MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*

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Homework 12

Problem 14-1
Show that covectors $\omega^1, \ldots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

Solution. $\boxed{\Longrightarrow}$ First, suppose the covectors are linearly dependent. Then there exist scalars $c_1, \ldots, c_k$, not all zero, such that

$$c_1\omega^1 + \cdots + c_k\omega^k = 0.$$

Without loss of generality, assume $c_1 \neq 0$. Then

$$\omega^1 = -\frac{c_2}{c_1}\omega^2 - \cdots - \frac{c_k}{c_1}\omega^k.$$

Substituting this into the wedge product

$$\begin{aligned}
\omega^1 \wedge \cdots \wedge \omega^k &= \left(-\frac{c_2}{c_1}\omega^2 - \cdots - \frac{c_k}{c_1}\omega^k\right) \wedge \omega^2 \wedge \cdots \wedge \omega^k \\
&= -\frac{c_2}{c_1}\omega^2 \wedge \omega^2 \wedge \cdots \wedge \omega^k - \cdots - \frac{c_k}{c_1}\omega^k \wedge \omega^2 \wedge \cdots \wedge \omega^k \\
&= 0,
\end{aligned}$$

since each term contains a repeated covector, and $\omega^i \wedge \omega^i = 0$ for any covector ω^i.
$\boxed{\Longleftarrow}$ Conversely, suppose $\omega^1 \wedge \cdots \wedge \omega^k = 0$. Let's extend $\omega^1, \ldots, \omega^k$ to a basis $\omega^1, \ldots, \omega^k, \omega^{k+1}, \ldots, \omega^n$ of the dual space. Each ω^i can be written in terms of the dual basis $e^1, \ldots, e^n$

$$\omega^i = \sum_{j=1}^{n} a_{ij} e^j$$

The wedge product $\omega^1 \wedge \cdots \wedge \omega^k$ can be written as a sum of basic k-forms $e^{i_1} \wedge \cdots \wedge e^{i_k}$ with coefficients given by $k \times k$ minors of the matrix (a_{ij}). Since this wedge product is zero, all these minors must be zero, which implies that the rows of (a_{ij}) are linearly dependent. Therefore, the covectors $\omega^1, \ldots, \omega^k$ are linearly dependent.
Therefore, covectors $\omega^1, \ldots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$. ∎

Problem 14-9
Let M, N be smooth manifolds, and suppose $\pi : M \to N$ is a surjective smooth submersion with connected fibers. We say that a tangent vector $v \in T_pM$ is **vertical** if $d\pi_p(v) = 0$. Suppose $\omega \in \Omega^k(M)$. Show that there exists $\eta \in \Omega^k(N)$ such that $\omega = \pi^*\eta$ if and only if $v \lrcorner\, \omega_p = 0$ and $v \lrcorner\, d\omega_p = 0$ for every $p \in M$ and every vertical vector $v \in T_pM$. [Hint: first, do the case in which $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is projection onto the first n coordinates.]

Solution. In this case, let's write coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ on $\mathbb{R}^{n+m}$ and $(x^1, \ldots, x^n)$ on $\mathbb{R}^n$. The vertical vectors are those of the form $\sum_{j=1}^m b_j \frac{\partial}{\partial y^j}$.
A general k-form ω on $\mathbb{R}^{n+m}$ can be written as

$$\omega = \sum_{I,J} f_{I,J}(x,y)\, \mathrm{d}x^I \wedge \mathrm{d}y^J,$$

where I and J are multi-indices with $|I| + |J| = k$. For ω to be a pullback, it must not contain any dy^j terms, so $f_{I,J} = 0$ for all $J \neq \emptyset$. Moreover, the coefficients cannot depend on y. Therefore

$$\omega = \sum_{|I|=k} f_I(x)\,\mathrm{d}x^I = \pi^\eta,$$

where $\eta = \sum_{|I|=k} f_I(x)\,\mathrm{d}x^I$ is a k-form on $\mathbb{R}^n$.

- **For necessity:** If $\omega = \pi^*\eta$, then ω has no $\mathrm{d}y^j$ terms, so contracting with vertical vectors gives zero. Also, $\mathrm{d}\omega = \pi^* d\eta$ has at most one $\mathrm{d}y^j$ in each term, so contracting with vertical vectors again gives zero.
- **For sufficiency:** If $v \lrcorner \omega = 0$ for all vertical vectors, then ω cannot have any terms with more than one $\mathrm{d}y^j$. If also $v \lrcorner \mathrm{d}\omega = 0$, then the coefficients cannot depend on y.
- **For the general case:** Let $p \in M$. Near p, we can choose coordinates $(x^1, \ldots, x^n)$ on N and extend them to coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ on M such that π looks like the projection $\mathbb{R}^{n+m} \to \mathbb{R}^n$ locally.

