PHYS 662 - Quantum Field Theory I

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Contents

Homework 1

Problem 1 - Translation Invariance on a Lattice

1. Consider the following generalization of the model we discussed in class

$$\mathcal{H} = \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} \lambda_{ij} q_i q_j,$$

where λ_{ij} depends only on $i - j$.

   (a) Show that the model is translationally invariant.

   (b) Find the normal modes.

2. Consider the case where the index i runs over a two-dimensional lattice and λ_{ij} is non-zero only for nearest neighbors and is independent of location

$$\lambda_{ij} = \delta_{j,j+1}\lambda.$$

   (a) Find the normal modes.

   (b) Find the dispersion relation.

Solution. 1. (a) The model is translationally invariant if the map

$$T : \begin{pmatrix} x_i \\ p_i \end{pmatrix} \to \begin{pmatrix} x_{i+1} \\ p_{i+1} \end{pmatrix}$$

yields the same Hamiltonian. We have

$$\begin{aligned} T(\mathcal{H}) &= T\left(\sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} \lambda_{ij} q_i q_j\right) \\ &= \sum_i \frac{p_{i+1}^2}{2m} + \frac{1}{2}\sum_{i,j} \lambda_{i+1,j+1} q_{i+1} q_{j+1} \\ &= \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} \lambda_{i,j} q_i q_j, \end{aligned}$$

where $\lambda_{i+1,j+1}$ is only dependent on $(i+1) - (j+1) = i - j$.

Thus, our model is translationally invariant.

(b) Consider the Fourier modes

$$q_i = \frac{1}{\sqrt{N}}\sum_k \mathrm{e}^{\mathrm{i}kx_k} q_k \qquad \Longleftrightarrow \qquad q_k = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}kx_i} q_i$$

$$p_i = \frac{1}{\sqrt{N}}\sum_k \mathrm{e}^{\mathrm{i}kx_k} p_k \qquad \Longleftrightarrow \qquad p_k = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}kx_i} p_i$$

We will find the Fourier modes of the momentum part first

$$
\begin{aligned}
\sum_i \frac{p_i^2}{2m} &= \frac{1}{2m}\sum_{i=1}^{N}\left(\frac{1}{\sqrt{N}}\sum_k \mathrm{e}^{\mathrm{i}kx_i}p_k\right)\left(\frac{1}{\sqrt{N}}\sum_{k'} \mathrm{e}^{\mathrm{i}k'x_i}p_{k'}\right)\\
&= \frac{1}{2mN}\sum_{i=1}^{N}\sum_k\sum_{k'} \mathrm{e}^{\mathrm{i}(k+k')x_i}p_k p_{k'}\\
&= \frac{N}{2mN}\sum_k\sum_{k'}\delta(k+k')p_k p_{k'}\\
&= \frac{1}{2m}\sum_k p_k p_{-k}.
\end{aligned}
$$

Now, we will find the Fourier modes of the position part. We have

$$
\begin{aligned}
\frac{1}{2}\sum_{i,j}\lambda_{ij}q_i q_j &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}q_i q_j\\
&= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\lambda_{ij}\left(\frac{1}{\sqrt{N}}\sum_k \mathrm{e}^{\mathrm{i}kx_i}q_k\right)\left(\frac{1}{\sqrt{N}}\sum_{k'} \mathrm{e}^{\mathrm{i}k'x_j}q_{k'}\right)\\
&= \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_k\sum_{k'}\lambda_{ij}\mathrm{e}^{\mathrm{i}kx_i}\mathrm{e}^{\mathrm{i}k'x_j}q_k q_{k'}\\
&= \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_k\sum_{k'}\lambda_{ij}\mathrm{e}^{\mathrm{i}kx_i}\mathrm{e}^{\mathrm{i}k'x_i}\mathrm{e}^{-\mathrm{i}k'x_i}\mathrm{e}^{\mathrm{i}k'x_j}q_k q_{k'}\\
&= \frac{1}{2N}\sum_{i=1}^{N}\sum_k\sum_{k'}\mathrm{e}^{\mathrm{i}(k+k')x_i}q_k q_{k'}\sum_{j=1}^{N}\lambda(j-i)\mathrm{e}^{\mathrm{i}k'(x_j-x_i)}\\
&= \frac{1}{2N}\sum_{i=1}^{N}\sum_k\sum_{k'}\mathrm{e}^{\mathrm{i}(k+k')x_i}q_k q_{k'}\tilde{\lambda}(-k')\\
&= \frac{N}{2N}\sum_k\sum_{k'}\delta(k+k')q_k q_{k'}\tilde{\lambda}(-k')\\
&= \frac{1}{2}\sum_k\tilde{\lambda}(k)q_k q_{-k}.
\end{aligned}
$$

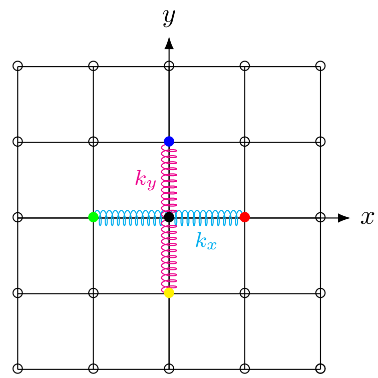
As a worded explanation of what happened: we Fourier transformed our coordinates to momentum space, inserted an identity $1 = \mathrm{e}^{\mathrm{i}k'x_i}\mathrm{e}^{-\mathrm{i}k'x_i}$, factored x_i's in the exponents in one exponential and took the difference of x_j and x_i in another. Then, the term $\sum_{j=1}^{N}\lambda(j-i)\mathrm{e}^{\mathrm{i}k'(x_j-x_i)}$ is the Fourier transform of the λ_{ij} we started with. Finally, the Fourier expansion of a δ-function is defined by $\sum_{i=1}^{N}\mathrm{e}^{\mathrm{i}(k+k')x_i} = N\delta(k+k')$, which leads us to the final result.
Finally, our Fourier transformed Hamiltonian is given by

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2m}\sum_k p_k p_{-k} + \frac{1}{2}\sum_k\tilde{\lambda}(k)q_k q_{-k}\\
&= \frac{1}{2m}\sum_k p_k p_{-k} + \frac{m}{2}\sum_k\omega_k^2 q_k q_{-k},
\end{aligned}
$$

where

$$
\omega_k^2 = \frac{1}{m}\tilde{\lambda}(k) = \frac{1}{m}\sum_{j=1}^{N}\lambda(j-i)\mathrm{e}^{-\mathrm{i}k(x_j-x_i)}.
$$

2. (a) The index i running over a two-dimensional lattice is the same as two indices i and j running each on a one-dimensional line. We have that $\lambda_{i,j}$ is non-zero only for nearest neighbor interactions, which means it is interacting either with the particle to its right, to its left, to the top, or to the bottom, and not multiple at the same time. The figure below should help illustrate that.



From these facts, we can simplify our Hamiltonian to

$$
\begin{aligned}
\mathcal{H} &= \sum_{i,j} \frac{p_{i,j}^2}{2m} + \frac{1}{2} \sum_{i,j,k,\ell} \lambda_{k,\ell} q_{i,j} q_{k,\ell} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{p_{i,j}^2}{2m} + \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} \sum_{k=i-1}^{i+1} \sum_{\ell=j-1}^{j+1} \lambda_{k,\ell} q_{i,j} q_{k,\ell} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{p_{i,j}^2}{2m} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda \left(q_{i,j} q_{i+1,j} + q_{i,j} q_{i-1,j} + q_{i,j} q_{i,j+1} + q_{i,j} q_{i,j-1} \right),
\end{aligned}
$$

where $\lambda_{k,\ell}$ refers to the spring constant **between** $q_{i,j}$ and $q_{k,\ell}$, since we will always center our point at (i,j), so we relieve ourselves from this redundancy in subscript notation. Additionally, we note that when $i = k$ and $j = \ell$, then $\lambda_{i,j} = 0$ since that refers to the constant between a point and itself, and that $\lambda_{i\pm1,j\pm1} = 0$.

Consider the Fourier modes

$$
\begin{aligned}
q_{i,j} &= \frac{1}{N} \sum_{k_x} \sum_{k_y} \mathrm{e}^{\mathrm{i}(k_x x_i + k_y y_j)} q_{k_x,k_y} \qquad \Longleftrightarrow \qquad q_{k_x,k_y} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}(k_x x_i + k_y y_j)} q_{i,j} \\
p_{i,j} &= \frac{1}{N} \sum_{k_x} \sum_{k_y} \mathrm{e}^{\mathrm{i}(k_x x_i + k_y y_j)} p_{k_x,k_y} \qquad \Longleftrightarrow \qquad p_{k_x,k_y} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}(k_x x_i + k_y y_j)} p_{i,j}
\end{aligned}
$$

where we assume that $k_x = k_y$, but we will keep the subscripts there for now.

We now apply the Fourier modes to $p_{i,j}p_{m,n}$, getting

$$
\begin{aligned}
\sum_{i,j}\frac{p_{i,j}^2}{2m} &= \frac{1}{2m}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{1}{N}\sum_{k_x}\sum_{k_y}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}p_{k_x,k_y}\right)\left(\frac{1}{N}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_x'x_i+k_y'y_j)}p_{k_x',k_y'}\right)\\
&= \frac{1}{2mN^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i+k_y'y_j)}p_{k_x,k_y}p_{k_x',k_y'}\\
&= \frac{1}{2mN^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_x+k_x')x_i}\mathrm{e}^{\mathrm{i}(k_y+k_y')y_j}p_{k_x,k_y}p_{k_x',k_y'}\\
&= \frac{1}{2mN^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\left(\sum_{i=1}^{N}\mathrm{e}^{\mathrm{i}(k_x+k_x')x_i}\right)\left(\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}(k_y+k_y')y_j}\right)p_{k_x,k_y}p_{k_x',k_y'}\\
&= \frac{1}{2mN^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\left(N\delta(k_x+k_x')\right)\left(N\delta(k_y+k_y')\right)p_{k_x,k_y}p_{k_x',k_y'}\\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y}p_{k_x,k_y}p_{-k_x,-k_y}.
\end{aligned}
$$

After applying the Fourier modes to $q_{i,j}q_{m,n}$, we get

$$
\begin{aligned}
q_{i,j}q_{m,n} &= \left(\frac{1}{N}\sum_{k_x}\sum_{k_y}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}q_{k_x,k_y}\right)\left(\frac{1}{N}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_x'x_i'+k_y'y_j')}q_{k_x',k_y'}\right)\\
&= \frac{1}{N^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i'+k_y'y_j')}q_{k_x,k_y}q_{k_x',k_y'}.
\end{aligned}
$$

We will now insert an identity, defined by

$$
\mathrm{e}^{\mathrm{i}(k_x'x_i+k_y'y_j)}\mathrm{e}^{-\mathrm{i}(k_x'x_i+k_y'y_j)} = 1.
$$

$$
\begin{aligned}
q_{i,j}q_{m,n} &= \frac{1}{N^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i'+k_y'y_j')}q_{k_x,k_y}q_{k_x',k_y'}\\
&= \frac{1}{N^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i+k_y'y_j)}\mathrm{e}^{-\mathrm{i}(k_x'x_i+k_y'y_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i'+k_y'y_j')}q_{k_x,k_y}q_{k_x',k_y'}\\
&= \frac{1}{N^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}\left[(k_x+k_x')x_i+(k_y+k_y')y_j\right]}\mathrm{e}^{\mathrm{i}(k_x'(x_i'-x_i)+k_y'(y_j'-y_j))}q_{k_x,k_y}q_{k_x',k_y'}.
\end{aligned}
$$

We can notice that, if and when this is done for the entire Hamiltonian, the addition summations over i and j will make the first two exponential terms turn in δ-functions for k_x and k_y, which will end up giving us $k_x' = -k_x$ and $k_y' = -k_y$.

For the second exponential term, the x_i' will, in reality, be either x_{i+1} pr x_{i-1}. The points (i,j) and $(i\pm1,j)$ or $(i,j\pm1)$ are at a distance of ϵ away from each other. This means, for $q_{i,j}q_{i\pm1,j}$, we can write $x_{i\pm1} = x_i \pm \epsilon$, and for $q_{i,j}q_{i,j\pm1}$, we can write $y_{j\pm1} = y_j \pm \epsilon$.

After doing that to all the pairs of points we have, we get

$$
\begin{aligned}
H &= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} \\
&\quad + \frac{\lambda}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_x}\sum_{k_y}\sum_{k'_x}\sum_{k'_y} \mathrm{e}^{\mathrm{i}\left[(k_x+k'_x)x_i+(k_y+k'_y)y_j\right]}\left[\mathrm{e}^{\mathrm{i}\left[k'_x(x_{i+1}-x_i)+k'_y(y_j-y_j)\right]}q_{k_x,k_y}q_{k'_x,k'_y}\right. \\
&\qquad + \mathrm{e}^{\mathrm{i}\left[k'_x(x_{i-1}-x_i)+k'_y(y_j-y_j)\right]}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad + \mathrm{e}^{\mathrm{i}\left[k'_x(x_i-x_i)+k'_y(y_{j+1}-y_j)\right]}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad \left. + \mathrm{e}^{\mathrm{i}\left[k'_x(x_i-x_i)+k'_y(y_{j-1}-y_j)\right]}q_{k_x,k_y}q_{k'_x,k'_y}\right] \\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} \\
&\quad + \frac{\lambda}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_x}\sum_{k_y}\sum_{k'_x}\sum_{k'_y} \mathrm{e}^{\mathrm{i}\left[(k_x+k'_x)x_i+(k_y+k'_y)y_j\right]}\left[\mathrm{e}^{\mathrm{i}k'_x\epsilon}q_{k_x,k_y}q_{k'_x,k'_y}\right. \\
&\qquad + \mathrm{e}^{-\mathrm{i}k'_x\epsilon}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad + \mathrm{e}^{\mathrm{i}k'_y\epsilon}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad \left. + \mathrm{e}^{-\mathrm{i}k'_y\epsilon}q_{k_x,k_y}q_{k'_x,k'_y}\right] \\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} \\
&\quad + \frac{\lambda}{2N^2}\sum_{k_x}\sum_{k_y}\sum_{k'_x}\sum_{k'_y}\left(\sum_{i=1}^{N}\mathrm{e}^{\mathrm{i}(k_x+k'_x)x_i}\right)\left(\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}(k_y+k'_y)y_j}\right) \\
&\qquad \times q_{k_x,k_y}q_{k'_x,k'_y}\left[\mathrm{e}^{\mathrm{i}k'_x\epsilon}+\mathrm{e}^{-\mathrm{i}k'_x\epsilon}+\mathrm{e}^{\mathrm{i}k'_y\epsilon}+\mathrm{e}^{-\mathrm{i}k'_y\epsilon}\right] \\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} \\
&\quad + \frac{\lambda}{2N^2}\sum_{k_x}\sum_{k_y}\sum_{k'_x}\sum_{k'_y}\left(N\delta(k_x+k'_x)\right)\left(N\delta(k_y+k'_y)\right)q_{k_x,k_y}q_{k'_x,k'_y}\left[2\cos(k'_x\epsilon)+2\cos(k'_y\epsilon)\right] \\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} + \frac{\lambda}{2}\sum_{k_x}\sum_{k_y} q_{k_x,k_y}q_{-k_x,-k_y}\left[2\cos(k_x\epsilon)+2\cos(k_y\epsilon)\right] \\
&= \sum_{k_x}\sum_{k_y}\left[\frac{1}{2m}p_{k_x,k_y}p_{-k_x,-k_y} + \frac{1}{2}2\lambda\left(\cos(k_x\epsilon)+\cos(k_y\epsilon)\right)q_{k_x,k_y}q_{-k_x,-k_y}\right] \\
&= \sum_{k_x}\sum_{k_y}\left[\frac{1}{2m}p_{k_x,k_y}p_{-k_x,-k_y} + \frac{1}{2}m\omega_k^2 q_{k_x,k_y}q_{-k_x,-k_y}\right]
\end{aligned}
$$

From this, we can say that

$$\omega_k^2 = \frac{2\lambda}{m}\left(\cos(k_x\epsilon)+\cos(k_y\epsilon)\right) = \omega_{k_x}^2 + \omega_{k_y}^2,$$

where

$$\omega_{k_x}^2 = \frac{2\lambda}{m}\cos(k_x\epsilon) \quad \text{and} \quad \omega_{k_y}^2 = \frac{2\lambda}{m}\cos(k_y\epsilon).$$

(b) To find the dispersion relation, we must plug in the normal modes in the equation of motion of the system. To do that, we first have to find the Hamilton-Jacobi equations by

$$\dot{q}_k = \{q_k, \mathcal{H}\},$$
$$\dot{p}_k = \{p_k, \mathcal{H}\}.$$

The Poisson bracket is given by

$$\{f, g\} = \sum_{\ell=1}^{N} \left(\frac{\partial f}{\partial q_\ell} \frac{\partial g}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell} \frac{\partial g}{\partial q_\ell} \right).$$

Finding $\dot{q}_k$: We have

$$\begin{aligned}
\dot{q}_k &= \{q_k, \mathcal{H}\} \\
&= \sum_{\ell=1}^{N} \left(\frac{\partial q_k}{\partial q_\ell} \frac{\partial \mathcal{H}}{\partial p_\ell} - \frac{\partial q_k}{\partial p_\ell} \frac{\partial \mathcal{H}}{\partial q_\ell} \right) \\
&= \sum_{\ell=1}^{N} \left(\delta_{k\ell} \frac{p_{-\ell}}{m} - 0 \right) \\
&= \frac{p_{-k}}{m}.
\end{aligned}$$

