PHYS 663 - Quantum Field Theory II

An Introduction to Quantum Fied Theory by *Peskin and Schroeder*
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Homework 5

Problem 1
Compute the β-function for the gauge coupling of Yang-Mills coupled to a complex scalar field in representation R at 1-loop. Compute it again for Yang-Mills coupled to N_f Dirac Fermions in representations R_i and N_s complex scalars in R'_j.

Solution. We consider Yang-Mills theory coupled to a complex scalar field in representation R

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi),$$

where $D_\mu = \partial_\mu - igA^a_\mu T^a_R$ is the covariant derivative with T^a_R being the generators in representation R.
To calculate the β-function, I need to determine the counterterms that renormalize the gauge coupling. The relevant diagrams contributing to the gauge field self-energy at one loop are

- Pure gauge and ghost loops (same as in pure Yang-Mills)
- Scalar loop contributions

For pure Yang-Mills, the contribution to the gauge field two-point function is:

$$\Pi^{ab}_{\mu\nu}|_{\text{YM}} = \delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu}) \frac{g^2}{16\pi^2} \frac{11}{3} C_2(G) \frac{1}{\epsilon},$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation.
For the complex scalar loop (which has two real degrees of freedom), the contribution is

$$\Pi^{ab}_{\mu\nu}|_{\phi} = -\delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu}) \frac{g^2}{16\pi^2} \frac{1}{3} T(R) \frac{1}{\epsilon},$$

where $T(R)$ is the trace normalization factor for representation R: $\text{Tr}(T^a_R T^b_R) = T(R)\delta^{ab}$.
The total divergent part of the gauge field self-energy is

$$\Pi^{ab}_{\mu\nu} = \delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu}) \frac{g^2}{16\pi^2} \left(\frac{11}{3} C_2(G) - \frac{1}{3} T(R) \right) \frac{1}{\epsilon}.$$

From this, we can derive the β-function

$$\beta(g) = -\frac{g^3}{16\pi^2} \left(\frac{11}{3} C_2(G) - \frac{1}{3} T(R) \right).$$

Now, we extend this to Yang-Mills theory coupled to N_f Dirac fermions in representations R_i and N_s complex scalars in representations R'_j

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \sum_{i=1}^{N_f} \bar{\psi}_i (i\gamma^\mu D_\mu) \psi_i + \sum_{j=1}^{N_s} (D_\mu \phi_j)^\dagger (D^\mu \phi_j)$$

The additional contributions to the gauge field self-energy are

1. **Fermion loops:** Each Dirac fermion contributes

$$\Pi^{ab}_{\mu\nu}|_{\psi_i} = -\delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu}) \frac{g^2}{16\pi^2} \frac{4}{3} T(R_i) \frac{1}{\epsilon}.$$

2. **Scalar loops:** Each complex scalar contributes

$$\Pi^{ab}_{\mu\nu}|_{\phi_j} = -\delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\frac{g^2}{16\pi^2}\frac{1}{3}T(R'_j)\frac{1}{\epsilon}.$$

The total divergent part is

$$\Pi^{ab}_{\mu\nu} = \delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\frac{g^2}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}\sum_{i=1}^{N_f} T(R_i) - \frac{1}{3}\sum_{j=1}^{N_s} T(R'_j)\right)\frac{1}{\epsilon}$$

This gives the β-function

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}\sum_{i=1}^{N_f} T(R_i) - \frac{1}{3}\sum_{j=1}^{N_s} T(R'_j)\right)$$

For special cases

- In SU(N) gauge theory with fields in the fundamental representation, $C_2(G) = N$ and $T(R) = 1/2$
- For QCD with N_f flavors of quarks, the β-function is

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right)$$

This β-function determines how the coupling constant evolves with energy scale and is crucial for understanding asymptotic freedom in non-Abelian gauge theories. ■

Problem 2
Write down an action noncommutative Yang-Mills theory on a lattice as a sum over plaquettes. What is the path integral measure? Compute the Haar integrals

$$\int \mathrm{d}U U_{ij}$$
$$\int \mathrm{d}U U_{ij} U_{kl}$$
$$\int \mathrm{d}U U_{ij} U^*_{kl}$$

and use them to argue that the vacuum expectation of Wilson loop operators W_c at infinite coupling is

$$\langle 0|W_c|0\rangle \sim \left(\frac{1}{g^2}\right)^{\frac{A}{a^2}},$$

where A is the area of the Wilson loop and a is the lattice spacing.

Solution. On a noncommutative space, the coordinate operators satisfy

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is a constant antisymmetric matrix. In the continuum, this noncommutativity is implemented through the Moyal-Weyl star-product

$$(f \star g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}\right) f(x+\xi)g(x+\eta)\bigg|_{\xi=\eta=0}.$$

On a d-dimensional lattice with $x \in a\mathbb{Z}^d$, the noncommutativity appears in the product rule for lattice fields

$$(f \star g)(x) = \sum_{y,z} f(y)g(z)K(x,y,z),$$

where $K(x,y,z)$ is a kernel incorporating the noncommutative phase structure. We define the star-commutator as

$$[f,g]_\star = f\star g - g\star f.$$

Gauge fields on the lattice are encoded in link variables

$$U_\mu(x) = \exp_\star(iaA_\mu(x)) = 1 + iaA_\mu(x) + \frac{(ia)^2}{2!}A_\mu(x)\star A_\mu(x) + \cdots,$$

which live on the link connecting site x to $x + a\hat{\mu}$. These transform under local star-unitary transformations as

$$U_\mu(x) \to \Omega(x)\star U_\mu(x)\star\Omega^\dagger(x+a\hat{\mu}),$$

where $\Omega(x)\star\Omega^\dagger(x) = \Omega^\dagger(x)\star\Omega(x) = 1$.
The plaquette variable, which measures the curvature around an elementary square, is defined for a plaquette in the μ-ν plane based at x as:

$$U_{\mu\nu}(x) = U_\mu(x)\star U_\nu(x+a\hat{\mu})\star U^\dagger_\mu(x+a\hat{\nu})\star U^\dagger_\nu(x).$$

