PHYS 601 - Methods of Theoretical Physics II

Ralph Razzouk

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rlphrazz@gmail.com

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Homework 1

Problem 1
Find the shortest distance between two points in polar coordinates, i.e., using the line element $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$.

Solution. Given $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$, we have

$$\begin{aligned}
I &= \int_{(x_1,y_1)}^{(x_2,y_2)} \mathrm{d}s \\
&= \int_{(r_1,\theta_1)}^{(r_2,\theta_2)} \sqrt{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2} \\
&= \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2} \\
&= \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \sqrt{r^2 + r'^2}
\end{aligned}$$

We are interested in the shortest distance between these two points; hence, we need to minimize the integrand. Define the integrand to be the function $f(\theta)$. Plugging in f in the Euler-Lagrange equation will give us the result we need. The Euler-Lagrange equation is given by

$$\frac{\partial f}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial f}{\partial r'}\right) = 0$$

By using the chain rule on the $\frac{\mathrm{d}}{\mathrm{d}\theta}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\partial}{\partial \theta} + r'\frac{\partial}{\partial r} + r''\frac{\partial}{\partial r'}$$

Replacing, we get

$$\frac{\partial f}{\partial r} - \frac{\partial}{\partial \theta}\frac{\partial f}{\partial r'} - r'\frac{\partial}{\partial r}\frac{\partial f}{\partial r'} - r''\frac{\partial^2 f}{\partial r'^2} = 0$$

Plugging in our function f defined above

$$\frac{\partial f}{\partial r} = \frac{r}{(r^2 + r'^2)^{\frac{1}{2}}}, \qquad \frac{\partial f}{\partial r'} = \frac{r'}{(r^2 + r'^2)^{\frac{1}{2}}}$$

$$\begin{aligned}
\implies \frac{r}{(r^2 + r'^2)^{\frac{1}{2}}} - 0 + \frac{rr'^2}{(r^2 + r'^2)^{\frac{3}{2}}} - r''\left[\frac{1}{(r^2 + r'^2)^{\frac{1}{2}}} - \frac{r'^2}{(r^2 + r'^2)^{\frac{3}{2}}}\right] &= 0 \\
\implies \frac{r - r''}{(r^2 + r'^2)^{\frac{1}{2}}} + \frac{rr'^2 + r'^2 r''}{(r^2 + r'^2)^{\frac{3}{2}}} &= 0 \\
\implies \frac{(r - r'')(r^2 + r'^2) + rr'^2 + r'^2 r''}{(r^2 + r'^2)^{\frac{3}{2}}} &= 0 \\
\implies \frac{r^3 + 2rr'^2 - r^2 r''}{(r^2 + r'^2)^{\frac{3}{2}}} &= 0 \\
\implies \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{\frac{3}{2}}} &= 0
\end{aligned}$$

This term is the same term used to describe the curvature. It is only zero when r is very large, i.e., for a straight line. Thus, the shortest distance between two points is, as expected, a straight line. ■

Problem 2
Inverse Isoperimetric Problem (Isoareametric Problem): Prove that of all simple closed curves enclosing a given area, the least perimeter is possessed by the circle.

Solution. Given a fixed constant area, we want to show that the perimeter of our closed domain is the least when our enclosure is a circle. The area is given by

$$A = \int \mathrm{d}a = \int_0^{2\pi} \int_0^r r' \, \mathrm{d}r' \, \mathrm{d}\theta = \frac{1}{2} \int_0^{2\pi} r^2 \, \mathrm{d}\theta$$

The perimeter of a closed curve in polar coordinates is given by

$$\begin{aligned} P = \oint \mathrm{d}s &= \oint \sqrt{\mathrm{d}r^2 + r^2 \, \mathrm{d}\theta^2} \\ &= \oint \mathrm{d}\theta \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2} \\ &= \oint \mathrm{d}\theta \sqrt{r^2 + r'^2}. \end{aligned}$$

As usual, our intentions are to extremize (specifically minimize) the integrand. We define $f(r, \theta) = \frac{1}{2} r^2$ to be the integrand function of our constraint given by the fixed area A and $g(r, \theta) = \sqrt{r^2 + r'^2}$ to be the integrand function of our perimeter integral P. Let $h(r, \theta) = f + \lambda g$, where λ is a Lagrange multiplier. Replacing in the Euler-Lagrange equation, we get

$$\frac{\partial h}{\partial r} - \frac{\partial}{\partial \theta} \frac{\partial h}{\partial r'} - r' \frac{\partial}{\partial r} \frac{\partial h}{\partial r'} - r'' \frac{\partial^2 h}{\partial r'^2} = 0$$

$$\frac{\partial h}{\partial r} = \frac{\lambda r}{(r^2 + r'^2)^{\frac{1}{2}}} + r, \qquad \frac{\partial h}{\partial r'} = \frac{\lambda r'}{(r^2 + r'^2)^{\frac{1}{2}}}$$

$$\implies \frac{\lambda r}{(r^2 + r'^2)^{\frac{1}{2}}} + r - 0 + \frac{\lambda r r'^2}{(r^2 + r'^2)^{\frac{3}{2}}} - \lambda r'' \left[\frac{1}{(r^2 + r'^2)^{\frac{1}{2}}} - \frac{r'^2}{(r^2 + r'^2)^{\frac{3}{2}}} \right] = 0$$

$$\implies \frac{r^2 + 2r'^2 - r r''}{(r^2 + r'^2)^{\frac{3}{2}}} = -\frac{1}{\lambda}$$

This is the curvature equation for a circle with radius λ. Thus, a fixed area has a minimal perimeter when the enclosure is a circle. ∎

Problem 3
In all our discussions so far on finding the function f for which

$$I = \int_{x_1}^{x_2} f \,\mathrm{d}x$$

is an extremum, it has been assumed that f depends on x, y, and y', that is, $f = f(x, y, y')$. Show that, if f also depends on $y'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, and for fixed end points at which y and y' are prescribed, the Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

Solution. Let $f = f(x, \mathbf{y}),\ \mathbf{y} = (y, y', y'')$ be our function. We want to find the Euler-Lagrange equation for f, i.e. the equation that minimizes the definite integral of our function, given by

$$I = \int_{x_1}^{x_2} f(x, y, y', y'') \,\mathrm{d}x$$

Define $y_\epsilon(x) = y(x) + \epsilon\eta(x)$, with $\eta(x_1) = \eta(x_2) = 0$ as our boundary conditions. Our integral can be rewritten as follows

$$I_\epsilon = \int_{x_1}^{x_2} f(x, y_\epsilon, y'_\epsilon, y''_\epsilon) \,\mathrm{d}x$$

To extremize our integral, we require that

$$\left.\frac{\mathrm{d}I(\epsilon)}{\mathrm{d}x}\right|_{\epsilon\to 0} = 0$$

$$\begin{aligned}
\implies \frac{\mathrm{d}I(\epsilon)}{\mathrm{d}x} &= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \epsilon} + \frac{\partial f}{\partial y''}\frac{\partial y''}{\partial \epsilon}\right] \mathrm{d}x \\
&= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' + \frac{\partial f}{\partial y''}\eta''\right] \mathrm{d}x
\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}
\frac{\mathrm{d}I(\epsilon)}{\mathrm{d}x} &= \left.\frac{\partial f}{\partial y'}\eta\right|_{x_1}^{x_2} + \left.\frac{\partial f}{\partial y'}\eta'\right|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\eta - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y''}\right)\eta'\right] \mathrm{d}x \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y''}\right)\eta\right|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\eta + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right)\eta\right] \mathrm{d}x \\
&= \int_{x_1}^{x_2} \eta \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right)\right] \mathrm{d}x \\
&= 0
\end{aligned}$$

Since $\eta(x) \neq 0$ is an arbitrary function, then the integral is extremized when the integrand is equal to zero

$$\implies \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

Thus, we have obtained the Euler-Lagrange equation that also accounts for the second derivative of y. ■

Problem 4
We proved in class that the curve which encloses the most area for a given perimeter was a circle. To do this, we demonstrated that this curve was characterized by a constant curvature $1/\lambda$ everywhere. Obtain the same result by using the Euler-Lagrange equation to solve for $r = r(\theta)$ or $\theta = \theta(r)$ directly. [Hint: Use the fact that $f(\theta, r, r')$ and $g(\theta, r, r')$ are independent of θ.]

Solution. Starting with the same setup as Problem 2, we have

$$A = \frac{1}{2}\int_0^{2\pi} r^2 \, \mathrm{d}\theta$$

$$P = \oint \mathrm{d}\theta \sqrt{r^2 + r'^2}.$$

We define $f(r,\theta) = \frac{1}{2}r^2$ to be the integrand function of our constraint given by the fixed area A and $g(r,\theta) = \sqrt{r^2 + r'^2}$ to be the integrand function of our perimeter integral P. Let $h(r,\theta) = f + \lambda g$, where λ is a Lagrange multiplier.
Our Euler-Lagrange equation is given by

$$\frac{\partial h}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial h}{\partial r'}\right) = 0.$$

We can notice that the function h is independent of θ explicitly, and thus, we may use the following

$$\frac{\partial h}{\partial \theta} = -\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r'\frac{\partial h}{\partial r'} - h\right) = 0$$

$$\implies r'\frac{\partial h}{\partial r'} - h = \text{constant} = c_1$$

Replacing our function, we get

$$r'\frac{\lambda r'}{(r^2 + r'^2)^{\frac{1}{2}}} - \frac{1}{2}r^2 - \lambda(r^2 + r'^2)^{\frac{1}{2}} = c_1$$

$$\frac{-\lambda r^2}{(r^2 + r'^2)^{\frac{1}{2}}} = c_1 + \frac{1}{2}r^2$$

$$r' = \sqrt{\frac{4\lambda^2 r^4}{(r^2 + 2c_1)^2} - r^2}$$

$$\mathrm{d}\theta = \left(\frac{4\lambda^2 r^4}{(r^2 + 2c_1)^2} - r^2\right)^{-\frac{1}{2}} \mathrm{d}r$$

$$\theta = \int \left(\frac{4\lambda^2 r^4}{(r^2 + 2c_1)^2} - r^2\right)^{-\frac{1}{2}} \mathrm{d}r$$

The solution to this integral is extremely messy and assumes a lot of constraints on our parameters. If we assume that $c_1 = 0$, then the integral becomes

$$\theta = \arcsin\left(\frac{r}{2\lambda}\right) + c_2$$

Assuming again that $c_2 = 0$, we get

$$r(\theta) = 2\lambda\sin(\theta)$$

Converting to Cartesian coordinates, we have

$$\sqrt{x^2 + y^2} = 2\lambda\frac{y}{\sqrt{x^2 + y^2}}$$

$$x^2 + y^2 - 2\lambda y = 0$$

$$x^2 + (y - \lambda)^2 = \lambda^2$$

which is an equation of a circle, centered at $(0, \lambda)$, with radius λ. ■

Problem 5
In connection with Problem 4 above, show that the curvature K in polar coordinate system is

$$K = \left| \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} \right|,$$

where $r' = \mathrm{d}r/\,\mathrm{d}\theta$ and $r'' = \mathrm{d}^2 r/\,\mathrm{d}\theta^2$.

Solution. We know the general formula of the curvature is given by

$$K = \left| \frac{y''}{(1 + y'^2)^{\frac{3}{2}}} \right|$$

We will use the conversion from Cartesian to polar to express y' and y'' and arrive at the requested form. In connection with Problem 4, the radius is a function of θ, $r(\theta)$.
We have

$$\begin{cases} x = r(\theta)\cos(\theta) \\ y = r(\theta)\sin(\theta) \end{cases}$$

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^{-1} = \frac{r'\sin(\theta) + r\cos(\theta)}{r'\cos(\theta) - r\sin(\theta)}$$

$$y''(x) = \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}y'}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}y'}{\mathrm{d}\theta}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^{-1} = \frac{r^2 + 2r'^2 - r''r}{(r'\cos(\theta) - r\sin(\theta))^3}$$

Replacing in K, we get

$$\begin{aligned} K &= \left| \frac{\frac{r^2+2r'^2-r''r}{(r'\cos(\theta)-r\sin(\theta))^3}}{\left(1 + \left(\frac{r'\sin(\theta)+r\cos(\theta)}{r'\cos(\theta)-r\sin(\theta)}\right)^2\right)^{\frac{3}{2}}} \right| \\ &= \left| \frac{\frac{r^2+2r'^2-r''r}{(r'\cos(\theta)-r\sin(\theta))^3}}{\left(\frac{r^2+r'^2}{(r'\cos(\theta)-r\sin(\theta))^2}\right)^{\frac{3}{2}}} \right| \\ &= \left| \frac{r^2 + 2r'^2 - r''r}{(r^2 + r'^2)^{\frac{3}{2}}} \right|. \end{aligned}$$

■

Homework 2

Problem 1
Fermat's Principle If the velocity u of light is given by the continuous function $u = u(y)$, the actual light path connecting (x_1, y_1) and (x_2, y_2) in a plane is the one which extremizes the time integral

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\mathrm{d}s}{u(y)}$$

(a) Derive Snell's Law from Fermat's Principle: Prove that $\frac{\sin(\phi)}{u}$ is a constant, where ϕ is the angle between the tangent of the light path and a vertical line at that point.

(b) Suppose that light travels in the xy-plane in such a way that its speed is proportional to y. Prove that the light rays emitted from any point are circles with their centers on the x-axis.

