Homework 4
Due Wednesday, February 14th

Problem 1

Recall the advection equation:

$$\partial_t u + a\partial_x u = 0, \tag{1}$$

where a is a constant. In class, I discussed this equation and briefly told you how it's solved, but I'd like to see you give it a try yourself. For general initial conditions $u(x,0) = f_0(x)$, find the general solution $u(x,t)$ for this equation by any means you like.

Problem 2

In class we derived the following equations starting from Euler's equations:

$$\begin{aligned}
\dot{\rho} + (v\cdot\nabla)\rho + \rho(\nabla\cdot v) &= 0 && (2)\\
\dot{\vec{v}} + (v\cdot\nabla)\vec{v} + \vec{\nabla}P/\rho &= 0 && (3)\\
\dot{P} + (v\cdot\nabla)P + \gamma P(\nabla\cdot v) &= 0 && (4)
\end{aligned}$$

Define the quantity $s \equiv \ln(P/\rho^\gamma)$. Show the following is true:

$$\dot{s} + (v\cdot\nabla)s = 0. \tag{5}$$

Does this mean that entropy is conserved? What conditions are necessary for this to be true?

Problem 3

Work out a second-order ODE describing the density as a function of radius in a star (as in, if this ODE were solved, the solution would be $\rho(r)$). Use the following two assumptions: First, the star is in hydrostatic equilibrium. Second, assume a polytropic equation of state $P = K\rho^\gamma$. It is often conventional to define $\gamma \equiv 1 + 1/n$ for this problem (and it simplifies the resulting equations).

There are actually exact solutions for a few values of n but I won't ask you to derive them. You can try if you want though!

Problem 4

Imagine you have a star in hydrostatic equilibrium, with mass M and radius R.

a) Estimate the average density and pressure inside the star.

b) Now imagine the radius of the star is gently stretched out by a factor α:

$$R \to \alpha R, \tag{6}$$

but the mass is kept fixed. What is the new density and pressure?

c) Define P_g to be the pressure necessary to maintain hydrostatic equilibrium. It is important to understand that this number should change differently from P as the star is stretched by α. Calculate the new value of P_g after stretching by α.

d) The ratio P/P_g tells us what direction the star will move after this change; if $P/P_g > 1$, pressure is larger than necessary for equilibrium and the star will expand. Likewise, if $P/P_g < 1$ the star will want to contract.

Use this ratio to determine whether the star is stable to being stretched or compressed. How is stability conditional on the value of γ?