PHYS 601 - Methods of Theoretical Physics II

Mathematical Methods for Physicists by *Arfken, Weber, Harris*
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Homework 5

Problem 1
In class, we have derived Stirling's formula for real S,

$$S! \approx \sqrt{2\pi S} S^S \mathrm{e}^{-S}.$$

Show that the same result holds for arbitrary complex z as well, namely

$$\Gamma(z+1) = z! \approx \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z}.$$

Solution. To derive Stirling's approximation for complex numbers, we must first define the complex logarithm. The complex logarithm is well-defined modulo $2\pi i$ (so that we don't have to worry about the branches of $\ln(z)$), as

$$\ln(AB) \equiv \ln(A) + \ln(B) \pmod{2\pi i}.$$

The gamma function is single-valued, but Stirling's approximation contains $\sqrt{z}$ and z^z, which are both multi-valued in the complex plane, and hence, Stirling's approximation will not be analytic around zero. To fix this, we will restrict z to the right half of the complex plane, where we can specify the principal branch of $\ln(z)$, $\sqrt{z}$, and z^z. Let $z = re^{i\theta}$, then $\ln(z) = \ln(r) + i\theta$, where $\theta \in (-\pi, \pi)$. Taking the logarithm of both sides and expanding, we have

$$\begin{aligned}
\ln(\Gamma(z)) &\approx \frac{\ln(2\pi)}{2} + \left(z - \frac{1}{2}\right)\ln(z) - z \\
&= \frac{\ln(2\pi)}{2} + r\mathrm{e}^{i\theta}(\ln(r) + i\theta) - \frac{\ln(r) + i\theta}{2} - r\mathrm{e}^{i\theta} \\
&= \frac{\ln(2\pi)}{2} - \frac{\ln(r) + i\theta}{2} + r\mathrm{e}^{i\theta}(\ln(r) + i\theta - 1) \\
&= \frac{\ln(2\pi)}{2} - \frac{\ln(r) + i\theta}{2} + r(\cos(\theta) + i\sin(\theta))(\ln(r) + i\theta - 1) \\
&= \frac{\ln(2\pi)}{2} - \frac{\ln(r) + i\theta}{2} + r\ln(r)\cos(\theta) + ir\theta\cos(\theta) - r\cos(\theta) + ir\ln(r)\sin(\theta) - r\theta\sin(\theta) - ir\sin(\theta),
\end{aligned}$$

which gives us

$$\begin{aligned}
\Re(\ln(\Gamma(z))) &= \frac{\ln(2\pi)}{2} - \frac{\ln(r)}{2} + r\ln(r)\cos(\theta) - r\cos(\theta) - r\theta\sin(\theta), \\
\Im(\ln(\Gamma(z))) &= -\frac{\theta}{2} + r\ln(r)\sin(\theta) - r\sin(\theta) + r\theta\cos(\theta).
\end{aligned}$$

If we pick a fixed non-zero value for θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and take $r \to +\infty$, the term $r\ln(r)\sin(\theta)$ in the expression for the imaginary part dominates, so the imaginary part of the gamma function will become arbitrarily large and the gamma function will just spiral around the origin an infinite number of times as z tends to infinity along the non-trivial ray.

Although the gamma function is single-valued in the entire plane, its logarithm becomes multi-valued if we analytically continue it around the poles of the gamma function. It is true that any two values of the logarithm will differ by a multiple of $2\pi i$, but that is not true for Stirling's approximation, as seen above. Applying the exponential to the logarithm of the gamma function wipes any ambiguity, but not so for the approximation.

Consider

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Gamma(x) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2} = \frac{1}{2z^2} - \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\pi\cot(\pi\zeta)}{(z+\zeta)^2}\,\mathrm{d}\zeta$$

by contour integration. Assuming $\Re(z) > 0$, we can split the integral and use substitution to get

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\pi\cot(\pi\zeta)}{(z+\zeta)^2}\,\mathrm{d}\zeta &= \frac{1}{2}\int_0^\infty \cot(\pi i\eta)\left(\frac{1}{(z+\eta)^2}-\frac{1}{(z-\eta)^2}\right)\mathrm{d}\eta \\ &= -\int_0^\infty \coth(\pi\eta)\frac{2\eta z}{(z^2+\eta)^2}\,\mathrm{d}\eta.\end{aligned}$$

Substituting back, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Gamma(x) = \frac{1}{z}+\frac{1}{2z^2}+\int_0^\infty \frac{1}{\mathrm{e}^{2\pi\eta}-1}\frac{4\eta z}{(z^2+\eta^2)^2}\,\mathrm{d}\eta.$$