Solution. (a) The time integral we have to extremize is given by

$$T = \int_C \mathrm{d}t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\mathrm{d}s}{u(y)} = \int F(x, y, y')\,\mathrm{d}s.$$

We have that

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \implies \mathrm{d}s = \mathrm{d}y\sqrt{1 + (x')^2}.$$

The velocity of light $u(y)$ in the given medium is given by

$$u(y) = \frac{c}{n(y)},$$

where $n(y)$ is the index of refraction of the medium as a function of y. Replacing in our integral, we have

$$T = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\sqrt{1 + (x')^2}}{\frac{c}{n(y)}}\,\mathrm{d}y = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{n(y)}{c}\sqrt{1 + (x')^2}\,\mathrm{d}y$$

$$\implies F(x, y, y') = \frac{n(y)}{c}\sqrt{1 + (x')^2}.$$

Plugging in F into the Euler-Lagrange equations, we get

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$0 - \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{n(y)x'}{c\sqrt{1 + (x')^2}}\right) = 0$$

$$\frac{n(y)x'}{c\sqrt{1 + (x')^2}} = \text{constant} \implies \frac{\sin(\phi)}{u(y)} = \text{constant}.$$

(b) If $u(y) \propto y \implies n(y) \propto y^{-1} \implies n(y) = ay^{-1}$. We have that

$$
\begin{aligned}
\frac{x'}{\sqrt{1+(x')^2}}\frac{a}{cy} &= \text{constant} \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{cy}{c_1}\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \\
\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 &= \left(\frac{cy}{c_1}\right)^2\left(1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2\right) \\
\left(\frac{c_1}{cy}\right)^2 &= \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1 \\
\left(\frac{c_1}{cy}\right)^2 - 1 &= \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{\left(\frac{c_1}{cy}\right)^2 - 1} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sqrt{c_1^2-(cy)^2}}{cy} \\
\mathrm{d}x &= \frac{cy}{\sqrt{c_1^2-(cy)^2}}\,\mathrm{d}y \\
x(y) &= -\frac{\sqrt{c_1^2-(cy)^2}}{c} + x_0 \\
c(x-x_0) &= -\sqrt{c_1^2-(cy)^2} \\
c^2(x-x_0)^2 + (cy)^2 &= c_1^2 \\
(x-x_0)^2 + y^2 &= \left(\frac{c_1}{c}\right)^2,
\end{aligned}
$$

which is a family of circles, centered at $(x_0, 0) \in x$-axis, with radius $\frac{c_1}{c}$.

■

Problem 2

The Lagrangian density $\tilde{L}$ which generates a given set of Euler-Lagrange equations is not unique. Prove this result by showing that adding a divergence to $\tilde{L}$ does not alter the Euler-Lagrange equations. Specifically, let

$$\tilde{L} = \tilde{L}\left(x_k, w_j, \frac{\partial w_j}{\partial x_k}\right); \quad \tilde{L}' = \tilde{L} + \sum_k \frac{\partial f_k}{\partial x_k} \tag{1}$$

where $f_k = f_k(w_j)$, and $j = 1, \ldots, m$, $k = 1, \ldots, n$. Then show that $\tilde{L}'$ and $\tilde{L}$ lead to the same Euler-Lagrange equations.

Solution. Let $\tilde{L}' = \tilde{L} + \sum_k \frac{\partial f_k}{\partial x_k}$. Using the chain rule, we have

$$\sum_k \frac{\partial f_k}{\partial x_k} = \sum_{i,k} \frac{\partial f_k}{\partial \phi_i}\frac{\partial \phi_i}{\partial x_k}.$$

The Euler-Lagrange equation of this system states

$$\frac{\partial \tilde{L}'}{\partial \phi_j} - \sum_k \frac{\partial}{\partial x_k}\frac{\partial \tilde{L}'}{\partial\left(\frac{\partial \phi_j}{\partial x_k}\right)} = 0.$$

Computing each term separately, we have

- First term:

$$\begin{aligned}
\frac{\partial \tilde{L}'}{\partial \phi_j} &= \frac{\partial \tilde{L}}{\partial \phi_j} + \frac{\partial}{\partial \phi_j} \sum_k \frac{\partial f_k}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \phi_j} + \frac{\partial}{\partial \phi_j} \sum_{i,k} \frac{\partial f_k}{\partial \phi_i} \frac{\partial \phi_i}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \phi_j} + \sum_{i,k} \frac{\partial^2 f_k}{\partial \phi_j \partial \phi_i} \frac{\partial \phi_i}{\partial x_k}.
\end{aligned}$$

- Second term:

$$\begin{aligned}
\frac{\partial \tilde{L}'}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} &= \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \frac{\partial}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} \sum_k \frac{\partial f_k}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \frac{\partial}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} \sum_{i,k} \frac{\partial f_k}{\partial \phi_i} \frac{\partial \phi_i}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \sum_{i,k} (0) \left(\frac{\partial \phi_i}{\partial x_k}\right) + \sum_{i,k} \frac{\partial f_k}{\partial \phi_i} \delta_{ij} \delta_{kl} \\
&= \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \frac{\partial f_l}{\partial \phi_j}.
\end{aligned}$$

- Derivative of second term:

$$\begin{aligned}
\sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}'}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} &= \sum_l \frac{\partial}{\partial x_k} \left[\frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \frac{\partial f_l}{\partial \phi_j} \right] \\
&= \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \sum_l \frac{\partial}{\partial x_k} \frac{\partial f_l}{\partial \phi_j} \\
&= \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \sum_{i,l} \frac{\partial}{\partial \phi_i} \frac{\partial \phi_i}{\partial x_k} \frac{\partial f_l}{\partial \phi_j} \\
&= \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \sum_{i,l} \frac{\partial^2 f_l}{\partial \phi_i \partial \phi_j} \frac{\partial \phi_i}{\partial x_k}.
\end{aligned}$$

Replacing in the Euler-Lagrange equation, we get

$$\begin{aligned}
\frac{\partial \tilde{L}'}{\partial \phi_j} - \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}'}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} &= \frac{\partial \tilde{L}}{\partial \phi_j} + \sum_{i,k} \frac{\partial^2 f_k}{\partial \phi_j \partial \phi_i} \frac{\partial \phi_i}{\partial x_k} - \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} - \sum_{i,l} \frac{\partial^2 f_l}{\partial \phi_i \partial \phi_j} \frac{\partial \phi_i}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \phi_j} - \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)}.
\end{aligned}$$

Thus, the Euler-Lagrange equations are the same and invariant under the addition of a divergence. ■

Problem 3
Show that, if $\psi(x)$ and $\bar{\psi}(x)$ are taken as two independent functions, the Lagrangian density ($\bar{\psi} = \psi^*$ and $\dot{\psi} = \partial\psi/\partial t$)

$$\tilde{L} = \frac{\hbar^2}{2m}\nabla\psi\nabla\bar{\psi} + V\psi\bar{\psi} - \frac{i\hbar}{2}(\bar{\psi}\dot{\psi} - \psi\dot{\bar{\psi}})$$

leads to the time-independent Schrodinger equation

$$H\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

and the complex conjugate of this equation.

Solution. We have that $\bar{\psi} = \psi^*$ and $\dot{\psi} = \partial\psi/\partial t$. The Euler-Lagrange equations for ψ and ψ^* are

$$\frac{\partial L}{\partial\psi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi)}\right) = 0,$$

$$\frac{\partial L}{\partial\psi^*} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi^*)}\right) = 0,$$

where $\mu = (0, 1, 2, 3) = (t, x, y, z)$.
We first find all the partial derivatives

- First:
$$\frac{\partial L}{\partial\psi^*} = \frac{\partial\left(V\psi\psi^* - \frac{i\hbar}{2}\psi^*\dot{\psi}\right)}{\partial\psi^*} = V\psi - \frac{i\hbar}{2}\dot{\psi}.$$

- Second:
$$\frac{\partial L}{\partial(\partial_0\psi^*)} = -\frac{i\hbar}{2}\frac{\partial(-\psi\dot{\psi}^*)}{\partial(\partial_0\psi^*)} = \frac{i\hbar}{2}\psi.$$

- Third:
$$\frac{\partial L}{\partial(\partial_i\psi^*)} = \frac{\hbar^2}{2m}\frac{\partial(\nabla\psi\nabla\psi^*)}{\partial(\partial_i\psi^*)} = \frac{\hbar^2}{2m}\nabla\psi e_i,$$

  where $e_i = \{e_1, e_2, e_3\}$.

Replacing in the Euler-Lagrange equations, we get

$$\frac{\partial L}{\partial\psi^*} - \partial_0\left(\frac{\partial L}{\partial(\partial_0\psi^*)}\right) - \partial_i\left(\frac{\partial L}{\partial(\partial_i\psi^*)}\right) = 0$$

$$V\psi - \frac{i\hbar}{2}\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2}\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\nabla^2\psi = 0$$

$$\implies -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\implies \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi,$$

as needed. ■

Homework 3

Problem 1
For each differential equation below, find all the singularities (including those at infinity) and state whether each is regular or irregular.

NAME	EXPRESSION
Hypergeometric	$x(x-1)y'' + [(1+a+b)x - c]\,y' + aby = 0$
Legendre	$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$
Chebyshev	$(1-x^2)y'' - xy' + n^2y = 0$
Confluent Hypergeometric	$xy'' + (c-x)y' - ay = 0$
Laguerre	$xy'' + (1-x)y' + ay = 0$
Bessel	$x^2y'' + xy' + (x^2-n^2)y = 0$
Simple Harmonic Oscillator	$y'' + \omega^2y = 0$
Hermite	$y'' - 2xy' + 2\alpha y = 0$

Solution. Consider the general form of a second-order ordinary differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

An ordinary differential equation is said to have singularities when the highest order term has zeroes or when the lower order terms have poles.

- A point x_0 is said to be an **ordinary point** if $P(x)$ and $Q(x)$ are analytic at $x = x_0$.
- A point x_0 is said to be a **singular point** if $P(x)$ and $Q(x)$ are not analytic at $x = x_0$.
  - A point x_0 is said to be a **regular singular point** if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at x_0.
  - Otherwise, x_0 is said to be an **irregular singular point**.

There remains one region of interest, which is as $x \to \infty$. To study the ODE at infinity, we make a variable change of $z = \frac{1}{x}$. Now, we study what happens at $z = 0$. Under such a transformation, the transformed forms of $P(x)$ and $Q(x)$ will have to be studied using the analysis above. Accordingly, we have

$$\tilde{P}(z) = \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \quad \text{and} \quad \tilde{Q}(z) = \frac{Q\left(\frac{1}{z}\right)}{z^4}$$

- **Hypergeometric:**

  $$x(x-1)y'' + [(1+a+b)x - c]\,y' + aby = 0$$

  The above ODE reduces to

  $$y'' + \frac{(1+a+b)x - c}{x(x-1)}y' + \frac{ab}{x(x-1)}y = 0,$$

  where

  $$P(x) = \frac{(1+a+b)x - c}{x(x-1)} \qquad \text{and} \qquad Q(x) = \frac{ab}{x(x-1)}.$$

  The hypergeometric ODE has interesting points at $x_0 = 0, 1, \infty$.

  For $x_0 = 0$:

  - The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.

– The quantity $(x-x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0}\\ &= \left.\frac{(1+a+b)x-c}{x-1}\right|_{x=0}\\ &= c \sim \text{finite.}\end{aligned}$$

– The quantity $(x-x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= x^2Q(x)\big|_{x=0}\\ &= \left.\frac{abx}{x-1}\right|_{x=0}\\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 = 1$:

– The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.

– The quantity $(x-x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1}\\ &= \left.\frac{(1+a+b)x-c}{x}\right|_{x=1}\\ &= 1+a+b-c \sim \text{finite.}\end{aligned}$$

– The quantity $(x-x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= (x-1)^2Q(x)\big|_{x=1}\\ &= \left.\frac{ab(x-1)}{x}\right|_{x=1}\\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2}\\ &= \frac{2}{z} - \frac{\frac{(1+a+b)\frac{1}{z}-c}{\frac{1}{z}\left(\frac{1}{z}-1\right)}}{z^2}\\ &= \frac{2}{z} - \frac{1+a+b-cz}{z\left(1-z\right)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4}\\ &= \frac{\frac{ab}{\frac{1}{z}\left(\frac{1}{z}-1\right)}}{z^4}\\ &= \frac{ab}{z^2\left(1-z\right)}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - \frac{1 + a + b - cz}{1 - z}\Bigg|_{z=0} \\ &= 1 - a - b \sim \text{finite.}\end{aligned}$$

- The quantity $(z - z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z - z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{ab}{(1 - z)}\Bigg|_{z=0} \\ &= ab \sim \text{finite.}\end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Legendre:**

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0$$

The above ODE reduces to

$$y'' - \frac{2x}{1 - x^2}y' + \frac{\ell(\ell + 1)}{1 - x^2}y = 0,$$

where

$$P(x) = -\frac{2x}{1 - x^2} = -\frac{2x}{(1 - x)(1 + x)} \qquad \text{and} \qquad Q(x) = \frac{\ell(\ell + 1)}{1 - x^2} = \frac{\ell(\ell + 1)}{(1 - x)(1 + x)}.$$

The Legendre ODE has interesting points at $x_0 = \pm 1, \infty$.

For $x_0 = -1$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)\big|_{x_0} &= (x + 1)P(x)\big|_{x=-1} \\ &= -\frac{2x}{1 - x}\Bigg|_{x=-1} \\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2Q(x)\big|_{x_0} &= (x + 1)^2Q(x)\big|_{x=-1} \\ &= \frac{\ell(\ell + 1)(x + 1)}{1 - x}\Bigg|_{x=-1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = -1$ is a regular singular point.

