PHYS 630 - Advanced Electricity and Magnetism

Student: **Ralph Razzouk**

Homework 2

Problem 1
A spherical, static charge distribution has a charge density

$$\rho(r) = \begin{cases} \left(1 - \frac{r}{r_0}\right)\rho_0, & r \le r_0 \\ 0, & r \ge r_0. \end{cases}$$

(a) Find the electric potential. (Normalize the potential so that $\phi(r_0) = 0$.) By differentiating the electric potential, find the electric field.

(b) Find the total charge $q(r)$ (charge within r), and find the electric field using Gauss' theorem. Compare the results.

(c) Find the total electromagnetic energy $\varepsilon = \int \frac{E^2}{8\pi}\,\mathrm{d}V$. Also, verify $\varepsilon = \frac{1}{2}\int \rho\phi\,\mathrm{d}V$.

Hint: Do not forget about the surface term.

Solution. (a) Since our charge distribution is spherically symmetric and has no dependence on θ and φ, then our Laplacian operator ∇^2 can be written as

$$\nabla^2\Box = \frac{1}{r^2}\partial_r\left(r^2\partial_r\Box\right).$$

Using that, the electric potential is given by

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) = -4\pi\rho.$$

Now, we will solve the differential equation for both cases: when $r \le r_0$ and when $r \ge r_0$, and make sure both solutions are equivalent at $r = r_0$.

$\boxed{\mathbf{For\ r \le r_0:}}$

$$\begin{aligned}
\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) &= -4\pi\rho \\
\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) &= -4\pi\rho_0\left(1 - \frac{r}{r_0}\right) \\
\partial_r\left(r^2\partial_r\phi\right) &= -4\pi\rho_0\left(r^2 - \frac{r^3}{r_0}\right) \\
r^2\partial_r\phi &= -4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^4}{4r_0}\right) + c_1 \\
\partial_r\phi &= -4\pi\rho_0\left(\frac{r}{3} - \frac{r^2}{4r_0}\right) + \frac{c_1}{r^2} \\
\phi(r) &= -4\pi\rho_0\left(\frac{r^2}{6} - \frac{r^3}{12r_0}\right) - \frac{c_1}{r} + c_2 \\
\phi(r) &= -\frac{2\pi\rho_0}{3}\left(r^2 - \frac{r^3}{2r_0}\right) - \frac{c_1}{r} + c_2,
\end{aligned}$$

where c_1 and c_2 are arbitrary constants. Since there is no point charge in the center, then $c_1 = 0$. Additionally, we want to have $\phi(r_0) = 0$, which means $c_2 = \frac{\pi\rho_0}{3}r_0^2$. Thus, we have

$$\phi_{r\le r_0}(r) = \frac{\pi\rho_0}{3}\left(\frac{r^3}{r_0} - 2r^2 + r_0^2\right).$$

$\boxed{\textbf{For } \mathbf{r \geq r_0}\textbf{:}}$

$$\begin{aligned}
\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) &= -4\pi\rho \\
\partial_r\left(r^2\partial_r\phi\right) &= 0 \\
r^2\partial_r\phi &= c_1 \\
\partial_r\phi &= \frac{c_1}{r^2} \\
\phi(r) &= -\frac{c_1}{r} + c_2,
\end{aligned}$$

where c_1 and c_2 are arbitrary constants. Since we are outside the charged sphere, c_1 will end up being the total charge density of the charged sphere, *i.e.*

$$-c_1 = \int_{\mathcal{V}} \rho_0 \,\mathrm{d}V = 4\pi\rho_0 \int_0^{r_0} r^2 \,\mathrm{d}r = \frac{4\pi\rho_0}{3} r_0^3.$$

Additionally, we set c_2 so that $\phi(r_0) = 0$, which gives us $c_2 = \frac{4\pi\rho_0}{3} r_0^2$. Thus, we have

$$\phi_{r\geq r_0}(r) = \frac{4\pi\rho_0}{3}\left(r_0^2 + \frac{r_0^3}{r}\right).$$

