PHYS 601 - Methods of Theoretical Physics II

Mathematical Methods for Physicists by *Arfken, Weber, Harris*

Student: **Ralph Razzouk**

Homework 3

Problem 1

For each differential equation below, find all the singularities (including those at infinity) and state whether each is regular or irregular.

NAME	EXPRESSION
Hypergeometric	$x(x-1)y'' + [(1+a+b)x - c]\, y' + aby = 0$
Legendre	$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$
Chebyshev	$(1-x^2)y'' - xy' + n^2 y = 0$
Confluent Hypergeometric	$xy'' + (c-x)y' - ay = 0$
Laguerre	$xy'' + (1-x)y' + ay = 0$
Bessel	$x^2y'' + xy' + (x^2 - n^2)y = 0$
Simple Harmonic Oscillator	$y'' + \omega^2 y = 0$
Hermite	$y'' - 2xy' + 2\alpha y = 0$

Solution. Consider the general form of a second-order ordinary differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

An ordinary differential equation is said to have singularities when the highest order term has zeroes or when the lower order terms have poles.

- A point x_0 is said to be an **ordinary point** if $P(x)$ and $Q(x)$ are analytic at $x = x_0$.
- A point x_0 is said to be a **singular point** if $P(x)$ and $Q(x)$ are not analytic at $x = x_0$.
  - A point x_0 is said to be a **regular singular point** if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at x_0.
  - Otherwise, x_0 is said to be an **irregular singular point**.

There remains one region of interest, which is as $x \to \infty$. To study the ODE at infinity, we make a variable change of $z = \frac{1}{x}$. Now, we study what happens at $z = 0$. Under such a transformation, the transformed forms of $P(x)$ and $Q(x)$ will have to be studied using the analysis above. Accordingly, we have

$$\tilde{P}(z) = \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \quad \text{and} \quad \tilde{Q}(z) = \frac{Q\left(\frac{1}{z}\right)}{z^4}$$

- **Hypergeometric:**

$$x(x-1)y'' + [(1+a+b)x - c]\, y' + aby = 0$$

The above ODE reduces to

$$y'' + \frac{(1+a+b)x - c}{x(x-1)}y' + \frac{ab}{x(x-1)}y = 0,$$

where

$$P(x) = \frac{(1+a+b)x - c}{x(x-1)} \qquad \text{and} \qquad Q(x) = \frac{ab}{x(x-1)}.$$

The hypergeometric ODE has interesting points at $x_0 = 0, 1, \infty$.

For $x_0 = 0$**:**

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0}\\ &= \left.\frac{(1+a+b)x-c}{x-1}\right|_{x=0}\\ &= c \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= x^2Q(x)\big|_{x=0}\\ &= \left.\frac{abx}{x-1}\right|_{x=0}\\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 = 1$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1}\\ &= \left.\frac{(1+a+b)x-c}{x}\right|_{x=1}\\ &= 1+a+b-c \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= (x-1)^2Q(x)\big|_{x=1}\\ &= \left.\frac{ab(x-1)}{x}\right|_{x=1}\\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2}\\ &= \frac{2}{z} - \frac{\frac{(1+a+b)\frac{1}{z}-c}{\frac{1}{z}\left(\frac{1}{z}-1\right)}}{z^2}\\ &= \frac{2}{z} - \frac{1+a+b-cz}{z\left(1-z\right)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4}\\ &= \frac{\frac{ab}{\frac{1}{z}\left(\frac{1}{z}-1\right)}}{z^4}\\ &= \frac{ab}{z^2\left(1-z\right)}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - \frac{1 + a + b - cz}{1 - z}\Bigg|_{z=0} \\ &= 1 - a - b \sim \text{finite.}\end{aligned}$$

- The quantity $(z - z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z - z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{ab}{(1 - z)}\Bigg|_{z=0} \\ &= ab \sim \text{finite.}\end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Legendre:**

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0$$

The above ODE reduces to

$$y'' - \frac{2x}{1 - x^2}y' + \frac{\ell(\ell + 1)}{1 - x^2}y = 0,$$

where

$$P(x) = -\frac{2x}{1 - x^2} = -\frac{2x}{(1 - x)(1 + x)} \qquad \text{and} \qquad Q(x) = \frac{\ell(\ell + 1)}{1 - x^2} = \frac{\ell(\ell + 1)}{(1 - x)(1 + x)}.$$

The Legendre ODE has interesting points at $x_0 = \pm 1, \infty$.

