PHYS 580 - Computational Physics

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Homework 2

Problem 1
For the driven, nonlinear pendulum, investigate how the Poincare section depends on the strength of the linear dissipation term. Use the same physics parameters as the calculation in Fig. 3.6 of the Giordano-Nakanishi book.

(a) First set $F_D = 0.5, q = 0.5$, which puts you in the periodic regime, and then vary q to see how the Poincare section changes.

(b) Next, repeat the same from a starting point $F_D = 1.2, q = 0.5$ in the chaotic regime. In both cases, keep all other physics parameters fixed as you vary q.

Solution. For this problem, we investigate how the Poincare sections of a driven nonlinear pendulum depend on the dissipation strength (q). We examine two distinct regimes characterized by different driving force amplitudes.

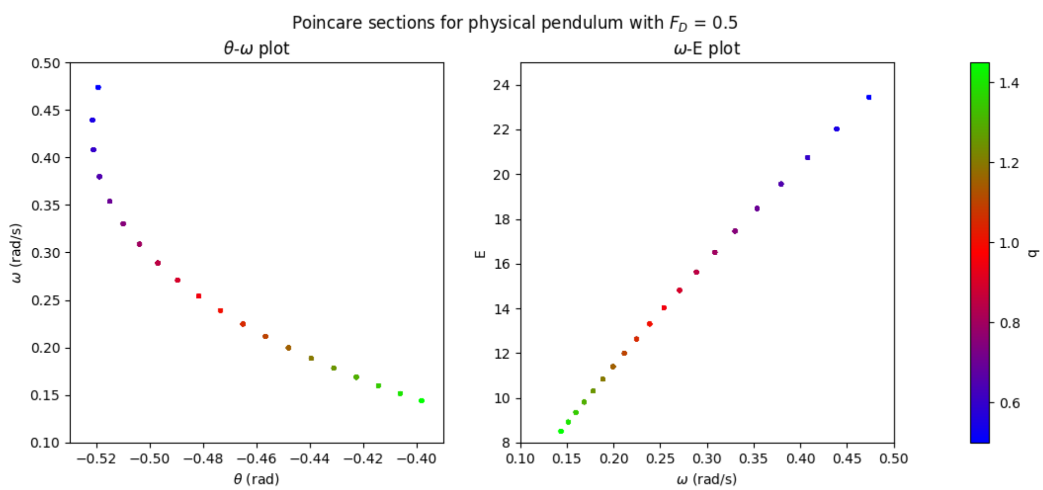

Figure 1: Plots showing Poincare sections for the physical pendulum with $F_D = 0.5$.

First, we analyze the case with $F_D = 0.5$ and varying q from 0.5 to 1.5. The θ-ω plot reveals a highly ordered structure, indicative of periodic motion. As q increases (shown by the color gradient from blue to green), we observe

- A systematic shift of the Poincare points towards smaller absolute values of both θ and ω
- A clear single-point structure for each q value, confirming stable periodic orbits
- No signs of period doubling or chaos

The corresponding ω-E plot provides additional insight

- Energy decreases monotonically with increasing q
- Points form a smooth curve suggesting a continuous transition between states
- Higher dissipation (larger q) leads to lower energy states, as expected physically

