PHYS 601 - Methods of Theoretical Physics II

Mathematical Methods for Physicists by *Arfken, Weber, Harris*
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Homework 1

Problem 1
Find the shortest distance between two points in polar coordinates, i.e., using the line element $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$.

Proof. Given $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$, we have

$$\begin{aligned} I &= \int_{(x_1,y_1)}^{(x_2,y_2)} \mathrm{d}s \\ &= \int_{(r_1,\theta_1)}^{(r_2,\theta_2)} \sqrt{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2} \\ &= \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2} \\ &= \int_{\theta_1}^{\theta_2} \mathrm{d}\theta \sqrt{r^2 + r'^2} \end{aligned}$$

We are interested in the shortest distance between these two points; hence, we need to minimize the integrand. Define the integrand to be the function $f(\theta)$. Plugging in f in the Euler-Lagrange equation will give us the result we need. The Euler-Lagrange equation is given by

$$\frac{\partial f}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial f}{\partial r'}\right) = 0$$

By using the chain rule on the $\frac{\mathrm{d}}{\mathrm{d}\theta}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\partial}{\partial\theta} + r'\frac{\partial}{\partial r} + r''\frac{\partial}{\partial r'}$$

Replacing, we get

$$\frac{\partial f}{\partial r} - \frac{\partial}{\partial\theta}\frac{\partial f}{\partial r'} - r'\frac{\partial}{\partial r}\frac{\partial f}{\partial r'} - r''\frac{\partial^2 f}{\partial r'^2} = 0$$

Plugging in our function f defined above

$$\frac{\partial f}{\partial r} = \frac{r}{(r^2 + r'^2)^{\frac{1}{2}}}, \qquad \frac{\partial f}{\partial r'} = \frac{r'}{(r^2 + r'^2)^{\frac{1}{2}}}$$

$$\begin{aligned} \implies \frac{r}{(r^2+r'^2)^{\frac{1}{2}}} - 0 + \frac{rr'^2}{(r^2+r'^2)^{\frac{3}{2}}} - r''\left[\frac{1}{(r^2+r'^2)^{\frac{1}{2}}} - \frac{r'^2}{(r^2+r'^2)^{\frac{3}{2}}}\right] &= 0 \\ \implies \frac{r - r''}{(r^2+r'^2)^{\frac{1}{2}}} + \frac{rr'^2 + r'^2 r''}{(r^2+r'^2)^{\frac{3}{2}}} &= 0 \\ \implies \frac{(r-r'')(r^2+r'^2) + rr'^2 + r'^2 r''}{(r^2+r'^2)^{\frac{3}{2}}} &= 0 \\ \implies \frac{r^3 + 2rr'^2 - r^2 r''}{(r^2+r'^2)^{\frac{3}{2}}} &= 0 \\ \implies \frac{r^2 + 2r'^2 - rr''}{(r^2+r'^2)^{\frac{3}{2}}} &= 0 \end{aligned}$$

This term is the same term used to describe the curvature. It is only zero when r is very large, i.e., for a straight line. Thus, the shortest distance between two points is, as expected, a straight line. ■

Problem 2
Inverse Isoperimetric Problem (Isoareametric Problem): Prove that of all simple closed curves enclosing a given area, the least perimeter is possessed by the circle.

Proof. Given a fixed constant area, we want to show that the perimeter of our closed domain is the least when our enclosure is a circle. The area is given by

$$A = \int \mathrm{d}a = \int_0^{2\pi} \int_0^r r' \,\mathrm{d}r' \,\mathrm{d}\theta = \frac{1}{2} \int_0^{2\pi} r^2 \,\mathrm{d}\theta$$

The perimeter of a closed curve in polar coordinates is given by

$$\begin{aligned} P = \oint \mathrm{d}s &= \oint \sqrt{\mathrm{d}r^2 + r^2 \,\mathrm{d}\theta^2} \\ &= \oint \mathrm{d}\theta \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2} \\ &= \oint \mathrm{d}\theta \sqrt{r^2 + r'^2}. \end{aligned}$$

As usual, our intentions are to extremize (specifically minimize) the integrand. We define $f(r,\theta) = \frac{1}{2}r^2$ to be the integrand function of our constraint given by the fixed area A and $g(r,\theta) = \sqrt{r^2 + r'^2}$ to be the integrand function of our perimeter integral P. Let $h(r,\theta) = f + \lambda g$, where λ is a Lagrange multiplier.