For $x_0 = -1$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)|_{x_0} &= (x + 1)P(x)|_{x=-1} \\ &= -\frac{2x}{1 - x}\Bigg|_{x=-1} \\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2Q(x)\Big|_{x_0} &= (x + 1)^2Q(x)\Big|_{x=-1} \\ &= \frac{\ell(\ell + 1)(x + 1)}{1 - x}\Bigg|_{x=-1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = -1$ is a regular singular point.

For $x_0 = 1$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1} \\ &= -\frac{2x}{1+x}\bigg|_{x=1} \\ &= -1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2 Q(x)\big|_{x_0} &= (x-1)^2 Q(x)\big|_{x=1} \\ &= \frac{\ell(\ell+1)(x-1)}{1+x}\bigg|_{x=1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-\frac{2\left(\frac{1}{z}\right)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^2} \\ &= \frac{2}{z} + \frac{2}{z(z-1)(z+1)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{\ell(\ell+1)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^4} \\ &= \frac{\ell(\ell+1)}{z^2(z-1)(z+1)}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{2}{(z-1)(z+1)}\bigg|_{z=0} \\ &0 \sim \text{finite.}\end{aligned}$$

- The quantity $(z - z_0)^2 \tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z - z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{\ell(\ell+1)}{(z-1)(z+1)}\bigg|_{z=0} \\ &= -\ell(\ell+1) \sim \text{finite.}\end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Chebyshev:**
$$(1-x^2)y'' - xy' + n^2 y = 0$$

  The above ODE reduces to
$$y'' - \frac{x}{1-x^2}y' + \frac{n^2}{1-x^2}y = 0,$$
  where
$$P(x) = -\frac{x}{1-x^2} = -\frac{x}{(1-x)(1+x)} \qquad \text{and} \qquad Q(x) = \frac{n^2}{1-x^2} = \frac{n^2}{(1-x)(1+x)}.$$

  The Chebyshev ODE has interesting points at $x_0 = \pm 1, \infty$.

  For $x_0 = -1$:

  - The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
  - The quantity $(x - x_0)P(x)$, evaluated at x_0, is
$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= (x+1)P(x)\big|_{x=-1} \\ &= -\frac{x}{1-x}\bigg|_{x=-1} \\ &= \frac{1}{2} \sim \text{finite.}\end{aligned}$$
  - The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is
$$\begin{aligned}(x-x_0)^2 Q(x)\big|_{x_0} &= (x+1)^2 Q(x)\big|_{x=-1} \\ &= \frac{(x+1)n^2}{1-x}\bigg|_{x=-1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

  Thus, $x_0 = -1$ is a regular singular point.

  For $x_0 = 1$:

  - The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.
  - The quantity $(x - x_0)P(x)$, evaluated at x_0, is
$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1} \\ &= -\frac{x}{1+x}\bigg|_{x=1} \\ &= -\frac{1}{2} \sim \text{finite.}\end{aligned}$$
  - The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is
$$\begin{aligned}(x-x_0)^2 Q(x)\big|_{x_0} &= (x-1)^2 Q(x)\big|_{x=1} \\ &= \frac{(x-1)n^2}{1+x}\bigg|_{x=1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

  Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-\frac{\left(\frac{1}{z}\right)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^2} \\ &= \frac{2}{z} + \frac{1}{z(z-1)(z+1)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{n^2}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^4} \\ &= \frac{n^2}{z^2(z-1)(z+1)}.\end{aligned}$$

– The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
– The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{1}{(z-1)(z+1)}\Bigg|_{z=0} \\ &= 1 \sim \text{finite}.\end{aligned}$$

– The quantity $(z - z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{n^2}{(z-1)(z+1)}\Bigg|_{z=0} \\ &= -n^2 \sim \text{finite}.\end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Confluent Hypergeometric:**

$$xy'' + (c - x)y' - ay = 0$$

The above ODE reduces to

$$y'' + \frac{c-x}{x}y' - \frac{a}{x}y = 0,$$

where

$$P(x) = \frac{c-x}{x} \qquad \text{and} \qquad Q(x) = -\frac{a}{x}.$$

The Confluent Hypergeometric ODE has interesting points at $x_0 = 0, \infty$.
For $x_0 = 0$:

– The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.

