PHYS 601 - Methods of Theoretical Physics II

Mathematical Methods for Physicists by *Arfken, Weber, Harris*
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Homework 9

Problem 1
[Arfken pp. 738-739] Solve the integral equation

$$\phi(x) = x + \frac{1}{2}\int_{-1}^{+1} (t - x)\phi(t)\,\mathrm{d}t$$

for $\phi(x)$ using the Neumann series method.

Solution. Considering a generic Fredholm equation, we have $f(x) = x$, $\lambda = \frac{1}{2}$, $a = -1$, $b = 1$, and $K(x,t) = t - x$.

Solving this by a Neumann series solution, we take

$$\begin{aligned}
\phi_0(x) &\equiv f(x) \\
\phi_1(x) &\equiv f(x) + \lambda \int_a^b K(x,t_1) f(t_1)\,\mathrm{d}t_1 \\
\phi_2(x) &\equiv f(x) + \lambda \int_a^b K(x,t_1) f(t_1)\,\mathrm{d}t_1 + \lambda^2 \int_a^b \int_a^b K(x,t_1) K(t_1,t_2) f(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\vdots \\
\phi_N(x) &\equiv \sum_{n=0}^{N} \lambda^n u_n(x),
\end{aligned}$$

where

$$\begin{aligned}
u_0(x) &= f(x) \\
u_1(x) &= \int_a^b K(x,t_1) f(t_1)\,\mathrm{d}t_1 \\
u_2(x) &= \int_a^b \int_a^b K(x,t_1) K(t_1,t_2) f(t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&\vdots \\
u_n(x) &= \int_a^b \int_a^b \cdots \int_a^b K(x,t_1) K(t_1,t_2) \cdots K(t_{n-1},t_n) f(t_n)\,\mathrm{d}t_n\,\mathrm{d}t_{n-1} \cdots \mathrm{d}t_1.
\end{aligned}$$

The Neumann series solution is then

$$\begin{aligned}
\phi(x) &= \lim_{N\to\infty} \phi_N(x) \\
&= \lim_{N\to\infty} \sum_{n=0}^{N} \lambda^n u_n(x) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} (t_1 - x)(t_2 - t_1)\cdots(t_n - t_{n-1}) f(t_n)\,\mathrm{d}t_n\,\mathrm{d}t_{n-1}\cdots\mathrm{d}t_1 \\
&= x + \frac{1}{2}\int_{-1}^{1}(t_1 - x)t_1\,\mathrm{d}t_1 + \left(\frac{1}{2}\right)^2 \int_{-1}^{1}\int_{-1}^{1}(t_1 - x)(t_2 - t_1)t_2\,\mathrm{d}t_2\,\mathrm{d}t_1 + \cdots \\
&= x + \frac{1}{2}\left[\frac{t_1^3}{3} - \frac{t_1^2}{2}x\right]_{-1}^{1} + \left(\frac{1}{2}\right)^2 \int_{-1}^{1}\left[\frac{t_2^3}{3}t_1 - \frac{t_2^2}{2}t_1^2 - \frac{t_2^3}{3}x + \frac{t_2^2}{2}t_1 x\right]_{-1}^{1}\mathrm{d}t_1 + \cdots \\
&= x + \frac{1}{2}\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^2 \int_{-1}^{1}\left[-t_1^2 - \frac{2}{3}x\right]\mathrm{d}t_1 + \cdots \\
&= x + \frac{1}{2}\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^2 \left[-\frac{t_1^3}{3} - \frac{2}{3}xt_1\right]_{-1}^{1} + \cdots \\
&= x + \frac{1}{2}\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^2 \left(-\frac{4}{3}x\right) + \cdots \\
&= x + \frac{1}{2}\left(\frac{2}{3}\right) + \left(\frac{1}{2}\right)^2 \left(\frac{2}{3}\right)(-2x) + \cdots \\
&= x\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n}\left(\frac{2}{3}\right)^{n}(-2)^n + \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n+1}\left(\frac{2}{3}\right)^{n+1}(-2)^n \\
&= x\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n}\left(-\frac{4}{3}\right)^{n} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n}\left(\frac{2}{3}\right)^{n}(-2)^n \\
&= x\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n}\left(-\frac{4}{3}\right)^{n} + \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n}\left(-\frac{4}{3}\right)^{n} \\
&= x\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n} + \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n} \\
&= \left(x + \frac{1}{3}\right)\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^{n} \\
&= \left(x + \frac{1}{3}\right)\frac{1}{1+\frac{1}{3}} \\
&= \frac{3}{4}\left(x + \frac{1}{3}\right).
\end{aligned}$$