Replacing in the Euler-Lagrange equation, we get

$$\frac{\partial h}{\partial r} - \frac{\partial}{\partial \theta}\frac{\partial h}{\partial r'} - r'\frac{\partial}{\partial r}\frac{\partial h}{\partial r'} - r''\frac{\partial^2 h}{\partial r'^2} = 0$$

$$\frac{\partial h}{\partial r} = \frac{\lambda r}{(r^2 + r'^2)^{\frac{1}{2}}} + r, \qquad \frac{\partial h}{\partial r'} = \frac{\lambda r'}{(r^2 + r'^2)^{\frac{1}{2}}}$$

$$\implies \frac{\lambda r}{(r^2 + r'^2)^{\frac{1}{2}}} + r - 0 + \frac{\lambda r r'^2}{(r^2 + r'^2)^{\frac{3}{2}}} - \lambda r''\left[\frac{1}{(r^2 + r'^2)^{\frac{1}{2}}} - \frac{r'^2}{(r^2 + r'^2)^{\frac{3}{2}}}\right] = 0$$

$$\implies \frac{r^2 + 2r'^2 - r r''}{(r^2 + r'^2)^{\frac{3}{2}}} = -\frac{1}{\lambda}$$

This is the curvature equation for a circle with radius λ. Thus, a fixed area has a minimal perimeter when the enclosure is a circle. ∎

Problem 3
In all our discussions so far on finding the function f for which

$$I = \int_{x_1}^{x_2} f \, \mathrm{d}x$$

is an extremum, it has been assumed that f depends on x, y, and y', that is, $f = f(x, y, y')$. Show that, if f also depends on $y'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, and for fixed end points at which y and y' are prescribed, the Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

Proof. Let $f = f(x, \mathbf{y})$, $\mathbf{y} = (y, y', y'')$ be our function. We want to find the Euler-Lagrange equation for f, i.e. the equation that minimizes the definite integral of our function, given by

$$I = \int_{x_1}^{x_2} f(x, y, y', y'') \, \mathrm{d}x$$

Define $y_\epsilon(x) = y(x) + \epsilon\eta(x)$, with $\eta(x_1) = \eta(x_2) = 0$ as our boundary conditions. Our integral can be rewritten as follows

$$I_\epsilon = \int_{x_1}^{x_2} f(x, y_\epsilon, y'_\epsilon, y''_\epsilon) \, \mathrm{d}x$$

To extremize our integral, we require that

$$\left.\frac{\mathrm{d}I(\epsilon)}{\mathrm{d}x}\right|_{\epsilon \to 0} = 0$$

$$\begin{aligned}
\implies \frac{\mathrm{d}I(\epsilon)}{\mathrm{d}x} &= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \epsilon} + \frac{\partial f}{\partial y''}\frac{\partial y''}{\partial \epsilon}\right] \mathrm{d}x \\
&= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta' + \frac{\partial f}{\partial y''}\eta''\right] \mathrm{d}x
\end{aligned}$$

Using integration by parts, we get

$$\begin{aligned}
\frac{\mathrm{d}I(\epsilon)}{\mathrm{d}x} &= \left.\frac{\partial f}{\partial y'}\eta\right|_{x_1}^{x_2} + \left.\frac{\partial f}{\partial y'}\eta'\right|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\eta - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y''}\right)\eta'\right] \mathrm{d}x \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y''}\right)\eta\right|_{x_1}^{x_2} + \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\eta + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right)\eta\right] \mathrm{d}x \\
&= \int_{x_1}^{x_2} \eta \left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right)\right] \mathrm{d}x \\
&= 0
\end{aligned}$$

Since $\eta(x) \neq 0$ is an arbitrary function, then the integral is extremized when the integrand is equal to zero

$$\implies \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial f}{\partial y''}\right) = 0.$$

Thus, we have obtained the Euler-Lagrange equation that also accounts for the second derivative of y. ■

Problem 4
We proved in class that the curve which encloses the most area for a given perimeter was a circle. To do this, we demonstrated that this curve was characterized by a constant curvature $1/\lambda$ everywhere. Obtain the same result by using the Euler-Lagrange equation to solve for $r = r(\theta)$ or $\theta = \theta(r)$ directly. [Hint: Use the fact that $f(\theta, r, r')$ and $g(\theta, r, r')$ are independent of θ.]

Proof. Starting with the same setup as Problem 2, we have

$$A = \frac{1}{2}\int_0^{2\pi} r^2 \,\mathrm{d}\theta$$

$$P = \oint \mathrm{d}\theta \sqrt{r^2 + r'^2}.$$

We define $f(r,\theta) = \frac{1}{2}r^2$ to be the integrand function of our constraint given by the fixed area A and $g(r,\theta) = \sqrt{r^2 + r'^2}$ to be the integrand function of our perimeter integral P. Let $h(r,\theta) = f + \lambda g$, where λ is a Lagrange multiplier.

