MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*

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Homework 1

Problem 1

- Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.
- Show that $\mathbb{RP}^n$ is compact. [Hint: show that the restriction of π to $\mathbb{S}^n$ is surjective.]

Solution.

- $\boxed{\textbf{Hausdorff}}$ Let $[x]$ and $[y]$ be two distinct points of $\mathbb{RP}^n$, *i.e.* two distinct 1-dimensional subspaces of $\mathbb{R}^{n+1}$, that are spanned by the unit vectors x and y, respectively. Since $\mathbb{S}^n$ is Hausdorff, it is not hard to see that we can find pairwise disjoint open sets $U, \bar{U}, V, \bar{V} \subseteq \mathbb{S}^n$, such that $x \in U$, $-x \in \bar{U}$, $y \in V$, and $-y \in \bar{V}$, where $\bar{A} = -a$ such that $a \in A$. Let $\hat{U} = U \cup \bar{U}$ and $\hat{V} = V \cup \bar{V}$. We define

$$\begin{aligned} \varphi : \mathbb{R}^n - \{0\} &\to \mathbb{S}^n \\ p &\mapsto \frac{p}{||p||}. \end{aligned}$$

We claim that $U = \pi\left(\varphi^{-1}\left(\hat{U}\right)\right)$ and $V = \pi\left(\varphi^{-1}\left(\hat{V}\right)\right)$ are open, disjoint, and contain $[x]$ and $[y]$, respectively. It is clear that $[x] \in U$ and $[y] \in V$. Let $[p] \in U \cap V$, then $[p] = \pi(u) = \pi(v)$ for some $u \in \varphi^{-1}\left(\hat{U}\right)$ and $v \in \varphi^{-1}\left(\hat{V}\right)$. Then $u = \lambda v$ for some $\lambda \in \mathbb{R} - \{0\}$. Hence, $\varphi(u) = \pm\varphi(v)$. However, this implies that $\varphi(u) \in \hat{U} \cap \hat{V}$, which is a contradiction. We conclude that $\mathbb{RP}^n$ is Hausdorff.

  $\boxed{\textbf{Second-countable}}$ Let $\mathcal{B}$ be a countable basis for $\mathbb{R}^n$. We claim that $\pi(\mathcal{B}) = \{\pi(B) \mid B \in \mathcal{B}\}$ is a basis for $\mathbb{RP}^n$. Define

$$\begin{aligned} f_t : \mathbb{R}^n - \{0\} &\to \mathbb{R}^n - \{0\} \\ p &\mapsto tp, \end{aligned}$$

  for every $t \in \mathbb{R}^n - \{0\}$. Note that f_t is continuous and has a continuous inverse $f_{\frac{1}{t}}$. Hence, if U is open, then $f_t(U)$ is also open. We claim that $\pi^{-1}(\pi(U)) = \bigcup_{t \in \mathbb{R}-\{0\}} f_t(U)$ for every open set $U \in \mathbb{R}^n - \{0\}$. Let $p \in \pi^{-1}(\pi(U))$. Then $\pi(p) \in \pi(U)$, implying that there is a $u \in U$ such that u spans the same vector space as p. Hence, $p = \lambda u$, for some non-zero λ, and therefore $p \in \bigcup_{t \in \mathbb{R}-\{0\}} f_t(U)$.

  Conversely, suppose that $p \in \bigcup_{t \in \mathbb{R}-\{0\}} f_t(U)$. Then $p = f_\lambda(u) = \lambda u$, for some $u \in U$ and some non-zero λ. Hence, $\pi(p) = \pi(\lambda u) = \pi(u)$, so $p \in \pi^{-1}(\pi(U))$, proving the claim.

  Showing that $\mathbb{RP}^n$ is second-countable is equivalent to proving that $\pi(\mathcal{B})$ is a basis. Let $[p] \in \pi(B_1) \cap \pi(B_2)$ for two basis sets $B_1, B_2 \in \mathcal{B}$. Then $p \in \pi^{-1}(\pi(B_1)) \cap \pi^{-1}(\pi(B_2))$, which is open by our previous work. Since this set is non-empty, there is a basis set B_3 contained in $\pi^{-1}(\pi(B_1)) \cap \pi^{-1}(\pi(B_2))$. Then $\pi(B_3) \subseteq \pi(B_1) \cap \pi(B_2)$, showing that $\pi(\mathcal{B})$ is a basis.

  Since the condition of locally Euclidean was shown in the textbook, all three conditions are met, and therefore $\mathbb{RP}^n$ is a topological n-manifold.

