MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*
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Homework 5

Problem 5-6
Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M, v \in T_xM, |v| = 1\}.$$

It is called the **unit tangent bundle of M**. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$. (Used on p. 147.)

Solution. Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, then TM is an embedded $(2m)$-dimensional submanifold of $T\mathbb{R}^n$. Consider the smooth map

$$\begin{aligned} \Phi : TM &\to \mathbb{R}, \\ v &\mapsto |v|. \end{aligned}$$

Notice that the differential of Φ in the level set $\Phi^{-1}(1)$ has at least rank 1 since there is no element in the level set that maps to the origin. Additionally, the dimension of the codomain of Φ is $\dim(\mathbb{R}) = 1$, so the dimension of the level set $\Phi^{-1}(1)$ is $\dim(M) - \dim(\mathbb{R}) = 2m - 1$. Thus, the differential $\mathrm{d}\Phi$ is surjective for all points $p \in M$, and $\Phi^{-1}(1)$ is then a regular level set.
Then, by the regular level set theorem, every regular level set of a smooth map is a closed embedded submanifold whose codimension is equal to the dimension of the range, which is one in this case.
Consider now the unit tangent bundle UM. Notice that UM is the level set $\Phi^{-1}(1)$, which we showed to be an embedded $(2m-1)$-dimensional submanifold of TM. Therefore, UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n$. ■

Problem 5-7
Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x, y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

Solution. Both the domain and codomain are smooth manifolds, and the function F is a smooth map since it a polynomial. Given $F(x, y) = x^3 + xy + y^3$, we can compute the differential of F, given by

$$\begin{aligned} \mathrm{d}F(x,y) &= \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 3x^2 + y & x + 3y^2 \end{pmatrix}. \end{aligned}$$

The constant rank level set theorem guarantees that each level set of F is a closed embedded submanifold of codimension equal to the constant rank in M. This will not necessarily be true when the domain contains a point where $\mathrm{d}F$ has rank 0, *i.e.* where both partial derivatives are zero.
Then, solving for these points, we have

$$\begin{aligned} \mathrm{d}F(x,y) = \begin{pmatrix} 0 & 0 \end{pmatrix} &\implies \begin{cases} 3x^2 + y = 0 & \implies y = -3x^2 \\ x + 3y^2 = 0 & \implies x + 3\left(-3x^2\right)^2 = x + 27x^4 = 0 \end{cases} \\ &\implies x(27x^3 + 1) = 0 \\ &\implies x = 0 \quad \text{or} \quad x = -\frac{1}{3} \\ &\implies (x,y) = (0,0) \quad \text{or} \quad (x,y) = \left(-\frac{1}{3}, -\frac{1}{3}\right). \end{aligned}$$

Plugging in these values in F, we have

$$F(0,0) = 0 \quad \text{and} \quad F\left(-\frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{27}.$$

Then, unless F is equal to one of those two preceding values, the level set will always be an embedded submanifold. Thus, for all $c \notin \{0, \frac{1}{27}\}$, the level set $F^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.
We will now check whether or not the level sets are embedded submanifolds at the critical points.

- **For** $(x, y) = (0, 0)$:

  The level set $F^{-1}(0)$ is the folium of Descartes. A sketch of what that looks like is shown in the figures below.

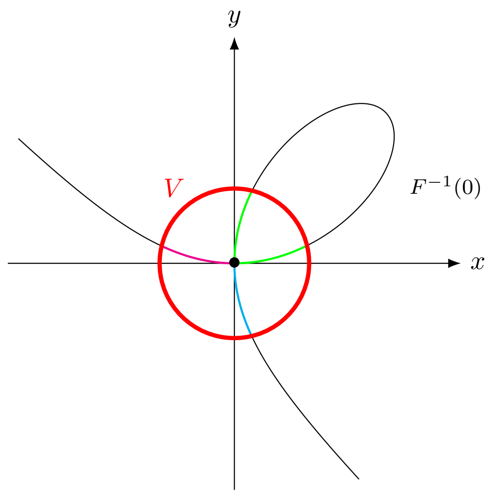

Figure 1: Connected components after removing the point $(0, 0)$.

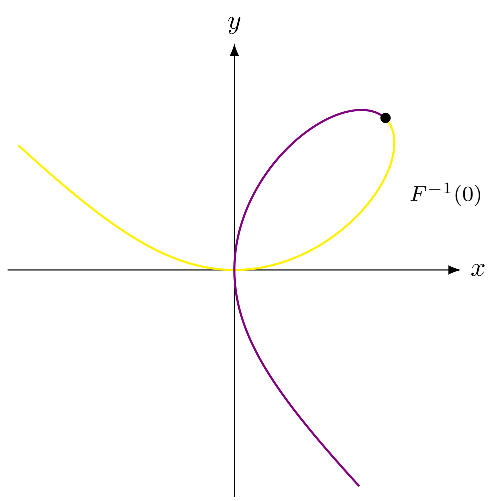

Figure 2: Connected components after removing any point $(x, y) \neq (0, 0)$.

We first note that $(0,0)$ is a non-degenerate critical point of F. Notice that $F^{-1}(0)$ does not have the discrete topology, then it cannot be a zero-dimensional embedded submanifold. Additionally, notice that $F^{-1}(0) \setminus (0,0)$ has three connected components, but $F^{-1}(0) \setminus (x,y)$, for any $(x,y) \neq (0,0)$, has only two connected components, then $F^{-1}(0)$ cannot be homeomorphic to $\mathbb{R}$, so it cannot be a one-dimensional embedded submanifold. Finally, $F^{-1}(0)$ is closed and not equal to $\mathbb{R}^2$, so it cannot be a two-dimensional embedded submanifold.

