PHYS 601 - Methods of Theoretical Physics II

Mathematical Methods for Physicists by *Arfken, Weber, Harris*

Student: **Ralph Razzouk**

Homework 4

Problem 1

The first four Legendre polynomials are

$$P_0(x) = 1, \quad P_2 = \frac{1}{2}(3x^2 - 1),$$
$$P_1(x) = x, \quad P_3 = \frac{1}{2}(5x^3 - 3x).$$

Obtain these four polynomials by each of the following methods:

(a) Generating function,

(b) Rodrigues' formula,

(c) Schmidt orthogonalization,

(d) Series solution.

Solution. (a) The generating function of the Legendre polynomials is given by

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

The Taylor expansion of the left hand-side has the form of

$$g(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left.\frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t)\right|_{t=0} t^n.$$

By comparison, we have

$$P_n(x) = \frac{1}{n!} \left.\frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t)\right|_{t=0}.$$

- **For $n = 0$:**

$$\begin{aligned} P_0(x) &= \frac{1}{0!} \left.\frac{\mathrm{d}^0}{\mathrm{d}t^0} g(x,t)\right|_{t=0} \\ &= g(x,t)|_{t=0} \\ &= \left.\frac{1}{\sqrt{1-2xt+t^2}}\right|_{t=0} \\ &= 1. \end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned} P_1(x) &= \frac{1}{1!} \left.\frac{\mathrm{d}^1}{\mathrm{d}t^1} g(x,t)\right|_{t=0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t} g(x,t)\right|_{t=0} \\ &= \left.\frac{x-t}{(1-2tx+t^2)^{\frac{3}{2}}}\right|_{t=0} \\ &= x. \end{aligned}$$

- **For $n = 2$:**

$$\begin{aligned}
P_2(x) &= \frac{1}{2!}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(x,t)\bigg|_{t=0} \\
&= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(x,t)\bigg|_{t=0} \\
&= \frac{1}{2}\frac{2t^2 - 4xt + 3x^2 - 1}{(1-2xt+t^2)^{\frac{5}{2}}}\bigg|_{t=0} \\
&= \frac{1}{2}(3x^2 - 1).
\end{aligned}$$

- **For $n = 3$:**

$$\begin{aligned}
P_3(x) &= \frac{1}{3!}\,\frac{\mathrm{d}^3}{\mathrm{d}t^3} g(x,t)\bigg|_{t=0} \\
&= \frac{1}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3} g(x,t)\bigg|_{t=0} \\
&= \frac{1}{6}\frac{3\,(x-t)\left(2t^2 - 4xt + 5x^2 - 3\right)}{(1-2xt+t^2)^{\frac{7}{2}}}\bigg|_{t=0} \\
&= \frac{1}{2}(5x^3 - 3x).
\end{aligned}$$

(b) The Rodrigues' formula for the Legendre polynomials is given by

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

Calculating, we have

- **For $n = 0$:**

$$\begin{aligned}
P_0(x) &= \frac{1}{2^0 0!}\frac{\mathrm{d}^0}{\mathrm{d}x^0}(x^2-1)^0 \\
&= 1.
\end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned}
P_1(x) &= \frac{1}{2^1 1!}\frac{\mathrm{d}^1}{\mathrm{d}x^1}(x^2-1)^1 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1) \\
&= x.
\end{aligned}$$

- **For $n = 2$:**

$$\begin{aligned}
P_2(x) &= \frac{1}{2^2 2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 \\
&= \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^4 - 2x^2 + 1) \\
&= \frac{1}{8}(12x^2 - 4) \\
&= \frac{1}{2}(3x^2 - 1).
\end{aligned}$$

- **For $n = 3$:**

$$\begin{aligned} P_3(x) &= \frac{1}{2^3 3!}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2-1)^3 \\ &= \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^6 - 3x^4 + 3x^2 - 1) \\ &= \frac{1}{48}(120x^3 - 72x) \\ &= \frac{1}{2}(5x^3 - 3x). \end{aligned}$$

(c) We define an inner product

$$\langle f|g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$$

and a set of functions

$$u_n(x) = x^n,$$

where n is a non-negative integer.