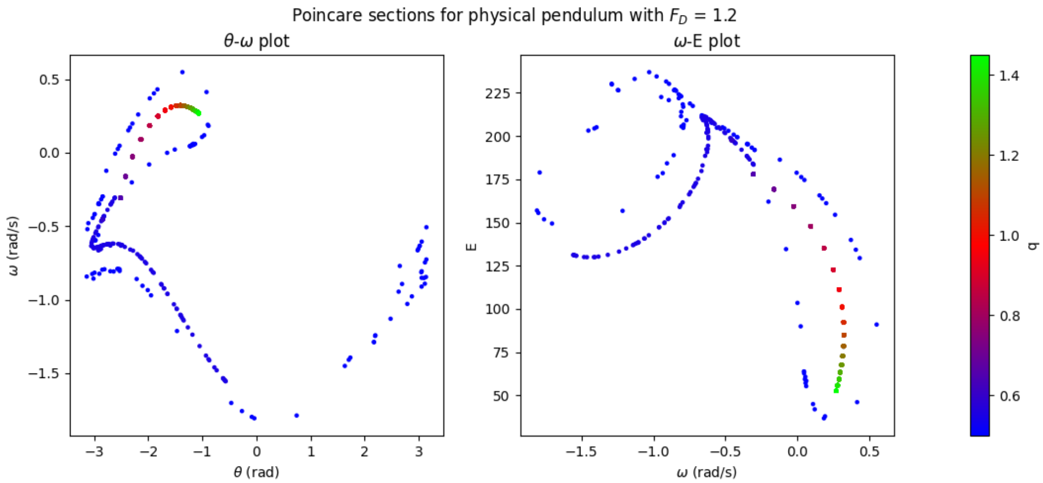

Figure 2: Plots showing Poincare sections for the physical pendulum with $F_D = 1.2$.

For $F_D = 1.2$, we enter a fundamentally different regime. The θ-ω phase space shows:

- Complex structure at low q (blue points) indicating chaotic behavior
- Multiple scattered points forming intricate patterns
- As q increases, the system transitions from chaos to more ordered motion
- The phase space coverage is much larger compared to $F_D = 0.5$

The ω-E plot for this chaotic regime demonstrates

- Significantly higher energy states compared to $F_D = 0.5$
- A complex energy distribution at low q values
- Gradual organization of points as dissipation increases
- Clear evidence of the system exploring multiple energy states when in chaos

This analysis reveals how dissipation can act as a control parameter, capable of transitioning the system from chaos to order in the strongly driven case ($F_D = 1.2$), while maintaining ordered behavior but with decreasing amplitude in the weakly driven case ($F_D = 0.5$). The Poincare sections effectively visualize these transitions and provide insight into the system's long-term behavior. ■

Problem 3.4
For simple harmonic motion, the general form for the equation of motion is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k\,\mathrm{sgn(x)}|x|^\alpha,$$

with $\alpha = 1$, where the sign function is

$$\mathrm{sgn}(x) = \begin{cases} +1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

This has the same form as (3.2), although the variables have different names. Begin by writing a program that uses the Euler-Cromer method to solve for x as a function of time according to (3.9), with $\alpha = 1$ (for convenience you can take $k = 1$). The subroutine described in this section can be modified to accomplish this. Show that the period of the oscillations is independent of the amplitude of the motion. This is a key feature of simple harmonic motion.
Then expand your program to treat the case $\alpha = 3$. This is an example of an anharmonic oscillator. Calculate the period of the oscillations for several different amplitudes (amplitudes in the range of 0.2 to 1 are good choices), and show that the period of the motion now depends on the amplitude. Give an intuitive argument to explain why the period becomes longer as the amplitude is decreased.

Solution. We investigate the behavior of a general oscillator described by the equation:

$$\frac{d^2 x}{dt^2} = -k\,\mathrm{sgn}(x)|x|^\alpha,$$

where $k = 1$ for simplicity. Using the Euler-Cromer method for numerical integration, we analyze two cases: $\alpha = 1$ corresponding to simple harmonic motion, and $\alpha = 3$ representing an anharmonic oscillator.

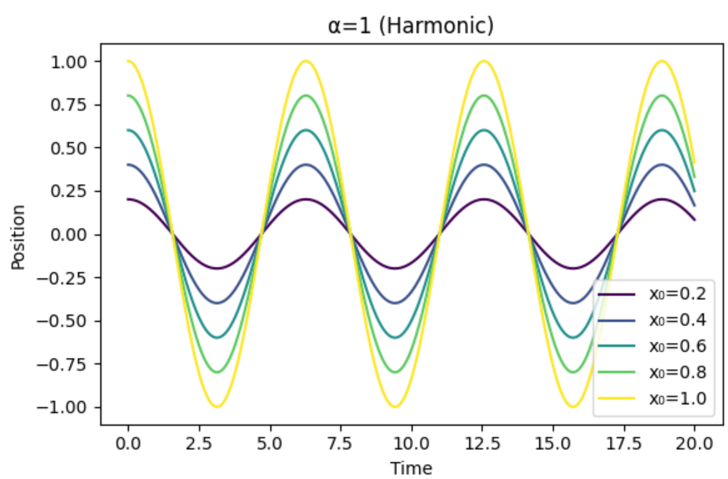

Figure 3: Plot showing the displacement for the harmonic ($\alpha = 1$) case.

