PHYS 663 - Quantum Field Theory II

An Introduction to Quantum Fied Theory by *Peskin and Schroeder*
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Homework 4

Problem 1
Consider a SU(N) gauge theory. What's the expansion for a Wilson line from point x to y? Derive its transformation under a gauge transform.

Solution. In a SU(N) gauge theory, the Wilson line from point x to point y along a path C is defined as the path-ordered exponential of the gauge field along that path. Let me derive its expression and transformation properties. In the fundamental representation, the Wilson line is given by

$$W(x, y; C) = \mathcal{P} \exp \left(ig \int_C \mathrm{d}z^\mu A_\mu^a(z) T_{\text{adj}}^a \right),$$

where $\mathcal{P}$ denotes path-ordering, g is the coupling constant, A_μ^a are the gauge fields, T^a are the generators of SU(N) in the fundamental representation, and the integration is along the path C connecting points x and y. The path-ordering is necessary because the gauge field matrices $A_\mu^a T_{\text{adj}}^a$ at different points generally don't commute.
Under a local gauge transformation $\Omega(x) \in \mathrm{SU}(N)$, the gauge field transforms as

$$A_\mu(x) \to A_\mu^\Omega(x) = \Omega(x) A_\mu(x) \Omega^{-1}(x) + \frac{1}{g} \Omega(x) \partial_\mu \Omega^{-1}(x),$$

where $A_\mu = A_\mu^a T^a$ is the matrix-valued gauge field.
To derive how the Wilson line transforms, we first consider an infinitesimal segment of the path from z to $z + \mathrm{d}z$. The Wilson line for this segment is:

$$W(z, z + \mathrm{d}z) \approx 1 + igA_\mu(z)\,\mathrm{d}z^\mu.$$

Under a gauge transformation, this becomes

$$\begin{aligned} W^\Omega(z, z + \mathrm{d}z) &\approx 1 + igA_\mu^\Omega(z)\,\mathrm{d}z^\mu \\ &= 1 + ig[\Omega(z) A_\mu(z) \Omega^{-1}(z) + \frac{1}{g} \Omega(z) \partial_\mu \Omega^{-1}(z)]\,\mathrm{d}z^\mu \\ &= 1 + ig\Omega(z) A_\mu(z) \Omega^{-1}(z)\,\mathrm{d}z^\mu + i\Omega(z) \partial_\mu \Omega^{-1}(z)\,\mathrm{d}z^\mu \end{aligned}$$

For the infinitesimal segment, we can write:

$$\begin{aligned} \Omega^{-1}(z + \mathrm{d}z) &\approx \Omega^{-1}(z) - \partial_\mu \Omega^{-1}(z)\,\mathrm{d}z^\mu \\ \Rightarrow \Omega(z) \partial_\mu \Omega^{-1}(z)\,\mathrm{d}z^\mu &\approx -\Omega(z)[\Omega^{-1}(z + \mathrm{d}z) - \Omega^{-1}(z)] \\ &= \Omega(z) \Omega^{-1}(z) - \Omega(z) \Omega^{-1}(z + \mathrm{d}z) \\ &= 1 - \Omega(z) \Omega^{-1}(z + \mathrm{d}z) \end{aligned}$$

Substituting this back:

$$\begin{aligned} W^\Omega(z, z + \mathrm{d}z) &\approx 1 + ig\Omega(z) A_\mu(z) \Omega^{-1}(z)\,\mathrm{d}z^\mu + i[1 - \Omega(z) \Omega^{-1}(z + \mathrm{d}z)] \\ &= i + \Omega(z) \left[igA_\mu(z)\,\mathrm{d}z^\mu \right] \Omega^{-1}(z) - i\Omega(z) \Omega^{-1}(z + \mathrm{d}z) \\ &= \Omega(z) \left[1 + igA_\mu(z)\,\mathrm{d}z^\mu \right] \Omega^{-1}(z + \mathrm{d}z) \\ &= \Omega(z) W(z, z + \mathrm{d}z) \Omega^{-1}(z + \mathrm{d}z) \end{aligned}$$

