PHYS 660 - Quantum Mechanics I

Modern Quantum Mechanics by *J. J. Sakurai*

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Homework 5

Based on problems 3.12, 3.15, 3.18, 3.20 of Sakurai's book.

Problem 1
An angular momentum eigenstate $|\ell, \ell_z = \ell\rangle$ is rotated by an infinitesimal angle $\alpha \ll 1$ about the y-axis. If, in the new state, we measure $\hat{\ell}_z$, what is the probability of obtaining $\ell_z = \ell$? Find the answer up to terms of order α^2.
Note: Perform the calculation *without* using the explicit form of the $d^{(j)}_{\ell_z', \ell_z}$ matrix.

Solution. The quantum rotation operator $\hat{D}$ is given by

$$\hat{D}(\hat{\mathbf{n}}, \theta) = \mathrm{e}^{-i\theta \frac{\hat{\mathbf{n}}\cdot\mathbf{J}}{\hbar}} = \mathrm{e}^{-i\theta\hat{\mathbf{n}}\cdot\mathbf{j}}.$$

A rotation by an infinitesimal angle $\alpha \ll 1$ is given by the rotation operator

$$\begin{aligned}\hat{D}(\hat{\mathbf{y}}, \alpha) &= \mathrm{e}^{-i\alpha\hat{\ell}_y} \\ &= \sum_{n=0}^{\infty} \frac{(-i\alpha)^n}{n!}\hat{\ell}_y^n \\ &= \mathbb{I} - i\alpha\hat{\ell}_y - \frac{\alpha^2}{2}\hat{\ell}_y^2 + \mathcal{O}(\alpha^3).\end{aligned}$$

Now, we will rewrite the operator above in terms of $\hat{\ell}_\pm = \hat{\ell}_x \pm i\hat{\ell}_y \implies \hat{\ell}_y = \frac{\hat{\ell}_+ - \hat{\ell}_-}{2i}$. Additionally, we will use the following facts

$$\begin{aligned}&\hat{\ell}_\pm |\ell, \ell_z\rangle = \sqrt{\ell(\ell+1) - \ell_z(\ell_z \pm 1)}\, |\ell, \ell_z \pm 1\rangle, \\ &\hat{\ell}^2 |\ell, \ell_z\rangle = \ell(\ell+1)\, |\ell, \ell_z\rangle, \\ &\hat{\ell}_z |\ell, \ell_z\rangle = \ell_z\, |\ell, \ell_z\rangle.\end{aligned}$$

Additionally,

$$\hat{\ell}_+ |\ell, \ell_z = \ell\rangle = \sqrt{\ell(\ell+1) - \ell(\ell+1)}\, |\ell, \ell+1\rangle = 0.$$

Replacing in the Taylor expansion of the exponential, we have

$$\begin{aligned}\hat{D}(\hat{\mathbf{y}}, \alpha) &= \mathbb{I} - i\alpha\hat{\ell}_y - \frac{\alpha^2}{2}\hat{\ell}_y^2 + \mathcal{O}(\alpha^3) \\ &= \mathbb{I} - i\alpha\left(\frac{\hat{\ell}_+ - \hat{\ell}_-}{2i}\right) - \frac{\alpha^2}{2}\left(\frac{\hat{\ell}_+ - \hat{\ell}_-}{2i}\right)^2 + \mathcal{O}(\alpha^3) \\ &= \mathbb{I} - \frac{\alpha}{2}\left(\hat{\ell}_+ - \hat{\ell}_-\right) + \frac{\alpha^2}{8}\left(\hat{\ell}_+ - \hat{\ell}_-\right)^2 + \mathcal{O}(\alpha^3) \\ &= \mathbb{I} - \frac{\alpha}{2}\left(\hat{\ell}_+ - \hat{\ell}_-\right) + \frac{\alpha^2}{8}\left(\hat{\ell}_+^2 - \hat{\ell}_+\hat{\ell}_- - \hat{\ell}_-\hat{\ell}_+ + \hat{\ell}_-^2\right) + \mathcal{O}(\alpha^3).\end{aligned}$$

