PHYS 617 - Statistical Mechanics

A Modern Course in Statistical Physics by *Linda E. Reichl*

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Homework 4

Problem 1

Recall the advection equation:

$$\partial_t u + a\partial_x u = 0,$$

where a is a constant. In class, I discussed this equation and briefly told you how it's solved, but I'd like to see you give it a try yourself. For general initial conditions $u(x,0) = f_0(x)$, find the general solution $u(x,t)$ for this equation by any means you like.

Solution. To solve the advection equation, we use the method of separation of variables. Assume that u has the form $u(x,t) = X(x)T(t)$. Then, if we derive respectively and replace back into the advection equation, and dividing by $X(x)T(t)$, we get

$$\begin{aligned} \frac{T'}{T} + a\frac{X'}{X} &= 0 \\ \frac{T'}{T} &= -a\frac{X'}{X}. \end{aligned}$$

$X(x)$ in independent of t and $T(t)$ is independent of x, that means each hand-side is a constant, say λ. Solving each equation separately, we have

$$\begin{cases} \frac{X'}{X} = \lambda, \\ \frac{T'}{T} = -a\lambda, \end{cases} \implies \begin{cases} \frac{\mathrm{d}X}{X} = \lambda\,\mathrm{d}x, \\ \frac{\mathrm{d}T}{T} = -a\lambda\,\mathrm{d}t, \end{cases} \implies \begin{cases} \ln(X) = \lambda x + c_1, \\ \ln(T) = -a\lambda t + c_2, \end{cases} \implies \begin{cases} X(x) = A\mathrm{e}^{\lambda x}, \\ T(t) = B\mathrm{e}^{-a\lambda t}. \end{cases}$$

Then, we have that

$$u(x,t) = X(x)T(t) = \left(A\mathrm{e}^{\lambda x}\right)\left(B\mathrm{e}^{-a\lambda t}\right) = C\mathrm{e}^{\lambda(x-at)}.$$

Given the initial condition $u(x,0) = f_0(x)$, then

$$u(x,0) = f_0(x) = C\mathrm{e}^{\lambda x}.$$

Thus, the general solution is

$$u(x,t) = f_0(x)\mathrm{e}^{-a\lambda t}.$$

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Problem 2
In class we derived the following equations starting from Euler's equations:

$$\begin{aligned}\dot{\rho} + (v \cdot \nabla)\rho + \rho(\nabla \cdot v) &= 0, \\ \dot{\vec{v}} + (v \cdot \nabla)\vec{v} + \frac{1}{\rho}\vec{\nabla}P &= 0, \\ \dot{P} + (v \cdot \nabla)P + \gamma P(\nabla \cdot v) &= 0.\end{aligned}$$

Define the quantity $s \equiv \left(\frac{P}{\rho^\gamma}\right)$. Show the following is true:

$$\dot{s} + (v \cdot \nabla)s = 0.$$

Does this mean that entropy is conserved? What conditions are necessary for this to be true?

Solution. Using the defined quantity s, rewritten as $s = \ln(P) - \gamma\ln(\rho)$, we take the total time derivative of s. Then

$$\begin{aligned}\dot{s} &= \frac{\partial s}{\partial P}\frac{\partial P}{\partial t} + \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial t} \\ &= \frac{\partial s}{\partial P}\dot{P} + \frac{\partial s}{\partial \rho}\dot{\rho} \\ &= \left(\frac{1}{P}\right)\dot{P} + \left(-\frac{\gamma}{\rho}\right)\dot{\rho} \\ &= \frac{\dot{P}}{P} - \frac{\gamma\dot{\rho}}{\rho}.\end{aligned}$$

