PHYS 662 - Quantum Field Theory I

Student: **Ralph Razzouk**

Homework 6

Problem 1 - Double Well Potential
Consider a particle of mass m in a 1D potential $V(q) = \lambda(q^2 - q_0^2)^2$. Write down a path-integral in imaginary time that computes

$$\left\langle q_0, \frac{\tau}{2} \middle| -q_0, -\frac{\tau}{2} \right\rangle = \left\langle q_0 \middle| \mathrm{e}^{-\frac{H\tau}{\hbar}} \middle| -q_0 \right\rangle .$$

(a) What is the Euclidean Lagrangian? How is it different from the real-time Lagrangian?

(b) Write down the Euler-Lagrange equation for this Euclidean action.

(c) Compute the path-integral with this potential.

Solution. Recall that for imaginary time path integrals, we make the Wick rotation $t \to -\mathrm{i}\tau$. This transforms the quantum amplitude into a statistical partition function.

(a) The Euclidean Lagrangian is obtained from the regular Lagrangian by the Wick rotation. The regular Lagrangian is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} m\dot{q}^2 - V(q) \\ &= \frac{1}{2} m\dot{q}^2 - \lambda(q^2 - q_0^2)^2.\end{aligned}$$

Under $t \to -\mathrm{i}\tau$, we have $\dot{q} \to \mathrm{i}\frac{\mathrm{d}q}{\mathrm{d}\tau}$, giving us the Euclidean Lagrangian

$$\begin{aligned}\mathcal{L}_E &= \frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + V(q) \\ &= \frac{1}{2} m \left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + \lambda(q^2 - q_0^2)^2.\end{aligned}$$

The key difference is the sign change before the potential term, and that we're now working with derivatives with respect to τ instead of t.

(b) The Euler-Lagrange equation for the Euclidean action is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \mathcal{L}_E}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}_E}{\partial q} = 0.$$

Substituting our Euclidean Lagrangian, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(m\frac{\mathrm{d}q}{\mathrm{d}\tau}\right) = -\frac{\partial V}{\partial q} = -4\lambda q(q^2 - q_0^2).$$

This simplifies to

$$m\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} = -4\lambda q(q^2 - q_0^2).$$

(c) To compute the path integral, we need to evaluate

$$\left\langle q_0 \middle| \mathrm{e}^{-\frac{H\tau}{\hbar}} \middle| -q_0 \right\rangle = \mathcal{N} \int_{q(-\tau/2)=-q_0}^{q(\tau/2)=q_0} \mathcal{D}q(\tau) \exp\left(-\frac{1}{\hbar}\int_{-\tau/2}^{\tau/2} \mathcal{L}_E \,\mathrm{d}\tau\right)$$

The full path integral is

$$\mathcal{N}\int \mathcal{D}q(\tau)\exp\left(-\frac{1}{\hbar}\int_{-\tau/2}^{\tau/2}\left[\frac{1}{2}m\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2+\lambda(q^2-q_0^2)^2\right]\mathrm{d}\tau\right)$$

This path integral describes quantum tunneling between the two degenerate minima at $q=\pm q_0$ of the double-well potential. The exact solution involves instantons - solutions where the particle tunnels from $-q_0$ to q_0 in imaginary time.

The leading contribution to this path integral comes from a single instanton solution, which follows the classical equation of motion in imaginary time. The full result includes contributions from multi-instanton configurations, but the single instanton typically dominates for large τ.

For large τ, the amplitude behaves as

$$\left\langle q_0\middle|\mathrm{e}^{-\frac{H\tau}{\hbar}}\middle|-q_0\right\rangle \sim e^{-S_E[\text{instanton}]/\hbar},$$

where S_E[instanton] is the Euclidean action evaluated on the instanton solution.

■

Problem 2 - Free Propagator
Derive the path-integral expression for the propagator of a free particle in d spatial dimensions with the action $S_0=\frac{1}{2}m^2\int\dot{q}^2$ by discretizing time.

