MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*
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Homework 6

Submitting an incomplete homework as I had been absent for a few lectures and didn't manage to read up on the topics missed.

> **Problem 6-2**
> Prove Theorem 6.18 (the Whitney immersion theorem) in the special case $\partial M = \varnothing$. [Hint: without loss of generality, assume that M is an embedded n-dimensional submanifold of $\mathbb{R}^{2n+1}$. Let $UM \subseteq T\mathbb{R}^{2n+1}$ be the unit tangent bundle of M (Problem 5-6), and let $G : UM \to \mathbb{RP}^{2n}$ be the map $G(x, v) = [v]$. Use Sard's theorem to conclude that there is some $v \in \mathbb{R}^{2n+1} \backslash \mathbb{R}^{2n}$ such that $[v]$ is not in the image of G, and show that the projection from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$ with kernel $\mathbb{R}v$ restricts to an immersion of M into $\mathbb{R}^{2n}$.]

Solution. The Whitney Immersion Theorem states that every smooth manifold M can be smoothly immersed into some Euclidean space $\mathbb{R}^n$, for sufficiently large N. We aim to prove this for the case $\partial M = \varnothing$. We use the hint given.
Without loss of generality, assume that M is an embedded n-dimensional submanifold of $\mathbb{R}^{n+1}$. Let UM be the unit tangent bundle of M and let $G : UM \to \mathbb{RP}^{2n}$ be the map $G(x, v) = [v]$. Since $\partial M = \varnothing$. then UM is a smooth manifold of dimension $2n$ and $\mathbb{RP}^{2n}$ is a smooth manifold of dimension $2n - 1$.
By Sard's theorem, almost every point of $\mathbb{RP}^{2n}$ is a regular value of G, then the preimage of any regular value is a smooth submanifold of UM of dimension 1. Thus, there exists some $v \in \mathbb{R}^{2n+1} \setminus \mathbb{R}^{2n}$ such that $[v]$ is not in the image of G.
Let $p : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ be the projection with kernel $\mathbb{R}v$. Then p restricts to an immersion of M into $\mathbb{R}^{2n}$.
Let i be an injective smooth map that embeds the n-dimensional submanifold M into $\mathbb{R}^{2n+1}$ and let X be a vector field on M. Then $\mathrm{d}_\pi(M)$ is a linear subspace of $\mathbb{R}^{2n}$ of dimension n and $\mathrm{d}_\pi(x)$ is a linear map from $\mathrm{d}_\pi(M)$ to $\mathbb{R}^{2n}$ that is injective at every point of M.
Therefore, $\mathrm{d}_\pi(X)$ has full rank n at every point of M, which means that $p \circ i$ is an immersion of M into $\mathbb{R}^{2n}$. ∎

> **Problem 6-5**
> Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. Show that M has a tubular neighborhood U with the following property: for each $y \in U, r(y)$ is the unique point in M closest to y, where $r : U \to M$ is the retraction defined in Proposition 6.25. [Hint: first show that if $y \in \mathbb{R}^n$ has a closest point $x \in M$, then $(y - x) \perp T_xM$. Then, using the notation of the proof of Theorem 6.24, show that for each $x \in M$, it is possible to choose $\delta > 0$ such that every $y \in E(V_\delta(x))$ has a closest point in M, and that point is equal to $r(y)$.]

Solution. Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold and let $y \in \mathbb{R}^n$ and define the map

$$\begin{aligned} d : \mathbb{R}^n &\to \mathbb{R} \\ p &\mapsto |y - p|. \end{aligned}$$

Let $x \in M$ be a closest point to $y \in \mathbb{R}^n$. The directional derivative of d in the direction of v is

$$\nabla d \cdot v|_x = \frac{y - x}{|y - x|} \cdot v = 0.$$

Thus $y - x \perp T_xM$.
Incomplete...

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Problem 6-10
Suppose $F : N \to M$ is a smooth map that is transverse to an embedded submanifold $X \subseteq M$, and let $W = F^{-1}(X)$. For each $p \in W$, show that $T_pW = (dF_p)^{-1}\left(T_{F(p)}X\right)$. Conclude that if two embedded submanifolds $X, X' \subseteq M$ intersect transversely, then $T_p(X \cap X') = T_pX \cap T_pX'$ for every $p \in X \cap X'$. (Used on p. 146)

Solution. Let $X \subseteq M$ and $W = F^{-1}(X)$. It is enough to check that W is a submanifold of M in a neighborhood $X \in W$. Let (V, ψ) be a local coordinate chart of N adapted to X around $X = F_p$, then $\psi : V \to \mathbb{R}^{n+k}$ and $\psi(V \cap X) = \psi(V) \cap \mathbb{R}^n$, where $n = \dim(X)$.
Let $\pi_2 : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be the standard projection and let $g = \pi_1 \circ \psi$, then $g : V \to \mathbb{R}^k$ is a submersion and $g^{-1}(0) = V \cap X$.
Incomplete... ■