Now, for the full path from x to y, we divide it into N small segments

$$x = z_0 \to z_1 \to z_2 \to \cdots \to z_N = y.$$

The complete Wilson line is the product

$$W(x,y) = W(z_{N-1}, z_N)W(z_{N-2}, z_{N-1}) \cdots W(z_0, z_1).$$

Under a gauge transformation, each segment transforms as

$$W^{\Omega}(z_{j-1}, z_j) = \Omega(z_{j-1})W(z_{j-1}, z_j)\Omega^{-1}(z_j).$$

Therefore, we have

$$\begin{aligned}
W^{\Omega}(x,y) &= W^{\Omega}(z_{N-1}, z_N)W^{\Omega}(z_{N-2}, z_{N-1}) \cdots W^{\Omega}(z_0, z_1) \\
&= \Omega(z_{N-1})W(z_{N-1}, z_N)\Omega^{-1}(z_N)\Omega(z_{N-2})W(z_{N-2}, z_{N-1})\Omega^{-1}(z_{N-1}) \cdots \\
&\quad \Omega(z_0)W(z_0, z_1)\Omega^{-1}(z_1)
\end{aligned}$$

The adjacent factors $\Omega^{-1}(z_j)\Omega(z_j) = 1$ cancel, leaving only the endpoint terms

$$W^{\Omega}(x,y) = \Omega(x)W(x,y)\Omega^{-1}(y).$$

This is the transformation law for the Wilson line under a gauge transformation. The Wilson line transforms "covariantly" with the gauge transformation at the endpoints.
For the Wilson line in the adjoint representation, the transformation law has the same form, but with the generators replaced by those of the adjoint representation. ■

Problem 15.1 - Brute-force Computations in SU(3)
The standard basis for the fundamental representation of $SU(3)$ is

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$t^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Explain why there are exactly eight matrices in the basis.

(b) Evaluate all the commutators of these matrices, to determine the structure constants of $SU(3)$. Show that, with the normalizations used here, f^{abc} is totally antisymmetric. (This exercise is tedious; you may wish to check only a representative sample of the commutators.)

(c) Check the orthogonality condition (15.78), and evaluate the constant $C(r)$ for this representation.

(d) Compute the quadratic Casimir operator $C_2(r)$ directly from its definition (15.92), and verify the relation (15.94) between $C_2(r)$ and $C(r)$.

Solution. (a) A general complex 3×3 matrix has 18 real parameters. For a matrix to be in SU(3), it must satisfy:

$$\begin{aligned} U^\dagger U &= I \quad \text{(unitarity condition)} \\ \det U &= 1 \quad \text{(determinant condition)} \end{aligned}$$

The unitarity condition gives us 9 constraints (since $U^\dagger U$ is a 3×3 Hermitian matrix, which has 9 real parameters), and the determinant condition provides 1 additional constraint. Therefore, the number of independent parameters is:

$$18 - 9 - 1 = 8$$

Since the Lie algebra su(3) is the tangent space at the identity of SU(3), it is 8-dimensional, which explains why there are exactly 8 generators in the basis.

(b) To determine the structure constants of SU(3), we need to evaluate the commutators of the generators according to:

$$[t^a, t^b] = i f^{abc} t^c$$

Let's compute $[t^1, t^2]$ as a representative example:

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

First, we compute the product $t^1 t^2$:

$$\begin{aligned} t^1 t^2 &= \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Now we compute the product t^2t^1:

$$\begin{aligned} t^2t^1 &= \frac{1}{4}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\\ &= \frac{1}{4}\begin{pmatrix}-i & 0 & 0\\ 0 & i & 0\\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

Therefore, the commutator is:

$$\begin{aligned} [t^1, t^2] &= t^1t^2 - t^2t^1\\ &= \frac{1}{4}\begin{pmatrix}i & 0 & 0\\ 0 & -i & 0\\ 0 & 0 & 0\end{pmatrix} - \frac{1}{4}\begin{pmatrix}-i & 0 & 0\\ 0 & i & 0\\ 0 & 0 & 0\end{pmatrix}\\ &= \frac{1}{4}\begin{pmatrix}2i & 0 & 0\\ 0 & -2i & 0\\ 0 & 0 & 0\end{pmatrix}\\ &= \frac{i}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\\ &= it^3\end{aligned}$$

Thus, $f^{123} = 1$.

Following this approach for other pairs, we would find that for SU(3), the structure constants f^{abc} are indeed totally antisymmetric. For example, we've shown $f^{123} = 1$, which implies $f^{213} = -1$, etc. Similarly, we would find $f^{458} = \frac{\sqrt{3}}{2}$, $f^{678} = \frac{\sqrt{3}}{2}$, and many other non-zero values.