For $\alpha = 1$, we observe several key features:

- The oscillations maintain perfect periodicity for all initial amplitudes
- The period remains constant regardless of the initial displacement
- All solutions are in phase, indicating synchronous motion
- The amplitude of oscillation is conserved, showing no numerical dissipation

- Each curve follows a sinusoidal pattern, characteristic of simple harmonic motion

This behavior demonstrates the defining characteristic of simple harmonic motion: the period's independence from amplitude. This property arises because the restoring force is linearly proportional to displacement, making the motion isochronous.

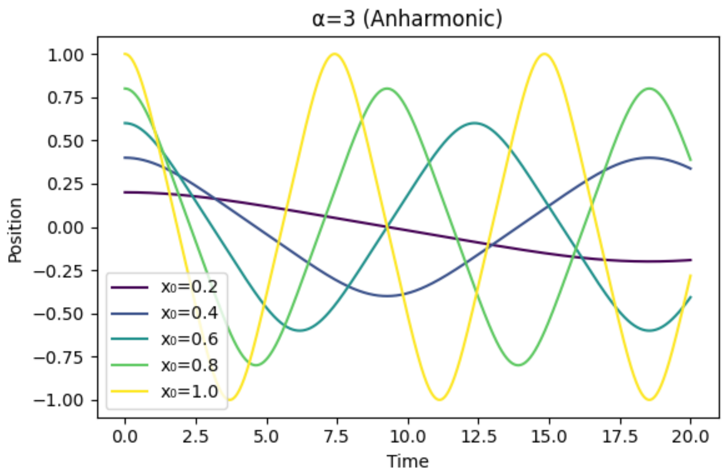

Figure 4: Plot showing the displacement for the anharmonic ($\alpha = 3$) case.

The anharmonic case ($\alpha = 3$) exhibits markedly different behavior:

- Different initial amplitudes lead to distinct periods of oscillation
- Smaller amplitudes ($x_0 = 0.2, 0.4$) result in significantly longer periods
- Larger amplitudes ($x_0 = 0.8, 1.0$) show faster oscillations
- The oscillations maintain their respective amplitudes but become increasingly out of phase
- The waveform deviates from pure sinusoidal motion

The inverse relationship between period and amplitude in the anharmonic case can be understood physically: for small displacements, the cubic restoring force ($|x|^3$) is much weaker than in the linear case, resulting in slower motion and longer periods. Conversely, at larger displacements, the cubic term yields a much stronger force than the linear case, producing faster oscillations.

This analysis demonstrates how nonlinearity in the restoring force fundamentally alters the oscillator's behavior, breaking the isochronous property characteristic of simple harmonic motion. The Euler-Cromer method effectively captures these dynamics while maintaining energy conservation, as evidenced by the stable amplitude of oscillations in both cases. ■

Problem 3.5
For the anharmonic oscillator of (3.9) in the previous exercise, it is possible to analytically obtain the period of oscillation for general values of α in terms of certain special functions.

(a) Do this, and describe how the relationship between the period and the amplitude depends on the value of α. Can you give a physical interpretation to the finding?

(b) Also check how well the periods computed in Problem 2 agree with the analytic results and discuss whether the difference you find is within the expected accuracy of the numerical calculation.

Hint: If you multiply both sides of (3.9) by $\frac{dx}{dt}$, you can integrate with respect to t. This then leads to a relation between the velocity and x.