Applying this operator to the state $|\ell, \ell_z\rangle$, we have

$$\begin{aligned}
\hat{D}(\hat{\mathbf{y}}, \alpha)\,|\ell, \ell_z\rangle &= \left[\mathbb{I} - \frac{\alpha}{2}\left(\hat{\ell}_+ - \hat{\ell}_-\right) + \frac{\alpha^2}{8}\left(\hat{\ell}_+^2 - \hat{\ell}_+\hat{\ell}_- - \hat{\ell}_-\hat{\ell}_+ + \hat{\ell}_-^2\right) + \mathcal{O}(\alpha^3)\right]|\ell, \ell_z\rangle \\
&= \left[\mathbb{I} - \frac{\alpha}{2}\hat{\ell}_+ + \frac{\alpha}{2}\hat{\ell}_- + \frac{\alpha^2}{8}\hat{\ell}_+^2 - \frac{\alpha^2}{8}\hat{\ell}_+\hat{\ell}_- - \frac{\alpha^2}{8}\hat{\ell}_-\hat{\ell}_+ + \frac{\alpha^2}{8}\hat{\ell}_-^2 + \mathcal{O}(\alpha^3)\right]|\ell, \ell_z\rangle \\
&= |\ell, \ell_z\rangle - \frac{\alpha}{2}\hat{\ell}_+\,|\ell, \ell_z\rangle + \frac{\alpha}{2}\hat{\ell}_-\,|\ell, \ell_z\rangle + \frac{\alpha^2}{8}\hat{\ell}_+^2\,|\ell, \ell_z\rangle - \frac{\alpha^2}{8}\hat{\ell}_+\hat{\ell}_-\,|\ell, \ell_z\rangle - \frac{\alpha^2}{8}\hat{\ell}_-\hat{\ell}_+\,|\ell, \ell_z\rangle + \frac{\alpha^2}{8}\hat{\ell}_-^2\,|\ell, \ell_z\rangle \\
&= |\ell, \ell_z\rangle - \frac{\alpha}{2}\sqrt{\ell(\ell+1) - \ell_z(\ell_z+1)}\,|\ell, \ell_z+1\rangle + \frac{\alpha}{2}\sqrt{\ell(\ell+1) - \ell_z(\ell_z-1)}\,|\ell, \ell_z-1\rangle \\
&\quad + \frac{\alpha^2}{8}\sqrt{\ell(\ell+1) - (\ell_z+1)(\ell_z+2)}\sqrt{\ell(\ell+1) - \ell_z(\ell_z+1)}\,|\ell, \ell_z+2\rangle \\
&\quad - \frac{\alpha^2}{8}\sqrt{\ell(\ell+1) - \ell_z(\ell_z-1)}\sqrt{\ell(\ell+1) - \ell_z(\ell_z-1)}\,|\ell, \ell_z\rangle \\
&\quad - \frac{\alpha^2}{8}\sqrt{\ell(\ell+1) - \ell_z(\ell_z+1)}\sqrt{\ell(\ell+1) - \ell_z(\ell_z+1)}\,|\ell, \ell_z\rangle \\
&\quad + \frac{\alpha^2}{8}\sqrt{\ell(\ell+1) - (\ell_z-1)(\ell_z-2)}\sqrt{\ell(\ell+1) - \ell_z(\ell_z-1)}\,|\ell, \ell_z-2\rangle \\
&= |\ell, \ell_z\rangle - \frac{\alpha}{2}\left[\sqrt{\ell(\ell+1) - \ell_z(\ell_z+1)}\,|\ell, \ell_z+1\rangle - \sqrt{\ell(\ell+1) - \ell_z(\ell_z-1)}\,|\ell, \ell_z-1\rangle\right] \\
&\quad + \frac{\alpha^2}{8}\left[\sqrt{\ell(\ell+1) - (\ell_z+1)(\ell_z+2)}\sqrt{\ell(\ell+1) - \ell_z(\ell_z+1)}\,|\ell, \ell_z+2\rangle\right. \\
&\quad \left. -2\left[\ell^2 + \ell - \ell_z^2\right]|\ell, \ell_z\rangle + \sqrt{\ell(\ell+1) - (\ell_z-1)(\ell_z-2)}\sqrt{\ell(\ell+1) - \ell_z(\ell_z-1)}\,|\ell, \ell_z-2\rangle\right].
\end{aligned}$$

Now, we apply the operator to the state when $\ell_z = \ell$, and knowing that $\ell_+\,|\ell, \ell\rangle = 0$, then

$$\hat{D}(\hat{\mathbf{y}}, \alpha)\,|\ell, \ell\rangle = \left(1 - \frac{\alpha^2}{4}\ell\right)|\ell, \ell\rangle + \frac{\alpha}{2}\sqrt{2\ell}\,|\ell, \ell-1\rangle + \frac{\alpha^2}{4}\sqrt{2\ell(\ell-1)}\,|\ell, \ell-2\rangle\,.$$

Thus, the probability of obtaining $\ell_z = \ell$ after a rotation by α is

$$\begin{aligned}
P\left(\ell_z = \ell \mid \hat{D}(\hat{\mathbf{y}}, \alpha)\right) &= \left|\left\langle \ell, \ell \middle| \hat{D}(\hat{\mathbf{y}}, \alpha) \middle| \ell, \ell \right\rangle\right|^2 \\
&= \left| |\ell, \ell\rangle \left[\left(1 - \frac{\alpha^2}{4}\ell\right)|\ell, \ell\rangle + \frac{\alpha}{2}\sqrt{2\ell}\,|\ell, \ell-1\rangle + \frac{\alpha^2}{4}\sqrt{2\ell(\ell-1)}\,|\ell, \ell-2\rangle\right]\right|^2 \\
&= \left|\left(1 - \frac{\alpha^2}{4}\ell\right)\right|^2 \\
&= 1 - \frac{\alpha^2}{2}\ell + \frac{\alpha^4}{16}\ell^2 \\
&= 1 - \frac{\alpha^2}{2}\ell + \mathcal{O}(\alpha^4).
\end{aligned}$$

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Problem 2
The wave function of a particle subjected to a spherically symmetric potential $V(r)$ is given by

$$\psi(\vec{r}) = (x + y + 3z)f(r).$$

(a) Is ψ an eigenfunction of L^2? If so, what is the value of ℓ? If not, what are the possible values of ℓ we may obtain when L^2 is measured?

