

PHYS 617 - Statistical Mechanics

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Spring 2024
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Homework 1

Problem 1

Estimate the number of electrons in the sun.

Note: When I ask you to “estimate” a number, I want zero significant digits. Your answer should just be 10 to some power (and you get full credit even if you’re one power off). Accordingly, your derivation should avoid any complexity at a higher level of detail than the order-of-magnitude estimate you are looking for.

Solution. The mass of the Sun $m_{\odot} \approx 10^{30}$ kg and the mass of a proton $m_p \approx 10^{-27}$ kg. Since the Sun is primarily composed of hydrogen (1 proton, 1 electron), and since the mass of a proton is much larger than that of an electron, then the number of electrons in the Sun can be estimated as

$$\# e^- \in \text{Sun} \approx \frac{m_{\odot}}{m_p} = \frac{10^{30}}{10^{-27}} = 10^{57} \text{ kg.}$$

■

Problem 2

Show that the diffusion equation

$$\dot{u} - \nu u'' = 0$$

is a conservation law. Define the total “charge” between two points a and b :

$$Q \equiv \int_a^b u \, dx$$

and show that the rate of change of Q in time is determined only by the flux of charge through the points a and b . What is this flux (rate of flow of u) and can it be interpreted physically?

Solution. We have defined

$$Q = \int_a^b u(x, t) \, dx.$$

Calculating the rate of change of Q in time, we have

$$\begin{aligned} \frac{dQ}{dt} &= \frac{d}{dt} \left(\int_a^b u(x, t) \, dx \right) \\ &= \int_a^b \frac{\partial u(x, t)}{\partial t} \, dx \\ &= \nu \int_a^b \frac{\partial^2 u(x, t)}{\partial x^2} \, dx \\ &= \nu \left. \frac{\partial u(x, t)}{\partial x} \right|_a^b \\ &= \nu \left[\frac{\partial u(b, t)}{\partial x} - \frac{\partial u(a, t)}{\partial x} \right] \\ &= \nu [u'(b, t) - u'(a, t)]. \end{aligned}$$

The flux of charge through the boundaries at a and b can be defined as

$$J_a \equiv \nu \frac{\partial u(a, t)}{\partial x}, \quad J_b \equiv \nu \frac{\partial u(b, t)}{\partial x}.$$

Thus, we have

$$\frac{dQ}{dt} = J_b - J_a.$$

The flux is the rate of flow of u and can be interpreted physically as the rate of flow of the material through the boundaries of the system. The flux is positive when the material is flowing into the system and negative when the material is flowing out of the system. Thus, the rate of change of Q is determined only by the flux of charge through the boundaries. ■

Problem 3

Estimate how long it takes a photon to escape the sun. Assume a photon mean free path given by λ and the radius of the sun is given by R .

Note: when I ask you to “estimate” a formula (rather than a number) the result should not include any dimensionless coefficients like π or 2. I am only looking for how the answer scales as a function of the different variables in the system.

Solution. The problem of finding out the time it takes for a photon to escape the sun (from the center) is equivalent to a random walk of that photon. In other words, the time τ taken by the photon to reach the surface is the time t it takes in each step multiplied by the number of those steps N taken, given by

$$\tau = tN.$$

The time t for each step is given by

$$t = \frac{\lambda}{c}$$

and the number of steps N taken is given by the linear distance, which is the radius of the Sun R , covered by random walk, given by

$$N = \frac{R^2}{\lambda^2}.$$

Putting everything together, we get

$$\tau = tN = \frac{\lambda}{c} \frac{R^2}{\lambda^2} = \frac{R^2}{\lambda c}.$$

■

Problem 4

Solve the diffusion equation for an initial condition given by a single Fourier mode:

$$\rho(x, 0) = e^{ikx}$$

What is $\rho(x, t)$?

Solution. The diffusion equation is given by

$$\dot{\rho}(x, t) - \nu \rho''(x, t) = 0.$$

Let

$$r(k, t) = \mathcal{F}[\rho(x, t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \rho(x, t) e^{ikx} dx$$

be the Fourier transform of $\rho(x, t)$. Then, our PDE becomes

$$\dot{r}(k, t) + \nu k^2 r(k, t) = 0.$$

The solution to the ODE above is given by

$$r(k, t) = r(k, 0) e^{-\nu k^2 t}$$

where

$$r(k, 0) = \mathcal{F}[\rho(x, 0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \rho(x, 0) e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \left. \frac{e^{2ikx}}{2ik} \right|_{-\infty}^0 = \frac{1}{2ik\sqrt{2\pi}}$$

We now have

$$r(k, t) = \frac{1}{2ik\sqrt{2\pi}} e^{-\nu k^2 t}.$$

Taking the inverse Fourier transform, we get

$$\begin{aligned} \rho(x, t) &= [r(k, t)]^{-T} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r(k, t) e^{-ikx} dk \\ &= \frac{1}{4ik\pi} \int_{-\infty}^{\infty} e^{-\nu k^2 t} e^{-ikx} dk. \end{aligned}$$

Using the following property

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}} e^c$$

with $a = \nu t$, $b = -ix$, and $c = 0$, we get

$$\rho(x, t) = \frac{1}{4ik\pi} \left(\sqrt{\frac{\pi}{\nu t}} e^{\frac{-x^2}{4\nu t}} \right) = \frac{1}{2ik\sqrt{4\pi\nu t}} e^{\frac{-x^2}{4\nu t}}.$$

■

Problem 5

Derive the next-order correction to Stirling's approximation:

$$\ln(N!) = N \ln(N) - N + f(N)$$

where $f(N)$ is the correction. Do this by finding a relationship between $f(N)$ and $f(N+1)$ and setting that (approximately) to

$$f(N+1) - f(N) \approx \frac{df}{dN}$$

Make a plot of the ratio of $N!$ to its approximation and show that it asymptotes to a constant:

$$\frac{N!}{\text{Appx.}} = e^{\ln(N!) - N \ln(N) + N - f(N)}$$

Make a plot of the above formula for the $f(N)$ you computed and $f(N) = 0$ with no correction, to show the difference. I highly recommend keeping it in the exponential form, as $N!$ and N^N are both very large and will not be representable as standard floating-point numbers for $N > \sim 100$; so take the difference $\ln(N!) - N \ln(N)$ first before exponentiating.

Depending on the detail of your calculation, your ratio should asymptote to a constant which may not be equal to unity. Double-check the formula by looking up Stirling's Approximation on google (assuming you haven't already) and check that the constant term you get is consistent with the overall dimensionless constant in the most detailed formula you can find on the internet.

Solution. Computing the derivative's approximation

$$\begin{aligned} \frac{df}{dN} &\approx f(N+1) - f(N) \\ &= \ln((N+1)!) - (N+1) \ln(N+1) + (N+1) - \ln(N!) + N \ln(N) - N \\ &= \ln\left(\frac{(N+1)!}{N!}\right) + \ln\left(\frac{N^N}{(N+1)^{N+1}}\right) + 1 \\ &= \ln(N+1) + \ln\left(\frac{N^N}{(N+1)^{N+1}}\right) + 1 \\ &= N \ln\left(\frac{N}{N+1}\right) + 1 \end{aligned}$$

Taking the integral, we get

$$\begin{aligned} f(N) &= \int (N+1) \ln\left(\frac{N}{N+1}\right) + 1 \, dN \\ &= \frac{\ln(|N+1|) + N^2 \ln\left(\frac{N}{N+1}\right) + N}{2} \end{aligned}$$

Plotting the two functions, with the computed $f(N)$ and with $f(N) = 0$, we get

```
import matplotlib.pyplot as plt
import numpy as np
from scipy.special import factorial

N = np.arange(0, 170, 1)
f = ( np.log( np.abs(N+1)) + N**2 * np.log( N / (N+1) ) + N ) /2

asympt = np.exp( np.log(factorial(N)) - N*np.log(N) + N - f)
asympt_0 = np.exp(np.log(factorial(N)) - N*np.log(N) + N)
```

```
plt.plot(N, asymp, 'b', N, asymp_value, 'r--')  
plt.plot(N, asymp_0, 'g')  
plt.show()
```

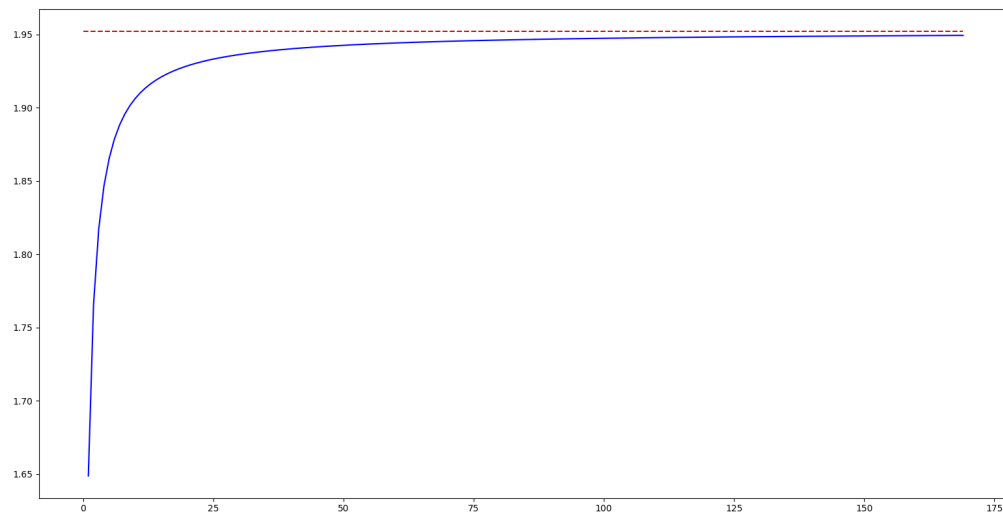
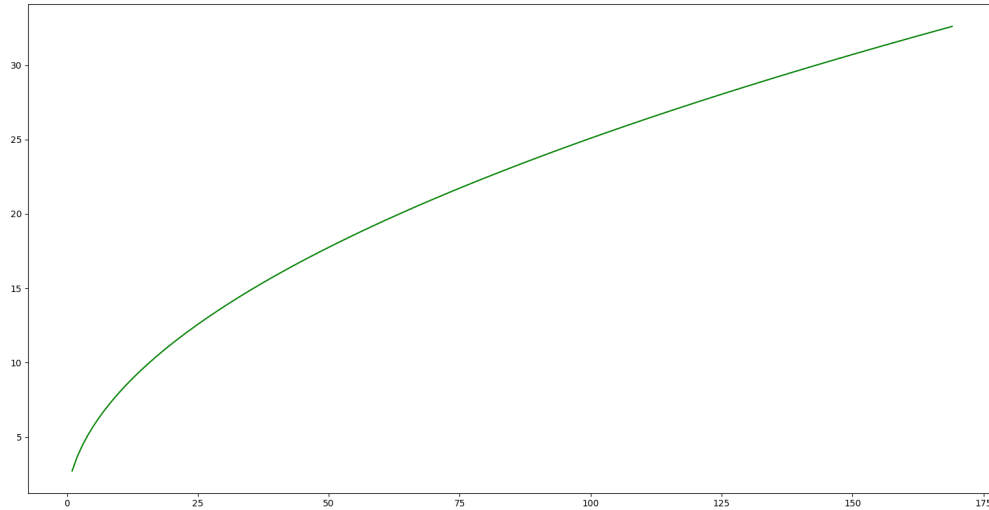


Figure 1: Plot of the ratio with the computed $f(N)$

■

Figure 2: Plot of the ratio with $f(N) = 0$

Homework 2

Problem 1

Derive the multiplicity factor $\Omega(E, V)$ and the equation of state for a relativistic gas of particles. Recall the formula relating energy and momentum

$$\epsilon^2 = p^2 c^2 + m^2 c^4$$

and assume the system is ultra-relativistic, so that $p^2 c^2 \gg m^2 c^4$.

What is the total energy E of the system as a function of the temperature T ?

What is the pressure P as a function of the temperature?

What is the adiabatic index γ describing this system? (i.e. $P = Kn^\gamma$ for adiabatic expansion or compression, where n is the number density).

Solution. Assume our system has N particles in 3 dimensions. We know the description for that system is equivalent to $3N$ particles in 1 dimension or 1 particle in $3N$ dimensions.

Since the system is ultra-relativistic, then $p^2 c^2 \gg m^2 c^4 \implies \epsilon = pc \implies p = \frac{\epsilon}{c}$. Additionally, since this is a system of $3N$ particles in 1 dimension, each with energy ϵ_i , then the total energy of the system E is given by

$$E = \sum_{i=1}^{3N} \epsilon_i = c \sum_{i=1}^{3N} p_i.$$

The possible positions is given by the hypervolume of the position space, which is $\frac{V^N}{N!}$. The possible momenta is given by the hypervolume of the momentum space, which is $\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2}-1)!} R^{3N-1}$, where $R = \sum_i p_i = \frac{E}{c}$.

Then, the multiplicity factor is

$$\begin{aligned}
 \Omega(E, V) &= \frac{(\# \text{ possible positions})(\# \text{ possible momenta})}{h^{3N}} \\
 &= \frac{(\text{area of position space})(\text{area of momentum space})}{h^{3N}} \\
 &= \frac{\left(\frac{V^N}{N!}\right) \left(\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2}-1)!} R^{3N-1}\right)}{h^{3N}} \\
 &= \frac{\left(\frac{V^N}{N!}\right) \left(\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2}-1)!} \left(\frac{E}{c}\right)^{3N-1}\right)}{h^{3N}} \\
 &\sim \frac{\left(\frac{V}{N}\right)^N \left(2 \left(\frac{2\pi}{3N}\right)^{\frac{3N}{2}} \left(\frac{E}{c}\right)^{3N}\right)}{h^{3N}} \quad (\text{since } N \gg 1) \\
 &\sim \frac{\left(\frac{V}{N}\right)^N \left(\frac{E}{c\sqrt{N}}\right)^{3N}}{h^{3N}},
 \end{aligned}$$

and

$$\ln(\Omega) \sim N \ln\left(\frac{V}{N}\right) + 3N \ln\left(\frac{E}{c\sqrt{N}}\right) - 3N \ln(h).$$

To derive an equation for the total energy E of the system as a function of temperature T , we first find

$$\begin{aligned}
 \beta &= \frac{1}{k_B T} = \frac{\partial \ln(\Omega)}{\partial E} \\
 &= \frac{\partial}{\partial E} \left[N \ln\left(\frac{V}{N}\right) + 3N \ln\left(\frac{E}{c\sqrt{N}}\right) - 3N \ln(h) \right] \\
 &= \frac{3N}{E},
 \end{aligned}$$

$$\implies E = 3N k_B T.$$

In other words, the average energy per particle is $3k_B T$, which is significantly greater than the classical average energy per particle which is $k_B T$.

To derive an equation for the pressure as a function of temperature T , we first find

$$\begin{aligned}
 \beta P &= \frac{\partial \ln(\Omega)}{\partial V} \\
 &= \frac{\partial}{\partial V} \left[N \ln\left(\frac{V}{N}\right) + 3N \ln\left(\frac{E}{c\sqrt{N}}\right) - 3N \ln(h) \right] \\
 &= \frac{N}{V},
 \end{aligned}$$

$$\implies P = \frac{N k_B T}{V}.$$

In other words, the relativistic gas of particles can still be treated as an ideal gas.

To find the adiabatic index γ , this means $dS = 0$, as our system is in adiabatic expansion or compression and the overall entropy does not change. We have

$$S = k_B \ln(\Omega) \sim k_B \left[N \ln\left(\frac{V}{N}\right) + 3N \ln\left(\frac{E}{c\sqrt{N}}\right) - 3N \ln(h) \right],$$

then

$$\begin{aligned}dS &= \frac{\partial S}{\partial E} dE + \frac{\partial S}{\partial V} dV \\&= \frac{3Nk_B}{E} dE + \frac{Nk_B}{V} dV \\&= 0 \\&\implies \frac{3}{E} dE = -\frac{1}{V} dV \\&\implies 3 \ln(E) = -\ln(V) + \text{const.} \\&\implies E^3 \propto V^{-1} \\&\implies \frac{E}{V} \propto V^{-\frac{4}{3}},\end{aligned}$$

and the adiabatic index $\gamma = \frac{4}{3}$. ■

Problem 2

Van Der Waals came up with a method for describing a system of hard spheres with a physical size R_0 . The only difference to the multiplicity factor is that each hard sphere removes some volume $V_0 \sim R_0^3$ from the system, such that the total volume that the particles have available to them is actually $V - NV_0$, instead of V . Derive the multiplicity factor and equation of state for this system (assuming a monotonic, non-relativistic gas, and answering the same questions as in Problem 1 except show that it is not really consistent with a simple γ -law; derive a relationship between pressure and density for this system while undergoing adiabatic changes in density and pressure. There should be some unusual behavior when the number density n is close to some n_{crit} . How do we interpret this behavior and what is the value of n_{crit} ?