$\blacksquare$

Problem 2
Explicitly exhibit the form of the resolvent kernel $\tilde{K}(x,t;\lambda)$ for the above equation.

Solution. From the previous problem, we have

$$u_n(x) = \int_a^b \int_a^b \cdots \int_a^b K(x,t)K(t,t_1)\cdots K(t_{n-2},t_{n-1})f(t_{n-1})\,\mathrm{d}t_{n-1}\,\mathrm{d}t_{n-2}\cdots\mathrm{d}t_1\,\mathrm{d}t.$$

We will now play around with the dummy integration variables in an attempt to, instead of integrating over t_{n-1}, t_{n-2}, $\cdots$, then t, to doing so in the reverse order. Now, it is important to note that, given $K(x,t) = t - x$, the kernel K is not symmetric, *i.e.* $K(x,t) \neq K(t,x)$, so we have to be careful with reordering. After reordering, we get

$$\begin{aligned}
u_n(x) &= \int_a^b \left[\int_a^b \int_a^b \cdots \int_a^b K(x,t)K(t,t_1)\cdots K(t_{n-2},t_{n-1})\,\mathrm{d}t_{n-2}\,\mathrm{d}t_{n-3}\cdots\mathrm{d}t_1\,\mathrm{d}t\right] f(t_{n-1})\,\mathrm{d}t_{n-1} \\
&= \int_a^b \left[\int_a^b \int_a^b \cdots \int_a^b K(x,t_1)K(t_1,t_2)\cdots K(t_{n-1},t)\,\mathrm{d}t_{n-1}\,\mathrm{d}t_{n-2}\cdots\mathrm{d}t_2\,\mathrm{d}t_1\right] f(t)\,\mathrm{d}t \\
&= \int_a^b K_n(x,t)f(t)\,\mathrm{d}t,
\end{aligned}$$

where we define

$$K_n(x,t) = \int_a^b \int_a^b \cdots \int_a^b K(x,t_1)K(t_1,t_2)\cdots K(t_{n-1},t)\,\mathrm{d}t_{n-1}\,\mathrm{d}t_{n-2}\cdots\mathrm{d}t_2\,\mathrm{d}t_1.$$

Due to the nested integration, it might be hard to see the point of defining K_n. What we did is redefine the integrand of the last integration without $f(t_n)$, and since every other variable was integrated over already, the integrand will only depend on x and t.
Now, we can write

$$\begin{aligned}
\phi(x) &= \lim_{N\to\infty}\sum_{n=0}^{N}\lambda^n u_n(x) \\
&= f(x) + \sum_{n=0}^{\infty}\lambda^n \int_a^b K_n(x,t)f(t)\,\mathrm{d}t \\
&= f(x) + \int_a^b \left(\sum_{n=0}^{\infty}\lambda^n K_n(x,t)\right) f(t)\,\mathrm{d}t \\
&= f(x) + \int_a^b \tilde{K}(x,t;\lambda)f(t)\,\mathrm{d}t,
\end{aligned}$$

where $\tilde{K}$ is the resolvent kernel defined to be

$$\tilde{K}(x,t;\lambda) = \sum_{n=0}^{\infty}\lambda^n K_n(x,t).$$

Computing the first few K_n, with $a = -1$ and $b = 1$, we have

- For $n = 0$, we have
$$K_0(x,t) = f(x).$$
- For $n = 1$, we have
$$K_1(x,t) = K(x,t) = t - x.$$

