PHYS 603 - Methods of Theoretical Physics III

Lie Algebras in Particle Physics by *H. Georgi*
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Homework 5

Problem 6.B
Suppose that the raising lowering operators of some Lie algebra satisfy

$$[E_\alpha, E_\beta] = N E_{\alpha+\beta}$$

for some nonzero N. Calculate

$$[E_\alpha, E_{-\alpha-\beta}]\,.$$

Please assume that the operators in question are normalized by the condition

$$\mathrm{Tr}\left(E_\alpha E_{-\alpha}\right) = \lambda > 0,$$

where the trace is in the adjoint, and λ is the same for all the root vectors α.

Solution. Given that $[E_\alpha, E_\beta] = N E_{\alpha+\beta}$, we want to calculate $[E_\alpha, E_{-\alpha-\beta}]$. Observe that $\alpha+(-\alpha-\beta) = -\beta$. We have

$$\begin{aligned}
[E_\alpha, E_{-\alpha-\beta}] &= E_\alpha E_{-\alpha-\beta} - E_{-\alpha-\beta} E_\alpha \\
&= -N E_{\alpha+(-\alpha-\beta)} \\
&= -N E_{-\beta}.
\end{aligned}$$

We can verify this using the trace invariance property

$$\begin{aligned}
\mathrm{Tr}([E_\alpha, E_\beta] E_{-\alpha-\beta}) &= N\lambda \\
\mathrm{Tr}(E_\alpha [E_\beta, E_{-\alpha-\beta}]) &= -N\lambda
\end{aligned}$$

Therefore, $[E_\alpha, E_{-\alpha-\beta}] = -N E_{-\beta}$. ■

Problem 7.B
Show that T_1, T_2 and T_3 generate an $SU(2)$ subalgebra of $SU(3)$. Every representation of $SU(3)$ must also be a representation of the subalgebra. However, the irreducible representations of $SU(3)$ are not necessarily irreducible under the subalgebra. How does the the representation generated by the Gell-Mann matrices transform under this subalgebra. That is, reduce, if necessary, the three dimensional representation into representations which are irreducible under the subalgebra and state which irreducible representations appear in the reduction. Then answer the same question for the adjoint representation of $SU(3)$.
Note that the problem has two parts: decomposition of the 3-dimensional representation and of the adjoint. Note also that in this problem T_a denote the matrices of the defining, rather than adjoint, representation of SU(3), see Eq. (7.6) of the textbook.
A good way to approach this problem is through the highest weight (HW) construction. We know how the generator T_3 of the SU(2) subalgebra acts in both of the SU(3) irreps in question. So, we can find in each case the eigenstate(s) of T_3 with the largest eigenvalue and, starting from those as HW states, the entire corresponding irreps of the SU(2). Then, we can similarly consider the action of T_3 on the remaining states.

Solution. The generators T_1, T_2, T_3 satisfy the commutation relations

$$[T_i, T_j] = i\epsilon_{ijk} T_k \quad (i, j, k = 1, 2, 3),$$

with $T_i = \frac{1}{2}\sigma_i$ containing the Pauli matrices, forming an $SU(2)$ subalgebra of $SU(3)$.
Consider the 3-dimensional fundamental representation with an arbitrary vector

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.$$

The SU(2) transformation is characterized by

$$U = \mathrm{e}^{\mathrm{i}\theta_i T_i} = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \quad u \in \mathrm{SU}(2).$$

This transformation reveals a crucial decomposition: the first two components (ψ_1, ψ_2) mix, forming a 2-dimensional doublet, while ψ_3 remains invariant as a singlet.
Using the highest weight method, T_3 provides eigenvalues

$$T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Longrightarrow \quad \begin{cases} \psi_1 \to \dfrac{1}{2} \\ \psi_2 \to -\dfrac{1}{2} \\ \psi_3 \to 0 \end{cases}$$

The highest weight state

$$\psi^{hw} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

generates the doublet, with the ψ_3 state forming a singlet. This leads to the representation decomposition

$$3 \to 2 \oplus 1.$$

For the adjoint representation, we analyze the generator combinations. First, define $E_{\pm 1,0} = T^{\pm} = T_1 \pm \mathrm{i}T_2$.
Their commutation with T_3 shows

$$\left[T_3, T^{\pm}\right] = \pm T^{\pm}.$$

These form a triplet with eigenvalues ± 1. For other generators, define $E_{\pm\frac{1}{2}} = T_4 \pm \mathrm{i}T_5$ and $E_{\pm\frac{1}{2}} = T_6 \pm \mathrm{i}T_7$.
Their commutators with T_3 reveal

$$\left[T_3, E_{\pm\frac{1}{2}}\right] = \pm\frac{1}{2}E_{\pm\frac{1}{2}}.$$

These generate two doublets. The T_8 generator commutes with T_3, forming a singlet. Thus, the adjoint representation decomposes as

$$8 \to 3 \oplus 2 \oplus 2 \oplus 1.$$

■