Thus,

$$\phi(r) = \begin{cases} \frac{\pi\rho_0}{3}\left(\frac{r^3}{r_0} - 2r^2 + r_0^2\right), & \text{for } r \leq r_0 \\ \frac{4\pi\rho_0}{3}\left(r_0^2 + \frac{r_0^3}{r}\right), & \text{for } r \geq r_0 \end{cases}$$

We now differentiate with respect to r using $E = -\nabla\phi$, then the electric field is

$$E(r) = \begin{cases} \frac{4\pi\rho_0}{3}\left(r - \frac{3r^2}{4r_0}\right), & \text{for } r \leq r_0 \\ \frac{4\pi\rho_0}{3}\frac{r_0^3}{r^2}, & \text{for } r \geq r_0 \end{cases}$$

(b) The total charge is given by

$$q(r) = \int_{\mathcal{V}} \rho(r')\,\mathrm{d}V'.$$

$\boxed{\textbf{For } \mathbf{r \leq r_0}\textbf{:}}$

$$\begin{aligned}
q(r) &= \int_{\mathcal{V}} \rho(r')\,\mathrm{d}V' \\
&= 4\pi\int_0^r \rho(r')r'^2\,\mathrm{d}r' \\
&= 4\pi\rho_0\int_0^r \left(1 - \frac{r'}{r_0}\right) r'^2\,\mathrm{d}r' \\
&= 4\pi\rho_0\int_0^r \left(r'^2 - \frac{r'^3}{r_0}\right)\mathrm{d}r' \\
&= 4\pi\rho_0\left[\frac{r'^3}{3} - \frac{r'^4}{4r_0}\right]_0^r \\
&= 4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^4}{4r_0}\right) \\
&= 4\pi\rho_0 r^3\left(\frac{1}{3} - \frac{r}{4r_0}\right).
\end{aligned}$$

$\boxed{\textbf{For } \mathbf{r \geq r_0}\textbf{:}}$

$$
\begin{aligned}
q(r) &= \int_{\mathcal{V}} \rho(r')\,\mathrm{d}V' \\
&= 4\pi \int_0^{r_0} \rho(r') r'^2\,\mathrm{d}r' \\
&= 4\pi\rho_0 \int_0^{r_0} \left(1 - \frac{r'}{r_0}\right) r'^2\,\mathrm{d}r' \\
&= 4\pi\rho_0 \int_0^{r_0} \left(r'^2 - \frac{r'^3}{r_0}\right) \mathrm{d}r' \\
&= 4\pi\rho_0 \left[\frac{r'^3}{3} - \frac{r'^4}{4r_0}\right]_0^{r_0} \\
&= 4\pi\rho_0 \left(\frac{r_0^3}{3} - \frac{r_0^3}{4}\right) \\
&= \frac{\pi\rho_0}{3} r_0^3.
\end{aligned}
$$

Thus,

$$
q(r) = \begin{cases} 4\pi\rho_0 r^3 \left(\frac{1}{3} - \frac{r}{4r_0}\right), & \text{for } r \leq r_0 \\ \frac{\pi\rho_0}{3} r_0^3, & \text{for } r \geq r_0 \end{cases}
$$

Gauss' theorem for electrostatics states

$$
\oint \mathbf{E} \cdot \mathrm{d}\mathbf{s} = 4\pi q(r)
$$

Finding the electric field using Gauss' theorem, we have

$\boxed{\textbf{For } \mathbf{r \leq r_0}\textbf{:}}$

$$
\begin{aligned}
\oint \mathbf{E} \cdot \mathrm{d}\mathbf{s} &= 4\pi q(r) \\
E(4\pi r^2) &= 4\pi \left[4\pi\rho_0 r^3 \left(\frac{1}{3} - \frac{r}{4r_0}\right)\right] \\
E &= 4\pi\rho_0 \left(\frac{r}{3} - \frac{r^2}{4r_0}\right).
\end{aligned}
$$

$\boxed{\textbf{For } \mathbf{r \geq r_0}\textbf{:}}$

$$
\begin{aligned}
\oint \mathbf{E} \cdot \mathrm{d}\mathbf{s} &= 4\pi q(r) \\
E(4\pi r^2) &= 4\pi \left(\frac{\pi\rho_0}{3} r_0^3\right) \\
E &= \frac{\pi\rho_0}{3} \frac{r_0^3}{r^2}.
\end{aligned}
$$

Thus,

$$
E(r) = \begin{cases} \frac{4\pi\rho_0}{3} \left(r - \frac{3r^2}{4r_0}\right), & \text{for } r \leq r_0 \\ \frac{\pi\rho_0}{3} \frac{r_0^3}{r^2}, & \text{for } r \geq r_0 \end{cases}
$$

which matches what we found in (a).