Solution. Assume our system has N particles in 3 dimensions (as before). Van Der Waals' method proposes to take $V \rightarrow V - NV_0$ to account for the volumes taken up by the hard spheres.

The possible positions is given by the hypervolume of the position space, which is $\frac{(V - NV_0)^N}{N!}$. The possible momenta is given by the hypervolume of the momentum space, which is $\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2} - 1)!} R^{3N-1}$, where $R = \sqrt{2mE}$. Then, the multiplicity factor is

$$\begin{aligned}
 \Omega(E, V) &= \frac{(\# \text{ possible positions})(\# \text{ possible momenta})}{h^{3N}} \\
 &= \frac{(\text{area of position space})(\text{area of momentum space})}{h^{3N}} \\
 &= \frac{\left(\frac{(V - NV_0)^N}{N!}\right) \left(\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2} - 1)!} R^{3N-1}\right)}{h^{3N}} \\
 &= \frac{\left(\frac{(V - NV_0)^N}{N!}\right) \left(\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2} - 1)!} (2mE)^{\frac{3N-1}{2}}\right)}{h^{3N}} \\
 &\sim \frac{\left(\frac{V - NV_0}{N}\right)^N \left(2 \left(\frac{2\pi}{3N}\right)^{\frac{3N}{2}} (2mE)^{\frac{3N}{2}}\right)}{h^{3N}} \quad (\text{since } N \gg 1) \\
 &\sim \frac{\left(\frac{V - NV_0}{N}\right)^N \left(\frac{mE}{N}\right)^{\frac{3N}{2}}}{h^{3N}},
 \end{aligned}$$

and

$$\ln(\Omega) \sim N \ln\left(\frac{V - NV_0}{N}\right) + \frac{3N}{2} \ln\left(\frac{mE}{N}\right) - 3N \ln(h).$$

To derive an equation for the total energy E of the system as a function of temperature T , we first find

$$\begin{aligned}
 \beta &= \frac{1}{k_B T} = \frac{\partial \ln(\Omega)}{\partial E} \\
 &= \frac{\partial}{\partial E} \left[N \ln\left(\frac{V - NV_0}{N}\right) + \frac{3N}{2} \ln\left(\frac{mE}{N}\right) - 3N \ln(h) \right] \\
 &= \frac{3N}{2E},
 \end{aligned}$$

$$\implies E = \frac{3}{2} N k_B T.$$

In other words, the average energy per particle is $\frac{3}{2} k_B T$.

To derive an equation for the pressure as a function of temperature T , we first find

$$\begin{aligned}\beta P &= \frac{\partial \ln(\Omega)}{\partial V} \\ &= \frac{\partial}{\partial V} \left[N \ln \left(\frac{V - NV_0}{N} \right) + \frac{3N}{2} \ln \left(\frac{mE}{N} \right) - 3N \ln(h) \right] \\ &= \frac{N}{V - NV_0}, \\ &\implies P = \frac{Nk_B T}{V - NV_0}.\end{aligned}$$

To find the adiabatic index γ , this means $dS = 0$, as our system is in adiabatic expansion or compression and the overall entropy does not change. We will show that this does not follow the simple γ -law. We have

$$S = k_B \ln(\Omega) \sim k_B \left[N \ln \left(\frac{V - NV_0}{N} \right) + \frac{3N}{2} \ln \left(\frac{mE}{N} \right) - 3N \ln(h) \right],$$

then

$$\begin{aligned}dS &= \frac{\partial S}{\partial E} dE + \frac{\partial S}{\partial V} dV \\ &= \frac{3Nk_B}{2E} dE + \frac{Nk_B}{V - NV_0} dV \\ &= 0 \\ &\implies \frac{3}{2} \frac{1}{E} dE = -\frac{1}{V - NV_0} dV \\ &\implies \frac{3}{2} \ln(E) = -\ln(V - NV_0) + \text{const.} \\ &\implies E^{\frac{3}{2}} \propto (V - NV_0)^{-1} \\ &\implies E \propto (V - NV_0)^{-\frac{2}{3}} \\ &\implies \frac{E}{V} \propto V^{-\frac{5}{3}} \left(1 - \frac{NV_0}{V} \right)^{-\frac{2}{3}},\end{aligned}$$

which doesn't follow the adiabatic γ -law.

Instead, we will derive a relationship between pressure and density for this system undergoing adiabatic changes in pressure and density. First, denote by $n = \frac{N}{V}$ the particle density. We will rewrite the equation for entropy as

$$S = k_B \ln(\Omega) \sim k_B \left[N \ln \left(\frac{k_B T}{P} \right) + \frac{3N}{2} \ln \left(\frac{3mP}{2} \right) + \frac{3N}{2} \ln \left(\frac{1}{n} - V_0 \right) - 3N \ln(h) \right],$$

We have

$$\begin{aligned}dS &= \frac{\partial S}{\partial P} dP + \frac{\partial S}{\partial n} dn \\ &= \left(-\frac{Nk_B}{P} + \frac{3Nk_B}{2P} \right) dP + \frac{3Nk_B}{2} \frac{1}{\frac{1}{n} - V_0} \left(-\frac{1}{n^2} \right) dn \\ &= 0 \\ &\implies \frac{1}{P} dP = \frac{3}{n - V_0 n^2} dn \\ &\implies \ln(P) = -3 \ln \left(\frac{1}{n} - V_0 \right) + \text{const.} \\ &\implies P \propto \left(\frac{1}{n} - V_0 \right)^{-3}.\end{aligned}$$

For $n = n_{crit} = \frac{1}{V_0}$, an unusual behavior occurs. This is because entire volume is filled with these hard spheres and there is no more free space, i.e. the entire space is filled with particles. ■

Problem 3

Imagine you have a room filled with N gas particles. What is the probability that if you observe this room at some instant, all of the particles will be in the left half of the room, with none in the right half (suffocating half the class)?

If N is Avogadro's number, how small is this probability? Are we safe from spontaneous death?

Solution. Initially, one could say that, assuming the probability of finding a particle anywhere in the room is uniform, then the probability of finding it in one-half of the room is 0.5. If we have N particles, then the probability would just be $(0.5)^N$. Although this is a really good estimate for all practical purposes, this reasoning is flawed, as we are assuming the particles are independent and will just pop into existence at sometime without any collisions happening, which is unrealistic because collisions are significant at the scale involved.

The number of collisions a particle experiences is

$$n = \frac{L/2}{\lambda},$$

where L is the length of the room and λ is the mean free path. Let's assume, as stated before, that there is a 0.5 probability that, with each collision, the particle will bounce with an angle which directs it towards the half of the room we are interested in. Then, considering that there are $N/2$ particles in half-room, the probability of them spontaneously moving to the our half of the room at once is

$$P(N) = (0.5^n)^{\frac{N}{2}} = 0.5^{\frac{LN}{4\lambda}}.$$

Letting $N = N_A = 6.022 \times 10^{23}$, and considering an average classroom with $L = 10$ m and a mean free path of a particle in the air to be $\lambda = 64$ nm, we have

$$P(N_A) = 0.5^{\frac{10(6.022 \times 10^{23})}{4(64 \times 10^{-9})}} \approx 0.5^{2.35 \times 10^{31}} \approx 10^{-7 \times 10^{30}},$$

which is unimaginably small. So yes, I'm pretty sure we are safe from spontaneous death. ■

Problem 4

Estimate the temperature below which you do not excite a single quantum rotational mode in a diatomic gas (as a function of the molecular moment of inertia I).

For molecular Hydrogen, where the hydrogen atoms are separated by about 3 Angstroms, what is this temperature in Kelvin?

Solution. New degrees of freedom open up at higher temperatures because, with the possible exception of translational kinetic energy, degrees of freedom are quantized. Due to quantum mechanics, the molecules can only vibrate/rotate/get excited with certain discrete energies, and there is a lowest energy at which this happens. The particular energy of this "lowest excited state" determines the temperature at which the degree of freedom "turns on" or "freezes", by which we mean that it is accessible to a large number of particles in the ensemble (in reality, there will nearly always be a few highly excited particles at any temperature simply due to the nature of the statistical distribution of the particles, but in most situations this tiny fraction is irrelevant). As such, the temperature at which new degrees of freedom turn on is highly dependent on the specific material that is being examined.

Every degree of freedom in a molecule accounts for $\frac{1}{2}k_B T$. A diatomic gas molecule has an energy of $\frac{5}{2}k_B T$: 3 translational degrees of freedom accounting for $\frac{3}{2}k_B T$ and 2 rotational degrees of freedom accounting for $\frac{2}{2}k_B T$. We need to find the temperature T at which the rotational degrees of freedom get "frozen". Since we are talking about one particle, then we can take $N = 1$. Thus, the energy as to which are are prescribed is

$$E = k_B T = \frac{I\omega^2}{2} \implies T = \frac{I\omega^2}{2k_B},$$

where I is the moment of inertia, and we get the spacing of energy levels for this degree of freedom from the quantization rule $I\omega \sim n\hbar$. For the ground state ($n = 1$), we have that $I\omega \sim \hbar$. Then

$$T = \frac{I \left(\frac{\hbar}{I}\right)^2}{2k_B} = \frac{\hbar^2}{2k_B I} = \frac{4 \times 10^{-46}}{I} \sim \frac{10^{-46}}{I}.$$

For a hydrogen molecule, we have two hydrogen atoms, each of mass $m_H = 1.67 \times 10^{-27}$ kg, and they are separated at a distance of $d = 3 \text{ \AA} = 3 \times 10^{-10}$ m, then

$$I = MR^2 = 2m_H \left(\frac{d}{2}\right)^2 = 7.515 \times 10^{-47} \text{ kg m}^2 \sim 10^{-47} \text{ kg m}^2,$$

and, thus,

$$T \sim 10 \text{ K}.$$

■

Problem 5

Estimate the temperature of the interior of the sun, knowing the mass $M_\odot \sim 10^{33}$ g and radius $R_\odot \sim 10^{11}$ cm. Assume the pressure in the sun is sufficient to keep it supported against gravitational collapse.

Note: Be careful when googling this number. It might give you a much smaller answer than your estimate! Don't trust google.

Solution. Since the pressure in the sun is sufficient to keep it supported against gravitational collapse, then we know the equation of hydrostatic equilibrium holds, stating

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}.$$

Assume the sun is purely made of hydrogen atoms. At the center, the hydrogen atoms are fully ionized, so that there are two free particles (one proton, one electron) for every hydrogen mass. The formula for heat pressure is

$$P_h = \frac{2Nk_B T}{V} = 2nk_B T,$$

where n is the number density of hydrogen atoms and the factor of 2 accounts for the fact that there are two free particles per hydrogen atom.

To estimate n , we assume that the sun has uniform density. The total number of hydrogen atoms in the sun is then

$$N = \frac{M_\odot}{m_H}$$

The density of hydrogen atoms is the total number divided by the volume of the sun

$$n = \frac{N}{V} = \frac{\frac{M_\odot}{m_H}}{\frac{4}{3}\pi R_\odot^3} = \frac{3M_\odot}{4\pi R_\odot^3 m_H}$$

Estimating the inward pressure due to gravity, we have that the force of the gravitational attraction between two masses M_1 and M_2 is given by Newton's law of gravitation

$$F = G \frac{M_1 M_2}{d^2},$$

where d is the distance between the two masses.

Now, for the sun, there's only one mass, M_\odot , and it's attracting itself. So let's simply replace both M_1 and M_2 by M_\odot . Moreover, there's only one characteristic distance in the picture, and that's the radius, R_\odot of the sun itself.

Pressure is force divided by area, which is the surface area of the sun, $A = 4\pi R_\odot^2$. Dividing the force of gravity by A to find an expression for the pressure exerted by gravity, we get

$$P_g = G \frac{M_\odot^2}{4\pi R_\odot^4}$$

Setting $P_h = P_g$, we get

$$T = \frac{GM_\odot m_H}{6k_B R_\odot}$$

Estimating the internal temperature of the Sun, we get

$$T \sim \frac{(10^{-11})(10^{30})(10^{-27})}{(10^{-23})(10^9)} = 10^6 \text{ K.}$$

■

Problem 6

Pathria & Beale derive the multiplicity for an ideal gas using quantum mechanics. Imagine a 3D infinite square well with sides of length L . As we are all experts on quantum mechanics, we know the infinite square well admits energy eigenvalues of

$$\epsilon(n_x, n_y, n_z) = \frac{\pi^2 \hbar^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2).$$

Given the constraint that the total energy

$$E = \sum_{i=1}^N \epsilon_i$$

for a fixed total number of particles N estimate the multiplicity factor $\Omega(E, V)$, i.e. for a given energy, how many possible quantum states of N particles can have that energy? You can assume large N and large E (large enough total energy that the average energy per particle is way above the ground state, that will help a lot). You should only worry about dependence on E and V , don't worry about terms that look like N^N unless you really want to. It should be equivalent to what we derived in class.

Note: the full formula they give is:

$$\ln(\Omega) = N \ln \left(\frac{V}{N h^3} \left(\frac{4\pi m E}{3N} \right)^{\frac{3}{2}} \right) + \frac{5}{2} N.$$

Solution. Our system has N particles in 3 dimensions. The given constraint is that the total energy of the system E is given by

$$E = \sum_{i=1}^N \epsilon_i,$$

where $\epsilon_i = \frac{\pi^2 \hbar^2}{2mL^2} (n_x^2 + n_y^2 + n_z^2)_i$ is the energy of the i th particle.

The possible positions is given by the hypervolume of the position space, which is $\frac{V^N}{N!}$. The possible momenta is given by the hypervolume of the momentum space, which is $\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2}-1)!} R^{3N-1}$, where $R = \sqrt{2mE}$.

Then, the multiplicity factor is

$$\begin{aligned}
 \Omega(E, V) &= \frac{(\# \text{ possible positions})(\# \text{ possible momenta})}{h^{3N}} \\
 &= \frac{(\text{area of position space})(\text{area of momentum space})}{h^{3N}} \\
 &= \frac{\left(\frac{V^N}{N!}\right) \left(\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2}-1)!} R^{3N-1}\right)}{h^{3N}} \\
 &= \frac{\left(\frac{V^N}{N!}\right) \left(\frac{2\pi^{\frac{3N}{2}}}{(\frac{3N}{2}-1)!} (2mE)^{\frac{3N-1}{2}}\right)}{h^{3N}} \\
 &\sim \frac{\left(\frac{Ve}{N}\right)^N \left(\frac{4\pi emE}{3N}\right)^{\frac{3N}{2}}}{h^{3N}} \quad (\text{since } N \gg 1)
 \end{aligned}$$

and

$$\ln(\Omega) \sim N \ln \left(\frac{V}{Nh^3} \left(\frac{4\pi mE}{3N} \right)^{\frac{3}{2}} \right) + \frac{5}{2}N.$$

■

Homework 3

Problem 1

- (a) Show that the ratio of specific heats in a gas can be determined by the slope of an isentropic path through (P, V) space:

$$\frac{C_P}{C_V} = - \left(\frac{\partial \ln(P)}{\partial \ln(V)} \right)_S.$$

You may assume a multiplicity factor $\Omega(E, V) \propto E^{\alpha N} V^N$.

- (b) Show generally that a gas with an adiabatic equation of state

$$P = (\gamma - 1) \frac{E}{V}$$

obeys a polytropic equation of state under adiabatic transformations (i.e. when entropy is kept fixed):

$$P = K n^\gamma,$$

where $n = N/V$ is the number density. Again, you may assume the system has multiplicity $\Omega \propto E^{\alpha N} V^N$, as above.

