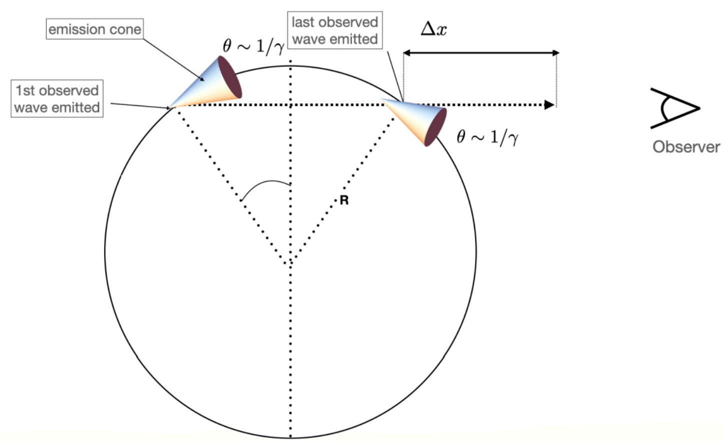
PHYS 630 - Advanced Electricity and Magnetism

Student: **Ralph Razzouk**

Homework 9

Problem 1

A particle is moving relativistically with Lorentz factor $\gamma \gg 1$ along a circle of radius R, as shown in the figure below. By aberration, a particle emits mostly within an angle $\sim 1/\gamma$ with respect to its instantaneous velocity. Estimate a distance Δx and corresponding time $\Delta x/c$ between first and last moment that the observer sees the particle. (An electromagnetic signal is first emitted at the "first observed wave emitted", and propagates with speed c. The particle nearly catches up with its own radiation, and so at the point "last observed wave emitted", it is behind the first emitted wave by Δx.)



Solution. Consider a particle moving in a circle at relativistic speed with Lorentz factor is $\gamma \gg 1$. The radiation is beamed within angle $\theta \sim 1/\gamma$.

The arc length $\Delta\ell$ taken by the particle is given by

$$\Delta\ell = 2\theta R \sim \frac{2R}{\gamma}.$$

and the time Δt that passes for the particle after taking the arc length path is

$$\Delta t = \frac{\Delta\ell}{v} \sim \frac{2R}{\gamma v}.$$

Thus, the distance between the first and last moment the observer sees the particle is

$$\begin{aligned}
\Delta x &= c\Delta t - v\Delta t \\
&= \frac{2Rc}{\gamma v} - \frac{2R}{\gamma} \\
&= \frac{2Rc}{\gamma v}\left(1 - \frac{v}{c}\right)
\end{aligned}$$

and the time between the first and last moment the observer sees the particle is

$$\Delta\tau = \frac{\Delta x}{c} = \frac{2R}{\gamma v}\left(1 - \frac{v}{c}\right).$$

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Problem 2 - Magneto-dipolar Emission
A sphere of radius R carries dipolar magnetic field B_0 and rotates with spin frequency Ω. Estimate the emitted power. [Hint: Check Landau & Lifshitz, vol 2, eq. (71.5)]

Solution. Landau and Lifshitz, Volume 2, Eq. 71.5 states that the total radiation I is given by

$$I = \frac{2}{3c^3}\ddot{d}^2 + \frac{1}{180c^5}\dddot{D}_{\alpha\beta}^2 + \frac{2}{3c^3}\ddot{m}^2,$$

where the terms corresponds to dipole radiation, quadrupole radiation, and magnetic dipole radiation, respectively.
To get the power, we have to integrate the radiation. Since we are considering a dipolar magnetic field, we only consider the last term. Additionally, we know that the magnetic field has the form

$$|\mathbf{B_0}| = \frac{\mu_0}{2}\frac{IR^2}{(z^2+R^2)^{\frac{3}{2}}}.$$

For $r \gg R$, we have

$$\begin{aligned}|\mathbf{B_0}| &= \frac{\mu_0}{2}\frac{IR^2}{z^3} \\ &= \frac{\mu_0}{4\pi}\frac{2I\pi R^2}{r^3} \\ &= \frac{\mu_0}{4\pi}\frac{2IA}{r^3} \\ &= \frac{\mu_0}{4\pi}\frac{2}{r^3}m,\end{aligned}$$

where A is the area of the loop. From the previous, we can write $\mathbf{m} \sim r^3\mathbf{B_0}$.
The magnetic dipole moment can be expressed in terms of its angular position as

$$m = m_0\cos(\Omega t)\hat{\mathbf{n}},$$

where m_0 is the magnitude of the dipole, Ω is the spin frequency, and $\hat{\mathbf{n}}$ is along $\hat{z}$. Taking the second time derivative, we have

$$\ddot{m} = -m_0\Omega^2\cos(\Omega t)\hat{z} = -\Omega^2 m.$$

To calculate power, we only need the magnitude of the magnetic dipole radiation; hence, $|\ddot{m}| = \Omega^2 m$. Thus, we have

$$\begin{aligned}P &= \frac{2}{3c^3}|\ddot{m}^2| \\ &= \frac{2}{3c^3}\left(\Omega^2 m\right)^2 \\ &= \frac{2}{3c^3}\left(\Omega^2 r^3\mathbf{B_0}\right)^2 \\ &= \frac{2}{3c^3}\Omega^4 r^6 {\mathbf{B_0}}^2.\end{aligned}$$

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Problem 3
In the problem above, the mass of the sphere is M (so that its moment of inertia is $I \approx (2/5)MR^2$). An observer can measure spin Ω and the rate of change $\dot{\Omega}$. Find the magnetic field on the surface in terms of M, R, Ω and $\dot{\Omega}$.

Solution. The rotational energy is given by

$$E = \frac{1}{2} I \Omega^2.$$

We know that $\frac{\mathrm{d}E}{\mathrm{d}t} = -P$, so we calculate the time derivative of the energy. We have

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} = I\Omega\dot{\Omega} &= -P \\ &= \frac{2}{3c^3}\Omega^4 r^6 \mathbf{B_0}^2. \end{aligned}$$

Thus, the magnetic field on the surface is

$$\begin{aligned} |\mathbf{B_0}| &= \sqrt{\frac{3c^3 I\Omega\dot{\Omega}}{2\Omega^4 r^6}} \\ &= \frac{c}{r^3}\sqrt{\frac{3cI\dot{\Omega}}{2\Omega^3}} \\ &= \frac{c}{\Omega r^3}\sqrt{\frac{3cI\dot{\Omega}}{2\Omega}}. \end{aligned}$$

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