ASTR 562 - High-Energy Astrophysics

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Spring 2024

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Contents

Homework 1

Problem 1
Radiation: Make a table having a row for each of the electromagnetic wavebands (radio, sub-mm, IR, Optical, UV, X-ray, and Gamma ray). Then add columns describing:

1. whether light from those wavebands can be detected from the ground or space,
2. what technology is used to detect this radiation, and
3. what complications there are in doing astronomy from this waveband.

Solution. The table is as follows

- **Radio**:
  1. Ground
  2. Single radio antennae and large interferometer arrays
  3. Reflection of low frequency radio waves by the plasma of the ionosphere and by the interplanetary and interstellar plasma for frequencies less than 1 MHz
- **Sub-mm**:
  1. Space
  2. Single element detectors (heterodyne receivers or bolometers)
  3. H_2O and CO_2 in Earth's atmosphere
- **Infrared**:
  1. Space
  2. Infrared detector arrays
  3. Strong thermal emitters of infrared radiation in Earth's atmosphere
- **Optical**:
  1. Ground
  2. CCD detector arrays
  3. Short wavelength absorption by the ozone in the upper atmosphere
- **Ultraviolet**:
  1. Space
  2. Ultraviolet spectrographs
  3. Ozone and molecular absorption
- **X-ray**:
  1. Space
  2. Proportional counters and scintillation detectors are used as well as other devices such as CCDs
  3. Photoelectric absorption by the atoms which make up the molecular gases of the atmosphere
- **Gamma ray**:
  1. Space
  2. Scintillation detectors
  3. Photoelectric absorption (between 100 keV and 1 MeV) and Compton scattering and electron–positron pair production at higher energies

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Problem 2
Relativity: Cosmic rays that hit the Earth's atmosphere produce muons as secondary particles (rest mass mc^2 of 105 MeV). They are produced typically at a height of 6 km and have a lifetime of 2200 ns before they decay into electrons and neutrinos. How is it possible that they are detected on the ground? Give a lower limit on their energy. (See Appendix for help).

Solution. Muons are very energetic as they are produced in the Earth's atmosphere at are moving at a velocity very close to the speed of light c with $\beta \approx 0.9999$. Classically, to find the distance covered by muons after being produced, we use

$$d = vt = (299, 792, 458)(2.2 \times 10^{-6}) = 659.5\,\text{m}$$

which is much less than the distance between their point of production and the ground. Since they are travelling at relativistic speeds, we must account for the time dilation they experience. In fact, the classical answer mixes up reference frames

- $\tau = 2.2 \times 10^{-6}\,\text{s}$ is the time in the muon's frame of reference.
- $d = 6\,\text{km}$ is the length of the atmosphere in Earth's frame of reference.

Thus, relativistically, the time them muon experiences is

$$t = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{2.2 \times 10^{-6}}{\sqrt{1 - 0.9999^2}} = 70.712(2.2 \times 10^{-6}) = 1.55 \times 10^{-4}\,\text{s}$$

which gives

$$d = vt = (299, 792, 458)(1.55 \times 10^{-4}) = 46.5\,\text{km} > 6\,\text{km},$$

and, hence, them muon does reach the ground. Equivalently, the same approach can be done using length contraction to explain this.
In the ultra-relativistic case, we find that the total energy $\approx$ kinetic energy $= \gamma m_0 c^2 = 70.712(105) = 7.424\,\text{GeV}$. ■

Problem 3
Plasmas: What is the Debye length for the plasma in the intracluster medium ($kT = 10^8\,\text{K}$, $n = 10^{-3}\,\text{cm}^{-3}$)? What is the Debye length for the plasma in the solar coronae ($kT = 10^7\,\text{K}$, $n = 10^{15}\,\text{cm}^{-3}$)? How about a molecular cloud ($kT = 10\,\text{K}$, $n = 10^7\,\text{cm}^{-3}$)?

