PHYS 662 - Quantum Field Theory I

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Homework 5

Problem 1 - Propagator as Green Function

(a) Show that both the advanced and retarded propagators are Green functions to the Klein-Gordon operator.

(b) Show that the Feynman propagator is a Green function for the Klein-Gordon operator.

Solution. (a) The Klein-Gordon operator is given by

$$\Box + m^2 = \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2.$$

A Green function $G(x)$ satisfies the following relation

$$(\Box + m^2)G(x) = -\delta^4(x).$$

The advanced (G^+) and retarded (G^-) propagators in position space are

$$G^\pm(x) = \mp\frac{\theta(\pm x^0)}{2\pi}\delta(x^2),$$

where $x^2 = (x^0)^2 - \mathbf{x}^2$ is the spacetime interval.

Applying the Klein-Gordon operator to $G^\pm$, we get

$$(\Box + m^2)G^\pm(x) = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\left[\mp\frac{\theta(\pm x^0)}{2\pi}\delta(x^2)\right].$$

Using the chain rule and product rule, we get

- **Time derivative terms:**

$$\frac{\partial^2}{\partial t^2}[\theta(\pm x^0)\delta(x^2)] = \theta(\pm x^0)\frac{\partial^2\delta(x^2)}{\partial t^2} \pm \delta'(x^0)\frac{\partial\delta(x^2)}{\partial t} + \delta(x^0)\frac{\partial^2\delta(x^2)}{\partial t^2}$$

- **Spatial derivative terms:**

$$\nabla^2[\theta(\pm x^0)\delta(x^2)] = \theta(\pm x^0)\nabla^2\delta(x^2).$$

We use the key identity

$$\Box\delta(x^2) = -2\pi\delta^4(x).$$

Combining terms, we have

$$\begin{aligned}\Box[\theta(\pm x^0)\delta(x^2)] &= \theta(\pm x^0)\Box\delta(x^2) \pm \delta'(x^0)\frac{\partial\delta(x^2)}{\partial t} + \delta(x^0)\frac{\partial^2\delta(x^2)}{\partial t^2} \\ &= -2\pi\theta(\pm x^0)\delta^4(x) \pm \text{terms with } \delta'(x^0).\end{aligned}$$

The terms with $\delta'(x^0)$ and its derivatives cancel out due to the properties of distributions. Thus,

$$(\Box + m^2)G^\pm(x) = -\delta^4(x).$$

The m^2 term vanishes when acting on the delta function because $\delta(x^2)$ is supported only on the light cone where $x^2 = 0$.

Therefore, both G^+ and G^- are indeed Green functions of the Klein-Gordon operator.

(b) The Feynman propagator in momentum space is given by

$$\tilde{G}_F(p) = \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon}.$$

To prove it is a Green function, we need to show that

$$(\Box + m^2) G_F(x) = -\delta^4(x).$$

We start from the momentum space representation and transform to position space

$$G_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{i}}{p^2 - m^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}p\cdot x}.$$

Applying the Klein-Gordon operator in position space is equivalent to multiplying by $(p^2 - m^2)$ in momentum space, hence

$$\begin{aligned}
(\Box + m^2) G_F(x) &= \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{i}(p^2 - m^2)}{p^2 - m^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}p\cdot x} \\
&= \int \frac{d^4p}{(2\pi)^4} \mathrm{i} \left(1 - \frac{\mathrm{i}\epsilon}{p^2 - m^2 + \mathrm{i}\epsilon}\right) e^{-\mathrm{i}p\cdot x} \\
&= \int \frac{d^4p}{(2\pi)^4} \mathrm{i} e^{-\mathrm{i}p\cdot x} - \int \frac{d^4p}{(2\pi)^4} \frac{\epsilon}{p^2 - m^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}p\cdot x}.
\end{aligned}$$

The first term yields

$$\int \frac{d^4p}{(2\pi)^4} \mathrm{i} e^{-\mathrm{i}p\cdot x} = -\delta^4(x).$$

For the second term, we can show that as $\epsilon \to 0$

$$\lim_{\epsilon \to 0} \int \frac{d^4p}{(2\pi)^4} \frac{\epsilon}{p^2 - m^2 + \mathrm{i}\epsilon} e^{-\mathrm{i}p\cdot x} = 0.$$

This follows because the integrand is bounded by ϵ multiplied by a function that has a $1/p^2$ falloff and the exponential factor oscillates rapidly for large p.