We now generate a set of orthonormal functions $\phi_n(x)$ using the Gram-Schmidt orthogonalization process. We have

- **For $n = 0$:**

$$\begin{aligned} \phi_0(x) &= \frac{u_0(x)}{\sqrt{\langle u_0|u_0\rangle}} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned} \psi_1(x) &= u_1(x) - \langle \phi_0|u_1\rangle\,\phi_0(x) \\ &= x - \frac{1}{2}\int_{-1}^{1} x\,\mathrm{d}x \\ &= x. \end{aligned}$$

Normalizing, we have

$$\phi_1(x) = \frac{\psi_1(x)}{\sqrt{\langle\psi_1|\psi_1\rangle}} = \frac{\psi_1(x)}{\sqrt{\int_{-1}^{1} x^2\,\mathrm{d}x}} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}x$$

- **For $n = 2$:**

$$\begin{aligned} \psi_2(x) &= u_2(x) - \langle \phi_0|u_2\rangle\,\phi_0(x) - \langle \phi_1|u_2\rangle\,\phi_1(x) \\ &= x^2 - \frac{1}{2}\int_{-1}^{1} x^2\,\mathrm{d}x - \frac{3}{2}x\int_{-1}^{1} x^3\,\mathrm{d}x \\ &= x^2 - \frac{1}{3}. \end{aligned}$$

Normalizing, we have

$$\phi_2(x) = \frac{\psi_2(x)}{\sqrt{\langle\psi_2|\psi_2\rangle}} = \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2\mathrm{d}x}} = \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2-1).$$

- **For** $n = 3$**:**

$$\begin{aligned}
\psi_3(x) &= u_3(x) - \langle \phi_0 | u_3 \rangle \, \phi_0(x) - \langle \phi_1 | u_3 \rangle \, \phi_1(x) - \langle \phi_2 | u_3 \rangle \, \phi_2(x) \\
&= x^3 - \frac{1}{2} \int_{-1}^{1} x^3 \, \mathrm{d}x - \frac{3}{2} x \int_{-1}^{1} x^4 \, \mathrm{d}x - \frac{5}{8}(3x^2 - 1) \int_{-1}^{1} (3x^2 - 1) x^3 \, \mathrm{d}x \\
&= x^3 - \frac{3}{2} x \int_{-1}^{1} x^4 \, \mathrm{d}x \\
&= x^3 - \frac{3}{5} x.
\end{aligned}$$

Normalizing, we have

$$\phi_3(x) = \frac{\psi_3(x)}{\sqrt{\langle \psi_3 | \psi_3 \rangle}} = \frac{x^3 - \frac{3}{5}x}{\sqrt{\int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)^2 \mathrm{d}x}} = \frac{1}{2}\sqrt{\frac{7}{2}}(5x^3 - 3x).$$

The Legendre polynomials $P_n(x)$ would then be

$$P_n(x) = \sqrt{\frac{2}{2n+1}} \phi_n(x)$$

(d) We seek solutions of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_\lambda \neq 0.$$

Deriving and replacing back into the Legendre ODE, we get

$$(1 - x^2) \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - 2x \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1} + n(n+1) \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - \sum_{\lambda=0}^{\infty} a_\lambda \left[(k+\lambda)(k+\lambda+1) - n(n+1)\right] x^{k+\lambda} = 0$$

Setting $\lambda = 0$, we get

- **Lowest order** x^{k-2}: This gives us the indicial equation

$$a_0 k(k-1) = 0 \implies k = 0, 1.$$

- **First order** x^{k-1}:

$$a_1 (k+1) k = 0 \implies \begin{cases} a_1 \text{ arbitrary,} & \text{if } k = 0, \\ a_1 = 0, & \text{if } k = 1, \end{cases}$$

- **General order** x^{k+j}:

$$a_{j+2}(k+j+2)(k+j+1) - a_j \left[(k+j)(k+j+1) - n(n+1)\right] = 0$$

$$a_{j+2} = a_j \frac{(k+j)(k+j+1) - n(n+1)}{(k+j+1)(k+j+2)}$$

  - **For** $k = 0$**:** We have

$$a_{j+2} = a_j \frac{j(j+1) - n(n+1)}{(j+1)(j+2)}.$$

Since a_1 is arbitrary for $k = 0$, then we can set it to zero. Then we get a solution explicitly for even powers and

$$y_{2n} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda}$$

- **For** $k = 1$**:** We have

$$a_{j+2} = a_j \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)}.$$

Since $a_1 = 0$ for $k = 1$, then we get a solution explicitly for odd powers and