Solution. The equation for the Debye length is given by

$$\lambda_D = 69 \left(\frac{T}{n_e} \right)^{\frac{1}{2}}\ \text{m}.$$

- For plasma in the intracluster medium:

$$\lambda_D = 69 \left(\frac{10^8}{10^{-3} \cdot 10^{-6}} \right)^{\frac{1}{2}} = 69 \left(10^{8.5}\right) = 2.181 \times 10^{10}\,\text{m}.$$

- For plasma in the solar coronae:

$$\lambda_D = 69 \left(\frac{10^7}{10^{15} \cdot 10^{-6}} \right)^{\frac{1}{2}} = 69 \left(10^{-1}\right) = 6.9\,\text{m}.$$

- For plasma in the molecular cloud:

$$\lambda_D = 69 \left(\frac{10}{10^7 \cdot 10^{-6}} \right)^{\frac{1}{2}} = 69 \left(10^{8.5}\right) = 69\,\text{m}.$$

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Problem 4
Observing Photons: The light from a faint star has an energy flux of $10^{-7}\,\mathrm{ergs\,cm^{-2}\,s^{-1}}$. Assuming that the light has a wavelength of 5×10^{-5} cm estimate the number of photons from this star that enter a human eye in one second.

Solution. The radius of the average pupil opening of the human eye is around $r_p = 1\,\mathrm{mm} = 10^{-3}\,\mathrm{m}$. Thus, the surface area of the pupil is $A_p = \pi r_p^2 = \pi \times 10^{-6}\,\mathrm{m}^2$. The duration for which we are measuring for is $t = 1\,\mathrm{s}$.
The energy of each photon with a wavelength of $\lambda = 5 \times 10^{-5}\,\mathrm{cm}$ is given by

$$E = \frac{hc}{\lambda} = 3.972 \times 10^{-21}\,\mathrm{J}.$$

The number of photons N entering the pupil of the human eye is then

$$N = \frac{\Phi A_p t}{E} = \frac{\left(1\,\mathrm{J\,m^{-2}\,s^{-1}}\right)\left(\pi \times 10^{-6}\,\mathrm{m}^2\right)(1\,\mathrm{s})}{3.972 \times 10^{-21}\,\mathrm{J}} = 7.909 \times 10^{14}\ \text{photons}.$$

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Homework 2

Problem 1
Data Analysis: Describe in detail what is meant by analyzing the: a) energy, b) arrival time, and c) spatial position of photons. Mention the complications to building a perfect telescope in each case. Then describe how one might perform data analysis using all measurements of a photon together (spatially-resolved time-dependent spectroscopy).

Solution. Analyzing the energy, arrival time, and spatial position of photons is crucial in various fields such as astronomy, quantum mechanics, and medical imaging. Each aspect provides valuable information about the source of the photons and the environment through which they have traveled. Here's a detailed explanation of each aspect and the complications associated with building a perfect telescope for each:

(a) **Energy Analysis:** The energy of a photon is determined by its frequency or wavelength. Analyzing the energy spectrum of photons allows us to identify the chemical composition, temperature, and physical processes occurring in the source.

Complications:

- Building a perfect telescope for energy analysis requires high spectral resolution to differentiate between closely spaced energy levels.
- Detectors must be sensitive across a wide range of wavelengths to capture the entire energy spectrum.
- Calibration is critical to accurately determine the energy of detected photons, accounting for instrumental effects and atmospheric absorption.

(b) **Arrival Time Analysis:** Arrival time analysis involves determining the precise time when photons reach the detector. This information can reveal dynamic processes such as pulsations, explosions, or interactions with matter.

Complications:

- Achieving nanosecond or even femtosecond timing resolution requires advanced timing electronics and fast detectors.
- Background noise and timing jitter can obscure the true arrival time of photons, necessitating sophisticated data processing algorithms.
- Atmospheric effects and dispersion introduce delays, which must be corrected to accurately determine arrival times.

