PHYS 630 - Advanced Electricity and Magnetism

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Homework 8

Problem 1

A charged particle (charge e and mass m) is in motion in circularly polarized electromagnetic wave propagating along magnetic field. A circularly wave of amplitude $E_w = B_w$ propagates along the guiding magnetic field B_0. Find the velocity of the particle v (non-relativistic). Keep in mind that there are two possible circular polarizations, two possible signs of charge, and two possible directions of wave propagating (along and against the guide field).

[Hint: Let the guide field be along z. Write electromagnetic field of the wave (check that Maxwell's equations are satisfied). Assume that at each moment $v_z = 0$. Let a particle be located at $z = 0$. Assume that at each moment velocity is aligned or counter-aligned with the waves magnetic field, and hence perpendicular to the electric field (energy and value of velocity are conserved).]

Solution. We need to find the velocity of the particle $\mathbf{v}$ (non-relativistic). We make the following assumptions inspired by the hint:

- The guide field is along the z-axis.
- At each moment, $v_z = 0$.
- The particle is located at $z = 0$.
- The particle's velocity is counter-aligned with the wave's magnetic field, and hence perpendicular to the electric field (energy and value of velocity are conserved).

The electromagnetic field of the circularly polarized wave propagating along the z-axis and the guiding magnetic field $\mathbf{B}_0$ is given by:

$$\begin{aligned}
\mathbf{B}_0 &= (0,\, 0,\, B_0)\,, \\
\mathbf{E} &= E_\omega\left(\sin(\omega t - k_z z),\, -\cos(\omega t - k_z z),\, 0\right), \\
\mathbf{B} &= B_\omega\left(\cos(\omega t - k_z z),\, \sin(\omega t - k_z z),\, 0\right).
\end{aligned}$$

The force equation for a charged particle in this electromagnetic field is given by

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times (\mathbf{B} + \mathbf{B}_0)\right).$$

Assuming the particle's velocity is counter-aligned with the wave's magnetic field, we have

$$\mathbf{v} = -\left(\cos(\omega t),\, \sin(\omega t),\, 0\right),$$

and then $\mathbf{v} \times \mathbf{B} = 0$. Thus

$$\begin{aligned}
m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}_0\right) &\implies \begin{cases} m\dfrac{\mathrm{d}v_x}{\mathrm{d}t} = e\left(E_x + \dfrac{v_y}{c}B_0\right) \\ m\dfrac{\mathrm{d}v_y}{\mathrm{d}t} = e\left(E_y - \dfrac{v_x}{c}B_0\right) \end{cases} \\
&\implies \begin{cases} \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \dfrac{e}{m}\left(E_w\cos(\omega t) + \dfrac{v_y}{c}B_0\right) \\ \dfrac{\mathrm{d}v_y}{\mathrm{d}t} = \dfrac{e}{m}\left(E_w\sin(\omega t) - \dfrac{v_x}{c}B_0\right) \end{cases} \\
&\implies \begin{cases} \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \dfrac{eE_w}{m}\cos(\omega t) + \dfrac{eB_0}{mc}v_y \\ \dfrac{\mathrm{d}v_y}{\mathrm{d}t} = \dfrac{eE_w}{m}\sin(\omega t) - \dfrac{eB_0}{mc}v_x \end{cases}
\end{aligned}$$

The first equation rewritten gives

$$v_y = \frac{mc}{eB_0}\frac{\mathrm{d}v_x}{\mathrm{d}t} - \frac{cE_w}{B_0}\cos(\omega t),$$

and now taking the derivative, we get

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{mc}{eB_0}\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} + \frac{c\omega E_w}{B_0}\sin(\omega t).$$

Replacing in the second equation gives

$$\begin{aligned}
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_w}{m}\sin(\omega t) - \frac{eB_0}{mc}v_x \\
\frac{mc}{eB_0}\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} + \frac{c\omega E_w}{B_0}\sin(\omega t) &= \frac{eE_w}{m}\sin(\omega t) - \frac{eB_0}{mc}v_x \\
\frac{mc}{eB_0}\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} + \frac{eB_0}{mc}v_x + \left(\frac{c\omega E_w}{B_0} - \frac{eE_w}{m}\right)\sin(\omega t) &= 0 \\
\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} + \left(\frac{eB_0}{mc}\right)^2 v_x + \frac{eE_w}{m}\left(\omega - \frac{eB_0}{mc}\right)\sin(\omega t) &= 0 \\
\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} + \omega_0^2 v_x &= \frac{eE_w}{m}\left(\omega_0 - \omega\right)\sin(\omega t),
\end{aligned}$$

where

$$\omega_0 = \frac{eB_0}{mc}.$$

The solution to the second-order differential equation above is

$$\begin{aligned}
v_x(t) &= \frac{\frac{eE_\omega}{m}(\omega_0 - \omega)}{\omega_0^2 - \omega^2}\sin(\omega t) + c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t) \\
v_x(t) &= \frac{eE_\omega}{m(\omega_0 + \omega)}\sin(\omega t) + c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t).
\end{aligned}$$

Using v_x, we compute v_y by

$$\begin{aligned}
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_\omega}{m}\sin(\omega t) - \frac{eB_0}{mc}v_x \\
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_\omega}{m}\sin(\omega t) - \frac{eB_0}{mc}\left(\frac{eE_\omega}{m(\omega_0 + \omega)}\sin(\omega t) + c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t)\right) \\
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_\omega}{m}\sin(\omega t) - w_0\frac{eE_\omega}{m(\omega_0 + \omega)}\sin(\omega t) - w_0 c_1\sin(\omega_0 t) - w_0 c_2\cos(\omega_0 t) \\
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_\omega}{m}\left(1 - \frac{\omega_0}{\omega_0 + \omega}\right)\sin(\omega t) - w_0 c_1\sin(\omega_0 t) - w_0 c_2\cos(\omega_0 t) \\
v_y(t) &= -\frac{eE_\omega}{m\omega}\left(\frac{\omega}{\omega_0 + \omega}\right)\cos(\omega t) + c_1\cos(\omega_0 t) - c_2\sin(\omega_0 t) + c_3 \\
v_y(t) &= -\frac{eE_\omega}{m\left(\omega_0 + \omega\right)}\cos(\omega t) + c_1\cos(\omega_0 t) - c_2\sin(\omega_0 t) + c_3.
\end{aligned}$$

For completion, we note again that

$$v_z(t) = 0.$$

The velocity v will be the magnitude, which will be

$$v = \frac{eE_\omega}{m(\omega_0 + \omega)}.$$

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