Solution. For the anharmonic oscillator described by

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k\,\mathrm{sgn}(x)|x|^\alpha,$$

we can find the analytical solution by multiplying both sides by $\frac{\mathrm{d}x}{\mathrm{d}t}$ and integrating with respect to time:

$$\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k\,\mathrm{sgn}(x)|x|^\alpha \frac{\mathrm{d}x}{\mathrm{d}t}.$$

This leads to a relationship between velocity and position, ultimately yielding the period:

$$T = 4\sqrt{\frac{\pi}{2k(\alpha+1)}}\,x_0^{\frac{1-\alpha}{2}}\,\frac{\Gamma\left(\frac{1}{\alpha+1}\right)}{\Gamma\left(\frac{1}{\alpha+1}+\frac{1}{2}\right)},$$

where $\Gamma(z)$ is the gamma function.

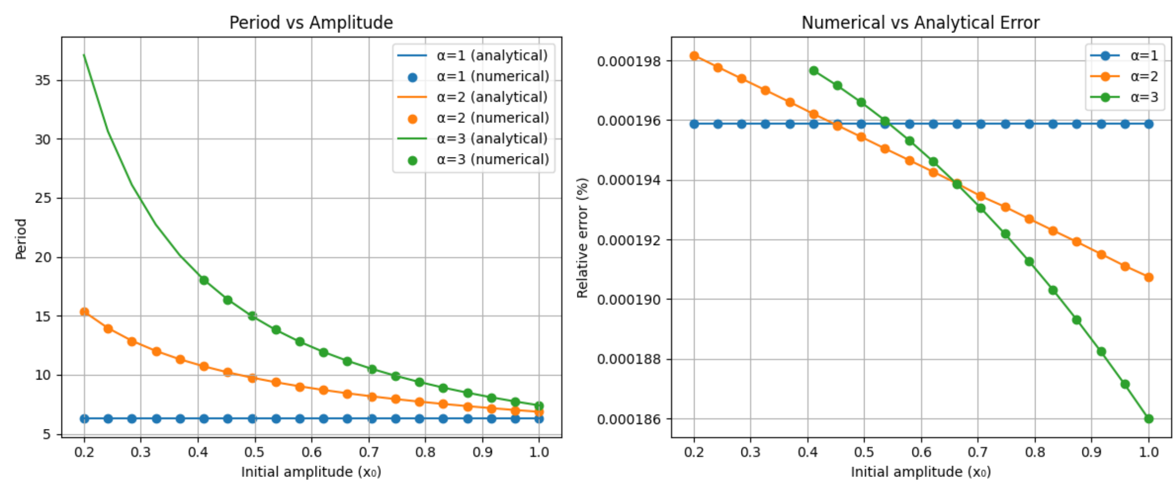

Figure 5: Plots showing the period variation as a function of the initial amplitude and their errors for each value of α.

Our analysis reveals several key features

- **For $\alpha = 1$ (simple harmonic oscillator):**
  - The period remains constant (≈ 6.28 units) regardless of amplitude
  - This independence reflects the system's isochronous nature

- The numerical solution shows excellent agreement with theory

- **For** $\alpha = 2$**:**
  - Period decreases with increasing amplitude
  - The relationship follows $T \propto x_0^{-0.5}$
  - The effect is more pronounced at small amplitudes

- **For** $\alpha = 3$**:**
  - Shows the strongest amplitude dependence
  - Period varies as $T \propto x_0^{-1}$
  - At small amplitudes ($x_0 = 0.2$), the period is nearly 6 times longer than at $x_0 = 1.0$

The comparison between numerical and analytical solutions reveals remarkable accuracy. The relative errors are consistently below 0.0002 (0.02%), with several notable characteristics:

- For $\alpha = 1$, the error remains nearly constant
- For $\alpha > 1$, errors tend to decrease with increasing amplitude
- The maximum error (≈ 0.000198) occurs for $\alpha = 2$ at small amplitudes