$$y_{2n+1} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda+1}$$

- **For** $n = 0$**:** We have

$$y_0 = P_0(x) = a_0.$$

Set $a_0 = 1$, then

$$P_0(x) = 1.$$

- **For** $n = 1$**:** We have

$$y_1 = P_1(x) = a_0 x.$$

Set $a_0 = 1$, then

$$P_1(x) = x.$$

- **For** $n = 2$**:** We have

$$y_2 = P_2(x) = a_0 - 3a_0 x^2.$$

Set $a_0 = -\frac{1}{2}$, then

$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$

- **For** $n = 3$**:** We have

$$y_3 = P_3(x) = a_0 x - \frac{5}{3} a_0 x^3.$$

Set $a_0 = -\frac{3}{2}$, then

$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

■

Problem 2
The Hermite differential equation is $H_n'' - 2xH_n' + 2nH_n = 0$.

(a) Solve this equation by series solution and show that it terminates for integral values of n.

(b) Use the series solution to generate the first four Hermite polynomials which are

$$H_0(x) = 1, \qquad H_2 = 4x^2 - 2,$$
$$H_1(x) = 2x, \quad H_3 = 8x^3 - 12x.$$

(c) Obtain the first four Hermite polynomials using the generating function which is

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

(d) Using the generating function, derive the recurrence relations

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,$$
$$H_n'(x) - 2nH_{n-1}(x) = 0.$$

(e) Using the result of part (d), verify that the $H_n(x)$ defined by the generating function obeys the Hermite differential equation.

Solution. (a) We seek solutions of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_\lambda \neq 0.$$

Deriving and replacing back into the Legendre ODE, we get

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} - 2x\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)x^{k+\lambda-1} + 2n\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} - \sum_{\lambda=0}^{\infty} 2a_\lambda (k+\lambda-n)x^{k+\lambda} = 0$$

Setting $\lambda = 0$, we get

- **Lowest order** x^{k-2}: This gives us the indicial equation

$$a_0 k(k-1) = 0 \implies k = 0, 1.$$

- **First order** x^{k-1}:

$$a_1(k+1)k = 0 \implies \begin{cases} a_1 \text{ arbitrary}, & \text{if } k = 0, \\ a_1 = 0, & \text{if } k = 1, \end{cases}$$

- **General order** x^{k+j}:

$$a_{j+2}(k+j+2)(k+j+1) - 2a_j(k+j-n) = 0$$

$$a_{j+2} = a_j \frac{2(k+j-n)}{(k+j+1)(k+j+2)}$$

- **For $k = 0$:** We have
$$a_{j+2} = a_j \frac{2(j-n)}{(j+1)(j+2)}.$$
Since a_1 is arbitrary for $k = 0$, then we can set it to zero. Then we get a solution explicitly for even powers and
$$y_{2n} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda}$$
Notice, since j is even from the recurrence relation, that if n is also even, then there will be some term that is zero which terminates the series.
- **For $k = 1$:** We have
$$a_{j+2} = a_j \frac{2(j+1-n)}{(j+2)(j+3)}.$$
Since $a_1 = 0$ for $k = 1$, then we get a solution explicitly for odd powers and
$$y_{2n+1} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda+1}$$
Notice, since j is odd from the recurrence relation, that if n is also odd, then there will be some term that is zero which terminates the series.

(b)
- **For $n = 0$:** We have
$$y_0 = H_0(x) = a_0.$$
Set $a_0 = 1$, then
$$H_0(x) = 1.$$
- **For $n = 1$:** We have
$$y_1 = H_1(x) = a_0 x.$$
Set $a_0 = 2$, then
$$H_1(x) = 2x.$$
- **For $n = 2$:** We have
$$y_2 = H_2(x) = a_0 - 2a_0 x^2.$$
Set $a_0 = -2$, then
$$H_2(x) = 4x^2 - 2.$$
- **For $n = 3$:** We have
$$y_3 = H_3(x) = a_0 x - \frac{2}{3} a_0 x^3.$$
Set $a_0 = -12$, then
$$H_3(x) = 8x^3 - 12x.$$