(c) The electrostatic energy is given by

$$
\begin{aligned}
\varepsilon &= \int_{\mathcal{V}} \frac{E^2}{8\pi}\,\mathrm{d}V \\
&= \frac{1}{8\pi}\int_0^\infty \int_0^\pi \int_0^{2\pi} E^2 r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{2}\int_0^\infty E^2 r^2\,\mathrm{d}r \\
&= \frac{1}{2}\left[\int_0^{r_0}\left(\frac{4\pi\rho_0}{3}\left(r-\frac{3r^2}{4r_0}\right)\right)^2 r^2\,\mathrm{d}r + \int_{r_0}^\infty\left(\frac{4\pi\rho_0}{3}\frac{r_0^3}{r^2}\right)^2 r^2\,\mathrm{d}r\right] \\
&= \frac{1}{2}\left[\frac{16\pi^2\rho_0^2}{9}\int_0^{r_0}\left(r-\frac{3r^2}{4r_0}\right)^2 r^2\,\mathrm{d}r + \frac{16\pi^2\rho_0^2}{9}\int_{r_0}^\infty\left(\frac{r_0^3}{r^2}\right)^2 r^2\,\mathrm{d}r\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\int_0^{r_0}\left(r-\frac{3r^2}{4r_0}\right)^2 r^2\,\mathrm{d}r + \int_{r_0}^\infty\left(\frac{r_0^3}{r^2}\right)^2 r^2\,\mathrm{d}r\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\int_0^{r_0}\left(r^2-\frac{6r^3}{4r_0}+\frac{9r^4}{16r_0^2}\right) r^2\,\mathrm{d}r + \int_{r_0}^\infty\left(\frac{r_0^6}{r^4}\right) r^2\,\mathrm{d}r\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\int_0^{r_0}\left(r^4-\frac{6r^5}{4r_0}+\frac{9r^6}{16r_0^2}\right)\mathrm{d}r + \int_{r_0}^\infty\left(\frac{r_0^6}{r^2}\right)\mathrm{d}r\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\left[\frac{r^5}{5}-\frac{r^6}{4r_0}+\frac{9r^7}{112r_0^2}\right]_0^{r_0} + \left[-\frac{r_0^6}{r}\right]_{r_0}^\infty\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\left(\frac{r_0^5}{5}-\frac{r_0^6}{4r_0}+\frac{9r_0^7}{112r_0^2}\right) + \left(0+\frac{r_0^6}{r_0}\right)\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left(\frac{r_0^5}{5}-\frac{r_0^5}{4}+\frac{9r_0^5}{112}+r_0^5\right) \\
&= \frac{8\pi^2\rho_0^2}{9}\left(\frac{577r_0^5}{560}\right) \\
&= \frac{577\pi^2\rho_0^2}{630}r_0^5.
\end{aligned}
$$

Additionally, we can write $E^2 = E \cdot E = E \cdot (-\nabla\phi)$. By replacing in the equation for the electrostatic energy, we get

$$
\begin{aligned}
\varepsilon &= \int_{\mathcal{V}} \frac{E^2}{8\pi}\,\mathrm{d}V \\
&= -\frac{1}{8\pi}\int_{\mathcal{V}} \mathbf{E}\cdot\nabla\phi\,\mathrm{d}V \\
&= -\frac{1}{8\pi}\int_{\mathcal{V}} \nabla\cdot(\phi\mathbf{E})\,\mathrm{d}V + \frac{1}{8\pi}\int_{\mathcal{V}}\phi(\nabla\cdot\mathbf{E})\,\mathrm{d}V \\
&= \frac{1}{8\pi}\oint_{\mathcal{S}}\phi\mathbf{E}\cdot\mathrm{d}\mathbf{S} + \frac{1}{2}\int_{\mathcal{V}}\phi\rho\,\mathrm{d}V.
\end{aligned}
$$

