PHYS 601 - Methods of Theoretical Physics II

Mathematical Methods for Physicists by *Arfken, Weber, Harris*

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Homework 2

Problem 1

Fermat's Principle If the velocity u of light is given by the continuous function $u = u(y)$, the actual light path connecting (x_1, y_1) and (x_2, y_2) in a plane is the one which extremizes the time integral

$$I = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\mathrm{d}s}{u(y)}$$

(a) Derive Snell's Law from Fermat's Principle: Prove that $\frac{\sin(\phi)}{u}$ is a constant, where ϕ is the angle between the tangent of the light path and a vertical line at that point.

(b) Suppose that light travels in the xy-plane in such a way that its speed is proportional to y. Prove that the light rays emitted from any point are circles with their centers on the x-axis.

Proof. (a) The time integral we have to extremize is given by

$$T = \int_C \mathrm{d}t = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\mathrm{d}s}{u(y)} = \int F(x, y, y')\,\mathrm{d}s.$$

We have that

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \implies \mathrm{d}s = \mathrm{d}y\sqrt{1 + (x')^2}.$$

The velocity of light $u(y)$ in the given medium is given by

$$u(y) = \frac{c}{n(y)},$$

where $n(y)$ is the index of refraction of the medium as a function of y. Replacing in our integral, we have

$$T = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{\sqrt{1 + (x')^2}}{\frac{c}{n(y)}}\,\mathrm{d}y = \int_{(x_1,y_1)}^{(x_2,y_2)} \frac{n(y)}{c}\sqrt{1 + (x')^2}\,\mathrm{d}y$$

$$\implies F(x, y, y') = \frac{n(y)}{c}\sqrt{1 + (x')^2}.$$

Plugging in F into the Euler-Lagrange equations, we get

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial F}{\partial x'}\right) = 0$$

$$0 - \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{n(y)x'}{c\sqrt{1 + (x')^2}}\right) = 0$$

$$\frac{n(y)x'}{c\sqrt{1 + (x')^2}} = \text{constant} \implies \frac{\sin(\phi)}{u(y)} = \text{constant}.$$

(b) If $u(y) \propto y \implies n(y) \propto y^{-1} \implies n(y) = ay^{-1}$. We have that

$$\begin{aligned}
\frac{x'}{\sqrt{1+(x')^2}}\frac{a}{cy} &= \text{constant} \\
\frac{\mathrm{d}x}{\mathrm{d}y} &= \frac{cy}{c_1}\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \\
\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 &= \left(\frac{cy}{c_1}\right)^2\left(1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2\right) \\
\left(\frac{c_1}{cy}\right)^2 &= \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 1 \\
\left(\frac{c_1}{cy}\right)^2 - 1 &= \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sqrt{\left(\frac{c_1}{cy}\right)^2 - 1} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\sqrt{c_1^2-(cy)^2}}{cy} \\
\mathrm{d}x &= \frac{cy}{\sqrt{c_1^2-(cy)^2}}\,\mathrm{d}y \\
x(y) &= -\frac{\sqrt{c_1^2-(cy)^2}}{c} + x_0 \\
c(x-x_0) &= -\sqrt{c_1^2-(cy)^2} \\
c^2(x-x_0)^2 + (cy)^2 &= c_1^2 \\
(x-x_0)^2 + y^2 &= \left(\frac{c_1}{c}\right)^2,
\end{aligned}$$

which is a family of circles, centered at $(x_0, 0) \in x$-axis, with radius $\frac{c_1}{c}$.

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Problem 2
The Lagrangian density $\tilde{L}$ which generates a given set of Euler-Lagrange equations is not unique. Prove this result by showing that adding a divergence to $\tilde{L}$ does not alter the Euler-Lagrange equations. Specifically, let

$$\tilde{L} = \tilde{L}\left(x_k, w_j, \frac{\partial w_j}{\partial x_k}\right); \quad \tilde{L}' = \tilde{L} + \sum_k \frac{\partial f_k}{\partial x_k} \tag{1}$$

where $f_k = f_k(w_j)$, and $j = 1, \ldots, m$, $k = 1, \ldots, n$. Then show that $\tilde{L}'$ and $\tilde{L}$ lead to the same Euler-Lagrange equations.

Proof. Let $\tilde{L}' = \tilde{L} + \sum_k \frac{\partial f_k}{\partial x_k}$. Using the chain rule, we have

$$\sum_k \frac{\partial f_k}{\partial x_k} = \sum_{i,k} \frac{\partial f_k}{\partial \phi_i}\frac{\partial \phi_i}{\partial x_k}.$$

The Euler-Lagrange equation of this system states

$$\frac{\partial \tilde{L}'}{\partial \phi_j} - \sum_k \frac{\partial}{\partial x_k}\frac{\partial \tilde{L}'}{\partial\left(\frac{\partial \phi_j}{\partial x_k}\right)} = 0.$$