Solution. (a) We have

$$C_V = \left(\frac{\partial Q}{\partial T} \right)_V \quad \text{and} \quad C_P = \left(\frac{\partial Q}{\partial T} \right)_P.$$

Assuming that $\Omega \propto E^{\alpha N} V^N$, then

$$\ln(\Omega) = \alpha N \ln(E) + N \ln(V).$$

From that, we can find the following

$$\begin{aligned} \beta &= \left(\frac{\partial \ln(\Omega)}{\partial E} \right)_V = \frac{\alpha N}{E} \implies E = \alpha N k_B T = \alpha P V, \\ \beta P &= \left(\frac{\partial \ln(\Omega)}{\partial v} \right)_E = \frac{N}{V} \implies P V = N k_B T \implies T = \frac{P V}{N k_B}. \end{aligned}$$

The entropy of our system is given by

$$S = k_B \ln(\Omega) = \alpha N k_B \ln(E) + N k_B \ln(V). \quad (1)$$

Computing the differentials, we have

$$\begin{aligned} dE &= T dS - P dV \\ dQ &= T dS = dE + P dV \\ \implies \frac{dQ}{dT} &= \frac{\partial E}{\partial T} + P \frac{\partial V}{\partial T} \end{aligned}$$

Computing the heat capacities, we have

$$\begin{aligned} C_V &= \left(\frac{\partial Q}{\partial T} \right)_V = \left(\frac{\partial E}{\partial T} \right)_V = \alpha N k_B \\ C_P &= \left(\frac{\partial Q}{\partial T} \right)_P = \left(\frac{\partial E}{\partial T} \right)_P + P \left(\frac{\partial V}{\partial T} \right)_P = \alpha N k_B + P \frac{N k_B}{P} = (\alpha + 1) N k_B \end{aligned}$$

The change in entropy dS is given by

$$\begin{aligned}
 dS &= \left(\frac{\partial S}{\partial E} \right) dE + \left(\frac{\partial S}{\partial V} \right) dV \\
 &= \frac{\alpha N k_B}{E} dE + \frac{N k_B}{V} dV \\
 &= \frac{\alpha N k_B}{E} \left(\left(\frac{\partial E}{\partial V} \right) dV + \left(\frac{\partial E}{\partial P} \right) dP \right) + \frac{N k_B}{V} dV \\
 &= \frac{\alpha N k_B}{\alpha P V} (\alpha P dV + \alpha V dP) + \frac{N k_B}{V} dV \\
 &= \frac{\alpha N k_B}{V} dV + \frac{\alpha N k_B}{P} dP + \frac{N k_B}{V} dV \\
 &= \frac{(\alpha + 1) N k_B}{V} dV + \frac{\alpha N k_B}{P} dP.
 \end{aligned}$$

Since we are moving on an isentropic path, then the change in entropy is zero. Hence

$$\begin{aligned}
 dS = 0 &= \frac{(\alpha + 1) N k_B}{V} dV + \frac{\alpha N k_B}{P} dP \\
 \implies \frac{dP}{P} &= -\frac{\alpha + 1}{\alpha} \frac{dV}{V} \\
 d \ln(P) &= -\frac{\alpha + 1}{\alpha} d \ln(V) \\
 \frac{d \ln(P)}{d \ln(V)} &= -\frac{\alpha + 1}{\alpha} = -\frac{C_P}{C_V}.
 \end{aligned}$$

(b) We are given an equation of state

$$P = (\gamma - 1) \frac{E}{V}.$$

Assuming that $\Omega \propto E^{\alpha N} V^N$, then

$$\ln(\Omega) = \alpha N \ln(E) + N \ln(V).$$

and we have

$$\begin{aligned}
 \beta &= \left(\frac{\partial \ln(\Omega)}{\partial E} \right)_V = \frac{\alpha N}{E} \implies E = \alpha N k_B T = \alpha P V, \\
 \beta P &= \left(\frac{\partial \ln(\Omega)}{\partial V} \right)_E = \frac{N}{V} \implies P V = N k_B T \implies P = n k_B T.
 \end{aligned}$$

From the equation of state, we have

$$P V = (\gamma - 1) E = \frac{1}{\gamma} E \implies \gamma = \frac{\alpha + 1}{\alpha}.$$

The change in entropy dS is given by

$$\begin{aligned}
 dS &= \left(\frac{\partial S}{\partial E} \right) dE + \left(\frac{\partial S}{\partial V} \right) dV \\
 &= \frac{\alpha N k_B}{E} dE + \frac{N k_B}{V} dV \\
 &= \frac{\alpha N k_B}{E} \left(\left(\frac{\partial E}{\partial V} \right) dV + \left(\frac{\partial E}{\partial P} \right) dP \right) + \frac{N k_B}{V} dV \\
 &= \frac{\alpha N k_B}{\alpha P V} (\alpha P dV + \alpha V dP) + \frac{N k_B}{V} dV \\
 &= \frac{\alpha N k_B}{V} dV + \frac{\alpha N k_B}{P} dP + \frac{N k_B}{V} dV \\
 &= \frac{(\alpha + 1) N k_B}{V} dV + \frac{\alpha N k_B}{P} dP.
 \end{aligned}$$

Under adiabatic transformations, we have $dS = 0$, hence

$$\begin{aligned}
 dS &= 0 \\
 \frac{dP}{P} &= -\frac{\alpha + 1}{\alpha} \frac{dV}{V} \\
 \ln(P) &= -\frac{\alpha + 1}{\alpha} \ln(V) + \text{constant} \\
 PV^{\frac{\alpha + 1}{\alpha}} &= \text{constant} \\
 P \left(\frac{N}{n} \right)^{\frac{\alpha + 1}{\alpha}} &= \text{constant} \\
 P &= K n^\gamma,
 \end{aligned}$$

where $K = \text{constant}$. ■

Problem 2

Let's think about the atmosphere. Atoms are bound by gravity, but they still seem to float some distance above Earth (rather than all staying on the ground, at minimum energy). We can figure out the distribution of atoms with height, using the magic of estimation.

- (a) First, assume a given mean free path λ for molecules (assuming a single species with mass m) and assume the atmosphere is some fixed temperature T . Estimate the diffusion constant ν in the atmosphere.
- (b) Atoms are always trying to fall in earth's gravity but they keep having collisions. Estimate the time Δt between collisions.
- (c) In the time Δt , gravity can give molecules a net drift velocity $v_d \sim -g\Delta t$, before the molecule is scattered and its velocity is totally random again. So, there is a downward flux of atoms given by $F = nv_d$, where n is the number density. This is cancelled by an equal and opposite diffusive flux (computed from homework 1, remember?). In equilibrium, these fluxes cancel (so that $\dot{n} = 0$); set the fluxes to cancel and use this fact to compute $n(z)$, the number density in the atmosphere, as a function of z . (call n_0 the number density at the base of the Earth). Assume g is a constant (not dependent on z).

Solution. (a) The diffusion equation is given by

$$\frac{\partial \rho}{\partial t} = \nu \frac{\partial^2 \rho}{\partial x^2}.$$

We can say, by dimensional analysis, that

$$\frac{1}{\Delta t} \sim \frac{\nu}{\lambda^2} \implies \nu \sim \lambda v_0,$$

where $v_0 \equiv \frac{\lambda}{\Delta t}$.

We define the density as $\rho = \frac{M}{V} = \frac{Nm}{V}$. Since energy is conserved, then the kinetic energy of a group of particles is equal to their free energy. We have

$$\begin{aligned} \frac{Nm v_0^2}{2} &= \frac{5}{2} N k_B T \\ m v_0^2 &= 5 k_B T \\ v_0 &= \sqrt{\frac{5 k_B T}{m}} \\ v_0 &\sim \sqrt{\frac{k_B T}{m}} \\ \implies \nu &\sim \lambda v_0 \sim \lambda \sqrt{\frac{k_B T}{m}}. \end{aligned}$$

(b) From before, we have

$$\Delta t \sim \frac{\lambda}{v_0} \sim \lambda \sqrt{\frac{m}{k_B T}}.$$

(c) We are given a drift velocity $v_d \sim -g\Delta t$ and a downward flux of atoms $F = nv_d$, where n is the number density.

In equilibrium, we have another flux $J = \nu \frac{\partial n}{\partial z}$. The change in the number density as a function of

vertical height is the difference between both fluxes and should be zero. Then

$$\begin{aligned}\frac{\partial n}{\partial z} &= J - F = 0 \\ \Rightarrow \nu \frac{\partial n}{\partial z} - n v_d &= 0 \\ \nu \frac{\partial n}{\partial z} + n g \Delta t &= 0 \\ \frac{dn}{n} &= -\frac{g \Delta t}{\nu} dz \\ \ln \left(\frac{n}{n_0} \right) &= -\frac{mg}{k_B T} z \\ n(z) &= n_0 e^{-\frac{mgz}{k_B T}}.\end{aligned}$$

■

Problem 3

- (a) Show that you can attain the results of problem (2c) by assuming that the chemical potential gets an external component

$$\mu = \mu_0(n, T) + \mu_{\text{ext}}(z),$$

where

$$\mu_{\text{ext}}(z) = mgz,$$

and assuming $\mu = \text{constant}$ in the atmosphere. Why would we add such a term to the chemical potential and why would we assume that μ is a constant throughout the atmosphere?

- (b) The atmosphere is roughly a 4-to-1 mixture Nitrogen and Oxygen. By what factor does the ratio n_{N_2}/n_{O_2} change when I go 10 km above the ground? You can assume the atmosphere has $T = 300$ Kelvin.

Solution. (a) The multiplicity of this system is given by

$$\Omega(E, V, N) = f(N) E^{\frac{3N}{2}} V^N,$$

where

$$f(N) = \left(\frac{2\pi m}{h^2} \right)^{\frac{3N}{2}} \frac{1}{N! \left(\frac{3N}{2} \right)!}.$$

Then, by taking the logarithm and deriving with respect to N , keeping E and V constant, we get the chemical potential given by

$$\beta \mu_0 = \left(\frac{\partial \ln(\Omega)}{\partial N} \right)_{E, V} = k_B T \ln(n),$$

where $n = \frac{N}{V}$ is the number density.

We have that

$$\begin{aligned}\mu &= \mu_0(n, T) + \mu_{\text{ext}}(z) \\ &= k_B T \ln(n) + f(T) + mgz \\ &= \text{const.}\end{aligned}$$

Since the chemical potential is constant throughout the atmosphere, then

$$\begin{aligned}\frac{d\mu}{dz} &= 0 \\ \frac{k_B T}{n} \frac{dn}{dz} + mg &= 0 \\ \frac{dn}{n} &= \frac{mg}{k_B T} dz \\ n(z) &= n_0 e^{\frac{mgz}{k_B T}}.\end{aligned}$$

(b) Given that the atmosphere is roughly a 4-to-1 mixture of Nitrogen and Oxygen, then

$$n_{\text{N}_2}(0) = 4n_{\text{O}_2}(0),$$

where $m_{\text{N}_2} = 28$ amu and $m_{\text{O}_2} = 32$ amu. Then

$$\begin{aligned}n_{\text{N}_2}(z) &= n_{\text{N}_2}(0) e^{-\frac{m_{\text{N}_2} gz}{k_B T}}, \\ n_{\text{O}_2}(z) &= n_{\text{O}_2}(0) e^{-\frac{m_{\text{O}_2} gz}{k_B T}}.\end{aligned}$$

Taking the ratio, we have

$$\frac{n_{\text{N}_2}(z)}{n_{\text{O}_2}(z)} = 4e^{-\frac{gz}{k_B T}(m_{\text{N}_2} - m_{\text{O}_2})}.$$

Going up 10 km, we get a change of

$$\frac{n_{\text{N}_2}(10)}{n_{\text{O}_2}(10)} = 4e^{-\frac{(9.81)(10)}{(1.38 \times 10^{-23})(300)}(28-32)1.67 \times 10^{-23}} = 4e^{\frac{4(9.81)(1.67)}{(1.38)(30)}} = 19.475.$$

■

Homework 4

Problem 1

Recall the advection equation:

$$\partial_t u + a \partial_x u = 0,$$

where a is a constant. In class, I discussed this equation and briefly told you how it's solved, but I'd like to see you give it a try yourself. For general initial conditions $u(x, 0) = f_0(x)$, find the general solution $u(x, t)$ for this equation by any means you like.

Solution. To solve the advection equation, we use the method of separation of variables. Assume that u has the form $u(x, t) = X(x)T(t)$. Then, if we derive respectively and replace back into the advection equation, and dividing by $X(x)T(t)$, we get

$$\begin{aligned} \frac{T'}{T} + a \frac{X'}{X} &= 0 \\ \frac{T'}{T} &= -a \frac{X'}{X}. \end{aligned}$$

$X(x)$ is independent of t and $T(t)$ is independent of x , that means each hand-side is a constant, say λ . Solving each equation separately, we have

$$\begin{cases} \frac{X'}{X} = \lambda, \\ \frac{T'}{T} = -a\lambda, \end{cases} \implies \begin{cases} \frac{dX}{X} = \lambda dx, \\ \frac{dT}{T} = -a\lambda dt, \end{cases} \implies \begin{cases} \ln(X) = \lambda x + c_1, \\ \ln(T) = -a\lambda t + c_2, \end{cases} \implies \begin{cases} X(x) = Ae^{\lambda x}, \\ T(t) = Be^{-a\lambda t}. \end{cases}$$

Then, we have that

$$u(x, t) = X(x)T(t) = (Ae^{\lambda x}) (Be^{-a\lambda t}) = Ce^{\lambda(x-at)}.$$

Given the initial condition $u(x, 0) = f_0(x)$, then

$$u(x, 0) = f_0(x) = Ce^{\lambda x}.$$

Thus, the general solution is

$$u(x, t) = f_0(x)e^{-a\lambda t}.$$

■

Problem 2

In class we derived the following equations starting from Euler's equations:

$$\begin{aligned}\dot{\rho} + (v \cdot \nabla)\rho + \rho(\nabla \cdot v) &= 0, \\ \dot{\vec{v}} + (v \cdot \nabla)\vec{v} + \frac{1}{\rho}\vec{\nabla}P &= 0, \\ \dot{P} + (v \cdot \nabla)P + \gamma P(\nabla \cdot v) &= 0.\end{aligned}$$

Define the quantity $s \equiv \left(\frac{P}{\rho^\gamma}\right)$. Show the following is true:

$$\dot{s} + (v \cdot \nabla)s = 0.$$

Does this mean that entropy is conserved? What conditions are necessary for this to be true?

Solution. Using the defined quantity s , rewritten as $s = \ln(P) - \gamma \ln(\rho)$, we take the total time derivative of s . Then

$$\begin{aligned}\dot{s} &= \frac{\partial s}{\partial P} \frac{\partial P}{\partial t} + \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial t} \\ &= \frac{\partial s}{\partial P} \dot{P} + \frac{\partial s}{\partial \rho} \dot{\rho} \\ &= \left(\frac{1}{P}\right) \dot{P} + \left(-\frac{\gamma}{\rho}\right) \dot{\rho} \\ &= \frac{\dot{P}}{P} - \frac{\gamma \dot{\rho}}{\rho}.\end{aligned}$$

Now, we calculate the second term

$$\begin{aligned}(v \cdot \nabla)s &= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)s \\ &= v_x \frac{\partial s}{\partial x} + v_y \frac{\partial s}{\partial y} + v_z \frac{\partial s}{\partial z} \\ &= v_x \left(\frac{\partial s}{\partial P} \frac{\partial P}{\partial x} + \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial x}\right) + v_y \left(\frac{\partial s}{\partial P} \frac{\partial P}{\partial y} + \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial y}\right) + v_z \left(\frac{\partial s}{\partial P} \frac{\partial P}{\partial z} + \frac{\partial s}{\partial \rho} \frac{\partial \rho}{\partial z}\right) \\ &= v_x \left(\frac{1}{P} \frac{\partial P}{\partial x} - \frac{\gamma}{\rho} \frac{\partial \rho}{\partial x}\right) + v_y \left(\frac{1}{P} \frac{\partial P}{\partial y} - \frac{\gamma}{\rho} \frac{\partial \rho}{\partial y}\right) + v_z \left(\frac{1}{P} \frac{\partial P}{\partial z} - \frac{\gamma}{\rho} \frac{\partial \rho}{\partial z}\right) \\ &= \frac{1}{P} \left(v_x \frac{\partial P}{\partial x} + v_y \frac{\partial P}{\partial y} + v_z \frac{\partial P}{\partial z}\right) - \frac{\gamma}{\rho} \left(v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z}\right) \\ &= \frac{1}{P} (v \cdot \nabla)P - \frac{\gamma}{\rho} (v \cdot \nabla)\rho \\ &= -\frac{1}{P} \left(\dot{P} + \gamma P(\nabla \cdot v)\right) + \frac{\gamma}{\rho} (\dot{\rho} + \rho(\nabla \cdot v)) \\ &= -\frac{\dot{P}}{P} - \gamma(\nabla \cdot v) + \frac{\gamma \dot{\rho}}{\rho} + \gamma(\nabla \cdot v) \\ &= -\frac{\dot{P}}{P} + \frac{\gamma \dot{\rho}}{\rho}.\end{aligned}$$

Thus,

$$\dot{s} + (v \cdot \nabla)s = 0.$$

No, this does not mean that entropy is conserved. A conservation law takes the form of

$$\dot{s} + \nabla \cdot j_s = 0.$$

Thus, for entropy to be conserved, the following condition must be satisfied

$$\nabla \cdot j_s = (v \cdot \nabla)s.$$

■

Problem 3

Work out a second-order ODE describing the density as a function of radius in a star (as in, if this ODE were solved, the solution would be $\rho(r)$). Use the following two assumptions: First, the star is in hydrostatic equilibrium. Second, assume a polytropic equation of state $P = K\rho^\gamma$. It is often conventional to define $\gamma \equiv 1 + \frac{1}{n}$ for this problem (and it simplifies the resulting equations).