(b) What are the probabilities for the particle to be found in various $\hat{\ell}_z$ eigenstates?

(c) Suppose it is known that $\psi(\vec{r})$ is an energy eigenfunction with eigenvalue E. Indicate how to find $V(r)$.

Solution. (a) We will first rewrite ψ in terms of spherical coordinates

$$\begin{aligned}\psi(\mathbf{r}) &= \left(r\sin(\theta)\cos(\varphi) + r\sin(\theta)\sin(\varphi) + 3r\cos(\theta)\right)f(r)\\ &= \left(\sin(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi) + 3\cos(\theta)\right)rf(r).\end{aligned}$$

We will now analyze the spherical harmonics of the wave-function we have. First, we can directly see it cannot be $\ell = 0$ since there is no constant term. Additionally, all terms depend on each variable, θ or φ, at most once, so it would be wise to check $\ell = 1$. From a lookup table of spherical harmonics, we have

$$\begin{aligned}Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin(\theta)\mathrm{e}^{-i\varphi} \implies \sin(\theta)\mathrm{e}^{-i\varphi} = 2\sqrt{\frac{2\pi}{3}}Y_1^{-1},\\ Y_1^{0}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\theta) \implies \cos(\theta) = 2\sqrt{\frac{\pi}{3}}Y_1^{0},\\ Y_1^{1}(\theta,\varphi) &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin(\theta)\mathrm{e}^{i\varphi} \implies \sin(\theta)\mathrm{e}^{i\varphi} = -2\sqrt{\frac{2\pi}{3}}Y_1^{1}.\end{aligned}$$

Replacing in our wave-function, we have

$$\begin{aligned}\psi(\mathbf{r}) &= \left(\sin(\theta)\cos(\varphi) + \sin(\theta)\sin(\varphi) + 3\cos(\theta)\right)rf(r)\\ &= \left[\sin(\theta)\left(\frac{\mathrm{e}^{i\varphi}+\mathrm{e}^{-i\varphi}}{2}\right) + \sin(\theta)\left(\frac{\mathrm{e}^{i\varphi}-\mathrm{e}^{-i\varphi}}{2i}\right) + 3\left(2\sqrt{\frac{\pi}{3}}Y_1^0\right)\right]rf(r)\\ &= \left[\frac{1}{2}\left(\sin(\theta)\mathrm{e}^{i\varphi} + \sin(\theta)\mathrm{e}^{-i\varphi}\right) - \frac{i}{2}\left(\sin(\theta)\mathrm{e}^{i\varphi} - \sin(\theta)\mathrm{e}^{-i\varphi}\right) + 2\sqrt{3\pi}Y_1^0\right]rf(r)\\ &= \left[\frac{1}{2}\left(-2\sqrt{\frac{2\pi}{3}}Y_1^1 + 2\sqrt{\frac{2\pi}{3}}Y_1^{-1}\right) - \frac{i}{2}\left(-2\sqrt{\frac{2\pi}{3}}Y_1^1 - 2\sqrt{\frac{2\pi}{3}}Y_1^{-1}\right) + 2\sqrt{3\pi}Y_1^0\right]rf(r)\\ &= \left[-\sqrt{\frac{2\pi}{3}}Y_1^1 + \sqrt{\frac{2\pi}{3}}Y_1^{-1} + i\sqrt{\frac{2\pi}{3}}Y_1^1 + i\sqrt{\frac{2\pi}{3}}Y_1^{-1} + 2\sqrt{3\pi}Y_1^0\right]rf(r)\\ &= \left[(i-1)\sqrt{\frac{2\pi}{3}}Y_1^1 + 2\sqrt{3\pi}Y_1^0 + (i+1)\sqrt{\frac{2\pi}{3}}Y_1^{-1}\right]rf(r).\end{aligned}$$

Thus, $\psi(\mathbf{r})$ is an eigenfunction of L^2 with $\ell = 1$.

(b) Before calculating the probabilities, we must first normalize the wave-function. To do so, we have to use the orthogonality property of the complex spherical harmonics, given by

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell_1}^{\ell_{z,1}}\left(Y_{\ell_2}^{\ell_{z,2}}\right)^*\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi = \delta_{\ell_1,\ell_2}\delta_{\ell_{z,1},\ell_{z,2}}.$$

