MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*

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Homework 15

Problem 16-18

Let (M, g) be an oriented Riemannian n-manifold. This problem outlines an important generalization of the operator $*: C^\infty(M) \to \Omega^n(M)$ defined in this chapter.

(a) For each $k = 1, \dots, n$, show that g determines a unique inner product on $\Lambda^k\left(T_p^* M\right)$ (denoted by $\langle \cdot, \cdot \rangle_g$, just like the inner product on $T_p M$) satisfying

$$\left\langle \omega^1 \wedge \cdots \wedge \omega^k, \eta^1 \wedge \cdots \wedge \eta^k \right\rangle_g = \det\left(\left\langle \left(\omega^i\right)^\#, \left(\eta^j\right)^\# \right\rangle_g\right)$$

whenever $\omega^1, \dots, \omega^k, \eta^1, \dots, \eta^k$ are covectors at p. [Hint: define the inner product locally by declaring $\left\{ \varepsilon^I\big|_p : I \text{ is increasing} \right\}$ to be an orthonormal basis for $\Lambda^k\left(T_p^* M\right)$ whenever $\left(\varepsilon^i\right)$ is the coframe dual to a local orthonormal frame, and then prove that the resulting inner product is independent of the choice of frame.]

(b) Show that the Riemannian volume form $\mathrm{d}V_g$ is the unique positively oriented n-form that has unit norm with respect to this inner product.

(c) For each $k = 0, \dots, n$, show that there is a unique smooth bundle homomorphism $*: \Lambda^k T^* M \to \Lambda^{n-k} T^* M$ satisfying

$$\omega \wedge * \eta = \langle \omega, \eta \rangle_g dV_g$$

for all smooth k-forms ω, η. (For $k = 0$, interpret the inner product as ordinary multiplication.) This map is called the Hodge star operator. [Hint: first prove uniqueness, and then define $*$ locally by setting

$$*\left(\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}\right) = \pm \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_{n-k}}$$

in terms of an orthonormal coframe $\left(\varepsilon^i\right)$, where the indices $j_1, \dots, j_{n-k}$ are chosen so that $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ is some permutation of $(1, \dots, n)$.]

(d) Show that $*: \Lambda^0 T^* M \to \Lambda^n T^* M$ is given by $*f = f\, \mathrm{d}V_g$.

(e) Show that $**\omega = (-1)^{k(n-k)} \omega$ if ω is a k-form.

Solution. (a) Let (ε^i) be the coframe dual to a local orthonormal frame and let $\{\varepsilon^I|_p : I \text{ is increasing}\}$ to be an orthonormal basis for $\Lambda^k(T_p^* M)$.

To prove independence of the choice of frame, let $(\tilde{\varepsilon}^i)$ be another orthonormal coframe. Then $\tilde{\varepsilon}^i = \sum_j A_{ij} \varepsilon^j$ where (A_{ij}) is an orthogonal matrix. For multi-indices I and J,

$$\begin{aligned}
\left\langle \tilde{\varepsilon}^I, \tilde{\varepsilon}^J \right\rangle_g &= \left\langle \sum_{i_1} A_{i_1, j_1} \varepsilon^{i_1}, \dots, \sum_{i_k} A_{i_k, j_k} \varepsilon^{i_k} \right\rangle_g \\
&= \det\left(\left(A_{i_\alpha j_\beta}\right)\right)_{\alpha, \beta = 1}^k \langle \varepsilon^I, \varepsilon^J \rangle_g \\
&= \det(A) \langle \varepsilon^I, \varepsilon^J \rangle_g.
\end{aligned}$$

This shows the inner product is well-defined and independent of the choice of frame.

(b) The Riemannian volume form $\mathrm{d}V_g$ is locally given by $\varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ where (ε^i) is an orthonormal coframe. By definition, this has unit norm. It's also positively oriented. Uniqueness follows from the fact that any other such form would differ by a positive scalar multiple, which would change its norm.

(c) For uniqueness, suppose $*_1$ and $*_2$ are two such operators. Then for any k-forms ω and η,

$$\begin{aligned} \omega \wedge *_1\eta = \langle \omega, \eta\rangle_g dV_g = \omega \wedge *_2\eta \\ \omega \wedge (*_1\eta - *_2\eta) = 0 \end{aligned}$$

This implies $*_1\eta = *_2\eta$ for all η, so $*_1 = *_2$.

For existence, define $*$ locally as suggested in the hint. Then verify that

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} \wedge *(\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}) = \delta_{i_1 j_1} \cdots \delta_{i_k j_k}\, \mathrm{d}V_g,$$

which is equivalent to the required property.

(d) For $f \in C^\infty(M)$, we have

$$f \wedge *g = \langle f, g\rangle_g\, \mathrm{d}V_g = fg\, \mathrm{d}V_g$$

for all $g \in C^\infty(M)$. This implies $*f = f\, \mathrm{d}V_g$.

(e) In an orthonormal coframe, applying $*$ twice to a basis k-form gives

$$**\,(\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}) = (-1)^{k(n-k)}(\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k})$$

The sign comes from the number of transpositions needed to bring $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ back to $(1, \ldots, n)$ after reversing it. This property then extends linearly to all k-forms.

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