Thus, $F^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$.

- **For** $(x,y) = \left(-\frac{1}{3}, -\frac{1}{3}\right)$:

The level set $F^{-1}\left(\frac{1}{27}\right)$ can be factorized and written as

$$
\begin{aligned}
F(x,y) &= \frac{1}{27} \\
x^3 + y^3 + xy - \frac{1}{27} &= 0 \\
x^3 + y^3 + 3xy(x+y) - 3xy(x+y) + xy - \left(\frac{1}{3}\right)^3 &= 0 \\
(x+y)^3 - 3xy\left(x + y - \frac{1}{3}\right) - \left(\frac{1}{3}\right)^3 &= 0 \\
(x+y)^3 - \left(\frac{1}{3}\right)^3 - 3xy\left(x + y - \frac{1}{3}\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left((x+y)^2 + \frac{1}{3}(x+y) + \frac{1}{9}\right) - 3xy\left(x + y - \frac{1}{3}\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left((x+y)^2 + \frac{1}{3}(x+y) + \frac{1}{9} - 3xy\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left(y^2 + 2xy + x^2 + \frac{1}{3}x + \frac{1}{3}y + \frac{1}{9} - 3xy\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left(y^2 - xy + \frac{1}{3}y + x^2 + \frac{1}{3} + \frac{1}{9}\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left(y^2 - \left(x - \frac{1}{3}\right)y + \left(x^2 + \frac{1}{3}x + \frac{1}{9}\right)\right) &= 0 \\
\implies x + y - \frac{1}{3} = 0 \qquad \text{or} \qquad y^2 - \left(x - \frac{1}{3}\right)y + \left(x^2 + \frac{1}{3}x + \frac{1}{9}\right) &= 0.
\end{aligned}
$$

The first condition is simple that the points (x,y) belong to the straight line $y = -x + \frac{1}{3}$. To find out what the second condition refers to visually, we solve the quadratic equation for y in terms of x.

$$
\begin{aligned}
y_{\pm} &= \frac{\left(x - \frac{1}{3}\right) \pm \sqrt{\left(x - \frac{1}{3}\right)^2 - 4(1)\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right)}}{2(1)} \\
&= \frac{\left(x - \frac{1}{3}\right) \pm \sqrt{x^2 - \frac{2}{3}x + \frac{1}{9} - 4x^2 - \frac{4}{3}x - \frac{4}{9}}}{2} \\
&= \frac{\left(x - \frac{1}{3}\right) \pm \sqrt{-3x^2 - 2x - \frac{1}{3}}}{2},
\end{aligned}
$$

where we require real roots, which implies that

$$
\begin{aligned}
-3x^2 - 2x - \frac{1}{3} &\geq 0 \\
x^2 + \frac{2}{3}x + \frac{1}{9} &\leq 0 \\
\left(x + \frac{1}{3}\right)^2 &\leq 0,
\end{aligned}
$$

which has a discriminant equal to zero, and hence, x is a double root at $x = -\frac{1}{3}$. Replacing in the solution for y, we also get that the discriminant is null and y is also is a double root of value $y = -\frac{1}{3}$. Thus, $F^{-1}\left(\frac{1}{27}\right)$ is the embedding of the disjoint union between a zero-dimensional submanifold and a one-dimensional submanifold, but these manifolds must have the same dimension everywhere, and thus, $F^{-1}\left(\frac{1}{27}\right)$ is not an embedded submanifold of $\mathbb{R}^2$.

■

Problem 5-19
Suppose $S \subseteq M$ is an embedded submanifold and $\gamma : J \to M$ is a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if S is not embedded.

Solution. Let M be m-dimensional and S be k-dimensional, with $k \leq m$. Since S is an embedded submanifold of M, then, for every point $p \in S$, there exists an open neighborhood U in M centered at p such that $\varphi : U \cap S \to \mathbb{R}^k$ is a smooth embedding.
The differential of γ at t, $\gamma'(t)$, is an element of the tangent space $T_{\gamma(t)}M$. To show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$, for all $t \in J$, we need to show that $\gamma'(t)$ lies in the image of the differential $\mathrm{d}\varphi(p)$ of the embedding φ at the point $p = \gamma(t) \in S$.
Consider a slice chart (U, φ) as defined above. Since $\gamma(t) \in S$, then it is also in U. Let $p = \gamma(t)$ and consider the composition map $\varphi \circ \gamma : J \to \mathbb{R}^k$. By the chain rule, the differential of the composition map is

$$
\mathrm{d}(\varphi \circ \gamma)(t) = \mathrm{d}\varphi(p)\gamma'(t).
$$

Since φ is an embedding, its differential is an injective linear map from T_pM to $\mathbb{R}^k$. Since $\gamma(t)$ is in S, then $\varphi \circ \gamma(t)$ is a constant function of t, and its derivative with respect to t is null. Thus, $\mathrm{d}\varphi(p)\gamma'(t)$ is the kernel of the linear map, and is therefore a subspace of $T_{\gamma(t)}S$.
A counterexample to this, given that S is not embedded, is by letting the curve to be

$$
\begin{aligned}
\gamma : \mathbb{R} &\to \mathbb{R}^2, \\
t &\mapsto (t^2, t^3).
\end{aligned}
$$

and S to be the x-axis. Then γ lies entirely in S, but at $t = 0$, the differential $\gamma'(0) = (0, 0)$ is not in the subspace $T_{\gamma(0)}S = \mathrm{span}\{(1, 0)\}$, because S is not an embedded submanifold of $\mathbb{R}^2$ at the origin since it is not locally diffeomorphic to a Euclidean space in any neighborhood of the origin. ■