(c) **Spatial Position Analysis:** Spatial position analysis involves determining the origin or direction of incoming photons. This is crucial for imaging objects in space, locating sources of radiation, and studying the distribution of matter.

Complications:

- Achieving high spatial resolution requires large aperture telescopes with precise optics to minimize aberrations.
- Atmospheric turbulence causes image distortion, limiting the achievable spatial resolution.
- Detector imperfections and electronic noise can introduce errors in determining the spatial position of photons.

Performing data analysis using spatially-resolved time-dependent spectroscopy: Spatially-resolved time-dependent spectroscopy combines information from all three aspects (energy, arrival time, and spatial position) to gain a comprehensive understanding of photon emissions. Sophisticated data analysis techniques such as Fourier transforms, deconvolution, and maximum likelihood estimation are employed to extract meaningful information from the data. By analyzing photons together, we can study how energy, timing,

and spatial distribution correlate with each other, providing insights into the underlying physical processes. Data from different detectors or instrument channels are correlated to reconstruct the complete spatiotemporal spectrum of the source. However, this approach requires careful calibration, synchronization, and computational resources to handle the large volumes of data generated.
In summary, analyzing the energy, arrival time, and spatial position of photons is essential for understanding various phenomena in astronomy and other fields. Building a perfect telescope for each aspect is challenging due to technical limitations and environmental factors. Spatially-resolved time-dependent spectroscopy offers a powerful approach to extracting comprehensive information from photon data, but it requires sophisticated instrumentation and data analysis techniques. ■

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Problem 2
Photons scatter many times in the interior of the sun. The cross-section for that scattering is approximately given by the Thomson Cross-Section ($\sigma_T = 6.65 \times 10^{-25}\,\mathrm{cm}^2$). Calculate the mean free path of a photon in the Sun assuming that the density is about $150\,\mathrm{g\,cm^{-3}}$ and the plasma is fully ionized and contains 25% Helium and 75% Hydrogen.

Solution. The mean free path is the typical distance a particle will travel between interactions. The mean free path for electron scattering in the sun is

$$\ell_{es} = \frac{1}{n_e \sigma_T},$$

where n_e is the number density of electrons and σ_T is the Thomson cross-section.
Assuming that the plasma is fully ionized, containing 25% Helium and 75% Hydrogen, and that there is one electron per atom of mass m_{H} and two electrons per atom of mass m_{He}, then

$$n_e = \frac{\rho}{0.75 m_{\mathrm{H}} + 0.25 m_{\mathrm{He}}} = \frac{0.15}{0.75(1.67\times10^{-27}) + 0.25(2)(1.67\times10^{-27})} = 7.185\times10^{25}\,\mathrm{cm}^{-3}.$$

The mean free path for electron scattering in the sun is then

$$\ell_{es} = \frac{1}{n_e\sigma_T} = \frac{1}{(7.185\times10^{25})\,(6.65\times10^{-25})} = 2.1\times10^{-2}\,\mathrm{cm}.$$

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Problem 3
The spatial energy density per unit frequency interval of blackbody radiation is given by

$$u_\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{\mathrm{e}^{\frac{h\nu}{k_B T}} - 1}$$

(a) Write down an expression for the total energy of a photon gas of volume V (Hint: this will be an integral over frequency).

(b) Approximate the denominator by $\mathrm{e}^{\frac{h\nu}{k_B T}} - 1 \approx \mathrm{e}^{\frac{h\nu}{k_B T}}$ and evaluate the integral in part (a).