For $x_0 = 1$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1} \\ &= -\frac{2x}{1+x}\bigg|_{x=1} \\ &= -1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= (x-1)^2Q(x)\big|_{x=1} \\ &= \frac{\ell(\ell+1)(x-1)}{1+x}\bigg|_{x=1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-\frac{2\left(\frac{1}{z}\right)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^2} \\ &= \frac{2}{z} + \frac{2}{z(z-1)(z+1)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{\ell(\ell+1)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^4} \\ &= \frac{\ell(\ell+1)}{z^2(z-1)(z+1)}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{2}{(z-1)(z+1)}\bigg|_{z=0} \\ 0 &\sim \text{finite.}\end{aligned}$$

- The quantity $(z - z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{\ell(\ell+1)}{(z-1)(z+1)}\bigg|_{z=0} \\ &= -\ell(\ell+1) \sim \text{finite.}\end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Chebyshev:**

$$(1-x^2)y'' - xy' + n^2y = 0$$

The above ODE reduces to

$$y'' - \frac{x}{1-x^2}y' + \frac{n^2}{1-x^2}y = 0,$$

where

$$P(x) = -\frac{x}{1-x^2} = -\frac{x}{(1-x)(1+x)} \qquad \text{and} \qquad Q(x) = \frac{n^2}{1-x^2} = \frac{n^2}{(1-x)(1+x)}.$$

The Chebyshev ODE has interesting points at $x_0 = \pm 1, \infty$.

For $x_0 = -1$**:**

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= (x+1)P(x)\big|_{x=-1} \\ &= -\frac{x}{1-x}\bigg|_{x=-1} \\ &= \frac{1}{2} \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= (x+1)^2Q(x)\big|_{x=-1} \\ &= \frac{(x+1)n^2}{1-x}\bigg|_{x=-1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = -1$ is a regular singular point.

For $x_0 = 1$**:**

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1} \\ &= -\frac{x}{1+x}\bigg|_{x=1} \\ &= -\frac{1}{2} \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= (x-1)^2Q(x)\big|_{x=1} \\ &= \frac{(x-1)n^2}{1+x}\bigg|_{x=1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-\frac{\left(\frac{1}{z}\right)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^2} \\ &= \frac{2}{z} + \frac{1}{z(z-1)(z+1)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{n^2}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^4} \\ &= \frac{n^2}{z^2(z-1)(z+1)}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{1}{(z-1)(z+1)}\Bigg|_{z=0} \\ &= 1 \sim \text{finite}.\end{aligned}$$

- The quantity $(z - z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{n^2}{(z-1)(z+1)}\Bigg|_{z=0} \\ &= -n^2 \sim \text{finite}.\end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Confluent Hypergeometric:**

$$xy'' + (c - x)y' - ay = 0$$

The above ODE reduces to

$$y'' + \frac{c-x}{x}y' - \frac{a}{x}y = 0,$$

where

$$P(x) = \frac{c-x}{x} \qquad \text{and} \qquad Q(x) = -\frac{a}{x}.$$

The Confluent Hypergeometric ODE has interesting points at $x_0 = 0, \infty$.

For $x_0 = 0$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.

– The quantity $(x-x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)|_{x_0} &= xP(x)|_{x=0}\\ &= c-x|_{x=0}\\ &= c \sim \text{finite.}\end{aligned}$$

– The quantity $(x-x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= x^2Q(x)\big|_{x=0}\\ &= -ax|_{x=0}\\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2}\\ &= \frac{2}{z} - \frac{\frac{c-\frac{1}{z}}{\frac{1}{z}}}{z^2}\\ &= \frac{2}{z} - \frac{cz-1}{z^2},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4}\\ &= \frac{-\frac{a}{\frac{1}{z}}}{z^4}\\ &= -\frac{a}{z^3}.\end{aligned}$$

– The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
– The quantity $(z-z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0}\\ &= 2 - \frac{cz-1}{z}\bigg|_{z=0}\\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

- **Laguerre:**

$$xy'' + (1-x)y' + ay = 0$$

The above ODE reduces to

$$y'' + \frac{1-x}{x}y' + \frac{a}{x}y = 0,$$

where

$$P(x) = \frac{1-x}{x} \qquad \text{and} \qquad Q(x) = \frac{a}{x}.$$

The Laguerre ODE has interesting points at $x_0 = 0, \infty$.
For $x_0 = 0$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0} \\ &= 1 - x\big|_{x=0} \\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2 Q(x)\big|_{x_0} &= x^2 Q(x)\big|_{x=0} \\ &= ax\big|_{x=0} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{\frac{1-\frac{1}{z}}{\frac{1}{z}}}{z^2} \\ &= \frac{2}{z} - \frac{z-1}{z^2},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{a}{\frac{1}{z}}}{z^4} \\ &= \frac{a}{z^3}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - \frac{z-1}{z}\Big|_{z=0} \\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

- **Bessel:**

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

The above ODE reduces to

$$y'' + \frac{1}{x}y' + \frac{x^2 - n^2}{x^2}y = 0,$$

where

$$P(x) = \frac{1}{x} \qquad \text{and} \qquad Q(x) = \frac{x^2 - n^2}{x^2}.$$

The Bessel ODE has interesting points at $x_0 = 0, \infty$.

For $x_0 = 0$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0} \\ &= 1\big|_{x=0} \\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2 Q(x)\Big|_{x_0} &= x^2 Q(x)\Big|_{x=0} \\ &= x^2 - n^2\Big|_{x=0} \\ &= -n^2 \sim \text{finite.}\end{aligned}$$

  Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{z}{z^2} \\ &= \frac{2}{z} - \frac{1}{z},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{\left(\frac{1}{z}\right)^2 - zn^2}{\left(\frac{1}{z}\right)^2}}{z^4} \\ &= \frac{1 - n^2 z^2}{z^4}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - 1\big|_{z=0} \\ &= 1 \sim \text{finite.}\end{aligned}$$

– The quantity $(z-z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0}\\ &= \frac{1-n^2z^2}{z^2}\Big|_{z=0}\\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

Simple Harmonic Oscillator:

$$y'' + \omega^2 y = 0$$

The above ODE is already in a reduced for where

$$P(x) = 0 \qquad \text{and} \qquad Q(x) = \omega^2.$$

The simple harmonic oscillator ODE has interesting points at $x_0 = \infty$.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2}\\ &= \frac{2}{z},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4}\\ &= \frac{\omega^2}{z^4}.\end{aligned}$$

– The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.

– The quantity $(z-z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0}\\ &= 2\big|_{z=0}\\ &= 2 \sim \text{finite}.\end{aligned}$$

– The quantity $(z-z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0}\\ &= \frac{\omega^2}{z^2}\Big|_{z=0}\\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

Hermite:

$$y'' - 2xy' + 2\alpha y = 0$$

The above ODE is already in a reduced for where

$$P(x) = -2x \qquad \text{and} \qquad Q(x) = 2\alpha.$$

The Hermite ODE has interesting points at $x_0 = \infty$.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-2\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} + \frac{2}{z^3},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{2\alpha}{z^4}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{2}{z^2}\Big|_{z=0} \\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

To summarize, we have

NAME	SINGULARITIES
Hypergeometric	Regular at $x = 0, 1, \infty$
Legendre	Regular at $x = \pm 1, \infty$
Chebyshev	Regular at $x = \pm 1, \infty$
Confluent Hypergeometric	Regular at $x = 0$ and irregular at $x = \infty$
Laguerre	Regular at $x = 0$ and irregular at $x = \infty$
Bessel	Regular at $x = 0$ and irregular at $x = \infty$
Simple Harmonic Oscillator	Irregular at $x = \infty$
Hermite	Irregular at $x = \infty$

■

Problem 2
For some of the above equations, $q(x) = 0$ when expressed in Sturm-Liouiville form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right]0 - \left[q(x) - \lambda w(x)\right]y = 0.$$

When $\lambda = 0$ also, the Sturm-Liouiville equation has a solution $y(x)$ determined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p(x)},$$

(a) Show this.

(b) Use this result to produce a second solution [in addition to those given on the sheet distributed in class] to the Legendre, Laguerre, and Hermite equations.

Solution. (a) Given the Sturm-Liouiville form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right]0 - \left[q(x) - \lambda w(x)\right]y = 0,$$

where $q(x) = \lambda = 0$, then we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right] = 0 \qquad \Longrightarrow \qquad p(x)y' &= \text{constant} \\ \Longrightarrow \qquad y' &= \frac{\text{constant}}{p(x)} \\ &= \frac{1}{p(x)},\end{aligned}$$

where the last step was done by absorbing the constant into $p(x)$.

(b)

- **Legendre:** The Legendre ODE is given by

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0,$$

which can be rewritten in the Sturm-Liouiville form as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)y'\right] + \ell(\ell + 1)y = 0,$$

where $p(x) = (1 - x^2)$, $q(x) = 0$, and $\lambda = \ell(\ell + 1)$.
By setting $\ell = 0$ and using the property in part (a), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p(x)} = \frac{1}{1 - x^2}$$

$$y = \int \frac{1}{1 - x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) + C,$$

which is the second solution of the Legendre ODE.

- **Laguerre:** The Laguerre ODE is given by

$$xy'' + (1 - x)y' + ay = 0.$$

To rewrite this in the Sturm-Liouiville form, we need to use an integrating factor $I(x)$, which would have initially been in the equation but was cancelled for its non-zero value. Let

$$xIy'' + (1 - x)Iy' + aIy = 0$$

be our new ODE. We require that $\frac{\mathrm{d}}{\mathrm{d}x}(xI) = (1-x)I$, and so we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(xI) = I + x\frac{\mathrm{d}I}{\mathrm{d}x} &= (1-x)I \\ \implies \frac{\mathrm{d}I}{\mathrm{d}x} &= -I \\ \implies I &= e^{-x}.\end{aligned}$$

Thus, our Laguerre ODE becomes

$$xe^{-x}y'' + (1-x)e^{-x}y' + ae^{-x}y = 0.$$

Rewriting in the Sturm-Liouiville form, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{-x}y'\right] + ae^{-x}y = 0,$$

where $p(x) = xe^{-x}$, $q(x) = 0$, and ae^{-x}.
By setting $ae^{-x} = 0 \implies a = 0$, since $e^{-x} \neq 0$, and using the property in part (a), we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{p(x)} = \frac{1}{xe^{-x}} \\ y &= \int \frac{1}{xe^{-x}}\,\mathrm{d}x = Ei(x) + C,\end{aligned}$$

which is the second solution of the Laguerre ODE, where $Ei(x)$ is defined to be the exponential integral.

- **Hermite:** The Hermite ODE is given by

$$y'' - 2xy' + 2\alpha y = 0.$$

To rewrite this in the Sturm-Liouiville form, we need to use an integrating factor $I(x)$, which would have initially been in the equation but was cancelled for its non-zero value. Let

$$Iy'' - 2xIy' + 2\alpha Iy = 0$$

be our new ODE. We require that $\frac{\mathrm{d}}{\mathrm{d}x}(I) = -2xI$, and so we have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}x} &= -2xI \\ \implies I &= e^{-x^2}.\end{aligned}$$

Thus, our Hermite ODE becomes

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2\alpha e^{-x^2}y = 0$$

Rewriting in the Sturm-Liouiville form, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x^2}y'\right] + 2\alpha e^{-x^2}y = 0,$$

where $p(x) = e^{-x^2}$, $q(x) = 0$, and $2\alpha e^{-x^2}$.
By setting $2\alpha e^{-x^2} = 0 \implies \alpha = 0$, since $e^{-x^2} \neq 0$, and using the property in part (a), we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{p(x)} = \frac{1}{e^{-x^2}} \\ y &= \int e^{x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2} + C,\end{aligned}$$

which is the second solution of the Hermite ODE, where $\mathrm{erfi}(x)$ is defined to be the imaginary error function.

■

Homework 4

Problem 1
The first four Legendre polynomials are

$$P_0(x) = 1, \quad P_2 = \frac{1}{2}(3x^2 - 1),$$
$$P_1(x) = x, \quad P_3 = \frac{1}{2}(5x^3 - 3x).$$

Obtain these four polynomials by each of the following methods:

(a) Generating function,

(b) Rodrigues' formula,

(c) Schmidt orthogonalization,

(d) Series solution.

Solution. (a) The generating function of the Legendre polynomials is given by

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n.$$

The Taylor expansion of the left hand-side has the form of

$$g(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t)\bigg|_{t=0} t^n.$$

By comparison, we have

$$P_n(x) = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t)\bigg|_{t=0}.$$

- **For** $n = 0$**:**

$$\begin{aligned} P_0(x) &= \frac{1}{0!} \frac{\mathrm{d}^0}{\mathrm{d}t^0} g(x,t)\bigg|_{t=0} \\ &= g(x,t)|_{t=0} \\ &= \frac{1}{\sqrt{1-2xt+t^2}}\bigg|_{t=0} \\ &= 1. \end{aligned}$$

- **For** $n = 1$**:**

$$\begin{aligned} P_1(x) &= \frac{1}{1!} \frac{\mathrm{d}^1}{\mathrm{d}t^1} g(x,t)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} g(x,t)\bigg|_{t=0} \\ &= \frac{x-t}{(1-2tx+t^2)^{\frac{3}{2}}}\bigg|_{t=0} \\ &= x. \end{aligned}$$

- **For** $n = 2$**:**

$$\begin{aligned} P_2(x) &= \frac{1}{2!}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2}g(x,t)\bigg|_{t=0} \\ &= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}g(x,t)\bigg|_{t=0} \\ &= \frac{1}{2}\frac{2t^2-4xt+3x^2-1}{(1-2xt+t^2)^{\frac{5}{2}}}\Bigg|_{t=0} \\ &= \frac{1}{2}(3x^2-1). \end{aligned}$$

- **For** $n = 3$**:**

$$\begin{aligned} P_3(x) &= \frac{1}{3!}\,\frac{\mathrm{d}^3}{\mathrm{d}t^3}g(x,t)\bigg|_{t=0} \\ &= \frac{1}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3}g(x,t)\bigg|_{t=0} \\ &= \frac{1}{6}\frac{3\,(x-t)\left(2t^2-4xt+5x^2-3\right)}{(1-2xt+t^2)^{\frac{7}{2}}}\Bigg|_{t=0} \\ &= \frac{1}{2}(5x^3-3x). \end{aligned}$$

(b) The Rodrigues' formula for the Legendre polynomials is given by

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

Calculating, we have

- **For** $n = 0$**:**

$$\begin{aligned} P_0(x) &= \frac{1}{2^0 0!}\frac{\mathrm{d}^0}{\mathrm{d}x^0}(x^2-1)^0 \\ &= 1. \end{aligned}$$

- **For** $n = 1$**:**

$$\begin{aligned} P_1(x) &= \frac{1}{2^1 1!}\frac{\mathrm{d}^1}{\mathrm{d}x^1}(x^2-1)^1 \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1) \\ &= x. \end{aligned}$$

- **For** $n = 2$**:**

$$\begin{aligned} P_2(x) &= \frac{1}{2^2 2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 \\ &= \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^4-2x^2+1) \\ &= \frac{1}{8}(12x^2-4) \\ &= \frac{1}{2}(3x^2-1). \end{aligned}$$

- **For $n = 3$:**

$$\begin{aligned}
P_3(x) &= \frac{1}{2^3 3!}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2-1)^3\\
&= \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^6-3x^4+3x^2-1)\\
&= \frac{1}{48}(120x^3-72x)\\
&= \frac{1}{2}(5x^3-3x).
\end{aligned}$$

(c) We define an inner product

$$\langle f|g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$$

and a set of functions

$$u_n(x) = x^n,$$

where n is a non-negative integer.