This high accuracy validates our numerical approach using the Euler-Cromer method. The decreasing errors with amplitude for $\alpha > 1$ can be understood physically: larger amplitudes lead to stronger restoring forces, making the numerical integration more stable.
The period-amplitude relationship can be interpreted physically through the restoring force's nonlinearity. For $\alpha > 1$, small displacements experience much weaker restoring forces than in the linear case, resulting in longer periods. Conversely, large displacements encounter stronger forces, leading to faster oscillations. This effect becomes more pronounced as α increases, explaining the stronger amplitude dependence for $\alpha = 3$. ■

Problem 3.13
Write a program to calculate and compare the behavior of two, nearly identical pendulums. Use it to calculate the divergence of two nearby trajectories in the chaotic regime, as in Figure 3.7, and make a qualitative estimate of the corresponding Lyapunov exponent from the slope of a plot of $\log(\Delta\theta)$ as a function of t.

Solution. To investigate the sensitivity to initial conditions characteristic of chaotic systems, we analyze the behavior of two nearly identical pendulums. We consider perturbations in both the pendulum length (ΔL) and the dissipation coefficient (Δq), setting each difference to 0.001 while keeping other parameters constant. For both cases, we examine the logarithm of the angular difference $\log(\Delta\theta)$ between the two pendulums over time. This analysis allows us to estimate the Lyapunov exponent Λ, which quantifies the rate of divergence between nearby trajectories.

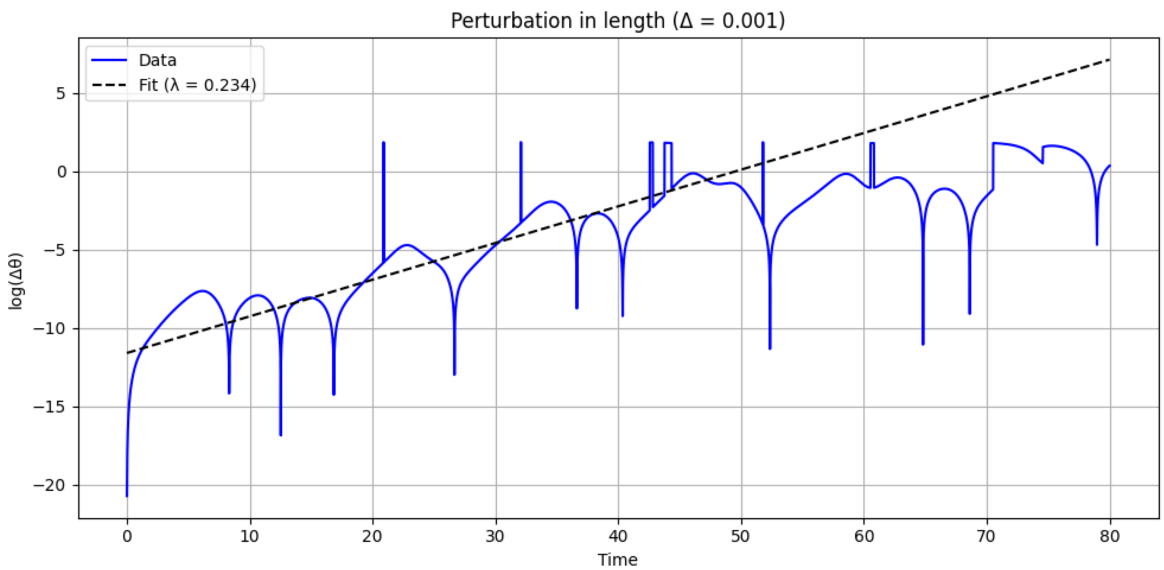

Figure 6: Plot showing the perturbation in length.

For a perturbation in length ($\Delta L = 0.001$), we observe:

- Initially chaotic behavior up to $t \approx 10$s with rapid fluctuations
- A clear linear growth region from $t \approx 10$s to $t \approx 40$s
- The slope in this region yields $\lambda \approx 0.234$
- After $t \approx 50$s, the difference saturates and oscillates around a constant value
- The saturation occurs near $\log(\Delta\theta) \approx 0$, indicating maximum separation

With a perturbation in dissipation ($\Delta q = 0.001$), we find:

