# PHYS 601 - Methods of Theoretical Physics II

**Mathematical Methods for Physicists** by *Arfken, Weber, Harris*

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## Homework 8

**Problem 1**

Show that the Green's function for the 2-dimensional Laplace over the entire 2-dimensional space is

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi} \ln \left[ (x - x')^2 + (y - y')^2 \right]^{\frac{1}{2}},$$

where, $\mathbf{r} = (x, y)$ and $\mathbf{r}' = (x', y')$.

*Solution.* Consider

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}'),$$

where $\mathbf{r} = (x, y)$ and $\mathbf{r}' = (x', y')$.

The boundary condition of our system is given by $G(r, r') \to 0$ as $r \to \infty$. Making a coordinate transformation by moving our system to the source given by $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Now, the system is circularly symmetric and the Green's function only depends on $R = |\mathbf{r} - \mathbf{r}'|$. Thus, we have

$$\nabla^2 G(R) = -\delta(\mathbf{R}) = -\frac{1}{2\pi R}\delta(R).$$

In polar coordinates, we have

$$\begin{aligned} \frac{\partial^2 G}{\partial R^2} + \frac{1}{R}\frac{\partial G}{\partial R} + \frac{1}{R^2}\frac{\partial^2 G}{\partial \theta^2} &= -\frac{1}{2\pi R}\delta(R) \\ \frac{\mathrm{d}^2 G}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}G}{\mathrm{d}R} &= -\frac{1}{2\pi R}\delta(R). \end{aligned}$$

- **For** $R > 0$**:** We have

$$\frac{\mathrm{d}^2 G}{\mathrm{d}R^2} + \frac{1}{R}\frac{\mathrm{d}G}{\mathrm{d}R} = 0$$

$$\frac{\mathrm{d}^2 G}{\mathrm{d}R^2} = -\frac{1}{R}\frac{\mathrm{d}G}{\mathrm{d}R}$$

  Let $H \equiv \frac{\mathrm{d}G}{\mathrm{d}R}$. Then

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}R} &= -\frac{1}{R}H \\ \frac{\mathrm{d}H}{H} &= -\frac{\mathrm{d}R}{R} \\ \ln(H) &= -\ln(R) + C \\ H &= \frac{A}{R} \\ \frac{\mathrm{d}G}{\mathrm{d}R} &= \frac{A}{R} \\ G &= A\ln(R) + B. \end{aligned}$$

  Applying the boundary condition, we have $G(R) \to 0$ as $R \to \infty$. Thus $B = 0$ and we have

$$G = A\ln(R).$$

We have that $\nabla^2 G(R) = -\frac{1}{2\pi R}\delta(R)$. Integrating twice, we have

$$\begin{aligned}\iint \nabla^2 G(R)\,\mathrm{d}A &= -\frac{1}{2\pi}\iint \frac{1}{R}\delta(R)\,\mathrm{d}A \\ &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^r \frac{1}{R}\delta(R)R\,\mathrm{d}R\,\mathrm{d}\phi \\ &= -1.\end{aligned}$$

Additionally, by using the divergence theorem, we have

$$\begin{aligned}\iint \nabla^2 G(R)\,\mathrm{d}A &= \iint \nabla\cdot(\nabla G(R))\,\mathrm{d}A \\ &= \oint \nabla G(R)\big|_{R=r}\cdot\mathrm{d}\mathbf{s} \\ &= \int_0^{2\pi}\left(\hat{R}\frac{\partial}{\partial R}(A\ln(R))\right)\bigg|_{R=r}\cdot\left(\hat{R}r\,\mathrm{d}\theta\right) \\ &= \int_0^{2\pi}\frac{\partial}{\partial R}(A\ln(R))\bigg|_{R=r} r\,\mathrm{d}\theta \\ &= \int_0^{2\pi}\frac{A}{R}\bigg|_{R=r} r\,\mathrm{d}\theta \\ &= \int_0^{2\pi} A\,\mathrm{d}\theta \\ &= 2\pi A.\end{aligned}$$

Thus, $2\pi A = -1 \implies A = -\frac{1}{2\pi}$, giving us

$$G(R) = -\frac{1}{2\pi}\ln(R).$$

Therefore,

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{2\pi}\ln\left[\sqrt{(x-x')^2+(y-y')^2}\right].$$

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**Problem 2**
Write down the solution to the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = f(x,y)$$

in terms of $G(\mathbf{r},\mathbf{r}')$.