Finding $\dot{p}_k$: We have

$$\begin{aligned}
\dot{p}_k &= \{p_k, \mathcal{H}\} \\
&= \sum_{\ell=1}^{N} \left(\frac{\partial p_k}{\partial q_\ell} \frac{\partial \mathcal{H}}{\partial p_\ell} - \frac{\partial p_k}{\partial p_\ell} \frac{\partial \mathcal{H}}{\partial q_\ell} \right) \\
&= \sum_{\ell=1}^{N} (0 - \delta_{k\ell} \lambda(\cos(k_x \epsilon) + \cos(k_y \epsilon)) q_{-\ell}) \\
&= -\lambda(\cos(k_x \epsilon) + \cos(k_y \epsilon)) q_{-k}.
\end{aligned}$$

Now that we have these expressions, we can use them to reach a second-order differential equation and solve it. From the equation of $\dot{p}_k$, we have

$$\dot{p}_{-k} = -\lambda(\cos(k_x \epsilon) + \cos(k_y \epsilon)) q_k,$$

and then, using a rearrangement of the equation for $\dot{q}_k$, we have

$$\ddot{q}_k = -\frac{k}{m}(\cos(k_x \epsilon) + \cos(k_y \epsilon)) q_k.$$

Plugging in a solution of the form

$$\begin{aligned}
q_k &= \sum_\omega \mathrm{e}^{-\mathrm{i}\omega t} q_{k,\omega} \\
\implies \dot{q}_k &= \sum_\omega (-\mathrm{i}\omega) \mathrm{e}^{-\mathrm{i}\omega t} q_{k,\omega} \\
\implies \ddot{q}_k &= -\sum_\omega \omega^2 \mathrm{e}^{-\mathrm{i}\omega t} q_{k,\omega} \\
&= -\sum_\omega \frac{\lambda}{m} (\cos(k_x \epsilon) + \cos(k_y \epsilon)) \mathrm{e}^{-\mathrm{i}\omega t} q_{k,\omega}.
\end{aligned}$$

Therefore, the dispersion relation is given by

$$\omega^2 - \frac{\lambda}{m}(\cos(k_x \epsilon) + \cos(k_y \epsilon)) = 0.$$

■

Problem 2 - One-dimensional Heisenberg Model
Consider a one-dimensional array of ferromagnets with $\vec{S}_i$ a vector degree of freedom on each site. Take the Hamiltonian

$$\mathcal{H} = -J\sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

with $J > 0$. The operators $\vec{S}_i$ satisfy the $SU(2)$ algebra

$$\left[\left(\vec{S}_i\right)_\alpha, \left(\vec{S}_j\right)_\beta\right] = \mathrm{i}\delta_{i,j}\epsilon^{\alpha\beta\gamma}\left(\vec{S}_j\right)^\gamma.$$

(a) Write down the classical equations of motion for $\vec{S}_i$ variables as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{S}_i = \cdots.$$

(b) Treat spins as classical variables and take the continuum limit to find the equations of motion of the low energy field.

Solution. (a) Using the Hamilton-Jacobi equation, we have

$$\begin{aligned}
\dot{\vec{S}}_i &= \{\vec{S}_i, \mathcal{H}\} \\
&= -\frac{\mathrm{i}}{\hbar}\left[\vec{S}_i, \mathcal{H}\right] \\
&= -\frac{\mathrm{i}}{\hbar}\left[\vec{S}_i, -J\sum_j \vec{S}_j \cdot \vec{S}_{j+1}\right] \\
&= J\frac{\mathrm{i}}{\hbar}\left[\vec{S}_i, \sum_j \vec{S}_j \cdot \vec{S}_{j+1}\right].
\end{aligned}$$

To break this down component-wise, we can write

$$\begin{aligned}
\left(\dot{\vec{S}}_i\right)_\alpha &= J\frac{\mathrm{i}}{\hbar}\left[\left(\vec{S}_i\right)_\alpha, \sum_j\sum_\beta \left(\vec{S}_j\right)_\beta \left(\vec{S}_{j+1}\right)_\beta\right] \\
&= J\frac{\mathrm{i}}{\hbar}\sum_j\sum_\beta \left[\left(\vec{S}_i\right)_\alpha, \left(\vec{S}_j\right)_\beta \left(\vec{S}_{j+1}\right)_\beta\right] \\
&= J\frac{\mathrm{i}}{\hbar}\sum_j\sum_\beta \left(\left(\vec{S}_j\right)_\beta \left[\left(\vec{S}_i\right)_\alpha, \left(\vec{S}_{j+1}\right)_\beta\right] + \left[\left(\vec{S}_i\right)_\alpha, \left(\vec{S}_j\right)_\beta\right]\left(\vec{S}_{j+1}\right)_\beta\right) \\
&= J\frac{\mathrm{i}}{\hbar}\sum_j\sum_\beta\sum_\gamma \left(\mathrm{i}\delta_{i,j+1}\epsilon^{\alpha\beta\gamma}\left(\vec{S}_j\right)_\beta \left(\vec{S}_{j+1}\right)_\gamma + \mathrm{i}\delta_{i,j}\epsilon^{\alpha\beta\gamma}\left(\vec{S}_j\right)_\gamma \left(\vec{S}_{j+1}\right)_\beta\right) \\
&= J\frac{\mathrm{i}}{\hbar}\sum_\beta\sum_\gamma \mathrm{i}\epsilon^{\alpha\beta\gamma}\left(\left(\vec{S}_{i-1}\right)_\beta \left(\vec{S}_i\right)_\gamma + \left(\vec{S}_i\right)_\gamma \left(\vec{S}_{i+1}\right)_\beta\right) \\
&= -\frac{J}{\hbar}\sum_\beta\sum_\gamma \epsilon^{\alpha\beta\gamma}\left(\left(\vec{S}_{i-1}\right)_\beta + \left(\vec{S}_{i+1}\right)_\beta\right)\left(\vec{S}_i\right)_\gamma \\
&= -\frac{J}{\hbar}\left[\left(\vec{S}_{i-1} + \vec{S}_{i+1}\right) \times \vec{S}_i\right]_\alpha.
\end{aligned}$$

Therefore,

$$\dot{\vec{S}}_i = \frac{J}{\hbar}\vec{S}_i \times \left(\vec{S}_{i+1} + \vec{S}_{i-1}\right).$$

(b) We introduce a lattice spacing a and a spin field $\vec{S}(x,t)$ at position x and time t such that $x = na$. We define

$$\delta(na,t) \equiv \delta_n(t).$$

Assume a small value of a and then we expand in a Taylor series around $x = na$. We have

$$\begin{aligned} S_{n+1}(t) + S_{n-1}(t) &= S(x+a,t) + S(x-a,t) \\ &= \left(S(x,t) + \frac{\partial S(x,t)}{\partial x} a + \frac{\partial^2 S(x,t)}{\partial x^2} a^2 + \cdots \right) \\ &\quad + \left(S(x,t) + \frac{\partial S(x,t)}{\partial x}(-a) + \frac{\partial^2 S(x,t)}{\partial x^2}(-a)^2 + \cdots \right) \\ &= 2S(x,t) + \frac{\partial^2 S(x,t)}{\partial x^2} a^2 + \mathcal{O}(a^4). \end{aligned}$$

We now replace the previous equation in the final result obtained in part (a), getting

$$\begin{aligned} \dot{\vec{S}}_i &= \frac{J}{\hbar} \vec{S}_i \times \left(\vec{S}_{i+1} + \vec{S}_{i-1} \right) \\ &= \frac{J}{\hbar} \vec{S}_i \times \left(2\vec{S}_i + \frac{\partial^2 \vec{S}_i}{\partial x^2} a^2 \right) \\ &= \frac{J}{\hbar} \vec{S}_i \times \frac{\partial^2 \vec{S}_i}{\partial x^2} a^2. \end{aligned}$$

The continuum limit is then achieved by taking $a \to 0$.

■

Homework 2

Problem 1 - Action Principle for a Relativistic Particle
Consider a free particle moving along $x^\mu(t)$ in Minkowski spacetime

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu.$$

Take as the action

$$S = -m\int \mathrm{d}s = -m\int \mathrm{d}t\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right).$$

Vary it and find the equations of motion.

Solution. We start by dividing by $(\mathrm{d}t)^2$ on both sides to get a time derivative of the infinitesimal path, and taking $\eta_{\mu,\nu}$ to be the Minkowski metric, we have

$$\begin{aligned}
\frac{(\mathrm{d}s)^2}{(\mathrm{d}t)^2} &= \frac{1}{(\mathrm{d}t)^2}\left(\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right)\\
\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= \frac{1}{(\mathrm{d}t)^2}\left[(\mathrm{d}t)^2 - \left((\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2\right)\right]\\
\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= 1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\\
\frac{\mathrm{d}s}{\mathrm{d}t} &= \sqrt{1 - \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 - \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\\
\dot{s} &= \sqrt{1 - (\dot{x} + \dot{y} + \dot{z})}\\
\dot{s} &= \sqrt{\dot{x}_\mu\dot{x}^\mu} = \sqrt{\partial_0 x_\mu \partial_0 x^\mu}.
\end{aligned}$$

Our action becomes

$$S = -m\int_{t_i}^{t_f} \sqrt{\dot{x}_\mu\dot{x}^\mu}\,\mathrm{d}t,$$

where $\mathcal{L}(x^\mu, \dot{x}^\mu, t) = -m\sqrt{\dot{x}_\mu\dot{x}^\mu}$ is the Lagrangian. The Euler-Lagrange equations are given by

$$\frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = 0.$$

- **Varying** x^μ:
$$\frac{\partial\mathcal{L}}{\partial x^\mu} = 0.$$
- **Varying** $\dot{x}^\mu$:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right) = -m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{2\dot{x}^\mu}{2\sqrt{\dot{x}_\mu\dot{x}^\mu}}\right) = -m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}_\mu\dot{x}^\mu}}\right).$$

Then, plugging these into the Euler-Lagrange equation, we have

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}^\mu}{\sqrt{\dot{x}_\mu\dot{x}^\mu}}\right) = 0 \quad\implies\quad \dot{x}^\mu = \text{constant}.$$

■

Problem 2 - Coulomb Gauge
Consider classical Maxwell fields.

(a) Expand the Lagrangian density in the presence of a current in terms of $A^\mu = (\varphi, \vec{A})$, where φ is the electrostatic potential and $\vec{A}$ is the vector potential satisfying the Coulomb gauge $\vec{\nabla} \cdot \vec{A}(x) = 0$.

(b) Vary the action with respect to φ and $\vec{A}$ to find the equations of motion.

(c) Find the solution to the equations of motion using Fourier transform.

Solution. (a) Considering classical Maxwell fields, the Lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu.$$

The field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

By replacing that in the Lagrangian, we get

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4} \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + J^\mu A_\mu \\
&= -\frac{1}{4} \left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu\right] + J^\mu A_\mu \\
&= -\frac{1}{2} \left[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right] + J^\mu A_\mu \\
&= -\frac{1}{2} \left[\left(\partial_t A_\nu\right)\left(\partial_t A^\nu\right) - \left(\partial_x A_\nu\right)\left(\partial_x A^\nu\right) - \left(\partial_y A_\nu\right)\left(\partial_y A^\nu\right) - \left(\partial_z A_\nu\right)\left(\partial_z A^\nu\right)\right. \\
&\quad \left. - \left(\partial_t A_\mu\right)\left(\partial^\mu A_t\right) + \left(\partial_x A_\mu\right)\left(\partial^\mu A_x\right) + \left(\partial_y A_\mu\right)\left(\partial^\mu A_y\right) + \left(\partial_z A_\mu\right)\left(\partial^\mu A_z\right)\right] - J^\mu A_\mu \\
&= -\frac{1}{2} \left[\cancel{\left(\partial_t A_t\right)\left(\partial_t A_t\right)} - \left(\partial_t A_x\right)\left(\partial_t A_x\right) - \left(\partial_t A_y\right)\left(\partial_t A_y\right) - \left(\partial_t A_z\right)\left(\partial_t A_z\right)\right. \\
&\quad - \left(\partial_x A_t\right)\left(\partial_x A_t\right) + \cancel{\left(\partial_x A_x\right)\left(\partial_x A_x\right)} + \left(\partial_x A_y\right)\left(\partial_x A_y\right) + \left(\partial_x A_z\right)\left(\partial_x A_z\right) \\
&\quad - \left(\partial_y A_t\right)\left(\partial_y A_t\right) + \left(\partial_y A_x\right)\left(\partial_y A_x\right) + \cancel{\left(\partial_y A_y\right)\left(\partial_y A_y\right)} + \left(\partial_y A_z\right)\left(\partial_y A_z\right) \\
&\quad - \left(\partial_z A_t\right)\left(\partial_z A_t\right) + \left(\partial_z A_x\right)\left(\partial_z A_x\right) + \left(\partial_z A_y\right)\left(\partial_z A_y\right) + \cancel{\left(\partial_z A_z\right)\left(\partial_z A_z\right)} \\
&\quad - \cancel{\left(\partial_t A_t\right)\left(\partial_t A_t\right)} + \left(\partial_t A_x\right)\left(\partial_x A_t\right) + \left(\partial_t A_y\right)\left(\partial_y A_t\right) + \left(\partial_t A_z\right)\left(\partial_z A_t\right) \\
&\quad + \left(\partial_x A_t\right)\left(\partial_t A_x\right) - \cancel{\left(\partial_x A_x\right)\left(\partial_x A_x\right)} - \left(\partial_x A_y\right)\left(\partial_y A_x\right) - \left(\partial_x A_z\right)\left(\partial_z A_x\right) \\
&\quad + \left(\partial_y A_t\right)\left(\partial_t A_y\right) - \left(\partial_y A_x\right)\left(\partial_x A_y\right) - \cancel{\left(\partial_y A_y\right)\left(\partial_y A_y\right)} - \left(\partial_y A_z\right)\left(\partial_z A_y\right) \\
&\quad \left. + \left(\partial_z A_t\right)\left(\partial_t A_z\right) - \left(\partial_z A_x\right)\left(\partial_x A_z\right) - \left(\partial_z A_y\right)\left(\partial_y A_z\right) - \cancel{\left(\partial_z A_z\right)\left(\partial_z A_z\right)}\right] - J^\mu A_M \\
&= -\frac{1}{2} \Big[- \left(\partial_t A_x - \partial_x A_t\right)^2 - \left(\partial_t A_y - \partial_y A_t\right)^2 - \left(\partial_t A_z - \partial_z A_t\right)^2 + \\
&\quad + \left(\partial_x A_y - \partial_y A_x\right)^2 + \left(\partial_x A_z - \partial_z A_x\right)^2 + \left(\partial_y A_z - \partial_z A_y\right)^2\Big] - J^\mu A_\mu \\
&= \frac{1}{2} \left[\left|\partial_t \mathbf{A} + \nabla\varphi\right|^2 - \left|\nabla \times \mathbf{A}\right|^2\right] - J^\mu A_\mu \\
&= \frac{1}{2} \left(\left|\partial_t \mathbf{A} + \nabla\varphi\right|^2 - \left|\nabla \times \mathbf{A}\right|^2\right) - (\rho\varphi - \mathbf{j} \cdot \mathbf{A}).
\end{aligned}$$

(b) The action of this system is given by

$$\begin{aligned}
S &= \int \mathcal{L} \, \mathrm{d}^4 x \\
&= \int \left[\frac{1}{2} \left(\left|\partial_t \mathbf{A} + \nabla\varphi\right|^2 - \left|\nabla \times \mathbf{A}\right|^2\right) - (\rho\varphi - \mathbf{j} \cdot \mathbf{A})\right] \mathrm{d}^4 x.
\end{aligned}$$

Varying the action, we get

$$\begin{aligned}\delta S &= \int \Big[\frac{1}{2}\left(2(\nabla\varphi)\cdot(\nabla(\delta\varphi)) + 2(\nabla\varphi)\cdot(\partial_t(\delta\mathbf{A})) + 2(\nabla(\delta\varphi))\cdot(\partial_t\mathbf{A})\right.\\ &\qquad \left.+2(\partial_t\mathbf{A})\cdot(\partial_t(\delta\mathbf{A})) - 2(\nabla\times\mathbf{A})\cdot(\nabla\times\delta\mathbf{A})\right) - \rho\delta\varphi + \mathbf{j}\cdot\delta\mathbf{A}\Big]\,\mathrm{d}^4x\\ &= \int \left[\left(-\nabla^2\varphi - \rho\right)\delta\varphi + \left(-\partial_t^2\mathbf{A} + \nabla^2\mathbf{A} - \partial_t(\nabla\varphi) + \mathbf{j}\right)\delta\mathbf{A}\right]\mathrm{d}^4x,\end{aligned}$$

which give us the following equations of motion

$$\nabla^2\varphi = -\rho,$$
$$\left(-\partial_t^2 + \nabla^2\right)\mathbf{A} = \nabla(\partial_t\varphi) - \mathbf{j}.$$

(c) Let $\nabla^2\varphi(\mathbf{x},t) = -\rho(\mathbf{x},t)$. Then

$$\begin{aligned}\nabla^2\int G(\mathbf{x},t,\mathbf{x}',t')\rho(\mathbf{x}',t')\,\mathrm{d}^3x'\,\mathrm{d}t' &= \nabla^2\int G(\mathbf{x}',\mathbf{x})\delta(t'-t)\rho(\mathbf{x}',t')\,\mathrm{d}^3x'\,\mathrm{d}t'\\ &= -\int \delta(\mathbf{x}'-\mathbf{x})\delta(t'-t)\rho(\mathbf{x}',t')\,\mathrm{d}^3x'\,\mathrm{d}t',\end{aligned}$$

where $G(\mathbf{x},t,\mathbf{x}',t')\rho(\mathbf{x}',t') = G(\mathbf{x}',\mathbf{x})\delta(t'-t)$ and $\nabla^2 G(\mathbf{x}',\mathbf{x}) = -\delta(\mathbf{x}'-\mathbf{x})$.
Consider the Fourier transform