Solution. (a) The total energy of a photon gas of volume V is given by

$$E = \int_0^\infty u_\nu V\,\mathrm{d}\nu = \frac{8\pi V}{c^3}\int_0^\infty \frac{h\nu^3}{\mathrm{e}^{\frac{h\nu}{k_B T}} - 1}\,\mathrm{d}\nu.$$

(b) Approximating the denominator by $\mathrm{e}^{\frac{h\nu}{k_B T}} - 1 \approx \mathrm{e}^{\frac{h\nu}{k_B T}}$, we have

$$E = \frac{8\pi V}{c^3}\int_0^\infty \frac{h\nu^3}{\mathrm{e}^{\frac{h\nu}{k_B T}}}\,\mathrm{d}\nu$$

Letting $x = \frac{h\nu}{k_B T}$, $\mathrm{d}x = \frac{h}{k_B T}\,\mathrm{d}\nu$, we have

$$\begin{aligned}
E &= \frac{8\pi V k_B T}{c^3} \int_0^\infty \frac{\left(\frac{k_B T}{h} x\right)^3}{\mathrm{e}^x}\,\mathrm{d}x \\
&= \frac{8\pi V (k_B T)^4}{(hc)^3} \int_0^\infty x^3 \mathrm{e}^{-x}\,\mathrm{d}x \\
&= \frac{8\pi V (k_B T)^4}{(hc)^3} \left[-\left(x^3 + 3x^2 + 6x + 6\right)\mathrm{e}^{-x}\right]_0^\infty \\
&= \frac{48\pi V (k_B T)^4}{(hc)^3}.
\end{aligned}$$

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Homework 3

Problem 1
Bremsstrahlung: The total power emitted from thermal bremsstrahlung we said was

$$\frac{\mathrm{d}E}{\mathrm{d}V\,\mathrm{d}t} = \frac{32\pi e^6}{3hmc^3}\left(\frac{2\pi k_B}{3m}\right)^{\frac{1}{2}} Z^2 n_e n_{ion} T^{\frac{1}{2}},$$

ignoring the Gaunt factor. Now, the energy per unit volume of a thermal plasma is nk_BT. What is the cooling time of a plasma due to bremsstrahlung (i.e. the time it takes the plasma to lose its energy due to radiation)? How long does it take a 10^8 K plasma with number density $10^{-2}\,\mathrm{cm}^{-3}$ to cool down (typical of plasma in a cluster of galaxies)?

Solution. A typical astrophysical plasma is made up of Hydrogen ($Z = 1$, ions and free electrons). In that case, we can assume that the number of free electrons and the number of ions distributed among the shared volume is the same, *i.e.* $n_e = n_{ion}$. Given that the energy per unit volume of a thermal plasma is nk_BT, where $n = n_e + n_{ion}$, we have

$$\frac{\mathrm{d}E}{\mathrm{d}V\,\mathrm{d}t} = P = \frac{E}{t} \implies t = \frac{E}{P} = \frac{(n_e + n_{ion})k_BT}{P}.$$

Computing the value of P, we have

$$P = \frac{32\pi e^6}{3hmc^3}\left(\frac{2\pi k_B}{3m}\right)^{\frac{1}{2}} Z^2 n_e n_{ion} T^{\frac{1}{2}} = 2.4\times 10^{-27} n_e n_{ion} T^{\frac{1}{2}} \quad \left[\mathrm{erg\,s^{-1}\,cm^{-3}}\right]$$

The time needed to cool down is then

$$\begin{aligned} t &= \frac{(n_e + n_{ion})k_BT}{2.4\times 10^{-27}\times n_e n_{ion} T^{\frac{1}{2}}} \\ &= \frac{2nk_BT^{\frac{1}{2}}}{2.4\times 10^{-27}\times n^2} \\ &= \frac{2k_BT^{\frac{1}{2}}}{2.4\times 10^{-27}\times n} \\ &= 6\times 10^3\frac{T^{\frac{1}{2}}}{n} \quad \text{[years]}. \end{aligned}$$

For a plasma with temperature 10^8 K and number density $10^{-2}\,\mathrm{cm}^{-3}$ to cool down, the time needed is

$$t = 6\times 10^3\frac{(10^8)^{\frac{1}{2}}}{10^{-2}} = 6\times 10^9 \text{ years}.$$

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Problem 2
Synchrotron Radiation: Given that the synchrotron power for a single electron in a magnetic field is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 u_B,$$

calculate the time it takes an electron to cool from synchrotron emission. How does it depend on energy? How long does it take a 1 keV electron to cool down in a 1 μG magnetic field (typical of our own galaxy)?

