SM ASS - 7

① ⓐ For a relativistic gas

$$\epsilon_p = \sqrt{p^2c^2+m^2c^4}$$

taking $c=1, m=1$ $\quad \epsilon_p = \sqrt{p^2+1}$

The partition function is

$$Z(\beta) = \int e^{-\beta\sqrt{p^2+1}} \frac{d^3p\, d^3x}{h^3}$$

$$= \frac{4\pi V}{h^3} \int e^{-\beta\sqrt{p^2+1}} p^2 dp$$

For calculating $Z(\beta)$ at different β we create a linespace of β from

ⓑ $$C_V = \frac{\partial}{\partial T}\left(-\frac{\partial \ln Z}{\partial \beta}\right) = \frac{\partial \beta}{\partial T}\frac{\partial}{\partial \beta}\left(\frac{\partial \ln Z}{\partial \beta}\right) = -\frac{1}{T^2}\frac{\partial E}{\partial \beta}$$

$$= -\beta^2 \frac{\partial E}{\partial \beta}$$

Expected $T_{tran} \to kT = mc^2 \to T \sim 1$ $(k = m = c = 1)$

1.5 (non relativistic) $\xrightarrow{T\sim 1}$ 3 (relativistic)

② $$Z = \int \frac{d^3x\, d^3p}{h^3} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} e^{-\beta\left(\frac{p^2}{2m} + \frac{\hbar^2}{2I} j(j+1)\right)}$$

All different m values are degenerate.

$\therefore$ There is a factor of $2j+1$ in the sum

$$\to Z = \frac{V}{h^3} \int d^3p\, e^{-\beta p^2/2m} \sum_{j=0}^{\infty} (2j+1) e^{-\frac{\beta\hbar^2}{2I} j(j+1)}$$

$$= V\left(\frac{2\pi m}{\beta h^2}\right)^{3/2} \sum_{j=0}^{\infty} (2j+1) e^{-\frac{\beta\hbar^2}{2I} j(j+1)}$$

We calculate E & C_V similarly

③ The partition function is given by

$$Z = 1 + e^{-\beta\Delta E}$$

$$E = -\frac{\partial \ln Z}{\partial \beta} = -\frac{-\Delta E\, e^{-\beta\Delta E}}{1 + e^{-\beta\Delta E}}$$

$$= \frac{\Delta E\, e^{-\beta\Delta E}}{1 + e^{-\beta\Delta E}}$$

$$= \frac{\Delta E}{1 + e^{\beta\Delta E}}$$

$$\frac{\partial E}{\partial \beta} = \frac{\partial(\beta\Delta E)}{\partial \beta} \cdot \frac{\partial E}{\partial(\beta\Delta E)} = \Delta E \cdot \Delta E \cdot \left(-\frac{e^{\beta\Delta E}}{(1+e^{\beta\Delta E})^2}\right)$$

$$C_V = -\beta^2 \frac{\partial E}{\partial \beta} = \beta^2(\Delta E)^2 \frac{e^{\beta\Delta E}}{(1+e^{\beta\Delta E})^2}$$

$$= \frac{x^2 e^x}{(1+e^x)^2}$$

④ We fit the given data with

$$C_V = 3Nk\left(\frac{\epsilon}{kT}\right)^2 \frac{e^{\epsilon/kT}}{(e^{\epsilon/kT}-1)^2}$$

