PHYS 603 - Methods of Theoretical Physics III

Lie Algebras in Particle Physics by *H. Georgi*
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Homework 3

Problem 1
Consider the group $SL(2, \mathbf{C})$ of complex 2×2 matrices with unit determinant and the transformation of an Hermitian matrix

$$H = \begin{pmatrix} ct+z & x-\mathrm{i}y \\ x+\mathrm{i}y & ct-z \end{pmatrix}$$

into another Hermitian matrix H' via

$$H' = M^\dagger H M. \tag{1}$$

where $M \in SL(2, \mathbb{C})$.

(a) Verify that this transformation preserves the value of $c^2t^2 - x^2 - y^2 - z^2$.

(b) Consider the 1-parametric subgroup of $SL(2, \mathbb{C})$ formed by matrices of the form

$$B(b) = \begin{pmatrix} \mathrm{e}^b & 0 \\ 0 & \mathrm{e}^{-b} \end{pmatrix}$$

where b is a real parameter. Verify that, for such B, $M = B$ in (1) describes a Lorentz transformation (boost) in the z direction and find the velocity of the reference frame that corresponds to such a boost.

(c) Verify that the multiplication law $B(b)B(b') = B(b+b')$ corresponds to the relativistic law of addition of velocities, for the special case when both velocities are in the z direction.

Solution. (a) Let M be an arbitrary $SL(2, \mathbb{C})$ matrix. We can write

$$\begin{aligned} H' &= M^\dagger H M \\ &= M^\dagger \begin{pmatrix} ct+z & x-\mathrm{i}y \\ x+\mathrm{i}y & ct-z \end{pmatrix} M. \end{aligned}$$

Since H' is Hermitian, we can write it in the same form with transformed coordinates

$$H' = \begin{pmatrix} ct'+z' & x'-\mathrm{i}y' \\ x'+\mathrm{i}y' & ct'-z' \end{pmatrix}.$$

Now we note that $\det(H) = (ct)^2 - x^2 - y^2 - z^2$ and similarly for H'. Then

$$\begin{aligned} \det(H') &= \det(M^\dagger H M) \\ &= \det(M^\dagger)\det(H)\det(M) \\ &= \det(H), \end{aligned}$$

where we used that $\det(M) = 1$ for $M \in SL(2, \mathbb{C})$. Therefore, the given transformation preserves the value of

$$(ct')^2 - (x')^2 - (y')^2 - (z')^2 = c^2t^2 - x^2 - y^2 - z^2.$$

(b) For the boost matrix $B(b)$, we have

$$\begin{aligned} H' &= B^\dagger(b) H B(b) \\ &= \begin{pmatrix} \mathrm{e}^b & 0 \\ 0 & \mathrm{e}^{-b} \end{pmatrix} \begin{pmatrix} ct+z & x-\mathrm{i}y \\ x+\mathrm{i}y & ct-z \end{pmatrix} \begin{pmatrix} \mathrm{e}^b & 0 \\ 0 & \mathrm{e}^{-b} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{2b}(ct+z) & \mathrm{e}^0(x-\mathrm{i}y) \\ \mathrm{e}^0(x+\mathrm{i}y) & \mathrm{e}^{-2b}(ct-z) \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$\begin{cases} ct'+z' &= \mathrm{e}^{2b}(ct+z) \\ ct'-z' &= \mathrm{e}^{-2b}(ct-z) \end{cases}$$

Adding and subtracting these equations gives

$$\begin{aligned} ct' &= ct\cosh(2b) + z\sinh(2b), \\ z' &= ct\sinh(2b) + z\cosh(2b). \end{aligned}$$

This is indeed a Lorentz boost in the z-direction with velocity

$$v = c\tanh(2b).$$

(c) Consider two successive boosts $B(b)B(b')$. From part (b), we know the velocities are

$$\begin{aligned} v_1 &= c\tanh(2b), \\ v_2 &= c\tanh(2b'). \end{aligned}$$

The composition property $B(b)B(b') = B(b+b')$ means the resulting velocity is

$$v = c\tanh(2(b+b')) = c\tanh(2b+2b').$$

Using the hyperbolic tangent addition formula

$$\tanh(A+B) = \frac{\tanh A + \tanh B}{1+\tanh A \tanh B},$$

we get

$$v = c\frac{\frac{v_1}{c}+\frac{v_2}{c}}{1+\frac{v_1 v_2}{c^2}} = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}.$$

which is precisely the relativistic velocity addition formula for parallel velocities.