Thus, we have

$$\begin{aligned}
\iiint \psi^*(\mathbf{r})\psi(\mathbf{r})\,\mathrm{d}\mathbf{r} &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi^*(\mathbf{r})\psi(\mathbf{r})r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left[(-i-1)\sqrt{\frac{2\pi}{3}}\left(Y_1^1\right)^* + 2\sqrt{3\pi}\left(Y_1^0\right)^* + (-i+1)\sqrt{\frac{2\pi}{3}}\left(Y_1^{-1}\right)^*\right] \\
&\qquad \left[(i-1)\sqrt{\frac{2\pi}{3}}Y_1^1 + 2\sqrt{3\pi}Y_1^0 + (i+1)\sqrt{\frac{2\pi}{3}}Y_1^{-1}\right] r^4 f^2(r)\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \left[(-i-1)(i-1)\frac{2\pi}{3} + 12\pi + (-i+1)(i+1)\frac{2\pi}{3}\right]\cdot\left(\int_0^{\infty} r^4 f^2(r)\,\mathrm{d}r\right) \\
&= \frac{44\pi}{3}\int_0^{\infty} r^4 f^2(r)\,\mathrm{d}r \\
&= 1 \\
&\implies \int_0^{\infty} r^4 f^2(r)\,\mathrm{d}r = \frac{3}{44\pi}.
\end{aligned}$$

The normalized wave-function Ψ is then

$$\begin{aligned}
\Psi(\theta,\varphi) &= \sqrt{\frac{3}{44\pi}}\left[(i-1)\sqrt{\frac{2\pi}{3}}Y_1^1 + 2\sqrt{3\pi}Y_1^0 + (i+1)\sqrt{\frac{2\pi}{3}}Y_1^{-1}\right] \\
&= \frac{i-1}{\sqrt{22}}Y_1^1 + \frac{3}{\sqrt{11}}Y_1^0 + \frac{i+1}{\sqrt{22}}Y_1^{-1}.
\end{aligned}$$

Thus, the probabilities for the particle to be found in various $\hat{\ell}_z$ states are

- For $\ell_z = 1$:

$$\begin{aligned}
\mathbb{P}(\ell_z = 1) &= \left|\int_0^{2\pi}\int_0^{\pi}\left(Y_1^1\right)^*\Psi(\theta,\varphi)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi\right|^2 \\
&= \left|\frac{i-1}{\sqrt{22}}\right|^2 \\
&= \frac{1}{11}.
\end{aligned}$$

- For $\ell_z = 0$:

$$\begin{aligned}
\mathbb{P}(\ell_z = 0) &= \left|\int_0^{2\pi}\int_0^{\pi}\left(Y_1^0\right)^*\Psi(\theta,\varphi)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi\right|^2 \\
&= \left|\frac{3}{\sqrt{11}}\right|^2 \\
&= \frac{9}{11}.
\end{aligned}$$

- For $\ell_z = -1$:

$$\begin{aligned}
\mathbb{P}(\ell_z = -1) &= \left|\int_0^{2\pi}\int_0^{\pi}\left(Y_1^{-1}\right)^*\Psi(\theta,\varphi)\sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\varphi\right|^2 \\
&= \left|\frac{i+1}{\sqrt{22}}\right|^2 \\
&= \frac{1}{11}.
\end{aligned}$$

(c) We have the Hamiltonian operator $\hat{H}$ given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\mathbf{r}) = -\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}).$$

To find $V(\mathbf{r})$, we have to solve the eigenvalue problem given by

$$\hat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}).$$

This eigenvalue problem is easier to solve if we use the original Cartesian coordinate system form of $\psi(\mathbf{r})$. Solving, we have

$$\begin{aligned}
\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right)\psi(\mathbf{r}) &= E\psi(\mathbf{r}) \\
-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) &= E\psi(\mathbf{r}) \\
(V(\mathbf{r}) - E)\psi(\mathbf{r}) &= \frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) \\
V(\mathbf{r}) - E &= \frac{\hbar^2}{2m}\frac{1}{\psi(\mathbf{r})}\nabla^2\psi(\mathbf{r}) \\
V(\mathbf{r}) &= E + \frac{\hbar^2}{2m}\frac{1}{\psi(\mathbf{r})}\nabla^2\psi(\mathbf{r}) \\
&= E + \frac{\hbar^2}{2m}\frac{1}{(x+y+3z)f(r)}\nabla^2\left[(x+y+3z)f(r)\right].
\end{aligned}$$

We have

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\psi(\mathbf{r}) &= \frac{\partial^2}{\partial x^2}\left[(x+y+3z)f(r)\right] \\
&= \frac{\partial}{\partial x}\left[f(r) + (x+y+3z)\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}x}\right] \\
&= \frac{\partial}{\partial x}\left[f(r) + (x+y+3z)\frac{\mathrm{d}f}{\mathrm{d}r}\frac{x}{r}\right] \\
&= \left[\frac{\mathrm{d}f}{\mathrm{d}r}\frac{x}{r} + \frac{\mathrm{d}f}{\mathrm{d}r}\frac{x}{r} + (x+y+3z)\left(\frac{\mathrm{d}^2f}{\mathrm{d}r^2}\left(\frac{x}{r}\right)^2 + \frac{\mathrm{d}f}{\mathrm{d}r}\frac{1}{r} + \frac{\mathrm{d}f}{\mathrm{d}r}\left(-\frac{x}{r^2}\frac{x}{r}\right)\right)\right] \\
&= \left[\frac{2x}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + (x+y+3z)\left(\frac{x^2}{r^2}\frac{\mathrm{d}^2f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{x^2}{r^3}\frac{\mathrm{d}f}{\mathrm{d}r}\right)\right].
\end{aligned}$$