- For $n = 2$, we have

$$\begin{aligned}
K_2(x,t) &= \int_a^b K(x,t_1)K(t_1,t)\,\mathrm{d}t_1 \\
&= \int_{-1}^1 (t_1 - x)(t - t_1)\,\mathrm{d}t_1 \\
&= \int_{-1}^1 -t_1^2 + (t+x)t_1 - xt\,\mathrm{d}t_1 \\
&= -\frac{t_1^3}{3} + (t+x)\frac{t_1^2}{2} - xtt_1\Big|_{-1}^1 \\
&= -\frac{2}{3} - 2xt.
\end{aligned}$$

- For $n = 3$, we have

$$\begin{aligned}
K_3(x,t) &= \int_a^b \int_a^b K(x,t_1)K(t_1,t_2)K(t_2,t)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_{-1}^1 \int_{-1}^1 (t_1 - x)(t_2 - t_1)(t - t_2)\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_{-1}^1 (t_1 - x)\left[-\frac{t_2^3}{3} + (t+t_1)\frac{t_2^2}{2} - t_1tt_2\right]_{-1}^1 \mathrm{d}t_1 \\
&= \int_{-1}^1 (t_1 - x)\left(-\frac{2}{3} - 2t_1t\right)\mathrm{d}t_1 \\
&= \int_{-1}^1 \left(-2t_1^2t + \left(2tx - \frac{2}{3}\right)t_1 + \frac{2}{3}x\right)\mathrm{d}t_1 \\
&= -\frac{2}{3}t_1^3t + \left(2tx - \frac{2}{3}\right)\frac{t_1^2}{2} + \frac{2}{3}xt_1\Big|_{-1}^1 \\
&= -\frac{4}{3}t + \frac{4}{3}x.
\end{aligned}$$

- For $n = 4$, we have

$$\begin{aligned}
K_4(x,t) &= \int_a^b \int_a^b \int_a^b K(x,t_1)K(t_1,t_2)K(t_2,t_3)K(t_3,t)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (t_1 - x)(t_2 - t_1)(t_3 - t_2)(t - t_3)\,\mathrm{d}t_3\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \frac{8}{3}xt + \frac{8}{9}.
\end{aligned}$$

Replacing with the above and $\lambda = \frac{1}{2}$, we have

$$\begin{aligned}
\tilde{K}\left(x,t;\frac{1}{2}\right) &= \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n K_n(x,t) \\
&= \frac{1}{2}(t-x) + \left(\frac{1}{2}\right)^2\left(-\frac{2}{3} - 2xt\right) + \left(\frac{1}{2}\right)^3\left(-\frac{4}{3}t + \frac{4}{3}x\right) + \left(\frac{1}{2}\right)^4\left(\frac{8}{3}xt + \frac{8}{9}\right) + \cdots \\
&= \frac{1}{2}\left(\left(t - x - xt - \frac{1}{3}\right) - \frac{1}{3}\left(t - x - xt - \frac{1}{3}\right) + \cdots\right) \\
&= \frac{1}{2}\left(t - x - xt - \frac{1}{3}\right)\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n \\
&= \frac{1}{2}\left(t - x - xt - \frac{1}{3}\right)\frac{1}{1+\frac{1}{3}} \\
&= \frac{3}{8}\left(t - x - xt - \frac{1}{3}\right).
\end{aligned}$$

Now, we can check if this resolvent kernel does yield the same solution we got in Problem 1. To do so, we substitute $\tilde{K}$ in ϕ, and get

$$\begin{aligned}
\phi(x) &= f(x) + \int_a^b \tilde{K}\left(x,t;\frac{1}{2}\right) f(t)\,\mathrm{d}t \\
&= x + \frac{3}{8}\int_{-1}^{1}\left(t - x - xt - \frac{1}{3}\right) t\,\mathrm{d}t \\
&= \frac{3}{4}x + \frac{1}{4},
\end{aligned}$$

which matches with our findings from before. ■

Problem 3
[Arfken p. 792] Solve the equation

$$\phi(x) = x + \frac{1}{2}\int_{-1}^{+1}(t-x)\phi(t)\,\mathrm{d}t$$

by the separable kernel method.