Our Euler-Lagrange equation is given by

$$\frac{\partial h}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\partial h}{\partial r'}\right) = 0.$$

We can notice that the function h is independent of θ explicitly, and thus, we may use the following

$$\frac{\partial h}{\partial \theta} = -\frac{\mathrm{d}}{\mathrm{d}\theta}\left(r'\frac{\partial h}{\partial r'} - h\right) = 0$$

$$\implies r'\frac{\partial h}{\partial r'} - h = \text{constant} = c_1$$

Replacing our function, we get

$$r'\frac{\lambda r'}{(r^2 + r'^2)^{\frac{1}{2}}} - \frac{1}{2}r^2 - \lambda(r^2 + r'^2)^{\frac{1}{2}} = c_1$$

$$\frac{-\lambda r^2}{(r^2 + r'^2)^{\frac{1}{2}}} = c_1 + \frac{1}{2}r^2$$

$$r' = \sqrt{\frac{4\lambda^2 r^4}{(r^2 + 2c_1)^2} - r^2}$$

$$\mathrm{d}\theta = \left(\frac{4\lambda^2 r^4}{(r^2 + 2c_1)^2} - r^2\right)^{-\frac{1}{2}} \mathrm{d}r$$

$$\theta = \int \left(\frac{4\lambda^2 r^4}{(r^2 + 2c_1)^2} - r^2\right)^{-\frac{1}{2}} \mathrm{d}r$$

The solution to this integral is extremely messy and assumes a lot of constraints on our parameters. If we assume that $c_1 = 0$, then the integral becomes

$$\theta = \arcsin\left(\frac{r}{2\lambda}\right) + c_2$$

Assuming again that $c_2 = 0$, we get

$$r(\theta) = 2\lambda \sin(\theta)$$

Converting to Cartesian coordinates, we have

$$\sqrt{x^2 + y^2} = 2\lambda \frac{y}{\sqrt{x^2 + y^2}}$$

$$x^2 + y^2 - 2\lambda y = 0$$

$$x^2 + (y - \lambda)^2 = \lambda^2$$

which is an equation of a circle, centered at $(0, \lambda)$, with radius λ. ■

Problem 5
In connection with Problem 4 above, show that the curvature K in polar coordinate system is

$$K = \left| \frac{r^2 + 2r'^2 - rr''}{(r^2 + r'^2)^{3/2}} \right|,$$

where $r' = \mathrm{d}r/\,\mathrm{d}\theta$ and $r'' = \mathrm{d}^2 r/\,\mathrm{d}\theta^2$.

Proof. We know the general formula of the curvature is given by

$$K = \left| \frac{y''}{(1 + y'^2)^{\frac{3}{2}}} \right|$$

We will use the conversion from Cartesian to polar to express y' and y'' and arrive at the requested form. In connection with Problem 4, the radius is a function of θ, $r(\theta)$.

We have

$$\begin{cases} x = r(\theta)\cos(\theta) \\ y = r(\theta)\sin(\theta) \end{cases}$$

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}\theta}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^{-1} = \frac{r'\sin(\theta) + r\cos(\theta)}{r'\cos(\theta) - r\sin(\theta)}$$

$$y''(x) = \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}y'}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}y'}{\mathrm{d}\theta}\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^{-1} = \frac{r^2 + 2r'^2 - r''r}{(r'\cos(\theta) - r\sin(\theta))^3}$$

Replacing in K, we get

$$\begin{aligned} K &= \left| \frac{\frac{r^2+2r'^2-r''r}{(r'\cos(\theta)-r\sin(\theta))^3}}{\left(1 + \left(\frac{r'\sin(\theta)+r\cos(\theta)}{r'\cos(\theta)-r\sin(\theta)}\right)^2\right)^{\frac{3}{2}}} \right| \\ &= \left| \frac{\frac{r^2+2r'^2-r''r}{(r'\cos(\theta)-r\sin(\theta))^3}}{\left(\frac{r^2+r'^2}{(r'\cos(\theta)-r\sin(\theta))^2}\right)^{\frac{3}{2}}} \right| \\ &= \left| \frac{r^2 + 2r'^2 - r''r}{(r^2 + r'^2)^{\frac{3}{2}}} \right|. \end{aligned}$$

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