We now generate a set of orthonormal functions $\phi_n(x)$ using the Gram-Schmidt orthogonalization process. We have

- **For $n = 0$:**

$$\begin{aligned}
\phi_0(x) &= \frac{u_0(x)}{\sqrt{\langle u_0|u_0\rangle}}\\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned}
\psi_1(x) &= u_1(x) - \langle\phi_0|u_1\rangle\,\phi_0(x)\\
&= x - \frac{1}{2}\int_{-1}^{1} x\,\mathrm{d}x\\
&= x.
\end{aligned}$$

Normalizing, we have

$$\phi_1(x) = \frac{\psi_1(x)}{\sqrt{\langle\psi_1|\psi_1\rangle}} = \frac{\psi_1(x)}{\sqrt{\int_{-1}^{1} x^2\,\mathrm{d}x}} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}x$$

- **For $n = 2$:**

$$\begin{aligned}
\psi_2(x) &= u_2(x) - \langle\phi_0|u_2\rangle\,\phi_0(x) - \langle\phi_1|u_2\rangle\,\phi_1(x)\\
&= x^2 - \frac{1}{2}\int_{-1}^{1} x^2\,\mathrm{d}x - \frac{3}{2}x\int_{-1}^{1} x^3\,\mathrm{d}x\\
&= x^2 - \frac{1}{3}.
\end{aligned}$$

Normalizing, we have

$$\phi_2(x) = \frac{\psi_2(x)}{\sqrt{\langle\psi_2|\psi_2\rangle}} = \frac{x^2-\frac{1}{3}}{\sqrt{\int_{-1}^{1}\left(x^2-\frac{1}{3}\right)^2\mathrm{d}x}} = \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2-1).$$

- **For $n = 3$:**

$$
\begin{aligned}
\psi_3(x) &= u_3(x) - \langle \phi_0 | u_3 \rangle \phi_0(x) - \langle \phi_1 | u_3 \rangle \phi_1(x) - \langle \phi_2 | u_3 \rangle \phi_2(x) \\
&= x^3 - \frac{1}{2} \int_{-1}^{1} x^3 \, \mathrm{d}x - \frac{3}{2} x \int_{-1}^{1} x^4 \, \mathrm{d}x - \frac{5}{8}(3x^2 - 1) \int_{-1}^{1} (3x^2 - 1) x^3 \, \mathrm{d}x \\
&= x^3 - \frac{3}{2} x \int_{-1}^{1} x^4 \, \mathrm{d}x \\
&= x^3 - \frac{3}{5} x.
\end{aligned}
$$

Normalizing, we have

$$
\phi_3(x) = \frac{\psi_3(x)}{\sqrt{\langle \psi_3 | \psi_3 \rangle}} = \frac{x^3 - \frac{3}{5}x}{\sqrt{\int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)^2 \mathrm{d}x}} = \frac{1}{2} \sqrt{\frac{7}{2}} (5x^3 - 3x).
$$

The Legendre polynomials $P_n(x)$ would then be

$$
P_n(x) = \sqrt{\frac{2}{2n+1}} \phi_n(x)
$$

(d) We seek solutions of the form

$$
y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_\lambda \neq 0.
$$

Deriving and replacing back into the Legendre ODE, we get

$$
(1 - x^2) \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - 2x \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1} + n(n+1) \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0
$$

$$
\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - \sum_{\lambda=0}^{\infty} a_\lambda \left[(k+\lambda)(k+\lambda+1) - n(n+1)\right] x^{k+\lambda} = 0
$$

Setting $\lambda = 0$, we get

- **Lowest order x^{k-2}:** This gives us the indicial equation

$$
a_0 k(k-1) = 0 \implies k = 0, 1.
$$

- **First order x^{k-1}:**

$$
a_1 (k+1) k = 0 \implies \begin{cases} a_1 \text{ arbitrary}, & \text{if } k = 0, \\ a_1 = 0, & \text{if } k = 1, \end{cases}
$$

- **General order x^{k+j}:**

$$
a_{j+2}(k+j+2)(k+j+1) - a_j \left[(k+j)(k+j+1) - n(n+1)\right] = 0
$$

$$
a_{j+2} = a_j \frac{(k+j)(k+j+1) - n(n+1)}{(k+j+1)(k+j+2)}
$$

  - **For $k = 0$:** We have

$$
a_{j+2} = a_j \frac{j(j+1) - n(n+1)}{(j+1)(j+2)}.
$$

Since a_1 is arbitrary for $k = 0$, then we can set it to zero. Then we get a solution explicitly for even powers and

$$
y_{2n} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda}
$$

- **For** $k = 1$**:** We have
$$a_{j+2} = a_j \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)}.$$
Since $a_1 = 0$ for $k = 1$, then we get a solution explicitly for odd powers and
$$y_{2n+1} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda+1}$$
- **For** $n = 0$**:** We have
$$y_0 = P_0(x) = a_0.$$
Set $a_0 = 1$, then
$$P_0(x) = 1.$$
- **For** $n = 1$**:** We have
$$y_1 = P_1(x) = a_0 x.$$
Set $a_0 = 1$, then
$$P_1(x) = x.$$
- **For** $n = 2$**:** We have
$$y_2 = P_2(x) = a_0 - 3a_0 x^2.$$
Set $a_0 = -\frac{1}{2}$, then
$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$
- **For** $n = 3$**:** We have
$$y_3 = P_3(x) = a_0 x - \frac{5}{3} a_0 x^3.$$
Set $a_0 = -\frac{3}{2}$, then
$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

■

Problem 2
The Hermite differential equation is $H_n'' - 2xH_n' + 2nH_n = 0$.

(a) Solve this equation by series solution and show that it terminates for integral values of n.

(b) Use the series solution to generate the first four Hermite polynomials which are

$$H_0(x) = 1, \qquad H_2 = 4x^2 - 2,$$
$$H_1(x) = 2x, \quad H_3 = 8x^3 - 12x.$$

(c) Obtain the first four Hermite polynomials using the generating function which is

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

(d) Using the generating function, derive the recurrence relations

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,$$

$$H_n'(x) - 2nH_{n-1}(x) = 0.$$

(e) Using the result of part (d), verify that the $H_n(x)$ defined by the generating function obeys the Hermite differential equation.

Solution. (a) We seek solutions of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_\lambda \neq 0.$$

Deriving and replacing back into the Legendre ODE, we get

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} - 2x\sum_{\lambda=0}^{\infty} a_\lambda(k+\lambda)x^{k+\lambda-1} + 2n\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} - \sum_{\lambda=0}^{\infty} 2a_\lambda(k+\lambda-n)x^{k+\lambda} = 0$$

Setting $\lambda = 0$, we get

- **Lowest order** x^{k-2}: This gives us the indicial equation

$$a_0 k(k-1) = 0 \implies k = 0, 1.$$

- **First order** x^{k-1}:

$$a_1(k+1)k = 0 \implies \begin{cases} a_1 \text{ arbitrary}, & \text{if } k = 0, \\ a_1 = 0, & \text{if } k = 1, \end{cases}$$

- **General order** x^{k+j}:

$$a_{j+2}(k+j+2)(k+j+1) - 2a_j(k+j-n) = 0$$

$$a_{j+2} = a_j \frac{2(k+j-n)}{(k+j+1)(k+j+2)}$$

- **For $k = 0$:** We have
$$a_{j+2} = a_j \frac{2(j-n)}{(j+1)(j+2)}.$$
Since a_1 is arbitrary for $k = 0$, then we can set it to zero. Then we get a solution explicitly for even powers and
$$y_{2n} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda}$$
Notice, since j is even from the recurrence relation, that if n is also even, then there will be some term that is zero which terminates the series.
- **For $k = 1$:** We have
$$a_{j+2} = a_j \frac{2(j+1-n)}{(j+2)(j+3)}.$$
Since $a_1 = 0$ for $k = 1$, then we get a solution explicitly for odd powers and
$$y_{2n+1} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda+1}$$
Notice, since $j + 1$ is odd from the recurrence relation, that if n is also odd, then there will be some term that is zero which terminates the series.

(b)
- **For $n = 0$:** We have
$$y_0 = H_0(x) = a_0.$$
Set $a_0 = 1$, then
$$H_0(x) = 1.$$
- **For $n = 1$:** We have
$$y_1 = H_1(x) = a_0 x.$$
Set $a_0 = 2$, then
$$H_1(x) = 2x.$$
- **For $n = 2$:** We have
$$y_2 = H_2(x) = a_0 - 2a_0 x^2.$$
Set $a_0 = -2$, then
$$H_2(x) = 4x^2 - 2.$$
- **For $n = 3$:** We have
$$y_3 = H_3(x) = a_0 x - \frac{2}{3} a_0 x^3.$$
Set $a_0 = -12$, then
$$H_3(x) = 8x^3 - 12x.$$

(c) The generating function of the Hermite polynomials is given by
$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$
The Taylor expansion of g has the form of
$$g(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t) \right|_{t=0} t^n.$$
By comparison, we have
$$H_n(x) = \left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t) \right|_{t=0}.$$

- **For $n = 0$:**

$$\begin{aligned} H_0(x) &= \frac{\mathrm{d}^0}{\mathrm{d}t^0} g(x,t)\bigg|_{t=0} \\ &= g(x,t)|_{t=0} \\ &= e^{-t^2+2tx}\Big|_{t=0} \\ &= 1. \end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned} H_1(x) &= \frac{\mathrm{d}^1}{\mathrm{d}t^1} g(x,t)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} g(x,t)\bigg|_{t=0} \\ &= 2(x-t)e^{-t^2+2tx}\Big|_{t=0} \\ &= 2x. \end{aligned}$$

- **For $n = 2$:**

$$\begin{aligned} H_2(x) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} g(x,t)\bigg|_{t=0} \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} e^{-t^2+2tx}\bigg|_{t=0} \\ &= (2(x-t))^2 e^{-t^2+2tx} - 2e^{-t^2+2tx}\Big|_{t=0} \\ &= 4x^2 - 2. \end{aligned}$$

- **For $n = 3$:**

$$\begin{aligned} H_3(x) &= \frac{\mathrm{d}^3}{\mathrm{d}t^3} g(x,t)\bigg|_{t=0} \\ &= \frac{\mathrm{d}^3}{\mathrm{d}t^3} e^{-t^2+2tx}\bigg|_{t=0} \\ &= (2(x-t))^3 e^{-t^2+2tx} - 8(x-t)e^{-t^2+2tx} - 4(x-t)e^{-t^2+2tx}\Big|_{t=0} \\ &= 8x^3 - 12x. \end{aligned}$$

(d) The generating function of the Hermite polynomials is given by

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

- Deriving both sides with respect to t, we get

$$\frac{\partial g(x,t)}{\partial t} = 2(x-t)e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!},$$

which implies that

$$\begin{aligned} 2(x-t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} \\ \sum_{n=0}^{\infty} 2xH_n(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} &= \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} \end{aligned}$$

$$\sum_{m=1}^{\infty} 2H_{m-1}(x)\frac{t^m}{(m-1)!} - \sum_{m=0}^{\infty} 2xH_m(x)\frac{t^m}{m!} + \sum_{m=-1}^{\infty} H_m(x)\frac{t^m}{m!} = 0$$
$$\sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!} - \sum_{m=0}^{\infty} 2xH_m(x)\frac{t^m}{m!} + \sum_{m=-1}^{\infty} H_{m+1}(x)\frac{t^m}{m!} = 0$$
$$\implies H_{n+1} - 2xH_n + 2nH_{n-1} = 0.$$

- Deriving both sides with respect to x, we get

$$\frac{\partial g(x,t)}{\partial x} = 2te^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!},$$

which implies that

$$\sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!}$$
$$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} = 0$$

$$\sum_{m=0}^{\infty} H_m'(x)\frac{t^m}{m!} - \sum_{m=1}^{\infty} 2H_{m-1}(x)\frac{t^m}{(m-1)!} = 0$$
$$\sum_{m=0}^{\infty} H_m'(x)\frac{t^m}{m!} - \sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!} = 0$$
$$\implies H_n' - 2nH_{n-1} = 0.$$

(e) Replacing in the Hermite differential equation, we have

$$\begin{aligned}
H_n'' - 2xH_n' + 2nH_n & \\
&= 2nH_{n-1}' - 4nxH_{n-1} + 2nH_n \\
&= 4n^2H_{n-2}' - 4nxH_{n-1} + 2nH_n \\
&= 0.
\end{aligned}$$

∎

Problem 3
Use the generating function for the Bessel functions,

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n,$$

to obtain the following recurrence relations

(a) $J_{n-1} + J_{n+1} = \frac{2n}{x}J_n$,

(b) $J_{n-1} - J_{n+1} = 2J_n'$,

(c) $J_{n-1} - \frac{n}{x}J_n = J_n'$,

(d) $J_{n+1} - \frac{n}{x}J_n = -J_n'$.