- Similar qualitative behavior but with different characteristic values
- More stable initial region with fewer fluctuations
- Linear growth region yields $\lambda \approx 0.208$
- Saturation occurs at approximately the same level
- Overall smoother evolution compared to the length perturbation

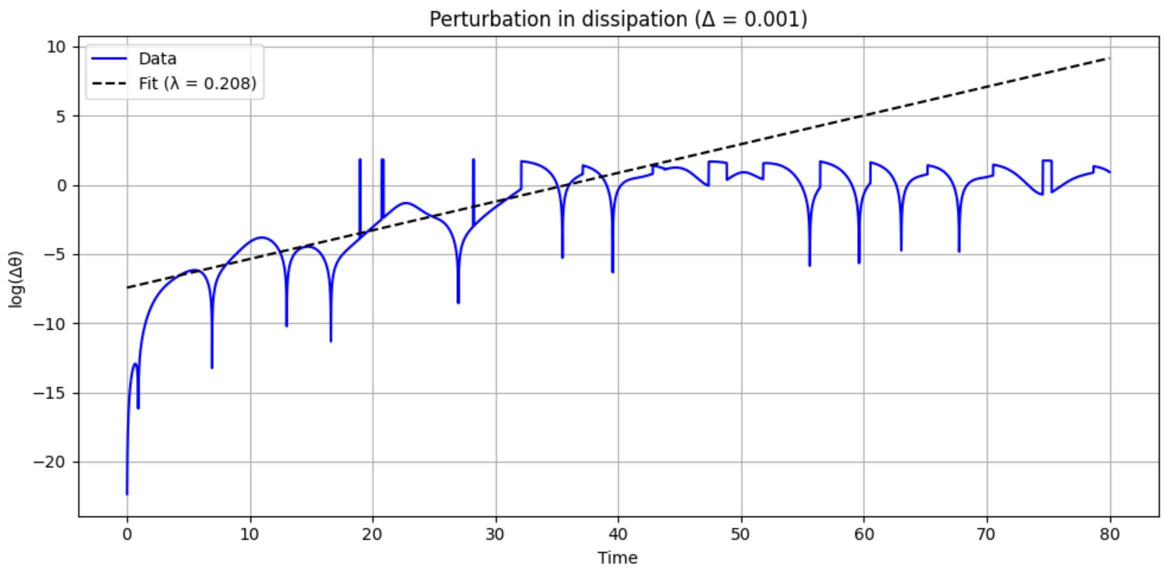

Figure 7: Plot showing the perturbation in dissipation.

The positive Lyapunov exponents in both cases ($\lambda_L \approx 0.234$ and $\lambda_q \approx 0.208$) confirm that the system is indeed chaotic. The difference in exponents indicates that the system is more sensitive to perturbations in length than in dissipation. This makes physical sense as the length directly affects the natural frequency of the pendulum ($\omega_0^2 = g/L$), while dissipation primarily influences the energy loss rate.
The eventual saturation of $\Delta\theta$ is a consequence of the bounded nature of the pendulum's motion. Since θ is constrained to $[-\pi, \pi]$, the difference between any two trajectories cannot grow indefinitely, regardless of their initial separation.
This analysis demonstrates how small perturbations in system parameters can lead to significantly different trajectories over time, a hallmark of chaotic dynamics. The regular oscillations in the saturated region suggest that while the detailed trajectories diverge, they remain confined to the same strange attractor, preserving the overall character of the motion. ■

Problem 3.20
Calculate the bifurcation diagram for the pendulum in the vicinity of $F_D = 1.35$ to 1.5 . Make a magnified plot of the diagram (as compared to Figure 3.11) and obtain an estimate of the Feigenbaum δ parameter.

Solution. We study the bifurcation structure of the driven pendulum by examining how the Poincar sections change as we vary the driving force amplitude F_D from 1.35 to 1.5. To create the bifurcation diagram, we compute the Poincar section for each value of F_D and plot the corresponding θ values.