Now, we calculate the second term

$$\begin{aligned}(v \cdot \nabla)s &= \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)s \\ &= v_x\frac{\partial s}{\partial x} + v_y\frac{\partial s}{\partial y} + v_z\frac{\partial s}{\partial z} \\ &= v_x\left(\frac{\partial s}{\partial P}\frac{\partial P}{\partial x} + \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial x}\right) + v_y\left(\frac{\partial s}{\partial P}\frac{\partial P}{\partial y} + \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial y}\right) + v_z\left(\frac{\partial s}{\partial P}\frac{\partial P}{\partial z} + \frac{\partial s}{\partial \rho}\frac{\partial \rho}{\partial z}\right) \\ &= v_x\left(\frac{1}{P}\frac{\partial P}{\partial x} - \frac{\gamma}{\rho}\frac{\partial \rho}{\partial x}\right) + v_y\left(\frac{1}{P}\frac{\partial P}{\partial y} - \frac{\gamma}{\rho}\frac{\partial \rho}{\partial y}\right) + v_z\left(\frac{1}{P}\frac{\partial P}{\partial z} - \frac{\gamma}{\rho}\frac{\partial \rho}{\partial z}\right) \\ &= \frac{1}{P}\left(v_x\frac{\partial P}{\partial x} + v_y\frac{\partial P}{\partial y} + v_z\frac{\partial P}{\partial z}\right) - \frac{\gamma}{\rho}\left(v_x\frac{\partial \rho}{\partial x} + v_y\frac{\partial \rho}{\partial y} + v_z\frac{\partial \rho}{\partial z}\right) \\ &= \frac{1}{P}(v \cdot \nabla)P - \frac{\gamma}{\rho}(v \cdot \nabla)\rho \\ &= -\frac{1}{P}\left(\dot{P} + \gamma P(\nabla \cdot v)\right) + \frac{\gamma}{\rho}\left(\dot{\rho} + \rho(\nabla \cdot v)\right) \\ &= -\frac{\dot{P}}{P} - \gamma(\nabla \cdot v) + \frac{\gamma\dot{\rho}}{\rho} + \gamma(\nabla \cdot v) \\ &= -\frac{\dot{P}}{P} + \frac{\gamma\dot{\rho}}{\rho}.\end{aligned}$$

Thus,

$$\dot{s} + (v \cdot \nabla)s = 0.$$

No, this does not mean that entropy is conserved. A conservation law takes the form of

$$\dot{s} + \nabla \cdot j_s = 0.$$

Thus, for entropy to be conserved, the following condition must be satisfied

$$\nabla \cdot j_s = (v \cdot \nabla)s.$$

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Problem 3
Work out a second-order ODE describing the density as a function of radius in a star (as in, if this ODE were solved, the solution would be $\rho(r)$). Use the following two assumptions: First, the star is in hydrostatic equilibrium. Second, assume a polytropic equation of state $P = K\rho^\gamma$. It is often conventional to define $\gamma \equiv 1 + \frac{1}{n}$ for this problem (and it simplifies the resulting equations).

There are actually exact solutions for a few values of n but I won't ask you to derive them. You can try if you want though!

Solution. Assuming we are in hydrostatic equilibrium, then

$$\frac{1}{\rho}\vec{\nabla}P = \vec{g},$$

which is equivalent to

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{Gm_{\mathrm{enc}}\rho}{r^2}$$

in polar coordinates, where $m_{\mathrm{enc}}(r) = \int_0^r 4\pi(r')^2\rho\,\mathrm{d}r'$, or equivalently,

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2\rho.$$

We are given a polytropic equation of state $P = K\rho^\gamma$, then

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}r} &= K\gamma\rho^{\gamma-1}\frac{\mathrm{d}\rho}{\mathrm{d}r}\\ &= K\left(1+\frac{1}{n}\right)\rho^{\frac{1}{n}}\frac{\mathrm{d}\rho}{\mathrm{d}r}.\end{aligned}$$

Replacing in the equation of hydrostatic equilibrium, we have

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{Gm_{\mathrm{enc}}\rho}{r^2}\\
K\gamma\rho^{\frac{1}{n}}\frac{\mathrm{d}\rho}{\mathrm{d}r} &= -\frac{4\pi G\rho}{r^2}\int_0^r (r')^2\rho\,\mathrm{d}r'\\
\frac{K\gamma}{4\pi G}r^2\rho^{\frac{1}{n}-1}\frac{\mathrm{d}\rho}{\mathrm{d}r} &= -\int_0^r (r')^2\rho\,\mathrm{d}r'\\
\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{K\gamma}{4\pi G}r^2\rho^{\frac{1}{n}-1}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right] &= -\frac{\mathrm{d}}{\mathrm{d}r}\left[\int_0^r (r')^2\rho\,\mathrm{d}r'\right]\\
\frac{K\gamma}{4\pi G}\left[2r\rho^{\frac{1}{n}-1}\frac{\mathrm{d}\rho}{\mathrm{d}r} + r^2\left(\frac{1}{n}-1\right)\rho^{\frac{1}{n}-2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + r^2\rho^{\frac{1}{n}-1}\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}\right] &= -r^2\rho\\
\gamma\left[2\rho^{\frac{1}{n}}\frac{\mathrm{d}\rho}{\mathrm{d}r} + r\left(\frac{1}{n}-1\right)\rho^{\frac{1}{n}-1}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + r\rho^{\frac{1}{n}}\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2}\right] &= -\frac{4\pi G}{K}r\rho^2\\
\frac{2}{r}\rho^{\gamma-1}\frac{\mathrm{d}\rho}{\mathrm{d}r} + (\gamma-2)\,\rho^{\gamma-2}\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)^2 + \rho^{\gamma-1}\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2} &= -\frac{4\pi G}{K\gamma}\rho^2\\
\rho^{\gamma-1}\frac{\mathrm{d}^2\rho}{\mathrm{d}r^2} + \left[\frac{2}{r}\rho^{\gamma-1} + (\gamma-2)\,\rho^{\gamma-2}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right]\frac{\mathrm{d}\rho}{\mathrm{d}r} &= -\frac{4\pi G}{K\gamma}\rho^2.\end{aligned}$$

