PHYSICS 601

Homework Assignment 3

1. For each differential equation below, find all the singularities (including those at infinity) and state whether each is regular or irregular.

NAME	EXPRESSION
Hypergeometric	$x(x-1)y'' + [(1+a+b)x - c]y' + aby = 0$
Legendre	$(1-x^2)y'' - 2xy' + l(l+1)y = 0$
Chebyshev	$(1-x^2)y'' - xy' + n^2y = 0$
Confluent Hypergeometric	$xy'' + (c-x)y' - ay = 0$
Laguerre	$xy'' + (1-x)y' + ay = 0$
Bessel	$x^2y'' + xy' + (x^2-n^2)y = 0$
Simple Harmonic Oscillator	$y'' + \omega^2 y = 0$
Hermite	$y'' - 2xy' + 2\alpha y = 0$

2. For some of the above equations, $q(x) = 0$ when expressed in Sturm-Liouiville form:

$$\frac{d}{dx}[p(x)y'] - [q(x) - \lambda w(x)]y = 0 .$$

When $\lambda = 0$ also, the Sturm-Liouiville equation has a solution $y(x)$ determined by

$$\frac{dy}{dx} = \frac{1}{p(x)} .$$

a) Show this.

b) Use this result to produce a second solution [in addition to those given on the sheet distributed in class] to the Legendre, Laguerre, and Hermite equations.

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Homework Assignment 4

1. The first four Legendre polynomials are

$$P_0(x) = 1, \qquad P_2(x) = \frac{1}{2}(3x^2 - 1),$$
$$P_1(x) = x, \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

Obtain these four polynomials by each of the following methods:

a) Generating function,

b) Rodrigues' formula,

c) Schmidt orthogonalization,

d) Series solution.

2. The Hermite differential equation is $H_n'' - 2xH_n' + 2nH_n = 0$.

a) Solve this equation by series solution and show that it terminates for integral values of n.

b) Use the series solution to generate the first four Hermite polynomials which are

$$H_0(x) = 1, \qquad H_2(x) = 4x^2 - 2,$$
$$H_1(x) = 2x, \qquad H_3(x) = 8x^3 - 12x.$$

c) Obtain the first four Hermite polynomials using the generating function which is

$$g(x, t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

d) Using the generating function derive the recurrence relations:

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,$$
$$H_n'(x) - 2nH_{n-1}(x) = 0.$$

e) Using the result of part (d) verify that the $H_n(x)$ defined by the generating function obeys the Hermite differential equation.

3. Use the generating function for the Bessel functions,

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n ,$$

to obtain the following recurrence relations:

a) $J_{n-1} + J_{n+1} = \dfrac{2n}{x} J_n ,$

b) $J_{n-1} - J_{n+1} = 2J_n' ,$

c) $J_{n-1} - \dfrac{n}{x} J_n = J_n' ,$

d) $J_{n+1} - \dfrac{n}{x} J_n = -J_n' .$

e) Using the above results verify that J_n satisfies Bessel's equation,

$$x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n = 0 .$$

f) Verify that the series solution

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

satisfies the same equation.