There are actually exact solutions for a few values of n but I won't ask you to derive them. You can try if you want though!

Solution. Assuming we are in hydrostatic equilibrium, then

$$\frac{1}{\rho} \vec{\nabla} P = \vec{g},$$

which is equivalent to

$$\frac{dP}{dr} = -\frac{Gm_{\text{enc}}\rho}{r^2}$$

in polar coordinates, where $m_{\text{enc}}(r) = \int_0^r 4\pi(r')^2 \rho dr'$, or equivalently,

$$\frac{dm}{dr} = 4\pi r^2 \rho.$$

We are given a polytropic equation of state $P = K\rho^\gamma$, then

$$\begin{aligned} \frac{dP}{dr} &= K\gamma\rho^{\gamma-1} \frac{d\rho}{dr} \\ &= K \left(1 + \frac{1}{n}\right) \rho^{\frac{1}{n}} \frac{d\rho}{dr}. \end{aligned}$$

Replacing in the equation of hydrostatic equilibrium, we have

$$\begin{aligned} \frac{dP}{dr} &= -\frac{Gm_{\text{enc}}\rho}{r^2} \\ K\gamma\rho^{\frac{1}{n}} \frac{d\rho}{dr} &= -\frac{4\pi G\rho}{r^2} \int_0^r (r')^2 \rho dr' \\ \frac{K\gamma}{4\pi G} r^2 \rho^{\frac{1}{n}-1} \frac{d\rho}{dr} &= -\int_0^r (r')^2 \rho dr' \\ \frac{d}{dr} \left[\frac{K\gamma}{4\pi G} r^2 \rho^{\frac{1}{n}-1} \frac{d\rho}{dr} \right] &= -\frac{d}{dr} \left[\int_0^r (r')^2 \rho dr' \right] \\ \frac{K\gamma}{4\pi G} \left[2r\rho^{\frac{1}{n}-1} \frac{d\rho}{dr} + r^2 \left(\frac{1}{n} - 1 \right) \rho^{\frac{1}{n}-2} \left(\frac{d\rho}{dr} \right)^2 + r^2 \rho^{\frac{1}{n}-1} \frac{d^2\rho}{dr^2} \right] &= -r^2 \rho \\ \gamma \left[2\rho^{\frac{1}{n}} \frac{d\rho}{dr} + r \left(\frac{1}{n} - 1 \right) \rho^{\frac{1}{n}-1} \left(\frac{d\rho}{dr} \right)^2 + r\rho^{\frac{1}{n}} \frac{d^2\rho}{dr^2} \right] &= -\frac{4\pi G}{K} r\rho^2 \\ \frac{2}{r} \rho^{\gamma-1} \frac{d\rho}{dr} + (\gamma-2) \rho^{\gamma-2} \left(\frac{d\rho}{dr} \right)^2 + \rho^{\gamma-1} \frac{d^2\rho}{dr^2} &= -\frac{4\pi G}{K\gamma} \rho^2 \\ \rho^{\gamma-1} \frac{d^2\rho}{dr^2} + \left[\frac{2}{r} \rho^{\gamma-1} + (\gamma-2) \rho^{\gamma-2} \frac{d\rho}{dr} \right] \frac{d\rho}{dr} &= -\frac{4\pi G}{K\gamma} \rho^2. \end{aligned}$$

■

Problem 4

Imagine you have a star in hydrostatic equilibrium, with mass M and radius R .

- (a) Estimate the average density and pressure inside the star.
- (b) Now imagine the radius of the star is gently stretched out by a factor α :

$$R \rightarrow \alpha R,$$

but the mass is kept fixed. What is the new density and pressure?

- (c) Define P_g to be the pressure necessary to maintain hydrostatic equilibrium. It is important to understand that this number should change differently from P as the star is stretched by α . Calculate the new value of P_g after stretching by α .
- (d) The ratio P/P_g tells us what direction the star will move after this change; if $P/P_g > 1$, pressure is larger than necessary for equilibrium and the star will expand. Likewise, if $P/P_g < 1$ the star will want to contract.

Use this ratio to determine whether the star is stable to being stretched or compressed. How is stability conditional on the value of γ ?

Solution. Assuming we are in hydrostatic equilibrium, then

$$\frac{1}{\rho} \vec{\nabla} P = \vec{g},$$

which is equivalent to

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

in polar coordinates, where $M(r) = \int_0^R 4\pi(r')^2 \rho dr'$, or equivalently,

$$\frac{dM}{dr} = 4\pi R^2 \rho.$$

- (a) The average density $\bar{\rho}$ inside the star is

$$\begin{aligned} \bar{\rho} &= \frac{M}{V} = \frac{3M}{4\pi R^3} \\ \implies \bar{\rho} &\sim \frac{M}{R^3}. \end{aligned}$$

The average pressure \bar{P} inside the star is

$$\begin{aligned} \frac{1}{\bar{\rho}} \nabla \bar{P} &= G \frac{M}{R^2}, \\ \bar{P} &= G \frac{M\bar{\rho}}{R} = G \frac{3M^2}{4\pi R^4} \\ \implies \bar{P} &\sim \frac{GM^2}{R^4}. \end{aligned}$$

- (b) After gently stretching the star and undergoing the transformation $R \rightarrow \alpha R$, assuming α is dimensionless, then the new density ρ_α is

$$\rho_\alpha = \frac{M}{V_\alpha} = \frac{\bar{\rho}}{\alpha^3} = \frac{3M}{4\pi(\alpha R)^3},$$

and the new pressure P_α is given by the adiabatic relation $PV^\gamma = \text{constant}$, so that

$$\begin{aligned} P_\alpha V_\alpha^\gamma &= \bar{P} V^\gamma \\ P_\alpha \left(\frac{4}{3} \pi (\alpha R)^3 \right)^\gamma &= \bar{P} \left(\frac{4}{3} \pi R^3 \right)^\gamma \\ P_\alpha \alpha^{3\gamma} &= \bar{P} \\ P_\alpha &= \frac{\bar{P}}{\alpha^{3\gamma}} = \frac{GM^2}{\alpha^{3\gamma} R^4} \end{aligned}$$

(c) To remain in hydrostatic equilibrium after stretching by α , the pressure P_g must be given by

$$\begin{aligned} \frac{1}{\rho_\alpha} \vec{\nabla} P_g &= \frac{GM}{(\alpha R)^2} \\ P_g &= \frac{GM \rho_\alpha}{\alpha R} = \frac{GM^2}{(\alpha R)^4}. \end{aligned}$$

(d) Taking the ratio of pressures, we have

$$\frac{P_\alpha}{P_g} = \frac{\alpha^4}{\alpha^{3\gamma}} = \alpha^{4-3\gamma}.$$

- If $\frac{P_\alpha}{P_g} > 1$, then $\alpha^{4-3\gamma} > 1 \implies 4 - 3\gamma > 0 \implies \gamma < \frac{4}{3}$. In this case the star will undergo expansion until it reaches hydrostatic equilibrium, if possible.
- If $\frac{P_\alpha}{P_g} < 1$, then $\alpha^{4-3\gamma} < 1 \implies 4 - 3\gamma < 0 \implies \gamma > \frac{4}{3}$. In this case the star will undergo compression until it reaches hydrostatic equilibrium, if possible.

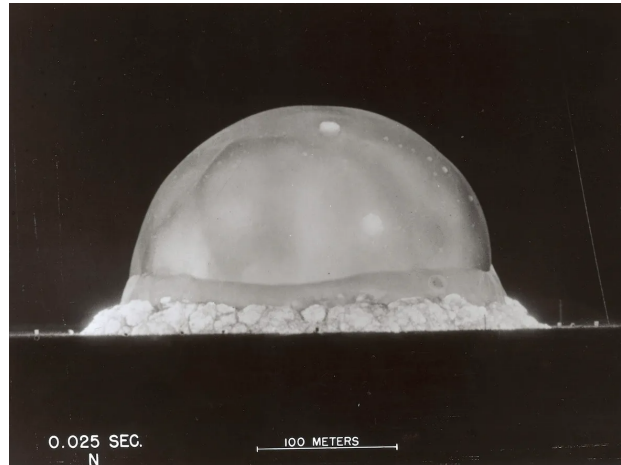
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Homework 5

Problem 1

The figure below was published in Time magazine in September of 1945. It is an image of an atomic test. At the time of publication, this was extremely new military technology and the explosion energy was classified by the US government.

Estimate the explosion energy. Express your answer in tons of TNT, as (for some reason) those are the units the military preferred. 1 ton of TNT is about 4×10^9 Joules or 4×10^{16} erg.



Solution. The **Taylor–von Neumann–Sedov** blast wave describes the shock wave generated by a powerful explosion. Consider an explosion releasing a significant amount of energy E within a confined space and short time frame. This results in a robust spherical shock wave expanding outward from the explosion's center. When the shock wave has traversed a large distance relative to the explosion's size, a self-similar solution is sought to characterize the flow. Here, details regarding the explosion's size and duration become negligible, with only the energy released E dictating the shock wave's behavior. With high precision, it can be assumed that the explosion occurred instantaneously at a single point (e.g., the origin $r = 0$) at time $t = 0$. Within this self-similar region, the shock wave remains immensely strong, such that the pressure P_1 behind it significantly surpasses the pressure ahead of it (typically atmospheric pressure), denoted as P_2 , which can be disregarded in analysis. While the pressure of the undisturbed gas may be insignificant, the density of this gas, denoted as ρ_0 , cannot be ignored due to finite density differences across strong shock waves, as per the Rankine–Hugoniot conditions. This approximation entails setting $p_0 = 0$ and the corresponding sound speed $c_0 = 0$, yet retaining a non-zero density, i.e., $\rho_0 \neq 0$.

The only parameters available at our disposal are the energy E and the undisturbed gas density ρ_0 . The only non-dimensional combination available from r , t , ρ_0 , and E is

$$R = \beta \left(\frac{Et^2}{\rho_0} \right)^{\frac{1}{5}}.$$

We have that

$$R \sim \left(\frac{Et^2}{\rho_0} \right)^{\frac{1}{5}} \implies E \sim \frac{\rho_0 R^5}{t^2} = \frac{(1.293 \text{ kg m}^{-3})(160 \text{ m})^5}{(0.025 \text{ s})^2} = 2.169 \times 10^{14} \text{ J} = 54,232 \text{ tons of TNT}.$$

■

Problem 2

- (a) In class, we discussed the condition for convective instability, being that the entropy has a negative vertical gradient. Please explain this reasoning again in your own words.
- (b) Assuming hydrostatic equilibrium, show that this implies a maximally steep temperature gradient in the atmosphere, beyond which instability occurs. What is the steepest negative temperature gradient allowed in Earth's atmosphere, in °C/km? Don't forget the Earth's atmosphere is a diatomic gas, made mostly out of nitrogen molecules, with $m = 28m_p$.

Solution. (a) Convective instability refers to a situation where a fluid (such as air or water) becomes unstable and starts to convect or circulate due to changes in temperature or density. The condition for convective instability is closely related to the concept of entropy, which is a measure of disorder or randomness in a system. In the context of convective instability, when we say that entropy has a negative vertical gradient, we're essentially saying that entropy decreases as you move upward in the fluid. This means that the fluid becomes more ordered or less random as you go up. Now, imagine a scenario where you have a parcel of fluid that's slightly warmer (and thus less dense) than the surrounding fluid. If you lift this parcel upwards, it will be in an environment where entropy is decreasing with height (negative gradient). As the parcel rises, it finds itself in a region where the fluid around it is becoming more ordered (lower entropy). Since nature tends towards disorder (higher entropy), the parcel is going to have a tendency to keep rising, leading to convective motion. This process is often observed in the atmosphere, where warm air rising from the Earth's surface encounters cooler air aloft. The warm air parcel, being less dense, continues to rise due to convective instability until it reaches a level where its temperature is similar to the surrounding air, at which point it may stop rising or spread out horizontally.

- (b) Assuming hydrostatic equilibrium, we have that $P = K\rho^\gamma$. We also know that convective instability occurs when

$$\frac{\partial}{\partial z} \left(\frac{P}{\rho^\gamma} \right) > 0.$$

Under the assumption that we are dealing with an ideal gas, then $PV = nk_B T \implies P = \frac{\rho k_B}{m} T$, and

we have

$$\begin{aligned}
 \frac{\partial T}{\partial z} &= \frac{\partial T}{\partial P} \frac{\partial P}{\partial z} + \frac{\partial T}{\partial \rho} \frac{\partial \rho}{\partial z} \\
 &= \left(\frac{m}{\rho k_B} \right) \frac{\partial P}{\partial z} + \left(-\frac{mP}{\rho^2 k_B} \right) \frac{\partial \rho}{\partial z} \\
 &= \left(\frac{T}{P} \right) \frac{\partial P}{\partial z} + \left(-\frac{T}{\rho} \right) \frac{\partial \rho}{\partial z} \\
 &= \frac{T}{P} \left[\frac{\partial P}{\partial z} - \frac{P}{\rho} \frac{\partial \rho}{\partial z} \right] \\
 &= \frac{T}{P} \left[\frac{\partial P}{\partial z} - \frac{P}{\rho} \frac{\partial \rho}{\partial P} \frac{\partial P}{\partial z} \right] \\
 &= \frac{T}{P} \left[\frac{\partial P}{\partial z} - \frac{P}{\rho} \left(\frac{1}{K\gamma} \left(\frac{P}{K} \right)^{\frac{1}{\gamma}-1} \right) \frac{\partial P}{\partial z} \right] \\
 &= \frac{T}{P} \left[\frac{\partial P}{\partial z} - \frac{1}{\rho\gamma} \left(\frac{P}{K} \right)^{\frac{1}{\gamma}} \frac{\partial P}{\partial z} \right] \\
 &= \frac{T}{P} \left[\frac{\partial P}{\partial z} - \frac{1}{\gamma} \frac{\partial P}{\partial z} \right] \\
 &= \frac{T}{P} \frac{\partial P}{\partial z} \left(\frac{\gamma-1}{\gamma} \right) \\
 &= \frac{T}{P} (-g\rho) \left(\frac{\gamma-1}{\gamma} \right) \\
 &= \frac{T}{P} \left(-g \frac{Pm}{k_B T} \right) \left(\frac{\gamma-1}{\gamma} \right) \\
 &= -\frac{mg}{k_B} \left(\frac{\gamma-1}{\gamma} \right).
 \end{aligned}$$

In Earth's atmosphere, assuming abundance in Nitrogen, the steepest negative temperature gradient is

$$\begin{aligned}
 \frac{\partial T}{\partial z} &= -\frac{mg}{k_B} \left(\frac{\gamma-1}{\gamma} \right) \\
 &= -\frac{28m_p(9.81)}{1.38 \times 10^{-23}} \left(\frac{\frac{7}{5}-1}{\frac{7}{5}} \right) \\
 &= -9.49 \times 10^{-3} \text{ } ^\circ\text{C m}^{-1} \\
 &= -9.49 \text{ } ^\circ\text{C km}^{-1}.
 \end{aligned}$$

■

Problem 3

Imagine a black hole with mass M embedded in a uniform gas. Far from this mass, as $r \rightarrow \infty$, the density $\rho \rightarrow \rho_0$ and sound speed $c_s \rightarrow c_0$ are constants. Near the hole, gas will accrete until it reaches a steady-state, spherically symmetric solution as a function of r . For this problem, consider a “black hole” just to be an accreting point mass; interestingly, relativity never enters into this calculation.

- (a) Estimate the location of the “sonic point”; i.e. the distance from the black hole where the accretion flow becomes supersonic, and sound waves can no longer travel upstream.
- (b) Estimate the accretion rate \dot{M} by determining its value at the sonic point.

Solution. (a) The speed of sound is very important in fluids as it is the speed at which waves and force propagate. In case of the Bondi flow, when the speed of the accreting gas is lower than v_c , pressure waves can travel “counter-current” and influence the external flow. Past v_c , inside the radius r_c , the inward flow is so fast that no fluid pressure wave/force manage to travel outside, since it travels slower than the medium itself. So all fluid dynamics of the star are confined inside this radius now. It is a similar principle as the black hole, its event horizon for light waves being equivalent to the Bondi critical point for sound waves.

