PHYS 580 - Computational Physics

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Lab 3

> **Problem 1**
> Take the linear oscillator with no damping and no driving force (*i.e.* completely harmonic). Investigate the stability and accuracy of three numerical methods: Euler, Euler-Cromer, and Runge-Kutta 2nd order. Demonstrate your conclusions with a few chosen parameters and plots. Be sure to include the cumulative (global) error analyses for both θ and E (energy) in your report, comparing the different methods to each other and to the theoretical expectations. Do NOT use Matlab built-in functions or Python libraries for Euler, RK, etc. integrators; Rather, code the respective algorithms into your programs explicitly.

Solution. We consider the system of differential equations for a simple harmonic oscillator:

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \omega \\ \frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\omega_0^2\theta\end{aligned}$$

where $\omega_0^2 = g/L$. The analytical solution with initial conditions $\theta(0) = \theta_0$ and $\omega(0) = \omega_0$ is:

$$\begin{aligned}\theta(t) &= \theta_0 \cos(\omega_0 t) + \frac{\omega_0}{\omega_0}\sin(\omega_0 t) \\ \omega(t) &= -\omega_0\theta_0 \sin(\omega_0 t) + \omega_0 \cos(\omega_0 t)\end{aligned}$$

The total energy of the system at any point is

$$E = \frac{1}{2}mL^2\omega^2 + \frac{1}{2}mgL\theta^2.$$

Using these three numerical methods with $\Delta t = 0.05$, $\theta_0 = 0.5$ and $\omega_0 = 0$, we find the following results with their respective global errors relative to the analytical solution.
Analysis of the results reveals several key points:

1. The Euler method shows significant instability, with the amplitude increasing over time. This is reflected in the rapidly growing global error and energy that increases without bound.
2. The Euler-Cromer method demonstrates much better stability. While it has some error accumulation, the energy oscillates around a constant value rather than growing unbounded. This makes it particularly suitable for oscillatory systems.
3. The RK2 method shows the best accuracy in terms of global error for both θ and ω. However, it doesn't preserve energy as well as the Euler-Cromer method over long time periods.

The global error analysis shows that the Euler method has exponential error growth, while both Euler-Cromer and RK2 methods maintain better error control. The Euler-Cromer method, despite having slightly larger position errors than RK2, better preserves the physical properties of the system through its superior energy conservation. ■

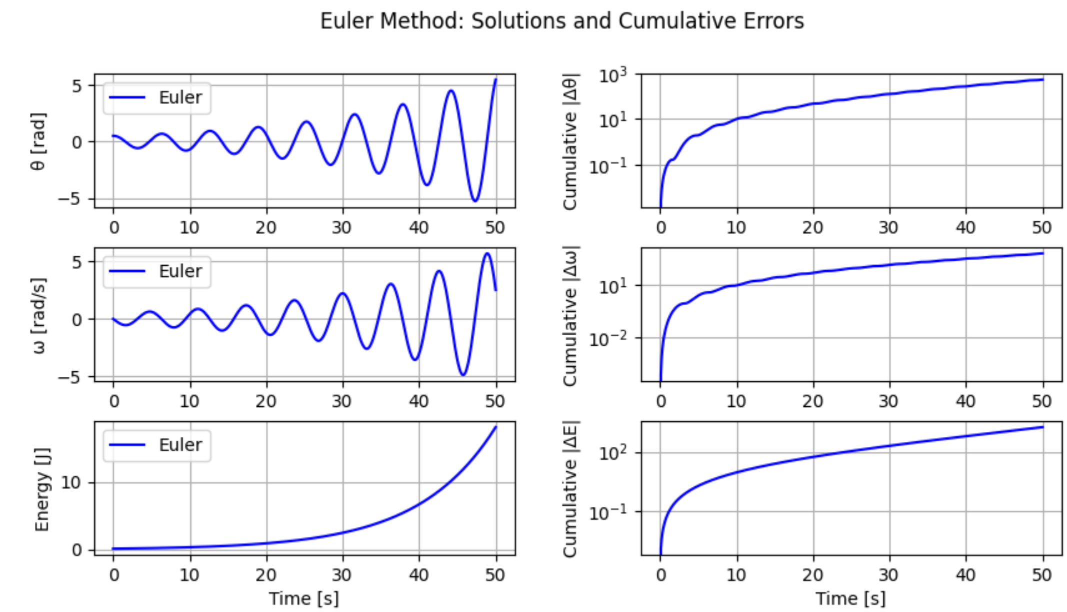

Figure 1: Numerical solution and error for SHO with the Euler Method. Left panels show θ, ω, and energy evolution. Right panels show cumulative errors.