Thus,

$$(\Box + m^2) G_F(x) = -\delta^4(x).$$

We can also understand this result by noting that the Feynman propagator can be written as

$$G_F(x) = \theta(x^0) G^+(x) + \theta(-x^0) G^-(x),$$

where $G^\pm$ are the advanced and retarded propagators.

Since we've already shown that $G^\pm$ are Green functions

$$(\Box + m^2) G^\pm(x) = -\delta^4(x).$$

Using the properties of the step function, we have

$$\theta(x^0) + \theta(-x^0) = 1$$

It follows that

$$\begin{aligned}
(\Box + m^2) G_F(x) &= (\Box + m^2) \left[\theta(x^0) G^+(x) + \theta(-x^0) G^-(x)\right] \\
&= -\theta(x^0)\delta^4(x) - \theta(-x^0)\delta^4(x) \\
&= -\delta^4(x).
\end{aligned}$$

Therefore, the Feynman propagator is indeed a Green function of the Klein-Gordon operator.

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Problem 2 - Causality Propagator

(a) Calculate the two point function of massive free fields in $d+1$ dimensions (hint: you might need Bessel functions).

(b) Calculate the commutator.

Solution. (a) Consider the action of the field operators on the vacuum states, given by

$$\begin{aligned}
\phi(x)\left|\Omega\right\rangle &= \int \left(\mathrm{e}^{-\mathrm{i}k_\mu x^\mu} a_{\mathbf{k}}\left|\Omega\right\rangle + \mathrm{e}^{\mathrm{i}k_\mu x^\mu} a^\dagger_{\mathbf{k}}\left|\Omega\right\rangle\right)\widetilde{\mathrm{d}k} \\
&= \int \mathrm{e}^{\mathrm{i}k_\mu x^\mu} a^\dagger_{\mathbf{k}}\left|\Omega\right\rangle \widetilde{\mathrm{d}k}, \\
\left\langle\Omega\right|\phi(x) &= \int \left(\left\langle\Omega\right| a_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}k_\mu x^\mu} + \left\langle\Omega\right| a^\dagger_{\mathbf{k}}\mathrm{e}^{\mathrm{i}k_\mu x^\mu}\right)\widetilde{\mathrm{d}k} \\
&= \int \left\langle\Omega\right| a_{\mathbf{k}}\mathrm{e}^{-\mathrm{i}k_\mu x^\mu}\widetilde{\mathrm{d}k}.
\end{aligned}$$

The expectation value is

$$\begin{aligned}
\left\langle\Omega\middle|a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\middle|\Omega\right\rangle &= \left\langle\Omega\middle|\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}\right]\middle|\Omega\right\rangle + \left\langle\Omega\middle|a^\dagger_{\mathbf{k}'}a_{\mathbf{k}}\middle|\Omega\right\rangle \\
&= (2\pi)^d\left(2\omega_{\mathbf{k}'}\right)\delta^d\left(\mathbf{k}-\mathbf{k}'\right).
\end{aligned}$$

The two point function becomes

$$\begin{aligned}
D(x-y) &= \langle\Omega|\phi(x)\phi(y)|\Omega\rangle \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d\left(2\omega_{\mathbf{k}}\right)}\frac{\mathrm{d}^d k'}{(2\pi)^d\left(2\omega_{\mathbf{k}'}\right)}\left\langle\Omega\middle|a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\middle|\Omega\right\rangle \mathrm{e}^{-\mathrm{i}k_\mu x^\mu + \mathrm{i}k'_\mu y^\mu} \\
&= \int \frac{\mathrm{d}^d k\,\mathrm{d}^d k'}{(2\pi)^d\left(2\omega_{\mathbf{k}}\right)}\delta^d\left(\mathbf{k}-\mathbf{k}'\right)\mathrm{e}^{-\mathrm{i}k_\mu x^\mu + \mathrm{i}k'_\mu{}^\mu y^\mu}\Big|_{k^0=\omega_{\mathbf{k}},\,k'^0=\omega_{\mathbf{k}'}} \\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d\left(2\omega_{\mathbf{k}}\right)}\mathrm{e}^{-\mathrm{i}k_\mu\left(x^\mu-y^\mu\right)}\Big|_{k^0=\omega_{\mathbf{k}}}.
\end{aligned}$$

Since two point function is a scalar, it can be further simplified by transforming to other frames. This necessitates us splitting the calculation into two cases, depending on the separation of the events x and y.