Integrating with respect to z, and then performing integration by parts, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\Gamma(x) &= C+\ln(z)-\frac{1}{2z}-\int_0^\infty \frac{1}{\mathrm{e}^{2\pi\eta}-1}\frac{2\eta}{z^2+\eta^2}\,\mathrm{d}\eta \\ &= C+\ln(z)-\frac{1}{2z}+\frac{1}{\pi}\int_0^\infty \ln(1-\mathrm{e}^{2\pi\eta})\frac{z^2-\eta^2}{(z^2+\eta^2)^2}\,\mathrm{d}\eta\end{aligned}$$

Integrating again, we get

$$\ln(\Gamma(x)) = C'+Cz+\left(z-\frac{1}{2}\right)\ln(z)+\frac{1}{\pi}\int_0^\infty \frac{z}{z^2+\eta^2}\ln\left(\frac{1}{1-\mathrm{e}^{-2\pi\eta}}\right)\mathrm{d}\eta,$$

which is essentially Stirling's formula with some constants of integration and limit relations

$$\ln(\Gamma(x)) = C'+Cz+z\ln(z)-\frac{1}{2}\ln(z)+J(z),$$

where $J(z)$ is the integral term.
Notice that $J(z)\to 0$ as $z\to\infty$ with $\Re(z)\geq c$, for some fixed real positive c.
The constant C is determined from the functional equation or recursive relation

$$z\Gamma(z)=\Gamma(z+1) \iff \ln(z)+\ln(\Gamma(z)) = \ln(\Gamma(z+1))$$

by letting z tend to infinity. We get that $C=-1$.
The constant C' is determined from the property

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$$

by letting z tend to infinity. We get that $C'=\frac{1}{2}\ln(2\pi)$.
Therefore,

$$\begin{aligned}\ln(\Gamma(x)) &\approx \frac{1}{2}\ln(2\pi)-z+z\ln(z)-\frac{1}{2}\ln(z), \\ &\implies \Gamma(z)\approx\sqrt{2\pi}z^{z+\frac{1}{2}}e^{-z}.\end{aligned}$$

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Problem 2
Repeat the derivation given in class of the result

$$I=\Gamma(a)\Gamma(1-a)=\int_0^\infty \frac{u^{a-1}}{1+u}\,\mathrm{d}u=\frac{\pi}{\sin(\pi a)}.$$

Use the substitution $u=\mathrm{e}^x$ to write down I in the form

$$I=\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,\mathrm{d}x.$$

Integrate around the rectangular contour with the four corners at R, $R + 2\pi i$, $-R + 2\pi i$, and $-R$. Determine that the residue at $z = i\pi$ is $-\mathrm{e}^{ia\pi}$.

Solution. From the definition of the gamma function, we have

$$
\begin{aligned}
\Gamma(a)\Gamma(1-a) &= \int_0^\infty \frac{u^{a-1}}{1+u}\,\mathrm{d}u \\
&= \int_{-\infty}^\infty \frac{\mathrm{e}^{(a-1)x}}{1+\mathrm{e}^x}\mathrm{e}^x\,\mathrm{d}x \\
&= \int_{-\infty}^\infty \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^x}\,\mathrm{d}x
\end{aligned}
$$

Theorem 1. (Cauchy Residue Theorem) If f is holomorphic inside and on the simple closed contour C, except for a finite set of points of isolated singularities z_k inside C, then

$$
\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^n \operatorname{res}(f, z_k).
$$

We have singularities at $z = \pm\pi i$. Our contour C is a rectangular region in the upper-half of the complex plane, so it only encloses $z = \pi i$. Using the Cauchy residue theorem, we have

$$
\begin{aligned}
\oint_C \frac{\mathrm{e}^{az}}{1+\mathrm{e}^z}\,\mathrm{d}z &= 2\pi i\ \operatorname{res}\left[\left.\frac{\mathrm{e}^{az}}{1+\mathrm{e}^z}\right|_{z=\pi i}\right] \\
&= 2\pi i \lim_{z\to\pi i}\left((z-\pi i)\frac{\mathrm{e}^{az}}{1+\mathrm{e}^z}\right) \\
&= 2\pi i \lim_{z\to\pi i}\left(\frac{\mathrm{e}^{az} + a(z-\pi i)\mathrm{e}^{az}}{\mathrm{e}^z}\right) \\
&= -2\pi i \mathrm{e}^{a\pi i}.
\end{aligned}
$$