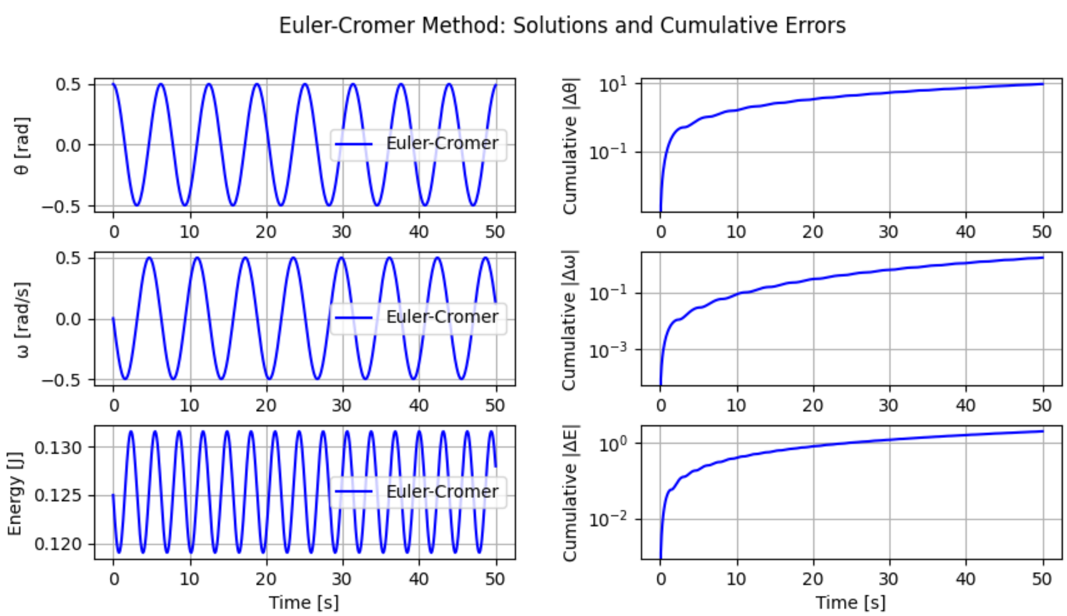

Figure 2: Numerical solution and error for SHO with the Euler-Cromer Method. Note the stable energy oscillations.

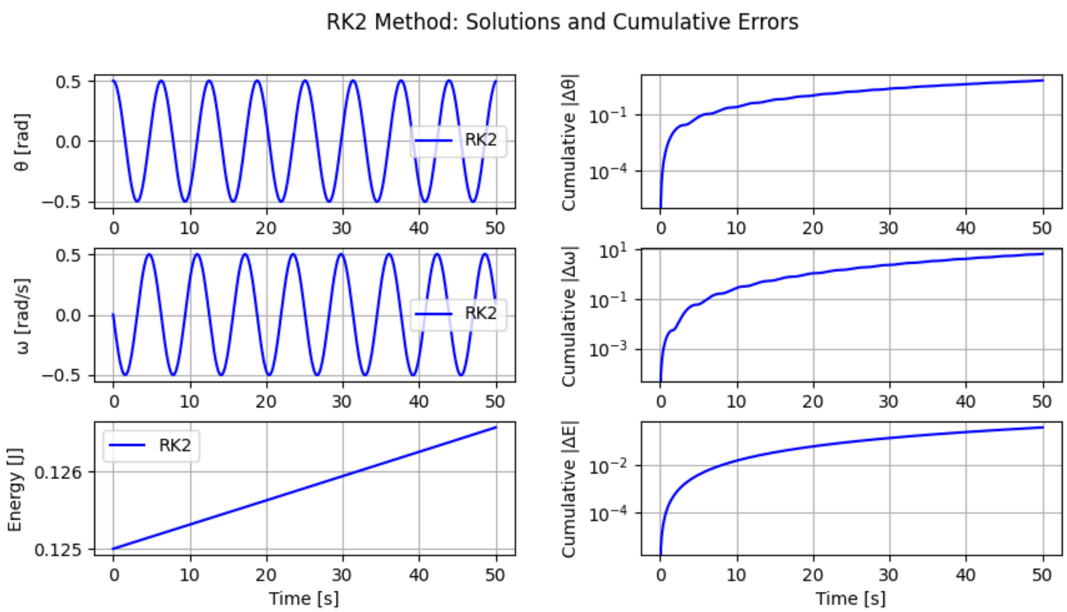

Figure 3: Numerical solution and error for SHO with the RK2 Method, demonstrating better error control.