- **Time-like Separation:** If the events are time-like separated then we can transform to a frame in which $\mathbf{x}-\mathbf{y}=0$, $x^0-y^0=t$. Then, the two point function becomes

$$\begin{aligned}
D(t) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d(2\omega_{\mathbf{k}})}\mathrm{e}^{-\mathrm{i}\omega_k t} \\
&= \int \frac{k^{d-1}\,\mathrm{d}k\,\mathrm{d}\Omega}{(2\pi)^d\left(2\omega_{\mathbf{k}}\right)}\mathrm{e}^{-\mathrm{i}\omega_k t} \\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_0^\infty \frac{k^{d-1}}{\omega_k}\mathrm{e}^{-\mathrm{i}\omega_k t}\,\mathrm{d}k.
\end{aligned}$$

Performing a change of variables from k to $E\equiv\omega_k$. Since $E^2=k^2+m^2$ and $k\geq 0$, we get

$$k=\sqrt{E^2-m^2},\quad E\,\mathrm{d}E = k\,\mathrm{d}k.$$

Thus, the integral becomes

$$D(t) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_m^\infty \left(E^2 - m^2\right)^{\frac{d-2}{2}} \mathrm{e}^{-\mathrm{i}Et}\,\mathrm{d}E.$$

We can represent this in terms of Hankel functions. Define a variable z as $E = zm$. Then

$$D(t) = \frac{m^{d-1}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)} \int_1^\infty \mathrm{e}^{-\mathrm{i}mtz} \left(z^2 - 1\right)^{\frac{d-2}{2}} \mathrm{d}z.$$

The Hankel function of the first kind has the integral representation

$$H_\nu^{(1)}(-t) = -\frac{2}{\sqrt{\pi}} \frac{\mathrm{i}}{\Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{t}{2}\right)^\nu \int_1^\infty \mathrm{e}^{-itz} \left(z^2 - 1\right)^{\nu - \frac{1}{2}} \mathrm{d}z.$$

Taking $\nu = \frac{d-1}{2}$, the two point function can be written in terms of the Hankel function of the first kind as

$$D(t) = \frac{\mathrm{i}}{4} \left(\frac{m}{2\pi t}\right)^{\frac{d-1}{2}} H^{(1)}_{\frac{d-1}{2}}(-mt).$$

- **Space-like Separation:** If the events are space-like separated, then we can transform to a frame in which $\mathbf{x} - \mathbf{y} = \mathbf{r}$, $x^0 - y^0 = 0$. Then, the two point function becomes

$$D(\mathbf{r}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d \left(2\omega_{\mathbf{k}}\right)} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

For the $3+1$-dimensional case. we transform to a frame where $\mathbf{x} - \mathbf{y} = r\hat{z}$, $x^0 - y^0 = 0$. Then, the two point function becomes

$$D(\mathbf{r}) = \int \frac{k^2\,\mathrm{d}k \sin(\theta)\,\mathrm{d}\theta\,\mathrm{d}\phi}{(2\pi)^3 \left(2\omega_{\mathbf{k}}\right)} \mathrm{e}^{\mathrm{i}kr\cos(\theta)} = \frac{1}{(2\pi)^2} \int_0^\infty \frac{k^2\,\mathrm{d}k}{2\sqrt{k^2 + m^2}} \int_0^\pi \mathrm{d}\theta \sin(\theta) \mathrm{e}^{\mathrm{i}kr\cos(\theta)}.$$

The angular integral can easily be evaluated by the substitution $x = \cos(\theta)$ as

$$\frac{1}{2}\int_0^\pi \mathrm{d}\theta \sin\theta \mathrm{e}^{\mathrm{i}kr\cos(\theta)} = -\frac{1}{2}\int_1^{-1} \mathrm{d}x \mathrm{e}^{\mathrm{i}krx} = \frac{1}{2\mathrm{i}kr}\left(\mathrm{e}^{\mathrm{i}kr} - \mathrm{e}^{-\mathrm{i}kr}\right) = \frac{\sin(kr)}{kr}.$$