*Solution.* Since $\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}')$, we have

$$\begin{aligned}\nabla^2 u(x,y) &= f(x,y) \\ &= \iint \delta(\mathbf{r}-\mathbf{r}')f(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\ &= \iint -\nabla^2 G(\mathbf{r},\mathbf{r}')f(x',y')\,\mathrm{d}x'\,\mathrm{d}y' \\ &= \nabla^2\left[\iint -G(\mathbf{r},\mathbf{r}')f(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\right].\end{aligned}$$

Thus,

$$u(x,y) = -\iint G(\mathbf{r},\mathbf{r}')f(x',y')\,\mathrm{d}x'\,\mathrm{d}y'.$$

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**Problem 3**
$\psi(x,t)$ satisfies the 1-dimensional Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\psi}{\partial t},$$

with initial condition $\psi(x,0)=\delta(x)$, and boundary condition

$$\frac{\partial\psi}{\partial x}\left(-\frac{L}{2},t\right) = \frac{\partial\psi}{\partial x}\left(\frac{L}{2},t\right) = 0.$$

Show by the method of separation of variables that

$$\psi(x,t) = \frac{1}{L}\sum_{n=-\infty}^{\infty}\cos\left(\frac{2n\pi x}{L}\right)\exp\left[-i\frac{\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t\right].$$

*Solution.* Applying the method of separation of variables, we consider $\psi(x,t)=X(x)T(t)$. Replacing in the Schrodinger equation, we have

$$\begin{aligned}
i\hbar\frac{\partial\psi}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\\
i\hbar X\frac{\partial T}{\partial t} &= -\frac{\hbar^2}{2m}T\frac{\partial^2 X}{\partial x^2}\\
i\hbar\frac{1}{T}\frac{\partial T}{\partial t} &= -\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2 X}{\partial x^2}.
\end{aligned}$$

The left hand side is dependent on $t$ only and the right hand side is dependent of $x$ only. Thus, they must both be equal to some constant $E$. Then

$$\begin{cases} i\hbar\frac{1}{T}\frac{\partial T}{\partial t} = E \\ -\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = E \end{cases} \implies \begin{cases} \frac{\partial T(t)}{\partial t} = -\frac{iE}{\hbar}T \\ \frac{\partial^2 X(x)}{\partial x^2} = -\frac{2mE}{\hbar^2}X \end{cases} \implies \begin{cases} T(t) = c_1\mathrm{e}^{-\frac{iE}{\hbar}t} \\ X(x) = c_2\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}x} + c_3\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}x} \end{cases}$$

Thus,

$$\begin{aligned}
\psi(x,t) &= \left(c_2\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}x} + c_3\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}x}\right)c_1\mathrm{e}^{-\frac{iE}{\hbar}t}\\
&= \left(A\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}x} + B\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}x}\right)\mathrm{e}^{-\frac{iE}{\hbar}t}.
\end{aligned}$$

Applying the boundary conditions, we have

$$
\begin{aligned}
\frac{\partial \psi}{\partial x}\left(-\frac{L}{2}, t\right) &= \frac{\partial \psi}{\partial x}\left(\frac{L}{2}, t\right) = 0 \\
i\frac{\sqrt{2mE}}{\hbar}\left(A\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} - B\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}}\right)\mathrm{e}^{-\frac{iE}{\hbar}t} &= i\frac{\sqrt{2mE}}{\hbar}\left(A\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} - B\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}}\right)\mathrm{e}^{-\frac{iE}{\hbar}t} \\
A\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} - B\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} &= A\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} - B\mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}\frac{L}{2}} \\
A - B\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}L} &= A\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}L} - B \\
\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}L} &= 1 \\
\frac{\sqrt{2mE}}{\hbar}L &= 2n\pi, \quad n \in \mathbb{Z}^+ \\
\lambda_n \equiv \frac{\sqrt{2mE}}{\hbar} &= \frac{2n\pi}{L} \\
E &= \frac{\hbar^2}{2m}\left(\frac{2n\pi}{L}\right)^2,
\end{aligned}
$$

and $A = B$. We take positive values of $n$ only to generate distinct solutions since we have a symmetry across $x = 0$.
Thus,

$$
\begin{aligned}
\psi_n(x,t) &= A\left(\mathrm{e}^{i\frac{\sqrt{2mE}}{\hbar}x} + \mathrm{e}^{-i\frac{\sqrt{2mE}}{\hbar}x}\right)\mathrm{e}^{-\frac{iE}{\hbar}t} \\
&= 2A\cos\left(\frac{\sqrt{2mE}}{\hbar}x\right)\mathrm{e}^{-\frac{iE}{\hbar}t} \\
&= C_n\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i}{\hbar}\frac{\hbar^2}{2m}\left(\frac{2n\pi}{L}\right)^2 t} \\
&= C_n\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t}
\end{aligned}
$$

which gives us

$$
\psi(x,t) = \sum_{n=0}^{\infty} C_n\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t}.
$$

Applying the initial condition, we have

$$
\psi(x,0) = \sum_{n=0}^{\infty} C_n\cos\left(\frac{2n\pi x}{L}\right) = \delta(x).
$$

Then, we have

$$
C_n = \frac{2}{L}\int_{-L/2}^{L/2}\cos\left(\frac{2n\pi x}{L}\right)\delta(x)\,\mathrm{d}x = \frac{2}{L}.
$$

Therefore,

$$
\psi(x,t) = \frac{2}{L}\sum_{n=0}^{\infty}\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t} = \frac{1}{L}\sum_{n=-\infty}^{\infty}\cos\left(\frac{2n\pi x}{L}\right)\mathrm{e}^{-\frac{i\hbar}{2m}\left(\frac{2n\pi}{L}\right)^2 t}.
$$

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