Problem 2
For the linear oscillator, turn on some damping and/or driving force. Using the method and parameters you determined in Problem 1 to be adequate, demonstrate the overdamped, underdamped, and resonant regimes (cf. Section 3.2 of the textbook). Set $L = 9.8$ m to simplify things a bit.

Solution. For this problem, we modify our equations to include both damping (β) and driving force terms

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}\theta}{\mathrm{d}t} + \omega_0^2\theta = F_D\sin(\Omega_D t),$$

where F_D is the driving force amplitude and Ω_D is its frequency. The solution consists of two parts:

1. A complementary function $\theta_c(t)$ which depends on the damping coefficient, and
2. A particular solution $\theta_p(t) = D\sin(\Omega_D t + \delta)$ which introduces forced oscillations.

The results show that both RK2 and Euler-Cromer methods provide stable solutions, while the Euler method exhibits some numerical instabilities. After the initial transient period, the system settles into a steady-state oscillation where the energy input from the driving force balances the energy loss from damping.
For resonance investigation, we set the driving frequency to:

$$\Omega_D = \omega_R = \sqrt{\omega_0^2 - 2\beta^2}.$$

Finally, we examine the three damping regimes by removing the driving force ($F_D = 0$):
The three damping regimes show distinctly different behaviors:

- **Underdamped ($\beta < \omega_0$):** Shows oscillatory decay
- **Critical damping ($\beta = \omega_0$):** Returns to equilibrium in minimum time without oscillation
- **Overdamping ($\beta > \omega_0$):** Shows slower exponential decay without oscillation

In all cases, the energy monotonically decreases due to the damping term, until the system either reaches equilibrium (undriven case) or settles into steady-state oscillations (driven case). The Euler-Cromer method proves particularly effective for these simulations due to its stability and energy-conserving properties when no damping is present. ■