- $\boxed{\textbf{Compactness}}$ We have to notice that any element $[x] \in \mathbb{RP}^n$, with an arbitrary representation $x \in \mathbb{R}^{n+1}$, has another normalized representation, given by

$$\tilde{x} \equiv \frac{x}{||x||},$$

  since

$$[x] = \pi(x) = \pi\left(\frac{x}{||x||}\right) = [\tilde{x}],$$

which lies on the unit sphere $\mathbb{S}^n$.

Now, consider the restriction $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ of $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$ to the unit sphere $\mathbb{S}^n$. By the previous argument, $\pi|_{\mathbb{S}^n}$ must be surjective, *i.e.* $\pi(\mathbb{S}^n) = \mathbb{RP}^n$.

Since $\mathbb{S}^n$ is compact, and the image of a compact set under a continuous function is compact, then $\mathbb{RP}^n$ is compact.

$\blacksquare$

Problem 2
Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the ***line with two origins***.)

Solution. Let

$$\begin{aligned} \pi : X &\to M \\ x &\mapsto [x] \end{aligned}$$

be the natural projection, sending each point to its equivalence class. This map defines the quotient topology of M. Let p_1 and p_2 denote the upper origin $[(0, 1)]$ and the lower origin $[(0, -1)]$ of M, respectively.
$\boxed{\pi \textbf{ is an open map}}$ A result from general topology states

Lemma 1. A quotient map π is an open map if and only if

$$U \subset X \text{ is open } \implies \pi^{-1}(\pi(U)) \subset X \text{ is open.}$$

This property holds in the case of the line with two origins: fix an open subset $U \subset X$. Then it is easy to see that $\pi^{-1}(\pi(U)) = U \cup U^{\mathrm{R}}$ where U^{R} denotes the reflection of U about the x-axis. Both sets are open in X, making π an open map.
$\boxed{M \textbf{ is second countable}}$ Define an "open interval" in M to be the image of the corresponding open interval in $\mathbb{R}$ under the quotient map π as

$$\begin{aligned} ([a], 0) &\equiv \pi \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| \; a < x < 0 \right\} = \pi \left\{ \begin{pmatrix} x \\ -1 \end{pmatrix} \middle| \; a < x < 0 \right\} \\ (0, [b]) &\equiv \pi \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| \; 0 < x < b \right\} = \pi \left\{ \begin{pmatrix} x \\ -1 \end{pmatrix} \middle| \; 0 < x < b \right\}. \end{aligned}$$

These "open intervals" are open in M since π is an open map. Imitating the basis $\{(-q, q) \mid q \in \mathbb{Q}_+\}$ of the real line, we can define a countable basis $\mathcal{B}$ of M as

$$\mathcal{B} \equiv \left\{ \begin{array}{c} ([-q], 0) \cup \{p_1\} \cup (0, [q]) \\ ([-q], 0) \cup \{p_2\} \cup (0, [q]) \end{array} \middle| \; q \in \mathbb{Q}_+ \right\}$$

It is easy to verify that $\mathcal{B}$ indeed constitutes a basis for M.
$\boxed{M \textbf{ is locally Euclidean}}$ For each point $[p] \in M$, we should find an open neighbourhood $U \subset M$ containing $[p]$ for which there is a homeomorphism $\varphi : U \to \mathbb{R}$.
The trick is to notice that the upper part of X can generate the quotient line with the upper origin p_1 and the lower part of X can generate the quotient line with the lower origin p_2 (i.e., any point $[p] \in M$ is contained in $\pi(\mathbb{R} \times \{1\}) = M - \{p_2\}$ or in $\pi(\mathbb{R} \times \{-1\}) = M - \{p_1\}$). In other words, two large sets already satisfy the desired open neighbourhood condition: $M - \{p_2\}$ and $M - \{p_1\}$. Both sets are open since π is open.
Such separation results in additional properties. By construction, we can find at least one x such that $p \sim (x, 1)$ for all $p \in M - \{p_2\}$ and $p \sim (x, -1)$ for all $p \in M - \{p_1\}$. Conversely, if there is one more x' with $p \sim (x', 1)$ or $p \sim (x', -1)$, respectively, we must necessarily have $x = x'$. This allows us to write our

equivalence classes $[p]$ in either $M - \{p_2\}$ or $M - \{p_1\}$ as $[x]$ for some scalars $x \in \mathbb{R}$ without any problems (which is the first hint that our manifold is one-dimensional). Using this property, we define

$$\begin{aligned} \varphi_1 : M - \{p_2\} &\to \mathbb{R} \\ [x] &\mapsto x \end{aligned}$$

and

$$\begin{aligned} \varphi_2 : M - \{p_1\} &\to \mathbb{R} \\ [x] &\mapsto x. \end{aligned}$$