– The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)|_{x_0} &= xP(x)|_{x=0} \\ &= c - x|_{x=0} \\ &= c \sim \text{finite.}\end{aligned}$$

– The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2Q(x)\big|_{x_0} &= x^2Q(x)\big|_{x=0} \\ &= -ax|_{x=0} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{\frac{c-\frac{1}{z}}{\frac{1}{z}}}{z^2} \\ &= \frac{2}{z} - \frac{cz-1}{z^2},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{-\frac{a}{\frac{1}{z}}}{z^4} \\ &= -\frac{a}{z^3}.\end{aligned}$$

– The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
– The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - \frac{cz-1}{z}\Bigg|_{z=0} \\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

- **Laguerre:**

$$xy'' + (1 - x)y' + ay = 0$$

The above ODE reduces to

$$y'' + \frac{1-x}{x}y' + \frac{a}{x}y = 0,$$

where

$$P(x) = \frac{1-x}{x} \qquad \text{and} \qquad Q(x) = \frac{a}{x}.$$

The Laguerre ODE has interesting points at $x_0 = 0, \infty$.
For $x_0 = 0$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0} \\ &= 1 - x\big|_{x=0} \\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2 Q(x)\big|_{x_0} &= x^2 Q(x)\big|_{x=0} \\ &= ax\big|_{x=0} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{\frac{1-\frac{1}{z}}{\frac{1}{z}}}{z^2} \\ &= \frac{2}{z} - \frac{z-1}{z^2},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{a}{\frac{1}{z}}}{z^4} \\ &= \frac{a}{z^3}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - \frac{z-1}{z}\bigg|_{z=0} \\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

- **Bessel:**

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

The above ODE reduces to

$$y'' + \frac{1}{x}y' + \frac{x^2 - n^2}{x^2}y = 0,$$

where
$$P(x) = \frac{1}{x} \qquad \text{and} \qquad Q(x) = \frac{x^2 - n^2}{x^2}.$$

The Bessel ODE has interesting points at $x_0 = 0, \infty$.

For $x_0 = 0$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is
$$\begin{aligned}(x - x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0} \\ &= 1\big|_{x=0} \\ &= 1 \sim \text{finite.}\end{aligned}$$
- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is
$$\begin{aligned}(x - x_0)^2 Q(x)\big|_{x_0} &= x^2 Q(x)\big|_{x=0} \\ &= x^2 - n^2\big|_{x=0} \\ &= -n^2 \sim \text{finite.}\end{aligned}$$

  Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have
$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{z}{z^2} \\ &= \frac{2}{z} - \frac{1}{z},\end{aligned}$$

and
$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{\left(\frac{1}{z}\right)^2 - zn^2}{\left(\frac{1}{z}\right)^2}}{z^4} \\ &= \frac{1 - n^2 z^2}{z^4}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is
$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - 1\big|_{z=0} \\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(z-z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} = z^2\tilde{Q}(z)\Big|_{z=0} = \frac{1-n^2z^2}{z^2}\Big|_{z=0} \to \infty.$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

Simple Harmonic Oscillator:

$$y'' + \omega^2 y = 0$$

The above ODE is already in a reduced for where

$$P(x) = 0 \qquad \text{and} \qquad Q(x) = \omega^2.$$

The simple harmonic oscillator ODE has interesting points at $x_0 = \infty$.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\tilde{P}(z) = \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} = \frac{2}{z},$$

and

$$\tilde{Q}(z) = \frac{Q\left(\frac{1}{z}\right)}{z^4} = \frac{\omega^2}{z^4}.$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z-z_0)\tilde{P}(z)$, evaluated at z_0, is

$$(z-z_0)\tilde{P}(z)\Big|_{z_0} = z\tilde{P}(z)\Big|_{z=0} = 2|_{z=0} = 2 \sim \text{finite}.$$

- The quantity $(z-z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} = z^2\tilde{Q}(z)\Big|_{z=0} = \frac{\omega^2}{z^2}\Big|_{z=0} \to \infty.$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

Hermite:

$$y'' - 2xy' + 2\alpha y = 0$$

The above ODE is already in a reduced for where

$$P(x) = -2x \qquad \text{and} \qquad Q(x) = 2\alpha.$$

The Hermite ODE has interesting points at $x_0 = \infty$.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-2\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} + \frac{2}{z^3},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{2\alpha}{z^4}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{2}{z^2}\Big|_{z=0} \\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

To summarize, we have

NAME	SINGULARITIES
Hypergeometric	Regular at $x = 0, 1, \infty$
Legendre	Regular at $x = \pm 1, \infty$
Chebyshev	Regular at $x = \pm 1, \infty$
Confluent Hypergeometric	Regular at $x = 0$ and irregular at $x = \infty$
Laguerre	Regular at $x = 0$ and irregular at $x = \infty$
Bessel	Regular at $x = 0$ and irregular at $x = \infty$
Simple Harmonic Oscillator	Irregular at $x = \infty$
Hermite	Irregular at $x = \infty$

■

Problem 2
For some of the above equations, $q(x) = 0$ when expressed in Sturm-Liouiville form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right]0 - \left[q(x) - \lambda w(x)\right]y = 0.$$

When $\lambda = 0$ also, the Sturm-Liouiville equation has a solution $y(x)$ determined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p(x)},$$

(a) Show this.