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\nabla^2\int \tilde{G}(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\,\mathrm{d}^3k = -\frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\,\mathrm{d}^3k,$$

then we can conclude that $\tilde{G}(\mathbf{k}) = \frac{1}{k^2}$.
Replacing, we have

$$G(\mathbf{x}',\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{1}{k^2}e^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}\,\mathrm{d}^3k.$$

Let $\mathrm{d}^3k = k^2\sin(\theta)\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi$, which means $\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}) = k\,|\mathbf{x}'-\mathbf{x}|\cos(\theta)$. Thus,

$$\begin{aligned}G(\mathbf{x}',\mathbf{x}) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\lim_{\epsilon\to 0}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{k^2\sin(\theta)e^{ik\cdot|\mathbf{x}'-\mathbf{x}|\cos(\theta)}}{k^2+\epsilon^2}\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \frac{2\pi}{(2\pi)^{\frac{3}{2}}}\lim_{\epsilon\to 0}\int_0^{\infty}\frac{k^2}{k^2+\epsilon^2}\int_0^{\pi}\sin(\theta)e^{ik\cdot|\mathbf{x}'-\mathbf{x}|\cos(\theta)}\,\mathrm{d}\theta\,\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\lim_{\epsilon\to 0}\int_0^{\infty}\frac{k^2}{k^2+\epsilon^2}\left(\frac{e^{ik\cdot|\mathbf{x}'-\mathbf{x}|} - e^{-ik\cdot|\mathbf{x}'-\mathbf{x}|}}{ik\,|\mathbf{x}'-\mathbf{x}|}\right)\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{i\,|\mathbf{x}'-\mathbf{x}|}\lim_{\epsilon\to 0}\int_0^{\infty}\frac{k\left(e^{ik\cdot|\mathbf{x}'-\mathbf{x}|} - e^{-ik\cdot|\mathbf{x}'-\mathbf{x}|}\right)}{(k-i\epsilon)(k+i\epsilon)}\,\mathrm{d}k\\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{i\,|\mathbf{x}'-\mathbf{x}|}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{ke^{ik\cdot|\mathbf{x}'-\mathbf{x}|}}{(k-i\epsilon)(k+i\epsilon)}\,\mathrm{d}k\\ &= \frac{1}{|\mathbf{x}'-\mathbf{x}|}.\end{aligned}$$

Thus,

$$\varphi(\mathbf{x},t) = \int\frac{\rho(\mathbf{x}',t)}{|\mathbf{x}'-\mathbf{x}|}\,\mathrm{d}^3x'.$$

■

Problem 3 - Spontaneous Symmetry Breaking
Consider a complex relativistic scalar field with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \Phi^\dagger\right)\left(\partial^\mu \Phi\right) + \lambda\left(\Phi^\dagger \Phi - \phi_0\right)^2.$$

(a) Find the classical equations of motion.

(b) Consider constant field configurations and find the minima of action.

(c) Expand the field $\Phi(x^\mu)$ around this minimum to the second-order and find the mass of the excitations.

(d) Is there a massless mode?

Solution. (a) The classical equations of motion are given by the Euler-Lagrange equation for scalar fields given by

$$\frac{\partial \mathcal{L}}{\partial \Phi(x)} - D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi(x))} = 0.$$

Since we have two dynamical variables, Φ and $\Phi^\dagger$, we have to vary with respect to each of them to derive the equations of motion, *i.e.* we will have to take two Euler-Lagrange equations, one for Φ and another for $\Phi^\dagger$. When varying the first, the second must be considered as a constant, and vice versa.

- **For** Φ:
  - **First Term**:
  $$\frac{\partial \mathcal{L}}{\partial \Phi} = \lambda(2(\Phi^\dagger)^2\Phi - 2\phi_0\Phi^\dagger) = 2\lambda\Phi^\dagger(\Phi^\dagger\Phi - \phi_0).$$
  - **Second Term**:
  $$\begin{aligned} D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} &= -\frac{1}{2} D_\mu \left(\partial^\mu \Phi^\dagger\right) \\ &= -\frac{1}{2}\left(\partial_\mu \partial^\mu \Phi^\dagger + \delta^\mu_\nu \partial_\mu \partial^\nu \Phi^\dagger\right) \\ &= -\partial_\mu \partial^\mu \Phi^\dagger. \end{aligned}$$
  - **Euler-Lagrange**:
  $$\begin{gathered} \frac{\partial \mathcal{L}}{\partial \Phi} - D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0 \\ 2\lambda\Phi^\dagger(\Phi^\dagger\Phi - \phi_0) + \partial_\mu\partial^\mu\Phi^\dagger = 0. \end{gathered}$$
- **For** $\Phi^\dagger$:
  - **First Term**:
  $$\frac{\partial \mathcal{L}}{\partial \Phi^\dagger} = \lambda(2\Phi^\dagger\Phi^2 - 2\phi_0\Phi) = 2\lambda\Phi(\Phi^\dagger\Phi - \phi_0).$$
  - **Second Term**:
  $$\begin{aligned} D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\dagger)} &= -\frac{1}{2} D_\mu \left(\partial^\mu \Phi\right) \\ &= -\frac{1}{2}\left(\partial_\mu \partial^\mu \Phi + \delta^\mu_\nu \partial_\mu \partial^\nu \Phi\right) \\ &= -\partial_\mu \partial^\mu \Phi. \end{aligned}$$
  - **Euler-Lagrange**:
  $$\begin{gathered} \frac{\partial \mathcal{L}}{\partial \Phi^\dagger} - D_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\dagger)} = 0 \\ 2\lambda\Phi(\Phi^\dagger\Phi - \phi_0) + \partial_\mu\partial^\mu\Phi = 0. \end{gathered}$$

Thus, the two equations of motions are given by

$$\begin{cases} 2\lambda\Phi^\dagger(\Phi^\dagger\Phi-\phi_0)+\partial_\mu\partial^\mu\Phi^\dagger & =0, \\ 2\lambda\Phi(\Phi^\dagger\Phi-\phi_0)+\partial_\mu\partial^\mu\Phi & =0. \end{cases}$$

(b) If $\Phi = \text{constant}$, then $\partial^\mu\Phi = 0$, and we have

$$\begin{aligned} 2\lambda\Phi^\dagger(\Phi^\dagger\Phi-\phi_0) &= 0, \\ \implies \Phi = 0, \qquad \text{or} \qquad \Phi^\dagger\Phi &= \phi_0 \\ \implies \Phi &= \sqrt{\phi_0}\mathrm{e}^{\mathrm{i}\theta}. \end{aligned}$$

The minimum of the action is then $S_{\min} = 0$.

(c) If

$$\begin{aligned} \Phi &= \sqrt{\phi_0}\mathrm{e}^{\mathrm{i}\theta}+\Psi, \\ \Phi^\dagger &= \sqrt{\phi_0}\mathrm{e}^{-\mathrm{i}\theta}+\Psi^\dagger, \end{aligned}$$

where $|\Psi$ is small, then the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right)+\lambda\left[\left(\sqrt{\phi_0}\mathrm{e}^{-\mathrm{i}\theta}+\Psi^\dagger\right)\left(\sqrt{\phi_0}\mathrm{e}^{\mathrm{i}\theta}+\Psi\right)-\phi_0\right]^2 \\ &= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right)+\lambda\left[\left(\phi_0+\sqrt{\phi_0}\mathrm{e}^{-\mathrm{i}\theta}\Psi+\sqrt{\phi_0}\mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger+\Psi^\dagger\Psi\right)-\phi_0\right]^2 \\ &= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right)+\lambda\left[\sqrt{\phi_0}\left(\mathrm{e}^{-\mathrm{i}\theta}\Psi+\mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger\right)+\Psi^\dagger\Psi\right]^2 \\ &= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right)+\lambda\left[\phi_0\left(\mathrm{e}^{-\mathrm{i}\theta}\Psi+\mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger\right)^2+2\sqrt{\phi_0}\left(\mathrm{e}^{-\mathrm{i}\theta}\Psi+\mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger\right)\Psi^\dagger\Psi+\left(\Psi^\dagger\Psi\right)^2\right] \\ &= -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right)+\lambda\left[\phi_0\left(\mathrm{e}^{-2\mathrm{i}\theta}\Psi^2+2\Psi\Psi^\dagger+\mathrm{e}^{2\mathrm{i}\theta}\left(\Psi^\dagger\right)^2\right)+2\sqrt{\phi_0}\left(\mathrm{e}^{-\mathrm{i}\theta}\Psi+\mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger\right)\Psi^\dagger\Psi+\left(\Psi^\dagger\Psi\right)^2\right]. \end{aligned}$$

Invoking the fact that Ψ is small, we will only consider terms up to the second-order in Ψ and ignore those of higher orders. Then, we have

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\Psi^\dagger\right)\left(\partial^\mu\Psi\right)+\lambda\phi_0\left(\mathrm{e}^{-2\mathrm{i}\theta}\Psi^2+2\Psi\Psi^\dagger+\mathrm{e}^{2\mathrm{i}\theta}\left(\Psi^\dagger\right)^2\right).$$

We define the following

$$\begin{cases} \varphi_1 = \mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger+\mathrm{e}^{-\mathrm{i}\theta}\Psi, \\ \varphi_2 = \mathrm{e}^{\mathrm{i}\theta}\Psi^\dagger-\mathrm{e}^{-\mathrm{i}\theta}\Psi, \end{cases} \iff \begin{cases} \Psi = \mathrm{e}^{\mathrm{i}\theta}\left(\varphi_1-\varphi_2\right), \\ \Psi^\dagger = \mathrm{e}^{-\mathrm{i}\theta}\left(\varphi_1+\varphi_2\right). \end{cases}$$

Replacing in the Lagrangian, we have

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2}\left(\partial_\mu\left(\mathrm{e}^{-\mathrm{i}\theta}\left(\varphi_1+\varphi_2\right)\right)\right)\left(\partial^\mu\left(\mathrm{e}^{\mathrm{i}\theta}\left(\varphi_1-\varphi_2\right)\right)\right) \\ &+\lambda\phi_0\left[\mathrm{e}^{-2\mathrm{i}\theta}\left(\mathrm{e}^{\mathrm{i}\theta}\left(\varphi_1-\varphi_2\right)\right)^2+2\left(\mathrm{e}^{\mathrm{i}\theta}\left(\varphi_1-\varphi_2\right)\right)\left(\mathrm{e}^{-\mathrm{i}\theta}\left(\varphi_1+\varphi_2\right)\right)+\mathrm{e}^{2\mathrm{i}\theta}\left(\mathrm{e}^{-\mathrm{i}\theta}\left(\varphi_1+\varphi_2\right)\right)^2\right] \\ = &-\frac{1}{2}\left(\partial_\mu\left(\varphi_1+\varphi_2\right)\right)\left(\partial^\mu\left(\varphi_1-\varphi_2\right)\right)+\lambda\phi_0\left[\left(\varphi_1-\varphi_2\right)^2+2\left(\varphi_1-\varphi_2\right)\left(\varphi_1+\varphi_2\right)+\left(\varphi_1+\varphi_2\right)^2\right] \\ = &-\frac{1}{2}\left(\partial_\mu\varphi_1+\partial_\mu\varphi_2\right)\left(\partial^\mu\varphi_1-\partial^\mu\varphi_2\right)+\lambda\phi_0\left[\left(\varphi_1-\varphi_2\right)+\left(\varphi_1+\varphi_2\right)\right]^2 \\ = &-\frac{1}{2}\left(\partial_\mu\varphi_1\partial^\mu\varphi_1-\partial_\mu\varphi_2\partial^\mu\varphi_2\right)+4\lambda\phi_0\varphi_1^2. \end{aligned}$$

$$\frac{m_1^2}{2} = 4\lambda\phi_0, \quad \text{and} \quad \frac{m_2^2}{2} = 0,$$
$$m_1 = \sqrt{8\lambda\phi_0}, \quad \text{and} \quad m_2 = 0.$$

(d) From part (c), we can see the massless mode is m_2.

■

Homework 3

Problem 1 - Algebra of Charges
Consider the Lagrangian

$$\mathcal{L} = \partial_\mu \Phi_a^\dagger \partial^\mu \Phi_a - V\left(\Phi_a\right)$$

and suppose it is symmetric under the transformation

$$\phi_a \to \phi^a + \mathrm{i}\theta_k \left(\hat{\lambda}_k\right)^a_b \phi^b + \mathcal{O}(\theta_k^2)$$

(a) Write down the conserved currents and conserved charges.

(b) Use canonical commutation relations

$$\left[\pi_a\left(\vec{x}\right), \phi_b\left(\vec{y}\right)\right] = -\mathrm{i}\delta^d\left(\vec{x}-\vec{y}\right)\delta_a^b$$

to compute the commutators $\left[J_0^a\left(\vec{x}\right), J_0^b\left(\vec{y}\right)\right]$, $\left[Q^a, J_0^b\left(\vec{x}\right)\right]$, and $\left[Q^a, Q^b\right]$.

Solution. (a) We are given the infinitesimal transformation

$$\delta\phi_a(\theta) = \mathrm{i}\theta_k(\lambda_k)_{ab}\phi_b,$$

that leaves the Lagrangian invariant. From that, the conserved current is

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\delta\phi_a + \delta\phi_a^\dagger \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a^\dagger)} = J_k^\mu \theta_k,$$

where

$$J_k^\mu = \mathrm{i}\left[\left(\partial_\mu \phi_a^\dagger\right)(\lambda_k)_{ab}\phi_b - \phi_a^\dagger(\lambda_k)_{ab}^\dagger\left(\partial_\mu\phi_b\right)\right]$$

represents the component of the current corresponding to the kth generator of the symmetry. Since a general symmetry transformation can have multiple independent components, each corresponding to a different generator of the symmetry group, then we have a family of conserved currents J_k^μ, one for each generator k. The current labeled with k is tied to the specific transformation parameterized by θ_k.

Then, the conserved charge is

$$\begin{aligned} Q_k &= \int J_k^0 \,\mathrm{d}^d x \\ &= \mathrm{i}\int \left[\left(\partial_\mu \phi_a^\dagger\right)(\lambda_k)_{ab}\phi_b - \phi_a^\dagger(\lambda_k)_{ab}^\dagger\left(\partial_\mu\phi_b\right)\right]\mathrm{d}^d x. \end{aligned}$$

(b) Define the canonical momenta as

$$\pi_a = \frac{\partial\mathcal{L}}{\partial(\partial_t \phi_a)} = \frac{\partial \phi_a^\dagger}{\partial t}.$$

Then the canonical commutation relations are given by

$$\begin{aligned} \left[\pi_a\left(\mathbf{x}\right), \phi_b\left(\mathbf{y}\right)\right] = \left[\pi_a^\dagger\left(\mathbf{x}\right), \phi_b^\dagger\left(\mathbf{y}\right)\right] &= -\mathrm{i}\delta^d\left(\mathbf{x}-\mathbf{y}\right)\delta_a^b, \\ \left[\pi_a^\dagger\left(\mathbf{x}\right), \phi_b\left(\mathbf{y}\right)\right] = \left[\pi_a\left(\mathbf{x}\right), \phi_b^\dagger\left(\mathbf{y}\right)\right] &= 0. \end{aligned}$$

Combining these commutation relations, and using the identity

$$[AB, CD] = A[B,C]D + [A,C]BD + CA[B,D] + C[A,D]B$$

we get the following four results:

- $$\begin{aligned}&\left[\phi_a^\dagger(\mathbf{x})\frac{\partial\phi_b}{\partial t}(\mathbf{x}),\frac{\partial\phi_c^\dagger}{\partial t}(\mathbf{y})\phi_d(\mathbf{y})\right]\\&=\left[\phi_a^\dagger(\mathbf{x})\pi_b^\dagger(\mathbf{x}),\pi_c(\mathbf{y})\phi_d(\mathbf{y})\right]\\&=\phi_a^\dagger(\mathbf{x})[\pi_b^\dagger(\mathbf{x}),\pi_c(\mathbf{y})]\phi_d(\mathbf{y})+[\phi_a^\dagger(\mathbf{x}),\pi_c(\mathbf{y})]\pi_b^\dagger(\mathbf{x})\phi_d(\mathbf{y})\\&\quad+\pi_c(\mathbf{y})\phi_a^\dagger(\mathbf{x})[\pi_b^\dagger(\mathbf{x}),\phi_d(\mathbf{y})]+\pi_c(\mathbf{y})[\phi_a^\dagger,\phi_d(\mathbf{y})]\pi_b^\dagger(\mathbf{x})\\&=0.\end{aligned}$$