The location of the sonic point for the black hole would be a Schwarzschild radius away from the center. The Schwarzschild radius is given by

$$R_s = \frac{2GM}{c^2} \sim \frac{GM}{c^2}.$$

- (b) Since we have mass density conservation, then $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$. Taking the volume integral of this quantity (assuming it is constant in time), we have

$$\begin{aligned} \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV &= \frac{\partial}{\partial t} \int_V \rho dV + \oint_A (\rho \mathbf{v}) \cdot d\mathbf{A} \\ &= \frac{\partial}{\partial t} M - \rho v \oint_A dA \\ &= \dot{M} - 4\pi r^2 \rho v \\ &= 0. \end{aligned}$$

Then

$$\dot{M} = 4\pi R_s^2 \rho v \sim R_s^2 \rho v \sim \frac{(GM)^2 \rho}{c^3}.$$

■

Problem 4

2D turbulence behaves fundamentally differently from 3D turbulence. The reason is that in 2D the “vorticity” $\omega = \nabla \times v$ obeys a conservation law (maybe I can show this at some point; it’s not too hard to derive). Formulate an analogous vorticity cascade argument (analogous to the energy cascade we derived in class) and derive the following:

- (a) How does velocity v of an eddy scale with its size ℓ ? (in 3D we saw $v \propto \ell^{\frac{1}{3}}$)
- (b) Given this, what is the slope of the power spectrum dE/dk of 2D turbulence (instead of the $k^{-\frac{5}{3}}$ law we found in 3D)?

Note: the most familiar example you might have of 2D turbulence is weather patterns on the surface of the earth (on scales much larger than the height of the atmosphere, so the flow is effectively 2D). Another example is convective zones on planets like Jupiter. The steeper power-law in 2D turbulence puts more energy in long-lived large-scale structures like hurricanes or Jupiter’s big red dot.

Solution. (a) Writing the velocity vector \mathbf{v} in cylindrical coordinates, we have $\mathbf{v} = v(r)\hat{\phi}$, then the vorticity ω is

$$\begin{aligned}\omega &= \nabla \times v \\ &= \frac{1}{r} \frac{\partial}{\partial r} (rv) \hat{\phi} \\ &= \frac{1}{r} \left(v + r \frac{\partial v}{\partial r} \right) \hat{\phi} \\ &= \left(\frac{v}{r} + \frac{\partial v}{\partial r} \right) \hat{\phi}.\end{aligned}$$

In analogy to the energy cascade, we have that the vorticity flow (enstrophy) from scale i to $i+1$ will be constant. Hence

$$\omega = \nabla \times v = \text{constant} \implies \omega \sim \frac{v}{\ell} \implies v \propto \ell.$$

- (b) We have that the wave-number $k = \frac{1}{\ell} \implies \ell \propto k^{-1}$. Then, the slope of the power spectrum is

$$\frac{dE}{dk} \sim \frac{v^2}{k} \sim \frac{\ell^2}{k} \propto \ell^3.$$

■

Homework 6

Problem 1

- (a) Estimate the kinematic viscosity ν of a plasma made out of electrons and protons. You may assume the plasma has mass density ρ and temperature T .
- (b) Show for a plasma with characteristic mass M , size L , temperature T and velocity V , the Reynolds number is given by

$$Re \sim LV/\nu = C \left(\frac{M}{M_\odot} \right) \left(\frac{L}{R_\odot} \right)^{-2} \left(\frac{T}{1 \text{ K}} \right)^{-\frac{5}{2}} \left(\frac{V}{1 \text{ km/s}} \right).$$

What is your estimate for the coefficient C ? (this might help to explain why astrophysical flows are often treated as having zero viscosity).

Solution. (a) Considering a plasma, the mass of the protons will dominate the mass of electrons, so we can say $m \sim m_p$. We know that the diffusion constant ν of a particle is given by $\nu \sim \lambda v_{th}$. Additionally, for a plasma, the collision cross-section σ_c is given by

$$\sigma_c \sim b^2 \sim \left(\frac{e^2}{4\pi\epsilon_0 k_B T} \right)^2,$$

and the mean free path of a particle is given by

$$\lambda \sim \frac{1}{n\sigma_c}.$$

Then,

$$\begin{aligned} \nu &\sim \lambda v_{th} \\ &\sim \frac{1}{n\sigma_c} \sqrt{\frac{k_B T}{m}} \\ &\sim \frac{V}{N} \left(\frac{4\pi\epsilon_0}{e^2} k_B T \right)^2 \sqrt{\frac{k_B T}{m}} \\ &\sim \frac{M}{N\rho} \left(\frac{\epsilon_0}{e^2} \right)^2 \frac{(k_B T)^{\frac{5}{2}}}{\sqrt{m}} \\ &\sim \frac{m}{\rho} \left(\frac{\epsilon_0}{e^2} \right)^2 \frac{(k_B T)^{\frac{5}{2}}}{\sqrt{m}} \\ &\sim \left(\frac{\epsilon_0}{e^2} \right)^2 \frac{(k_B T)^{\frac{5}{2}} \sqrt{m}}{\rho}. \end{aligned}$$

(b) The Reynolds number Re is given by

$$\begin{aligned}
 Re &\sim \frac{LV}{\nu} \\
 &\sim LV \left(\frac{e^2}{\epsilon_0} \right)^2 \frac{\rho}{(k_B T)^{\frac{5}{2}} \sqrt{m}} \\
 &\sim LV \left(\frac{e^2}{\epsilon_0} \right)^2 \frac{\frac{m}{L^3}}{(k_B T)^{\frac{5}{2}} \sqrt{m}} \\
 &\sim \left(\frac{e^2}{\epsilon_0} \right)^2 \frac{\sqrt{m} V}{L^2 (k_B T)^{\frac{5}{2}}} \\
 &\sim \frac{1}{\sqrt{m}} \left(\frac{e^2}{\epsilon_0} \right)^2 \frac{m V}{L^2 (k_B T)^{\frac{5}{2}}} \\
 &\sim \frac{1}{\sqrt{m} k_B^{\frac{5}{2}}} \left(\frac{e^2}{\epsilon_0} \right)^2 \frac{m V}{L^2 (T)^{\frac{5}{2}}} \\
 &\equiv C M L^{-2} T^{-\frac{5}{2}} V.
 \end{aligned}$$

We can see that the constant has to be on the order of $\frac{e^4}{\sqrt{m} k_B^{\frac{5}{2}} \epsilon_0^2} \sim 10^{17}$, which is huge. From that, we can say that ν has to be extremely small, and hence, the astrophysical plasma can be assumed to have zero viscosity. ■

Problem 2

Compute the electric susceptibility χ_E of water vapor. Assume that a water molecule has a given electric dipole moment p and it can have any random orientation in the presence of an electric field. The susceptibility should be some function of number density n and temperature T . You may find it useful to know the following expansion of $\coth(x)$ for small x :

$$\coth(x) \approx \frac{1}{x} + \frac{1}{3}x + \dots$$

If it's been a while since your last E&M course, the electric susceptibility of a material is determined by its affinity for inducing polarization in the presence of an electric field

$$\vec{P} = \epsilon_0 \chi_E \vec{E},$$

where \vec{P} is the dipole moments per unit volume. Don't worry about the electric field produced by the dipoles themselves; \vec{E} is meant to represent the total electric field, so it is already taken into account.

Solution. The potential energy of the water vapor molecule due to an electric field is $\epsilon(\theta) = -\mathbf{P} \cdot \mathbf{E} = -PE \cos(\theta)$. Then the probability of having a given potential energy V for some angle θ is

$$\mathbb{P}(\theta) = \frac{1}{N} e^{-\frac{V}{k_B T}} = \frac{1}{N} e^{PE \cos(\theta)}.$$

We know that $\cos(\theta) \in [-1, 1]$, which means that the energy can vary from $-PE$ to $+PE$. Then the expected

value (average value) of the energy of the system is given by

$$\begin{aligned}
 \langle \epsilon \rangle &= \frac{\int_{-PE}^{PE} \epsilon e^{-\frac{\epsilon}{k_B T}} d\epsilon}{\int_{-PE}^{PE} e^{-\frac{\epsilon}{k_B T}} d\epsilon} \\
 &= \frac{-k_B T \epsilon e^{-\frac{\epsilon}{k_B T}} \Big|_{-PE}^{PE} + k_B T \int_{-PE}^{PE} e^{-\frac{\epsilon}{k_B T}} d\epsilon}{\int_{-PE}^{PE} e^{-\frac{\epsilon}{k_B T}} d\epsilon} \\
 &= \frac{-k_B T P E \left(e^{-\frac{PE}{k_B T}} + e^{\frac{PE}{k_B T}} \right) - (k_B T)^2 \left(e^{-\frac{PE}{k_B T}} - e^{\frac{PE}{k_B T}} \right)}{-k_B T \left(e^{-\frac{PE}{k_B T}} - e^{\frac{PE}{k_B T}} \right)} \\
 &= k_B T - \frac{e^{\frac{PE}{k_B T}} + e^{-\frac{PE}{k_B T}}}{e^{\frac{PE}{k_B T}} - e^{-\frac{PE}{k_B T}}} P E \\
 &= k_B T - P E \coth \left(\frac{PE}{k_B T} \right).
 \end{aligned}$$

Since $PE \ll k_B T$, then we can use the small-angle approximation for coth to get

$$\begin{aligned}
 \langle \epsilon \rangle &= k_B T - P E \coth \left(\frac{PE}{k_B T} \right) \\
 &\approx k_B T - P E \left(\frac{1}{\frac{PE}{k_B T}} + \frac{1}{3} \frac{PE}{k_B T} \right) \\
 &= -\frac{1}{3} \frac{(PE)^2}{k_B T}.
 \end{aligned}$$

For the polarization vector, we can write it as the average polarization of each particle multiplied by the number of particles, *i.e.*

$$\mathbf{P} = n \langle p \rangle.$$

We can now take $\langle P \rangle$ to be parallel to the electric field as

$$\begin{aligned}
 \langle \epsilon \rangle &= -\langle p \rangle E \implies \langle p \rangle = -\frac{\langle \epsilon \rangle}{E} \\
 -\frac{n \langle \epsilon \rangle}{E} &= \epsilon_0 \chi_E E \implies \chi_E = -\frac{n \langle \epsilon \rangle}{E^2 \epsilon_0} = \frac{np^2}{3\epsilon_0 k_B T}.
 \end{aligned}$$

■

Problem 3

- (a) Compute the single-particle partition function

$$z = \int \frac{d^3x d^3p}{h^3} e^{-\beta \epsilon_p}$$

for a single particle in a classical ideal monatomic gas.

- (b) Let's look at the quantum version. Imagine an infinite square well with sides of length
- L
- . Show that the partition function for this system, can be expressed as

$$z = \left(\sum_{n=1}^{\infty} e^{-\beta \epsilon_0 n^2} \right)^3$$

for some energy scale ϵ_0 .

Write a computer program to compute this explicitly as a function of $\beta \epsilon_0$ by approximating the infinite sum up to very large N . Make a plot of $z(\beta \epsilon_0)$, where

$$z(x) = \left(\sum_{n=1}^{\infty} e^{-x n^2} \right)^3$$

and examine the limiting cases $\beta \epsilon_0 \ll 1$ and $\beta \epsilon_0 \gg 1$. Does your solution agree with the classical result in the approximate limit? Plot the classical result alongside the quantum result. Make sure the plot is clear enough that I can actually see where these curves agree and where they disagree.

Solution. (a) The energy of a single particle is given by $\epsilon_p = \frac{p^2}{2m}$. Then the single-particle partition function is

$$\begin{aligned} z &= \int \frac{d^3x d^3p}{h^3} e^{-\beta \epsilon_p} \\ &= \frac{1}{h^3} \int_0^L d^3x \int_0^\infty e^{-\frac{\beta p^2}{2m}} d^3p \\ &= \frac{1}{h^3} (L^3) \int_0^\infty e^{-\frac{\beta p^2}{2m}} 4\pi p^2 dp \\ &= \frac{L^3}{h^3} \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}} \\ &= \left(\frac{2\pi m L^2}{\beta h^2} \right)^{\frac{3}{2}}. \end{aligned}$$

- (b) The energy in a quantum well is given by

$$\epsilon_n = \frac{n^2 \hbar^2}{8mL^2} = \epsilon_0 n^2.$$

In three dimensions, we can write $n^2 = n_x^2 + n_y^2 + n_z^2$, and we get

$$\begin{aligned}
 z &= \sum_{n=1}^{\infty} e^{-\beta \epsilon_n} \\
 &= \sum_{n=1}^{\infty} e^{-\beta \epsilon_0 n^2} \\
 &= \sum_{n_x, n_y, n_z}^{\infty} e^{-\beta \epsilon_0 (n_x^2 + n_y^2 + n_z^2)} \\
 &= \left(\sum_{n_x=1}^{\infty} e^{-\beta \epsilon_0 n_x^2} \right) \left(\sum_{n_y=1}^{\infty} e^{-\beta \epsilon_0 n_y^2} \right) \left(\sum_{n_z=1}^{\infty} e^{-\beta \epsilon_0 n_z^2} \right) \\
 &= \left(\sum_{n=1}^{\infty} e^{-\beta \epsilon_0 n^2} \right)^3.
 \end{aligned}$$

Setting $x = \beta \epsilon_0$, we have

$$z(x) = \left(\sum_{n=1}^{\infty} e^{-x n^2} \right)^3.$$

We can rewrite the partition function from part (a) as

$$z_1 = \left(\frac{2\pi m L^2}{\beta h^2} \right)^{\frac{3}{2}} = \left(\frac{\pi}{4\beta \epsilon_0} \right)^{\frac{3}{2}} = \left(\frac{\pi}{4x} \right)^{\frac{3}{2}}.$$

For large N (~ 1000), we have

```

import numpy as np
import matplotlib.pyplot as plt

N = 1000
x = np.linspace(0.005, 10, N)

z_classical = (np.pi / (4 * x))**(3/2)

n = 1
summand = 0
while (n < N):
    summand += np.exp(-x*n**2)
    n += 1
z_quantum = summand**3

plt.plot(x, z_classical, 'r')
plt.plot(x, z_quantum, 'b')
plt.show()

```

Listing 1: Code block of the plot shown

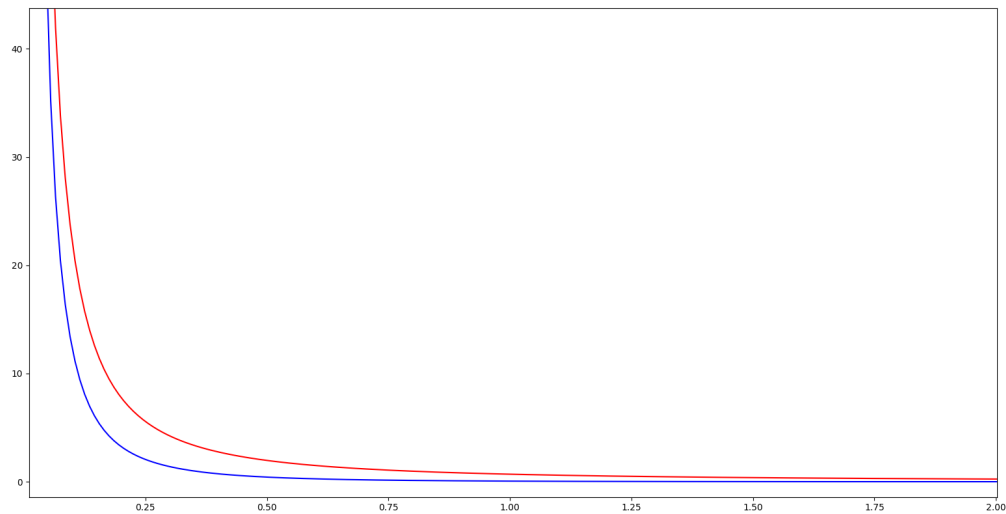


Figure 3: Plot showing the classical partition function (red) vs. the quantum partition function (blue)

For both functions, we have

- For $\beta\epsilon_0 \ll 1$, $z \rightarrow \infty$.
- For $\beta\epsilon_0 \gg 1$, $z \rightarrow 0$.

The main difference is that the quantum partition function decays asymptotically to zero faster than the classical partition function, and the latter is less singular at the origin than the former. ■

Homework 7

Problem 1

- (a) Write a computer program to compute the equation of state of a **relativistic** monatomic ideal gas. In practice, you can just make a plot of the energy E as a function of temperature for the gas.

