MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*

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Homework 3

Problem 3-1
Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

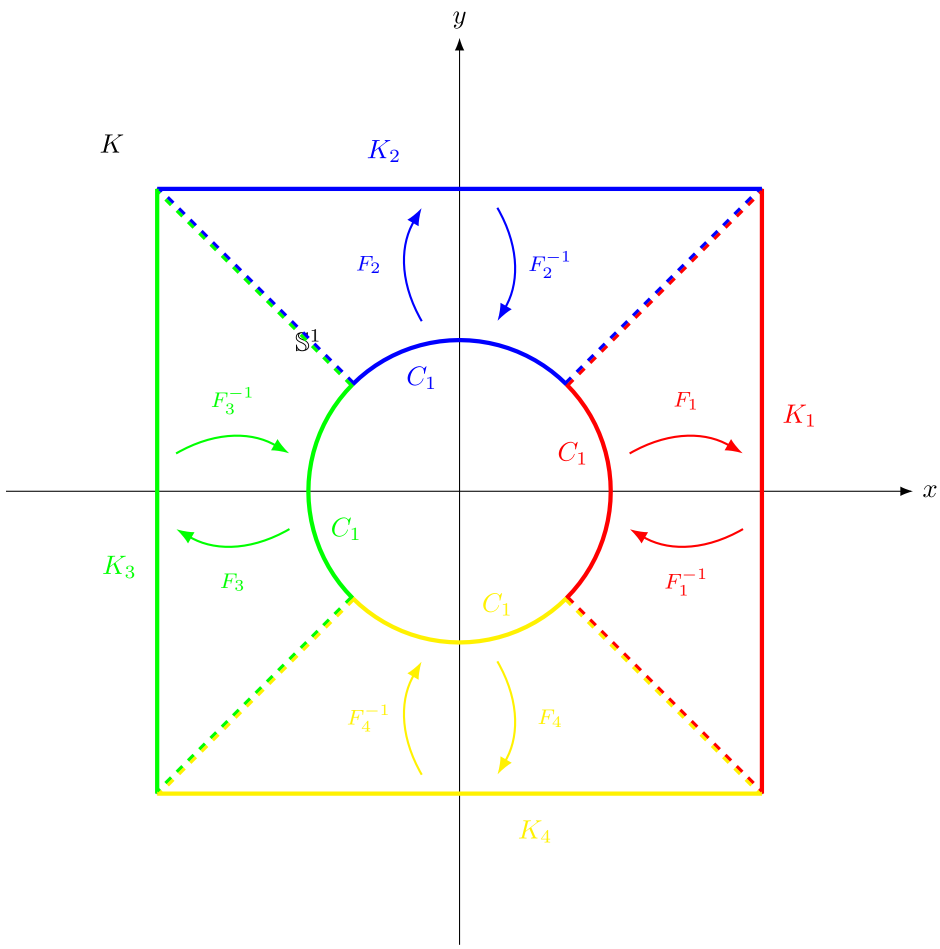
Solution. $\boxed{\implies}$ Suppose $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. We need to show that the anti-derivative is constant on each component of M. The coordinate representation of a function can be taken as the total derivative, which means all the partial derivatives are zero. If, for every $U \in M$ containing p, we have that the total derivative is zero, then its anti-derivative is going to be a constant, and that is for every point p and the small coordinate chart around p in the manifold. Since its constant around every coordinate chart around M then it will be constant around every component in M since connected components are path-connected components. Thus, F is constant on each component of M.
$\boxed{\impliedby}$ Suppose that F is constant on each component of M. The derivative of F at p, by definition, only depends on F in a neighborhood of p, which means that $\mathrm{d}F_p$ will be the derivative of a constant for each component of M, which is zero. Thus, $\mathrm{d}F_p$ is the zero map for each $p \in M$.
Therefore, $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M. ■

