MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*

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Homework 9

Problem 10-10
Suppose M is a compact smooth manifold and $E \to M$ is a smooth vector bundle of rank k. Use transversality to prove that E admits a smooth section σ with the following property: if $k > \dim(M)$, then σ is nowhere vanishing; while if $k \leq \dim(M)$, then the set of points where σ vanishes is a smooth compact codimension-k submanifold of M. Use this to show that M admits a smooth vector field with only finitely many singular points.

Solution. Let M be a compact smooth manifold of dimension m and let $E \to M$ be a smooth vector bundle of rank k.
First, note that E always admits a zero section, which we'll call $\xi : M \to E$. Let $M_0 := \xi(M) \subseteq E$ denote the image of this zero section, which is an embedded k-codimensional submanifold of E.
By the Homotopy Transversality Theorem, there exists a homotopy $H : M \times [0, 1] \to E$ such that:

(i) $H(\cdot, 0) = \xi(\cdot)$

(ii) $H(\cdot, 1)$ is transverse to M_0

Let $\sigma := H(\cdot, 1) : M \to E$ be our section that is transverse to M_0. We now consider two cases:

- **For** $k > \dim(M)$:
  In this case, dimensional considerations show that σ must be nowhere vanishing. Indeed, if σ vanishes at a point, this would create a transverse intersection between $\sigma(M)$ and M_0, but
  $$\dim(\sigma(M)) + \dim(M_0) - \dim(E) = m + m - (m + k) = 2m - (m + k) < 0.$$
  making such an intersection impossible.
- **For** $k \leq \dim(M)$:
  Let's analyze the set of points where σ vanishes. Note that
  $$\begin{aligned} \dim(E) &= k + m, \\ \operatorname{codim}(M_0) &= k, \\ \operatorname{codim}(\sigma(M)) &= k. \end{aligned}$$
  Since σ is transverse to M_0, their intersection $M_0 \cap \sigma(M)$ is a smooth submanifold with:
  $$\operatorname{codim}(M_0 \cap \sigma(M)) = \operatorname{codim}(M_0) + \operatorname{codim}(\sigma(M)) = k + k = 2k$$

Therefore, $M_0 \cap \sigma(M)$ is a smooth compact codimension-k submanifold of M.
To prove the final statement about vector fields, apply this result to the tangent bundle TM. Since $\dim(TM) = \dim(M)$, we get a vector field with zeros forming a 0-dimensional submanifold. By compactness of M, this must be a finite set of points, giving us a vector field with finitely many singular points.

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