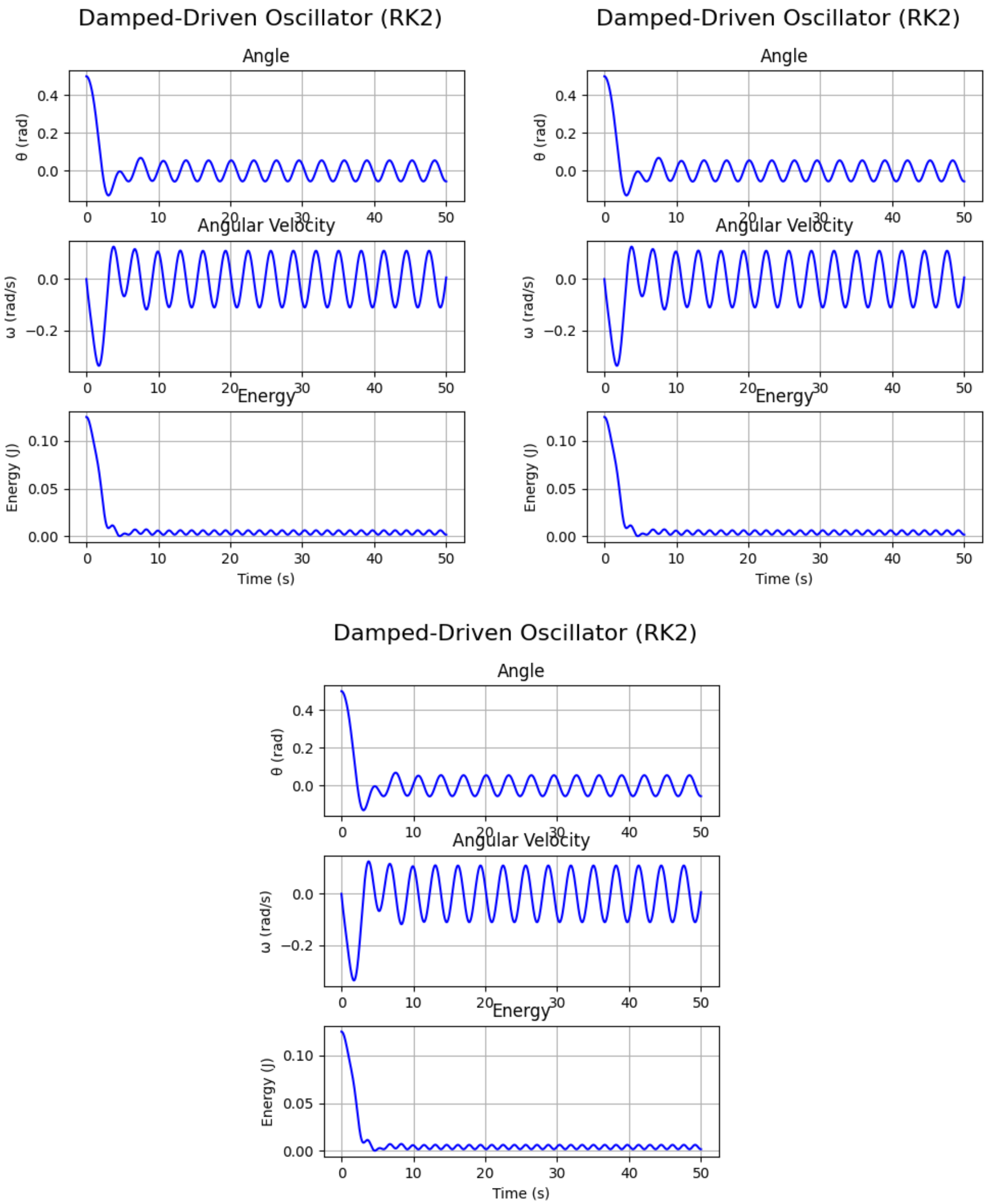

Figure 4: Numerical solutions using Euler, Euler-Cromer, and RK2 methods with parameters $L = 9.8$ m, $\beta = 0.5$, $F_D = 0.2$, and $\Omega_D = 2$. Note how all methods show initial transient behavior followed by steady-state oscillations.

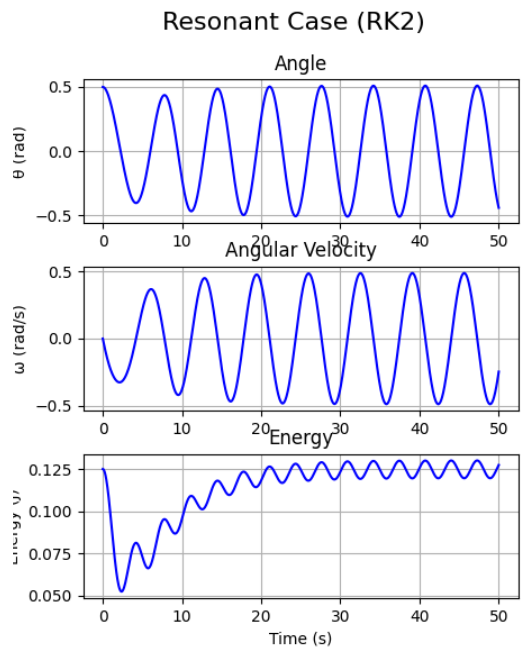

Figure 5: Resonant behavior using Euler-Cromer method with $L = 9.8$ m, $\beta = 0.2$, $F_D = 0.5$, and $\Omega_D = \omega_R$. Note the larger amplitude oscillations.