The action for noncommutative Yang-Mills theory on a lattice can then be written as

$$S_{\mathrm{NCYM}} = \frac{1}{g^2}\sum_{x\in a\mathbb{Z}^d}\sum_{\mu<\nu}\left(1 - \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_{\mu\nu}(x)\right),$$

or equivalently

$$S_{\text{NCYM}} = -\frac{1}{g^2} \sum_{x \in a\mathbb{Z}^d} \sum_{\mu<\nu} \text{Re}\,\text{Tr}\left[U_\mu(x) \star U_\nu(x+a\hat{\mu}) \star U_\mu^\dagger(x+a\hat{\nu}) \star U_\nu^\dagger(x)\right] + \text{const.}$$

In the continuum limit ($a \to 0$), expanding the star-exponential for the link variables shows that

$$U_{\mu\nu}(x) = 1 + ia^2 F_{\mu\nu}(x) + O(a^3),$$

where the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star.$$

This recovers the noncommutative Yang-Mills action in the continuum

$$S_{\text{NCYM}} \xrightarrow{a\to 0} \frac{1}{4g^2} \int d^d x\, \text{Tr}(F_{\mu\nu} \star F^{\mu\nu}).$$

The path integral measure for noncommutative lattice Yang-Mills is defined as the product over all lattice sites x and all directions μ of the Haar measures for the corresponding link variables

$$\mathcal{D}[U] = \prod_{x \in a\mathbb{Z}^d} \prod_{\mu=1}^{d} dU_\mu(x),$$

where $dU_\mu(x)$ is the bi-invariant Haar measure on the gauge group G, satisfying

$$d(\Omega U_\mu(x)) = dU_\mu(x) = d(U_\mu(x)\Omega), \quad \forall \Omega \in G.$$

This invariance ensures compatibility with noncommutative gauge transformations.
Let's evaluate the requested Haar integrals for $U \in U(N)$ or $SU(N)$ with dU being the normalized Haar measure.

1. **Single matrix element:**

$$\int dU\, U_{ij} = 0.$$

   This vanishes because there is no invariant tensor with a single fundamental index. Under any group translation, $U \to \Omega U$, the components transform, but the Haar measure is invariant. Thus, any fixed component must average to zero.

2. **Product of two matrix elements:**

$$\int dU\, U_{ij} U_{kl} = 0 \quad \text{(for general } N)$$

   For general N, this integral vanishes because there is no invariant tensor that can connect these indices in a way that preserves invariance under $U \to \Omega U$.

   For the special case of SU(2), we have:

$$\int dU\, U_{i_1 j_1} U_{i_2 j_2} = \frac{1}{2!} \epsilon_{i_1 i_2} \epsilon_{j_1 j_2}$$

3. **Matrix element and its conjugate:**

$$\int dU\, U_{ij} U^*_{kl} = \int dU\, U_{ij} U^\dagger_{lk} = \frac{1}{N} \delta_{ik} \delta_{jl}.$$

   This follows from the invariance of the Haar measure and the fact that $U_{ij} U^\dagger_{lk}$ transforms in a manner that allows for a non-zero result. The normalization factor $1/N$ is determined by contracting indices and using unitarity

$$\int dU\, U_{ij} U^\dagger_{jk} = \int dU\, (UU^\dagger)_{ik} = \int dU\, \delta_{ik} = \delta_{ik}.$$

This implies

$$\frac{1}{N}\delta_{ik}\delta_{jj} = \frac{1}{N}N\delta_{ik} = \delta_{ik}.$$

Therefore, $C = 1/N$.

The Wilson loop operator around a closed contour C is defined as

$$W_C = \frac{1}{N}\mathrm{Tr}\left(P\prod_{l\in C}U_l\right),$$

where P denotes path ordering along the contour C.
At infinite coupling ($g \to \infty$), the vacuum expectation value of the Wilson loop has the form

$$\langle 0|W_C|0\rangle = \int \mathcal{D}U\, W_C\, e^{-S},$$

where the partition function $Z = 1$ due to normalization of the Haar measure.
In the strong coupling limit, we expand the exponential

$$e^{-S} \approx 1 - \frac{1}{g^2}\sum_{\text{plaq}} \mathrm{Re}\,\mathrm{Tr}(U_{\text{plaq}}) + \dots.$$

The lowest non-vanishing contribution to $\langle 0|W_C|0\rangle$ comes from terms where each link variable U_l in the Wilson loop is paired with a corresponding $U_l^\dagger$ from the plaquettes in the expansion of e^{-S}. Due to the orthogonality properties of the Haar measure (specifically $\int dU\, U_{ij}U^*_{kl} = \frac{1}{N}\delta_{ik}\delta_{jl}$), these pairs must match exactly.
For a Wilson loop enclosing an area A, the minimum number of plaquettes needed to "tile" this area is $N_P = A/a^2$, where a is the lattice spacing. Each plaquette contributes a factor of $1/g^2$ from the action. Therefore

$$\langle 0|W_C|0\rangle \sim \left(\frac{1}{g^2}\right)^{A/a^2}.$$

This area law decay of Wilson loops at strong coupling is a signature of confinement in lattice gauge theories. The expectation value falls off exponentially with the area enclosed by the loop, demonstrating that creating widely separated quarks requires energy proportional to their separation. ■