We aim to show that $\varepsilon = \frac{1}{2}\int_{\mathcal{V}}\phi\rho\,\mathrm{d}V$, *i.e.* that $\frac{1}{8\pi}\oint_{\mathcal{S}}\phi\mathbf{E}\cdot\mathrm{d}\mathbf{S} = 0$.

We have

$$
\begin{aligned}
\frac{1}{2}\int_{\mathcal{V}} \phi\rho \,\mathrm{d}V &= \frac{4\pi}{2}\left[\int_0^{r_0} \phi\rho r^2 \,\mathrm{d}r + \int_{r_0}^{\infty} \phi\rho r^2 \,\mathrm{d}r\right] \\
&= 2\pi\left[\int_0^{r_0} \frac{\pi\rho_0}{3}\left(\frac{r^3}{r_0} - 2r^2 + r_0^2\right)\left(1 - \frac{r}{r_0}\right)\rho_0 r^2 \,\mathrm{d}r + 0\right] \\
&= \frac{2\pi^2\rho_0^2}{3}\int_0^{r_0}\left(\frac{r^3}{r_0} - 2r^2 + r_0^2\right)\left(1 - \frac{r}{r_0}\right) r^2 \,\mathrm{d}r \\
&= \frac{2\pi^2\rho_0^2}{3}\int_0^{r_0}\left(\frac{r^5}{r_0} - 2r^4 + r_0^2 r^2 - \frac{r^6}{r_0^2} + \frac{2r^5}{r_0} - r_0 r^3\right)\mathrm{d}r \\
&= \frac{2\pi^2\rho_0^2}{3}\int_0^{r_0}\left(\frac{3r^5}{r_0} - 2r^4 + r_0^2 r^2 - \frac{r^6}{r_0^2} - r_0 r^3\right)\mathrm{d}r \\
&= \frac{2\pi^2\rho_0^2}{3}\left[\left(\frac{3r^6}{6r_0} - \frac{2r^5}{5} + \frac{r_0^2 r^3}{3} - \frac{r^7}{7r_0^2} - \frac{r_0 r^4}{4}\right)\right]_0^{r_0} \\
&= \frac{2\pi^2\rho_0^2}{3}\left(\frac{3}{6} - \frac{2}{5} + \frac{1}{3} - \frac{1}{7} - \frac{1}{4}\right) r_0^5 \\
&= \frac{2\pi^2\rho_0^2}{3}\left(\frac{17}{420} r_0^5\right) \\
&= \frac{17\pi^2\rho_0^2}{630} r_0^5.
\end{aligned}
$$

■

Problem 2
Consider a charge q_0 surrounded by a cloud with charge density

$$\rho_e = -\frac{q_0}{\pi a^3} \mathrm{e}^{-\frac{2r}{a}}$$

Find the total charge of the system, the total potential $\phi(r)$, and the total electric field $\mathbf{E}(r)$. Plot the total potential $\phi(r)$ and compare with Coulomb.
Hint:

$$\int x^2 \mathrm{e}^{-\frac{x}{a}}\,\mathrm{d}x = -a\mathrm{e}^{-\frac{x}{a}}\left(2x^2 + 2ax + a^2\right).$$

Solution. The total charge Q enclosed is given by

$$\begin{aligned}
Q &= q_0 + \int_{\mathcal{V}} \rho_e \,\mathrm{d}V \\
&= q_0 + \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho_e r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= q_0 + 4\pi \int_0^{\infty} \left(-\frac{q_0}{\pi a^3}\mathrm{e}^{-\frac{2r}{a}}\right) r^2\,\mathrm{d}r \\
&= q_0 - \frac{q_0}{a^3}\int_0^{\infty} (2r)^2 \mathrm{e}^{-\frac{2r}{a}}\,\mathrm{d}r \\
&= q_0 - \frac{q_0}{a^3}\left|-a\left(2x^2+2ax+a^2\right)\mathrm{e}^{-\frac{2r}{a}}\right|_0^{\infty} \\
&= q_0 - \frac{q_0}{a^2}(a^2) \\
&= q_0 - q_0 \\
&= 0.
\end{aligned}$$