(e) Using the above results, verify that J_n satisfies Bessel's equation,

$$x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n = 0.$$

(e) Verify that the series solution

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

satisfies the same equation.

Solution. The generating function of the Bessel functions is given by

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n.$$

(a) Deriving both sides with respect to t, we get

$$\frac{\partial g(x,t)}{\partial t} = \frac{x}{2}\left(1+\frac{1}{t^2}\right)e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1},$$

which implies that

$$\frac{x}{2}\left(1+\frac{1}{t^2}\right)\sum_{-\infty}^{\infty} J_n(x)t^n = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1}$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^n + \frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^{n-2} = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1}$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^{n-2} - \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1} + \frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^n = 0$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_{m+1}(x)t^{m-1} - \sum_{-\infty}^{\infty} mJ_m(x)t^{m-1} + \frac{x}{2}\sum_{-\infty}^{\infty} J_{m-1}(x)t^{m-1} = 0$$

$$\implies J_{n+1} + J_{n-1} = \frac{2n}{x}J_n.$$

(b) Deriving both sides with respect to x, we get

$$\frac{\partial g(x,t)}{\partial x} = \frac{\left(t - \frac{1}{t}\right)}{2} e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n'(x) t^n,$$

which implies that

$$\begin{aligned} \frac{\left(t - \frac{1}{t}\right)}{2} \sum_{-\infty}^{\infty} J_n(x) t^n &= \sum_{-\infty}^{\infty} J_n'(x) t^n \\ \frac{1}{2}\sum_{-\infty}^{\infty} J_n(x) t^{n+1} - \frac{1}{2}\sum_{-\infty}^{\infty} J_n(x) t^{n-1} &= \sum_{-\infty}^{\infty} J_n'(x) t^n \end{aligned}$$

$$\begin{aligned} \sum_{-\infty}^{\infty} J_n'(x) t^n - \frac{1}{2}\sum_{-\infty}^{\infty} J_n(x) t^{n+1} + \frac{1}{2}\sum_{-\infty}^{\infty} J_n(x) t^{n-1} &= 0 \\ \sum_{-\infty}^{\infty} J_m'(x) t^m - \frac{1}{2}\sum_{-\infty}^{\infty} J_{m-1}(x) t^{m} + \frac{1}{2}\sum_{-\infty}^{\infty} J_{m+1}(x) t^{m} &= 0 \end{aligned}$$

$$\implies J_{n-1} - J_{n+1} = 2J_n'.$$

(c) Adding the two equations derived in parts (a) and (b), we get

$$J_{n-1} - \frac{n}{x} J_n = J_n'.$$

(d) Subtracting the two equations derived in parts (a) and (b), we get

$$J_{n+1} - \frac{n}{x} J_n = -J_n'.$$

(e) Bessel's equation is given by

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0.$$

Deriving and replacing, we get

$$\begin{aligned} x^2 J_n'' + x J_n' + (x^2 - n^2) J_n &= \frac{x^2}{2}(J_{n-1}' - J_{n+1}') + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\ &= \frac{x^2}{4}\left(J_{n-2} - 2J_n + J_{n+2}\right) + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\ &= \frac{x}{2}\left((n-1)J_{n-1} + (n+1)J_{n+1}\right) - x^2 J_n + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\ &= \frac{x}{2}\left((n-1)J_{n-1} + (n+1)J_{n+1} + J_{n-1} - J_{n+1}\right) - n^2 J_n \\ &= \frac{nx}{2}\left(J_{n-1} + J_{n+1}\right) - n^2 J_n \\ &= \frac{nx}{2}\frac{2n}{x} J_n - n^2 J_n \\ &= 0. \end{aligned}$$

(f) Given the series solution

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s}$$

Deriving and replacing, we get

$$\begin{aligned}
x^2 J_n'' + x J_n' + (x^2 - n^2) J_n &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)(n+2s-1)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(x^2-n^2)\left(\frac{x}{2}\right)^{n+2s} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)(n+2s-1)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad - \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}n^2\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2\left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left[(n+2s)(n+2s-1)+(n+2s)-n^2\right]\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2\left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2 s\,(n+s)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2\left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{(s-1)!(n+s-1)!}2^2\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2\left(\frac{x}{2}\right)^{n+2s+2} \\
&= 0.
\end{aligned}$$

■

Homework 5

Problem 1
In class, we have derived Stirling's formula for real S,

$$S! \approx \sqrt{2\pi S} S^S \mathrm{e}^{-S}.$$

Show that the same result holds for arbitrary complex z as well, namely

$$\Gamma(z+1) = z! \approx \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z}.$$

Solution. To derive Stirling's approximation for complex numbers, we must first define the complex logarithm. The complex logarithm is well-defined modulo $2\pi i$ (so that we don't have to worry about the branches of $\ln(z)$), as

$$\ln(AB) \equiv \ln(A) + \ln(B) \pmod{2\pi i}.$$

The gamma function is single-valued, but Stirling's approximation contains $\sqrt{z}$ and z^z, which are both multi-valued in the complex plane, and hence, Stirling's approximation will not be analytic around zero. To fix this, we will restrict z to the right half of the complex plane, where we can specify the principal branch of $\ln(z)$, $\sqrt{z}$, and z^z. Let $z = r\mathrm{e}^{i\theta}$, then $\ln(z) = \ln(r) + i\theta$, where $\theta \in (-\pi, \pi)$. Taking the logarithm of both sides and expanding, we have

$$\begin{aligned}
\ln(\Gamma(z)) &\approx \frac{\ln(2\pi)}{2} + \left(z - \frac{1}{2}\right)\ln(z) - z \\
&= \frac{\ln(2\pi)}{2} + r\mathrm{e}^{i\theta}(\ln(r) + i\theta) - \frac{\ln(r) + i\theta}{2} - r\mathrm{e}^{i\theta} \\
&= \frac{\ln(2\pi)}{2} - \frac{\ln(r) + i\theta}{2} + r\mathrm{e}^{i\theta}(\ln(r) + i\theta - 1) \\
&= \frac{\ln(2\pi)}{2} - \frac{\ln(r) + i\theta}{2} + r(\cos(\theta) + i\sin(\theta))(\ln(r) + i\theta - 1) \\
&= \frac{\ln(2\pi)}{2} - \frac{\ln(r) + i\theta}{2} + r\ln(r)\cos(\theta) + ir\theta\cos(\theta) - r\cos(\theta) + ir\ln(r)\sin(\theta) - r\theta\sin(\theta) - ir\sin(\theta),
\end{aligned}$$

which gives us

$$\Re(\ln(\Gamma(z))) = \frac{\ln(2\pi)}{2} - \frac{\ln(r)}{2} + r\ln(r)\cos(\theta) - r\cos(\theta) - r\theta\sin(\theta),$$
$$\Im(\ln(\Gamma(z))) = -\frac{\theta}{2} + r\ln(r)\sin(\theta) - r\sin(\theta) + r\theta\cos(\theta).$$

If we pick a fixed non-zero value for θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and take $r \to +\infty$, the term $r\ln(r)\sin(\theta)$ in the expression for the imaginary part dominates, so the imaginary part of the gamma function will become arbitrarily large and the gamma function will just spiral around the origin an infinite number of times as z tends to infinity along the non-trivial ray.

Although the gamma function is single-valued in the entire plane, its logarithm becomes multi-valued if we analytically continue it around the poles of the gamma function. It is true that any two values of the logarithm will differ by a multiple of $2\pi i$, but that is not true for Stirling's approximation, as seen above. Applying the exponential to the logarithm of the gamma function wipes any ambiguity, but not so for the approximation.

Consider

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Gamma(x) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} = \frac{1}{2z^2} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\pi\cot(\pi\zeta)}{(z+\zeta)^2}\,\mathrm{d}\zeta$$

by contour integration. Assuming $\Re(z) > 0$, we can split the integral and use substitution to get

$$\begin{aligned}
\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\pi\cot(\pi\zeta)}{(z+\zeta)^2}\,\mathrm{d}\zeta &= \frac{1}{2}\int_0^{\infty} \cot(\pi i\eta)\left(\frac{1}{(z+\eta)^2} - \frac{1}{(z-\eta)^2}\right)\mathrm{d}\eta \\
&= -\int_0^{\infty} \coth(\pi\eta)\frac{2\eta z}{(z^2+\eta)^2}\,\mathrm{d}\eta.
\end{aligned}$$

Substituting back, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Gamma(x) = \frac{1}{z} + \frac{1}{2z^2} + \int_0^\infty \frac{1}{\mathrm{e}^{2\pi\eta}-1}\frac{4\eta z}{(z^2+\eta^2)^2}\,\mathrm{d}\eta.$$

Integrating with respect to z, and then performing integration by parts, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\Gamma(x) &= C + \ln(z) - \frac{1}{2z} - \int_0^\infty \frac{1}{\mathrm{e}^{2\pi\eta}-1}\frac{2\eta}{z^2+\eta^2}\,\mathrm{d}\eta \\ &= C + \ln(z) - \frac{1}{2z} + \frac{1}{\pi}\int_0^\infty \ln(1-\mathrm{e}^{2\pi\eta})\frac{z^2-\eta^2}{(z^2+\eta^2)^2}\,\mathrm{d}\eta\end{aligned}$$

Integrating again, we get

$$\ln(\Gamma(x)) = C' + Cz + \left(z-\frac{1}{2}\right)\ln(z) + \frac{1}{\pi}\int_0^\infty \frac{z}{z^2+\eta^2}\ln\left(\frac{1}{1-\mathrm{e}^{-2\pi\eta}}\right)\mathrm{d}\eta,$$

which is essentially Stirling's formula with some constants of integration and limit relations

$$\ln(\Gamma(x)) = C' + Cz + z\ln(z) - \frac{1}{2}\ln(z) + J(z),$$

where $J(z)$ is the integral term.
Notice that $J(z) \to 0$ as $z \to \infty$ with $\Re(z) \geq c$, for some fixed real positive c.
The constant C is determined from the functional equation or recursive relation

$$z\Gamma(z) = \Gamma(z+1) \iff \ln(z) + \ln(\Gamma(z)) = \ln(\Gamma(z+1))$$

by letting z tend to infinity. We get that $C = -1$.
The constant C' is determined from the property

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

by letting z tend to infinity. We get that $C' = \frac{1}{2}\ln(2\pi)$.
Therefore,

$$\begin{aligned}\ln(\Gamma(x)) &\approx \frac{1}{2}\ln(2\pi) - z + z\ln(z) - \frac{1}{2}\ln(z), \\ \implies \Gamma(z) &\approx \sqrt{2\pi}z^{z+\frac{1}{2}}e^{-z}.\end{aligned}$$

■

Problem 2
Repeat the derivation given in class of the result

$$I = \Gamma(a)\Gamma(1-a) = \int_0^\infty \frac{u^{a-1}}{1+u}\,\mathrm{d}u = \frac{\pi}{\sin(\pi a)}.$$

Use the substitution $u = \mathrm{e}^x$ to write down I in the form

$$I = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,\mathrm{d}x.$$

Integrate around the rectangular contour with the four corners at R, $R+2\pi i$, $-R+2\pi i$, and $-R$. Determine that the residue at $z = i\pi$ is $-\mathrm{e}^{ia\pi}$.

Solution. From the definition of the gamma function, we have

$$\begin{aligned}\Gamma(a)\Gamma(1-a) &= \int_0^\infty \frac{u^{a-1}}{1+u}\,\mathrm{d}u \\ &= \int_{-\infty}^\infty \frac{\mathrm{e}^{(a-1)x}}{1+\mathrm{e}^x}\mathrm{e}^x\,\mathrm{d}x \\ &= \int_{-\infty}^\infty \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^x}\,\mathrm{d}x\end{aligned}$$

Theorem 1. (Cauchy Residue Theorem) If f is holomorphic inside and on the simple closed contour C, except for a finite set of points of isolated singularities z_k inside C, then

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^n \operatorname{res}(f, z_k).$$

We have singularities at $z = \pm\pi i$. Our contour C is a rectangular region in the upper-half of the complex plane, so it only encloses $z = \pi i$. Using the Cauchy residue theorem, we have

$$\begin{aligned}\oint_C \frac{\mathrm{e}^{az}}{1+\mathrm{e}^z}\,\mathrm{d}z &= 2\pi i\ \operatorname{res}\left[\left.\frac{\mathrm{e}^{az}}{1+\mathrm{e}^z}\right|_{z=\pi i}\right] \\ &= 2\pi i \lim_{z\to\pi i}\left((z-\pi i)\frac{\mathrm{e}^{az}}{1+\mathrm{e}^z}\right) \\ &= 2\pi i \lim_{z\to\pi i}\left(\frac{\mathrm{e}^{az} + a(z-\pi i)\mathrm{e}^{az}}{\mathrm{e}^z}\right) \\ &= -2\pi i\mathrm{e}^{a\pi i}.\end{aligned}$$

Breaking up the contour integral, we have

$$\begin{aligned}\oint_C \frac{\mathrm{e}^{az}}{1+\mathrm{e}^z}\,\mathrm{d}z &= I_1 + I_2 + I_3 + I_4 \\ &= \lim_{R\to\infty}\int_{-R}^R \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^x}\,\mathrm{d}x + i\lim_{R\to\infty}\int_0^{2\pi}\frac{\mathrm{e}^{aR+ayi}}{1+\mathrm{e}^{R+yi}}\,\mathrm{d}y \\ &\quad + \lim_{R\to\infty}\int_R^{-R}\frac{\mathrm{e}^{ax+2a\pi i}}{1+\mathrm{e}^{x+2\pi i}}\,\mathrm{d}x + i\lim_{R\to\infty}\int_{2\pi}^0 \frac{\mathrm{e}^{aR+ayi}}{1+\mathrm{e}^{R+yi}}\,\mathrm{d}y \\ &= I + 0 - \mathrm{e}^{2a\pi i}I + 0.\end{aligned}$$

Equating both sides, we have

$$\begin{aligned}
I &= \frac{-2\pi i \mathrm{e}^{a\pi i}}{1-\mathrm{e}^{2a\pi i}} \\
&= \frac{2\pi i \mathrm{e}^{a\pi i}}{\mathrm{e}^{2a\pi i}-1} \\
&= \frac{2\pi i}{\mathrm{e}^{a\pi i}-\mathrm{e}^{-a\pi i}} \\
&= \frac{\pi}{\sin(a\pi)}.
\end{aligned}$$

■

Homework 6

Problem 1
Consider the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

(a) Develop an asymptotic series for $\operatorname{erf}(x)$. [Hint: look at $1 - \operatorname{erf}(x)$ and develop this in a series by partial integration]

(b) Show that this series is not uniformly convergent.