Solution. We wish to find the propagator $\langle\mathbf{q}_f,t_f|\mathbf{q}_i,t_i\rangle$ for a free particle in d dimensions. Let us split the time interval t_f-t_i into n equal parts

$$\Delta t=\frac{t_f-t_i}{n},\quad t_{r+1}=t_r+\Delta t,\quad (\mathbf{q}_0,t_0)=(\mathbf{q}_i,t_i),\quad (\mathbf{q}_n,t_n)=(\mathbf{q}_f,t_f).$$

The transition element then becomes

$$\begin{aligned}\langle\mathbf{q}_f,t_f|\mathbf{q}_i,t_i\rangle &= \int\left(\prod_{r=1}^{n-1}\mathrm{d}\mathbf{q}_r\right)\left(\prod_{r=0}^{n-1}\langle\mathbf{q}_{r+1},t_{r+1}|\mathbf{q}_r,t_r\rangle\right)\\ &= \int\left(\prod_{r=1}^{n-1}\mathrm{d}\mathbf{q}_r\right)\left(\prod_{r=0}^{n-1}\left\langle\mathbf{q}_{r+1}\middle|e^{-\mathrm{i}H\Delta t}\middle|\mathbf{q}_r\right\rangle\right).\end{aligned}$$

Each matrix element can be simplified by inserting resolutions of identity in momentum space

$$\begin{aligned}\left\langle\mathbf{q}_{r+1}\middle|e^{-\mathrm{i}H\Delta t}\middle|\mathbf{q}_r\right\rangle &= \int\frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d}\left\langle\mathbf{q}_{r+1}\middle|e^{-\mathrm{i}H\Delta t}\middle|\mathbf{p}_r\right\rangle\langle\mathbf{p}_r|\mathbf{q}_r\rangle\\ &= \int\frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d}\langle\mathbf{q}_{r+1}|\mathbf{p}_r\rangle\langle\mathbf{p}_r|\mathbf{q}_r\rangle\exp\left(-\mathrm{i}H\left(\frac{\mathbf{q}_{r+1}+\mathbf{q}_r}{2},\mathbf{p}_r\right)\Delta t\right)\\ &= \int\frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d}\exp\left(\mathrm{i}\mathbf{p}_r\cdot(\mathbf{q}_{r+1}-\mathbf{q}_r)\right)\exp\left(-\mathrm{i}H\left(\frac{\mathbf{q}_{r+1}+\mathbf{q}_r}{2},\mathbf{p}_r\right)\Delta t\right).\end{aligned}$$

For a free particle, $H=\frac{\mathbf{p}^2}{2m}$, so the transition relation becomes

$$\langle\mathbf{q}_f,t_f|\mathbf{q}_i,t_i\rangle=\int\left(\prod_{r=0}^{n-1}\frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d}\right)\left(\prod_{r=1}^{n-1}\mathrm{d}\mathbf{q}_r\right)\exp\left(\mathrm{i}\sum_{r=0}^{n-1}\Delta t\left[\mathbf{p}_r\cdot\frac{\mathbf{q}_{r+1}-\mathbf{q}_r}{\Delta t}-\frac{\mathbf{p}_r^2}{2m}\right]\right).$$

Going to the continuum limit, this becomes

$$\langle\mathbf{q}_f,t_f||\mathbf{q}_i,t_i\rangle=\int\mathcal{D}\mathbf{p}\mathcal{D}\mathbf{q}\exp\left(\mathrm{i}\int\mathrm{d}t\left[\mathbf{p}\cdot\dot{\mathbf{q}}-\frac{\mathbf{p}^2}{2m}\right]\right).$$

The momentum integral is Gaussian and can be evaluated. For each r, we have

$$\int \frac{\mathrm{d}\mathbf{p}_r}{(2\pi)^d} \exp\left(-\mathrm{i}\Delta t\left[\frac{\mathbf{p}_r^2}{2m} - \mathbf{p}_r \cdot \frac{\mathbf{q}_{r+1} - \mathbf{q}_r}{\Delta t}\right]\right) = \left(\frac{m}{2\pi\mathrm{i}\Delta t}\right)^{d/2}.$$

Thus, the path integral becomes

$$\langle \mathbf{q}_f, t_f || \mathbf{q}_i, t_i\rangle = \left(\frac{m}{2\pi\mathrm{i}\Delta t}\right)^{nd/2} \int \left(\prod_{r=1}^{n-1} \mathrm{d}\mathbf{q}_r\right) \exp\left(\mathrm{i}\sum_{r=0}^{n-1} \Delta t \frac{m}{2}\left(\frac{\mathbf{q}_{r+1} - \mathbf{q}_r}{\Delta t}\right)^2\right).$$

Going to the continuum limit, this becomes

$$\langle \mathbf{q}_f, t_f || \mathbf{q}_i, t_i\rangle = \int \mathcal{D}\mathbf{q} \exp\left(\mathrm{i}\int \mathrm{d}t \frac{m}{2}\dot{\mathbf{q}}^2\right).$$

This can be solved to get the final result

$$\langle \mathbf{q}_f, t_f || \mathbf{q}_i, t_i\rangle = \left(\frac{m}{2\pi\mathrm{i}(t_f - t_i)}\right)^{d/2} \exp\left(\frac{\mathrm{i}m(\mathbf{q}_f - \mathbf{q}_i)^2}{2(t_f - t_i)}\right).$$

■

Problem 3 - Path-integral for Complex Scalar
Consider the action for a complex massive free scalar field $\Phi(x)$.