Substituting this back into the two point function, we get

$$D(\mathbf{r}) = \frac{1}{(2\pi)^2} \int_0^\infty \frac{k^2\,\mathrm{d}k}{\sqrt{k^2 + m^2}} \frac{\sin(kr)}{kr}.$$

As it turns out, this can be written in terms of Bessel functions. To see this, let us define the variable $z = kr$. Then, we get

$$D(\mathbf{r}) = \frac{1}{(2\pi r)^2} \int_0^\infty \mathrm{d}z \frac{z\sin(z)}{\sqrt{z^2 + m^2 r^2}}.$$

The modified Bessel function of the second kind has the integral representation

$$K_1(x) = -\frac{1}{x}\int_0^\infty \mathrm{d}z \frac{z\sin(z)}{\sqrt{z^2 + x^2}}.$$

Then, the two point function can be written in terms of modified Bessel functions of the second kind as

$$D(\mathbf{r}) = -\frac{m}{4\pi^2 r} K_1(mr).$$

(b) Let us evaluate the product of two fields by a mode expansion

$$\begin{aligned}\phi(x)\phi(y) &= \int \widetilde{\mathrm{d}k}\widetilde{\mathrm{d}k'}\left(\mathrm{e}^{-\mathrm{i}k_\mu x^\mu}a_{\mathbf{k}} + \mathrm{e}^{\mathrm{i}k_\mu x^\mu}a^\dagger_{\mathbf{k}}\right)\left(\mathrm{e}^{-\mathrm{i}k'_\mu y^\mu}a_{\mathbf{k}'} + \mathrm{e}^{\mathrm{i}k'^\mu y^\mu y^\mu}a^\dagger_{\mathbf{k}'}\right)\\
&= \int \widetilde{\mathrm{d}k}\widetilde{\mathrm{d}k'}\left(\mathrm{e}^{-\mathrm{i}\left(k_\mu x^\mu + k'_\mu y^\mu\right)}a_{\mathbf{k}}a_{\mathbf{k}'} + \mathrm{e}^{-\mathrm{i}\left(k_\mu x^\mu - k'_\mu y^\mu\right)}a_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\right)\\
&= \int \widetilde{\mathrm{d}k}\widetilde{\mathrm{d}k'}\left(\mathrm{e}^{\mathrm{i}\left(k_\mu x^\mu - k'_\mu y^\mu\right)}a^\dagger_{\mathbf{k}}a_{\mathbf{k}'} + \mathrm{e}^{\mathrm{i}\left(k_\mu x^\mu + k'_\mu y^\mu\right)}a^\dagger_{\mathbf{k}}a^\dagger_{\mathbf{k}'}\right).\end{aligned}$$

Then, the commutator becomes

$$\begin{aligned}[\phi(x),\phi(y)] &= \int \widetilde{\mathrm{d}k}\widetilde{\mathrm{d}k'}\left(\mathrm{e}^{-\mathrm{i}\left(k_\mu x^\mu - k'_\mu y^\mu\right)}\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}\right] - \mathrm{e}^{\mathrm{i}\left(k_\mu x^\mu - k'_\mu y^\mu\right)}\left[a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}\right]\right)\\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})}\frac{\mathrm{d}^d k'}{(2\pi)^d\,(2\omega_{\mathbf{k}'})}\left(\mathrm{e}^{-i\left(k_\mu x^\mu - k'_\mu y^\mu\right)}\left[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}\right] - \mathrm{e}^{i\left(k_\mu x^\mu - k'_\mu y^\mu\right)}\left[a_{\mathbf{k}'}, a^\dagger_{\mathbf{k}}\right]\right)\\
&= \int \frac{\mathrm{d}^d k\,\mathrm{d}^d k'}{(2\pi)^d\,(2\omega_{\mathbf{k}})}\delta^d\left(\mathbf{k}-\mathbf{k}'\right)\left(\mathrm{e}^{-i\left(k_\mu x^\mu - k_{\mu\mu} y^\mu\right)} - \mathrm{e}^{i\left(k_\mu x^\mu - k'_\mu y^\mu\right)}\right)\Bigg|_{k^0=\omega_{\mathbf{k}},k'^0=\omega_{\mathbf{k}'}}\\
&= \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})}\left(\mathrm{e}^{-\mathrm{i}k_\mu\left(x^\mu - y^\mu\right)} - \mathrm{e}^{\mathrm{i}k_\mu\left(x^\mu - y^\mu\right)}\right)\Bigg|_{k^0=\omega_{\mathbf{k}}}\\
&= D(x-y) - D(y-x).\end{aligned}$$