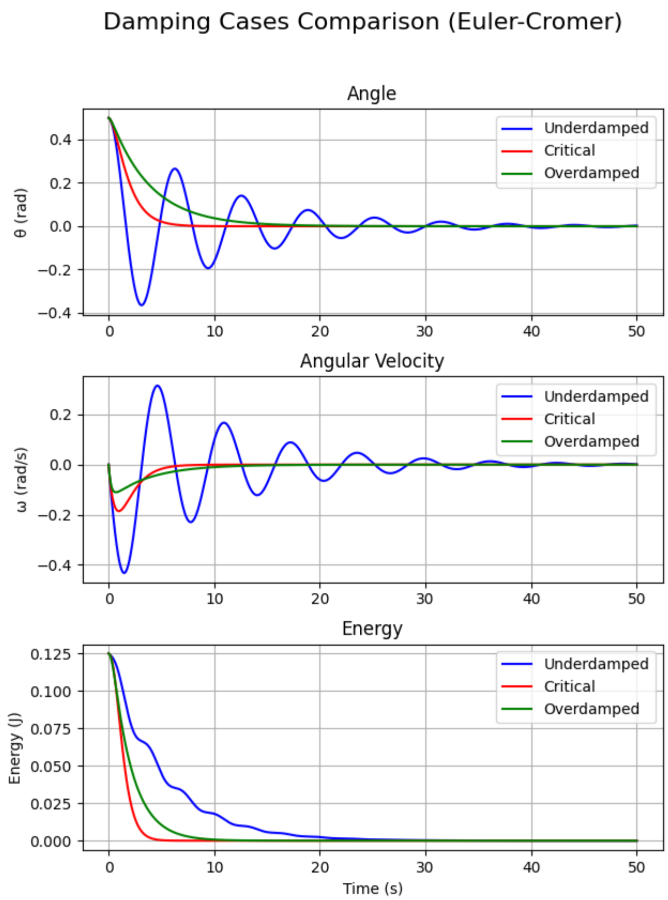

Figure 6: Comparison of underdamped ($\beta < \omega_0$), critically damped ($\beta = \omega_0$), and overdamped ($\beta > \omega_0$) cases using the Euler-Cromer method.

Problem 3
Take the physical oscillator with nonlinear equation of motion (i.e., the full $\sin(\theta)$ term), but without damping and driving force. Investigate the dependence, if any, of the period (if periodic) and wave form on the amplitude of oscillation. Compare with the exact result for the period:

$$T = 4\sqrt{\frac{L}{g}}K\left(\sin\left(\frac{\theta_m}{2}\right)\right) = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1}{16}\theta_m^2 + \frac{11}{3072}\theta_m^4 + \cdots\right).$$

Here, the right-hand side explicitly shows the first three terms of $K(a)$, the complete elliptic integral of 1st kind, when expanded into powers of the oscillation amplitude θ_m.

Solution. For the nonlinear case, we modify our system of equations to include the full $\sin\theta$ term

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \omega \\ \frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\omega_0^2 \sin\theta\end{aligned}$$

The energy for this nonlinear pendulum is

$$E = \frac{1}{2}mL^2\omega^2 + mgL(1 - \cos\theta).$$

Similar to Problem 1, we observe that

- The Euler method shows increasing amplitude and energy over time
- The Euler-Cromer method maintains stable energy oscillations
- The RK2 method provides accurate trajectories but with slight energy drift

For the period analysis, we used the Euler-Cromer method to calculate the oscillation period for different initial amplitudes. The results show excellent agreement with the analytical prediction, with relative errors remaining below 0.3% even for large amplitudes up to 80 degrees. The period increases with amplitude as expected from the analytical formula, demonstrating the nonlinear nature of the pendulum for large angles. Key observations from the period analysis:

- For small amplitudes ($\theta_m < 20$), the period is nearly constant, corresponding to the linear regime
- As amplitude increases, the period increases quadratically, following the analytical prediction
- The numerical results match the analytical formula within 0.3% error across the tested range
- The error generally increases with amplitude due to the accumulation of numerical errors over longer periods

The close agreement between numerical and analytical results validates both our implementation of the Euler-Cromer method and its suitability for studying nonlinear oscillatory systems. ∎