(a) Compute $Z_0[0]$ explicitly.

(b) Write down the action in the presence of sources for both $\Phi(x)$ and $\Phi^*(x)$.

(c) Evaluate the path-integral and compute the two-point function by taking derivatives with respect to J and setting $J = 0$.

(d) Show that differentiating the path-integral gives time-ordered correlators.

Solution. (a) For a complex scalar field, the action without sources is

$$S_0 = \int \mathrm{d}^{d+1}x \left[-\Phi^*(-\Box + m^2)\Phi\right].$$

Thus,

$$Z_0[0] = \int \mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left(\mathrm{i}\int \mathrm{d}^{d+1}x \left[-\Phi^*(-\Box + m^2)\Phi\right]\right).$$

For simplicity, let's convert this to real fields

$$\Phi = \frac{\phi_1 + \mathrm{i}\phi_2}{\sqrt{2}}.$$

The measure transforms as

$$\int \mathcal{D}\Phi\mathcal{D}\Phi^* = C_1 \int \mathcal{D}\phi_1\mathcal{D}\phi_2.$$

The action becomes

$$Z_0[0] = C_1 \int \mathcal{D}\phi_1\mathcal{D}\phi_2 \exp\left(-\frac{1}{2}\mathrm{i}\int \mathrm{d}^{d+1}x \left[\phi_1(-\Box + m^2)\phi_1 + \phi_2(-\Box + m^2)\phi_2\right]\right).$$

This is the product of two Gaussian integrals

$$Z_0[0] \propto \left[\det\left(\mathrm{i}(-\Box + m^2)\right)\right]^{-1}.$$

(b) With sources, the action becomes

$$S = \int \mathrm{d}^{d+1}x \left[-\Phi^*(-\Box + m^2)\Phi + J^*\Phi + \Phi^* J \right].$$

(c) Let's evaluate the path integral with sources

$$Z_0[J] = \int \mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left(\mathrm{i} \int \mathrm{d}^{d+1}x \left[-\Phi^*(-\Box + m^2)\Phi + J^*\Phi + \Phi^* J \right] \right).$$

To evaluate this, let us shift the field

$$\Phi \to \Phi + f(J).$$

For this to eliminate linear terms, we need

$$(-\Box + m^2) f(J) = J.$$

Thus,

$$f(J)(x) = -\int \mathrm{d}^{d+1}y \Delta_F(x-y)J(y).$$

After completing the square, we get

$$Z_0[J] = Z_0[0] \exp\left(-\mathrm{i} \int \mathrm{d}^{d+1}y J^*(x)\Delta_F(x-y)J(y) \right).$$

Taking functional derivatives, we have

$$\begin{aligned}
\frac{\delta Z_0[J]}{\mathrm{i}\delta J(x_1)} &= -Z_0[J] \int \mathrm{d}^{d+1}y J^*(y)\Delta_F(y - x_1), \\
\frac{\delta Z_0[J]}{\mathrm{i}\delta J^*(x_1)} &= -Z_0[J] \int \mathrm{d}^{d+1}y \Delta_F(x_1 - y)J(y).
\end{aligned}$$

Therefore, the two-point functions are

$$\begin{aligned}
\frac{1}{Z_0[J]} \frac{\delta^2 Z_0[J]}{(\mathrm{i}\delta J(x_1))(\mathrm{i}\delta J(x_2))}\Big|_{J=0} &= 0, \\
\frac{1}{Z_0[J]} \frac{\delta^2 Z_0[J]}{(\mathrm{i}\delta J^*(x_1))(\mathrm{i}\delta J^*(x_2))}\Big|_{J=0} &= 0, \\
\frac{1}{Z_0[J]} \frac{\delta^2 Z_0[J]}{(\mathrm{i}\delta J^*(x_1))(\mathrm{i}\delta J(x_2))}\Big|_{J=0} &= \mathrm{i}\Delta_F(x_1 - x_2), \\
\frac{1}{Z_0[J]} \frac{\delta^2 Z_0[J]}{(\mathrm{i}\delta J(x_1))(\mathrm{i}\delta J^*(x_2))}\Big|_{J=0} &= \mathrm{i}\Delta_F(x_2 - x_1).
\end{aligned}$$