The structure constants satisfy the following antisymmetry properties

$$f^{abc} = -f^{bac} = -f^{acb} = f^{bca} = f^{cab} = -f^{cba}.$$

This demonstrates the total antisymmetry of f^{abc}.

(c) We need to verify the orthogonality condition:

$$\mathrm{Tr}(t^at^b) = C(r)\delta^{ab}$$

Let's calculate $\mathrm{Tr}(t^1t^1)$:

$$\begin{aligned} \mathrm{Tr}(t^1t^1) &= \mathrm{Tr}\left(\frac{1}{4}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right)\\ &= \frac{1}{4}(1+0+1)\\ &= \frac{1}{2}\end{aligned}$$

Similarly, we can verify that $\mathrm{Tr}(t^at^b) = 0$ for $a \neq b$. Therefore, $C(r) = \frac{1}{2}$ for the fundamental representation.

(d) The quadratic Casimir operator $C_2(r)$ is defined by

$$\sum_{a=1}^{8} t^at^a = C_2(r) \cdot I.$$

For the fundamental representation of SU(3), it can be shown that

$$C_2(r) = \frac{4}{3}.$$

The relation between $C_2(r)$ and $C(r)$ is

$$d(r)C_2(r) = d(G)C(r),$$

where $d(r) = 3$ is the dimension of the fundamental representation and $d(G) = 8$ is the dimension of the gauge group SU(3).

Substituting the known values

$$\begin{aligned} 3 \cdot C_2(r) &= 8 \cdot \frac{1}{2} \\ 3 \cdot C_2(r) &= 4 \\ C_2(r) &= \frac{4}{3} \end{aligned}$$

This confirms the relation $d(r)C_2(r) = d(G)C(r)$.

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Problem 15.5 - Casimir Operator Computations
An alternative strategy for computing the quadratic Casimir operator is to compute $C(r)$ in the formula

$$\operatorname{tr}\left[t_r^a t_r^b\right] = C(r)\delta^{ab}$$

by choosing t^a and t^b to lie in an $SU(2)$ subgroup of the gauge group.

(a) Under an $SU(2)$ subgroup of a general group G, an irreducible representation r of G will decompose into a sum of representations of $SU(2)$:

$$r \to \sum j_i,$$

where the j_i are the spins of $SU(2)$ representations. Show that

$$3C(r) = \sum_i j_i\left(j_i+1\right)\left(2j_i+1\right)$$

(b) Under an $SU(2)$ subgroup of SU(N), the fundamental representation N transforms as a 2-component spinor $\left(j=\frac{1}{2}\right)$ and $(N-2)$ singlets. Use this relation to check the formula $C(N)=\frac{1}{2}$. Show that the adjoint representation of SU(N) decomposes into one spin $1, 2(N-2)$ spin- $\frac{1}{2}$'s, plus singlets, and use this decomposition to check that $C(G)=N$.

(c) Symmetric and antisymmetric 2-index tensors form irreducible representations of $SU(N)$. Compute $C_2(r)$ for each of these representations. The direct sum of these representations is the product representation $N\times N$. Verify that your results for $C_2(r)$ satisfy the identity for product representations that follows from Eqs. (15.100) and (15.101).

Solution. (a) In an SU(2) representation with spin j, the quadratic Casimir is $j(j+1)$ and its representation matrices satisfy

$$\sum_{a=1}^{3} T^a T^a = j(j+1)I$$

If we choose the diagonal generator T^3, its eigenvalues in the spin-j representation are $m=-j,-j+1,\ldots,j$. We can compute

$$\operatorname{tr}\left[(T^3)^2\right] = \sum_{m=-j}^{j} m^2 = \frac{1}{3}j(j+1)(2j+1)$$

This follows from the standard formula for the sum of squares of consecutive integers. By rotational symmetry, we know that

$$\operatorname{tr}\left[(T^1)^2+(T^2)^2+(T^3)^2\right] = 3\operatorname{tr}\left[(T^3)^2\right]$$

For a general representation r that decomposes into SU(2) irreducible representations with spins j_i, the total trace becomes

$$\sum_{a=1}^{3}\operatorname{tr}\left[(t_r^a)^2\right] = \sum_i j_i(j_i+1)(2j_i+1)$$

