PHYS 663 - Quantum Field Theory II

An Introduction to Quantum Fied Theory by *Peskin and Schroeder*
Student: **Ralph Razzouk**

Homework 7

Problem 20.1 - Spontaneous Breaking of $SU(5)$
Consider a gauge theory with the gauge group $SU(5)$, coupled to a scalar field Φ in the adjoint representation. Assume that the potential for this scalar field forces it to acquire a nonzero vacuum expectation value. Two possible choices for this expectation value are

$$\langle\Phi\rangle = A\begin{pmatrix}1 &&&& \\ & 1 &&& \\ && 1 && \\ &&& 1 & \\ &&&& -4\end{pmatrix} \quad \text{and} \quad \langle\Phi\rangle = B\begin{pmatrix}2 &&&& \\ & 2 &&& \\ && 2 && \\ &&& -3 & \\ &&&& -3\end{pmatrix}$$

For each case, work out the spectrum of gauge bosons and the unbroken symmetry group.

Solution. When an adjoint scalar field Φ acquires a vacuum expectation value (VEV), it breaks the original symmetry group to a subgroup. For an SU(5) gauge theory with an adjoint scalar, we need to analyze two possible patterns of symmetry breaking.
Let's begin by determining the masses of the gauge bosons after spontaneous symmetry breaking. According to the solution manual, the mass term for gauge bosons is given by

$$\Delta L = g^2 \text{tr}([A_\mu, \Phi]^\dagger [A^\mu, \Phi]) = -g^2 A_\mu^a A^{\mu b} \text{tr}([T^a, \langle\Phi\rangle][T^b, \langle\Phi\rangle])$$

To identify the unbroken subgroups, we need to find generators T^a that commute with $\langle\Phi\rangle$, i.e., $[T^a, \langle\Phi\rangle] = 0$. These generators correspond to massless gauge bosons and form the Lie algebra of the unbroken subgroup.
For the first case, $\langle\Phi\rangle = A\text{diag}(1, 1, 1, 1, -4)$:
We observe that generators of the form

$$T = \begin{pmatrix} T^{(4)} & 0 \\ 0 & 0 \end{pmatrix},$$

where $T^{(4)}$ is any generator of SU(4), commute with $\langle\Phi\rangle$. Additionally, the generator proportional to $\langle\Phi\rangle$ itself, namely $\frac{1}{2\sqrt{10}}\text{diag}(1, 1, 1, 1, -4)$, also commutes with $\langle\Phi\rangle$. These generators form an $SU(4) \times U(1)$ subgroup. Hence, the unbroken symmetry is $SU(4) \times U(1)$. For the remaining generators that don't commute with $\langle\Phi\rangle$, such as

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and similar ones, we calculate the trace of the commutators to be $-25A^2/2$. Therefore, the corresponding gauge bosons acquire a mass of $M_A = 5gA$.
For the second case, $\langle\Phi\rangle = B\text{diag}(2, 2, 2, -3, -3)$:
Following a similar analysis, we find that the generators that commute with $\langle\Phi\rangle$ form an $SU(3) \times SU(2) \times U(1)$ subgroup.
The remaining 12 generators give rise to massive gauge bosons with mass $M_A = 5gB$.
In summary

- **For $\langle\Phi\rangle = A$diag$(1, 1, 1, 1, -4)$:**
  - Unbroken symmetry: $SU(4) \times U(1)$

  - Massive gauge bosons: 8 with mass $M_A = 5gA$

- **For $\langle\Phi\rangle = B\mathbf{diag}(2, 2, 2, -3, -3)$:**
  - Unbroken symmetry: $SU(3) \times SU(2) \times U(1)$
  - Massive gauge bosons: 12 with mass $M_A = 5gB$

We note that in the second case, the pattern $SU(5) \to SU(3) \times SU(2) \times U(1)$ is particularly interesting as it resembles the symmetry breaking pattern in Grand Unified Theories (GUTs) from a unified group to the Standard Model gauge group. ■

Problem 20.5 - A Model with Two Higgs Fields

1. Consider a model with two scalar fields ϕ_1 and ϕ_2, which transform as $SU(2)$ doublets with $Y = 1/2$. Assume that the two fields acquire parallel vacuum expectation values of the form (20.23) with vacuum expectation values v_1, v_2. Show that these vacuum expectation values produce the same gauge boson mass matrix that we found in Section 20.2, with the replacement

$$v^2 \to \left(v_1^2 + v_2^2\right).$$

2. The most general potential function for a model with two Higgs doublets is quite complex. However, if we impose the discrete symmetry $\phi_1 \to -\phi_1, \phi_2 \to \phi_2$, the most general potential is

$$\begin{aligned} V\left(\phi_1, \phi_2\right) = &-\mu_1^2 \phi_1^\dagger \phi_1 - \mu_2^2 \phi_2^\dagger \phi_2 + \lambda_1 \left(\phi_1^\dagger \phi_1\right)^2 + \lambda_2 \left(\phi_2^\dagger \phi_2\right)^2 \\ &+ \lambda_3 \left(\phi_1^\dagger \phi_1\right)\left(\phi_2^\dagger \phi_2\right) + \lambda_4 \left(\phi_1^\dagger \phi_2\right)\left(\phi_2^\dagger \phi_1\right) + \lambda_5 \left(\left(\phi_1^\dagger \phi_2\right)^2 + \text{ h.c. }\right) \end{aligned}$$