Do this by computing the partition function $Z(\beta)$ by explicit numerical integral, and then taking the derivative of $\ln(Z)$ numerically (by sampling its value at neighboring points and measuring the slope). Show that you get the non-relativistic and ultra-relativistic limits correct and that it transitions at the right temperature.

- (b) Take a derivative of $E(T)$ with respect to temperature to get the heat capacity (at constant volume) as a function of temperature. Plot the molar heat capacity as a function of temperature. **Make sure both of the regimes along with the transition point are clearly visible in your plots of $E(T)$ and $C_V(T)$.**

Solution. (a) The energy of a relativistic particle is given by

$$\epsilon^2 = p^2 c^2 + m^2 c^4 \implies \epsilon = c \sqrt{p^2 + (mc)^2}.$$

The partition function Z is then given by

$$\begin{aligned} Z &= \sum_{\text{states}} e^{-\beta \epsilon_q} \\ &= \frac{1}{h^3} \int e^{-\beta \epsilon_q} d^3x d^3p \\ &= \frac{1}{h^3} \int e^{-\beta c \sqrt{p^2 + (mc)^2}} d^3x d^3p \\ &= \frac{L^3}{h^3} \int e^{-\beta c \sqrt{p^2 + (mc)^2}} d^3p \\ &= \frac{4\pi L^3}{h^3} \int p^2 e^{-\beta c \sqrt{p^2 + (mc)^2}} dp. \end{aligned}$$

We will now integrate this numerically since this is a tough integral to calculate in a closed form (even for symbolic calculators). We will use the function `quad` from the module `scipy.integrate` to integrate what we need and we will vary the temperature T from 0 K to 5 K. Additionally, we will use units where $c = h = k_B = m_p = 1$. Finally, we will assume a box of length $L = 1$ m.

- (b) To get the heat capacity (at fixed volume), we just take the derivative of the energy $E(T)$ with respect to the temperature T . We expect the transition point from non-relativistic to ultra-relativistic to be at the temperature T when $k_B T = mc^2 \implies T = 1$ K.

```

import numpy as np
import scipy.integrate as int
import matplotlib.pyplot as plt

##### CONSTANTS #####
h = 1          # Planck's constant (GeV/Hz)
c = 1          # Speed of light (in natural units)
k_B = 1        # Boltzmann constant (GeV/K)
m_p = 1        # Proton mass (GeV/c^2)
L = 1          # Length of our cube (m)

##### PROBLEM 1 #####
# TEMPERATURE => 1 K -> 5 K
T = np.linspace(0, 5, 1000)
beta = 1 / (k_B * T)

# INTEGRAL OF PARTITION FUNCTION
def relativisticPartitionFunction(b):
    integrand = int.quad(
        lambda p: p**2 * (np.exp(- b * c * np.sqrt(p**2 + (m_p*c)**2))), 0, np.inf
    )[0]
    return (4*np.pi*L**3)/h**3 * integrand

Z = []
for b in beta:
    z = relativisticPartitionFunction(b)
    Z.append(z)

E = - np.diff(np.log(Z)) / np.diff(beta)

plt.plot(T[:-1], E, 'r', label='Relativistic')
plt.plot(T, (3*k_B*T)/2, 'g', label='Non-Relativistic')
plt.plot(T, 3*k_B*T, 'b', label='Ultra-Relativistic')
# plt.plot(T[:-1], E_discrete, 'k', label='Discrete')
plt.legend(['Relativistic', 'Non-Relativistic', 'Ultra-Relativistic', 'Discrete'])
plt.xlabel('Temperature (K)')
plt.ylabel('Energy')
plt.title('Equation of State of Relativistic Monatomic Ideal Gas')
plt.show()

C_V = np.diff(E) / np.diff(T[:-1])

plt.plot(T[:-2], C_V, 'r', label='Relativistic')
plt.plot(T[:-2], 1.5*np.ones(len(C_V)), 'g', label='Non-Relativistic')
plt.plot(T[:-2], 3*np.ones(len(C_V)), 'b', label='Ultra-Relativistic')
plt.legend(['Relativistic', 'Non-Relativistic', 'Ultra-Relativistic', 'Discrete'])
plt.xlabel('Temperature (K)')
plt.ylabel('Molar Heat Capacity')
plt.title('Molar Heat Capacity of Relativistic Monatomic Ideal Gas')
plt.show()

```

Listing 2: Code for Problem 1

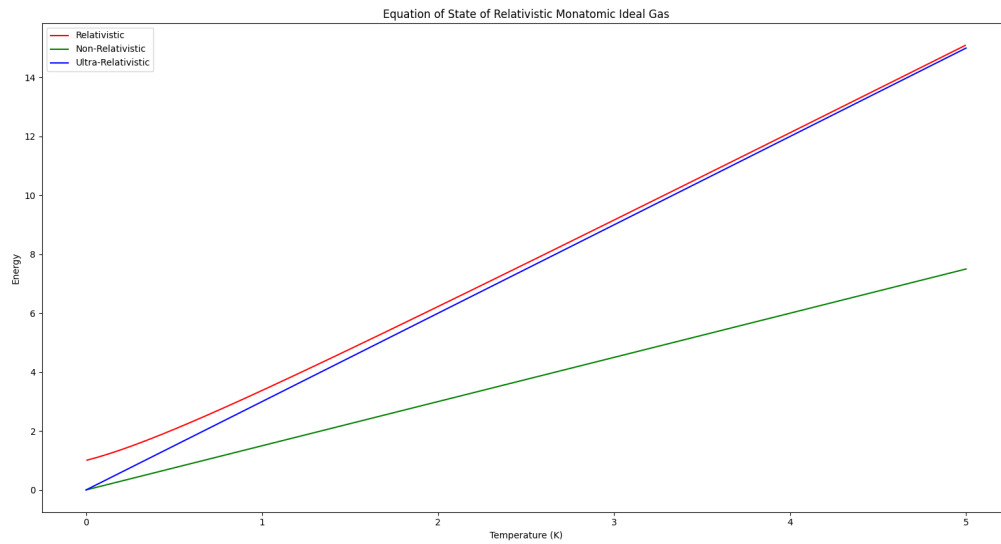


Figure 4: Plot of the energy E as a function of temperature T for the relativistic monatomic ideal gas.

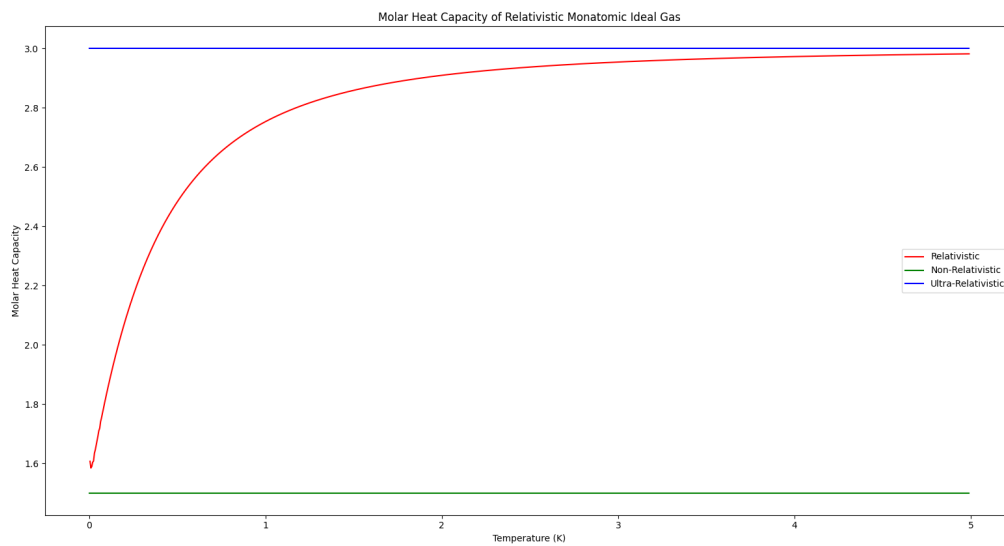


Figure 5: Plot of the molar heat capacity C_V as a function of temperature T for the relativistic monatomic ideal gas.

■

Problem 2

Write a computer program to compute the equation of state of a non-relativistic diatomic ideal gas. Again, just make a plot of $E(T)$. Treat the linear momentum integral classically, but include a full computation of the angular momentum term in quantum mechanics:

$$E(p, j) = \frac{p^2}{2m} + \frac{\hbar^2}{2I} j(j+1)$$

(the last term just being the eigenvalue of J^2). When performing the sum over quantum states, remember that there are $2j+1$ possible m eigenstates for every value of j , so the sum looks like:

$$Z = \int \frac{d^3x d^3p}{h^3} \sum_{j=0}^{\infty} \sum_{m=-j}^j e^{-\beta E(p, j)}$$

Once you have $E(T)$, once again compute the heat capacity $C_V(T)$ (again using numerical derivative). Check behavior at low and high temperatures.

Solution. The energy of a non-relativistic diatomic particle is given by

$$E(p, j) = \frac{p^2}{2m} + \frac{\hbar^2}{2I} j(j+1).$$

The partition function Z is then given by

$$Z = \frac{1}{h^3} \int \sum_{j=0}^{\infty} \sum_{m=-j}^j e^{-\beta E(p, j)} d^3x d^3p.$$

As mentioned in the problem statement, there are $2j+1$ possible m eigenstates for every j . Thus, all different m are $(2j+1)$ -degenerate. This gives us

$$\begin{aligned} Z &= \frac{1}{h^3} \int \sum_{j=0}^{\infty} (2j+1) e^{-\beta E(p, j)} d^3x d^3p \\ &= \frac{V}{h^3} \int \sum_{j=0}^{\infty} (2j+1) e^{-\beta \left(\frac{p^2}{2m} + \frac{\hbar^2}{2I} j(j+1) \right)} d^3p \\ &= \frac{V}{h^3} \sum_{j=0}^{\infty} (2j+1) e^{-\beta \frac{\hbar^2}{2I} j(j+1)} \int e^{-\beta \frac{p^2}{2m}} d^3p \\ &= \frac{V}{h^3} \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}} \sum_{j=0}^{\infty} (2j+1) e^{-\beta \frac{\hbar^2}{2I} j(j+1)}. \end{aligned}$$

We find $E(T)$ and $C_V(T)$ in a similar fashion to Problem 1.

```
##### PROBLEM 2 #####
T = np.linspace(0.02, 5, 1000)
beta = 1 / (k_B * T)

m_1 = m_p
m_2 = m_p
mu = (m_1*m_2)/(m_1 + m_2)
r = 0.1
I = mu * r**2

def monatomicPartitionFunction(b):
    coefficient = (L**3)/(h**3) * ((2 * np.pi * m_p) / b)**(3/2)
    summation = 0
    for j in range(0, 1000):
        summation += (2*j + 1) * np.exp(-b * (h/(2*np.pi))**2 * 1/(2*I) * j * (j + 1))

    return coefficient * summation

Z = []
for b in beta:
    z = monatomicPartitionFunction(b)
    Z.append(z)

E = - np.diff(np.log(Z)) / np.diff(beta)

plt.plot(T[:-1], E, 'r', label='Diatomic')
plt.plot(T[:-1], 1.5*k_B*T[:-1], 'g', label='3 dof')
plt.plot(T[:-1], 2.5*k_B*T[:-1], 'b', label='5 dof')
plt.legend(['Diatomic', '3 D.O.F.', '5 D.O.F.'])
plt.xlabel('Temperature $T$ [K]')
plt.ylabel('Energy $E$')
plt.title('Equation of State of Non-relativistic Diatomic Ideal Gas')
plt.show()

C_V = np.diff(E) / np.diff(T[:-1])

plt.plot(T[:-2], C_V, 'r', label='Diatomic')
plt.plot(T[:-2], 1.5*np.ones(len(T[:-2])), 'g', label='3 dof')
plt.plot(T[:-2], 2.5*np.ones(len(T[:-2])), 'b', label='5 dof')
plt.legend(['Diatomic', '3 D.O.F.', '5 D.O.F.'])
plt.xlabel('Temperature $T$ [K]')
plt.ylabel('Molar Heat Capacity $C_V$')
plt.title('Molar Heat Capacity of Non-relativistic Diatomic Ideal Gas')
plt.show()
```

Listing 3: Code for Problem 2

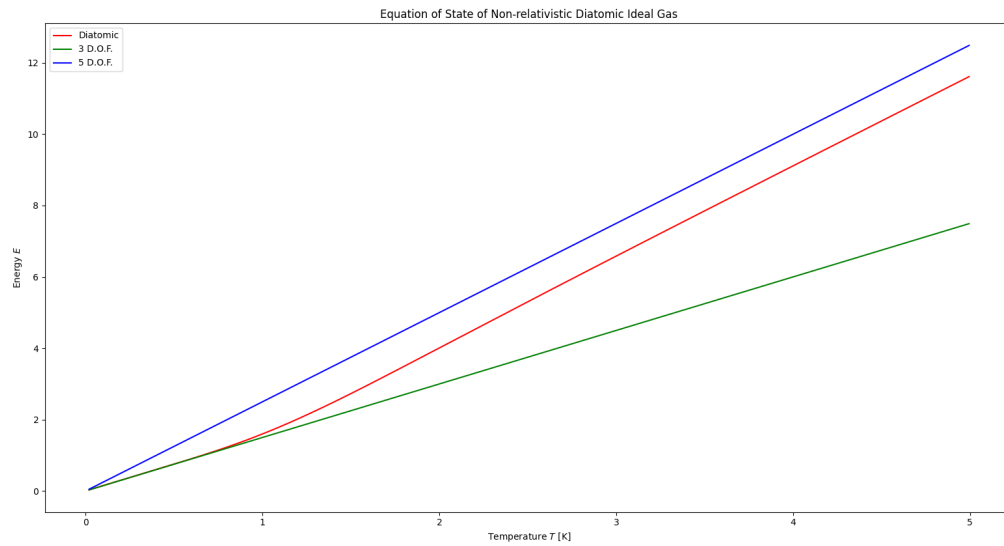


Figure 6: Plot of the energy E as a function of temperature T for a non-relativistic diatomic ideal gas.

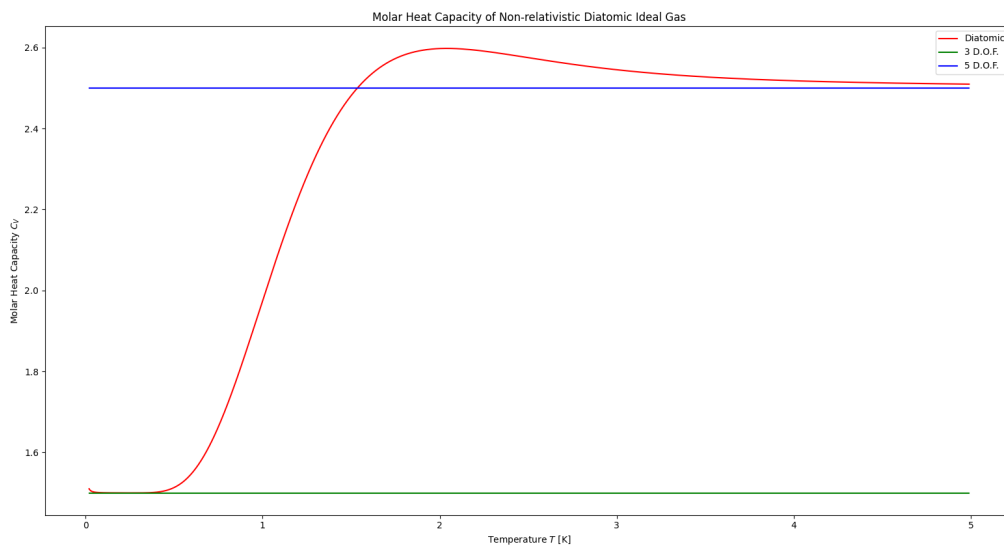


Figure 7: Plot of the heat capacity C_V as a function of temperature T for a non-relativistic diatomic ideal gas.

■

Problem 3

Compute the heat capacity per particle in a two-state system with energy separation ΔE . Make a plot of this heat capacity as a function of $\beta\Delta E$. Where is it comparable to the heat capacity you would estimate for a classical system like a gas?