Solution. (a) Note that the complimentary error function erfc is defined as

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

Developing an asymptotic expansion for the complimentary error function (and therefore also of the error function) for large x, we have

$$\begin{aligned}
\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t \\
&= \frac{2}{\sqrt{\pi}} \int_{x^2}^\infty \mathrm{e}^{-u} \frac{1}{2\sqrt{u}}\,\mathrm{d}u \qquad (u = t^2) \\
&= \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty u^{-\frac{1}{2}} \mathrm{e}^{-u}\,\mathrm{d}u.
\end{aligned}$$

Using tabular integration by parts, we have

$$\begin{array}{ccc}
u^{-\frac{1}{2}} & & \mathrm{e}^{-u} \\
 & \searrow + & \\
-\frac{1}{2}u^{-\frac{3}{2}} & & -\mathrm{e}^{-u} \\
 & \searrow - & \\
\frac{1}{2}\frac{3}{2}u^{-\frac{5}{2}} & & \mathrm{e}^{-u} \\
 & \searrow + & \\
\vdots & & -\mathrm{e}^{-u} \\
 & & \vdots \\
\frac{1}{2}\frac{3}{2}\cdots\frac{2n-1}{2}u^{-\frac{2n+1}{2}} & & \\
 & & (-1)^{n+1}\mathrm{e}^{-u}
\end{array}$$

which gives us

$$\begin{aligned}
\operatorname{erfc}(x) &= \frac{1}{\sqrt{\pi}} \left[\sum_{n=0}^\infty (-1)^{n+1} \frac{(2n-1)!!}{2^n} u^{-\frac{2n+1}{2}} \mathrm{e}^{-u} \right]_{x^2}^\infty \\
&= 0 - \frac{1}{\sqrt{\pi}} \left[\sum_{n=0}^\infty (-1)^{n+1} \frac{(2n-1)!!}{2^n} x^{-(2n+1)} \mathrm{e}^{-x^2} \right] \\
&= \frac{1}{x\sqrt{\pi}} \mathrm{e}^{-x^2} \sum_{n=0}^\infty (-1)^n \frac{(2n-1)!!}{2^n x^{2n}}.
\end{aligned}$$

Thus, the asymptotic expansion for $\mathrm{erf}(x)$ is

$$\mathrm{erf}(x) = 1 - \mathrm{erfc}(x) = 1 - \frac{1}{x\sqrt{\pi}}\mathrm{e}^{-x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n x^{2n}}.$$

(b) A sequence of functions $\{f_n\}$, $n = 1, 2, \dots$ is said to be uniformly convergent to f for a set E of values of x if, for each $\epsilon > 0$, an integer N can be found such that

$$|f_n(x) - f(x)| < \epsilon,$$

for $n \geq N$ and all $x \in E$.

A series converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by

$$\sum_{k=1}^{n} f_k(x) = S_n(x)$$

converges uniformly on E.

To test for uniform convergence, we can use Abel's uniform convergence test or the Weierstrass M-test. Our series looks like

$$\mathrm{erfc}(x) = \frac{1}{x\sqrt{\pi}}\mathrm{e}^{-x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n}\frac{1}{x^{2n}} = \frac{1}{x\sqrt{\pi}}\mathrm{e}^{-x^2}\sum_{n=0}^{\infty}a_n f_n(x).$$

The sequence of partial sums for the first few terms looks like

$$\left\{\frac{1}{x},\quad \frac{1}{x} - \frac{1}{2x^2},\quad \frac{1}{x} - \frac{1}{2x^2} + \frac{1\cdot 3}{4x^4},\quad \frac{1}{x} - \frac{1}{2x^2} + \frac{1\cdot 3}{4x^4} - \frac{1\cdot 3\cdot 5}{8x^6}, \cdots\right\}.$$

For the asymptotic sequence

$$\phi_n(x) = \frac{\mathrm{e}^{-x^2}}{x^{2n+1}} \quad \text{as } x \to \infty,$$

for a fixed N, we have that

$$\begin{aligned}
|S_{n+1} - S_n| &= \left|C\int_x^{\infty}\frac{e^{-t^2}}{t^{2n+2}}\,\mathrm{d}t\right| \\
&= \left|C\int_x^{\infty}\left(-\frac{1}{2t^{2n+3}}\right)\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t^2})\,\mathrm{d}t\right| \\
&\lesssim C\frac{e^{-x^2}}{x^{2n+3}} \\
&= \frac{C\phi_n(x)}{x^2},
\end{aligned}$$

where C is a constant depending only on n. Hence, the series diverges for any large x.

■

Problem 2
In class, we developed asymptotic expansions for $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$. Rederive the same expansions by a series of partial integrations. Hint: Recall that

$$\mathrm{Ci}(x) + i\mathrm{Si}(x) = -\int_x^\infty \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t.$$

Solution. Using tabular integration by parts, we have

$$\begin{array}{ccc}
t^{-1} & & \mathrm{e}^{it} \\
 & + & \\
-t^{-2} & & -i\mathrm{e}^{it} \\
 & - & \\
2t^{-3} & & -\mathrm{e}^{it} \\
 & + & \\
\vdots & & i\mathrm{e}^{it} \\
 & & \\
(-1)^n n! t^{-(n+1)} & & \vdots \\
 & & \\
 & & (-1)^{-\frac{n}{2}}\mathrm{e}^{it}
\end{array}$$

which gives us

$$\begin{aligned}
-\int_x^\infty \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t &= -\sum_{n=0}^\infty (-1)^{-\left(\frac{n+1}{2}\right)} n! \frac{e^{it}}{t^{n+1}}\Bigg|_x^\infty \\
&= \sum_{n=0}^\infty (-1)^{-\left(\frac{n+1}{2}\right)} n! \frac{e^{ix}}{x^{n+1}} \\
&= e^{ix}\left[\sum_{k=0}^\infty (-1)^{k+1}\frac{(2k+1)!}{x^{2k+2}} + \sum_{k=0}^\infty (-1)^{k+\frac{3}{2}}\frac{(2k)!}{x^{2k+1}}\right] \\
&= (\cos(x) + i\sin(x))\left[\sum_{k=0}^\infty (-1)^{k+1}\frac{(2k+1)!}{x^{2k+2}} + i\sum_{k=0}^\infty (-1)^{k+1}\frac{(2k)!}{x^{2k+1}}\right] \\
&= \frac{\sin(x)}{x}\sum_{n=0}^\infty (-1)^n\frac{(2n)!}{x^{2n}} - \frac{\cos(x)}{x^2}\sum_{n=0}^\infty (-1)^n\frac{(2n+1)!}{x^{2n}} \\
&\quad - i\left(\frac{\cos(x)}{x}\sum_{n=0}^\infty (-1)^n\frac{(2n)!}{x^{2n}} + \frac{\sin(x)}{x^2}\sum_{n=0}^\infty (-1)^n\frac{(2n+1)!}{x^{2n}}\right).
\end{aligned}$$

Therefore,

$$\mathrm{Ci}(x) = \frac{\sin(x)}{x}\sum_{n=0}^\infty (-1)^n\frac{(2n)!}{x^{2n}} - \frac{\cos(x)}{x^2}\sum_{n=0}^\infty (-1)^n\frac{(2n+1)!}{x^{2n}}$$

and

$$\mathrm{Si}(x) = -\frac{\cos(x)}{x}\sum_{n=0}^\infty (-1)^n\frac{(2n)!}{x^{2n}} - \frac{\sin(x)}{x^2}\sum_{n=0}^\infty (-1)^n\frac{(2n+1)!}{x^{2n}}.$$

■

Homework 7

Problem 1
In a spinning factory, a worker watches over several hundred spindles. As each spindle turns, the yarn breaks at chance due to irregularities in the tension, evenness of yarn, and etc. For purposes of quality control, it is important to know how frequently breaks occur. Assume that a given worker watches 800 spindles, and that the probability of a break during a given time interval τ is 0.005 for each spindle.

(a) Find the most probable number of breaks during the time interval τ.

(b) Find the probability that no more than 10 breaks will occur during τ. (Use Poisson distribution.)

Solution. (a) Let $n = 800$ and $p = 0.005$, then the most probable number of breaks during the time interval τ is $np = 4$.

(b) Since np is significantly smaller than n, then a Poisson distribution can be used. A Poisson distribution is given by

$$P(m) = \frac{\mu^m \mathrm{e}^{-\mu}}{m!},$$

where $\mu = np = 4$ is the mean.

The probability that no more than 10 breaks will occur is

$$\begin{aligned} P(m \leq 10) &= \sum_{m=0}^{10} P(m) \\ &= \sum_{m=0}^{10} \frac{\mu^m \mathrm{e}^{-\mu}}{m!} \\ &= \sum_{m=0}^{10} \frac{4^m \mathrm{e}^{-4}}{m!} \\ &= \mathrm{e}^{-4} \sum_{m=0}^{10} \frac{4^m}{m!} \\ &= \mathrm{e}^{-4}(54.4431) \\ &= 0.9971. \end{aligned}$$

■

Problem 2
For a Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{1}{2}\frac{(x-\bar{x})^2}{\sigma^2}}$$

centered at $\bar{x}$, compute

(a) the mean value $\langle x \rangle$ and

(b) the variance $\left\langle (x-\bar{x})^2 \right\rangle$.

(c) Show that $\sigma^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2$.

Solution. (a) The expectation value of a Gaussian distributed random variable is

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x p(x)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (t+\bar{x})\,\mathrm{e}^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[\int_{-\infty}^{\infty} t \mathrm{e}^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t + \bar{x} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t \right] \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[0 + \bar{x}(\sqrt{2\pi}\sigma) \right] \\
&= \bar{x}.
\end{aligned}$$

(b) The variance of a Gaussian distributed random variable is

$$\begin{aligned}
\left\langle (x-\bar{x})^2 \right\rangle &= \int_{-\infty}^{\infty} (x-\bar{x})^2 p(x)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (x-\bar{x})^2 \mathrm{e}^{-\frac{(x-\bar{x})^2}{2\sigma^2}}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} t^2 \mathrm{e}^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[-\sigma^2 t \mathrm{e}^{-\frac{t^2}{2\sigma^2}} \Big|_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{t^2}{2\sigma^2}}\,\mathrm{d}t \right] \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[0 + \sigma^2(\sqrt{2\pi}\sigma) \right] \\
&= \sigma^2.
\end{aligned}$$

(c) We have that

$$\begin{aligned}
\sigma^2 &= \left\langle (x-\bar{x})^2 \right\rangle \\
&= \left\langle x^2 - 2x\bar{x} + \bar{x}^2 \right\rangle \\
&= \left\langle x^2 \right\rangle - 2\left\langle x\bar{x} \right\rangle + \left\langle \bar{x}^2 \right\rangle \\
&= \left\langle x^2 \right\rangle - 2\bar{x}\left\langle x \right\rangle + \bar{x}^2 \left\langle 1 \right\rangle \\
&= \left\langle x^2 \right\rangle - 2\bar{x}\bar{x} + \bar{x}^2 \\
&= \left\langle x^2 \right\rangle - \bar{x}^2 \\
&= \left\langle x^2 \right\rangle - \langle x \rangle^2.
\end{aligned}$$

■

Problem 3
Starting with the binomial distribution, derive the Gaussian distribution. Make certain to clearly state all approximations that are made.