(b) Use this result to produce a second solution [in addition to those given on the sheet distributed in class] to the Legendre, Laguerre, and Hermite equations.

Solution. (a) Given the Sturm-Liouiville form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right]0 - \left[q(x) - \lambda w(x)\right]y = 0,$$

where $q(x) = \lambda = 0$, then we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right] = 0 \quad &\implies \quad p(x)y' = \text{constant}\\ &\implies \quad y' = \frac{\text{constant}}{p(x)}\\ &\qquad\quad\;\; = \frac{1}{p(x)},\end{aligned}$$

where the last step was done by absorbing the constant into $p(x)$.

(b)

- **Legendre:** The Legendre ODE is given by

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0,$$

which can be rewritten in the Sturm-Liouiville form as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)y'\right] + \ell(\ell + 1)y = 0,$$

where $p(x) = (1 - x^2)$, $q(x) = 0$, and $\lambda = \ell(\ell + 1)$.
By setting $\ell = 0$ and using the property in part (a), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p(x)} = \frac{1}{1 - x^2}$$

$$y = \int \frac{1}{1 - x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) + C,$$

which is the second solution of the Legendre ODE.

- **Laguerre:** The Laguerre ODE is given by

$$xy'' + (1 - x)y' + ay = 0.$$

To rewrite this in the Sturm-Liouiville form, we need to use an integrating factor $I(x)$, which would have initially been in the equation but was cancelled for its non-zero value. Let

$$xIy'' + (1 - x)Iy' + aIy = 0$$

be our new ODE. We require that $\frac{\mathrm{d}}{\mathrm{d}x}(xI) = (1-x)I$, and so we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(xI) = I + x\frac{\mathrm{d}I}{\mathrm{d}x} &= (1-x)I \\ \implies \frac{\mathrm{d}I}{\mathrm{d}x} &= -I \\ \implies I &= e^{-x}.\end{aligned}$$

Thus, our Laguerre ODE becomes

$$xe^{-x}y'' + (1-x)e^{-x}y' + ae^{-x}y = 0.$$

Rewriting in the Sturm-Liouiville form, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{-x}y'\right] + ae^{-x}y = 0,$$

where $p(x) = xe^{-x}$, $q(x) = 0$, and ae^{-x}.
By setting $ae^{-x} = 0 \implies a = 0$, since $e^{-x} \neq 0$, and using the property in part (a), we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{p(x)} = \frac{1}{xe^{-x}} \\ y &= \int \frac{1}{xe^{-x}}\,\mathrm{d}x = Ei(x) + C,\end{aligned}$$

which is the second solution of the Laguerre ODE, where $Ei(x)$ is defined to be the exponential integral.

- **Hermite:** The Hermite ODE is given by

$$y'' - 2xy' + 2\alpha y = 0.$$

To rewrite this in the Sturm-Liouiville form, we need to use an integrating factor $I(x)$, which would have initially been in the equation but was cancelled for its non-zero value. Let

$$Iy'' - 2xIy' + 2\alpha Iy = 0$$

be our new ODE. We require that $\frac{\mathrm{d}}{\mathrm{d}x}(I) = -2xI$, and so we have

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}x} &= -2xI \\ \implies I &= e^{-x^2}.\end{aligned}$$

Thus, our Hermite ODE becomes

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2\alpha e^{-x^2}y = 0$$

Rewriting in the Sturm-Liouiville form, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x^2}y'\right] + 2\alpha e^{-x^2}y = 0,$$

where $p(x) = e^{-x^2}$, $q(x) = 0$, and $2\alpha e^{-x^2}$.
By setting $2\alpha e^{-x^2} = 0 \implies \alpha = 0$, since $e^{-x^2} \neq 0$, and using the property in part (a), we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{p(x)} = \frac{1}{e^{-x^2}} \\ y &= \int e^{x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2} + C,\end{aligned}$$

which is the second solution of the Hermite ODE, where $\mathrm{erfi}(x)$ is defined to be the imaginary error function.

■