Computing each term separately, we have

- First term:

$$\begin{aligned}
\frac{\partial \tilde{L}'}{\partial \phi_j} &= \frac{\partial \tilde{L}}{\partial \phi_j} + \frac{\partial}{\partial \phi_j} \sum_k \frac{\partial f_k}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \phi_j} + \frac{\partial}{\partial \phi_j} \sum_{i,k} \frac{\partial f_k}{\partial \phi_i} \frac{\partial \phi_i}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \phi_j} + \sum_{i,k} \frac{\partial^2 f_k}{\partial \phi_j \partial \phi_i} \frac{\partial \phi_i}{\partial x_k}.
\end{aligned}$$

- Second term:

$$\begin{aligned}
\frac{\partial \tilde{L}'}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} &= \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \frac{\partial}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} \sum_k \frac{\partial f_k}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \frac{\partial}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} \sum_{i,k} \frac{\partial f_k}{\partial \phi_i} \frac{\partial \phi_i}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \sum_{i,k} (0) \left(\frac{\partial \phi_i}{\partial x_k}\right) + \sum_{i,k} \frac{\partial f_k}{\partial \phi_i} \delta_{ij} \delta_{kl} \\
&= \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \frac{\partial f_l}{\partial \phi_j}.
\end{aligned}$$

- Derivative of second term:

$$\begin{aligned}
\sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}'}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} &= \sum_l \frac{\partial}{\partial x_k} \left[\frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \frac{\partial f_l}{\partial \phi_j} \right] \\
&= \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \sum_l \frac{\partial}{\partial x_k} \frac{\partial f_l}{\partial \phi_j} \\
&= \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \sum_{i,l} \frac{\partial}{\partial \phi_i} \frac{\partial \phi_i}{\partial x_k} \frac{\partial f_l}{\partial \phi_j} \\
&= \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} + \sum_{i,l} \frac{\partial^2 f_l}{\partial \phi_i \partial \phi_j} \frac{\partial \phi_i}{\partial x_k}.
\end{aligned}$$

Replacing in the Euler-Lagrange equation, we get

$$\begin{aligned}
\frac{\partial \tilde{L}'}{\partial \phi_j} - \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}'}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} &= \frac{\partial \tilde{L}}{\partial \phi_j} + \sum_{i,k} \frac{\partial^2 f_k}{\partial \phi_j \partial \phi_i} \frac{\partial \phi_i}{\partial x_k} - \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)} - \sum_{i,l} \frac{\partial^2 f_l}{\partial \phi_i \partial \phi_j} \frac{\partial \phi_i}{\partial x_k} \\
&= \frac{\partial \tilde{L}}{\partial \phi_j} - \sum_l \frac{\partial}{\partial x_k} \frac{\partial \tilde{L}}{\partial \left(\frac{\partial \phi_j}{\partial x_k}\right)}.
\end{aligned}$$

Thus, the Euler-Lagrange equations are the same and invariant under the addition of a divergence. ■

Problem 3
Show that, if $\psi(x)$ and $\bar{\psi}(x)$ are taken as two independent functions, the Lagrangian density ($\bar{\psi} = \psi^*$ and $\dot{\psi} = \partial\psi/\partial t$)

$$\tilde{L} = \frac{\hbar^2}{2m}\nabla\psi\nabla\bar{\psi} + V\psi\bar{\psi} - \frac{i\hbar}{2}(\bar{\psi}\dot{\psi} - \psi\dot{\bar{\psi}})$$

leads to the time-independent Schrodinger equation

$$H\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

and the complex conjugate of this equation.

Proof. We have that $\bar{\psi} = \psi^*$ and $\dot{\psi} = \partial\psi/\partial t$. The Euler-Lagrange equations for ψ and ψ^* are

$$\frac{\partial L}{\partial \psi} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi)}\right) = 0,$$

$$\frac{\partial L}{\partial \psi^*} - \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\psi^*)}\right) = 0,$$

where $\mu = (0,1,2,3) = (t,x,y,z)$.

We first find all the partial derivatives

- First:
$$\frac{\partial L}{\partial \psi^*} = \frac{\partial\left(V\psi\psi^* - \frac{i\hbar}{2}\psi^*\dot{\psi}\right)}{\partial\psi^*} = V\psi - \frac{i\hbar}{2}\dot{\psi}.$$

- Second:
$$\frac{\partial L}{\partial(\partial_0\psi^*)} = -\frac{i\hbar}{2}\frac{\partial(-\psi\dot{\psi}^*)}{\partial(\partial_0\psi^*)} = \frac{i\hbar}{2}\psi.$$

- Third:
$$\frac{\partial L}{\partial(\partial_i\psi^*)} = \frac{\hbar^2}{2m}\frac{\partial(\nabla\psi\nabla\psi^*)}{\partial(\partial_i\psi^*)} = \frac{\hbar^2}{2m}\nabla\psi e_i,$$

  where $e_i = \{e_1, e_2, e_3\}$.

Replacing in the Euler-Lagrange equations, we get

$$\frac{\partial L}{\partial \psi^*} - \partial_0\left(\frac{\partial L}{\partial(\partial_0\psi^*)}\right) - \partial_i\left(\frac{\partial L}{\partial(\partial_i\psi^*)}\right) = 0$$

$$V\psi - \frac{i\hbar}{2}\frac{\partial\psi}{\partial t} - \frac{i\hbar}{2}\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\nabla^2\psi = 0$$

$$\implies -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

$$\implies \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi,$$

as needed. ■