Thus, after doing the same for y and z and adding the results together, we get

We have

$$\begin{aligned}
\nabla^2\psi(\mathbf{r}) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(\mathbf{r}) \\
&= \left[\frac{2x}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + (x+y+3z)\left(\frac{x^2}{r^2}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{x^2}{r^3}\frac{\mathrm{d}f}{\mathrm{d}r}\right)\right] \\
&\quad + \left[\frac{2y}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + (x+y+3z)\left(\frac{y^2}{r^2}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{y^2}{r^3}\frac{\mathrm{d}f}{\mathrm{d}r}\right)\right] \\
&\quad + \left[\frac{6z}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + (x+y+3z)\left(\frac{z^2}{r^2}\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{z^2}{r^3}\frac{\mathrm{d}f}{\mathrm{d}r}\right)\right] \\
&= (x+y+3z)\frac{2}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + (x+y+3z)\left[\left(\frac{x^2+y^2+z^2}{r^2}\right)\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{3}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \left(\frac{x^2+y^2+z^2}{r^3}\right)\frac{\mathrm{d}f}{\mathrm{d}r}\right] \\
&= (x+y+3z)\left[\frac{2}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{3}{r}\frac{\mathrm{d}f}{\mathrm{d}r} - \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}\right] \\
&= (x+y+3z)\left[\frac{4}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\right].
\end{aligned}$$

Therefore, the potential V is given by

$$\begin{aligned}
V(\mathbf{r}) &= E + \frac{\hbar^2}{2m}\frac{1}{(x+y+3z)f(r)}(x+y+3z)\left[\frac{4}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\right] \\
&= E + \frac{\hbar^2}{2m}\frac{1}{f(r)}\left[\frac{4}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\mathrm{d}^2 f}{\mathrm{d}r^2}\right].
\end{aligned}$$

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Problem 3
Consider an orbital angular momentum eigenstate $|\ell = 2, \ell_z = 0\rangle$. Suppose that this state is rotated by an angle β about the y-axis.
If we measure $\hat{\ell}_z$, what are the possible values we can obtain and what is the probability of measuring each of them?

Solution. If we measure $\hat{\ell}_z$, we can obtain $0, \pm 1, \pm 2$. The probability of obtaining each is given by

$$P(\ell_z) = \left|\langle 2, \ell_z | \mathcal{D}(R) | 2, 0\rangle\right|^2.$$

The quantum rotation operator $\hat{\mathcal{D}}(R)$ for a rotation by an angle β about the y-axis applied to the eigenstate $|\ell, \ell_z\rangle$ is given by

$$\hat{\mathcal{D}}(R)\,|\ell, \ell_z\rangle = \hat{\mathcal{D}}(\alpha = 0, \beta, \gamma = 0)\,|\ell, \ell_z\rangle = \sum_{m'} |\ell, m'\rangle\,\hat{\mathcal{D}}^{(\ell)}_{m', \ell_z}(\beta) = \sum_{m'} |\ell, m'\rangle \sqrt{\frac{4\pi}{5}} \left(Y_\ell^{m'}\right)^* (\beta, \ell_z).$$

For $\ell = 2$ and $\ell_z = 0$, we have

$$\begin{aligned}
\hat{\mathcal{D}}(R)\,|\ell, \ell_z\rangle &= \sum_{m'} |2, m'\rangle\,\hat{\mathcal{D}}^{(2)}_{m', 0}(\beta) \\
&= \sum_{m'} |2, m'\rangle \sqrt{\frac{4\pi}{5}} \left(Y_2^{m'}\right)^* (\beta, 0).
\end{aligned}$$

Thus, the probability of measuring each is given by

$$\begin{aligned}
P(\ell_z) &= \left|\langle 2, \ell_z | \mathcal{D}(R) | 2, 0\rangle\right|^2 \\
&= \left|\langle 2, \ell_z| \left(\sum_{m'} |2, m'\rangle \sqrt{\frac{4\pi}{5}} \left(Y_2^{\ell_z}\right)^* (\beta, 0)\right)\right|^2 \\
&= \left|\sqrt{\frac{4\pi}{5}} \left(Y_2^{\ell_z}\right)^* (\beta, 0)\,\delta_{m', \ell_z}\right|^2 \\
&= \left|\sqrt{\frac{4\pi}{5}} \left(Y_2^{\ell_z}\right)^* (\beta, 0)\right|^2 \\
&= \frac{4\pi}{5} \left|\left(Y_2^{\ell_z}\right)^* (\beta, 0)\right|^2 \\
&= \frac{4\pi}{5} \left|(-1)^{\ell_z} Y_2^{-\ell_z}(\beta, 0)\right|^2 \\
&= \frac{4\pi}{5} \left|Y_2^{-\ell_z}(\beta, 0)\right|^2.
\end{aligned}$$