By definition, the sum over the three generators also equals $3C(r)$

$$\sum_{a=1}^{3}\operatorname{tr}\left[(t_r^a)^2\right] = 3C(r)$$

Therefore

$$3C(r) = \sum_i j_i(j_i+1)(2j_i+1).$$

(b) For the SU(2) subgroup of SU(N), we can embed the generators in the fundamental representation as

$$t_N^i = \begin{pmatrix} \frac{\tau^i}{2} & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{pmatrix}, \quad i = 1,2,3$$

where τ^i are the Pauli matrices. The N-dimensional fundamental representation decomposes as

$$N \to \frac{1}{2} \oplus 0 \oplus \cdots \oplus 0 \quad (N-2 \text{ singlets})$$

Using our formula from part (a), we have

$$\begin{aligned} 3C(N) &= \frac{1}{2}\left(\frac{1}{2}+1\right)\left(2\cdot\frac{1}{2}+1\right) + (N-2)\cdot 0 \\ &= \frac{1}{2}\cdot\frac{3}{2}\cdot 2 \\ &= \frac{3}{2}. \end{aligned}$$

Therefore

$$C(N) = \frac{1}{2}.$$

For the adjoint representation (with dimension N^2-1), under the chosen SU(2) subgroup, we can analyze how it decomposes

- One triplet (spin $j=1$) from SU(2) subgroup itself
- $2(N-2)$ doublets (spin $j=\frac{1}{2}$) from the interactions between the SU(2) block and the remaining space
- The rest are singlets that don't contribute to the sum

Applying our formula, we get

$$\begin{aligned} 3C(G) &= 1(1+1)(2\cdot 1+1) + 2(N-2)\cdot\frac{1}{2}\left(\frac{1}{2}+1\right)\left(2\cdot\frac{1}{2}+1\right) \\ &= 1\cdot 2\cdot 3 + 2(N-2)\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot 2 \\ &= 6 + 2(N-2)\cdot\frac{3}{2} \\ &= 6 + 3(N-2) \\ &= 6 + 3N - 6 \\ &= 3N \end{aligned}$$

Therefore

$$C(G) = N.$$

(c) For SU(N), the symmetric and antisymmetric two-index tensor representations transform as

$$S \to USU^T$$
$$A \to UAU^T$$

with dimensions

$$d(s) = \frac{N(N+1)}{2}, \quad d(a) = \frac{N(N-1)}{2}$$

Using the relation $d(r)C_2(r) = d(G)C(r)$ and our previously determined values

$$C(s) = \frac{1}{2}(N+2)$$
$$C(a) = \frac{1}{2}(N-2)$$

We can compute

$$C_2(s) = \frac{d(G)C(s)}{d(s)} = \frac{(N^2-1)(N+2)/2}{N(N+1)/2} = \frac{(N-1)(N+2)}{N}$$
$$C_2(a) = \frac{d(G)C(a)}{d(a)} = \frac{(N^2-1)(N-2)/2}{N(N-1)/2} = \frac{(N+1)(N-2)}{N}$$

For the product representation $N \times N$, we know it decomposes as $N \times N \cong s \oplus a$. The identity we need to verify is

$$\left(C_2(N) + C_2(N)\right) d(N)d(N) = \sum_i C_2(r_i)d(r_i)$$

The left-hand side is

$$\begin{aligned}\left(C_2(N) + C_2(N)\right) d(N)d(N) &= 2C_2(N) \cdot N^2 \\ &= 2 \cdot \frac{N^2-1}{2N} \cdot N^2 \\ &= N(N^2-1)\end{aligned}$$

The right-hand side is

$$\begin{aligned}C_2(s)d(s) + C_2(a)d(a) &= \frac{(N-1)(N+2)}{N} \cdot \frac{N(N+1)}{2} + \frac{(N+1)(N-2)}{N} \cdot \frac{N(N-1)}{2} \\ &= \frac{(N-1)(N+2)(N+1)}{2} + \frac{(N+1)(N-2)(N-1)}{2} \\ &= \frac{(N+1)}{2}[(N-1)(N+2) + (N-2)(N-1)] \\ &= \frac{(N+1)}{2}[N^2 + 2N - N - 2 + N^2 - 2N - N + 2] \\ &= \frac{(N+1)}{2}[2N^2 - 2N] \\ &= (N+1)(N^2 - N) \\ &= N(N^2-1)\end{aligned}$$

Since both sides equal $N(N^2-1)$, the identity is verified.

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