(c) The generating function of the Hermite polynomials is given by
$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$
The Taylor expansion of g has the form of
$$g(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t) \right|_{t=0} t^n.$$
By comparison, we have
$$H_n(x) = \left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t) \right|_{t=0}.$$

- **For $n = 0$:**

$$\begin{aligned} H_0(x) &= \left.\frac{\mathrm{d}^0}{\mathrm{d}t^0} g(x,t)\right|_{t=0} \\ &= g(x,t)|_{t=0} \\ &= \left. e^{-t^2+2tx}\right|_{t=0} \\ &= 1. \end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned} H_1(x) &= \left.\frac{\mathrm{d}^1}{\mathrm{d}t^1} g(x,t)\right|_{t=0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t} g(x,t)\right|_{t=0} \\ &= \left. 2(x-t)e^{-t^2+2tx}\right|_{t=0} \\ &= 2x. \end{aligned}$$

- **For $n = 2$:**

$$\begin{aligned} H_2(x) &= \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(x,t)\right|_{t=0} \\ &= \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2} e^{-t^2+2tx}\right|_{t=0} \\ &= \left. (2(x-t))^2 e^{-t^2+2tx} - 2e^{-t^2+2tx}\right|_{t=0} \\ &= 4x^2 - 2. \end{aligned}$$

- **For $n = 3$:**

$$\begin{aligned} H_3(x) &= \left.\frac{\mathrm{d}^3}{\mathrm{d}t^3} g(x,t)\right|_{t=0} \\ &= \left.\frac{\mathrm{d}^3}{\mathrm{d}t^3} e^{-t^2+2tx}\right|_{t=0} \\ &= \left. (2(x-t))^3 e^{-t^2+2tx} - 8(x-t)e^{-t^2+2tx} - 4(x-t)e^{-t^2+2tx}\right|_{t=0} \\ &= 8x^3 - 12x. \end{aligned}$$

(d) The generating function of the Hermite polynomials is given by

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

- Deriving both sides with respect to t, we get

$$\frac{\partial g(x,t)}{\partial t} = 2(x-t)e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!},$$

which implies that

$$\begin{aligned} 2(x-t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} \\ \sum_{n=0}^{\infty} 2xH_n(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} &= \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} \end{aligned}$$

$$\sum_{m=1}^{\infty} 2H_{m-1}(x)\frac{t^m}{(m-1)!} - \sum_{m=0}^{\infty} 2xH_m(x)\frac{t^m}{m!} + \sum_{m=-1}^{\infty} H_m(x)\frac{t^m}{m!} = 0$$
$$\sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!} - \sum_{m=0}^{\infty} 2xH_m(x)\frac{t^m}{m!} + \sum_{m=-1}^{\infty} H_{m+1}(x)\frac{t^m}{m!} = 0$$
$$\implies H_{n+1} - 2xH_n + 2nH_{n-1} = 0.$$

- Deriving both sides with respect to x, we get

$$\frac{\partial g(x,t)}{\partial x} = 2te^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!},$$

which implies that

$$\sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!}$$
$$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} = 0$$

$$\sum_{m=0}^{\infty} H_m'(x)\frac{t^m}{m!} - \sum_{m=1}^{\infty} 2H_{m-1}(x)\frac{t^m}{(m-1)!} = 0$$
$$\sum_{m=0}^{\infty} H_m'(x)\frac{t^m}{m!} - \sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!} = 0$$
$$\implies H_n' - 2nH_{n-1} = 0.$$

(e) Replacing in the Hermite differential equation, we have

$$\begin{aligned} H_n'' - 2xH_n' + 2nH_n & \\ &= 2nH_{n-1}' - 4nxH_{n-1} + 2nH_n \\ &= 4n^2H_{n-2}' - 4nxH_{n-1} + 2nH_n \\ &= 0. \end{aligned}$$

■

Problem 3
Use the generating function for the Bessel functions,

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n,$$

to obtain the following recurrence relations

(a) $J_{n-1} + J_{n+1} = \frac{2n}{x} J_n$,

(b) $J_{n-1} - J_{n+1} = 2J_n'$,

(c) $J_{n-1} - \frac{n}{x} J_n = J_n'$,

(d) $J_{n+1} - \frac{n}{x} J_n = -J_n'$.

(e) Using the above results, verify that J_n satisfies Bessel's equation,

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0.$$

(e) Verify that the series solution

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

satisfies the same equation.