Solution. The time needed for an electron to cool down is given by

$$t(\gamma) = \frac{E}{\mathrm{d}E/\,\mathrm{d}t} = \frac{\gamma m_e c^2}{\frac{4}{3}\sigma_T c \beta^2 \gamma^2 u_B} = \frac{3 m_e c}{4\sigma_T \beta^2 \gamma u_B} = \frac{6 m_e c \pi}{\sigma_T \beta^2 \gamma B^2}.$$

For $\beta \to 1$, we have

$$t(\gamma) = 2.5 \times 10^3 \frac{1}{\gamma B^2}.$$

An electron with energy 1 keV is moving at a speed

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \implies v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2} = 1.87 \times 10^7 \text{ m/s}.$$

Then, we have

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \approx 1$$

Thus, the time needed is

$$t = 2.5 \times 10^3 (10^6)^2 = 2.5 \times 10^{15} \text{ s}.$$

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Homework 4

Problem 1
Supernovae: A supernovae explodes with kinetic energy of 2×10^{51} ergs s^{-1}. It ejects 2 Solar masses of material. What is the velocity of the ejecta right after the explosion? If the density of the ISM is 2×10^{-24} g cm^{-3}, how long does it take to sweep up another 2 Solar masses of plasma?

Solution. The energy from a supernova explosion is given by

$$E_{SN} = \frac{1}{2}v^2.$$

Rewriting to find the velocity right after the explosion, we have

$$\begin{aligned} v &= \sqrt{\frac{2E_{SN}}{m_{ej}}} \\ &= \sqrt{\frac{2\left(2 \times 10^{51}\right) \times 10^{-7}}{2\left(2 \times 10^{30}\right)}} \\ &= \sqrt{\frac{10^{44}}{10^{30}}} \\ &= \sqrt{10^{14}} \\ &= 10^7 \text{ m/s.} \end{aligned}$$

The mass swept up from the interstellar material by the ejecta is given by

$$M_{sweep} \sim \frac{4\pi}{3}\rho_{ISM} r^3.$$

The time taken for the ejecta to sweep up another 2 solar masses is given by $t \sim \frac{r}{v}$. To find the radius at which that happens, we have

$$\begin{aligned} r &\sim \left(\frac{3M_{sweep}}{4\pi\rho_{ISM}}\right)^{\frac{1}{3}} \\ &\sim \left(\frac{3\left(2 \times 10^{30}\right)}{4\pi\left(2 \times 10^{-21}\right)}\right)^{\frac{1}{3}} \\ &\sim \left(\frac{3 \times 10^{51}}{4\pi}\right)^{\frac{1}{3}} \\ &\sim 10^{17} \text{ m.} \end{aligned}$$

Thus, the time taken is

$$t \sim \frac{r}{v} \sim \frac{10^{17} \text{ m}}{10^7 \text{ m/s}} \sim 10^{10} \text{ s} \sim 317 \text{ years.}$$

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Problem 2
Neutron Stars: Imagine a star like the sun with mass of 2×10^{30} kg and radius of 7×10^8 m shrinks and becomes a neutron star with radius of 10 km. If the Sun normally rotates with a period of 26 days, what will be its period afterwords assuming conservation of momentum. If the sun has a magnetic field of about 40 Gauss what will be the magnetic field intensity afterwords assuming conservation of magnetic flux $\left(B 4\pi R^2\right)$.

