

Homework 8 – Fermion Week

Due Thursday, March 28th

$$n(\mu) = 2 \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{1 + e^{\beta(\epsilon_p - \mu)}} \quad (1)$$

$$P(\mu) = \frac{2}{\beta} \int \frac{d^3p}{(2\pi\hbar)^3} \ln(1 + e^{-\beta(\epsilon_p - \mu)}) \quad (2)$$

Problem 1

a) Take the above two integrals derived in class for $P(\mu)$ and $n(\mu)$ and compute them both (as functions of μ) in the limit of low temperature (if you can't remember how to do this, try plotting the integrands at low temperature for inspiration!). Recall $\epsilon_p = \frac{p^2}{2m_e}$.

b) What is a good estimate for μ for a white dwarf of $1M_\odot$ and radius of 10^4 km? Answer in eV.

c) Plot this parametrically on a plot of $\log P(\mu)$ vs $\log n(\mu)$. Allow μ to take on values in the vicinity of your estimate above; $0 < \mu < 10\mu_0$, where μ_0 is your estimate from part (b).

Problem 2

For non-zero temperature, you can just perform these integrals on the computer! Do so, and generate additional parametric curves of $(n(\mu), P(\mu))$ in the same plot. Be careful, when $T \neq 0$, μ is no longer the Fermi energy; in fact, you might need μ to take on negative values to reach sufficiently low densities. For values of the temperature, try $T = 10^5$ K, 10^6 K, 10^7 K, and maybe a few others. Thus, your plot should have several different curves of (n, P) at different temperatures. Congratulations! You've computed the equation of state of a white dwarf $P(n, T)$. Can you separate this into a “degeneracy pressure” component and a “gas pressure” component?

Problem 3

a) Repeat the simplified argument from class that gave us the estimate for the equation of state of a fermi gas at low temperature:

$$P \sim \frac{\hbar^2}{m_e} n_e^{5/3} \quad (3)$$

b) At high enough densities, the Fermi energy is sufficiently large that $\epsilon_p \neq p^2/2m$ and instead it is given by the relativistic formula ($\epsilon_p = pc$). Estimate $P(\rho)$ in the relativistic regime.

c) Estimate the density n and pressure P at which the equation of state transitions from non-relativistic to relativistic. Estimate the mass of a white dwarf at the transition point. Give your answer in solar masses.

d) (optional, but maybe fun?) Extend your code from Problem 2 to include the full relativistic expression $\epsilon_p = \sqrt{p^2 c^2 + m_e^2 c^4} - m_e c^2$, to get curves $P(\rho)$ that transition from the non-relativistic to the relativistic regime, also at non-zero temperature. If you want you can compare your result to some industry standard EOS codes used in astrophysics: https://cococubed.com/code_pages/eos.shtml