Solution. The partition function of a two-state system with energy separation of ΔE is given by

$$Z = 1 + e^{-\beta\Delta E}.$$

The energy E is then

$$\begin{aligned} E &= -\frac{\partial \ln(Z)}{\partial \beta} \\ &= -\frac{-\Delta E e^{-\beta\Delta E}}{1 + e^{-\beta\Delta E}} \\ &= \frac{\Delta E e^{-\beta\Delta E}}{1 + e^{-\beta\Delta E}} \\ &= \frac{\Delta E}{e^{\beta\Delta E} + 1}. \end{aligned}$$

The molar heat capacity is then

$$\begin{aligned} C_V &= \left(\frac{\partial E}{\partial T} \right)_V \\ &= \frac{\partial E}{\partial \beta} \frac{\partial \beta}{\partial T} \\ &= \frac{\partial}{\partial \beta} \left[\frac{\Delta E}{e^{\beta\Delta E} + 1} \right] \left(-\frac{1}{k_B T^2} \right) \\ &= \left(-\frac{(\Delta E)^2 e^{\beta\Delta E}}{(e^{\beta\Delta E} + 1)^2} \right) (-k_B \beta^2) \\ &= k_B \beta^2 (\Delta E)^2 \frac{e^{\beta\Delta E}}{(e^{\beta\Delta E} + 1)^2}. \end{aligned}$$

Letting $x = \beta\Delta E$, we have

$$C_V = k_B \frac{x^2 e^x}{(e^x + 1)^2}.$$

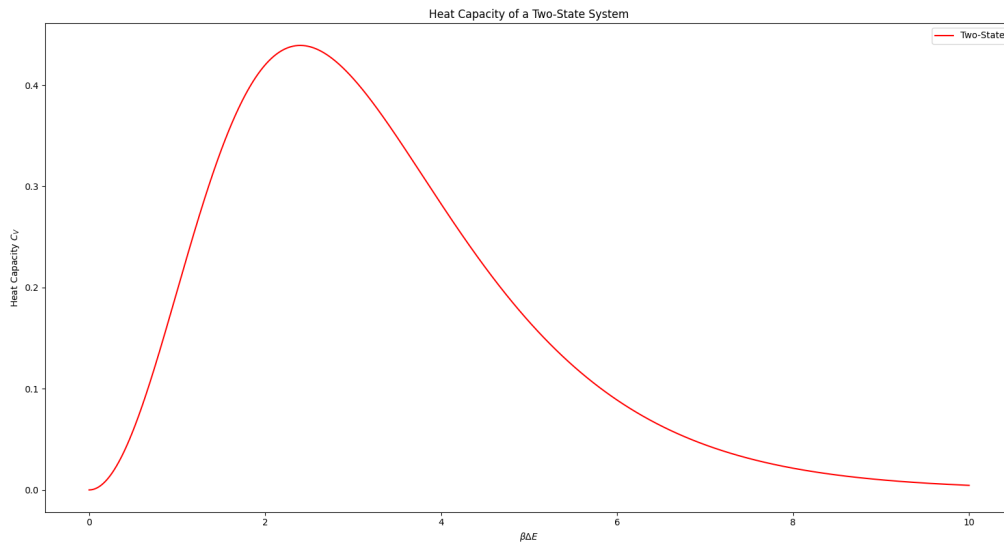
The heat capacity of a monatomic 1D gas is equal to $\frac{1}{2}$. Thus, when $\beta\Delta E \approx 2.4$, the heat capacity of the two-state system is comparable that of a monatomic 1D gas.

The graph of the heat capacity as a function of $\beta\Delta E$ is the following

```
##### PROBLEM 3 #####
x = np.linspace(0, 10, 1000)
C_V = k_B * x**2 * np.exp(x) / (np.exp(x) + 1)**2

plt.plot(x, C_V, 'r', label='Two-state')
plt.legend(['Two-State'])
plt.xlabel('$\\beta \\Delta E$')
plt.ylabel('Heat Capacity $C_V$')
plt.title('Heat Capacity of a Two-State System')
plt.show()
```

Listing 4: Code for Problem 3

Figure 8: Plot of the heat capacity C_V as a function of $\beta\Delta E$ for a two-state system.

■

Problem 4

Assuming I am technologically competent, I have attached a dataset to this assignment containing the molar heat capacity of diamond (in J/K) for a range of temperatures. Use this dataset to determine the characteristic vibrational frequency of carbon atoms in a diamond.

Are there parts of our theory of solids that predict the wrong answer?

Solution. The characteristic vibrational frequency is given by $\epsilon = \hbar\omega$. We use the equation for the heat capacity given by

$$C_V = 3Nk_B (\beta\epsilon)^2 \frac{e^{\beta\epsilon}}{(e^{\beta\epsilon} - 1)^2}.$$

Letting $a = \frac{\epsilon}{k_B}$ and $N = N_A$, we have

$$C_V = 3R \left(\frac{a}{T} \right)^2 \frac{e^{\frac{a}{T}}}{(e^{\frac{a}{T}} - 1)^2}.$$

By fitting our curve to the data, we find that $a = \frac{\epsilon}{k_B} \approx 1300 \text{ K} \implies \omega = \frac{1300k_B}{\hbar} = 1.7 \times 10^{14} \text{ Hz}$, which matches the experimental value. Thus, our theory of solids predicts the answer to a high degree of accuracy.

```
##### PROBLEM 4 #####
global T, C_P # Extracted from .dat file

a = 1300
N = 2
T_theory = np.linspace(0, 1100, 1000)
C_p_theory = 3 * N * k_B * (a/T_theory)**2 * np.exp(a/T_theory) / (np.exp(a/T_theory) - 1)**2

plt.scatter(T, C_p, c='k')
plt.plot(T_theory, C_p_theory, 'r')
plt.legend(['Data', 'Theory'])
plt.xlabel('Temperature $T$ [K]')
plt.ylabel('Molar Heat Capacity $C_P$ [cal/K]')
plt.title('Molar Heat Capacity of Diamond vs. Theoretical Prediction')
plt.show()
```

Listing 5: Code for Problem 4

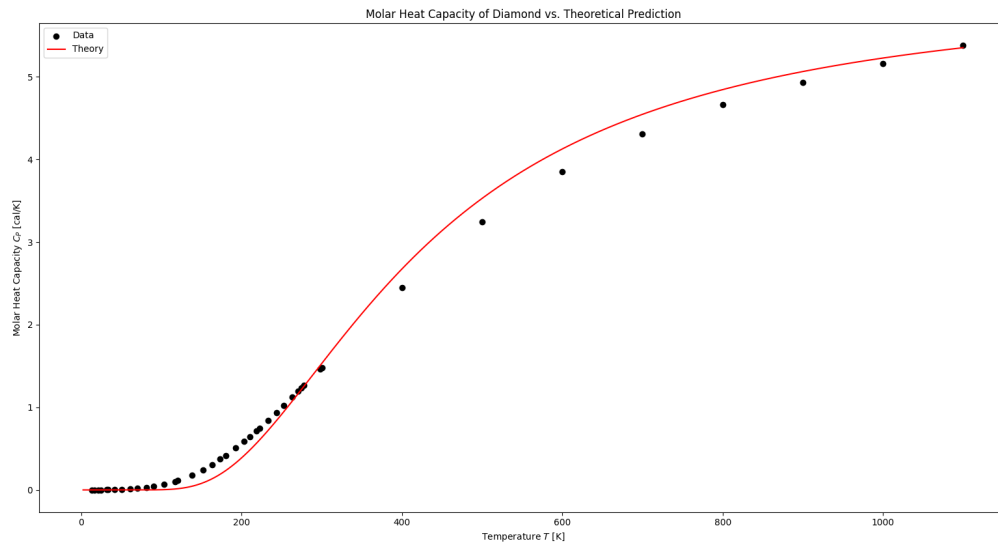


Figure 9: Plot of the experimental molar heat capacity C_V of diamond vs. the theoretical prediction.

■

Homework 8

$$n(\mu) = 2 \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{1 + e^{\beta(\epsilon_p - \mu)}}$$

$$P(\mu) = \frac{2}{\beta} \int \frac{d^3p}{(2\pi\hbar)^3} \ln \left(1 + e^{-\beta(\epsilon_p - \mu)} \right)$$

Problem 1

- (a) Take the above two integrals derived in class for $P(\mu)$ and $n(\mu)$ and compute them both (as functions of μ) in the limit of low temperature (if you can't remember how to do this, try plotting the integrands at low temperature for inspiration!). Recall $\epsilon_p = \frac{p^2}{2m_e}$.
- (b) What is a good estimate for μ for a white dwarf of mass of M_\odot and radius of 10^4 km? Write your answer in eV.
- (c) Plot this parametrically on a plot of $\log P(\mu)$ vs. $\log n(\mu)$. Allow μ to take on values in this vicinity of your estimate above; $0 < \mu < 10\mu_0$, where μ_0 is your estimate from part (b).

Solution. We have that $\epsilon_p = \frac{p^2}{2m_e}$.

- (a) • Calculating n , we have

$$\begin{aligned} n(\mu) &= 2 \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{1 + e^{\beta(\epsilon_p - \mu)}} \\ &= \frac{2}{(2\pi\hbar)^3} \int \frac{1}{1 + e^{\beta(\epsilon_p - \mu)}} d^3p \\ &= \frac{2}{(2\pi\hbar)^3} \int \frac{1}{1 + e^{\beta(\epsilon_p - \mu)}} 4\pi p^2 dp \\ &= \frac{1}{\pi^2 \hbar^3} \int_0^\infty \frac{p^2}{1 + e^{\beta(\epsilon_p - \mu)}} dp. \end{aligned}$$

For $T \rightarrow 0$, $\beta \rightarrow \infty$, which gives us

- When $\epsilon_p > \mu$, then $e^{\beta(\epsilon_p - \mu)} \rightarrow \infty$.
- When $\epsilon_p < \mu$, then $e^{\beta(\epsilon_p - \mu)} \rightarrow 0$.

Additionally, $\frac{p_0^2}{2m_e} = \mu \implies p_0 = \sqrt{2m_e\mu}$. Then

$$\begin{aligned} n(\mu) &= \frac{1}{\pi^2 \hbar^3} \int_0^{p_0} p^2 dp \\ &= \frac{p_0^3}{3\pi^2 \hbar^3} \\ &= \frac{(2m_e\mu)^{\frac{3}{2}}}{3\pi^2 \hbar^3}. \end{aligned}$$

- Calculating P , we have

$$\begin{aligned} P(\mu) &= \frac{2}{\beta} \int \frac{d^3p}{(2\pi\hbar)^3} \ln \left(1 + e^{-\beta(\epsilon_p - \mu)} \right) \\ &= \frac{2}{(2\pi\hbar)^3 \beta} \int \ln \left(1 + e^{-\beta(\epsilon_p - \mu)} \right) 4\pi p^2 dp \\ &= \frac{1}{\pi^2 \hbar^3 \beta} \int \ln \left(1 + e^{-\beta(\epsilon_p - \mu)} \right) p^2 dp. \end{aligned}$$

For $T \rightarrow 0$, $\beta \rightarrow \infty$, which gives us

```

import numpy as np
import matplotlib.pyplot as plt

##### CONSTANTS #####
hbar = 1.05 * 10**(-34)
k_B = 1.38 * 10**(-23)
m_e = 9.11 * 10**(-31)
mu_0 = 9 * 10**4
e = 1.6 * 10**(-19)

##### PROBLEM 1 #####
mu = np.linspace(0, 10*mu_0, 1000)

def n(mu):
    return (2 * m_e * mu)**(3/2) / (3 * np.pi**2 * hbar**3)

def p_0(mu):
    return np.sqrt(2 * m_e * mu)

def P(mu):
    return p_0(mu)**5 / (15 * m_e * np.pi**2 * hbar**3)

plt.plot(np.log10(n(mu * e)), np.log10(P(mu * e)), 'r')
plt.xlabel('$\ln(n(\mu))$')
plt.ylabel('$\ln(P(\mu))$')
plt.show()

```

Listing 6: Code for Problem 1 (c)

- When $\epsilon_p > \mu$, then $\ln(1 + e^{-\beta(\epsilon_p - \mu)}) \rightarrow 0$.
- When $\epsilon_p < \mu$, then $\ln(1 + e^{-\beta(\epsilon_p - \mu)}) \rightarrow \beta(\mu - \epsilon_p)$.

Additionally, $\frac{p_0^2}{2m_e} = \mu \implies p_0 = \sqrt{2m_e\mu}$. Then

$$\begin{aligned}
 P(\mu) &= \frac{1}{\pi^2 \hbar^3 \beta} \int_0^{p_0} p^2 \beta(\mu - \epsilon_p) dp \\
 &= \frac{1}{\pi^2 \hbar^3} \int_0^{p_0} p^2 \left(\mu - \frac{p^2}{2m_e} \right) dp \\
 &= \frac{1}{\pi^2 \hbar^3} \left(\frac{\mu p_0^3}{3} - \frac{p_0^5}{10m_e} \right) \\
 &= \frac{1}{\pi^2 \hbar^3} \left(\frac{p_0^5}{6m_e} - \frac{p_0^5}{10m_e} \right) \\
 &= \frac{p_0^5}{15m_e \pi^2 \hbar^3}.
 \end{aligned}$$

(b) We have that

$$\begin{aligned}
 n &= \frac{N}{V} \quad \text{and} \quad N = \frac{M_\odot}{m_p} = \frac{V(2m_e\mu)^{\frac{3}{2}}}{3\pi^2 \hbar^3} \\
 \implies \mu &= \frac{1}{2m_e} \left(\frac{3\pi^2 \hbar^3}{V} \frac{M_\odot}{m_p} \right)^{\frac{2}{3}} \sim 9 \times 10^4 \text{ eV}.
 \end{aligned}$$

(c) The code and graph we obtain are listed below.

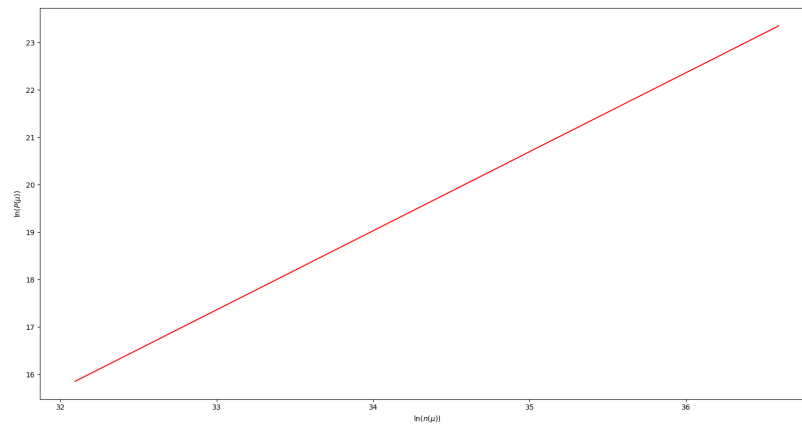


Figure 10: Plot of $\log P(\mu)$ vs. $\log n(\mu)$.

■

Problem 2

For non-zero temperature, you can just perform these integrals on the computer! Do so, and generate additional parametric curves of $(n(\mu), P(\mu))$ in the same plot. Be careful, when $T \neq 0$, μ is no longer the Fermi energy; in fact, you might need μ to take on negative values to reach sufficiently low densities. For values of the temperature, try $T = 10^5$ K, 10^6 K, 10^7 K, and maybe a few others. Thus, your plot should have several different curves of (n, P) at different temperatures. Congratulations! You've computed the equation of state of a white dwarf $P(n, T)$. Can you separate this into a “degeneracy pressure” component and a “gas pressure” component?

Solution. The code and plot that we get are listed below.
The code and graph we obtain are listed below.