Solution. Considering a generic Fredholm equation, we have $f(x) = x$, $\lambda = \frac{1}{2}$, $a = -1$, $b = 1$, and $K(x,t) = t - x$. The kernel K is separable, *i.e.* we can write K as

$$K(x,t) = \sum_{j=1}^{2} M_j(x)N_j(t) = t - x,$$

where

$$\begin{aligned} M_1(x) &= 1, \qquad N_1(t) = t \\ M_2(x) &= x, \qquad N_2(t) = -1. \end{aligned}$$

Now, replacing this new form of the kernel into the Fredholm integral equation, we have

$$\begin{aligned} \phi(x) &= x + \lambda\int_{-1}^{1}\sum_{j=1}^{2} M_j(x)N_j(t)\phi(t)\,\mathrm{d}t \\ &= x + \lambda\sum_{j=1}^{2} M_j(x)\int_{-1}^{1} N_j(t)\phi(t)\,\mathrm{d}t \\ &= x + \lambda\sum_{j=1}^{2} M_j(x)c_j. \end{aligned}$$

We now multiply both sides by $N_i(x)$ and integrate over x from -1 to 1, getting

$$\begin{aligned} \phi(x) &= x + \lambda\sum_{j=1}^{2} M_j(x)c_j \\ \int_{-1}^{1} N_i(x)\phi(x)\,\mathrm{d}x &= \int_{-1}^{1} xN_i(x)\,\mathrm{d}x + \lambda\int_{-1}^{1} N_i(x)\sum_{j=1}^{2} M_j(x)c_j\,\mathrm{d}x \\ c_i &= b_i + \lambda\sum_{j=1}^{2} c_j\left(\int_{-1}^{1} N_i(x)M_j(x)\right)\mathrm{d}x \\ c_i &= b_i + \lambda\sum_{j=1}^{2} c_j a_{ij}. \end{aligned}$$

In the previous set of equations, it is assumed that we are summing over i (two iterations), so if we consider the system we have now

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + \lambda\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda a_{11} & -\lambda a_{12} \\ -\lambda a_{21} & 1-\lambda a_{22} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

Now, we compute

$$a_{11} = \int_{-1}^{1} N_1(x)M_1(x)\,\mathrm{d}x = \int_{-1}^{1} x\,\mathrm{d}x = 0.$$

$$a_{12} = \int_{-1}^{1} N_1(x)M_2(x)\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \frac{2}{3}.$$

$$a_{21} = \int_{-1}^{1} N_2(x)M_1(x)\,\mathrm{d}x = \int_{-1}^{1} (-1)\,\mathrm{d}x = -2.$$

$$a_{22} = \int_{-1}^{1} N_2(x)M_2(x)\,\mathrm{d}x = \int_{-1}^{1} (-x)\,\mathrm{d}x = 0.$$

$$b_1 = \int_{-1}^{1} xN_1(x)\,\mathrm{d}x = \int_{-1}^{1} x^2\,\mathrm{d}x = \frac{2}{3}.$$

$$b_2 = \int_{-1}^{1} xN_2(x)\,\mathrm{d}x = \int_{-1}^{1} (-x)\,\mathrm{d}x = 0.$$

Thus, replacing, we have

$$\begin{pmatrix} 1 & -\frac{2}{3}\lambda \\ 2\lambda & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$$

Replacing $\lambda = \frac{1}{2}$, we have

$$\begin{pmatrix} 1 & -\frac{1}{3} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= \frac{1}{1+\frac{1}{3}}\begin{pmatrix} 1 & \frac{1}{3} \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix} \\
&= \frac{3}{4}\begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\end{aligned}$$

Therefore, our solution ϕ is

$$\begin{aligned}
\phi(x) &= x + \lambda\sum_{j=1}^{2} M_j(x)c_j \\
&= x + \lambda\left(M_1(x)c_1 + M_2(x)c_2\right) \\
&= x + \frac{1}{2}\left((1)\left(\frac{1}{2}\right) + (x)\left(-\frac{1}{2}\right)\right) \\
&= x + \frac{1}{4}\left(1 - x\right) \\
&= \frac{3}{4}x + \frac{1}{4},
\end{aligned}$$

as we found before. ■