- $$\begin{aligned}&\left[\frac{\partial\phi_a^\dagger}{\partial t}(\mathbf{x})\phi_b(\mathbf{x}),\phi_c^\dagger(\mathbf{y})\frac{\partial\phi_d}{\partial t}(\mathbf{y})\right]\\&=\left[\pi_a(\mathbf{x})\phi_b(\mathbf{x}),\phi_c^\dagger(\mathbf{y})\pi_d^\dagger(\mathbf{y})\right]\\&=\pi_a(\mathbf{x})[\phi_b(\mathbf{x}),\phi_c^\dagger(\mathbf{y})]\pi_d^\dagger(\mathbf{y})+[\pi_a(\mathbf{x}),\phi_c^\dagger(\mathbf{y})]\phi_b(\mathbf{x})\pi_d^\dagger(\mathbf{y})\\&\quad+\phi_c^\dagger(\mathbf{y})\pi_a(\mathbf{x})[\phi_b(\mathbf{x}),\pi_d^\dagger(\mathbf{y})]+\phi_c^\dagger(\mathbf{y})[\pi_a(\mathbf{x}),\pi_d^\dagger(\mathbf{y})]\phi_b(\mathbf{x})\\&=0.\end{aligned}$$

- $$\begin{aligned}&\left[\frac{\partial\phi_a^\dagger}{\partial t}(\mathbf{x})\phi_b(\mathbf{x}),\frac{\partial\phi_c^\dagger}{\partial t}(\mathbf{y})\phi_d(\mathbf{y})\right]\\&=[\pi_a(\mathbf{x})\phi_b(\mathbf{x}),\pi_c(\mathbf{y})\phi_d(\mathbf{y})]\\&=\pi_a(\mathbf{x})\left[\phi_b(\mathbf{x}),\pi_c(\mathbf{y})\right]\phi_d(\mathbf{y})+\pi_a(\mathbf{x})\pi_c(\mathbf{y})\left[\phi_b(\mathbf{x}),\phi_d(\mathbf{y})\right]\\&\quad+\left[\pi_a(\mathbf{x}),\pi_c(\mathbf{y})\right]\phi_d(\mathbf{y})\phi_b(\mathbf{x})+\pi_c(\mathbf{y})\left[\pi_a(\mathbf{x}),\phi_d(\mathbf{y})\right]\phi_b(\mathbf{x})\\&=\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\delta_{bc}\pi_a(\mathbf{x})\phi_d(\mathbf{y})-\delta_{ad}\pi_c(\mathbf{y})\phi_b(\mathbf{x})\right)\end{aligned}$$

- $$\begin{aligned}&\left[\phi_a^\dagger(\mathbf{x})\frac{\partial\phi_b}{\partial t}(\mathbf{x}),\phi_c^\dagger(\mathbf{y})\frac{\partial\phi_d}{\partial t}(\mathbf{y})\right]\\&=\left[\phi_a^\dagger(\mathbf{x})\pi_b^\dagger(\mathbf{x}),\phi_c^\dagger(\mathbf{y})\pi_d^\dagger(\mathbf{y})\right]\\&=\phi_a^\dagger(\mathbf{x})\left[\pi_b^\dagger(\mathbf{x}),\phi_c^\dagger(\mathbf{y})\right]\pi_d^\dagger(\mathbf{y})+\phi_a^\dagger(\mathbf{x})\phi_c^\dagger(\mathbf{y})\left[\pi_b^\dagger(\mathbf{x}),\pi_d^\dagger(\mathbf{y})\right]\\&\quad+\left[\phi_a^\dagger(\mathbf{x}),\phi_c^\dagger(\mathbf{y})\right]\pi_d^\dagger(\mathbf{y})\pi_b^\dagger(\mathbf{x})+\phi_c^\dagger(\mathbf{y})\left[\phi_a^\dagger(\mathbf{x}),\pi_d^\dagger(\mathbf{y})\right]\pi_b^\dagger(\mathbf{x})\\&=\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\delta_{ad}\phi_c^\dagger(\mathbf{y})\pi_b^\dagger(\mathbf{x})-\delta_{bc}\phi_a^\dagger(\mathbf{x})\pi_d^\dagger(\mathbf{y})\right)\end{aligned}$$

- The commutation relation for the conserved current is
$$\begin{aligned}&\left[J_k^0(\mathbf{x}),J_l^0(\mathbf{y})\right]\\&=\left[\left(\left(\partial_\mu\phi_a^\dagger\right)(\lambda_k)_{ab}\phi_b-\phi_a^\dagger(\lambda_k)_{ab}^\dagger\left(\partial_\mu\phi_b\right)\right),\left(\left(\partial_\mu\phi_a^\dagger\right)(\lambda_l)_{ab}\phi_b-\phi_a^\dagger(\lambda_l)_{ab}^\dagger\left(\partial_\mu\phi_b\right)\right)\right]\\&=\left[\partial_\mu\phi_a^\dagger(\mathbf{x})(\lambda_k)_{ab}\phi_b(\mathbf{x}),\partial_\mu\phi_c^\dagger(\mathbf{y})(\lambda_l)_{cd}\phi_d(\mathbf{y})\right]-\left[\partial_\mu\phi_a^\dagger(\mathbf{x})(\lambda_k)_{ab}\phi_b(\mathbf{x}),\phi_c^\dagger(\mathbf{y})(\lambda_l)_{cd}^\dagger\partial_\mu\phi_d(\mathbf{y})\right]\\&\quad-\left[\phi_a^\dagger(\mathbf{x})(\lambda_k)_{ab}^\dagger\partial_\mu\phi_b(\mathbf{x}),\partial_\mu\phi_c^\dagger(\mathbf{y})(\lambda_l)_{cd}\phi_d(\mathbf{y})\right]+\left[\phi_a^\dagger(\mathbf{x})(\lambda_k)_{ab}^\dagger\partial_\mu\phi_b(\mathbf{x}),\phi_c^\dagger(\lambda_l)_{cd}^\dagger(\mathbf{y})\partial_\mu\phi_d(\mathbf{y})\right]\\&=-\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\pi_a(\mathbf{x})\left(\lambda_k\right)_{ac}\left(\lambda_l\right)_{cb}\phi_b(\mathbf{y})-\pi_a(\mathbf{y})\left(\lambda_l\right)_{ac}\left(\lambda_k\right)_{cb}\phi_b(\mathbf{x})\right)\\&\quad-\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\phi_a^\dagger(\mathbf{y})\left(\lambda_l\right)_{ac}^\dagger\left(\lambda_k\right)_{cb}^\dagger\pi_b^\dagger(\mathbf{x})-\phi_a^\dagger(\mathbf{x})\left(\lambda_k\right)_{ac}^\dagger\left(\lambda_l\right)_{cb}^\dagger\pi_b^\dagger(\mathbf{y})\right)\\&=-\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\pi_a(\mathbf{x})\left[\lambda_k,\lambda_l\right]_{ab}\phi_b(\mathbf{x})+\phi_a^\dagger(\mathbf{x})\left[\lambda_k,\lambda_l\right]_{ab}^\dagger\pi_b^\dagger(\mathbf{x})\right).\end{aligned}$$

If λ_k are the generators of a Lie algebra with a Lie bracket given by $[\lambda_k, \lambda_l] = \mathrm{i} f_{kl}^m \lambda_m$, then

$$\begin{aligned}
\left[J_k^0(\mathbf{x}), J_l^0(\mathbf{y})\right] &= -\mathrm{i}\delta(\mathbf{x}-\mathbf{y}) \left(\pi_a(\mathbf{x}) \left[\lambda_k, \lambda_l\right]_{ab} \phi_b(\mathbf{x}) + \phi_a^\dagger(\mathbf{x}) \left[\lambda_k, \lambda_l\right]_{ab}^\dagger \pi_b^\dagger(\mathbf{x})\right) \\
&= \delta(\mathbf{x}-\mathbf{y}) f_{kl}^m \left(\pi_a(\mathbf{x}) \left(\lambda_m\right)_{ab} \phi_b(\mathbf{x}) + \phi_a^\dagger(\mathbf{x}) \left(\lambda_m\right)_{ab}^\dagger \pi_b^\dagger(\mathbf{x})\right) \\
&= -\mathrm{i}\delta(\mathbf{x}-\mathbf{y}) f_{kl}^m J_m^0(\mathbf{x}).
\end{aligned}$$

- The commutation relation for the conserved charge and current is

$$\begin{aligned}
\left[Q_k, J_l^0(\mathbf{x})\right] &= \int \left[J_k^0(\mathbf{y}), J_l^0(\mathbf{x})\right] \mathrm{d}^d y \\
&= -\mathrm{i} \int \delta(\mathbf{y}-\mathbf{x}) f_{kl}^m J_m^0(\mathbf{y}) \,\mathrm{d}^d y \\
&= -\mathrm{i} f_{kl}^m J_m^0(\mathbf{x}).
\end{aligned}$$

- The commutation relation for the conserved charges is

$$\begin{aligned}
\left[Q_k, Q_l\right] &= \int \left[Q_k, J_l^0(\mathbf{x})\right] \mathrm{d}^d x \\
&= -\mathrm{i} f_{kl}^m \int J_m^0(\mathbf{x}) \,\mathrm{d}^d x \\
&= -\mathrm{i} f_{kl}^m Q_m.
\end{aligned}$$

■

Problem 2 - Non-relativistic Galilean Group
Consider the non-relativistic Galilean group generated by rotations and non-relativistic boosts $\vec{x}^i \to \vec{x}^i + \vec{v}^i t$. Show that this algebra can be centrally extended with the central charge that's the total mass of the system.

Solution. The Lie algebra of the Galilean group is

$$\begin{aligned}
[J_i, J_j] &= \mathrm{i}\epsilon_{ijk} J_k \\
[J_i, K_j] &= \mathrm{i}\epsilon_{ijk} K_k \\
[J_i, P_j] &= \mathrm{i}\epsilon_{ijk} P_k \\
[K_i, \mathcal{H}] &= \mathrm{i} P_i \\
[P_i, P_j] = [K_i, P_j] = [K_i, K_j] = [J_i, \mathcal{H}] &= [P_i, \mathcal{H}] = 0,
\end{aligned}$$

where all indices run over $\{1, 2, 3\}$.
Let us try to centrally extend this algebra as follows

$$\begin{aligned}
&[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k + \mathrm{i} A_{ij}\mathbb{I} \qquad &[K_i, E] = \mathrm{i} P_i + \mathrm{i} D_i \mathbb{I} \qquad &[J_i, E] = \mathrm{i} W_i \mathbb{I} \\
&[J_i, K_j] = \mathrm{i}\epsilon_{ijk} K_k + \mathrm{i} B_{ij}\mathbb{I} \qquad &[K_i, P_j] = \mathrm{i} U_{ij}\mathbb{I} \qquad &[P_i, E] = \mathrm{i} X_i \mathbb{I} \\
&[J_i, P_j] = \mathrm{i}\epsilon_{ijk} P_k + i C_{ij}\mathbb{I} \qquad &[K_i, K_j] = \mathrm{i} V_{ij}\mathbb{I} \qquad &[P_i, P_j] = \mathrm{i} Y_{ij}\mathbb{I}
\end{aligned}$$

where A_{ij}, V_{ij}, and Y_{ij} are anti-symmetric in (i, j), while B_{ij}, C_{ij}, and U_{ij} are anti-symmetric in (i, j) for $i \neq j$. As it turns out, most of these variables can be eliminated. Some can be eliminated by using the Jacobi identity, and the others can be eliminated by redefining the generators.

For arbitrary α, β, we have

$$\begin{aligned}
0 &= \left[\left[\alpha K_i + \beta P_i, \alpha K_j + \beta P_j\right], J_k\right] + \left[\left[\alpha K_j + \beta P_j, J_k\right], \alpha K_i + \beta P_i\right] + \left[\left[J_k, \alpha K_i + \beta P_i\right], \alpha K_j + \beta P_j\right] \\
&= \mathrm{i}\left[\alpha^2 V_{ij}\mathbb{I} + \alpha\beta\left(U_{ij} + U_{ji}\right)\mathbb{I}, J_k\right] - \mathrm{i}\left[\epsilon_{kjm}\left(\alpha K_m + \beta P_m\right) + \left(\alpha B_{kj} + \beta C_{kj}\right)\mathbb{I}, \alpha K_i + \beta P_i\right] \\
&\quad + \mathrm{i}\left[\epsilon_{kim}\left(\alpha K_m + \beta P_m\right) + \left(\alpha B_{ki} + \beta C_{ki}\right)\mathbb{I}, \alpha K_j + \beta P_j\right] \\
&= \epsilon_{kjm}\left(\alpha^2 V_{mi} + \beta^2 Y_{mi} + \alpha\beta\left(U_{mi} - U_{im}\right)\right)\mathbb{I} - \epsilon_{kim}\left(\alpha^2 V_{mj} + \beta^2 Y_{mj} + \alpha\beta\left(U_{mj} - U_{jm}\right)\right)\mathbb{I} \\
&= \epsilon_{kjl}\epsilon_{kjm}\left(\alpha^2 V_{mi} + \beta^2 Y_{mi} + \alpha\beta\left(U_{mi} - U_{im}\right)\right)\mathbb{I} - \epsilon_{kjl}\epsilon_{kim}\left(\alpha^2 V_{mj} + \beta^2 Y_{mj} + \alpha\beta\left(U_{mj} - U_{jm}\right)\right)\mathbb{I} \\
&= 2\left(\alpha^2 V_{li} + \beta^2 Y_{li} + \alpha\beta\left(U_{li} - U_{il}\right)\right)\mathbb{I} - \left(\delta_{ji}\delta_{lm} - \delta_{jm}\delta_{li}\right)\left(\alpha^2 V_{mj} + \beta^2 Y_{mj} + \alpha\beta\left(U_{mj} - U_{jm}\right)\right)\mathbb{I} \\
&= \left(\alpha^2 V_{li} + \beta^2 Y_{li} + \alpha\beta\left(U_{li} - U_{il}\right)\right)\mathbb{I}.
\end{aligned}$$

Since α and β are arbitrary, we see that

$$V_{ij} = 0, \quad Y_{ij} = 0, \quad U_{ij} = U_{ji}.$$

Since U_{ij} is anti-symmetric for $i \neq j$, the last equation implies that we must have $U_{ij} = 0$ for $i \neq j$. That is, we must have $U_{ij} = M\delta_{ij}$.
Similarly, for arbitrary α, β, and γ, we have

$$\begin{aligned}
0 &= \left[\left[J_i, \alpha K_j + \beta P_j + \gamma J_j\right], E\right] + \left[\left[\alpha K_j + \beta P_j + \gamma J_j, E\right], J_i\right] + \left[\left[E, J_i\right], \alpha K_j + \beta P_j + \gamma J_j\right] \\
&= \mathrm{i}\alpha\left[\epsilon_{ijk}K_k + B_{ij}\mathbb{I}, E\right] + \mathrm{i}\beta\left[\epsilon_{ijk}P_k + C_{ij}\mathbb{I}, E\right] + \mathrm{i}\gamma\left[\epsilon_{ijk}J_k + A_{ij}, E\right] \\
&\quad + \mathrm{i}\alpha\left[P_j + D_j\mathbb{I}, J_i\right] + \mathrm{i}\beta\left[X_j\mathbb{I}, J_i\right] + i\gamma\left[W_j\mathbb{I}, J_i\right] - \mathrm{i}\left[W_i\mathbb{I}, \alpha K_j + \beta P_j + \gamma J_j\right] \\
&= \alpha\left(C_{ij} - \epsilon_{ijk}D_k\right)\mathbb{I} - \beta\epsilon_{ijk}X_k\mathbb{I} - \gamma\epsilon_{ijk}W_k\mathbb{I}.
\end{aligned}$$

As α, β, and γ are arbitrary, we immediately see that

$$X_i = 0, \quad W_i = 0, \quad C_{ij} = \epsilon_{ijk}D_k.$$

Let us define the auxiliary variables

$$\tilde{P}_i = P_i + D_i\mathbb{I} \quad , \quad \tilde{K}_i = K_i + \frac{1}{2}\epsilon_{ijk}B_{jk}\mathbb{I} \quad , \quad \tilde{J}_i = J_i + \frac{1}{2}\epsilon_{ijk}A_{jk}\mathbb{I}$$

Then, for $\tilde{K}_i$, we have

$$\begin{aligned}
\epsilon_{ijk}\tilde{K}_k &= \epsilon_{ijk}K_k + \frac{1}{2}\epsilon_{ijk}\epsilon_{kab}B_{ab}\mathbb{I} \\
&= \epsilon_{ijk}K_k + \frac{1}{2}\left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right)B_{ab}\mathbb{I} \\
&= \epsilon_{ijk}K_k + \frac{1}{2}\left(B_{ij} - B_{ji}\right)\mathbb{I} \\
&= \epsilon_{ijk}K_k + B_{ij}\mathbb{I}.
\end{aligned}$$

where the last equality follows since $i \neq j$. Similarly, for $\tilde{J}_i$, we get

$$\epsilon_{ijk}\tilde{J}_k = \epsilon_{ijk}J_k + A_{ij}\mathbb{I}.$$

Thus, the Lie algebra becomes

$$\left[\tilde{J}_i, \tilde{J}_j\right] = i\epsilon_{ijk}\tilde{J}_k, \quad \left[\tilde{J}_i, \tilde{K}_j\right] = i\epsilon_{ijk}\tilde{K}_k, \quad \left[\tilde{J}_i, \tilde{P}_j\right] = i\epsilon_{ijk}\tilde{P}_k,$$

$$\left[\tilde{K}_i, E\right] = i\tilde{P}_i, \quad \left[\tilde{K}_i, \tilde{P}_j\right] = iM\delta_{ij}\mathbb{I},$$

$$\left[\tilde{K}_i, \tilde{K}_j\right] = \left[\tilde{J}_i, E\right] = \left[\tilde{P}_i, E\right] = \left[\tilde{P}_i, \tilde{P}_j\right] = 0.$$

Thus, we can eliminate all the centrally extended variables except M. Therefore, the Galilean Lie algebra can be centrally extended by a single central charge M.