The total electric potential $\phi(r)$ of the system is given by

$$\begin{aligned}
\phi(r) &= \frac{q_0}{r} + \int_V \frac{\rho_c(r')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' \\
&= \frac{q_0}{r} + \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\left(-\frac{q_0}{\pi a^3}\mathrm{e}^{-\frac{2r'}{a}}\right)}{\sqrt{r^2+r'^2-2rr'\cos(\theta)}} r'^2 \sin(\theta)\,\mathrm{d}r'\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{q_0}{r} - 2\pi\int_0^{\pi}\int_0^{\infty} \frac{q_0}{\pi a^3}\frac{\mathrm{e}^{-\frac{2r'}{a}}}{\sqrt{r^2+r'^2-2rr'\cos(\theta)}} r'^2\sin(\theta)\,\mathrm{d}r'\,\mathrm{d}\theta \\
&= \frac{q_0}{r} - \frac{2q_0}{a^3}\int_0^{\pi}\int_0^{\infty} \frac{r'^2\sin(\theta)\mathrm{e}^{-\frac{2r'}{a}}}{\sqrt{r^2+r'^2-2rr'\cos(\theta)}}\,\mathrm{d}r'\,\mathrm{d}\theta \\
&= \frac{q_0}{r} - \frac{2q_0}{a^3}\left(\frac{a^2}{2}\right) \\
&= \frac{q_0}{r} - \frac{q_0}{a} \\
&= q_0\left(\frac{1}{r} - \frac{1}{a}\right).
\end{aligned}$$

The total electric field $\mathbf{E}(r)$ is given by

$$\begin{aligned}
E(r) &= -\nabla\phi(r) \\
&= -q_0\nabla\left(\frac{1}{r}-\frac{1}{a}\right) \\
&= \frac{q_0}{r^2}.
\end{aligned}$$

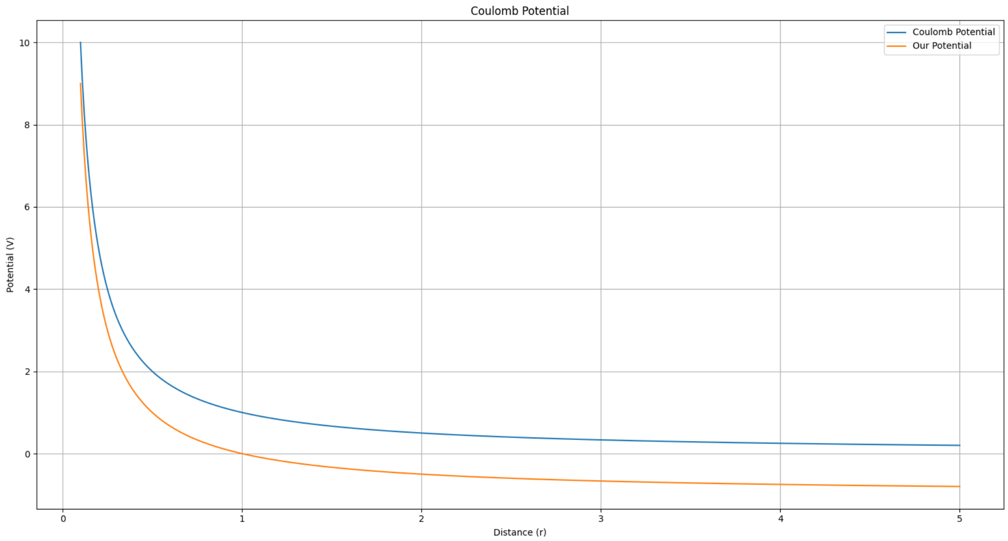

Figure 1: Plot of Our Potential vs. the Coulomb Potential

■