These functions are

- **(Bijective)** φ_1 is obviously surjective. To show injectivity, suppose that $\varphi_1([x]) = \varphi_1([x'])$ for some $[x], [x'] \in M - \{p_2\}$. Then $x = x'$ and so $[(x, 1)] = [(x', 1)]$. A similar argument works for φ_2.
- **(Continuous)** Let $\widetilde{U}$ be open in $\mathbb{R}$. The $\phi_1^{-1}(U)$ maps to $\pi(U \times \{1\})$ and $\phi_2^{-1}(U)$ to $\pi(U \times \{-1\})$, which are both open since π is an open map.

$\boxed{M \textbf{ is not Hausdorff}}$ We argue that the points $[(0, 1)]$ and $[(0, -1)]$ are not separable in M. Suppose, for the sake of contradiction, that we can find two disjoint, open sets U and U' containing $[(0, 1)]$ and $[(0, -1)]$, respectively. Since π is continuous, the sets $\pi^{-1}(U)$ and $\pi^{-1}(U')$ are two open subsets of X containing $(0, 1)$ and $(0, -1)$, respectively. We can then find $\varepsilon > 0$ small enough so that

$$\begin{aligned} (-\varepsilon, \varepsilon) \times \{1\} &\subseteq \pi^{-1}(U) \\ (-\varepsilon, \varepsilon) \times \{-1\} &\subseteq \pi^{-1}(U'). \end{aligned}$$

In other words, since the image of the preimage is a subset of the original set,

$$\begin{aligned} \pi((-\varepsilon, \varepsilon) \times \{1\}) &\subseteq U \\ \pi((-\varepsilon, \varepsilon) \times \{-1\}) &\subseteq U', \end{aligned}$$

which is a contradiction to the fact that $U \cap U' = \emptyset$.

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Problem 3
Let N denote the ***north pole*** $(0, \ldots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the ***south pole*** $(0, \ldots, 0, -1)$. Define the ***stereographic projection*** $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ by

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \backslash \{S\}$.

(a) For any $x \in \mathbb{S}^n \backslash \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13). Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called ***stereographic projection from the south pole***.)

(b) Show that σ is bijective, and

$$\sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\tilde{\sigma}$ are called ***stereographic coordinates***.)

(d) Show that this smooth structure is the same as the one defined in Example 1.31.

Solution. (a) Fix an arbitrary $x \in \mathbb{S}^n \backslash \{N\}$. A line passing through N and x is given by

$$\begin{aligned} L_{(N,x)} : \mathbb{R} &\to \mathbb{R}^{n+1} \\ t &\mapsto N + t(N - x) \end{aligned}$$

or in vector notation,

$$L_{(N,x)}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + t\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} x^1 \\ \vdots \\ x^n \\ x^{n+1} \end{bmatrix}\right) = \begin{bmatrix} -tx^1 \\ \vdots \\ -tx^n \\ 1 + t\left(1 - x^{n+1}\right) \end{bmatrix}$$

Figure 1: Stereographic projection from the north pole in case of $\mathbb{S}^1$.

Where does L exactly cross the x^{n+1}-axis? This happens at the value t_0 for which the $(n+1)$th component of $L\left(t_0\right)$ is zero. To derive it, we set

$$1 + t_0\left(1 - x^{n+1}\right) \overset{!}{=} 0 \implies t_0 = -\frac{1}{1 - x^{n+1}},$$

which results in

$$L(t_0) = \left[\frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}}\right] = [\sigma(x), 0].$$

Therefore, the intersection point is $(\sigma(x), 0)$.

(b) We compute the formula for $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$. First, we derive a formula for σ^2 by

$$\begin{aligned}
|\sigma(x)|^2 &= \frac{\left(x^1\right)^2 + \ldots + (x^n)^2}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{\left(x^1\right)^2 + \ldots + (x^n)^2 + \left(x^{n+1}\right)^2 - \left(x^{n+1}\right)^2}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{|x| - \left(x^{n+1}\right)^2}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{1 - \left(x^{n+1}\right)^2}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{\left(1 - x^{n+1}\right)\left(1 + x^{n+1}\right)}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{1 + x^{n+1}}{1 - x^{n+1}}.
\end{aligned}$$

