PHYS 603 - Methods of Theoretical Physics III

Lie Algebras in Particle Physics by *H. Georgi*

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Homework 8

The action of the lowering operators corresponding to the simple roots on the weights of the 3 of SU(3) is

$$E_{-\alpha^1} |_1\rangle = \frac{1}{\sqrt{2}} |_3\rangle\,, \quad E_{-\alpha^2} |_3\rangle = \frac{1}{\sqrt{2}} |_2\rangle$$

(all other applications being zero).

Problem 1

Recall that the highest-weight state of the 10 of SU(3) is

$$|HW\rangle = |_1\rangle |_1\rangle |_1\rangle\,.$$

(a) Find the states

$$\begin{aligned} |A\rangle &= \mathcal{N}_A E_{-\alpha^1} |HW\rangle\,, \\ |B\rangle &= \mathcal{N}_B E_{-\alpha^1} |A\rangle \end{aligned}$$

including positive normalization constants $\mathcal{N}_A$ and $\mathcal{N}_B$ chosen so that the norms of $|A\rangle$ and $|B\rangle$ are unity.

(b) Recall that any state $|D\rangle$ in 10 can be written as $|D\rangle = D^{ijk} |_i\rangle |_j\rangle |_k\rangle$ with a completely symmetric tensor D^{ijk}. Find these tensors for the states $|HW\rangle$, $|A\rangle$, and $|B\rangle$.

(c) Let

$$h^i{}_j = \mathrm{diag}(1, 1, -2)$$

be a tensor corresponding to an element in 8. Find the singlet S (there is only one) that can be made from D^{ijk}, its conjugate $\bar{D}_{lmn}$, and $h^p{}_q$ (using one copy of each) and compute the value of S for the three states listed in part (b).

Solution. (a) To find the states $|A\rangle$ and $|B\rangle$, we begin by applying the lowering operator $E_{-\alpha^1}$ to the highest weight state.

For $|A\rangle$, we compute:

$$E_{-\alpha^1} |HW\rangle = E_{-\alpha^1}(|_1\rangle |_1\rangle |_1\rangle).$$

Using the action of the lowering operator on each factor:

$$E_{-\alpha^1} |HW\rangle = (E_{-\alpha^1} |_1\rangle) |_1\rangle |_1\rangle + |_1\rangle (E_{-\alpha^1} |_1\rangle) |_1\rangle + |_1\rangle |_1\rangle (E_{-\alpha^1} |_1\rangle).$$

Substituting $E_{-\alpha^1} |_1\rangle = \frac{1}{\sqrt{2}} |_3\rangle$:

$$E_{-\alpha^1} |HW\rangle = \frac{1}{\sqrt{2}}(|_3\rangle |_1\rangle |_1\rangle + |_1\rangle |_3\rangle |_1\rangle + |_1\rangle |_1\rangle |_3\rangle).$$

To normalize this state, we compute

$$\langle A|A\rangle = \mathcal{N}_A^2 \cdot 3 = 1.$$

Therefore, we have

$$\mathcal{N}_A = \frac{1}{\sqrt{3}}.$$

The normalized state $|A\rangle$ is:

$$|A\rangle = \frac{1}{\sqrt{3}}(|_3\rangle\,|_1\rangle\,|_1\rangle + |_1\rangle\,|_3\rangle\,|_1\rangle + |_1\rangle\,|_1\rangle\,|_3\rangle).$$

$$E_{-\alpha^1}\,|A\rangle = \frac{1}{\sqrt{3}}E_{-\alpha^1}(|_3\rangle\,|_1\rangle\,|_1\rangle + |_1\rangle\,|_3\rangle\,|_1\rangle + |_1\rangle\,|_1\rangle\,|_3\rangle)$$

Since $E_{-\alpha^1}\,|_3\rangle = 0$ and $E_{-\alpha^1}\,|_1\rangle = \frac{1}{\sqrt{2}}\,|_3\rangle$, this becomes

$$E_{-\alpha^1}\,|A\rangle = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}(|_3\rangle\,|_3\rangle\,|_1\rangle + |_3\rangle\,|_1\rangle\,|_3\rangle + |_1\rangle\,|_3\rangle\,|_3\rangle)$$

The normalization constant is found from

$$\langle B|B\rangle = \mathcal{N}_B^2\cdot 3\cdot\frac{1}{6} = 1.$$

Therefore, we have

$$\mathcal{N}_B = \frac{1}{\sqrt{2}}\cdot\sqrt{6} = \sqrt{3}.$$

The normalized state $|B\rangle$ is

$$|B\rangle = \frac{1}{\sqrt{3}}(|_3\rangle\,|_3\rangle\,|_1\rangle + |_3\rangle\,|_1\rangle\,|_3\rangle + |_1\rangle\,|_3\rangle\,|_3\rangle).$$

(b) Any state $|D\rangle$ in the 10 representation can be written as $|D\rangle = D^{ijk}\,|_i\rangle\,|_j\rangle\,|_k\rangle$ with a completely symmetric tensor D^{ijk}.

