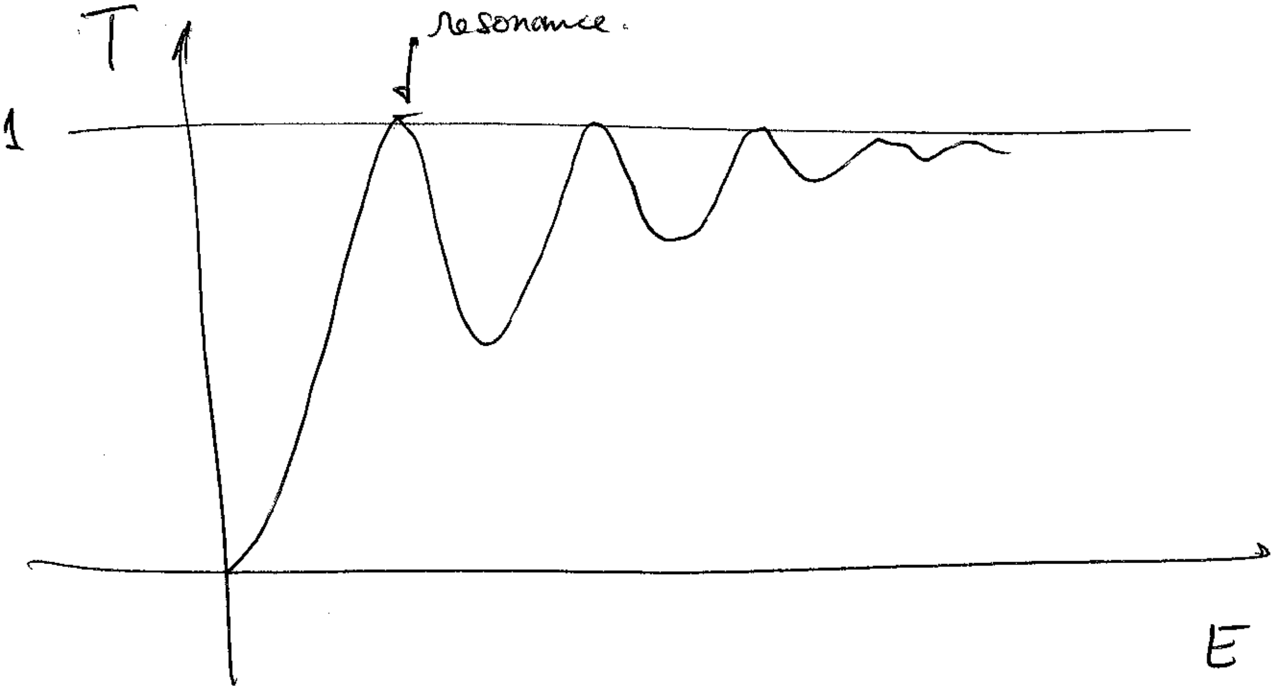
$$F = 0 \Rightarrow G = S_{11} B \qquad A = S_{21} B$$

$$T = |S_{11}|^2 \qquad R = |S_{21}|^2$$

$$T = \frac{4\tilde{k}^2 k^2}{(k^2+\tilde{k}^2)^2 s^2 ka + 4\tilde{k}^2 k^2 \underbrace{c^2 ka}_{1-s^2 ka}} = \frac{4\tilde{k}^2 k^2}{4\tilde{k}^2 k^2 + [\underbrace{(k^2+\tilde{k}^2)^2 - 4\tilde{k}^2 k^2}_{(k^2-\tilde{k}^2)^2 = k_0^4}] s^2 ka}$$

$$T = \frac{1}{1 + \frac{k_0^4}{4k^2\tilde{k}^2} s^2 ka}$$

$$R = \frac{k_0^2 s^2 ka}{4\tilde{k}^2 k^2 + k_0^4 s^2 ka} = \frac{\frac{k_0^2 s^2 ka}{4\tilde{k}^2 k^2}}{1 + \frac{k_0^4}{4k^2\tilde{k}^2} s^2 ka}.$$



$S_{ka} = 0$ $\boxed{ka = n\pi}$

$$\frac{\hbar^2 k^2}{2m} = V_0 + E \qquad \boxed{E = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 - V_0}$$

similar to bound states of square well. Fits wavelength

At resonance.

$$S = \frac{e^{-i\tilde{k}a}}{\cancel{2i\tilde{k}k}\, c_{ka}} \begin{pmatrix} \cancel{2i\tilde{k}k} & 0 \\ 0 & \cancel{2i\tilde{k}k} \end{pmatrix} = (-)^n e^{-i\tilde{k}a} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

like no potential except for a <u>phase</u>

Finally :

Suppose $\quad B = F$

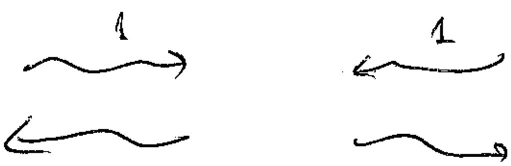
$$\begin{pmatrix} G \\ A \end{pmatrix} = B \begin{pmatrix} S_{11} + S_{12} \\ S_{21} + S_{22} \end{pmatrix}$$

$$S_{11} + S_{12} = \frac{e^{-i\tilde{k}a}}{(K^2 + \tilde{k}^2)\, sKa + 2i\tilde{k}K cKa} \left(2i\tilde{k}K - k_0^2\, sKa \right)$$

$$S_{21} + S_{22} = S_{11} + S_{12} \quad \Rightarrow \quad G = A.$$

$$\frac{G}{B} = \frac{e^{-i\tilde{k}a}\,(2i\tilde{k}K - k_0^2\, sKa)}{(K^2 + \tilde{k}^2)\, sKa + 2i\tilde{k}K cKa}$$

$$\left| \frac{G}{B} \right| = \frac{4\tilde{k}^2K^2 + k_0^4 s^2Ka}{4\tilde{k}^2K^2 + k_0^4 s^2Ka} = 1 \; \underline{!}$$



$e^{i\delta}$ ← phase shift

$$(2i\tilde{k}K - k_0^2\, sKa)^2 (-2i\tilde{k}K - k_0^2\, sKa) = -4\tilde{k}^2K^2 + k_0^4 s^2Ka - 4i\tilde{k}K k_0^2\, sKa$$

$x \leftrightarrow -x$ symmetry.

$A \leftrightarrow G$

$B \leftrightarrow F$

$$\begin{pmatrix} A \\ G \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} F \\ B \end{pmatrix}$$

but also

$$\begin{pmatrix} A \\ G \end{pmatrix} = \begin{pmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{pmatrix} \begin{pmatrix} F \\ B \end{pmatrix}$$

$$\Rightarrow \quad \boxed{S_{11} = S_{22}} \quad \boxed{S_{12} = S_{21}} \quad \checkmark$$