Using the above, we compute $\sigma^{-1} \circ \sigma(x)$

$$\begin{aligned}
\sigma^{-1} \circ \sigma(x) &= \sigma^{-1}\left(\frac{\left(x^1, \ldots x^n\right)}{1 - x^{n+1}}\right) \\
&= \frac{\left(2x^1, \ldots, 2x^n, \left(|\sigma(x)|^2 - 1\right)\left(1 - x^{n+1}\right)\right)}{\left(|\sigma(x)|^2 + 1\right)\left(1 - x^{n+1}\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n, \left(\frac{1+x^{n+1}}{1-x^{n+1}} - 1\right)\left(1 - x^{n+1}\right)\right)}{\left(\frac{1+x^{n+1}}{1-x^{n+1}} + 1\right)\left(1 - x^{n+1}\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n, \left(\frac{2x^{n+1}}{1-x^{n+1}}\right)\left(1 - x^{n+1}\right)\right)}{\left(\frac{2}{1-x^{n+1}}\right)\left(1 - x^{n+1}\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n, 2x^{n+1}\right)}{2} \\
&= \left(x^1, \ldots, x^{n+1}\right) \\
&= x.
\end{aligned}$$

Similarly, $\sigma \circ \sigma^{-1}$ is given by

$$\begin{aligned}
\sigma \circ \sigma^{-1}(x) &= \sigma\left(\frac{\left(2x^1, \ldots, 2x^n, |x|^2 - 1\right)}{|x|^2 + 1}\right) \\
&= \frac{\left(2x^1, \ldots, 2x^n\right)}{\left(|x|^2 + 1\right)\left(1 - \frac{|x|^2-1}{|x|^2+1}\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n\right)}{|x|^2 + 1 - \left(|x|^2 - 1\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n\right)}{2} \\
&= \left(x^1, \ldots, x^n\right) \\
&= x.
\end{aligned}$$

Thus, σ is a bijection.

(c) The transition map is given by

$$\begin{aligned}\tilde{\sigma} \circ \sigma^{-1} : \sigma(\mathbb{S}^n \backslash \{N, S\} &\to \tilde{\sigma}(\mathbb{S}^n \backslash \{N, S\}) \\ (u^1, \dots, u^n) &\mapsto \frac{(u^1, \dots, u^n)}{|u|}.\end{aligned}$$

To see why, we compute

$$\begin{aligned}\tilde{\sigma} \circ \sigma^{-1}(u^1, \dots, u^n) &= \tilde{\sigma}\left(\frac{\left(2u^1, \dots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}\right) \\ &= -\sigma\left((-1)\frac{\left(2u^1, \dots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}\right) \\ &= -\sigma\left(\frac{\left(2u^1, \dots, 2u^n, |u|^2 - 1\right)}{-|u|^2 - 1}\right) \\ &= -\frac{\left(2u^1, \dots, 2u^n\right)}{(-|u|^2 - 1)\left(1 - \frac{|u|^2 - 1}{-|u|^2 - 1}\right)} \\ &= -\frac{\left(2u^1, \dots, 2u^n\right)}{(-|u|^2 - 1)\left(1 + \frac{|u|^2 - 1}{|u|^2 + 1}\right)} \\ &= -\frac{\left(2u^1, \dots, 2u^n\right)}{(-|u|^2 - 1)\left(\frac{2|u|^2}{|u|^2 + 1}\right)} \\ &= -\frac{\left(2u^1, \dots, 2u^n\right)}{-2|u|^2} \\ &= \frac{\left(u^1, \dots, u^n\right)}{|u|^2} \\ &= \frac{u}{|u|^2}.\end{aligned}$$

We can skip a similar computation for its inverse by noticing that it is its own inverse

$$\begin{aligned}(\tilde{\sigma} \circ \sigma^{-1}) \circ (\tilde{\sigma} \circ \sigma^{-1}) &= \tilde{\sigma} \circ \sigma^{-1}\left(\frac{u}{|u|^2}\right) \\ &= \frac{\frac{u}{|u|^2}}{\left|\frac{u}{|u|^2}\right|^2} \\ &= \frac{\frac{u}{|u|^2}}{\frac{u^2}{|u|^4}} \\ &= \frac{\frac{u}{|u|^2}}{\frac{1}{|u|^2}} \\ &= u.\end{aligned}$$

In other words, the transition map is a diffeomorphism. Same goes for $\sigma \circ \tilde{\sigma}^{-1}$. Since $\mathbb{S}^n \backslash \{N\}$ and $\mathbb{S}^n \backslash \{S\}$ form an open cover of $\mathbb{S}^n$, we conclude that $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ define a smooth structure on $\mathbb{S}^n$.

(d) The charts from Example 1.31 can be grouped into three categories:

- $\left(U^+_{n+1}, \phi^+_{n+1}\right)$ which contain N.

