MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*
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Homework 13

Problem 16-4
Suppose M is an oriented compact smooth manifold with boundary. Show that there does not exist a retraction of M onto its boundary. [Hint: if the retraction is smooth, consider an orientation form on ∂M.]

Solution. Suppose, for contradiction, that there exists a smooth retraction $r : M \to \partial M$. Let ω be a volume form on M such that $\operatorname{supp}(\omega)$ is compact. By the properties of an orientation form, ω is a top-dimensional form on M. Define $\eta = r^*\omega$, the pullback of ω by the retraction r. Note that η is a top-dimensional form on ∂M. Since $r(x) = x$ for all $x \in \partial M$, we have $\eta = \omega|_{\partial M}$. Now, consider the integral $\int_{\partial M} \eta$. By the pullback property and the definition of r, we have

$$\begin{aligned} \int_{\partial M} \eta &= \int_{\partial M} r^*\omega \\ &= \int_M (r_*\eta) \\ &= \int_M 0. \end{aligned}$$

However, this contradicts the fact that ω is a volume form and thus has a non-zero integral over M.
Therefore, no such smooth retraction can exist. ■

Problem 16-5
Suppose M and N are oriented, compact, connected, smooth manifolds, and $F, G : M \to N$ are homotopic diffeomorphisms. Show that F and G are either both orientation-preserving or both orientation-reversing. [Hint: use Theorem 6.29 and Stokes's theorem on $M \times I$.]

Solution. Let $H : M \times I \to N$ be a homotopy between F and G, where $I = [0, 1]$. Consider the map $\det(\mathrm{d}H) : M \times I \to \mathbb{R}$. By Theorem 6.29, $\det(\mathrm{d}H)$ is continuous. Define

$$\alpha(t) = \int_M \det(\mathrm{d}H_t)\ \mathrm{d}V_M,$$

where $H_t(x) = H(x, t)$ and $\mathrm{d}V_M$ is the volume form on M. By Stokes's theorem and the fact that M is compact, $\alpha(t)$ is constant for all $t \in I$.

- At $t = 0$, $\alpha(0) = \int_M \det(\mathrm{d}F)\ \mathrm{d}V_M$, the sign of which determines whether F preserves or reverses orientation.
- At $t = 1$, $\alpha(1) = \int_M \det(\mathrm{d}G)\ \mathrm{d}V_M$, the sign of which determines the orientation of G.

Since $\alpha(t)$ is constant, $\det(\mathrm{d}F)$ and $\det(\mathrm{d}G)$ must have the same sign.
Therefore, F and G are either both orientation-preserving or both orientation-reversing. ■

Problem 16-6
The Hairy Ball Theorem: *There exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.* ("You cannot comb the hair on a ball.") Prove this by showing that the following are equivalent:

(a) There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.

(b) There exists a continuous map $V : \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$.

(c) The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\mathrm{Id}_{\mathbb{S}^n}$.

(d) The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.

(e) n is odd.

[Hint: use Problems 9-4, 15-3, and 16-5.]

Solution. The equivalence follows from several key observations:

- (a) $\implies$ (b): Suppose there exists a nowhere-vanishing vector field V on $\mathbb{S}^n$. Then for each $x \in \mathbb{S}^n$, we have $V(x) \neq 0$. Define a map

$$\begin{aligned} V' : \mathbb{S}^n &\to \mathbb{S}^n, \\ x &\mapsto V'(x) = \frac{V(x)}{|V(x)|}. \end{aligned}$$

  Then $V'(x)$ is a unit vector for all $x \in \mathbb{S}^n$, and since $V(x)$ is tangent to $\mathbb{S}^n$ at x, we have $V'(x) \perp x$ for all $x \in \mathbb{S}^n$.

- (b) $\implies$ (c): Suppose there exists a continuous map $V : \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ for all $x \in \mathbb{S}^n$. Define a homotopy

$$\begin{aligned} H : \mathbb{S}^n \times [0,1] &\to \mathbb{S}^n, \\ (x,t) &\mapsto H(x,t) = \cos(\pi t)x + \sin(\pi t)V(x). \end{aligned}$$

  Then $H(x,0) = x$ and $H(x,1) = -x = \alpha(x)$ for all $x \in \mathbb{S}^n$. Moreover, since $V(x) \perp x$, we have $|H(x,t)| = 1$ for all $(x,t) \in \mathbb{S}^n \times [0,1]$, so H is a homotopy between $\mathrm{Id}_{\mathbb{S}}^n$ and α.

- (c) $\implies$ (d): Suppose α is homotopic to $\mathrm{Id}_{\mathbb{S}}^n$. Then the induced maps on homology groups are equal, *i.e.* $\alpha_* = \mathrm{Id}_{H_n(\mathbb{S}^n)}$. Since $H_n(\mathbb{S}^n) \cong \mathbb{Z}$, this means that α_* acts as multiplication by some integer k on $H_n(\mathbb{S}^n)$. Since α is a homeomorphism, it must be that $k = \pm 1$. If $k = 1$, then α is orientation-preserving.

- (d) $\implies$ (e): Suppose α is orientation-preserving. Then the induced map on the top homology group is multiplication by 1. In particular, this means that the degree of α is 1 . Since the degree of α is also equal to $(-1)^{n+1}$, we must have n odd.

- (e) $\implies$ (a): Assume n is odd. We will construct a nowhere-vanishing vector field on $\mathbb{S}^n$. Define $V : \mathbb{S}^n \to \mathbb{R}^{n+1}$ by

$$V(x_1, \ldots, x_{n+1}) = (-x_2, x_1, -x_3, x_2, \ldots, (-1)^{n+1}x_n, x_{n-1}, (-1)^{n+2}x_{n+1}).$$

  We need to verify two properties:

  - $V(x)$ is tangent to $\mathbb{S}^n$ at x.
  - $V(x) \neq 0$ for all $x \in \mathbb{S}^n$.

  The first property follows from the construction. For the second, note that the coordinates of $V(x)$ have different signs, ensuring a non-zero vector.

  Thus, when n is odd, a nowhere-vanishing vector field exists on $\mathbb{S}^n$.

Therefore, we have showed that the statements are equivalent. ■