Breaking up the contour integral, we have

$$
\begin{aligned}
\oint_C \frac{\mathrm{e}^{az}}{1+\mathrm{e}^z}\,\mathrm{d}z &= I_1 + I_2 + I_3 + I_4 \\
&= \lim_{R\to\infty}\int_{-R}^{R} \frac{\mathrm{e}^{ax}}{1+\mathrm{e}^x}\,\mathrm{d}x + i\lim_{R\to\infty}\int_0^{2\pi} \frac{\mathrm{e}^{aR+ayi}}{1+\mathrm{e}^{R+yi}}\,\mathrm{d}y \\
&\quad + \lim_{R\to\infty}\int_{R}^{-R} \frac{\mathrm{e}^{ax+2a\pi i}}{1+\mathrm{e}^{x+2\pi i}}\,\mathrm{d}x + i\lim_{R\to\infty}\int_{2\pi}^{0} \frac{\mathrm{e}^{aR+ayi}}{1+\mathrm{e}^{R+yi}}\,\mathrm{d}y \\
&= I + 0 - \mathrm{e}^{2a\pi i} I + 0.
\end{aligned}
$$

Equating both sides, we have

$$
\begin{aligned}
I &= \frac{-2\pi i \mathrm{e}^{a\pi i}}{1-\mathrm{e}^{2a\pi i}} \\
&= \frac{2\pi i \mathrm{e}^{a\pi i}}{\mathrm{e}^{2a\pi i}-1} \\
&= \frac{2\pi i}{\mathrm{e}^{a\pi i}-\mathrm{e}^{-a\pi i}} \\
&= \frac{\pi}{\sin(a\pi)}.
\end{aligned}
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Homework 6

Problem 1
Consider the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

(a) Develop an asymptotic series for $\mathrm{erf}(x)$. [Hint: look at $1 - \mathrm{erf}(x)$ and develop this in a series by partial integration]

(b) Show that this series is not uniformly convergent.

Solution. (a) Note that the complimentary error function erfc is defined as

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t.$$

Developing an asymptotic expansion for the complimentary error function (and therefore also of the error function) for large x, we have

$$\begin{aligned}
\mathrm{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t \\
&= \frac{2}{\sqrt{\pi}} \int_{x^2}^\infty \mathrm{e}^{-u} \frac{1}{2\sqrt{u}}\,\mathrm{d}u \qquad (u = t^2) \\
&= \frac{1}{\sqrt{\pi}} \int_{x^2}^\infty u^{-\frac{1}{2}} \mathrm{e}^{-u}\,\mathrm{d}u.
\end{aligned}$$

Using tabular integration by parts, we have

$$\begin{array}{ccc}
u^{-\frac{1}{2}} & & \mathrm{e}^{-u} \\
 & + & \\
-\frac{1}{2}u^{-\frac{3}{2}} & & -\mathrm{e}^{-u} \\
 & - & \\
\frac{1}{2}\frac{3}{2}u^{-\frac{5}{2}} & & \mathrm{e}^{-u} \\
 & + & \\
 & & -\mathrm{e}^{-u} \\
\vdots & & \\
 & & \vdots \\
\frac{1}{2}\frac{3}{2}\cdots\frac{2n-1}{2}u^{-\frac{2n+1}{2}} & & \\
 & & (-1)^{n+1}\mathrm{e}^{-u}
\end{array}$$

which gives us

$$\begin{aligned}
\mathrm{erfc}(x) &= \frac{1}{\sqrt{\pi}} \left[\sum_{n=0}^\infty (-1)^{n+1} \frac{(2n-1)!!}{2^n} u^{-\frac{2n+1}{2}} \mathrm{e}^{-u} \right]_{x^2}^\infty \\
&= 0 - \frac{1}{\sqrt{\pi}} \left[\sum_{n=0}^\infty (-1)^{n+1} \frac{(2n-1)!!}{2^n} x^{-(2n+1)} \mathrm{e}^{-x^2} \right] \\
&= \frac{1}{x\sqrt{\pi}} \mathrm{e}^{-x^2} \sum_{n=0}^\infty (-1)^n \frac{(2n-1)!!}{2^n x^{2n}}.
\end{aligned}$$

Thus, the asymptotic expansion for $\operatorname{erf}(x)$ is

$$\operatorname{erf}(x) = 1 - \operatorname{erfc}(x) = 1 - \frac{1}{x\sqrt{\pi}} \mathrm{e}^{-x^2} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n x^{2n}}.$$

(b) A sequence of functions $\{f_n\}$, $n = 1, 2, \ldots$ is said to be uniformly convergent to f for a set E of values of x if, for each $\epsilon > 0$, an integer N can be found such that

$$|f_n(x) - f(x)| < \epsilon,$$

for $n \geq N$ and all $x \in E$.