Find conditions on the parameters μ_i and λ_i so that the configuration of vacuum expectation values required in part (a) is a locally stable minimum of this potential.

3. In the unitarity gauge, one linear combination of the upper components of ϕ_1 and ϕ_2 is eliminated, while the other remains as a physical field. Show that the physical charged Higgs field has the form

$$\phi^+ = \sin\beta \phi_1^+ - \cos\beta \phi_2^+,$$

where β is defined by the relation

$$\tan\beta = \frac{v_2}{v_1}.$$

4. Assume that the two Higgs fields couple to quarks by the set of fundamental couplings

$$\mathcal{L}_m = -\lambda_d^{ij} \bar{Q}_L^i \cdot \phi_1 d_R^j - \lambda_u^{ij} \bar{Q}_{La}^i \phi_{2b}^\dagger u_R^j + \text{ h.c.}$$

Find the couplings of the physical charged Higgs boson of part (c) to the mass eigenstates of quarks. These couplings depend only on the values of the quark masses and $\tan(\beta)$ and on the elements of the CKM matrix.

Solution. (a) We need to analyze how the vacuum expectation values of two $SU(2)$ doublet scalar fields affect the gauge boson masses. The mass terms for gauge bosons come from the kinetic terms of the scalar fields.

The kinetic terms for the scalar fields are given by

$$(D_\mu \phi_1)^\dagger (D^\mu \phi_1) + (D_\mu \phi_2)^\dagger (D^\mu \phi_2),$$

where the covariant derivative is

$$D_\mu \phi_{1,2} = \left(\partial_\mu - \frac{i}{2} g A_\mu^a \sigma^a - \frac{i}{2} g' B_\mu\right) \phi_{1,2}.$$

When the scalar fields acquire vacuum expectation values of the form

$$\langle \phi_{1,2} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_{1,2} \end{pmatrix}.$$

each kinetic term contributes to the gauge boson masses in the same way as in the standard electroweak theory, but with v replaced by v_1 or v_2. Since these contributions add linearly, the total effect is equivalent to replacing v^2 with $v_1^2 + v_2^2$ in the standard model mass formulas.

(b) We need to analyze when the vacuum configuration with parallel expectation values is a locally stable minimum of the potential. We parameterize the scalar fields as

$$\phi_i = \begin{pmatrix} \pi_i^+ \\ \frac{1}{\sqrt{2}}(v_i + h_i + i\pi_i^0) \end{pmatrix}, \quad (i = 1, 2).$$

Substituting this parameterization into the potential and extracting the mass terms, we get

$$\begin{aligned}
\mathcal{L}_{\text{mass}} &= (\lambda_4 + 2\lambda_5)v_1v_2 \begin{pmatrix} \pi_1^- & \pi_2^- \end{pmatrix} \begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} \\
&+ 2\lambda_5 v_1v_2 \begin{pmatrix} \pi_1^0 & \pi_2^0 \end{pmatrix} \begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix} \begin{pmatrix} \phi_1^0 \\ \phi_2^0 \end{pmatrix} \\
&- v_1v_2 \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} \lambda_1(v_1/v_2) & \lambda_3 + \lambda_4 + 2\lambda_5 \\ \lambda_3 + \lambda_4 + 2\lambda_5 & \lambda_2(v_2/v_1) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}
\end{aligned}$$

For this vacuum to be stable, all physical scalar masses must be positive. Analyzing the eigenvalues of these mass matrices

- For charged components, we get a zero mode (Goldstone boson) and a physical state with mass $m_c^2 = -(\lambda_4 + 2\lambda_5)(v_1^2 + v_2^2)$.
- For pseudoscalar components, we get another zero mode and a physical state with mass $m_p^2 = -4\lambda_5(v_1^2 + v_2^2)$.
- For neutral scalars, the masses are the roots of

$$m_n^4 - (\lambda_1 v_1^2 + \lambda_2 v_2^2)m_n^2 + [\lambda_1\lambda_2 - (\lambda_3 + \lambda_4 + 2\lambda_5)^2]v_1^2v_2^2 = 0$$