Again, just like the two point function, this can be further simplified depending on the separation between the events. For space-like separation, going to a frame in which $\mathbf{x}-\mathbf{y}=\mathbf{r}$, $x^0-y^0=0$ gives

$$[\phi(x),\phi(y)] = D(\mathbf{r}) - D(-\mathbf{r}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} - \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

But, if we perform the transformation $k \to -k$ on the second term, we get

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})}\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} = \int \frac{\mathrm{d}^d k}{(2\pi)^d\,(2\omega_{\mathbf{k}})}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}.$$

Thus,

$$[\phi(x),\phi(y)] = 0.$$

Hence, for space-like separation, the commutator exactly equals zero. Similarly, for time-like separation, we transform to a frame in which $\mathbf{x}-\mathbf{y}=0$, $x^0-y^0=t$ to get

$$\begin{aligned}[\phi(x),\phi(y)] &= D(t) - D(-t)\\
&= \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_m^\infty \left(E^2-m^2\right)^{\frac{d-2}{2}}\left(\mathrm{e}^{-\mathrm{i}Et} - \mathrm{e}^{\mathrm{i}Et}\right)\mathrm{d}E\\
&= -\frac{2\mathrm{i}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int_m^\infty \left(E^2-m^2\right)^{\frac{d-2}{2}}\sin(Et)\,\mathrm{d}E\\
&= \frac{\mathrm{i}}{4}\left(\frac{m}{2\pi t}\right)^{\frac{d-1}{2}}\left(H^{(1)}_{\frac{d-1}{2}}(-mt) - (-1)^{\frac{d-1}{2}}H^{(1)}_{\frac{d-1}{2}}(mt)\right).\end{aligned}$$

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Problem 3 - Charge Conjugation
Derive the action of the charge conjugation operator on a Dirac spinor.

Solution. Consider a Dirac spinor

$$\psi = \begin{pmatrix} \chi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{pmatrix}.$$

As the components transform under charge conjugation by

$$C\chi_\alpha C^{-1} = \xi_\alpha, \quad C\xi_\alpha C^{-1} = \chi_\alpha,$$

the Dirac spinor transforms under charge conjugation by

$$\psi^C = C\psi C^{-1} = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}.$$

This form of ψ^C suggests that there should be a way to represent ψ^C in terms of ψ. Let us try to see if that is actually the case. ψ to ψ^C shows that we need to transfer the adjoint from ξ to χ in ψ. This suggests that the adjoint of ψ should play a role. If we compute it, we get

$$\psi^\dagger = \begin{pmatrix} \chi^\dagger_{\dot{\alpha}} & \xi^\alpha \end{pmatrix}.$$

While this has the adjoint in the right terms, the positions of ξ and χ must be swapped. This can be achieved, if we remember than the γ^0 matrix can be written, in spinor indices, as

$$\gamma^0 = \begin{pmatrix} 0 & \delta^{\dot{\beta}}_{\dot{\alpha}} \\ \delta^\alpha_\beta & 0 \end{pmatrix}.$$

Then, if we multiply γ^0 to $\psi^\dagger$, we get

$$\bar{\psi} = \psi^\dagger\gamma^0 = \begin{pmatrix} \xi^\alpha & \chi^\dagger_{\dot{\alpha}} \end{pmatrix}.$$

This fixes the position, but the vector obtained yet is not in the right form to be equated to ψ^C. What we need is its transpose

$$\bar{\psi}^T = \begin{pmatrix} \xi^\alpha \\ \chi^\dagger_{\dot{\alpha}} \end{pmatrix}.$$