Solution. The binomial distribution is given by

$$P(m) = \binom{n}{m} p^m (1-p)^{n-m} = \frac{n!}{m!(n-m)!} p^m (1-p)^{n-m}.$$

Considering $n, np \to \infty$, with $m \sim np$, we apply Stirling's approximation on the binomial coefficient, giving us

$$\begin{aligned}
\frac{n!}{m!(n-m)!} &= \frac{\sqrt{2\pi n} n^n e^{-n}}{\sqrt{2\pi m} m^m e^{-m} \sqrt{2\pi (n-m)} (n-m)^{(n-m)} e^{-(n-m)}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{n^{n+\frac{1}{2}}}{m^{m+\frac{1}{2}} (n-m)^{(n-m)+\frac{1}{2}}} \\
&= \frac{1}{\sqrt{2\pi n}} \frac{n^{n+1}}{m^{m+\frac{1}{2}} (n-m)^{(n-m)+\frac{1}{2}}} \\
&= \frac{1}{\sqrt{2\pi n}} \left(\frac{m}{n}\right)^{-m-\frac{1}{2}} \left(1 - \frac{m}{n}\right)^{-n+m-\frac{1}{2}}.
\end{aligned}$$

Plugging the coefficient back into the binomial distribution, we get

$$\begin{aligned}
P(m) &= \frac{1}{\sqrt{2\pi n}} \left(\frac{m}{n}\right)^{-m-\frac{1}{2}} \left(1 - \frac{m}{n}\right)^{-n+m-\frac{1}{2}} p^m (1-p)^{n-m} \\
&= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-\left(m+\frac{1}{2}\right)\ln\left(\frac{m}{n}\right) - \left(n-m+\frac{1}{2}\right)\ln\left(1-\frac{m}{n}\right) + m\ln(p) + (n-m)\ln(1-p)}.
\end{aligned}$$

Let $x = m - np$, then

$$\begin{aligned}
P(x) &= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-\left(x+np+\frac{1}{2}\right)\ln\left(\frac{x}{n}+p\right) - \left(n-x-np+\frac{1}{2}\right)\ln\left(1-\frac{x}{n}-p\right) + (x+np)\ln(p) + (n(1-p)-x)\ln(1-p)} \\
&= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(np+x)\ln\left(1+\frac{x}{np}\right) - \frac{1}{2}\ln\left(p+\frac{x}{n}\right) - (n-np-x)\ln\left(1-\frac{x}{n-np}\right) - \frac{1}{2}\ln\left(1-p-\frac{x}{n}\right)}.
\end{aligned}$$

Now, we have $\frac{x}{n} \ll p$, in addition to performing a Taylor expansion of the logarithmic functions of the form $\ln(1+x)$, where x is small, giving us

$$\begin{aligned}
P(x) &= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(np+x)\ln\left(1+\frac{x}{np}\right) - \frac{1}{2}\ln(p) - (n-np-x)\ln\left(1-\frac{x}{n-np}\right) - \frac{1}{2}\ln(1-p)} \\
&= \frac{1}{\sqrt{2\pi n}} \mathrm{e}^{-(np+x)\ln\left(1+\frac{x}{np}\right) - (n(1-p)-x)\ln\left(1-\frac{x}{n(1-p)}\right) - \frac{1}{2}\ln(p(1-p))} \\
&= \frac{1}{\sqrt{2\pi n}} \frac{1}{\sqrt{p(1-p)}} \mathrm{e}^{-(np+x)\left(\frac{x}{np} - \frac{1}{2}\left(\frac{x}{np}\right)^2 + \cdots\right) - (n(1-p)-x)\left(-\frac{x}{n(1-p)} - \frac{1}{2}\left(\frac{x}{n(1-p)}\right)^2 + \cdots\right)} \\
&= \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-x + \frac{1}{2}\frac{x^2}{np} - \frac{x^2}{np} + O(3) + x + \frac{1}{2}\frac{x^2}{n(1-p)} - \frac{x^2}{n(1-p)} + O(3)} \\
&= \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-\frac{1}{2}\frac{x^2}{np} - \frac{1}{2}\frac{x^2}{n(1-p)} + O(3)} \\
&= \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-\frac{x^2}{2n}\left[\frac{1}{p} + \frac{1}{1-p}\right]} \\
&= \frac{1}{\sqrt{2\pi np(1-p)}} \mathrm{e}^{-\frac{1}{2}\frac{x^2}{np(1-p)}}.
\end{aligned}$$

Letting $\sigma = \sqrt{np(1-p)}$, we get

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\frac{x^2}{2\sigma^2}},$$

which is the Gaussian distribution. ■

Homework 8

Problem 1
Show that the Green's function for the 2-dimensional Laplace over the entire 2-dimensional space is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi} \ln \left[(x - x')^2 + (y - y')^2 \right]^{\frac{1}{2}},$$

where, $\mathbf{r} = (x, y)$ and $\mathbf{r}' = (x', y')$.

Solution. Consider

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),$$

where $\mathbf{r} = (x, y)$ and $\mathbf{r}' = (x', y')$.
The boundary condition of our system is given by $G(r, r') \to 0$ as $r \to \infty$. Making a coordinate transformation by moving our system to the source given by $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Now, the system is circularly symmetric and the Green's function only depends on $R = |\mathbf{r} - \mathbf{r}'|$. Thus, we have

$$\nabla^2 G(R) = -\delta(\mathbf{R}) = -\frac{1}{2\pi R}\delta(R).$$

In polar coordinates, we have

$$\begin{aligned} \frac{\partial^2 G}{\partial R^2} + \frac{1}{R}\frac{\partial G}{\partial R} + \frac{1}{R^2}\frac{\partial^2 G}{\partial \theta^2} &= -\frac{1}{2\pi R}\delta(R) \\ \frac{\mathrm{d}^2 G}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}G}{\mathrm{d}R} &= -\frac{1}{2\pi R}\delta(R). \end{aligned}$$

- **For $R > 0$:** We have

$$\begin{aligned} \frac{\mathrm{d}^2 G}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}G}{\mathrm{d}R} &= 0 \\ \frac{\mathrm{d}^2 G}{\mathrm{d}R^2} &= -\frac{1}{R}\frac{\mathrm{d}G}{\mathrm{d}R} \end{aligned}$$

  Let $H \equiv \frac{\mathrm{d}G}{\mathrm{d}R}$. Then

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}R} &= -\frac{1}{R}H \\ \frac{\mathrm{d}H}{H} &= -\frac{\mathrm{d}R}{R} \\ \ln(H) &= -\ln(R) + C \\ H &= \frac{A}{R} \\ \frac{\mathrm{d}G}{\mathrm{d}R} &= \frac{A}{R} \\ G &= A\ln(R) + B. \end{aligned}$$

  Applying the boundary condition, we have $G(R) \to 0$ as $R \to \infty$. Thus $B = 0$ and we have

$$G = A\ln(R).$$

  We have that $\nabla^2 G(R) = -\frac{1}{2\pi R}\delta(R)$. Integrating twice, we have

$$\begin{aligned} \iint \nabla^2 G(R)\,\mathrm{d}A &= -\frac{1}{2\pi}\iint \frac{1}{R}\delta(R)\,\mathrm{d}A \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^r \frac{1}{R}\delta(R) R\,\mathrm{d}R\,\mathrm{d}\phi \\ &= -1. \end{aligned}$$

Additionally, by using the divergence theorem, we have

$$\begin{aligned}\iint \nabla^2 G(R)\,\mathrm{d}A &= \iint \nabla\cdot(\nabla G(R))\,\mathrm{d}A \\ &= \oint \nabla G(R)\big|_{R=r}\cdot \mathrm{d}\mathbf{s} \\ &= \int_0^{2\pi} \left(\hat{R}\frac{\partial}{\partial R}(A\ln(R))\right)\bigg|_{R=r}\cdot\left(\hat{R}r\,\mathrm{d}\theta\right) \\ &= \int_0^{2\pi} \frac{\partial}{\partial R}(A\ln(R))\bigg|_{R=r} r\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{A}{R}\bigg|_{R=r} r\,\mathrm{d}\theta \\ &= \int_0^{2\pi} A\,\mathrm{d}\theta \\ &= 2\pi A.\end{aligned}$$

Thus, $2\pi A = -1 \implies A = -\frac{1}{2\pi}$, giving us

$$G(R) = -\frac{1}{2\pi}\ln(R).$$

Therefore,

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{2\pi}\ln\left[\sqrt{(x-x')^2+(y-y')^2}\right].$$

■

Problem 2
Write down the solution to the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f(x,y)$$

in terms of $G(\mathbf{r},\mathbf{r}')$.

Solution. Since $\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$, we have

$$\begin{aligned}\nabla^2 u(x,y) &= f(x,y) \\ &= \iint \delta(\mathbf{r}-\mathbf{r}')f(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\ &= \iint -\nabla^2 G(\mathbf{r},\mathbf{r}')f(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\ &= \nabla^2\left[\iint -G(\mathbf{r},\mathbf{r}')f(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\right].\end{aligned}$$

Thus,

$$u(x,y) = -\iint G(\mathbf{r},\mathbf{r}')f(x',y')\,\mathrm{d}x'\,\mathrm{d}y'.$$

■

Problem 3
$\psi(x,t)$ satisfies the 1-dimensional Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t},$$

with initial condition $\psi(x,0)=\delta(x)$, and boundary condition

$$\frac{\partial\psi}{\partial x}\left(-\frac{L}{2},t\right) = \frac{\partial\psi}{\partial x}\left(\frac{L}{2},t\right) = 0.$$

Show by the method of separation of variables that

$$\psi(x,t) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\cos\left(\frac{2n\pi x}{L}\right)\exp\left[-i\frac{\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t\right].$$

Solution. Applying the method of separation of variables, we consider $\psi(x,t)=X(x)T(t)$. Replacing in the Schrodinger equation, we have

$$\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \\ i\hbar X\frac{\partial T}{\partial t} &= -\frac{\hbar^2}{2m}T\frac{\partial^2 X}{\partial x^2} \\ i\hbar\frac{1}{T}\frac{\partial T}{\partial t} &= -\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2 X}{\partial x^2}. \end{aligned}$$

The left hand side is dependent on t only and the right hand side is dependent of x only. Thus, they must both be equal to some constant E. Then

$$\begin{cases} i\hbar\frac{1}{T}\frac{\partial T}{\partial t} = E \\ -\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = E \end{cases} \implies \begin{cases} \frac{\partial T(t)}{\partial t} = -\frac{iE}{\hbar}T \\ \frac{\partial^2 X(x)}{\partial x^2} = -\frac{2mE}{\hbar^2}X \end{cases} \implies \begin{cases} T(t) = c_1 \mathrm{e}^{-\frac{iE}{\hbar}t} \\ X(x) = c_2\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}x} + c_3\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}x} \end{cases}$$

Thus,

$$\begin{aligned} \psi(x,t) &= \left(c_2\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}x} + c_3\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}x}\right)c_1\mathrm{e}^{-\frac{iE}{\hbar}t} \\ &= \left(A\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}x} + B\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}x}\right)\mathrm{e}^{-\frac{iE}{\hbar}t}. \end{aligned}$$

Applying the boundary conditions, we have

$$\begin{aligned} \frac{\partial\psi}{\partial x}\left(-\frac{L}{2},t\right) &= \frac{\partial\psi}{\partial x}\left(\frac{L}{2},t\right) = 0 \\ i\frac{\sqrt{2mE}}{\hbar}\left(A\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} - B\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}}\right)\mathrm{e}^{-\frac{iE}{\hbar}t} &= i\frac{\sqrt{2mE}}{\hbar}\left(A\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} - B\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}}\right)\mathrm{e}^{-\frac{iE}{\hbar}t} \\ A\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} - B\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} &= A\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} - B\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} \\ A - B\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}L} &= A\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}L} - B \\ \mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}L} &= 1 \\ \frac{\sqrt{2mE}}{\hbar}L &= 2n\pi, \quad n\in\mathbb{Z}^+ \\ \lambda_n \equiv \frac{\sqrt{2mE}}{\hbar} &= \frac{2n\pi}{L} \\ E &= \frac{\hbar^2}{2m}\left(\frac{2n\pi}{L}\right)^2, \end{aligned}$$

and $A = B$. We take positive values of n only to generate distinct solutions since we have a symmetry across $x = 0$.
Thus,

$$\begin{aligned}
\psi_n(x,t) &= A\left(\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}x} + \mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}x}\right)\mathrm{e}^{-\frac{iE}{\hbar}t} \\
&= 2A\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right)\mathrm{e}^{-\frac{iE}{\hbar}t} \\
&= C_n\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(\frac{2n\pi}{L}\right)^2 t} \\
&= C_n\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t}
\end{aligned}$$

which gives us

$$\psi(x,t) = \sum_{n=0}^{\infty} C_n\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t}.$$

Applying the initial condition, we have

$$\psi(x,0) = \sum_{n=0}^{\infty} C_n\cos\left(\frac{2n\pi x}{L}\right) = \delta(x).$$

Then, we have

$$C_n = \frac{2}{L}\int_{-L/2}^{L/2}\cos\left(\frac{2n\pi x}{L}\right)\delta(x)\,\mathrm{d}x = \frac{2}{L}.$$

Therefore,

$$\psi(x,t) = \frac{2}{L}\sum_{n=0}^{\infty}\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t} = \frac{1}{L}\sum_{n=-\infty}^{\infty}\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t}.$$

■

Homework 9

Problem 1
[Arfken pp. 738-739] Solve the integral equation

$$\phi(x) = x + \frac{1}{2}\int_{-1}^{+1} (t-x)\phi(t)\,\mathrm{d}t$$

for $\phi(x)$ using the Neumann series method.

Solution. Considering a generic Fredholm equation, we have $f(x) = x$, $\lambda = \frac{1}{2}$, $a = -1$, $b = 1$, and $K(x,t) = t - x$.