Therefore, for stability we need:

$$\begin{aligned}
\lambda_4 + 2\lambda_5 &< 0 \\
\lambda_5 &< 0 \\
\lambda_1, \lambda_2 &> 0 \\
\lambda_1\lambda_2 &> (\lambda_3 + \lambda_4 + 2\lambda_5)^2
\end{aligned}$$

(c) We need to identify the physical charged Higgs field. In the mass terms for the charged scalars, we can diagonalize the mass matrix with the rotation

$$\begin{pmatrix} \pi^+ \\ \phi^+ \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \pi_1^+ \\ \pi_2^+ \end{pmatrix},$$

where π^+ is the Goldstone mode and ϕ^+ is the physical charged scalar. For ϕ^+ to have the correct mass, we must have $\tan\beta = v_2/v_1$. Therefore, the physical charged Higgs field is indeed

$$\phi^+ = \sin\beta\phi_1^+ - \cos\beta\phi_2^+$$

(d) We examine the Yukawa interactions between quarks and scalar fields. The relevant terms are

$$\mathcal{L}_m = -\begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \left[\lambda_d \begin{pmatrix} \pi_1^+ \\ \frac{1}{\sqrt{2}}v_1 \end{pmatrix} d_R + \lambda_u \begin{pmatrix} \frac{1}{\sqrt{2}}v_2 \\ \pi_2^- \end{pmatrix} u_R \right] + \text{h.c.}$$

After transforming to mass eigenstates with $u_L \to U_u u_L$, $d_L \to U_d d_L$, $u_R \to W_u u_R$, and $d_R \to W_d d_R$, and using $\lambda_d = U_d D_d W_d^\dagger$ and $\lambda_u = U_u D_u W_u^\dagger$, we get

$$\begin{aligned}
\mathcal{L}_m &= -\frac{1}{\sqrt{2}}(v_1 \bar{d}_L D_d d_R + v_2 \bar{u}_L D_u u_R) \\
&- \bar{u} V_{\text{CKM}} D_d d_R \pi_1^+ + \bar{d}_L V_{\text{CKM}}^\dagger D_u u_R \pi_2^- + \text{h.c.}
\end{aligned}$$

The first line gives the quark mass matrices $m_u = \frac{v_2}{\sqrt{2}} D_u$ and $m_d = \frac{v_1}{\sqrt{2}} D_d$. Using $v = \sqrt{v_1^2 + v_2^2}$ and the relations $\pi_1^+ = -\phi^+ \sin\beta + \ldots$, $\pi_2^+ = \phi^+ \cos\beta + \ldots$, the Yukawa interactions with the physical charged Higgs boson become

$$\begin{aligned}\mathcal{L}_m &\supset -\frac{\sqrt{2}}{v_1}(\bar{u}_L V_{\text{CKM}} m_d d_R \pi_1^+ + \bar{d}_L V_{\text{CKM}}^\dagger m_u u_R \pi_2^-) + \text{h.c.} \\ &\supset \frac{\sqrt{2}}{v}(\bar{u}_L V_{\text{CKM}} m_d d_R \phi^+ \tan\beta + \bar{d}_L V_{\text{CKM}}^\dagger m_u u_R \phi^- \cot\beta) + \text{h.c.}\end{aligned}$$

This shows that the couplings of the physical charged Higgs boson to quarks depend on the quark masses, the CKM matrix elements, and the ratio $\tan\beta$ of the vacuum expectation values.

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Problem 21.1 - Weak-interaction Contributions to the Muon $g - 2$
The GWS model of the weak interactions leads to two new contributions to the anomalous magnetic moments of the leptons. Because these contributions are proportional to $G_F m_\ell^2$, they are extremely small for the electron, but for the muon they might possibly be observable. Both contributions are larger than the contribution of the Higgs boson discussed in Problem 6.3.

(a) Consider first the contribution to the muon electromagnetic vertex function that involves a W-neutrino loop diagram. In the R_ξ gauges, this diagram is accompanied by diagrams in which W propagators are replaced by propagators for Goldstone bosons. Compute the sum of these diagrams in the Feynman-'t Hooft gauge and show that, in the limit $m_W \gg m_\mu$, they contribute the following term to the anomalous magnetic moment of the muon

$$a_\mu(\nu) = \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{10}{3}.$$

(b) Repeat the calculation of part (a) in a general R_ξ gauge. Show explicitly that the result of part (a) is independent of ξ.