Therefore, the probability of measuring each possibility is

- For $\ell_z = 0$:

$$\begin{aligned}
P(0) &= \frac{4\pi}{5} \left|Y_2^0(\beta, 0)\right|^2 \\
&= \frac{4\pi}{5} \left|\frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2(\beta) - 1)\right|^2 \\
&= \frac{1}{4}(3\cos^2(\beta) - 1)^2.
\end{aligned}$$

- For $\ell_z = \pm 1$:

$$\begin{aligned} P(\pm 1) &= \frac{4\pi}{5}\left|Y_2^{\mp 1}(\beta, 0)\right|^2 \\ &= \frac{4\pi}{5}\left|\pm\frac{1}{2}\sqrt{\frac{15}{2\pi}}\mathrm{e}^{\mp i\varphi}\sin(\beta)\cos(\beta)\right|^2 \\ &= \frac{3}{2}\sin^2(\beta)\cos^2(\beta). \end{aligned}$$

- For $\ell_z = \pm 2$:

$$\begin{aligned} P(\pm 2) &= \frac{4\pi}{5}\left|Y_2^{\mp 2}(\beta, 0)\right|^2 \\ &= \frac{4\pi}{5}\left|\frac{1}{4}\sqrt{\frac{15}{2\pi}}\mathrm{e}^{\mp 2i\varphi}\sin^2(\beta)\right|^2 \\ &= \frac{3}{8}\sin^4(\beta). \end{aligned}$$

As a sanity check, we can show that the sum of the probabilities is equal to 1, but sanity is a privilege we do not have. ■

Problem 4
Given two particles in angular momentum eigenstates $\ell_1 = 1$ and $\ell_2 = 1$, the total possible angular momentum is $\ell_T = 0, 1, 2$.
Without using the table, write all eigenstates of the total angular momentum $|\ell_1, \ell_2; \ell_T, \ell_z\rangle$ as a linear combination of the states of the basis $|\ell_1, \ell_2; \ell_{1z}, \ell_{2z}\rangle$

Solution. Given that $\ell_1 = 1$ and $\ell_2 = 1$, then $\ell_T = 0, 1, 2$ and there are $(2\ell_1 + 1)(2\ell_2 + 1) = 9$ possible states. For the simplicity of notation, we will note the following: $|\ell_{1z} = a, \ell_{2z} = b\rangle = |a, b\rangle$.

- For $\ell_T = 2$ and $\ell_z = 2$:
  The only way to get $|\ell_T = 2, \ell_z = 2\rangle$ is if $\ell_{1z} = 1$ and $\ell_{2z} = 1$, which gives
  $$|\ell_T = 2, \ell_z = 2\rangle = |1, 1\rangle \,.$$
- For $\ell_T = 2$ and $\ell_z = -2$:
  The only way to get $|\ell_T = 2, \ell_z = -2\rangle$ is if $\ell_{1z} = -1$ and $\ell_{2z} = -1$, which gives
  $$|\ell_T = 2, \ell_z = -2\rangle = |-1, -1\rangle \,.$$

From the state $|\ell_T = 2, \ell_z = 2\rangle$, we can apply the angular momentum lowering operator $\hat{\ell}_-$ to find the other states. From this, we have
$$\hat{\ell}_\pm |\ell, \ell_z\rangle = \sqrt{\ell(\ell+1) - \ell_z(\ell_z \pm 1)}\, |\ell, \ell_z \pm 1\rangle \,.$$