we name $a = \frac{\epsilon}{k}$

For $N = N_A$,

$$C_V = 3R\left(\frac{a}{T}\right)^2 \frac{e^{a/T}}{(e^{a/T}-1)^2}$$

By fitting we obtain

$a \simeq 1295\,K$

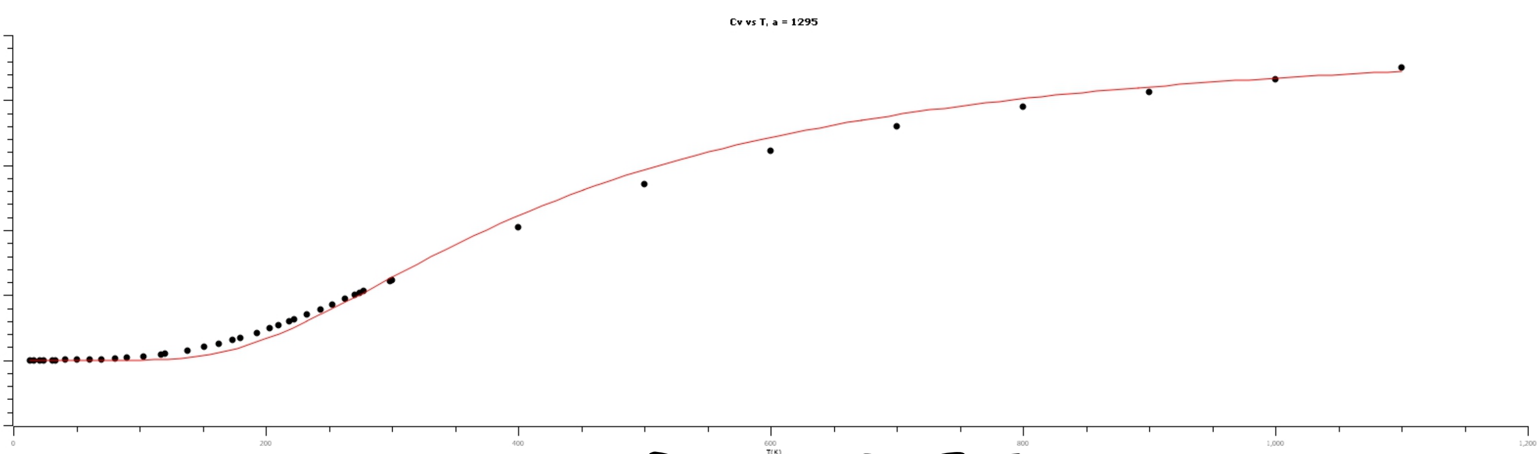
$\therefore \frac{\epsilon}{k} = 1295\,K$

$\to \frac{\hbar\omega}{k} = 1295\,K$

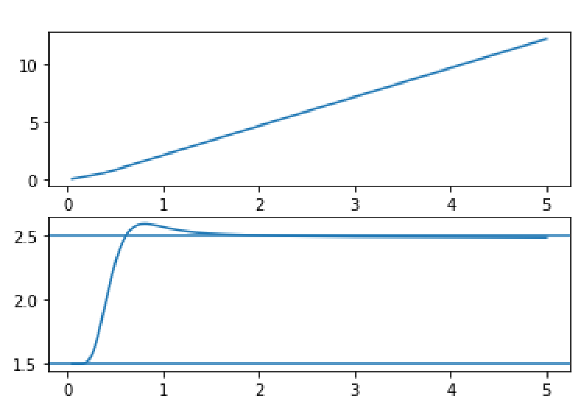
So matches with theory

$$\to \omega = \frac{1295 \times 1.38\times10^{-23}}{1.05 \times 10^{-34}}$$

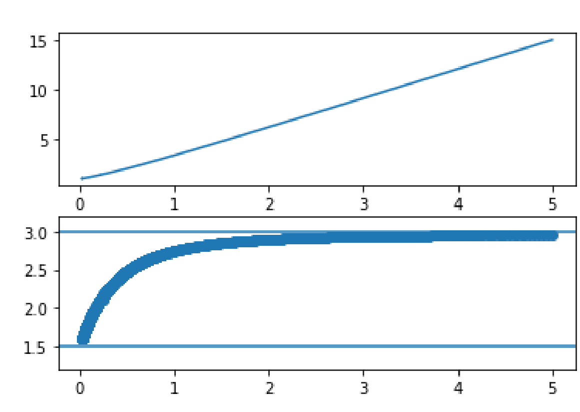
$$\simeq 1.7 \times 10^{14}\ Hz$$



DIAMOND FIT



DIATOMIC GAS



RELATIVISTIC GAS

C_V for 2 state system

C_V for monoatomic 1D gas