Solution. Converting the period to angular frequency, we have

$$\omega_i = \frac{2\pi}{T_i} = \frac{2\pi}{26 \times 60 \times 60 \times 24} = \frac{\pi}{1,123,200} = 2.797 \times 10^{-6} \text{ rad/s.}$$

Since there is no external torque being applied, we can apply the law of conservation of angular momentum. Clearly the star before and the neutron star are both spherical, then

$$\begin{aligned} L_i = L_f \implies \omega_f &= \frac{I_i \omega_i}{I_f} \\ &= \frac{\left(\frac{2}{5} M r_i^2\right)\omega_i}{\left(\frac{2}{5} M r_f^2\right)} \\ &= \frac{r_i^2 \omega_i}{r_f^2} \\ &= \frac{\left(7 \times 10^8\right)^2 \left(2.797 \times 10^{-6}\right)}{\left(10^4\right)^2} \\ &= \frac{1.37 \times 10^{12}}{10^8} \\ &= 1.37 \times 10^4 \text{ rad/s.} \end{aligned}$$

Then the period afterwards will be

$$T_f = \frac{2\pi}{\omega_f} = \frac{2\pi}{1.37 \times 10^4} = 4.586 \times 10^{-4} \text{ s.}$$

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Problem 3
Black Holes and Accretion: A ten Solar mass black hole swallows up a cloud of rock the mass of the Earth. Assume the rock falls from infinity and then fall directly reaches the event horizon at $\frac{2GM}{c^2}$. How much energy will be produced during this accretion event? How does this compare to the energy liberated if 0.7% of the rest mass of the rock was released during Hydrogen fusion?

Solution. The energy produced in an accretion event is given by

$$E_{\text{accretion}} = \frac{GMm}{R}, \quad \text{(from infinity to the surface).}$$

We have that $M = 10M_\odot = 2 \times 10^{31}$ kg, $m = 6 \times 10^{24}$ kg, and $r = \frac{2GM}{c^2}$, we have

$$\begin{aligned} E_{\text{accretion}} &= \frac{GMm}{\frac{2GM}{c^2}} \\ &= \frac{mc^2}{2} \\ &= \frac{\left(6 \times 10^{24}\right)(299792459)^2}{2} \\ &= 2.7 \times 10^{41} \text{ J.} \end{aligned}$$

The energy liberated if 0.7% of the rest mass was released during Hydrogen fusion is

$$\begin{aligned} E_{\text{fusion}} &= 0.007mc^2 \\ &= 0.007\left(6 \times 10^{24}\right)(299792459)^2 \\ &= 3.7 \times 10^{39} \text{ J.} \end{aligned}$$

Thus, the energy produced from the accretion of Earth's mass into a black hole's event horizon is 100 times larger than the energy produced by 0.7% of Earth's rest mass undergoing Hydrogen fusion. ■

Homework 5

Problem 1
(Gamma Ray Bursts) Assume a gamma ray burst has a flux of $2.6 \times 10^5 \,\mathrm{ergs\,s^{-1}\,cm^{-2}}$ for 1 second and is located at a distance of 1 Mpc. First, estimate the total energy of the burst if the emission occurred isotropically. Then, estimate the effective energy if all the emission occurred in two cones with opening angle of 10 degrees.

Solution. We have a gamma ray burst with a flux density of $\Phi = 2.6 \times 10^5 \,\mathrm{ergs\,cm^{-2}}$. The total energy of the burst if the emission occurred isotropically is given by

$$\begin{aligned} E_{\text{iso}} &= 4\pi D^2 \Phi \\ &= 4\pi (1\,\mathrm{Mpc})^2 (2.6 \times 10^5) \\ &= 4\pi (3.086 \times 10^{24})^2 (2.6 \times 10^5) \\ &= 3.11 \times 10^{55}\,\mathrm{ergs}. \end{aligned}$$