Solution. The generating function of the Bessel functions is given by

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n.$$

(a) Deriving both sides with respect to t, we get

$$\frac{\partial g(x,t)}{\partial t} = \frac{x}{2}\left(1+\frac{1}{t^2}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1},$$

which implies that

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right)\sum_{-\infty}^{\infty} J_n(x)t^n = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1}$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^n + \frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^{n-2} = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1}$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^{n-2} - \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1} + \frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^n = 0$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_{m+1}(x)t^{m-1} - \sum_{-\infty}^{\infty} mJ_m(x)t^{m-1} + \frac{x}{2}\sum_{-\infty}^{\infty} J_{m-1}(x)t^{m-1} = 0$$

$$\implies J_{n+1} + J_{n-1} = \frac{2n}{x} J_n.$$

(b) Deriving both sides with respect to x, we get

$$\frac{\partial g(x,t)}{\partial x} = \frac{\left(t-\frac{1}{t}\right)}{2} e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n'(x) t^n,$$

which implies that

$$\begin{aligned} \frac{\left(t-\frac{1}{t}\right)}{2} \sum_{-\infty}^{\infty} J_n(x) t^n &= \sum_{-\infty}^{\infty} J_n'(x) t^n \\ \frac{1}{2}\sum_{-\infty}^{\infty} J_n(x) t^{n+1} - \frac{1}{2}\sum_{-\infty}^{\infty} J_n(x) t^{n-1} &= \sum_{-\infty}^{\infty} J_n'(x) t^n \end{aligned}$$

$$\begin{aligned} \sum_{-\infty}^{\infty} J_n'(x) t^n - \frac{1}{2}\sum_{-\infty}^{\infty} J_n(x) t^{n+1} + \frac{1}{2}\sum_{-\infty}^{\infty} J_n(x) t^{n-1} &= 0 \\ \sum_{-\infty}^{\infty} J_m'(x) t^m - \frac{1}{2}\sum_{-\infty}^{\infty} J_{m-1}(x) t^{m} + \frac{1}{2}\sum_{-\infty}^{\infty} J_{m+1}(x) t^{m} &= 0 \end{aligned}$$

$$\implies J_{n-1} - J_{n+1} = 2J_n'.$$

(c) Adding the two equations derived in parts (a) and (b), we get

$$J_{n-1} - \frac{n}{x} J_n = J_n'.$$

(d) Subtracting the two equations derived in parts (a) and (b), we get

$$J_{n+1} - \frac{n}{x} J_n = -J_n'.$$

(e) Bessel's equation is given by

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0.$$

Deriving and replacing, we get

$$\begin{aligned} x^2 J_n'' + x J_n' + (x^2 - n^2) J_n &= \frac{x^2}{2}(J_{n-1}' - J_{n+1}') + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\ &= \frac{x^2}{4}\left(J_{n-2} - 2J_n + J_{n+2}\right) + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\ &= \frac{x}{2}\left((n-1)J_{n-1} + (n+1)J_{n+1}\right) - x^2 J_n + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\ &= \frac{x}{2}\left((n-1)J_{n-1} + (n+1)J_{n+1} + J_{n-1} - J_{n+1}\right) - n^2 J_n \\ &= \frac{nx}{2}\left(J_{n-1} + J_{n+1}\right) - n^2 J_n \\ &= \frac{nx}{2}\frac{2n}{x} J_n - n^2 J_n \\ &= 0. \end{aligned}$$

(f) Given the series solution

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

Deriving and replacing, we get

$$
\begin{aligned}
x^2 J_n'' + x J_n' + (x^2 - n^2) J_n &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} (n+2s)(n+2s-1) \left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} (n+2s) \left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} (x^2 - n^2) \left(\frac{x}{2}\right)^{n+2s} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} (n+2s)(n+2s-1) \left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} (n+2s) \left(\frac{x}{2}\right)^{n+2s} \\
&\quad - \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} n^2 \left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} 2^2 \left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left[(n+2s)(n+2s-1) + (n+2s) - n^2\right] \left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} 2^2 \left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} 2^2 s\,(n+s) \left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} 2^2 \left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{(s-1)!(n+s-1)!} 2^2 \left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} 2^2 \left(\frac{x}{2}\right)^{n+2s+2} \\
&= 0.
\end{aligned}
$$

$\blacksquare$