- For $\ell_T = 2$ and $\ell_z = 1$:
  We apply the $\hat{\ell}_-$ operator to $|\ell_T = 2, \ell_z = 2\rangle$ and get
  $$\begin{aligned}
  \hat{\ell}_- |\ell_T = 2, \ell_z = 2\rangle &= \left(\hat{\ell}_{1-} + \hat{\ell}_{2-}\right) |1, 1\rangle \\
  \hat{\ell}_- |\ell_T = 2, \ell_z = 2\rangle &= \hat{\ell}_{1-} |1, 1\rangle + \hat{\ell}_{2-} |1, 1\rangle \\
  2\, |\ell_T = 2, \ell_z = 1\rangle &= \sqrt{2}\, |0, 1\rangle + \sqrt{2}\, |1, 0\rangle \\
  |\ell_T = 2, \ell_z = 1\rangle &= \frac{1}{\sqrt{2}} \left(|0, 1\rangle + |1, 0\rangle\right).
  \end{aligned}$$
- For $\ell_T = 2$ and $\ell_z = 0$:
  We apply the $\hat{\ell}_-$ operator to $|\ell_T = 2, \ell_z = 1\rangle$ and get
  $$\begin{aligned}
  \hat{\ell}_- |\ell_T = 2, \ell_z = 1\rangle &= \left(\hat{\ell}_{1-} + \hat{\ell}_{2-}\right) \left(\frac{1}{\sqrt{2}} |0, 1\rangle + \frac{1}{\sqrt{2}} |1, 0\rangle\right) \\
  \hat{\ell}_- |\ell_T = 2, \ell_z = 1\rangle &= \frac{1}{\sqrt{2}} \hat{\ell}_{1-} |0, 1\rangle + \frac{1}{\sqrt{2}} \hat{\ell}_{1-} |1, 0\rangle + \frac{1}{\sqrt{2}} \hat{\ell}_{2-} |0, 1\rangle + \frac{1}{\sqrt{2}} \hat{\ell}_{2-} |1, 0\rangle \\
  \sqrt{6}\, |\ell_T = 2, \ell_z = 0\rangle &= \frac{1}{\sqrt{2}} \sqrt{2}\, |-1, 1\rangle + \frac{1}{\sqrt{2}} \sqrt{2}\, |0, 0\rangle + \frac{1}{\sqrt{2}} \sqrt{2}\, |0, 0\rangle + \frac{1}{\sqrt{2}} \sqrt{2}\, |1, -1\rangle \\
  |\ell_T = 2, \ell_z = 0\rangle &= \frac{1}{\sqrt{6}} \left(|1, -1\rangle + 2\, |0, 0\rangle + |-1, 1\rangle\right).
  \end{aligned}$$
- For $\ell_T = 2$ and $\ell_z = -1$:
  We apply the $\hat{\ell}_-$ operator to $|\ell_T = 2, \ell_z = 0\rangle$ and get
  $$\begin{aligned}
  \hat{\ell}_- |\ell_T = 2, \ell_z = 0\rangle &= \left(\hat{\ell}_{1-} + \hat{\ell}_{2-}\right) \frac{1}{\sqrt{6}} \left(|-1, 1\rangle + 2\, |0, 0\rangle + |1, -1\rangle\right) \\
  \hat{\ell}_- |\ell_T = 2, \ell_z = 0\rangle &= \frac{1}{\sqrt{6}} \left[\hat{\ell}_{1-} \left(|-1, 1\rangle + 2\, |0, 0\rangle + |1, -1\rangle\right) + \hat{\ell}_{2-} \left(|-1, 1\rangle + 2\, |0, 0\rangle + |1, -1\rangle\right)\right] \\
  \sqrt{6}\, |\ell_T = 2, \ell_z = -1\rangle &= \frac{1}{\sqrt{6}} \left[2\sqrt{2}\, |-1, 0\rangle + \sqrt{2}\, |0, -1\rangle + \sqrt{2}\, |-1, 0\rangle + 2\sqrt{2}\, |0, -1\rangle\right] \\
  |\ell_T = 2, \ell_z = -1\rangle &= \frac{1}{\sqrt{2}} \left(|-1, 0\rangle + |0, -1\rangle\right).
  \end{aligned}$$

- For $\ell_T = 2$ and $\ell_z = -2$:
  We apply the $\hat{\ell}_-$ operator to $|\ell_T = 2, \ell_z = 0\rangle$ and get

$$\hat{\ell}_- |\ell_T = 2, \ell_z = 0\rangle = \left(\hat{\ell}_{1-} + \hat{\ell}_{2-}\right) \frac{1}{\sqrt{6}} \left(|-1,1\rangle + 2\,|0,0\rangle + |1,-1\rangle\right)$$

$$\hat{\ell}_- |\ell_T = 2, \ell_z = 0\rangle = \frac{1}{\sqrt{6}} \left[\hat{\ell}_{1-} \left(|-1,1\rangle + 2\,|0,0\rangle + |1,-1\rangle\right) + \hat{\ell}_{2-} \left(|-1,1\rangle + 2\,|0,0\rangle + |1,-1\rangle\right)\right]$$

$$\sqrt{6}\,|\ell_T = 2, \ell_z = -1\rangle = \frac{1}{\sqrt{6}} \left[2\sqrt{2}\,|-1,0\rangle + \sqrt{2}\,|0,-1\rangle + \sqrt{2}\,|-1,0\rangle + 2\sqrt{2}\,|0,-1\rangle\right]$$

$$|\ell_T = 2, \ell_z = -1\rangle = \frac{1}{\sqrt{2}} \left(|-1,0\rangle + |0,-1\rangle\right).$$

- For $\ell_T = 1$ and $\ell_z = 1$:
  To do this, we can use the orthogonality property between different values of ℓ_T for the same value of ℓ_z. Taking a general case where

$$|\ell_T = 1, \ell_z = 1\rangle = A\,|1,0\rangle + B\,|0,1\rangle\,,$$

  where $|A|^2 + |B|^2 = 1$, we have

$$\langle \ell_T = 2, \ell_z = 1|\ell_T = 1, \ell_z = 1\rangle = \langle \ell_T = 2, \ell_z = 1|\left(A\,|1,0\rangle + B\,|0,1\rangle\right)$$

$$\langle \ell_T = 2, \ell_z = 1|\ell_T = 1, \ell_z = 1\rangle = \frac{1}{\sqrt{2}} \left(\langle 0,1| + \langle 1,0|\right)\left(A\,|1,0\rangle + B\,|0,1\rangle\right)$$

$$0 = \frac{1}{\sqrt{2}} (A + B)$$

$$\implies A = -B = \frac{1}{\sqrt{2}}.$$

  Thus,

$$|\ell_T = 1, \ell_z = 1\rangle = \frac{1}{\sqrt{2}} \left(|1,0\rangle - |0,1\rangle\right).$$