The effective energy if all the emission occurred in two cones of an opening angle is

$$E_{\text{eff}} = \frac{\Delta\Omega}{4\pi} E_{\text{iso}},$$

where $\Delta\Omega$ is the range of the solid angle, which, for small angles, can be estimated by

$$\begin{aligned} \Delta\Omega &= 2 \times \text{Cone Opening} \\ &= 2 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\alpha} \sin(\theta)\,\mathrm{d}\theta \\ &\approx 2 \int_0^{2\pi} \mathrm{d}\phi \int_0^{\alpha} \theta\,\mathrm{d}\theta \\ &= 2(2\pi)\left(\frac{\alpha^2}{2}\right) \\ &= 2\pi\alpha^2. \end{aligned}$$

Then, the effective energy for an opening angle of $10^\circ = 0.175\,\mathrm{rad}$ is

$$\begin{aligned} E_{\text{eff}} &= \frac{\Delta\Omega}{4\pi} E_{\text{iso}} \\ &= \frac{2\pi\alpha^2}{4\pi} E_{\text{iso}} \\ &= \frac{\alpha^2}{2} E_{\text{iso}} \\ &= \frac{(0.175)^2}{2} (3.11 \times 10^{55}) \\ &= 4.762 \times 10^{53}\,\mathrm{ergs}. \end{aligned}$$

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Problem 2
(Clusters of Galaxies) Consider the intracluster medium where there is some plasma with a density of 10^{-2} particles per sq. cm, and a temperature of 10^7 degrees. If the plasma is distributed in a sphere uniformly of 100 kpc, what would be the mass enclosed to achieve that temperature? Compare that to the mass of the plasma itself.

Solution. The ideal gas law states that

$$PV = Nk_BT \implies P = nk_BT.$$

Since the particles are distributed uniformly, *i.e.* in hydrostatic equilibrium, the pressure gradient balances the gravitational force and we can relate them by

$$\begin{aligned}
\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{GM}{r^2} \\
\frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{GM}{r^2}\rho \\
\frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{GM}{r^2}\frac{M}{\frac{4}{3}\pi r^3} \\
\frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{3GM^2}{4\pi r^5} \\
\mathrm{d}P &= -\frac{3GM^2}{4\pi r^5}\,\mathrm{d}r \\
P &= \frac{3GM^2}{16\pi r^4}.
\end{aligned}$$

Then, by equating them, we have

$$\frac{3GM^2}{16\pi r^4} = nk_BT \implies M = \sqrt{\frac{16\pi nk_BTr^4}{3G}}$$

Plugging in the values, we get the mass enclosed to achieve the given temperature, which is

$$\begin{aligned}
M &= \sqrt{\frac{16\pi nk_BTr^4}{3G}} \\
&= \sqrt{\frac{16\pi(10^{-2})(1.38\times10^{-23})(10^7)(3.09\times10^{21})^4}{3(6.67\times10^{-11})}} \\
&= \sqrt{\frac{6.324\times10^{69}}{2\times10^{-10}}} \\
&= \sqrt{3.162\times10^{79}} \\
&= 5.62\times10^{39}\,\mathrm{kg}.
\end{aligned}$$

Now to find the mass of the plasma itself, we will assume it is completely hydrogen. The mass of a hydrogen atom is essentially the mass of a proton m_p. The volume enclosed by a sphere with a radius of $r = 100$ kpc is

$$\begin{aligned}
V &= \frac{4}{3}\pi r^3 \\
&= \frac{4}{3}\pi(3.09\times10^{23})^3 \\
&= \frac{4}{3}\pi(2.95\times10^{69}) \\
&= 1.24\times10^{70}\,\mathrm{cm}^2.
\end{aligned}$$

The mass of a plasma itself is

$$\begin{aligned} m_{\text{plasma}} &= n m_p V \\ &= (10^{-2})(1.67 \times 10^{-27})(1.24 \times 10^{70}) \\ &= 2.07 \times 10^{41}\,\text{kg}. \end{aligned}$$

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