Solving this by a Neumann series solution, we take

$$\begin{aligned}
\phi_0(x) &\equiv f(x) \\
\phi_1(x) &\equiv f(x) + \lambda \int_a^b K(x,t_1) f(t_1)\,\mathrm{d}t_1 \\
\phi_2(x) &\equiv f(x) + \lambda \int_a^b K(x,t_1) f(t_1)\,\mathrm{d}t_1 + \lambda^2 \int_a^b \int_a^b K(x,t_1)K(t_1,t_2) f(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\vdots \\
\phi_N(x) &\equiv \sum_{n=0}^{N} \lambda^n u_n(x),
\end{aligned}$$

where

$$\begin{aligned}
u_0(x) &= f(x) \\
u_1(x) &= \int_a^b K(x,t_1) f(t_1)\,\mathrm{d}t_1 \\
u_2(x) &= \int_a^b \int_a^b K(x,t_1)K(t_1,t_2) f(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\vdots \\
u_n(x) &= \int_a^b \int_a^b \cdots \int_a^b K(x,t_1)K(t_1,t_2)\cdots K(t_{n-1},t_n) f(t_n)\,\mathrm{d}t_n\,\mathrm{d}t_{n-1}\cdots\mathrm{d}t_1.
\end{aligned}$$

The Neumann series solution is then

$$
\begin{aligned}
\phi(x) &= \lim_{N\to\infty} \phi_N(x) \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \lambda^n u_n(x) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} (t_1 - x)(t_2 - t_1)\cdots(t_n - t_{n-1}) f(t_n)\, \mathrm{d}t_n\, \mathrm{d}t_{n-1}\cdots \mathrm{d}t_1 \\
&= x + \frac{1}{2}\int_{-1}^{1} (t_1 - x)t_1\, \mathrm{d}t_1 + \left(\frac{1}{2}\right)^2 \int_{-1}^{1}\int_{-1}^{1} (t_1 - x)(t_2 - t_1)t_2\, \mathrm{d}t_2\, \mathrm{d}t_1 + \cdots \\
&= x + \frac{1}{2}\left[\frac{t_1^3}{3} - \frac{t_1^2}{2}x\right]_{-1}^{1} + \left(\frac{1}{2}\right)^2 \int_{-1}^{1} \left[\frac{t_2^3}{3}t_1 - \frac{t_2^2}{2}t_1^2 - \frac{t_2^3}{3}x + \frac{t_2^2}{2}t_1 x\right]_{-1}^{1} \mathrm{d}t_1 + \cdots \\
&= x + \frac{1}{2}\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^2 \int_{-1}^{1} \left[-t_1^2 - \frac{2}{3}x\right] \mathrm{d}t_1 + \cdots \\
&= x + \frac{1}{2}\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^2 \left[-\frac{t_1^3}{3} - \frac{2}{3}xt_1\right]_{-1}^{1} + \cdots \\
&= x + \frac{1}{2}\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^2 \left(-\frac{4}{3}x\right) + \cdots \\
&= x + \frac{1}{2}\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^2 \left(\frac{2}{3}\right)(-2x) + \cdots \\
&= x\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \left(\frac{2}{3}\right)^{n} (-2)^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n+1} \left(\frac{2}{3}\right)^{n+1} (-2)^n \\
&= x\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \left(-\frac{4}{3}\right)^{n} + \frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \left(\frac{2}{3}\right)^{n} (-2)^n \\
&= x\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} \left(-\frac{4}{3}\right)^{n} + \frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{n} \left(-\frac{4}{3}\right)^{n} \\
&= x\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^{n} + \frac{1}{3}\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^{n} \\
&= \left(x + \frac{1}{3}\right)\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^{n} \\
&= \left(x + \frac{1}{3}\right)\frac{1}{1 + \frac{1}{3}} \\
&= \frac{3}{4}\left(x + \frac{1}{3}\right).
\end{aligned}
$$

■

Problem 2
Explicitly exhibit the form of the resolvent kernel $\tilde{K}(x,t;\lambda)$ for the above equation.

Solution. From the previous problem, we have

$$u_n(x) = \int_a^b \int_a^b \cdots \int_a^b K(x,t)K(t,t_1)\cdots K(t_{n-2},t_{n-1})f(t_{n-1})\,\mathrm{d}t_{n-1}\,\mathrm{d}t_{n-2}\cdots \mathrm{d}t_1\,\mathrm{d}t.$$

We will now play around with the dummy integration variables in an attempt to, instead of integrating over t_{n-1}, t_{n-2}, $\cdots$, then t, to doing so in the reverse order. Now, it is important to note that, given $K(x,t) = t - x$, the kernel K is not symmetric, *i.e.* $K(x,t) \neq K(t,x)$, so we have to be careful with reordering. After reordering, we get

$$\begin{aligned}
u_n(x) &= \int_a^b \left[\int_a^b \int_a^b \cdots \int_a^b K(x,t)K(t,t_1)\cdots K(t_{n-2},t_{n-1})\,\mathrm{d}t_{n-2}\,\mathrm{d}t_{n-3}\cdots \mathrm{d}t_1\,\mathrm{d}t\right] f(t_{n-1})\,\mathrm{d}t_{n-1} \\
&= \int_a^b \left[\int_a^b \int_a^b \cdots \int_a^b K(x,t_1)K(t_1,t_2)\cdots K(t_{n-1},t)\,\mathrm{d}t_{n-1}\,\mathrm{d}t_{n-2}\cdots \mathrm{d}t_2\,\mathrm{d}t_1\right] f(t)\,\mathrm{d}t \\
&= \int_a^b K_n(x,t)f(t)\,\mathrm{d}t,
\end{aligned}$$

where we define

$$K_n(x,t) = \int_a^b \int_a^b \cdots \int_a^b K(x,t_1)K(t_1,t_2)\cdots K(t_{n-1},t)\,\mathrm{d}t_{n-1}\,\mathrm{d}t_{n-2}\cdots \mathrm{d}t_2\,\mathrm{d}t_1.$$

Due to the nested integration, it might be hard to see the point of defining K_n. What we did is redefine the integrand of the last integration without $f(t_n)$, and since every other variable was integrated over already, the integrand will only depend on x and t.
Now, we can write

$$\begin{aligned}
\phi(x) &= \lim_{N\to\infty} \sum_{n=0}^{N} \lambda^n u_n(x) \\
&= f(x) + \sum_{n=0}^{\infty} \lambda^n \int_a^b K_n(x,t)f(t)\,\mathrm{d}t \\
&= f(x) + \int_a^b \left(\sum_{n=0}^{\infty} \lambda^n K_n(x,t)\right) f(t)\,\mathrm{d}t \\
&= f(x) + \int_a^b \tilde{K}(x,t;\lambda)f(t)\,\mathrm{d}t,
\end{aligned}$$

where $\tilde{K}$ is the resolvent kernel defined to be

$$\tilde{K}(x,t;\lambda) = \sum_{n=0}^{\infty} \lambda^n K_n(x,t).$$

Computing the first few K_n, with $a = -1$ and $b = 1$, we have

- For $n = 0$, we have
$$K_0(x,t) = f(x).$$
- For $n = 1$, we have
$$K_1(x,t) = K(x,t) = t - x.$$

- For $n = 2$, we have

$$\begin{aligned}
K_2(x,t) &= \int_a^b K(x,t_1)K(t_1,t)\,\mathrm{d}t_1 \\
&= \int_{-1}^1 (t_1 - x)(t - t_1)\,\mathrm{d}t_1 \\
&= \int_{-1}^1 -t_1^2 + (t+x)t_1 - xt\,\mathrm{d}t_1 \\
&= -\frac{t_1^3}{3} + (t+x)\frac{t_1^2}{2} - xtt_1\bigg|_{-1}^1 \\
&= -\frac{2}{3} - 2xt.
\end{aligned}$$

- For $n = 3$, we have

$$\begin{aligned}
K_3(x,t) &= \int_a^b \int_a^b K(x,t_1)K(t_1,t_2)K(t_2,t)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_{-1}^1 \int_{-1}^1 (t_1 - x)(t_2 - t_1)(t - t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_{-1}^1 (t_1 - x)\left[-\frac{t_2^3}{3} + (t+t_1)\frac{t_2^2}{2} - t_1tt_2\right]_{-1}^1 \mathrm{d}t_1 \\
&= \int_{-1}^1 (t_1 - x)\left(-\frac{2}{3} - 2t_1t\right)\mathrm{d}t_1 \\
&= \int_{-1}^1 \left(-2t_1^2t + \left(2tx - \frac{2}{3}\right)t_1 + \frac{2}{3}x\right)\mathrm{d}t_1 \\
&= -\frac{2}{3}t_1^3t + \left(2tx - \frac{2}{3}\right)\frac{t_1^2}{2} + \frac{2}{3}xt_1\bigg|_{-1}^1 \\
&= -\frac{4}{3}t + \frac{4}{3}x.
\end{aligned}$$

- For $n = 4$, we have

$$\begin{aligned}
K_4(x,t) &= \int_a^b \int_a^b \int_a^b K(x,t_1)K(t_1,t_2)K(t_2,t_3)K(t_3,t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (t_1 - x)(t_2 - t_1)(t_3 - t_2)(t - t_3)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \frac{8}{3}xt + \frac{8}{9}.
\end{aligned}$$

Replacing with the above and $\lambda = \frac{1}{2}$, we have

$$\begin{aligned}
\tilde{K}\left(x, t; \frac{1}{2}\right) &= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n K_n(x,t) \\
&= \frac{1}{2}(t-x) + \left(\frac{1}{2}\right)^2 \left(-\frac{2}{3} - 2xt\right) + \left(\frac{1}{2}\right)^3 \left(-\frac{4}{3}t + \frac{4}{3}x\right) + \left(\frac{1}{2}\right)^4 \left(\frac{8}{3}xt + \frac{8}{9}\right) + \cdots \\
&= \frac{1}{2}\left(\left(t - x - xt - \frac{1}{3}\right) - \frac{1}{3}\left(t - x - xt - \frac{1}{3}\right) + \cdots\right) \\
&= \frac{1}{2}\left(t - x - xt - \frac{1}{3}\right) \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \\
&= \frac{1}{2}\left(t - x - xt - \frac{1}{3}\right) \frac{1}{1 + \frac{1}{3}} \\
&= \frac{3}{8}\left(t - x - xt - \frac{1}{3}\right).
\end{aligned}$$

Now, we can check if this resolvent kernel does yield the same solution we got in Problem 1. To do so, we substitute $\tilde{K}$ in ϕ, and get

$$\begin{aligned}
\phi(x) &= f(x) + \int_a^b \tilde{K}\left(x, t; \frac{1}{2}\right) f(t)\,\mathrm{d}t \\
&= x + \frac{3}{8}\int_{-1}^{1} \left(t - x - xt - \frac{1}{3}\right) t\,\mathrm{d}t \\
&= \frac{3}{4}x + \frac{1}{4},
\end{aligned}$$

which matches with our findings from before. ■

Problem 3
[Arfken p. 792] Solve the equation

$$\phi(x) = x + \frac{1}{2}\int_{-1}^{+1} (t-x)\phi(t)\,\mathrm{d}t$$

by the separable kernel method.

Solution. Considering a generic Fredholm equation, we have $f(x) = x$, $\lambda = \frac{1}{2}$, $a = -1$, $b = 1$, and $K(x,t) = t - x$. The kernel K is separable, *i.e.* we can write K as

$$K(x,t) = \sum_{j=1}^{2} M_j(x)N_j(t) = t - x,$$

where

$$\begin{aligned} M_1(x) &= 1, \qquad N_1(t) = t \\ M_2(x) &= x, \qquad N_2(t) = -1. \end{aligned}$$

Now, replacing this new form of the kernel into the Fredholm integral equation, we have

$$\begin{aligned} \phi(x) &= x + \lambda \int_{-1}^{1} \sum_{j=1}^{2} M_j(x)N_j(t)\phi(t)\,\mathrm{d}t \\ &= x + \lambda \sum_{j=1}^{2} M_j(x) \int_{-1}^{1} N_j(t)\phi(t)\,\mathrm{d}t \\ &= x + \lambda \sum_{j=1}^{2} M_j(x)c_j. \end{aligned}$$

We now multiply both sides by $N_i(x)$ and integrate over x from -1 to 1, getting

$$\begin{aligned} \phi(x) &= x + \lambda \sum_{j=1}^{2} M_j(x)c_j \\ \int_{-1}^{1} N_i(x)\phi(x)\,\mathrm{d}x &= \int_{-1}^{1} xN_i(x)\,\mathrm{d}x + \lambda \int_{-1}^{1} N_i(x) \sum_{j=1}^{2} M_j(x)c_j\,\mathrm{d}x \\ c_i &= b_i + \lambda \sum_{j=1}^{2} c_j \left(\int_{-1}^{1} N_i(x)M_j(x) \right) \mathrm{d}x \\ c_i &= b_i + \lambda \sum_{j=1}^{2} c_j a_{ij}. \end{aligned}$$

In the previous set of equations, it is assumed that we are summing over i (two iterations), so if we consider the system we have now

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$\begin{pmatrix} 1 - \lambda a_{11} & -\lambda a_{12} \\ -\lambda a_{21} & 1 - \lambda a_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Now, we compute

$$a_{11} = \int_{-1}^{1} N_1(x)M_1(x)\,\mathrm{d}x = \int_{-1}^{1} x\,\mathrm{d}x = 0.$$

$$a_{12} = \int_{-1}^{1} N_1(x)M_2(x)\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \frac{2}{3}.$$

$$a_{21} = \int_{-1}^{1} N_2(x)M_1(x)\,\mathrm{d}x = \int_{-1}^{1} (-1)\,\mathrm{d}x = -2.$$

$$a_{22} = \int_{-1}^{1} N_2(x)M_2(x)\,\mathrm{d}x = \int_{-1}^{1} (-x)\,\mathrm{d}x = 0.$$

$$b_1 = \int_{-1}^{1} xN_1(x)\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \frac{2}{3}.$$

$$b_2 = \int_{-1}^{1} xN_2(x)\,\mathrm{d}x = \int_{-1}^{1} (-x)\,\mathrm{d}x = 0.$$

Thus, replacing, we have

$$\begin{pmatrix} 1 & -\frac{2}{3}\lambda \\ 2\lambda & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$$

Replacing $\lambda = \frac{1}{2}$, we have

$$\begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \frac{1}{1+\frac{1}{3}} \begin{pmatrix} 1 & \frac{1}{3} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} \\ &= \frac{3}{4} \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

Therefore, our solution ϕ is

$$\begin{aligned} \phi(x) &= x + \lambda \sum_{j=1}^{2} M_j(x)c_j \\ &= x + \lambda \left(M_1(x)c_1 + M_2(x)c_2\right) \\ &= x + \frac{1}{2}\left((1)\left(\frac{1}{2}\right) + (x)\left(-\frac{1}{2}\right)\right) \\ &= x + \frac{1}{4}\left(1 - x\right) \\ &= \frac{3}{4}x + \frac{1}{4}, \end{aligned}$$

as we found before. ∎