MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*

Student: **Ralph Razzouk**

Homework 14

Problem 17-1
Let M be a smooth manifold with or without boundary, and let $\omega \in \Omega^p(M), \eta \in \Omega^q(M)$ be closed forms. Show that the de Rham cohomology class of $\omega \wedge \eta$ depends only on the cohomology classes of ω and η, and thus there is a well-defined bilinear map $\cup : H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \to H^{p+q}_{\mathrm{dR}}(M)$, called the **cup product**, given by $[\omega] \cup [\eta] = [\omega \wedge \eta]$.

Solution. Let $\omega' \in [\omega]$ and $\eta' \in [\eta]$ be other representatives of the same cohomology classes. Then

$$\omega' - \omega = \mathrm{d}\alpha, \text{ for some } \alpha \in \Omega^{p-1}(M)$$

and

$$\eta' - \eta = \mathrm{d}\beta, \text{ for some } \beta \in \Omega^{q-1}(M).$$

Consider $\omega' \wedge \eta'$, then

$$\begin{aligned} \omega' \wedge \eta' &= (\omega + \mathrm{d}\alpha) \wedge (\eta + \mathrm{d}\beta) \\ &= \omega \wedge \eta + \omega \wedge \mathrm{d}\beta + \mathrm{d}\alpha \wedge \eta + \mathrm{d}\alpha \wedge \mathrm{d}\beta. \end{aligned}$$

Recall that $\mathrm{d}^2 = 0$ and $d\alpha \wedge \mathrm{d}\beta = 0$ due to the graded anticommutativity of the exterior derivative. Also, $\mathrm{d}(\omega \wedge \mathrm{d}\beta) = \mathrm{d}\omega \wedge \mathrm{d}\beta = 0$ since ω is closed.
Therefore,

$$\omega' \wedge \eta' - \omega \wedge \eta = \mathrm{d}(\alpha \wedge \eta + \omega \wedge \beta).$$

This implies $[\omega' \wedge \eta'] = [\omega \wedge \eta]$ in $H^{p+q}_{\mathrm{dR}}(M)$, proving the cup product is well-defined. ∎

Problem 17-8
Suppose M is a compact, connected, orientable, smooth n-manifold.

(a) Show that there is a one-to-one correspondence between orientations of M and orientations of the vector space $H^n_{\mathrm{dR}}(M)$, under which the cohomology class of a smooth orientation form is an oriented basis for $H^n_{\mathrm{dR}}(M)$.

(b) Now suppose M and N are smooth n-manifolds with given orientations. Show that a diffeomorphism $F : M \to N$ is orientation preserving if and only if $F^* : H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$ is orientation preserving.

Solution. (a) Let $\omega \in \Omega^n(M)$ be an orientation form. Since M is compact and connected, $[\omega] \in H^n_{\mathrm{dR}}(M)$ is a non-zero basis element.

- **Injective:** Suppose two orientations ω_1 and ω_2 are mapped to the same cohomology class. Then $\omega_1 = f\omega_2$ for some non-zero smooth function f. Since f is non-zero everywhere (due to the connectedness of M), the sign of f determines the same orientation. This proves injectivity.
- **Surjective:** Suppose $v \in H^n_{\mathrm{dR}}(M)$ is a non-zero cohomology class. Since $\dim H^n_{\mathrm{dR}}(M) = 1$, there exists a unique smooth volume form ω' such that $[\omega'] = v$. The form ω' defines an orientation on M. This proves surjectivity.

(b) We will prove both directions of the if and only if statement.

$\boxed{\Longrightarrow}$ Assume F is orientation preserving. Let ω_N be a volume form representing the orientation of N. Since F is orientation preserving, $F^*\omega_N$ is a volume form on M representing the orientation of M. By the previous theorem on the correspondence between manifold and cohomology orientations, this means $F^*\omega_N$ corresponds to the same orientation in $H^n_{\mathrm{dR}}(M)$ as ω_N does in $H^n_{\mathrm{dR}}(N)$. In other words, $[F^{\omega}_N] = F^*$ in $H^n_{\mathrm{dR}}(M)$, which means $F^{\cdot} H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$ preserves orientation.

$\boxed{\Longleftarrow}$ Conversely, suppose $F^* : H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$ preserves orientation. Let ω_N be a volume form representing the orientation of N. Then $F^*\omega_N$ must correspond to the orientation of M under the correspondence we established in the previous theorem. By the properties of pullback and orientation, this means the sign of $\det(\mathrm{d}F)$ at each point must be positive. The condition that $\det(\mathrm{d}F)$ has a consistent positive sign is precisely the definition of an orientation-preserving diffeomorphism.

Therefore, a diffeomorphism F is orientation preserving if and only if F^* is orientation preserving.

■

Problem 17-13
Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-torus. Consider the two maps $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ given by $f(w, z) = (w, z)$ and $g(w, z) = (z, \bar{w})$. Show that f and g have the same degree, but are not homotopic. [Suggestion: consider the induced homomorphisms on the first cohomology group or the fundamental group.]

Solution. Consider the induced maps on $H^1(\mathbb{T}^2; \mathbb{R})$. Let α, β be the dual basis of the first cohomology group.

- For f^*, we have

$$f^*(\alpha) = \alpha,$$
$$f^*(\beta) = \beta.$$

- For g^*, we have

$$g^*(\alpha) = \beta,$$
$$g^*(\beta) = -\alpha.$$

Both maps have degree ± 1 on $H^1(\mathbb{T}^2; \mathbb{R})$. However, f and g are not homotopic. We will prove this by contradiction. Let f and g be homotopic, then their induced maps on $\pi_1(\mathbb{T}^2)$ would be conjugate. f_* is the identity on $\pi_1(\mathbb{T}^2)$ and g_* interchanges the generators of $\pi_1(\mathbb{T}^2)$.
Therefore, f and g are not homotopic. ■