A series converges uniformly on E if the sequence $\{S_n\}$ of partial sums defined by

$$\sum_{k=1}^{n} f_k(x) = S_n(x)$$

converges uniformly on E.

To test for uniform convergence, we can use Abel's uniform convergence test or the Weierstrass M-test. Our series looks like

$$\operatorname{erfc}(x) = \frac{1}{x\sqrt{\pi}} \mathrm{e}^{-x^2} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{2^n} \frac{1}{x^{2n}} = \frac{1}{x\sqrt{\pi}} \mathrm{e}^{-x^2} \sum_{n=0}^{\infty} a_n f_n(x).$$

The sequence of partial sums for the first few terms looks like

$$\left\{ \frac{1}{x}, \quad \frac{1}{x} - \frac{1}{2x^2}, \quad \frac{1}{x} - \frac{1}{2x^2} + \frac{1 \cdot 3}{4x^4}, \quad \frac{1}{x} - \frac{1}{2x^2} + \frac{1 \cdot 3}{4x^4} - \frac{1 \cdot 3 \cdot 5}{8x^6}, \cdots \right\}.$$

For the asymptotic sequence

$$\phi_n(x) = \frac{\mathrm{e}^{-x^2}}{x^{2n+1}} \quad \text{as } x \to \infty,$$

for a fixed N, we have that

$$\begin{aligned}
|S_{n+1} - S_n| &= \left| C \int_x^{\infty} \frac{e^{-t^2}}{t^{2n+2}} \, \mathrm{d}t \right| \\
&= \left| C \int_x^{\infty} \left(-\frac{1}{2t^{2n+3}} \right) \frac{\mathrm{d}}{\mathrm{d}t} (e^{-t^2}) \, \mathrm{d}t \right| \\
&\lesssim C \frac{e^{-x^2}}{x^{2n+3}} \\
&= \frac{C \phi_n(x)}{x^2},
\end{aligned}$$

where C is a constant depending only on n. Hence, the series diverges for any large x.

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Problem 2
In class, we developed asymptotic expansions for $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$. Rederive the same expansions by a series of partial integrations. Hint: Recall that

$$\mathrm{Ci}(x) + i\mathrm{Si}(x) = -\int_x^\infty \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t.$$

Solution. Using tabular integration by parts, we have

$$\begin{array}{ccc}
t^{-1} & & \mathrm{e}^{it} \\
 & + & \\
-t^{-2} & & -i\mathrm{e}^{it} \\
 & - & \\
2t^{-3} & & -\mathrm{e}^{it} \\
 & + & \\
\vdots & & i\mathrm{e}^{it} \\
 & & \\
(-1)^n n! t^{-(n+1)} & & \vdots \\
 & & \\
 & & (-1)^{-\frac{n}{2}}\mathrm{e}^{it}
\end{array}$$

which gives us

$$\begin{aligned}
-\int_x^\infty \frac{\mathrm{e}^{it}}{t}\,\mathrm{d}t &= -\sum_{n=0}^{\infty}(-1)^{-\left(\frac{n+1}{2}\right)} n! \frac{e^{it}}{t^{n+1}}\bigg|_x^\infty \\
&= \sum_{n=0}^{\infty}(-1)^{-\left(\frac{n+1}{2}\right)} n! \frac{e^{ix}}{x^{n+1}} \\
&= e^{ix}\left[\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(2k+1)!}{x^{2k+2}} + \sum_{k=0}^{\infty}(-1)^{k+\frac{3}{2}}\frac{(2k)!}{x^{2k+1}}\right] \\
&= (\cos(x) + i\sin(x))\left[\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(2k+1)!}{x^{2k+2}} + i\sum_{k=0}^{\infty}(-1)^{k+1}\frac{(2k)!}{x^{2k+1}}\right] \\
&= \frac{\sin(x)}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} - \frac{\cos(x)}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}} \\
&\quad - i\left(\frac{\cos(x)}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} + \frac{\sin(x)}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}}\right).
\end{aligned}$$

Therefore,

$$\mathrm{Ci}(x) = \frac{\sin(x)}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} - \frac{\cos(x)}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}}$$

and

$$\mathrm{Si}(x) = -\frac{\cos(x)}{x}\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{x^{2n}} - \frac{\sin(x)}{x^2}\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)!}{x^{2n}}.$$

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