■

Problem 3 - Adjoint Representation
Show that $\left(\hat{\lambda}_k\right)^{\ell}_{m} = \mathrm{i}C^{\ell}_{km}$ is a representation by checking the commutation relations for $\left(\hat{\lambda}_k\right)^{\ell}_{m}$.
The resulting identity is called the Jacobi identity. Why does it hold? What's the dimension of the adjoint representation?

Solution. Let us consider the matrices $\left(\hat{\lambda}_k\right)^{\ell}_{m} = \mathrm{i}C^{\ell}_{km}$, where C^{ℓ}_{km} are the structure constant of a Lie algebra. To verify that $\hat{\lambda}_k$ forms a representation of the Lie algebra, we need to check that they satisfy the same commutation relations as the Lie algebra generators.
A Lie algebra is defined by the commutation relation

$$\left[\hat{\lambda}_k, \hat{\lambda}_m\right] = \mathrm{i}C^{\ell}_{km}\hat{\lambda}_{\ell},$$

where the structure constants C^{ℓ}_{km} are anti-symmetric in (k, m). Computing the left-hand side, we have

$$\begin{aligned}\left[\hat{\lambda}_k, \hat{\lambda}_m\right]^{\ell}_{n} &= \left(\hat{\lambda}_k\hat{\lambda}_m - \hat{\lambda}_m\hat{\lambda}_k\right)^{\ell}_{n} \\ &= \mathrm{i}C^{p}_{kn}\mathrm{i}C^{\ell}_{mp} - \mathrm{i}C^{p}_{mn}\mathrm{i}C^{\ell}_{kp} \\ &= -C^{p}_{kn}C^{\ell}_{mp} + C^{p}_{mn}C^{\ell}_{kp}.\end{aligned}$$

Additionally, the elements of the Lie algebra satisfy the Jacobi identity given by

$$\left[\left[\hat{\lambda}_k, \hat{\lambda}_\ell\right], \hat{\lambda}_m\right] + \left[\left[\hat{\lambda}_\ell, \hat{\lambda}_m\right], \hat{\lambda}_k\right] + \left[\left[\hat{\lambda}_m, \hat{\lambda}_k\right], \hat{\lambda}_\ell\right] = 0.$$

By replacing in the Jacobi identity, we get the following relation

$$C^{n}_{k\ell}C^{p}_{mn} + C^{n}_{\ell m}C^{p}_{kn} + C^{n}_{mk}C^{p}_{\ell n}$$

Define the adjoint expression

$$\left(\hat{\lambda}_k\right)^{a}_{b} = \mathrm{i}C^{a}_{kb},$$

where $\left(\hat{\lambda}_k\right)^{a}_{b}$ is the matrix element of $\hat{\lambda}_k$. Then the commutation relation becomes

$$\begin{aligned}\left[\hat{\lambda}_k, \hat{\lambda}_\ell\right]^{a}_{b} &= \left(\hat{\lambda}_k\right)^{a}_{c}\left(\hat{\lambda}_\ell\right)^{c}_{b} - \left(\hat{\lambda}_\ell\right)^{a}_{c}\left(\hat{\lambda}_k\right)^{c}_{b} \\ &= \mathrm{i}C^{a}_{kc}\mathrm{i}C^{c}_{\ell b} - \mathrm{i}C^{a}_{\ell c}\mathrm{i}C^{c}_{kb} \\ &= -C^{a}_{kc}C^{c}_{\ell b} + C^{a}_{\ell c}C^{c}_{kb} \\ &= -C^{c}_{k\ell}C^{a}_{cb} \\ &= \mathrm{i}C^{c}_{k\ell}\left(\hat{\lambda}_c\right)^{a}_{b}.\end{aligned}$$

Thus, the adjoint expression satisfies the Lie algebra commutation relations and is a representation of the Lie algebra, and is hence called the adjoint representation.
The dimension of the adjoint representation is the same as the dimension of the Lie algebra itself, which is the number of independent generators $\left(\hat{\lambda}_c\right)^{a}_{b}$. If the Lie algebra has n generators, then the dimension of the adjoint representation is n. Particularly, there are range(a) values for a and range(b) values for b, both of which are equal to $\dim(G)$. Thus, the dimension of the adjoint representation is equal to the product, which is $\dim(G)^2$. ■

Homework 4

Problem 1 - Poincare Algebra
Consider the Poincare group $(\Lambda^\mu{}_\nu, a^\mu)$ in some representation $S(\Lambda, a)$. Expand the representation matrices around $\delta^\mu{}_\nu$ and derive the Poincare algebra by imposing the multiplication rule for the Poincare group.

Solution. Consider infinitesimal transformations around the identity

$$\begin{aligned}\Lambda^\mu{}_\nu &= \delta^\mu{}_\nu + \omega^\mu{}_\nu + O(\omega^2),\\ a^\mu &= \epsilon^\mu,\end{aligned}$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$ (anti-symmetry from Lorentz condition).
The representation matrices can be expanded as

$$S(\Lambda, a) = 1 + \frac{i}{2}\omega_{\mu\nu}J^{\mu\nu} + i\epsilon_\mu P^\mu + O(\omega^2, \epsilon^2),$$

where $J^{\mu\nu}$ are generators of Lorentz transformations and P^μ are generators of translations.
For two transformations (Λ_1, a_1) and (Λ_2, a_2), we have

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2).$$

Expanding to first order, we get

$$\begin{aligned}\Lambda^\mu_{1\,\nu} &= \delta^\mu{}_\nu + \omega^\mu_{1\,\nu},\\ \Lambda^\mu_{2\,\nu} &= \delta^\mu{}_\nu + \omega^\mu_{2\,\nu},\\ a_1^\mu &= \epsilon_1^\mu,\\ a_2^\mu &= \epsilon_2^\mu.\end{aligned}$$

From representation multiplication, we get

$$S(\Lambda_1, a_1)S(\Lambda_2, a_2) = S((\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2))$$

The left hand-side expands to

$$(1 + \frac{i}{2}\omega_{1\mu\nu}J^{\mu\nu} + i\epsilon_{1\mu}P^\mu)(1 + \frac{i}{2}\omega_{2\mu\nu}J^{\mu\nu} + i\epsilon_{2\mu}P^\mu),$$

and the right hand-side expands to

$$1 + \frac{i}{2}(\omega_1 + \omega_2)_{\mu\nu}J^{\mu\nu} + i(\epsilon_1 + \epsilon_2 + \omega_1\epsilon_2)_\mu P^\mu.$$

Comparing terms and using anti-symmetry of $\omega^{\mu\nu}$, we have

- For $[J^{\mu\nu}, J^{\rho\sigma}]$:
$$[J^{\mu\nu}, J^{\rho\sigma}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}).$$
- For $[J^{\mu\nu}, P^\rho]$:
$$[J^{\mu\nu}, P^\rho] = i(\eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu).$$
- For $[P^\mu, P^\nu]$:
$$[P^\mu, P^\nu] = 0.$$

Therefore, the complete Poincar algebra is

$$\begin{aligned}[J^{\mu\nu}, J^{\rho\sigma}] &= i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}),\\ [J^{\mu\nu}, P^\rho] &= i(\eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu),\\ [P^\mu, P^\nu] &= 0.\end{aligned}$$

These relations show that the Lorentz generators form a closed subalgebra, the translations commute with each other, and the Lorentz transformations act on translations as a vector representation.

∎

Problem 2 - Symmetries of Euclidean Field Theory
Consider a Euclidean flat space in $\mathbb{R}^{d+1}$. Work out the commutators of the Lie algebra of $SO(d+1)$. Identify the generators of translations and rotations.

Solution. Let's first identify the generators of the Lie algebra $SO(d+1)$. In $\mathbb{R}^{d+1}$, we can denote the generators as $M_{\mu\nu}$ where $\mu, \nu = 0, 1, \ldots, d$. These generators are antisymmetric: $M_{\mu\nu} = -M_{\nu\mu}$.
For rotations in $\mathbb{R}^{d+1}$, the generators $M_{\mu\nu}$ act on coordinates x^ρ as

$$(M_{\mu\nu})^\rho_\sigma = \mathrm{i}(\delta^\rho_\mu \delta_{\nu\sigma} - \delta^\rho_\nu \delta_{\mu\sigma}).$$

The commutation relations can be worked out by considering the action on coordinates

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho}),$$

where $\eta_{\mu\nu}$ is the Euclidean metric ($\delta_{\mu\nu}$ in this case).
In a flat Euclidean space, translations are not part of $SO(d+1)$. The translation generators P_μ would be additional generators that satisfy $[P_\mu, P_\nu] = 0$ and rotations $[M_{\mu\nu}, P_\rho] = i(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu)$.
The rotation generators $M_{\mu\nu}$ can be interpreted geometrically. They generate rotations in the μ-ν plane and there are $\frac{d(d+1)}{2}$ independent generators, each corresponding to a plane of rotation in $\mathbb{R}^{d+1}$.
The algebra $SO(d+1)$ is completely characterized by the commutation relations mentioned above. These are the fundamental relationships that define the structure constants of the Lie algebra.

■

Problem 3 - Proper Length
In $3+1$ dim, a Lorentz vector is the irreducible representation $\left(\frac{1}{2}, \frac{1}{2}\right)$. Write down its proper length in term of the spinor degrees of freedom.

Solution. Consider a Lorentz vector V^μ in $(3+1)$ dimensions. As a $(\frac{1}{2}, \frac{1}{2})$ representation, we can express it using the Pauli matrices $\sigma^\mu = (1, \vec{\sigma})$ as

$$V^{\alpha\dot{\beta}} = V^\mu (\sigma_\mu)^{\alpha\dot{\beta}},$$

where α is the left-handed spinor index and $\dot{\beta}$ is the right-handed spinor index.
The proper length squared of the vector can be written as

$$V_\mu V^\mu = V_0^2 - \vec{V}^2.$$

In terms of spinor components, this becomes

$$V_\mu V^\mu = -\frac{1}{2} V_{\alpha\dot{\beta}} V^{\alpha\dot{\beta}},$$

where the indices are raised and lowered with the ϵ tensor

$$V^{\alpha\dot{\beta}} = \epsilon^{\alpha\gamma} \epsilon^{\dot{\beta}\dot{\delta}} V_{\gamma\dot{\delta}}.$$

The factor of $-\frac{1}{2}$ appears due to the normalization of the Pauli matrices

$$\mathrm{Tr}(\sigma^\mu \sigma^\nu) = 2\eta^{\mu\nu}.$$

This demonstrates how the Lorentz-invariant length of a vector can be expressed purely in terms of its spinor components, making manifest the $SL(2, \mathbb{C})$ structure of the Lorentz group. ■

Problem 4
Show that $SU(2)$ is a double cover of $SO(3)$.

Solution. To show that $SU(2)$ is a double cover of $SO(3)$, we need to demonstrate that there exists a surjective homomorphism from $SU(2)$ to $SO(3)$ where each element in $SO(3)$ is associated with exactly two elements in $SU(2)$ (meaning the kernel of the homomorphism is $\{\pm I\}$), effectively creating a "two-to-one" mapping between the groups.
Let's establish the map $\phi : SU(2) \to SO(3)$. For any $U \in SU(2)$ and vector $\vec{v} \in \mathbb{R}^3$, we can form:

$$V = \vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$$

where σ_i are the Pauli matrices. The map ϕ is defined by,

$$\phi(U) : \vec{v} \mapsto \vec{v}' \quad \text{where} \quad V' = UVU^\dagger.$$

is a homomorphism. We will show that

(i) ϕ maps to $SO(3)$: V' is Hermitian since $(UVU^\dagger)^\dagger = UVU^\dagger$. Additionally, $\det(V') = \det(V)$ since $\det(U) = 1$. The trace is preserved $\mathrm{Tr}(V') = \mathrm{Tr}(V)$. Therefore, $\phi(U)$ preserves length and orientation.

(ii) ϕ is a homomorphism:

$$\begin{aligned} \phi(U_1)\phi(U_2)V &= U_1(U_2VU_2^\dagger)U_1^\dagger \\ &= (U_1U_2)V(U_1U_2)^\dagger \\ &= \phi(U_1U_2)V \end{aligned}$$

(iii) $\ker(\phi) = \{\pm I\}$: If $UVU^\dagger = V$ for all V, then U must commute with all Pauli matrices, but only scalar matrices commute with all Pauli matrices, and since $\det(U) = 1$, then $U = \pm I$.

(iv) ϕ is surjective: Any rotation in $SO(3)$ can be parameterized by axis $\hat{n}$ and angle θ. The corresponding $SU(2)$ element is $U = \exp(-\mathrm{i}\frac{\theta}{2}\hat{n} \cdot \vec{\sigma})$. This covers all of $SO(3)$, but with U and $-U$ giving the same rotation.

We have shown that $SU(2)$ maps onto $SO(3)$ via ϕ and the map is 2-to-1 since $\ker(\phi) = \{\pm I\}$. Therefore, $SU(2)$ is a double cover of $SO(3)$.
This relationship explains why spinors must rotate by 4π to return to their original state, while vectors only need 2π. ∎

Problem 5 - Clifford Algebra
Show that $\sigma^\mu_{\alpha\dot{\alpha}} \quad \bar{\sigma}^\mu_{\alpha i}$ satisfy the Clifford algebra

$$\begin{aligned} (\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_\alpha{}^\beta &= 2\eta^{\mu\nu}\delta_\alpha{}^\beta, \\ (\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} &= 2\eta^{\mu\prime}\delta^{\dot{\alpha}}{}_{\dot{\beta}}. \end{aligned}$$

Solution. Recall the explicit forms

$$\begin{aligned} \sigma^\mu &= (1,\, \vec{\sigma})_{\alpha\dot{\alpha}}, \\ \bar{\sigma}^\mu &= (1,\, -\vec{\sigma})^{\dot{\alpha}\alpha}, \end{aligned}$$

where $\vec{\sigma}$ are the Pauli matrices satisfying $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.
The properties we will apply are

- The anti-commutation relations of Pauli matrices: $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.
- The Minkowski metric: $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

- The identity $(\sigma^i)^2 = 1$ for each Pauli matrix.

(i) **The first relation**: Let us consider cases.

- **Case 1:** $\mu = \nu = 0$

$$\begin{aligned}(\sigma^0\bar{\sigma}^0 + \sigma^0\bar{\sigma}^0)_\alpha{}^\beta &= (1\cdot 1 + 1\cdot 1)_\alpha{}^\beta \\ &= 2\delta_\alpha{}^\beta \\ &= 2\eta^{00}\delta_\alpha{}^\beta.\end{aligned}$$

- **Case 2:** $\mu = 0, \nu = i$ (or vice versa)

$$\begin{aligned}(\sigma^0\bar{\sigma}^i + \sigma^i\bar{\sigma}^0)_\alpha{}^\beta &= (1\cdot(-\sigma^i) + \sigma^i\cdot 1)_\alpha{}^\beta \\ &= (-\sigma^i + \sigma^i)_\alpha{}^\beta \\ &= 0 \\ &= 2\eta^{0i}\delta_\alpha{}^\beta.\end{aligned}$$

- **Case 3:** $\mu = i, \nu = j$

$$\begin{aligned}(\sigma^i\bar{\sigma}^j + \sigma^j\bar{\sigma}^i)_\alpha{}^\beta &= (\sigma^i(-\sigma^j) + \sigma^j(-\sigma^i))_\alpha{}^\beta \\ &= -(\sigma^i\sigma^j + \sigma^j\sigma^i)_\alpha{}^\beta \\ &= -2\delta^{ij}\delta_\alpha{}^\beta \\ &= 2\eta^{ij}\delta_\alpha{}^\beta.\end{aligned}$$

(ii) **The second relation**: The proof follows analogously, with $\bar{\sigma}^\mu$ appearing first in each product. The key difference is the arrangement of dotted indices, but the algebraic structure remains the same due to the properties of the Pauli matrices.

Therefore, both Clifford algebra relations are satisfied for all values of the indices. ■

Problem 6

P^2 is a Lorentz-invariant quantity and commutes with all Poincare generators (it is a quadratic Casimir). The $3+1$ dim Poincare algebra has another quadratic casimir:

$$W^2 = W^\mu W_\mu,$$

where

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}$$

Show that

$$W^2 = \frac{1}{2}P^2 J^{\mu\nu}J_{\nu\mu} + P_\mu J^{\mu\nu}P_\lambda J^\lambda_\nu.$$

Apply this to massless particle states and write it in terms of A, B, and J (the generators we encountered in the little group of massless particles).