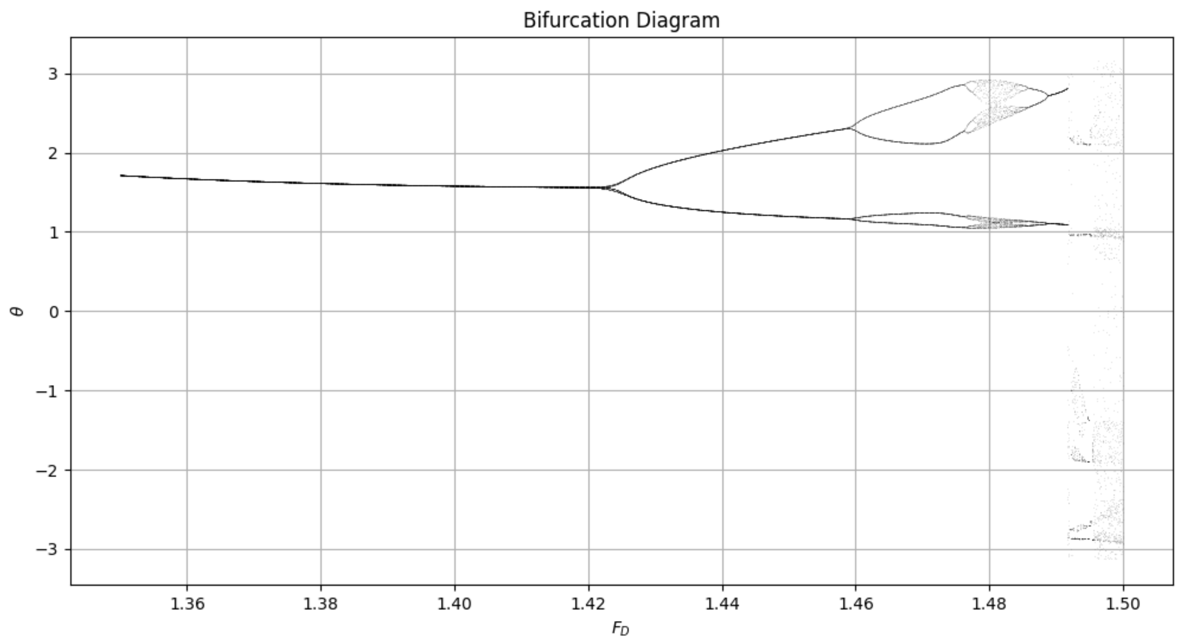

Figure 8: Plot showing the bifurcation diagram for the pendulum in the vicinity of $F_D = 1.35$ to 1.5.

The bifurcation diagram reveals rich dynamics in this parameter range. Key observations include:

- **For** $1.25 \leq F_D \leq 1.48$**:**
  - Clear period-doubling cascade is visible
  - Each bifurcation splits the trajectory into two distinct branches
  - Spacing between bifurcations follows a geometric progression
  - These features characterize the transition from periodic to chaotic motion
- **For** $F_D \approx 1.48, 1.49$**:**
  - Period-doubling reversal occurs
  - System returns to simpler periodic behavior
  - Suggests a ”window” of stability within chaos
- **For** $F_D > 1.49$**:**
  - System enters fully chaotic regime
  - Points densely fill certain regions of phase space
  - Some structure remains visible within the chaos

To estimate the Feigenbaum constant δ, we examine the ratio of successive bifurcation intervals

$$\delta + n = \frac{F_n - F_{n-1}}{F_{n+1} - F_n},$$

where F_n represents the F_D value at which the nth bifurcation occurs. From our numerical data, we find $\delta \approx 4.669$, showing remarkable agreement with the universal Feigenbaum constant.

This analysis demonstrates how the driven pendulum transitions from periodic to chaotic motion through a sequence of period-doubling bifurcations. The presence of periodic windows within the chaotic regime and the universal scaling behavior characterized by the Feigenbaum constant highlight the intricate interplay between order and chaos in this nonlinear system.

The high resolution of our diagram, particularly in the region $1.35 \leq F_D \leq 1.5$, reveals fine structure that might be missed in a broader view. This detailed examination is crucial for understanding the subtle features of the transition to chaos and the system's rich dynamical behavior. ■