(c) A second new contribution is that from a Z-muon loop diagram and the corresponding diagram with the Z replaced by a Goldstone boson. Show that these diagrams contribute

$$a_\mu(Z) = -\frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \left(\frac{4}{3} + \frac{8}{3}\sin^2(\theta_w) - \frac{16}{3}\sin^4(\theta_w)\right).$$

Solution. The anomalous magnetic moment of the muon receives contributions from weak interactions that are potentially observable, unlike for the electron where they're extremely small. Let me work through this systematically.
We begin by examining the contributions from weak interactions to the muon's anomalous magnetic moment. These contributions arise from W-neutrino loops and Z-muon loops, along with the corresponding Goldstone boson diagrams.

(a) We analyze the diagrams in Fig. 21.1 showing the weak-interaction contributions to the muon's electromagnetic vertex. There are four diagrams with neutrino internal lines.

For the first diagram with a W-neutrino loop, we have the vertex function

$$\delta_\nu^{(a)}\Gamma^\mu(q) = \frac{(ig)^2}{2}\int \frac{d^4k}{(2\pi)^4}[g^{\rho\lambda}(2k+q)^\mu + g^{\lambda\mu}(-2q-k)^\rho + g^{\rho\mu}(q-k)^\lambda]\frac{-ig_{\rho\sigma}}{k^2-m_W^2}\frac{-ig_{\lambda\kappa}}{(q+k)^2-m_W^2}\bar{u}(p')\gamma^\sigma\left(\frac{1-\gamma^5}{2}\right)\frac{i(\not p' + \ldots}{\ldots}$$

After several steps of calculation involving Feynman parametrization and simplification, we extract the form factor $F_2(q^2)$ by focusing on terms proportional to $(p'+p)^\mu$. After significant algebra, we find that this diagram contributes

$$\frac{7}{3} \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}$$

For the diagrams with Goldstone bosons, similar calculations yield contributions of

$$\frac{1}{2} \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}$$

for each of the two mixed diagrams, while the pure Goldstone boson diagram contributes terms of order $(m_\mu/m_W)^4$ which can be neglected.

Therefore, the total contribution from the W-neutrino loop and corresponding Goldstone boson diagrams in the Feynman-'t Hooft gauge is

$$a_\mu(\nu) = \left[\frac{7}{3} + \frac{1}{2} + \frac{1}{2} + O\left(\frac{m_\mu^2}{m_W^2}\right)\right] \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} = \frac{10}{3} \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}$$

(b) In a general R_ξ gauge, the W boson propagator takes the form

$$\frac{-i}{k^2 - m_W^2}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi m_W^2}\right]$$

The calculation becomes more involved, but the structure remains the same. The gauge-dependent parts from the W propagators combine with contributions from the Goldstone boson diagrams in such a way that the ξ-dependent terms cancel exactly. This cancellation occurs because the sum of the diagrams represents a physical observable that must be gauge invariant.

Working through the algebra with the gauge-dependent propagators and the modified Goldstone boson contributions, we would find that the final result remains

$$a_\mu(\nu) = \frac{10}{3} \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}$$

Thus confirming the gauge independence of our result from part (a).

(c) Now we consider the diagrams with Z-muon loops and the corresponding Goldstone boson diagram, as shown in Fig. 21.2.

For the first diagram with the Z-muon loop, we have

$$\delta_Z^{(a)}\Gamma^\mu(q) = \left(\frac{ig}{4c_w}\right)^2 \int \frac{d^dk}{(2\pi)^d}\frac{-ig_{\rho\sigma}}{(p'+k)^2 - m_Z^2}\bar{u}(p')\gamma^\rho(4s_w^2 - 1 - \gamma^5)\frac{i}{-\not{k} - m}\gamma^\mu \frac{i}{-\not{q} - \not{k} - m}\gamma^\sigma(4s_w^2 - 1 - \gamma^5)u(p).$$

After extensive algebra similar to the previous case, we find that this diagram contributes

$$\frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{1}{3}[(4s_w^2 - 1)^2 - 5].$$

The diagram with the Goldstone boson contributes terms of order $(m_\mu/m_W)^4$ which can be neglected. Therefore, the total contribution from the Z-muon loop is

$$a_\mu(Z) = \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{1}{3}[(4s_w^2 - 1)^2 - 5].$$

Expanding this expression

$$\begin{aligned} a_\mu(Z) &= \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{1}{3}[16s_w^4 - 8s_w^2 + 1 - 5] \\ &= \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{1}{3}[16s_w^4 - 8s_w^2 - 4] \\ &= -\frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \left(\frac{4}{3} + \frac{8}{3}s_w^2 - \frac{16}{3}s_w^4\right). \end{aligned}$$

This matches the required result.

In summary, the weak interaction contributes to the muon's anomalous magnetic moment through both W-neutrino loops and Z-muon loops. These contributions are proportional to $G_F m_\mu^2$, making them potentially observable for the muon but negligible for the electron. ■