This, in turn, is almost equal to ψ^C, but not exactly. The indices in this expression are in the wrong locations. To fix this, let us use the charge conjugation matrix

$$\mathcal{C} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$

Then, if we multiply the charge conjugation matrix to $\bar{\psi}^T$, we get

$$\mathcal{C}\bar{\psi}^T = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix} = \psi^C.$$

Thus, the charge conjugation operation is represented by

$$\psi^C = C\psi C^{-1} = \mathcal{C}\bar{\psi}^T.$$

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Problem 4 - Gordon Identities
Derive the identities

$$\bar{u}_{s'}\left(\vec{p}'\right)\left((p'+p)^\mu - 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 u_s(\vec{p}) = 0,$$
$$\bar{v}_{s'}\left(\vec{p}'\right)\left((p'+p)^\mu - 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 v_s(\vec{p}) = 0.$$

Solution. We have

$$\begin{aligned}
p^\mu + 2iS^{\mu\nu}p_\nu &= \frac{1}{2}\left(2\eta^{\mu\nu}\right)p_\nu - \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]p_\nu \\
&= \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\} - [\gamma^\mu, \gamma^\nu]\right)p_\nu \\
&= \not{p}\gamma^\mu \\
p^\mu - 2iS^{\mu\nu}p_\nu &= \frac{1}{2}\left(2\eta^{\mu\nu}\right)p_\nu + \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]p_\nu \\
&= \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]\right)p_\nu \\
&= \gamma^\mu \not{p}
\end{aligned}$$

The contents in the brackets of the Gordon identities can be simplified to give

$$\begin{aligned}
(p'+p)^\mu + 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu &= p'^\mu + 2\mathrm{i}S^{\mu\nu}p'_\nu + p^\mu - 2\mathrm{i}S^{\mu\nu}p_\nu \\
&= \not{p}''\gamma^\mu + \gamma^\mu\not{p} \\
\implies \left((p'+p)^\mu + 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 &= \left(\not{p}''\gamma^\mu + \gamma^\mu\not{p}\right)\gamma_5 \\
&= \not{p}''\gamma^\mu\gamma_5 - \gamma^\mu\gamma_5\not{p}
\end{aligned}$$

where the last equality follows because γ_5 anti-commutes with γ^μ. Using this and the equations of motion for u_s and v_s, we have

$$\begin{aligned}
\bar{u}_s(\mathbf{p})(\not{p}+m) = 0, &\quad (\not{p}+m)u_s(\mathbf{p}) = 0 \\
\bar{v}_s(\mathbf{p})(\not{p}-m) = 0, &\quad (\not{p}-m)v_s(\mathbf{p}) = 0
\end{aligned}$$

Deriving the Gordon identities, starting with the u_s case, we have

$$\begin{aligned}
\bar{u}_{s'}\left(\mathbf{p}'\right)\left((p'+p)^\mu + 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 u_s(\mathbf{p}) &= \bar{u}_{s'}\left(\mathbf{p}'\right)\left(\not{p}'\gamma^\mu\gamma_5 - \gamma^\mu\gamma_5\not{p}\right)u_s(\mathbf{p}) \\
&= \bar{u}_{s'}\left(\mathbf{p}'\right)\left(-m\gamma^\mu\gamma_5 + m\gamma^\mu\gamma_5\right)u_s(\mathbf{p}) \\
&= 0,
\end{aligned}$$

where the second equality followed from the equations of motion. The v_s case follows similarly

$$\begin{aligned}
\bar{v}_{s'}\left(\mathbf{p}'\right)\left((p'+p)^\mu + 2\mathrm{i}S^{\mu\nu}\left(p'-p\right)_\nu\right)\gamma_5 v_s(\mathbf{p}) &= \bar{v}_{s'}\left(\mathbf{p}'\right)\left(\not{p}'\gamma^\mu\gamma_5 - \gamma^\mu\gamma_5\not{p}\right)v_s(\mathbf{p}) \\
&= \bar{u}_{s'}\left(\mathbf{p}'\right)\left(m\gamma^\mu\gamma_5 - m\gamma^\mu\gamma_5\right)v_s(\mathbf{p}) \\
&= 0.
\end{aligned}$$

$\blacksquare$