$\blacksquare$

Problem 2
Consider the group $SL(2,\mathbb{R})$, parametrized as follows

$$M(x,y,\theta) = \begin{pmatrix} \sqrt{1+x^2+y^2}-x & y \\ y & \sqrt{1+x^2+y^2}+x \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Let us rename the parameters x, y, θ into α_a, $a = 1, 2, 3$, as follows: $x \equiv \alpha_1, y \equiv \alpha_2, \theta \equiv \alpha_3$.

(a) Find the generators X_a, $a = 1, 2, 3$, corresponding to this parametrization. Please do not include i in the definition of the generators, so they are real matrices.

(b) Compute all nonzero structure constants $f_{ab}{}^c$ of the algebra (defined here by $[X_a, X_b] = \sum_c f_{ab}{}^c X_c$).

(c) Consider the adjoint representation of the algebra, in which the generators are represented by operators $D(X_a)$ acting as follows:

$$D(X_a)\,|X_b\rangle = |[X_a, X_b]\rangle\,.$$

Compute the matrices of $T_a \equiv D(X_a)$ in the basis of $|X_b\rangle$. As a check of your calculation, verify that commutation relations among these matrices correctly represent the algebra.

Solution. (a) The generators can be found by taking derivatives at the identity

$$X_a = \left.\frac{\partial M(\alpha_1,\, \alpha_2,\, \alpha_3)}{\partial \alpha_a}\right|_{\alpha_1=\alpha_2=\alpha_3=0}.$$

- **For X_1:** we need the derivative with respect to x evaluated at zero, giving us

$$X_1 = \left.\frac{\partial M}{\partial \alpha_1}\right|_{\alpha_1=0} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- **For X_2:**

$$X_2 = \left.\frac{\partial M}{\partial \alpha_2}\right|_{\alpha_2=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- **For X_3:** we get the derivative of the rotation matrix, giving us

$$X_3 = \left.\frac{\partial M}{\partial \alpha_3}\right|_{\alpha_3=0} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(b) Let's compute the commutators directly. We have

$$[X_1, X_2] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} = -2X_3$$

$$[X_1, X_3] = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} = -2X_2$$

$$[X_2, X_3] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} = 2X_1$$

Therefore the non-zero structure constants are

$$\begin{cases} f_{12}{}^3 & = -2 \\ f_{13}{}^2 & = -2 \\ f_{23}{}^1 & = 2 \end{cases}$$

and their negatives with swapped indices due to antisymmetry: $f_{ab}{}^c = -f_{ba}{}^c$.

(c) In the adjoint representation, the matrices T_a have elements

$$(T_a)_{bc} = f_{ab}{}^{c}.$$

Therefore, we have

- **For X_1:** The action of operator $D(X_1)$ on X_α is

$$\begin{aligned} D(X_1)\,|X_1\rangle &= |[X_1, X_1]\rangle = 0 \\ D(X_1)\,|X_2\rangle &= |[X_1, X_2]\rangle = -2\,|X_3\rangle \\ D(X_1)\,|X_3\rangle &= |[X_1, X_3]\rangle = -2\,|X_2\rangle \end{aligned}$$

  From that, we have

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}.$$

- **For X_2:** The action of operator $D(X_2)$ on X_α is

$$\begin{aligned} D(X_2)\,|X_1\rangle &= |[X_2, X_1]\rangle = 2\,|X_3\rangle \\ D(X_2)\,|X_2\rangle &= |[X_2, X_2]\rangle = 0 \\ D(X_2)\,|X_3\rangle &= |[X_2, X_3]\rangle = 2\,|X_1\rangle \end{aligned}$$

  From that, we have

$$T_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

- **For X_3:** The action of operator $D(X_3)$ on X_α is

$$\begin{aligned} D(X_3)\,|X_1\rangle &= |[X_3, X_1]\rangle = 2\,|X_2\rangle \\ D(X_3)\,|X_2\rangle &= |[X_3, X_2]\rangle = -2\,|X_1\rangle \\ D(X_3)\,|X_3\rangle &= |[X_3, X_3]\rangle = 0 \end{aligned}$$

  From that, we have

$$T_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To verify these satisfy the same commutation relations, we compute the following commutators:

$$\begin{aligned} [T_1, T_2] = T_1T_2 - T_2T_1 &= \begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ -4 & 0 & -2 \\ 0 & 0 & -2 \end{pmatrix} = -2T_3 \\ [T_1, T_3] = T_1T_3 - T_3T_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -4 \\ 0 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix} = -2T_2 \\ [T_2, T_3] = T_2T_3 - T_3T_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{pmatrix} = 2T_1 \end{aligned}$$

Therefore, the commutation relations of the Lie algebra found in (b) are satisfied.

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