For $|HW\rangle = |_1\rangle\,|_1\rangle\,|_1\rangle$, the tensor components are

$$D^{ijk}_{HW} = \begin{cases} 1 & \text{if } i=j=k=1 \\ 0 & \text{otherwise} \end{cases}$$

For $|A\rangle = \frac{1}{\sqrt{3}}(|_3\rangle\,|_1\rangle\,|_1\rangle + |_1\rangle\,|_3\rangle\,|_1\rangle + |_1\rangle\,|_1\rangle\,|_3\rangle)$, the tensor components are

$$D^{ijk}_{A} = \begin{cases} \frac{1}{\sqrt{3}} & \text{if exactly one index is 3 and the rest are 1} \\ 0 & \text{otherwise} \end{cases}$$

For $|B\rangle = \frac{1}{\sqrt{3}}(|_3\rangle\,|_3\rangle\,|_1\rangle + |_3\rangle\,|_1\rangle\,|_3\rangle + |_1\rangle\,|_3\rangle\,|_3\rangle)$, the tensor components are

$$D^{ijk}_{B} = \begin{cases} \frac{1}{\sqrt{3}} & \text{if exactly two indices are 3 and one is 1} \\ 0 & \text{otherwise} \end{cases}$$

(c) To construct a singlet from D^{ijk}, its conjugate $\bar{D}_{lmn}$, and $h^p{}_q$, we need to contract all indices properly. Since all indices must be used, and considering symmetry properties, the singlet is:

$$S = D^{ijk}\bar{D}_{ljk}h^l{}_i.$$

We evaluate this singlet for each state:

- **For $|HW\rangle$:**

$$
\begin{aligned}
S_{HW} &= D^{111}\bar{D}_{111}{h^1}_1 \\
&= 1 \cdot 1 \cdot 1 = 1.
\end{aligned}
$$

- **For $|A\rangle$:**

$$
\begin{aligned}
S_A &= D^{311}\bar{D}_{311}{h^3}_3 + D^{131}\bar{D}_{131}{h^1}_1 + D^{113}\bar{D}_{113}{h^1}_1 \\
&= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot (-2) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot 1 + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot 1 \\
&= \frac{1}{3}(-2+1+1) = 0.
\end{aligned}
$$

- **For $|B\rangle$:**

$$
\begin{aligned}
S_B &= D^{331}\bar{D}_{331}{h^3}_3 + D^{313}\bar{D}_{313}{h^3}_3 + D^{133}\bar{D}_{133}{h^1}_1 \\
&= \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot (-2) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot (-2) + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \cdot 1 \\
&= \frac{1}{3}(-2) + \frac{1}{3}(-2) + \frac{1}{3} \\
&= \frac{1}{3}(-4+1) = -1.
\end{aligned}
$$

Therefore, the values of the singlet S for the three states are

$$
S_{HW} = 1, \quad S_A = 0, \quad S_B = -1.
$$

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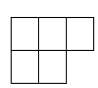
Problem 2
Decompose the following tensor products into direct sums of irreducible representations of SU(3) using Young tableaux:

(a) $(2,1) \otimes \overline{3}$, where $(2,1)$ is one of the 15-dimensional irreducible representations [the (n,m) notation for the irreducible representations is defined in Eq. (9.27) of the textbook],

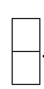
(b) $\overline{6} \otimes 6$. The dimensions of the irreducible representations are given by eq. (10.43) of the textbook.

Solution. (a) **For $(2,1) \otimes \overline{3}$:**

The Young tableau for $(2,1)$ is



and for $\overline{3}$ is



To decompose this tensor product, we label the boxes of $\overline{3}$ "a" and "b", where the first row would be entirely "a" and the second row entirely "b". We then add the "a" boxes to the tableau for $(2,1)$ in all the legal ways, then do the same for the "b" boxes. The main constraints are:

- Never two "a"'s in the same column.
- In the case of duplicate tableaux, only consider one.
- Read from right to left, top to bottom, and never accumulate more "a"'s than "b"'s.
- More than three boxes stacked on top of each other get removed since we are working modulo 3.

This gives us

$$
\begin{aligned}
(2,1) \otimes \overline{3} &\cong \ytableaushort{{}{}{},{}{}} \otimes \ytableaushort{a,b} \\
&(1) \to \ytableaushort{{}{}{}a,{}{}} \to \ytableaushort{{}{}{}a,{}{}b} \to (3,1) \cong 24 \\
&(2) \to \ytableaushort{{}{}{},{}{}a} \to \ytableaushort{{}{}{},{}{}a,b} \to \ytableaushort{{}{},{}a} \to (0,2) \cong \overline{6}
\end{aligned}
$$

Therefore, $(2,1) \otimes \overline{3} = 24 \oplus \overline{6}$.

(b) **For $\overline{6} \otimes 6$:**

The Young tableau for $\overline{6}$ is

$$\ydiagram{2,2}$$

and for 6 is

$$\ydiagram{2}.$$

This gives us

$$
\begin{aligned}
\overline{6} \otimes 6 &\cong \ydiagram{2,2} \otimes \ytableaushort{aa} \\
&(1) \to \ytableaushort{{}{}a,{}{}} \to \ytableaushort{{}{}aa,{}{}} \to (2,2) \cong 42 \\
&(2) \to \ytableaushort{{}{}a,{}{}} \to \ytableaushort{{}{}a,{}{},a} \to \ytableaushort{{}a,{}} \to (1,1) \cong 15 \\
&(3) \to \ytableaushort{{}{},{}{},a} \to \ytableaushort{{}{},{}{},aa} \to (0,0) \cong 1
\end{aligned}
$$

Therefore, $\overline{6} \otimes 6 = 42 \oplus 15 \oplus 1$. ■