- $\left(U_{n+1}^{-}, \phi_{n+1}^{-}\right)$ which contain S.
- $\left(U_i^{\pm}, \phi_i^{\pm}\right)$ for $i = 1, \dots, n$ which do not contain N and S.

We argue that each type of chart is smoothly compatible with σ; the case for $\tilde{\sigma}$ is similar. In the first and in the second case, we have

$$\phi_{n+1}^{\pm} \circ \sigma^{-1}\left(u^1, \dots, u^n\right) = \frac{\left(2u^1, \dots, 2u^n\right)}{|u|^2 + 1}$$

Both of these functions are smooth. The inverses are given by

$$\sigma \circ \left(\phi_{n+1}^{\pm}\right)^{-1}\left(u^1, \dots, u^n\right) = \frac{\left(u^1, \dots, u^n\right)}{1 \mp \sqrt{1 - |u|^2}}$$

Thus, $\sigma \circ \left(\phi_{n+1}^{-}\right)^{-1}$ is smooth. Since the domain $\left(\phi_{n+1}^{+}\right)\left(U_{n+1}^{+} \setminus \{N\}\right)$ does not include $\mathbf{0}$, $\sigma \circ \left(\phi_{n+1}^{+}\right)^{-1}$ is smooth.

For the third case, we have

$$\phi_i^{\pm} \circ \sigma^{-1}\left(u^1, \dots, u^n\right) = \frac{\left(2u^1, \dots, \widehat{2u^i}, \dots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}$$

which is smooth, and

$$\sigma \circ \left(\phi_i^{\pm}\right)^{-1}\left(u^1, \dots, u^n\right) = \frac{\left(u^1, \dots, u^{i-1}, \sqrt{1 - |u|^2}, u^i, \dots, u^{n-1}\right)}{1 - u^n}$$

which is also smooth, since $u^n \neq 1$ in $U_i^{\pm}$. In other words, the smooth atlas from Example 1.31 is smoothly compatible with the smooth atlas of stereographic projection.

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Problem 4
Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

Solution. We will show that if M is a smooth m-manifold and N is a smooth n-manifold with boundary, then $M \times N$ is a smooth manifold with boundary, and $\partial(M \times N) = M \times \partial N$.

Using the simpler result that finite products of smooth (boundary-less) manifolds are smooth manifolds, the problem can be reduced to the case where $k = 1$. The general case follows by induction.

First, $M \times N$ is Hausdorff and second-countable, since both M and N are. Given charts (U, ϕ) and (V, ψ) for M and N, respectively, we let $(U \times V, \phi \times \psi)$ be a chart for $M \times N$. The collection of all such charts then gives $M \times N$ the structure of a smooth manifold with boundary, as we now show.

Let $(U_1 \times V_1, \phi_1 \times \psi_1)$ and $(U_2 \times V_2, \phi_2 \times \psi_2)$ be two intersecting charts. Note that $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = \left(\phi_2 \circ \phi_1^{-1}\right) \times \left(\psi_2 \times \psi_1^{-1}\right)$ has domain $(\phi_1 \circ \psi_1)\left([U_1 \times V_1] \cap [U_2 \times V_2]\right) = \phi_1\left(U_1 \cap U_2\right) \times \psi_1\left(V_1 \cap V_2\right)$. Since $\psi_1\left(V_1 \cap V_2\right)$ can be extended to an open set such that $\psi_2 \circ \psi_1^{-1}$ is smooth, we have that the product charts are smoothly compatible. Thus, $M \times N$ is a smooth manifold with boundary.

Let $(x, y) \in \partial(M \times N)$. Then (x, y) is in the domain of some boundary chart $(U \times V, \phi \times \psi)$. Since $\phi(U)$ is open in $\mathbb{R}^m$, it follows that (V, ψ) must be a boundary chart. Moreover, $\psi(y)$ should lie on the boundary of $\mathbb{H}^n$, since otherwise there is an interior chart (V', ψ') whose domain contains y', and thus $(U \times V', \phi \times \psi')$ is an interior chart containing (x, y'). Thus, $y \in \partial N$, so $(x, y) \in M \times \partial N$. Conversely, let $(x, y) \in M \times \partial N$. Then there is a boundary chart (V, ψ) such that $y \in V$ and $\psi(y)$ is in $\partial \mathbb{H}^n$. Then, if u is in the domain of a chart (U, ϕ), then $(\phi \times \psi)(x, y) = (\phi(x), \psi(y)) \in \partial \mathbb{H}^{m+n}$, so $(x, y) \in \partial(M \times N)$. Hence, $\partial(M \times N) = M \times \partial N$. ■