(d) To show these give time-ordered correlators, note that functional derivatives give

$$\frac{\delta}{\mathrm{i}\delta J(y)} \int \mathcal{D}\Phi\mathcal{D}\Phi^* O_1(x_1)\cdots O_n(x_n)\mathrm{e}^{\mathrm{i}S} = \int \mathcal{D}\Phi\mathcal{D}\Phi^* O_1(x_1)\cdots O_n(x_n)\Phi^*(y)\mathrm{e}^{\mathrm{i}S}$$

When integrating over paths between initial state $\Phi_a(\mathbf{x})$ at $t = -T$ and final state $\Phi_b(\mathbf{x})$ at $t = T$, this gives

$$\langle \Phi_b | T(\Phi^*(x_1)\cdots\Phi(x_n)) | \Phi_a \rangle = \frac{1}{Z_0[J]} \prod_i \frac{\delta}{\mathrm{i}\delta J(x_i)} Z_0[J]\Big|_{J=0},$$

where T denotes time-ordering. Thus the functional derivatives automatically give time-ordered correlators.

■

Problem 4 - Connected Correlators
Consider the generating function $Z(J)$ for all correlators

$$\frac{\delta}{\mathrm{i}\delta J(x_1)}\cdots\frac{\delta}{\mathrm{i}\delta J(x_n)}Z(J)\bigg|_{J=0} = \langle\phi(x_1)\cdots\phi(x_n)\rangle\,.$$

Show that, if we define

$$W(J) = -\mathrm{i}\ln(Z(J)),$$

differentiating $W(J)$ with respect to $\mathrm{i}\delta J$ gives the connected correlators.

Solution. Let's prove this systematically. We can express $Z(J)$ as a sum over Feynman diagrams

$$Z(J) \propto \sum_{\Lambda} D_{\Lambda},$$

where $\{D_{\Lambda}\}$ is the set of all possible Feynman diagrams.
Most of these diagrams are disconnected - they are products of connected diagrams. Let us denote the set of connected diagrams by $\{C_{\lambda}\}$. Then each Feynman diagram can be written as

$$D_{\Lambda} = \prod_{\lambda}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}},$$

where n_{Λ}^{λ} counts the number of copies of diagram C_{λ} in D_{Λ}, and the factorial accounts for identical copies. Thus,

$$Z(J) \propto \sum_{\Lambda}\prod_{\lambda}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}}.$$

Let's split the sum into two pieces - one over diagrams $\tilde{\Lambda}$ with fixed numbers of all types except C_{λ_0}, and the rest over Λ'

$$\begin{aligned} Z(J) &\propto \sum_{\Lambda'}\sum_{\tilde{\Lambda}}\prod_{\lambda\neq\lambda_0}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}}\left(\frac{1}{n_{\Lambda}^{\lambda_0}!}(C_{\lambda_0})^{n_{\Lambda}^{\lambda_0}}\right) \\ &= \sum_{\Lambda'}\prod_{\lambda\neq\lambda_0}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}}\left(\sum_{\tilde{\Lambda}}\frac{1}{n_{\Lambda}^{\lambda_0}!}(C_{\lambda_0})^{n_{\Lambda}^{\lambda_0}}\right). \end{aligned}$$

The sum over $\tilde{\Lambda}$ is equivalent to summing over all possible values of $n_{\Lambda}^{\lambda_0}$

$$\sum_{\tilde{\Lambda}}\frac{1}{n_{\Lambda}^{\lambda_0}!}(C_{\lambda_0})^{n_{\Lambda}^{\lambda_0}} = \sum_{n_{\Lambda}^{\lambda_0}=0}^{\infty}\frac{1}{n_{\Lambda}^{\lambda_0}!}(C_{\lambda_0})^{n_{\Lambda}^{\lambda_0}} = \mathrm{e}^{C_{\lambda_0}}.$$

Thus,

$$Z(J) \propto \mathrm{e}^{C_{\lambda_0}}\sum_{\Lambda'}\prod_{\lambda\neq\lambda_0}\frac{1}{n_{\Lambda}^{\lambda}!}(C_{\lambda})^{n_{\Lambda}^{\lambda}}.$$

Repeating this process for all other indices λ

$$Z(J) \propto \prod_{\lambda}\mathrm{e}^{C_{\lambda}} = \mathrm{e}^{\sum_{\lambda}C_{\lambda}}.$$

Therefore,

$$W(J) = -\mathrm{i}\ln(Z(J)) = -\mathrm{i}\sum_{\lambda}C_{\lambda}.$$

This means when we differentiate $W(J)$ with respect to $\mathrm{i}\delta J$, we get the sum of only connected diagrams - the connected correlators. ∎