Solution. Starting with $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu J_{\alpha\beta}$, we have

$$\begin{aligned}W^2 &= W^\mu W_\mu \\ &= \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\rho\sigma\lambda}P_\nu J_{\alpha\beta}P^\rho J^{\sigma\lambda}.\end{aligned}$$

Using the identity for product of two Levi-Civita symbols, we get

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\rho\sigma\lambda} = -\delta^\nu_\rho\delta^\alpha_\sigma\delta^\beta_\lambda + \delta^\nu_\rho\delta^\alpha_\lambda\delta^\beta_\sigma + \delta^\nu_\sigma\delta^\alpha_\lambda\delta^\beta_\rho - \delta^\nu_\sigma\delta^\alpha_\rho\delta^\beta_\lambda + \delta^\nu_\lambda\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\nu_\lambda\delta^\alpha_\sigma\delta^\beta_\rho,$$

which gives

$$
\begin{aligned}
W^2 &= \frac{1}{4}[-(P\cdot P)(J^{\alpha\beta}J_{\beta\alpha}) + 2P_\mu J^{\mu\nu}P_\lambda J^\lambda_\nu] \\
&= \frac{1}{2}P^2 J^{\mu\nu}J_{\nu\mu} + P_\mu J^{\mu\nu}P_\lambda J^\lambda_\nu.
\end{aligned}
$$

For massless states ($P^2 = 0$), we choose the frame where

$$P^\mu = (E, 0, 0, E).$$

In terms of the little group generators, we have

$$
\begin{aligned}
J_{12} &= J \\
J_{1-} &= A \\
J_{2-} &= B
\end{aligned}
$$

where "$-$" refers to the light-cone combination $x^0 - x^3$.
Thus

$$
\begin{aligned}
W^2 &= P_\mu J^{\mu\nu}P_\lambda J^\lambda_\nu \\
&= E^2(J^2 + A^2 + B^2).
\end{aligned}
$$

For massless states, the first term vanishes due to $P^2 = 0$ and the remaining term gives the square of the helicity operator (J^2) plus contributions from the translational generators ($A^2 + B^2$). For physical states, we typically require $A = B = 0$, leaving only the helicity contribution.
Thus, for physical massless states with $A = B = 0$, we have

$$W^2 = E^2 J^2.$$

This is proportional to the square of the helicity, which is a key quantum number for massless particles. ■

Homework 5

Problem 1 - Propagator as Green Function

(a) Show that both the advanced and retarded propagators are Green functions to the Klein-Gordon operator.

(b) Show that the Feynman propagator is a Green function for the Klein-Gordon operator.

Solution. (a) The Klein-Gordon operator is given by

$$\Box + m^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2.$$

A Green function $G(x)$ satisfies the following relation

$$(\Box + m^2)G(x) = -\delta^4(x).$$

The advanced (G^+) and retarded (G^-) propagators in position space are

$$G^\pm(x) = \mp\frac{\theta(\pm x^0)}{2\pi}\delta(x^2),$$

where $x^2 = (x^0)^2 - \mathbf{x}^2$ is the spacetime interval.
Applying the Klein-Gordon operator to $G^\pm$, we get

$$(\Box + m^2)G^\pm(x) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\left[\mp\frac{\theta(\pm x^0)}{2\pi}\delta(x^2)\right].$$

Using the chain rule and product rule, we get

- **Time derivative terms:**

$$\frac{\partial^2}{\partial t^2}[\theta(\pm x^0)\delta(x^2)] = \theta(\pm x^0)\frac{\partial^2\delta(x^2)}{\partial t^2} \pm \delta'(x^0)\frac{\partial\delta(x^2)}{\partial t} + \delta(x^0)\frac{\partial^2\delta(x^2)}{\partial t^2}$$

- **Spatial derivative terms:**

$$\nabla^2[\theta(\pm x^0)\delta(x^2)] = \theta(\pm x^0)\nabla^2\delta(x^2).$$

We use the key identity

$$\Box\delta(x^2) = -2\pi\delta^4(x).$$

Combining terms, we have

$$\begin{aligned}\Box[\theta(\pm x^0)\delta(x^2)] &= \theta(\pm x^0)\Box\delta(x^2) \pm \delta'(x^0)\frac{\partial\delta(x^2)}{\partial t} + \delta(x^0)\frac{\partial^2\delta(x^2)}{\partial t^2}\\ &= -2\pi\theta(\pm x^0)\delta^4(x) \pm \text{terms with } \delta'(x^0).\end{aligned}$$

The terms with $\delta'(x^0)$ and its derivatives cancel out due to the properties of distributions. Thus,

$$(\Box + m^2)G^\pm(x) = -\delta^4(x).$$

The m^2 term vanishes when acting on the delta function because $\delta(x^2)$ is supported only on the light cone where $x^2 = 0$.

Therefore, both G^+ and G^- are indeed Green functions of the Klein-Gordon operator.

(b) The Feynman propagator in momentum space is given by

$$\tilde{G}_F(p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}.$$

To prove it is a Green function, we need to show that

$$(\Box + m^2)G_F(x) = -\delta^4(x).$$

We start from the momentum space representation and transform to position space

$$G_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}p\cdot x}.$$

Applying the Klein-Gordon operator in position space is equivalent to multiplying by $(p^2 - m^2)$ in momentum space, hence

$$\begin{aligned}(\Box + m^2)G_F(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{i}(p^2 - m^2)}{p^2 - m^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}p\cdot x} \\ &= \int \frac{d^4p}{(2\pi)^4} \mathrm{i}\left(1 - \frac{\mathrm{i}\epsilon}{p^2 - m^2 + \mathrm{i}\epsilon}\right) e^{-\mathrm{i}p\cdot x} \\ &= \int \frac{d^4p}{(2\pi)^4} \mathrm{i}e^{-\mathrm{i}p\cdot x} - \int \frac{d^4p}{(2\pi)^4} \frac{\epsilon}{p^2 - m^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}p\cdot x}.\end{aligned}$$

The first term yields

$$\int \frac{d^4p}{(2\pi)^4} \mathrm{i}e^{-\mathrm{i}p\cdot x} = -\delta^4(x).$$

For the second term, we can show that as $\epsilon \to 0$

$$\lim_{\epsilon\to 0} \int \frac{d^4p}{(2\pi)^4} \frac{\epsilon}{p^2 - m^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}p\cdot x} = 0.$$

This follows because the integrand is bounded by ϵ multiplied by a function that has a $1/p^2$ falloff and the exponential factor oscillates rapidly for large p.

Thus,

$$(\Box + m^2)G_F(x) = -\delta^4(x).$$

We can also understand this result by noting that the Feynman propagator can be written as

$$G_F(x) = \theta(x^0)G^+(x) + \theta(-x^0)G^-(x),$$

where $G^\pm$ are the advanced and retarded propagators.

Since we've already shown that $G^\pm$ are Green functions

$$(\Box + m^2)G^\pm(x) = -\delta^4(x).$$

Using the properties of the step function, we have

$$\theta(x^0) + \theta(-x^0) = 1$$

It follows that

$$\begin{aligned}(\Box + m^2)G_F(x) &= (\Box + m^2)\left[\theta(x^0)G^+(x) + \theta(-x^0)G^-(x)\right] \\ &= -\theta(x^0)\delta^4(x) - \theta(-x^0)\delta^4(x) \\ &= -\delta^4(x).\end{aligned}$$

Therefore, the Feynman propagator is indeed a Green function of the Klein-Gordon operator.

■

Problem 2 - Causality Propagator

(a) Calculate the two point function of massive free fields in $d+1$ dimensions (hint: you might need Bessel functions).

(b) Calculate the commutator.

Solution. (a) Consider the action of the field operators on the vacuum states, given by

$$\begin{aligned}
\phi(x)\left|\Omega\right\rangle &= \int \left(\mathrm{e}^{-\mathrm{i}k_\mu x^\mu} a_{\mathbf{k}} \left|\Omega\right\rangle + \mathrm{e}^{\mathrm{i}k_\mu x^\mu} a^\dagger_{\mathbf{k}} \left|\Omega\right\rangle\right) \widetilde{\mathrm{d}k} \\
&= \int \mathrm{e}^{\mathrm{i}k_\mu x^\mu} a^\dagger_{\mathbf{k}} \left|\Omega\right\rangle \widetilde{\mathrm{d}k}, \\
\left\langle\Omega\right| \phi(x) &= \int \left(\left\langle\Omega\right| a_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}k_\mu x^\mu} + \left\langle\Omega\right| a^\dagger_{\mathbf{k}} \mathrm{e}^{\mathrm{i}k_\mu x^\mu}\right) \widetilde{\mathrm{d}k} \\
&= \int \left\langle\Omega\right| a_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}k_\mu x^\mu} \widetilde{\mathrm{d}k}.
\end{aligned}$$

The expectation value is

$$\begin{aligned}
\left\langle\Omega\middle|a_{\mathbf{k}} a^\dagger_{\mathbf{k}'}\middle|\Omega\right\rangle &= \left\langle\Omega\middle|\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}\right]\middle|\Omega\right\rangle + \left\langle\Omega\middle|a^\dagger_{\mathbf{k}'} a_{\mathbf{k}}\middle|\Omega\right\rangle \\
&= (2\pi)^d \left(2\omega_{\mathbf{k}'}\right) \delta^d\left(\mathbf{k}-\mathbf{k}'\right).
\end{aligned}$$

The two point function becomes

$$\begin{aligned}
D(x-y) &= \left\langle\Omega|\phi(x)\phi(y)|\Omega\right\rangle \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d\left(2\omega_{\mathbf{k}}\right)} \frac{\mathrm{d}^d k'}{(2\pi)^d\left(2\omega_{\mathbf{k}'}\right)} \left\langle\Omega\middle|a_{\mathbf{k}} a^\dagger_{\mathbf{k}'}\middle|\Omega\right\rangle \mathrm{e}^{-\mathrm{i}k_\mu x^\mu + \mathrm{i}k'_\mu y^\mu} \\
&= \int \frac{\mathrm{d}^d k\, \mathrm{d}^d k'}{(2\pi)^d\left(2\omega_{\mathbf{k}}\right)} \delta^d\left(\mathbf{k}-\mathbf{k}'\right) \mathrm{e}^{-\mathrm{i}k_\mu x^\mu + \mathrm{i}k'_\mu{}^\mu y^\mu} \bigg|_{k^0=\omega_{\mathbf{k}}, k'^0=\omega_{\mathbf{k}'}} \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d\left(2\omega_{\mathbf{k}}\right)} \mathrm{e}^{-\mathrm{i}k_\mu\left(x^\mu-y^\mu\right)} \bigg|_{k^0=\omega_{\mathbf{k}}}.
\end{aligned}$$

Since two point function is a scalar, it can be further simplified by transforming to other frames. This necessitates us splitting the calculation into two cases, depending on the separation of the events x and y.

- **Time-like Separation:** If the events are time-like separated then we can transform to a frame in which $\mathbf{x}-\mathbf{y}=0$, $x^0-y^0=t$. Then, the two point function becomes

$$\begin{aligned}
D(t) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d(2\omega_{\mathbf{k}})} \mathrm{e}^{-\mathrm{i}\omega_k t} \\
&= \int \frac{k^{d-1}\,\mathrm{d}k\,\mathrm{d}\Omega}{(2\pi)^d\left(2\omega_{\mathbf{k}}\right)} \mathrm{e}^{-\mathrm{i}\omega_k t} \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_0^\infty \frac{k^{d-1}}{\omega_k} \mathrm{e}^{-\mathrm{i}\omega_k t}\,\mathrm{d}k.
\end{aligned}$$

  Performing a change of variables from k to $E \equiv \omega_k$. Since $E^2 = k^2 + m^2$ and $k \geq 0$, we get

$$k = \sqrt{E^2 - m^2}, \quad E\,\mathrm{d}E = k\,\mathrm{d}k.$$

Thus, the integral becomes

$$D(t) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_m^\infty \left(E^2 - m^2\right)^{\frac{d-2}{2}} \mathrm{e}^{-\mathrm{i}Et}\,\mathrm{d}E.$$

We can represent this in terms of Hankel functions. Define a variable z as $E = zm$. Then

$$D(t) = \frac{m^{d-1}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_1^\infty \mathrm{e}^{-\mathrm{i}mtz} \left(z^2 - 1\right)^{\frac{d-2}{2}} \mathrm{d}z.$$

The Hankel function of the first kind has the integral representation

$$H_\nu^{(1)}(-t) = -\frac{2}{\sqrt{\pi}}\frac{\mathrm{i}}{\Gamma\left(\nu + \frac{1}{2}\right)}\left(\frac{t}{2}\right)^\nu \int_1^\infty \mathrm{e}^{-itz}\left(z^2-1\right)^{\nu-\frac{1}{2}}\mathrm{d}z.$$

Taking $\nu = \frac{d-1}{2}$, the two point function can be written in terms of the Hankel function of the first kind as

$$D(t) = \frac{\mathrm{i}}{4}\left(\frac{m}{2\pi t}\right)^{\frac{d-1}{2}} H^{(1)}_{\frac{d-1}{2}}(-mt).$$

- **Space-like Separation:** If the events are space-like separated, then we can transform to a frame in which $\mathbf{x} - \mathbf{y} = \mathbf{r}$, $x^0 - y^0 = 0$. Then, the two point function becomes

$$D(\mathbf{r}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

For the $3+1$-dimensional case. we transform to a frame where $\mathbf{x} - \mathbf{y} = r\hat{z}$, $x^0 - y^0 = 0$. Then, the two point function becomes

$$D(\mathbf{r}) = \int \frac{k^2\,\mathrm{d}k\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi}{(2\pi)^3\,(2\omega_{\mathbf{k}})}\mathrm{e}^{\mathrm{i}kr\cos(\theta)} = \frac{1}{(2\pi)^2}\int_0^\infty \frac{k^2\,\mathrm{d}k}{2\sqrt{k^2+m^2}}\int_0^\pi \mathrm{d}\theta\sin(\theta)\mathrm{e}^{\mathrm{i}kr\cos(\theta)}.$$

The angular integral can easily be evaluated by the substitution $x = \cos(\theta)$ as

$$\frac{1}{2}\int_0^\pi \mathrm{d}\theta\sin\theta\mathrm{e}^{\mathrm{i}kr\cos(\theta)} = -\frac{1}{2}\int_1^{-1}\mathrm{d}x\mathrm{e}^{\mathrm{i}krx} = \frac{1}{2\mathrm{i}kr}\left(\mathrm{e}^{\mathrm{i}kr} - \mathrm{e}^{-\mathrm{i}kr}\right) = \frac{\sin(kr)}{kr}.$$

Substituting this back into the two point function, we get

$$D(\mathbf{r}) = \frac{1}{(2\pi)^2}\int_0^\infty \frac{k^2\,\mathrm{d}k}{\sqrt{k^2+m^2}}\frac{\sin(kr)}{kr}.$$

As it turns out, this can be written in terms of Bessel functions. To see this, let us define the variable $z = kr$. Then, we get

$$D(\mathbf{r}) = \frac{1}{(2\pi r)^2}\int_0^\infty \mathrm{d}z\frac{z\sin(z)}{\sqrt{z^2+m^2r^2}}.$$

The modified Bessel function of the second kind has the integral representation

$$K_1(x) = -\frac{1}{x}\int_0^\infty \mathrm{d}z\frac{z\sin(z)}{\sqrt{z^2+x^2}}.$$

Then, the two point function can be written in terms of modified Bessel functions of the second kind as

$$D(\mathbf{r}) = -\frac{m}{4\pi^2 r}K_1(mr).$$

(b) Let us evaluate the product of two fields by a mode expansion

$$
\begin{aligned}
\phi(x)\phi(y) &= \int \widetilde{\mathrm{d}k}\widetilde{\mathrm{d}k'} \left(\mathrm{e}^{-\mathrm{i}k_\mu x^\mu} a_{\mathbf{k}} + \mathrm{e}^{\mathrm{i}k_\mu x^\mu} a_{\mathbf{k}}^\dagger\right)\left(\mathrm{e}^{-\mathrm{i}k'_\mu y^\mu} a_{\mathbf{k}'} + \mathrm{e}^{\mathrm{i}k' y^\mu y^\mu} a_{\mathbf{k}'}^\dagger\right) \\
&= \int \widetilde{\mathrm{d}k}\widetilde{\mathrm{d}k'} \left(\mathrm{e}^{-\mathrm{i}\left(k_\mu x^\mu + k'_\mu y^\mu\right)} a_{\mathbf{k}} a_{\mathbf{k}'} + \mathrm{e}^{-\mathrm{i}\left(k_\mu x^\mu - k'_\mu y^\mu\right)} a_{\mathbf{k}} a_{\mathbf{k}'}^\dagger\right) \\
&= \int \widetilde{\mathrm{d}k}\widetilde{\mathrm{d}k'} \left(\mathrm{e}^{\mathrm{i}\left(k_\mu x^\mu - k'_\mu y^\mu\right)} a_{\mathbf{k}}^\dagger a_{\mathbf{k}'} + \mathrm{e}^{\mathrm{i}\left(k_\mu x^\mu + k'_\mu y^\mu\right)} a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger\right).
\end{aligned}
$$

Then, the commutator becomes

$$
\begin{aligned}
[\phi(x), \phi(y)] &= \int \widetilde{\mathrm{d}k}\widetilde{\mathrm{d}k'} \left(\mathrm{e}^{-\mathrm{i}\left(k_\mu x^\mu - k'_\mu y^\mu\right)} \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger\right] - \mathrm{e}^{\mathrm{i}\left(k_\mu x^\mu - k'_\mu y^\mu\right)} \left[a_{\mathbf{k}'}, a_{\mathbf{k}}^\dagger\right]\right) \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})} \frac{\mathrm{d}^d k'}{(2\pi)^d\,(2\omega_{\mathbf{k}'})} \left(\mathrm{e}^{-i\left(k_\mu x^\mu - k'_\mu y^\mu\right)} \left[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger\right] - \mathrm{e}^{i\left(k_\mu x^\mu - k'_\mu y^\mu\right)} \left[a_{\mathbf{k}'}, a_{\mathbf{k}}^\dagger\right]\right) \\
&= \int \frac{\mathrm{d}^d k\,\mathrm{d}^d k'}{(2\pi)^d\,(2\omega_{\mathbf{k}})} \delta^d\left(\mathbf{k} - \mathbf{k}'\right) \left(\mathrm{e}^{-i\left(k_\mu x^\mu - k_\mu{}_\mu y^\mu\right)} - \mathrm{e}^{i\left(k_\mu x^\mu - k'_\mu y^\mu\right)}\right)\Bigg|_{k^0=\omega_{\mathbf{k}}, k'^0=\omega_{\mathbf{k}'}} \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})} \left(\mathrm{e}^{-\mathrm{i}k_\mu\left(x^\mu - y^\mu\right)} - \mathrm{e}^{\mathrm{i}k_\mu\left(x^\mu - y^\mu\right)}\right)\Bigg|_{k^0=\omega_{\mathbf{k}}} \\
&= D(x-y) - D(y-x).
\end{aligned}
$$

Again, just like the two point function, this can be further simplified depending on the separation between the events. For space-like separation, going to a frame in which $\mathbf{x} - \mathbf{y} = \mathbf{r}$, $x^0 - y^0 = 0$ gives

$$
[\phi(x), \phi(y)] = D(\mathbf{r}) - D(-\mathbf{r}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.
$$

But, if we perform the transformation $k \to -k$ on the second term, we get

$$
\int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.
$$

Thus,

$$
[\phi(x), \phi(y)] = 0.
$$

Hence, for space-like separation, the commutator exactly equals zero. Similarly, for time-like separation, we transform to a frame in which $\mathbf{x} - \mathbf{y} = 0$, $x^0 - y^0 = t$ to get

$$
\begin{aligned}
[\phi(x), \phi(y)] &= D(t) - D(-t) \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_m^\infty \left(E^2 - m^2\right)^{\frac{d-2}{2}} \left(\mathrm{e}^{-\mathrm{i}Et} - \mathrm{e}^{\mathrm{i}Et}\right) \mathrm{d}E \\
&= -\frac{2\mathrm{i}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_m^\infty \left(E^2 - m^2\right)^{\frac{d-2}{2}} \sin(Et)\,\mathrm{d}E \\
&= \frac{\mathrm{i}}{4}\left(\frac{m}{2\pi t}\right)^{\frac{d-1}{2}} \left(H^{(1)}_{\frac{d-1}{2}}(-mt) - (-1)^{\frac{d-1}{2}} H^{(1)}_{\frac{d-1}{2}}(mt)\right).
\end{aligned}
$$

■

Problem 3 - Charge Conjugation
Derive the action of the charge conjugation operator on a Dirac spinor.