Problem 3-4
Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Solution. Take the tangent bundle $T\mathbb{S}^1$, which is the disjoint union of all the tangent spaces to the unit circle $\mathbb{S}^1$. Take the rotation of each tangent space from $T\mathbb{S}^1$ to, say, $(T\mathbb{S}^1)'$, by $\frac{\pi}{2}$. The rotation applied is clearly a diffeomorphism, since the rotation of every coordinate chart, for every point p, is equivalent to just changing the representation of the variables. By definition of a tangent bundle, the tangent space to every point p on the manifold $\mathbb{S}^1$ is isomorphic to $\mathbb{R}$ since it is one-dimensional. After rotating and equating every tangent space to $\mathbb{R}$, what we get is $\mathbb{S}^1 \times \mathbb{R}$. Geometrically, this is a cylinder of infinite height.
Therefore, $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. ■

Problem 3-5
Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F\left(\mathbb{S}^1\right) = K$, but there is no diffeomorphism with the same property. [Hint: let γ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $dF\left(\gamma'(t)\right)$ on the coordinate functions x and y.] (Used on p. 123.)

Solution. Consider the following figure



We define four curves $C_i \in \mathbb{S}^1 \subseteq \mathbb{R}^2$, for $i = 1, 2, 3, 4$, as follows

$$C_1 = \left\{(x, y) \mid |y| \le \frac{\sqrt{2}}{2} \text{ and } x = +\sqrt{1 - y^2}\right\},$$

$$C_2 = \left\{(x, y) \mid |x| \le \frac{\sqrt{2}}{2} \text{ and } y = +\sqrt{1 - x^2}\right\},$$

$$C_3 = \left\{(x, y) \mid |y| \le \frac{\sqrt{2}}{2} \text{ and } x = -\sqrt{1 - y^2}\right\},$$

$$C_4 = \left\{(x, y) \mid |x| \le \frac{\sqrt{2}}{2} \text{ and } y = -\sqrt{1 - x^2}\right\},$$

and the four sides of the square with side length 2 centered at the origin $K_i \in K \subseteq \mathbb{R}^2$, for $i = 1, 2, 3, 4$, as

$$\begin{aligned} K_1 &= \{(1, y) \mid -1 \leq y \leq 1\}, \\ K_2 &= \{(x, 1) \mid -1 \leq x \leq 1\}, \\ K_3 &= \{(-1, y) \mid -1 \leq y \leq 1\}, \\ K_4 &= \{(x, -1) \mid -1 \leq x \leq 1\}. \end{aligned}$$

We aim to find four functions that map each one of those arcs on $\mathbb{S}^1$ to their respective sides on the boundary of a square of side length 2 centered at the origin. We define our function F as

$$\begin{aligned} F &: \mathbb{R}^2 \to \mathbb{R}^2 \\ (x, y) \in C_i &\mapsto F_i(x, y) \in K_i, \end{aligned}$$

where

$$\begin{aligned} F_1(x, y) &= \frac{\sqrt{x^2 + y^2}}{|x|}(x, y), \\ F_2(x, y) &= \frac{\sqrt{x^2 + y^2}}{|y|}(x, y), \\ F_3(x, y) &= -\frac{\sqrt{x^2 + y^2}}{|x|}(x, y), \\ F_4(x, y) &= -\frac{\sqrt{x^2 + y^2}}{|y|}(x, y). \end{aligned}$$

Notice that the four curves C_i trace out the entirety of $\mathbb{S}^1$, *i.e.*

$$\bigcup_{i=1}^{4} C_i = \mathbb{S}^1,$$

and the four sides K_i trace out our square

$$\bigcup_{i=1}^{4} K_i = K.$$

By construction, we have that $F(\mathbb{S}^1) = K$. Since the four functions agree on their overlaps, then by the pasting lemma, we see that F is continuous on all the C_i's. We now have to show that the inverse is also continuous. We define the inverse of the F_i's as

$$\begin{aligned} F_1^{-1}(x, y) &= \frac{|x|}{\sqrt{x^2 + y^2}}(x, y), \\ F_2^{-1}(x, y) &= \frac{|y|}{\sqrt{x^2 + y^2}}(x, y), \\ F_3^{-1}(x, y) &= -\frac{|x|}{\sqrt{x^2 + y^2}}(x, y), \\ F_4^{-1}(x, y) &= -\frac{|y|}{\sqrt{x^2 + y^2}}(x, y). \end{aligned}$$