The conditions $v \lrcorner \omega = 0$ and $v \lrcorner \mathrm{d}\omega = 0$ for vertical vectors are tensorial, so they hold in one coordinate system if and only if they hold in all coordinate systems.
By our local result, in each coordinate neighborhood U_α, there exists $\eta_\alpha \in \Omega^k(\pi(U_\alpha))$ such that $\omega|U\alpha = \pi^\eta_\alpha$. These local forms must agree on overlaps because π is surjective and has connected fibers. Therefore, they piece together to give a global form $\eta \in \Omega^k(N)$ such that $\omega = \pi^\eta$.
The key insight is that the condition about contractions with vertical vectors forces the form to be "horizontal" (no dy terms) and "basic" (coefficients independent of y), which are exactly the conditions needed for a form to be a pullback. ■

Problem 15-3
Suppose $n \geq 1$, and let $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd. [Hint: consider the map $F : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ given by $F(x) = -x$, and use Corollary 15.34.] (Used on pp. 393, 435.)
(Since we havent introduced Riemannian manifolds yet, you should probably ignore the hint in the book. You might try to use the results of Ex. 15.13(b) instead, along with, say, stereographic coordinates.)

Solution. First, recall that a diffeomorphism is orientation-preserving if and only if its differential has positive determinant at every point. The antipodal map α is smooth, and at each point $x \in \mathbb{S}^n$, we need to compute $\det(\mathrm{d}\alpha_x)$. Using a stereographic projection $\phi : \mathbb{S}^n \setminus N \to \mathbb{R}^n$ from the north pole $N = (0, \ldots, 0, 1)$, given by

$$\phi(x^1, \ldots, x^{n+1}) = \left(\frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}} \right).$$

Consider the composition $\phi \circ \alpha \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. This is easier to analyze than α directly. For $y \in \mathbb{R}^n$, we write out this composition

$$(\phi \circ \alpha \circ \phi^{-1})(y) = -\frac{y}{|y|^2}$$

This is because the antipodal map of a point, when viewed through stereographic projection, is the inversion through the unit sphere scaled by -1. Computing the differential of this map at a point $y \in \mathbb{R}^n$, we have

$$\mathrm{d}(\phi \circ \alpha \circ \phi^{-1})_y(v) = -\frac{|y|^2 v - 2(y \cdot v)y}{|y|^4}$$

The determinant of this linear transformation is

$$\det(\mathrm{d}(\phi \circ \alpha \circ \phi^{-1})_y) = \frac{(-1)^n}{|y|^{2n}}$$

Since ϕ is a diffeomorphism (away from the north pole), the original map α is orientation-preserving if and only if this determinant is positive.

- When n is odd, $(-1)^n = -1$, and $|y|^{2n}$ is positive, so the determinant is negative.
- When n is even, $(-1)^n = 1$, and $|y|^{2n}$ is positive, so the determinant is positive.

However, we need to be careful: ϕ itself changes orientation. In fact, stereographic projection is orientation-preserving when n is even and orientation-reversing when n is odd (this can be proven by computing its Jacobian). Therefore, accounting for this, we have

- When n is odd, α is orientation-preserving.
- When n is even, α is orientation-reversing.

Therefore, α is orientation-preserving if and only if n is odd. ■

Problem 15-12
Show that every orientation-reversing diffeomorphism of $\mathbb{R}$ has a fixed point.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be an orientation-reversing diffeomorphism. Being orientation-reversing means that, for every point $x \in \mathbb{R}$, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) < 0.$$

This means f is strictly decreasing. Being a diffeomorphism means f is smooth and bijective.
Consider the function $g(x) = f(x) - x$. We will show this has a zero, which means f has a fixed point. Since f is strictly decreasing, $g'(x) = f'(x) - 1 < -1$ for all x. This means g is also strictly decreasing. Since f is bijective from $\mathbb{R}$ to $\mathbb{R}$ and strictly decreasing, then

$$\lim_{x\to\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

Thus,

$$\lim_{x\to\infty} g(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = \infty.$$

By the Intermediate Value Theorem, since g is continuous and takes both positive and negative values, there must exist some $c \in \mathbb{R}$ such that $g(c) = 0$. Thus, $f(c) = c$, so c is a fixed point of f.
Therefore, every orientation-reversing diffeomorphism of $\mathbb{R}$ has a fixed point. ■