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Problem 4
Imagine you have a star in hydrostatic equilibrium, with mass M and radius R.

(a) Estimate the average density and pressure inside the star.

(b) Now imagine the radius of the star is gently stretched out by a factor α:

$$R \to \alpha R,$$

but the mass is kept fixed. What is the new density and pressure?

(c) Define P_g to be the pressure necessary to maintain hydrostatic equilibrium. It is important to understand that this number should change differently from P as the star is stretched by α. Calculate the new value of P_g after stretching by α.

(d) The ratio P/P_g tells us what direction the star will move after this change; if $P/P_g > 1$, pressure is larger than necessary for equilibrium and the star will expand. Likewise, if $P/P_g < 1$ the star will want to contract.

Use this ratio to determine whether the star is stable to being stretched or compressed. How is stability conditional on the value of γ?

Solution. Assuming we are in hydrostatic equilibrium, then

$$\frac{1}{\rho}\vec{\nabla}P = \vec{g},$$

which is equivalent to

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2}$$

in polar coordinates, where $M(r) = \int_0^R 4\pi(r')^2\rho\,\mathrm{d}r'$, or equivalently,

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi R^2\rho.$$

(a) The average density $\bar{\rho}$ inside the star is

$$\bar{\rho} = \frac{M}{V} = \frac{3M}{4\pi R^3}$$
$$\implies \bar{\rho} \sim \frac{M}{R^3}.$$

The average pressure $\bar{P}$ inside the star is

$$\frac{1}{\bar{\rho}}\nabla\bar{P} = G\frac{M}{R^2},$$

$$\bar{P} = G\frac{M\bar{\rho}}{R} = G\frac{3M^2}{4\pi R^4}$$
$$\implies \bar{P} \sim \frac{GM^2}{R^4}.$$

(b) After gently stretching the star and undergoing the transformation $R \to \alpha R$, assuming α is dimensionless, then the new density ρ_α is

$$\rho_\alpha = \frac{M}{V_\alpha} = \frac{\bar{\rho}}{\alpha^3} = \frac{3M}{4\pi(\alpha R)^3},$$

and the new pressure P_α is given by the adiabatic relation $PV^\gamma = \text{constant}$, so that

$$\begin{aligned} P_\alpha V_\alpha^\gamma &= \bar{P} V^\gamma \\ P_\alpha \left(\frac{4}{3}\pi(\alpha R)^3\right)^\gamma &= \bar{P}\left(\frac{4}{3}\pi R^3\right)^\gamma \\ P_\alpha \alpha^{3\gamma} &= \bar{P} \\ P_\alpha &= \frac{\bar{P}}{\alpha^{3\gamma}} = \frac{GM^2}{\alpha^{3\gamma} R^4} \end{aligned}$$

(c) To remain in hydrostatic equilibrium after stretching by α, the pressure P_g must be given by

$$\frac{1}{\rho_\alpha}\vec{\nabla} P_g = \frac{GM}{(\alpha R)^2}$$

$$P_g = \frac{GM\rho_\alpha}{\alpha R} = \frac{GM^2}{(\alpha R)^4}.$$

(d) Taking the ratio of pressures, we have

$$\frac{P_\alpha}{P_g} = \frac{\alpha^4}{\alpha^{3\gamma}} = \alpha^{4-3\gamma}.$$

- If $\frac{P_\alpha}{P_g} > 1$, then $\alpha^{4-3\gamma} > 1 \implies 4 - 3\gamma > 0 \implies \gamma < \frac{4}{3}$. In this case the star will undergo expansion until it reaches hydrostatic equilibrium, if possible.
- If $\frac{P_\alpha}{P_g} < 1$, then $\alpha^{4-3\gamma} < 1 \implies 4 - 3\gamma < 0 \implies \gamma > \frac{4}{3}$. In this case the star will undergo compression until it reaches hydrostatic equilibrium, if possible.

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