Likewise, we have that the functions F_i^{-1} are also continuous. Thus, there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$.
We will show that there is no diffeomorphism with the property $F(\mathbb{S}^1) = K$ by contradiction.
Suppose F is a diffeomorphism. Let γ be a smooth curve whose image lies in $\mathbb{S}^1$, *i.e.*

$$\gamma : \mathbb{R} \to \mathbb{S}^1.$$

Now, $\frac{\mathrm{d}\gamma}{\mathrm{d}t} \equiv \gamma'(t)$ will be a vector on $\mathbb{S}^1$. Let $\gamma(t_0)$ be the point at which $F(\gamma(t_0)) = (1,1)$. Consider the action of $\mathrm{d}F(\gamma'(t))$ to the coordinate function x. For $t < t_0$, $\mathrm{d}F(\gamma'(t))x = 0$. For $t > t_0$, $\mathrm{d}F(\gamma'(t))x = \text{constant} \neq 0$. Thus, there was a discontinuous jump at $t = t_0$. But we know that F, γ, and the coordinate function x are smooth, and the action of a the derivative on a smooth function is smooth, then we have reached a contradiction.
Therefore, there is no diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$. ■

Problem 3-8
Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well-defined and bijective. (*Used on p. 72.*)

Solution. Let

$$\begin{aligned} \Psi : \mathcal{V}_pM &\to T_pM \\ [\gamma] &\mapsto \gamma'(0). \end{aligned}$$

Note that, by the derivative of the composition, we have $(f \circ \gamma)'(0) = \mathrm{d}f(\gamma(0))\gamma'(0) = \gamma'(0)f$.

- **Well-defined:**
  Let $[\gamma_1] = [\gamma_2]$, then

  $$\begin{aligned} (f \circ \gamma_1)'(0) &= (f \circ \gamma_2)'(0) \\ \gamma_1'(0)f &= \gamma_2'(0)f \\ \gamma_1'(0) &= \gamma_2'(0), \end{aligned}$$

  since the action of a differential operator on f is equal to the action of another differential operator on f, for all f, then the differential operators must be the same.
  Thus, $\gamma_1'(0) = \gamma_2'(0)$.
  Therefore, $\Psi[\gamma_1] = \Psi[\gamma_2]$ and Ψ is well-defined.

- **Bijective:**
  - **Injective:** Let $\Psi[\gamma_1] = \Psi[\gamma_2]$. Then, $\gamma_1'(0) = \gamma_2'(0)$, and we can apply both to the same function f, and, by definition, we will get the same value. We have

    $$\begin{aligned} \gamma_1'(0)f &= \gamma_2'(0)f \\ (f \circ \gamma_1)'(0) &= (f \circ \gamma_2)'(0), \end{aligned}$$

    and by the definition of the equivalence relation on γ, we have that $\gamma_1 \sim \gamma_2$, *i.e.* $[\gamma_1] = [\gamma_2]$.
    Thus, Ψ is injective.
  - **Surjective:** We need to show that, for all $v \in T_pM$, there exists $\gamma_v \in \mathcal{V}_pM$ such that $\Psi[\gamma_v] = v$. Let $\gamma_v \in \mathcal{V}_pM$ such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. The curve γ_v has an equivalence class, and the image of the equivalence under the map Ψ is equal to $\gamma_v'(0)$, which is v.
    Thus, Ψ is surjective.

  Therefore, Ψ is bijective.

■