- For $\ell_T = 1$ and $\ell_z = 0$:
  We apply the $\hat{\ell}_-$ operator to $|\ell_T = 1, \ell_z = 1\rangle$ and get

$$\hat{\ell}_- |\ell_T = 1, \ell_z = 1\rangle = \left(\hat{\ell}_{1-} + \hat{\ell}_{2-}\right) \frac{1}{\sqrt{2}} \left(|1,0\rangle - |0,1\rangle\right)$$

$$\hat{\ell}_- |\ell_T = 1, \ell_z = 1\rangle = \frac{1}{\sqrt{2}} \left[\hat{\ell}_{1-} \left(|1,0\rangle - |0,1\rangle\right) + \hat{\ell}_{2-} \left(|1,0\rangle - |0,1\rangle\right)\right]$$

$$\sqrt{2}\,|\ell_T = 1, \ell_z = 0\rangle = \frac{1}{\sqrt{2}} \left[\sqrt{2}\,|0,0\rangle - \sqrt{2}\,|-1,1\rangle + \sqrt{2}\,|1,-1\rangle - \sqrt{2}\,|0,0\rangle\right]$$

$$|\ell_T = 1, \ell_z = 0\rangle = \frac{1}{\sqrt{2}} \left(|1,-1\rangle - |-1,1\rangle\right).$$

- For $\ell_T = 1$ and $\ell_z = -1$:
  We apply the $\hat{\ell}_-$ operator to $|\ell_T = 1, \ell_z = 0\rangle$ and get

$$\hat{\ell}_- |\ell_T = 1, \ell_z = 0\rangle = \left(\hat{\ell}_{1-} + \hat{\ell}_{2-}\right) \frac{1}{\sqrt{2}} \left(|1,-1\rangle - |-1,1\rangle\right)$$

$$\hat{\ell}_- |\ell_T = 1, \ell_z = 0\rangle = \frac{1}{\sqrt{2}} \left[\hat{\ell}_{1-} \left(|1,-1\rangle - |-1,1\rangle\right) + \hat{\ell}_{2-} \left(|1,-1\rangle - |-1,1\rangle\right)\right]$$

$$\sqrt{2}\,|\ell_T = 1, \ell_z = -1\rangle = \frac{1}{\sqrt{2}} \left[\sqrt{2}\,|0,-1\rangle - \sqrt{2}\,|-1,0\rangle\right]$$

$$|\ell_T = 1, \ell_z = -1\rangle = \frac{1}{\sqrt{2}} \left(|0,-1\rangle - |-1,0\rangle\right).$$

- For $\ell_T = 0$ and $\ell_z = 0$:

  To do this, we can use the orthogonality property such that this state has to be orthogonal to all the other states we calculated. Taking a general case where

$$|\ell_T = 0, \ell_z = 0\rangle = A\,|1,-1\rangle + B\,|0,0\rangle + C\,|-1,1\rangle\,,$$

  where $|A|^2 + |B|^2 + |C|^2 = 1$, we have

  - Orthogonality with $\langle \ell_T = 2, \ell_z = 0|$:

$$\begin{aligned}
\langle \ell_T = 2, \ell_z = 0|\ell_T = 0, \ell_z = 0\rangle &= \langle \ell_T = 2, \ell_z = 0|\left(A\,|1,-1\rangle + B\,|0,0\rangle + C\,|-1,1\rangle\right) \\
\langle \ell_T = 2, \ell_z = 0|\ell_T = 0, \ell_z = 0\rangle &= \frac{1}{\sqrt{6}}\left(\langle 1,-1| + 2\,\langle 0,0| + \langle -1,1|\right)\left(A\,|1,-1\rangle + B\,|0,0\rangle + C\,|-1,1\rangle\right) \\
0 &= \frac{1}{\sqrt{6}}\left(A + 2B + C\right) \\
&\implies A + 2B + C = 0.
\end{aligned}$$

  - Orthogonality with $\langle \ell_T = 1, \ell_z = 0|$:

$$\begin{aligned}
\langle \ell_T = 1, \ell_z = 0|\ell_T = 0, \ell_z = 0\rangle &= \langle \ell_T = 1, \ell_z = 0|\left(A\,|1,-1\rangle + B\,|0,0\rangle + C\,|-1,1\rangle\right) \\
\langle \ell_T = 1, \ell_z = 0|\ell_T = 0, \ell_z = 0\rangle &= \frac{1}{\sqrt{2}}\left(\langle 1,-1| - \langle -1,1|\right)\left(A\,|1,-1\rangle + B\,|0,0\rangle + C\,|-1,1\rangle\right) \\
0 &= \frac{1}{\sqrt{2}}\left(A - C\right) \\
&\implies A - C = 0 \\
&\implies A = C \\
&\implies A = -B.
\end{aligned}$$

  Thus, $A = B = C = \frac{1}{\sqrt{3}}$, and we have

$$|\ell_T = 0, \ell_z = 0\rangle = \frac{1}{\sqrt{3}}\left(|1,-1\rangle + |0,0\rangle + |-1,1\rangle\right).$$

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