Solution. Consider a Dirac spinor

$$\psi = \begin{pmatrix} \chi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{pmatrix}.$$

As the components transform under charge conjugation by

$$C\chi_\alpha C^{-1} = \xi_\alpha, \quad C\xi_\alpha C^{-1} = \chi_\alpha,$$

the Dirac spinor transforms under charge conjugation by

$$\psi^C = C\psi C^{-1} = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}.$$

This form of ψ^C suggests that there should be a way to represent ψ^C in terms of ψ. Let us try to see if that is actually the case. ψ to ψ^C shows that we need to transfer the adjoint from ξ to χ in ψ. This suggests that the adjoint of ψ should play a role. If we compute it, we get

$$\psi^\dagger = \begin{pmatrix} \chi^\dagger_{\dot{\alpha}} & \xi^\alpha \end{pmatrix}.$$

While this has the adjoint in the right terms, the positions of ξ and χ must be swapped. This can be achieved, if we remember than the γ^0 matrix can be written, in spinor indices, as

$$\gamma^0 = \begin{pmatrix} 0 & \delta^{\dot{\beta}}_{\dot{\alpha}} \\ \delta^\alpha_\beta & 0 \end{pmatrix}.$$

Then, if we multiply γ^0 to $\psi^\dagger$, we get

$$\bar{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \xi^\alpha & \chi^\dagger_{\dot{\alpha}} \end{pmatrix}.$$

This fixes the position, but the vector obtained yet is not in the right form to be equated to ψ^C. What we need is its transpose

$$\bar{\psi}^T = \begin{pmatrix} \xi^\alpha \\ \chi^\dagger_{\dot{\alpha}} \end{pmatrix}.$$

This, in turn, is almost equal to ψ^C, but not exactly. The indices in this expression are in the wrong locations. To fix this, let us use the charge conjugation matrix

$$\mathcal{C} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$

Then, if we multiply the charge conjugation matrix to $\bar{\psi}^T$, we get

$$\mathcal{C}\bar{\psi}^T = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix} = \psi^C.$$

Thus, the charge conjugation operation is represented by

$$\psi^C = C\psi C^{-1} = \mathcal{C}\bar{\psi}^T.$$

■

Problem 4 - Gordon Identities
Derive the identities

$$\bar{u}_{s'}\left(\vec{p}'\right)\left((p'+p)^\mu - 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 u_s(\vec{p}) = 0,$$
$$\bar{v}_{s'}\left(\vec{p}'\right)\left((p'+p)^\mu - 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 v_s(\vec{p}) = 0.$$

Solution. We have

$$\begin{aligned}
p^\mu + 2iS^{\mu\nu}p_\nu &= \frac{1}{2}\left(2\eta^{\mu\nu}\right)p_\nu - \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]p_\nu \\
&= \frac{1}{2}\left(\left\{\gamma^\mu, \gamma^\nu\right\} - \left[\gamma^\mu, \gamma^\nu\right]\right)p_\nu \\
&= \not{p}\gamma^\mu \\
p^\mu - 2iS^{\mu\nu}p_\nu &= \frac{1}{2}\left(2\eta^{\mu\nu}\right)p_\nu + \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]p_\nu \\
&= \frac{1}{2}\left(\left\{\gamma^\mu, \gamma^\nu\right\} + \left[\gamma^\mu, \gamma^\nu\right]\right)p_\nu \\
&= \gamma^\mu \not{p}
\end{aligned}$$

The contents in the brackets of the Gordon identities can be simplified to give

$$\begin{aligned}
(p'+p)^\mu + 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu &= p'^\mu + 2\mathrm{i}S^{\mu\nu}p'_\nu + p^\mu - 2\mathrm{i}S^{\mu\nu}p_\nu \\
&= \not{p}''\gamma^\mu + \gamma^\mu\not{p} \\
\implies \left((p'+p)^\mu + 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 &= \left(\not{p}''\gamma^\mu + \gamma^\mu\not{p}\right)\gamma_5 \\
&= \not{p}''\gamma^\mu\gamma_5 - \gamma^\mu\gamma_5\not{p}
\end{aligned}$$

where the last equality follows because γ_5 anti-commutes with γ^μ. Using this and the equations of motion for u_s and v_s, we have

$$\begin{aligned}
\bar{u}_s(\mathbf{p})(\not{p}+m) = 0, &\quad (\not{p}+m)u_s(\mathbf{p}) = 0 \\
\bar{v}_s(\mathbf{p})(\not{p}-m) = 0, &\quad (\not{p}-m)v_s(\mathbf{p}) = 0
\end{aligned}$$

Deriving the Gordon identities, starting with the u_s case, we have

$$\begin{aligned}
\bar{u}_{s'}\left(\mathbf{p}'\right)\left((p'+p)^\mu + 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 u_s(\mathbf{p}) &= \bar{u}_{s'}\left(\mathbf{p}'\right)\left(\not{p}'\gamma^\mu\gamma_5 - \gamma^\mu\gamma_5\not{p}\right)u_s(\mathbf{p}) \\
&= \bar{u}_{s'}\left(\mathbf{p}'\right)\left(-m\gamma^\mu\gamma_5 + m\gamma^\mu\gamma_5\right)u_s(\mathbf{p}) \\
&= 0,
\end{aligned}$$

where the second equality followed from the equations of motion. The v_s case follows similarly

$$\begin{aligned}
\bar{v}_{s'}\left(\mathbf{p}'\right)\left((p'+p)^\mu + 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 v_s(\mathbf{p}) &= \bar{v}_{s'}\left(\mathbf{p}'\right)\left(\not{p}'\gamma^\mu\gamma_5 - \gamma^\mu\gamma_5\not{p}\right)v_s(\mathbf{p}) \\
&= \bar{u}_{s'}\left(\mathbf{p}'\right)\left(m\gamma^\mu\gamma_5 - m\gamma^\mu\gamma_5\right)v_s(\mathbf{p}) \\
&= 0.
\end{aligned}$$

■

Homework 6

Problem 1 - Double Well Potential
Consider a particle of mass m in a 1D potential $V(q) = \lambda(q^2 - q_0^2)^2$. Write down a path-integral in imaginary time that computes

$$\left\langle q_0, \frac{\tau}{2} \middle| -q_0, -\frac{\tau}{2} \right\rangle = \left\langle q_0 \middle| \mathrm{e}^{-\frac{H\tau}{\hbar}} \middle| -q_0 \right\rangle.$$

(a) What is the Euclidean Lagrangian? How is it different from the real-time Lagrangian?

(b) Write down the Euler-Lagrange equation for this Euclidean action.

(c) Compute the path-integral with this potential.

Solution. Recall that for imaginary time path integrals, we make the Wick rotation $t \to -\mathrm{i}\tau$. This transforms the quantum amplitude into a statistical partition function.

(a) The Euclidean Lagrangian is obtained from the regular Lagrangian by the Wick rotation. The regular Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}m\dot{q}^2 - V(q)\\ &= \frac{1}{2}m\dot{q}^2 - \lambda(q^2 - q_0^2)^2.\end{aligned}$$

Under $t \to -\mathrm{i}\tau$, we have $\dot{q} \to \mathrm{i}\frac{\mathrm{d}q}{\mathrm{d}\tau}$, giving us the Euclidean Lagrangian

$$\begin{aligned}\mathcal{L}_E &= \frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q)\\ &= \frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + \lambda(q^2 - q_0^2)^2.\end{aligned}$$

The key difference is the sign change before the potential term, and that we're now working with derivatives with respect to τ instead of t.

(b) The Euler-Lagrange equation for the Euclidean action is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}_E}{\partial\dot{q}}\right) - \frac{\partial\mathcal{L}_E}{\partial q} = 0.$$

Substituting our Euclidean Lagrangian, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(m\frac{\mathrm{d}q}{\mathrm{d}\tau}\right) = -\frac{\partial V}{\partial q} = -4\lambda q(q^2 - q_0^2).$$

This simplifies to

$$m\frac{\mathrm{d}^2q}{\mathrm{d}\tau^2} = -4\lambda q(q^2 - q_0^2).$$

(c) To compute the path integral, we need to evaluate

$$\left\langle q_0 \middle| \mathrm{e}^{-\frac{H\tau}{\hbar}} \middle| -q_0 \right\rangle = \mathcal{N}\int_{q(-\tau/2)=-q_0}^{q(\tau/2)=q_0} \mathcal{D}q(\tau)\exp\left(-\frac{1}{\hbar}\int_{-\tau/2}^{\tau/2}\mathcal{L}_E\,\mathrm{d}\tau\right)$$

The full path integral is

$$\mathcal{N}\int \mathcal{D}q(\tau)\exp\left(-\frac{1}{\hbar}\int_{-\tau/2}^{\tau/2}\left[\frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2+\lambda(q^2-q_0^2)^2\right]\mathrm{d}\tau\right)$$

This path integral describes quantum tunneling between the two degenerate minima at $q=\pm q_0$ of the double-well potential. The exact solution involves instantons - solutions where the particle tunnels from $-q_0$ to q_0 in imaginary time.

The leading contribution to this path integral comes from a single instanton solution, which follows the classical equation of motion in imaginary time. The full result includes contributions from multi-instanton configurations, but the single instanton typically dominates for large τ.

For large τ, the amplitude behaves as

$$\left\langle q_0\middle|\mathrm{e}^{-\frac{H\tau}{\hbar}}\middle|-q_0\right\rangle \sim e^{-S_E[\text{instanton}]/\hbar},$$

where S_E[instanton] is the Euclidean action evaluated on the instanton solution.

■

Problem 2 - Free Propagator
Derive the path-integral expression for the propagator of a free particle in d spatial dimensions with the action $S_0=\frac{1}{2}m^2\int\dot{q}^2$ by discretizing time.

Solution. We wish to find the propagator $\langle\mathbf{q}_f,t_f|\mathbf{q}_i,t_i\rangle$ for a free particle in d dimensions. Let us split the time interval t_f-t_i into n equal parts

$$\Delta t=\frac{t_f-t_i}{n},\quad t_{r+1}=t_r+\Delta t,\quad (\mathbf{q}_0,t_0)=(\mathbf{q}_i,t_i),\quad (\mathbf{q}_n,t_n)=(\mathbf{q}_f,t_f).$$

The transition element then becomes

$$\begin{aligned}\langle\mathbf{q}_f,t_f|\mathbf{q}_i,t_i\rangle &= \int\left(\prod_{r=1}^{n-1}\mathrm{d}\mathbf{q}_r\right)\left(\prod_{r=0}^{n-1}\langle\mathbf{q}_{r+1},t_{r+1}|\mathbf{q}_r,t_r\rangle\right)\\ &= \int\left(\prod_{r=1}^{n-1}\mathrm{d}\mathbf{q}_r\right)\left(\prod_{r=0}^{n-1}\left\langle\mathbf{q}_{r+1}\middle|e^{-\mathrm{i}H\Delta t}\middle|\mathbf{q}_r\right\rangle\right).\end{aligned}$$

Each matrix element can be simplified by inserting resolutions of identity in momentum space

$$\begin{aligned}\left\langle\mathbf{q}_{r+1}\middle|e^{-\mathrm{i}H\Delta t}\middle|\mathbf{q}_r\right\rangle &= \int\frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d}\left\langle\mathbf{q}_{r+1}\middle|e^{-\mathrm{i}H\Delta t}\middle|\mathbf{p}_r\right\rangle\langle\mathbf{p}_r|\mathbf{q}_r\rangle\\ &= \int\frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d}\langle\mathbf{q}_{r+1}|\mathbf{p}_r\rangle\langle\mathbf{p}_r|\mathbf{q}_r\rangle\exp\left(-\mathrm{i}H\left(\frac{\mathbf{q}_{r+1}+\mathbf{q}_r}{2},\mathbf{p}_r\right)\Delta t\right)\\ &= \int\frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d}\exp\left(\mathrm{i}\mathbf{p}_r\cdot(\mathbf{q}_{r+1}-\mathbf{q}_r)\right)\exp\left(-\mathrm{i}H\left(\frac{\mathbf{q}_{r+1}+\mathbf{q}_r}{2},\mathbf{p}_r\right)\Delta t\right).\end{aligned}$$

For a free particle, $H=\frac{\mathbf{p}^2}{2m}$, so the transition relation becomes

$$\langle\mathbf{q}_f,t_f|\mathbf{q}_i,t_i\rangle=\int\left(\prod_{r=0}^{n-1}\frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d}\right)\left(\prod_{r=1}^{n-1}\mathrm{d}\mathbf{q}_r\right)\exp\left(\mathrm{i}\sum_{r=0}^{n-1}\Delta t\left[\mathbf{p}_r\cdot\frac{\mathbf{q}_{r+1}-\mathbf{q}_r}{\Delta t}-\frac{\mathbf{p}_r^2}{2m}\right]\right).$$

Going to the continuum limit, this becomes

$$\langle\mathbf{q}_f,t_f||\mathbf{q}_i,t_i\rangle=\int\mathcal{D}\mathbf{p}\mathcal{D}\mathbf{q}\exp\left(\mathrm{i}\int\mathrm{d}t\left[\mathbf{p}\cdot\dot{\mathbf{q}}-\frac{\mathbf{p}^2}{2m}\right]\right).$$

The momentum integral is Gaussian and can be evaluated. For each r, we have

$$\int \frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d} \exp\left(-\mathrm{i}\Delta t\left[\frac{\mathbf{p}_r^2}{2m} - \mathbf{p}_r \cdot \frac{\mathbf{q}_{r+1}-\mathbf{q}_r}{\Delta t}\right]\right) = \left(\frac{m}{2\pi\mathrm{i}\Delta t}\right)^{d/2}.$$

Thus, the path integral becomes

$$\langle \mathbf{q}_f, t_f || \mathbf{q}_i, t_i\rangle = \left(\frac{m}{2\pi\mathrm{i}\Delta t}\right)^{nd/2} \int \left(\prod_{r=1}^{n-1} \mathrm{d}\mathbf{q}_r\right) \exp\left(\mathrm{i}\sum_{r=0}^{n-1} \Delta t \frac{m}{2}\left(\frac{\mathbf{q}_{r+1}-\mathbf{q}_r}{\Delta t}\right)^2\right).$$

Going to the continuum limit, this becomes

$$\langle \mathbf{q}_f, t_f || \mathbf{q}_i, t_i\rangle = \int \mathcal{D}\mathbf{q} \exp\left(\mathrm{i}\int \mathrm{d}t \frac{m}{2}\dot{\mathbf{q}}^2\right).$$

This can be solved to get the final result

$$\langle \mathbf{q}_f, t_f || \mathbf{q}_i, t_i\rangle = \left(\frac{m}{2\pi\mathrm{i}(t_f - t_i)}\right)^{d/2} \exp\left(\frac{\mathrm{i}m(\mathbf{q}_f - \mathbf{q}_i)^2}{2(t_f - t_i)}\right).$$

■

Problem 3 - Path-integral for Complex Scalar
Consider the action for a complex massive free scalar field $\Phi(x)$.

(a) Compute $Z_0[0]$ explicitly.

(b) Write down the action in the presence of sources for both $\Phi(x)$ and $\Phi^*(x)$.

(c) Evaluate the path-integral and compute the two-point function by taking derivatives with respect to J and setting $J = 0$.

(d) Show that differentiating the path-integral gives time-ordered correlators.