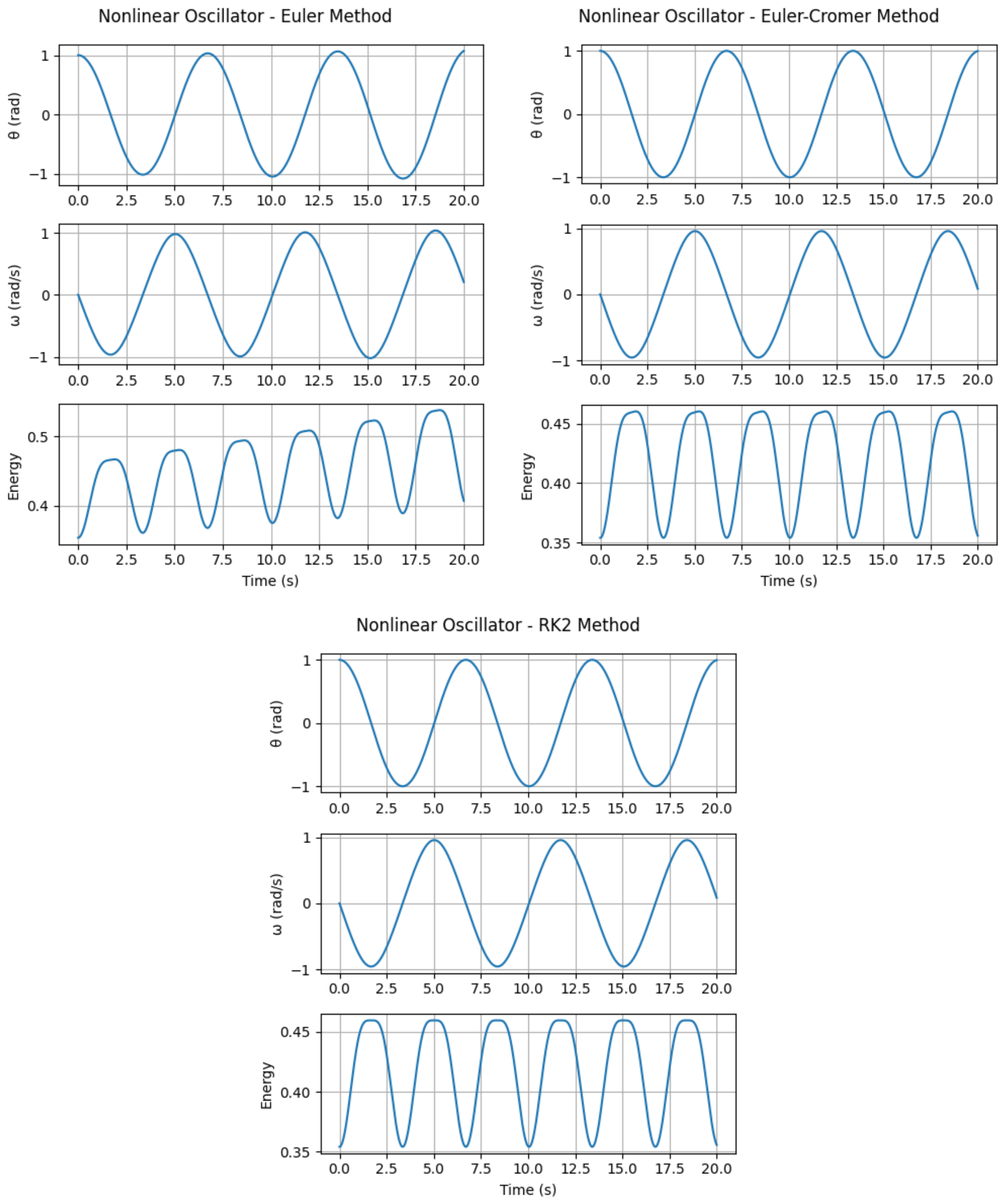

Figure 7: Numerical solutions using Euler, Euler-Cromer, and RK2 methods with $\theta_0 = 1.0$ rad. Note the differences in energy conservation among methods.

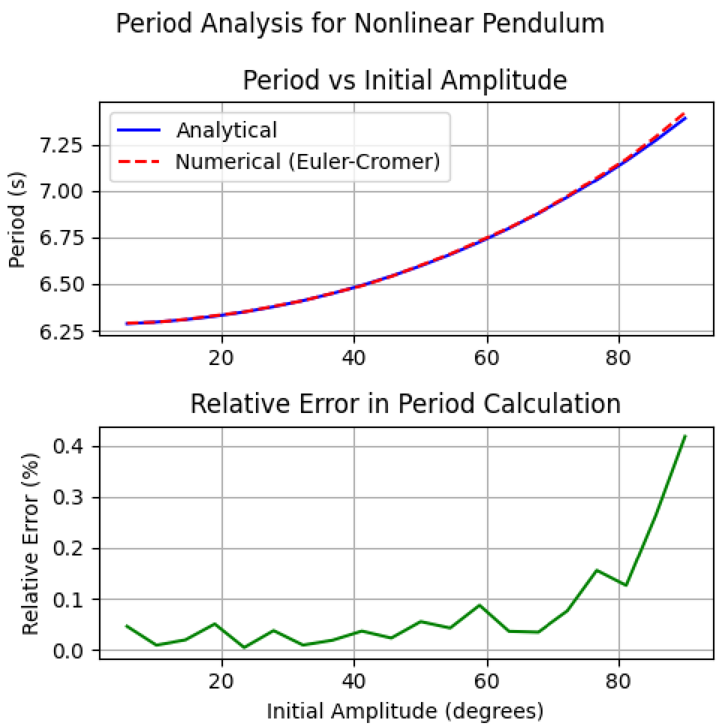

Figure 8: Comparison of numerical and analytical periods as a function of initial amplitude. The bottom panel shows the relative error between numerical and analytical results.