Solution. (a) For a complex scalar field, the action without sources is

$$S_0 = \int \mathrm{d}^{d+1}x \left[-\Phi^*(-\Box + m^2)\Phi\right].$$

Thus,

$$Z_0[0] = \int \mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left(\mathrm{i}\int \mathrm{d}^{d+1}x \left[-\Phi^*(-\Box + m^2)\Phi\right]\right).$$

For simplicity, let's convert this to real fields

$$\Phi = \frac{\phi_1 + \mathrm{i}\phi_2}{\sqrt{2}}.$$

The measure transforms as

$$\int \mathcal{D}\Phi\mathcal{D}\Phi^* = C_1 \int \mathcal{D}\phi_1\mathcal{D}\phi_2.$$

The action becomes

$$Z_0[0] = C_1 \int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\left(-\frac{1}{2}\mathrm{i}\int \mathrm{d}^{d+1}x \left[\phi_1(-\Box + m^2)\phi_1 + \phi_2(-\Box + m^2)\phi_2\right]\right).$$

This is the product of two Gaussian integrals

$$Z_0[0] \propto \left[\det\left(\mathrm{i}(-\Box + m^2)\right)\right]^{-1}.$$

(b) With sources, the action becomes

$$S = \int \mathrm{d}^{d+1}x \left[-\Phi^*(-\Box + m^2)\Phi + J^*\Phi + \Phi^* J\right].$$

(c) Let's evaluate the path integral with sources

$$Z_0[J] = \int \mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left(\mathrm{i}\int \mathrm{d}^{d+1}x \left[-\Phi^*(-\Box + m^2)\Phi + J^*\Phi + \Phi^* J\right]\right).$$

To evaluate this, let us shift the field

$$\Phi \to \Phi + f(J).$$

For this to eliminate linear terms, we need

$$(-\Box + m^2)f(J) = J.$$

Thus,

$$f(J)(x) = -\int \mathrm{d}^{d+1}y\Delta_F(x-y)J(y).$$

After completing the square, we get

$$Z_0[J] = Z_0[0]\exp\left(-\mathrm{i}\int \mathrm{d}^{d+1}yJ^*(x)\Delta_F(x-y)J(y)\right).$$

Taking functional derivatives, we have

$$\begin{aligned}\frac{\delta Z_0[J]}{\mathrm{i}\delta J(x_1)} &= -Z_0[J]\int \mathrm{d}^{d+1}yJ^*(y)\Delta_F(y-x_1),\\ \frac{\delta Z_0[J]}{\mathrm{i}\delta J^*(x_1)} &= -Z_0[J]\int \mathrm{d}^{d+1}y\Delta_F(x_1-y)J(y).\end{aligned}$$

Therefore, the two-point functions are

$$\begin{aligned}\frac{1}{Z_0[J]}\frac{\delta^2 Z_0[J]}{(\mathrm{i}\delta J(x_1))(\mathrm{i}\delta J(x_2))}\bigg|_{J=0} &= 0,\\ \frac{1}{Z_0[J]}\frac{\delta^2 Z_0[J]}{(\mathrm{i}\delta J^*(x_1))(\mathrm{i}\delta J^*(x_2))}\bigg|_{J=0} &= 0,\\ \frac{1}{Z_0[J]}\frac{\delta^2 Z_0[J]}{(\mathrm{i}\delta J^*(x_1))(\mathrm{i}\delta J(x_2))}\bigg|_{J=0} &= \mathrm{i}\Delta_F(x_1-x_2),\\ \frac{1}{Z_0[J]}\frac{\delta^2 Z_0[J]}{(\mathrm{i}\delta J(x_1))(\mathrm{i}\delta J^*(x_2))}\bigg|_{J=0} &= \mathrm{i}\Delta_F(x_2-x_1).\end{aligned}$$

(d) To show these give time-ordered correlators, note that functional derivatives give

$$\frac{\delta}{\mathrm{i}\delta J(y)}\int \mathcal{D}\Phi\mathcal{D}\Phi^* O_1(x_1)\cdots O_n(x_n)\mathrm{e}^{\mathrm{i}S} = \int \mathcal{D}\Phi\mathcal{D}\Phi^* O_1(x_1)\cdots O_n(x_n)\Phi^*(y)\mathrm{e}^{\mathrm{i}S}$$

When integrating over paths between initial state $\Phi_a(\mathbf{x})$ at $t=-T$ and final state $\Phi_b(\mathbf{x})$ at $t=T$, this gives

$$\langle\Phi_b|T(\Phi^*(x_1)\cdots\Phi(x_n))|\Phi_a\rangle = \frac{1}{Z_0[J]}\prod_i \frac{\delta}{\mathrm{i}\delta J(x_i)}Z_0[J]\Big|_{J=0},$$

where T denotes time-ordering. Thus the functional derivatives automatically give time-ordered correlators.

■

Problem 4 - Connected Correlators
Consider the generating function $Z(J)$ for all correlators

$$\frac{\delta}{\mathrm{i}\delta J(x_1)}\cdots\frac{\delta}{\mathrm{i}\delta J(x_n)}Z(J)\bigg|_{J=0} = \langle\phi(x_1)\cdots\phi(x_n)\rangle\,.$$

Show that, if we define

$$W(J) = -\mathrm{i}\ln(Z(J)),$$

differentiating $W(J)$ with respect to $\mathrm{i}\delta J$ gives the connected correlators.

Solution. Let's prove this systematically. We can express $Z(J)$ as a sum over Feynman diagrams

$$Z(J) \propto \sum_{\Lambda} D_{\Lambda},$$

where $\{D_{\Lambda}\}$ is the set of all possible Feynman diagrams.
Most of these diagrams are disconnected - they are products of connected diagrams. Let us denote the set of connected diagrams by $\{C_{\lambda}\}$. Then each Feynman diagram can be written as

$$D_{\Lambda} = \prod_{\lambda}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}},$$

where n_{Λ}^{λ} counts the number of copies of diagram C_{λ} in D_{Λ}, and the factorial accounts for identical copies. Thus,

$$Z(J) \propto \sum_{\Lambda}\prod_{\lambda}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}}.$$

Let's split the sum into two pieces - one over diagrams $\tilde{\Lambda}$ with fixed numbers of all types except C_{λ_0}, and the rest over Λ'

$$\begin{aligned}Z(J) &\propto \sum_{\Lambda'}\sum_{\tilde{\Lambda}}\prod_{\lambda\neq\lambda_0}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}}\left(\frac{1}{n_{\Lambda}^{\lambda_0}!}(C_{\lambda_0})^{n_{\Lambda}^{\lambda_0}}\right)\\ &= \sum_{\Lambda'}\prod_{\lambda\neq\lambda_0}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}}\left(\sum_{\tilde{\Lambda}}\frac{1}{n_{\Lambda}^{\lambda_0}!}(C_{\lambda_0})^{n_{\Lambda}^{\lambda_0}}\right).\end{aligned}$$

The sum over $\tilde{\Lambda}$ is equivalent to summing over all possible values of $n_{\Lambda}^{\lambda_0}$

$$\sum_{\tilde{\Lambda}}\frac{1}{n_{\Lambda}^{\lambda_0}!}(C_{\lambda_0})^{n_{\Lambda}^{\lambda_0}} = \sum_{n_{\Lambda}^{\lambda_0}=0}^{\infty}\frac{1}{n_{\Lambda}^{\lambda_0}!}(C_{\lambda_0})^{n_{\Lambda}^{\lambda_0}} = \mathrm{e}^{C_{\lambda_0}}.$$

Thus,

$$Z(J) \propto \mathrm{e}^{C_{\lambda_0}}\sum_{\Lambda'}\prod_{\lambda\neq\lambda_0}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}}.$$

Repeating this process for all other indices λ

$$Z(J) \propto \prod_{\lambda}\mathrm{e}^{C_{\lambda}} = \mathrm{e}^{\sum_{\lambda}C_{\lambda}}.$$

Therefore,

$$W(J) = -\mathrm{i}\ln(Z(J)) = -\mathrm{i}\sum_{\lambda}C_{\lambda}.$$

This means when we differentiate $W(J)$ with respect to $\mathrm{i}\delta J$, we get the sum of only connected diagrams - the connected correlators. ■

Homework 7

Problem 1
Consider a theory of real scalar fields A, B, and C with Lagrangian density

$$\begin{aligned}\delta = &-\frac{1}{2}\partial^\mu A\partial_\mu A - \frac{1}{2}m_A^2 A^2 \\ &-\frac{1}{2}\partial^\mu B\partial_\mu B - \frac{1}{2}m_B^2 B^2 \\ &-\frac{1}{2}\partial^\mu C\partial_\mu C - \frac{1}{2}m_C^2 C^2 \\ &+ gABC\end{aligned}$$

Write down the tree level amplitudes for these processes

$$\begin{array}{ll} AA \to AA & AA \to BC \\ AA \to AB & AB \to AB \\ AA \to BB & AB \to AC \end{array}$$

Solution. Let us start by analyzing the propagators and vertices from the given Lagrangian density. We can then use these to determine the tree-level amplitudes for the given scattering processes. Looking at the Lagrangian density, we can see that it consists of three Klein-Gordon terms and an interaction term. The Klein-Gordon terms lead to the propagators

$$\begin{aligned}\langle AA\rangle &= \frac{\mathrm{i}}{p^2 - m_A^2}, \\ \langle BB\rangle &= \frac{\mathrm{i}}{p^2 - m_B^2}, \\ \langle CC\rangle &= \frac{\mathrm{i}}{p^2 - m_C^2}.\end{aligned}$$

The interaction term $gABC$ leads to a single type of vertex where all three fields meet, with vertex factor ig. Importantly, this means no two fields of the same type can interact directly at a vertex.
For tree-level amplitudes in $\alpha\beta \to \gamma\delta$ scattering, we must have external legs connected by vertices and propagators. Due to the form of our interaction vertex, we can have three possible diagram topologies:

1. s-channel: Initial particles combine at a vertex, intermediate particle propagates, then splits at second vertex
2. t-channel: One initial particle exchanges with one final particle through an intermediate propagator
3. u-channel: Similar to t-channel but with different final state pairing

Let us now analyze each given process:

1. $AA \to AA$: Since our vertex requires three different fields, there are no allowed tree-level diagrams. Therefore
$$\mathcal{M}(AA \to AA) = 0.$$
2. $AA \to AB$: Again, no diagrams are possible since each vertex needs three different fields. Therefore
$$\mathcal{M}(AA \to AB) = 0.$$
3. $AA \to BB$: Here we can have t-channel and u-channel diagrams with a C propagator. Therefore
$$\mathcal{M}(AA \to BB) = -g^2\left(\frac{1}{(p_1 - k_1)^2 - m_C^2} + \frac{1}{(p_1 - k_2)^2 - m_C^2}\right).$$
where p_i are initial momenta and k_i are final momenta.

4. $AA \to BC$: No allowed diagrams since each vertex requires exactly three different fields. Therefore

$$\mathcal{M}(AA \to BC) = 0.$$

5. $AB \to AB$: Here we can have s-channel and u-channel diagrams with a C propagator. Therefore

$$\mathcal{M}(AB \to AB) = -g^2 \left(\frac{1}{(p_1 + p_2)^2 - m_C^2} + \frac{1}{(p_1 - k_2)^2 - m_C^2} \right).$$

6. $AB \to AC$: No allowed diagrams are possible. Therefore

$$\mathcal{M}(AB \to AC) = 0.$$

■

Problem 2 - Rutherford Scattering

The cross section for scattering an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. We treat the field $A_\mu(x)$ as a classical potential and consider the interaction

$$\hat{H}_I = \int \mathrm{d}^3x \; e\hat{\bar{\psi}}\gamma^\mu \hat{\psi} A_\mu,$$

where $\psi(x)$ is a quantized Dirac field.

(a) Compute the T-matrix for electron scattering off a localized classical potential to the lowest order.

(b) If $A_\mu(x)$ is time-independent, it is natural to define

$$\langle p'|\mathrm{i}T|p\rangle \equiv \mathrm{i}M(2\pi)\delta(E_f - E_i),$$

where E_i and E_f are the initial and final energies respectively.

Show that the cross section for scattering off a time-independent localized potential is

$$\mathrm{d}\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{\mathrm{d}^3p_f}{(2\pi)^3}\frac{1}{2E_f}\left|M\left(p_i - p_f\right)\right|^2 (2\pi)\,\delta(E_f - E_i),$$

where v_i is the initial velocity.

Integrate over $|p_f|$ to find $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

Solution. (a) We start with the interaction Hamiltonian

$$H_I = e\int \mathrm{d}^3x \; \bar{\psi}\gamma^\mu\psi A_\mu.$$

To find the T-matrix element to lowest order, we need to evaluate

$$\langle p'|\mathrm{i}T|p\rangle = \left\langle p' \middle| -\mathrm{i}\int \mathrm{d}^3x \; H_I(x) \middle| p \right\rangle.$$

For single electron states we have

$$\langle p'|\bar{\psi}\gamma^\mu\psi|p\rangle = \bar{u}(p')\gamma^\mu u(p)\mathrm{e}^{\mathrm{i}(p'-p)x}.$$

Therefore,

$$\langle p'|\mathrm{i}T|p\rangle = -\mathrm{i}e\int \mathrm{d}^3x \; A_\mu(x)\bar{u}(p')\gamma^\mu u(p)\mathrm{e}^{\mathrm{i}(p'-p)x} = -\mathrm{i}e\tilde{A}_\mu(p'-p)\bar{u}(p')\gamma^\mu u(p),$$

where $\tilde{A}_\mu(q)$ is the Fourier transform of $A_\mu(x)$.

(b) For a time-independent potential, we define

$$\langle p'|\mathrm{i}T|p\rangle = \mathrm{i}M(2\pi)\delta(E_f - E_i).$$

The differential cross section is given by

$$\mathrm{d}\sigma = \frac{1}{v_i}\frac{\mathrm{d}^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{2E_i}|M|^2(2\pi)\delta(E_f - E_i).$$

To integrate over $|p_f|$, we use the energy delta function. First, we write

$$\mathrm{d}^3 p_f = p_f^2\,\mathrm{d}p_f\,\mathrm{d}\Omega.$$

The delta function enforces $E_f = E_i$, which means $|p_f| = |p_i|$ for elastic scattering. Thus,

$$\delta(E_f - E_i) = \delta\left(\sqrt{p_f^2 + m^2} - \sqrt{p_i^2 + m^2}\right) = \frac{E_f}{p_f}\delta(p_f - p_i).$$

Integrating over p_f we have

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{v_i}\frac{p_i^2}{(2\pi)^3}\frac{1}{2E_i}\frac{1}{2E_i}|M|^2(2\pi)\frac{E_i}{p_i} \\ &= \frac{1}{(4\pi)^2}\frac{p_i}{v_i}|M|^2.\end{aligned}$$

Using $v_i = p_i/E_i$, we get our final result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{E_i}{(4\pi)^2}|M|^2.$$

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Problem 3
Any particle that couples to electrons can produce a correction to g − 2. Because the g − 2 agrees with QED to high accuracy, the corrections to g − 2 constrain properties of hypothetical particles.

(a) Consider the Higgs boson h which couples to electron as

$$H_{\text{int}} = \int \mathrm{d}^3x\ \frac{\lambda}{\sqrt{2}}h\bar{\psi}\psi.$$

Compute the contribution of a virtual Higgs boson to election's g − 2, in terms of λ and the mass of Higgs m_h.

(b) If the experimental value of $a = \mathrm{g} - 2$ agrees with a_{QED} up to 10^{-10}, what limit does this place on λ and m_h?

Solution. (a) The one-loop vertex correction from the Higgs boson is given by

$$-\mathrm{i}e\Gamma^\mu = \left(-\mathrm{i}\frac{\lambda}{\sqrt{2}}\right)^2 \mathrm{i}^3 \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{k^2 - m_h^2}\frac{(\not{p}' - \not{k} + m)}{(p' - k)^2 - m^2}\gamma^\mu\frac{(\not{p} - \not{k} + m)}{(p - k)^2 - m^2}.$$

Using Feynman parameters we can write

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ \delta(x + y + z - 1)\frac{1}{(Ax + By + Cz)^3}.$$

After the substitution $l = k - yp' - zp$, and keeping only terms relevant for g-2, we get

$$\Gamma^\mu = \frac{\lambda^2}{16\pi^2}\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\;\frac{\delta(x+y+z-1)}{xm_h^2 + (y+z)^2m^2 - yzq^2}\left[\gamma^\mu - \frac{2m^2}{q^2}((2-y-z)^2-2)\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right].$$

The g-2 factor is related to the form factor $F_2(0)$ by

$$\frac{g-2}{2} = F_2(0).$$

Therefore

$$a = \frac{g-2}{2} = \frac{\lambda^2}{8\pi^2}\int_0^1 \mathrm{d}Y\;\frac{2-(2-Y)^2}{(1-Y)m_h^2 + Y^2m^2},$$

where we changed variables to $Y = y + z$.
Evaluating this integral we get

$$a = \frac{\lambda^2}{8\pi^2}\frac{m^2}{m_h^2}.$$

(b) Given that the experimental value agrees with QED up to 10^{-10}, we must have

$$\frac{\lambda^2}{8\pi^2}\frac{m^2}{m_h^2} < 10^{-10}.$$

Solving for λ

$$\lambda < 10^{-5}\frac{m_h}{m}\sqrt{8\pi^2} \approx 10^{-4}\frac{m_h}{m}.$$

For a Higgs mass of $m_h \approx 125$ GeV and electron mass $m \approx 0.5$ MeV, this gives

$$\lambda < 2.5 \times 10^{-2}.$$

This puts a fairly stringent constraint on the coupling between the Higgs boson and electrons.

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