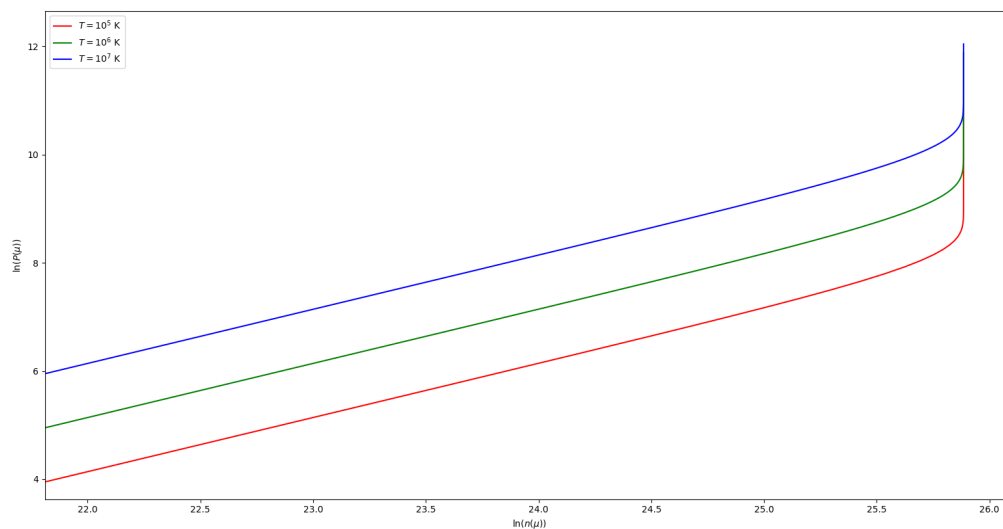


Figure 11: Plot of $\log P(\mu)$ vs. $\log n(\mu)$.

```

import numpy as np
import matplotlib.pyplot as plt

##### CONSTANTS #####
hbar = 1.05 * 10**(-34)
k_B = 1.38 * 10**(-23)
m_e = 9.11 * 10**(-31)
mu_0 = 9 * 10**4
e = 1.6 * 10**(-19)

##### PROBLEM 2 #####
T = np.array([10**5, 10**6, 10**7])
beta = 1 / (k_B * T)

def n(mu, beta):
    integrand = []
    for mu_var in mu:
        integrand.append(int.quad(
            lambda p: p**2 / (1 + np.exp(beta * (p**2/(2 * m_e) - mu_var))),
            0, 0.01*k_B)[0] / (np.pi**2 * hbar**3))
    return integrand

def P(mu, beta):
    integrand = []
    for mu_var in mu:
        integrand.append(int.quad(
            lambda p: p**2 * np.log(1 + np.exp(- beta * (p**2/(2 * m_e) - mu_var))),
            0, 0.01*k_B)[0] / (np.pi**2 * hbar**3 * beta))
    return integrand

mu = np.linspace(-mu_0, mu_0, 50000)
plt.plot(np.log10(n(mu*e, beta[0])), np.log10(P(mu*e, beta[0])), 'r', label="$T= {}$ K")

mu = np.linspace(-mu_0, mu_0, 50000)
plt.plot(np.log10(n(mu*e, beta[1])), np.log10(P(mu*e, beta[1])), 'g', label="$T= {}$ K")

mu = np.linspace(-mu_0, mu_0, 50000)
plt.plot(np.log10(n(mu*e, beta[2])), np.log10(P(mu*e, beta[2])), 'b', label="$T= {}$ K")

plt.xlabel('$\ln(n(\mu))$')
plt.ylabel('$\ln(P(\mu))$')
plt.show()

```

Listing 7: Code for Problem 2



Problem 3

- (a) Repeat the simplified argument from class that gave us the estimate for the equation of state of a Fermi gas at low temperature:

$$P \cong \frac{\hbar^2}{m_e} n_e^{\frac{5}{3}}.$$

- (b) At high enough densities, the Fermi energy is sufficiently large that $\epsilon_p \neq p^2/2m$ and instead it is given by the relativistic formula ($\epsilon_p = pc$). Estimate $P(\rho)$ in the relativistic regime.
- (c) Estimate the density n and pressure P at which the equation of state transitions from non-relativistic to relativistic. Estimate the mass of a white dwarf at the transition point. Give your answer in solar masses.
- (d) (optional, but maybe fun?) Extend your code from Problem 2 to include the full relativistic expression $\epsilon_p = \sqrt{p^2 c^2 + m_e^2 c^4} - m_e c^2$, to get curves $P(\rho)$ that transition from the non-relativistic to the relativistic regime, also at non-zero temperature. If you want you can compare your result to some industry standard EOS codes used in astrophysics: https://cococubed.com/code_pages/eos.shtml.

Solution. (a) From Heisenberg's uncertainty principle, we have $\Delta x \Delta p \sim \hbar$. Since $\Delta x \sim n_e^{-\frac{1}{3}}$, then $\Delta p \sim \hbar n_e^{\frac{1}{3}}$.

The pressure P can be written as energy per volume. Then

$$\begin{aligned} P &= \frac{(\Delta p)^2}{2m_e} n_e \\ &\sim \frac{\hbar^2 n_e^{\frac{2}{3}}}{2m_e} n_e \\ &\sim \frac{\hbar^2 n_e^{\frac{5}{3}}}{m_e}. \end{aligned}$$

- (b) Similarly, we write

$$\begin{aligned} P &= (\Delta p) c n_e \\ &\sim \hbar c n_e^{\frac{1}{3}} n_e \\ &\sim \hbar c n_e^{\frac{4}{3}} \\ &\sim \hbar c \left(\frac{\rho}{m_e} \right)^{\frac{4}{3}}. \end{aligned}$$

- (c) For a transition from non-relativistic to relativistic to occur, we need

$$\frac{(\Delta p)^2}{2m_e} \sim m_e c^2.$$

Working this out, we have

$$\Delta p \sim \sqrt{2} m_e c \sim m_e c.$$

Replacing in Heisenberg's uncertainty principle, we have

$$\Delta x \Delta p \sim \hbar$$

$$\Delta x \sim \frac{\hbar}{\sqrt{2}m_e c}.$$

Then

$$n = \frac{1}{(\Delta x)^3} = \frac{2^{\frac{3}{2}} m_e^3 c^3}{\hbar^3} \sim \left(\frac{m_e c}{\hbar} \right)^3.$$

Thus,

$$\begin{aligned} P &= (\Delta p) c n_e \\ &\sim m_e c^2 n_e \\ &\sim m_e c^2 \left(\frac{m_e c}{\hbar} \right)^3 \\ &\sim \frac{m_e^4 c^5}{\hbar^3}. \end{aligned}$$

Assuming hydrostatic equilibrium, we have

$$\begin{aligned} \frac{dP}{dR} &\sim \frac{P}{R} \sim \rho g \\ \Rightarrow \frac{P}{R} &\sim \frac{M}{R^3} \frac{GM}{R^2} \\ \Rightarrow P &\sim \frac{GM^2}{R^4}. \end{aligned}$$

From $\rho \sim n m_p \sim \frac{M}{R^3}$, we get $R \sim \left(\frac{M}{n m_p} \right)^{\frac{1}{3}}$. Then

$$\begin{aligned} P &\sim GM^2 \left(\frac{M}{n m_p} \right)^{-\frac{4}{3}} \\ \Rightarrow \frac{P}{G} &\sim M^{\frac{2}{3}} (n m_p)^{\frac{4}{3}} \\ \Rightarrow M^{\frac{2}{3}} &\sim \frac{P}{G (n m_p)^{\frac{4}{3}}} \\ \Rightarrow M &\sim \left(\frac{P}{G} \right)^{\frac{3}{2}} \frac{1}{(n m_p)^2}. \end{aligned}$$

Plugging in $P \sim 1.4 \times 10^{24}$ Pa and $n \sim 1.7 \times 10^{37} \text{ m}^{-3}$, we get

$$M \sim 4 \times 10^{30} \text{ kg} \sim 2M_{\odot}.$$

■

Homework 9

Problem 1

Consider a monatomic non-relativistic gas of bosons.

- (a) First, show the number density and energy density are described as follows:

$$n(\mu) = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1},$$

$$\mathcal{E}(\mu) = E/V = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\epsilon_p}{e^{\beta(\epsilon_p - \mu)} - 1},$$

with $\epsilon_p = p^2/2m$, where m is the boson mass.

- (b) Show in the limiting case $\mu \rightarrow -\infty$, you recover the classical result for a monatomic ideal gas:

$$\mathcal{E}_{\text{classical}} = \frac{3}{2}nkT.$$

- (c) In the limit $\mu \rightarrow 0$, $n(\mu) = n_{\text{crit}}(T)$. Compute this explicitly. You might have to look up an integral to do this problem; feel free to either leave your result in terms of a Riemann zeta function, or compute the decimal coefficient explicitly.

Solution. (a) The grand partition function is given by

$$\mathbb{Q} = \prod_q \left(\frac{1}{1 - e^{-\beta(\epsilon_q - \mu)}} \right).$$

Taking the logarithm, we have

$$\ln(\mathbb{Q}) = - \sum_q \ln \left(1 - e^{-\beta(\epsilon_q - \mu)} \right).$$

Now, we can find the number density and the energy density as follows

• **Number density:**

$$\begin{aligned} \beta N &= \partial_\mu \ln(\mathbb{Q}) \\ &= \frac{\partial}{\partial \mu} \left[- \sum_q \ln \left(1 - e^{-\beta(\epsilon_q - \mu)} \right) \right] \\ &= \sum_q \frac{\beta e^{-\beta(\epsilon_q - \mu)}}{1 - e^{-\beta(\epsilon_q - \mu)}} \\ &= \sum_q \frac{\beta}{e^{\beta(\epsilon_q - \mu)} - 1} \\ \Rightarrow N &= \sum_q \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1}. \end{aligned}$$

Making the jump from discrete to continuous, as we are assuming we have a gas of bosons, we

have

$$\begin{aligned}
 N &= \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} \frac{d^3x d^3p}{(2\pi\hbar)^3} \\
 &= V \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} \frac{d^3p}{(2\pi\hbar)^3} \\
 \Rightarrow n(\mu) &= \frac{N}{V} = \frac{1}{(2\pi\hbar)^3} \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} d^3p
 \end{aligned}$$

• **Energy density:**

$$\begin{aligned}
 E - \mu N &= -\partial_\beta \ln(\mathbb{Q}) \\
 &= -\frac{\partial}{\partial\beta} \left[-\sum_q \ln \left(1 - e^{-\beta(\epsilon_q - \mu)} \right) \right] \\
 &= \sum_q \frac{(\epsilon_q - \mu) e^{-\beta(\epsilon_q - \mu)}}{1 - e^{-\beta(\epsilon_q - \mu)}} \\
 &= \sum_q \frac{\epsilon_q - \mu}{e^{\beta(\epsilon_q - \mu)} - 1} \\
 \Rightarrow E &= \sum_q \frac{\epsilon_q - \mu}{e^{\beta(\epsilon_q - \mu)} - 1} + \mu N \\
 &= \sum_q \frac{\epsilon_q - \mu}{e^{\beta(\epsilon_q - \mu)} - 1} + \mu \sum_q \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} \\
 &= \sum_q \frac{\epsilon_q}{e^{\beta(\epsilon_q - \mu)} - 1}.
 \end{aligned}$$

Making the jump from discrete to continuous, as we are assuming we have a gas of bosons, we have

$$\begin{aligned}
 E &= \int \frac{\epsilon_q}{e^{\beta(\epsilon_q - \mu)} - 1} \frac{d^3x d^3p}{(2\pi\hbar)^3} \\
 &= V \int \frac{\epsilon_q}{e^{\beta(\epsilon_q - \mu)} - 1} \frac{d^3p}{(2\pi\hbar)^3} \\
 \Rightarrow \mathcal{E}(\mu) &= \frac{E}{V} = \frac{1}{(2\pi\hbar)^3} \int \frac{\epsilon_q}{e^{\beta(\epsilon_q - \mu)} - 1} d^3p.
 \end{aligned}$$

(b) If $\mu \rightarrow -\infty$, then $e^{\beta(\epsilon_q - \mu)} \gg 1$, and replacing the energy by $\frac{p^2}{2m}$, we have

$$\begin{aligned}
 n(-\infty) &= \frac{1}{(2\pi\hbar)^3} \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} d^3p \\
 &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{e^{\beta(\epsilon_q - \mu)}} 4\pi p^2 dp \\
 &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} 4\pi p^2 dp \\
 &= \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} dp \\
 &= \frac{1}{2\pi^2\hbar^3} \int_0^\infty \frac{p^2}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} dp \\
 &= \frac{1}{2\pi^2\hbar^3} \left(\frac{\sqrt{\pi} m^{\frac{3}{2}} e^{\beta\mu}}{\sqrt{2}\beta^{\frac{3}{2}}} \right) \\
 &= \frac{m^{\frac{3}{2}}}{2\sqrt{2}\pi^{\frac{3}{2}}\hbar^3\beta^{\frac{3}{2}}} e^{\beta\mu} \\
 &= \left(\frac{m}{2\pi\hbar^2\beta} \right)^{\frac{3}{2}} e^{\beta\mu}.
 \end{aligned}$$

The energy density is then

$$\begin{aligned}
 \mathcal{E}(-\infty) &= \frac{1}{(2\pi\hbar)^3} \int \frac{\epsilon_q}{e^{\beta(\epsilon_q - \mu)} - 1} d^3p \\
 &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{\epsilon_q}{e^{\beta(\epsilon_q - \mu)}} 4\pi p^2 dp \\
 &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{\frac{p^2}{2m}}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} 4\pi p^2 dp \\
 &= \frac{4\pi}{2m(2\pi\hbar)^3} \int_0^\infty \frac{p^4}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} dp \\
 &= \frac{1}{4m\pi^2\hbar^3} \int_0^\infty \frac{p^4}{e^{\beta\left(\frac{p^2}{2m} - \mu\right)}} dp \\
 &= \frac{1}{4m\pi^2\hbar^3} \left(\frac{3\sqrt{\pi} m^{\frac{5}{2}} e^{\beta\mu}}{\sqrt{2}\beta^{\frac{5}{2}}} \right) \\
 &= \frac{3m^{\frac{5}{2}}}{4\sqrt{2}\pi^{\frac{3}{2}}\hbar^3\beta^{\frac{5}{2}}} e^{\beta\mu} \\
 &= \frac{3}{2} k_B T \left(\frac{m}{2\pi\hbar^2\beta} \right)^{\frac{3}{2}} e^{\beta\mu} \\
 &= \frac{3}{2} n k_B T,
 \end{aligned}$$

yielding the classical result.

(c) If $\mu \rightarrow 0$, we have

$$\begin{aligned}
 n(0) &= \frac{1}{(2\pi\hbar)^3} \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} d^3p \\
 &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{e^{\beta\epsilon_q} - 1} d^3p \\
 &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{e^{\beta\epsilon_q} - 1} 4\pi p^2 dp \\
 &= \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2}{e^{\beta\frac{p^2}{2m}} - 1} dp \\
 &= \frac{1}{2\pi^2\hbar^3} \int_0^\infty \frac{p^2}{e^{\beta\frac{p^2}{2m}} - 1} dp.
 \end{aligned}$$

Now, let $x = \beta\frac{p^2}{2m} \implies dx = \beta\frac{p}{m} dp$. Then

$$\begin{aligned}
 n(0) \equiv n_{\text{crit}}(T) &= \frac{1}{2\pi^2\hbar^3} \frac{m}{\beta} \int_0^\infty \frac{\frac{2m}{\beta}x}{e^x - 1} \sqrt{\frac{\beta}{2mx}} dx \\
 &= \frac{1}{\sqrt{2}\pi^2\hbar^3} \left(\frac{m}{\beta}\right)^{\frac{3}{2}} \int_0^\infty \frac{x^{\frac{1}{2}}}{e^x - 1} dx \\
 &= \frac{1}{\sqrt{2}\pi^2\hbar^3} \left(\frac{m}{\beta}\right)^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right) \\
 &= \frac{1}{\sqrt{2}\pi^2\hbar^3} \left(\frac{m}{\beta}\right)^{\frac{3}{2}} \left(\frac{\sqrt{\pi}}{2}\right) (2.612).
 \end{aligned}$$

■

Problem 2

For μ in-between these limits, there is an analytic expression in terms of special functions, but we might as well do the integral numerically at that point. You are free to do the integrals either way (analytically or numerically). The goal will be to eliminate μ from the equations and make a plot of $\mathcal{E}(n, T)$.

- (a) For $T > T_{\text{crit}}$, first fix the temperature ($T = \text{const.}$) and plot $\mathcal{E}/\mathcal{E}_{\text{classical}}$ as a function of n . If you're having trouble figuring out how to do this, first compute $\mathcal{E}(\mu)$ and $n(\mu)$ and you should be able to make a parametric plot that puts $n(\mu)$ on the x-axis and $\mathcal{E}(\mu)$ on the y-axis.

You should be able to show with this plot that $\mathcal{E}/\mathcal{E}_{\text{classical}}$ is well-approximated by a linear function of n . Describe this function, using the limits $n = 0$ and $n = n_{\text{crit}}$ from Problem 1; this gives us a decent analytical description of $\mathcal{E}(n, T)$ in this regime with no free parameters.

- (b) For $T < T_{\text{crit}}$, the result is no longer dependent on n , and only on the temperature; explain why this is, and use this to compute $\mathcal{E}(T)$ for all $T < T_{\text{crit}}$.
- (c) Make a plot of $\mathcal{E}(T)/\mathcal{E}_{\text{classical}}(T)$, as a function of all T , both above and below the critical temperature. Is your result continuous and differentiable everywhere? (If you want, you can fix $n = N_A/\text{cm}^3$).
- (d) Finally, since you have an analytical expression everywhere, make a plot of the heat capacity, compared with the classical result $C_V/C_{V,\text{classical}}$. Is this function continuous and differentiable everywhere?

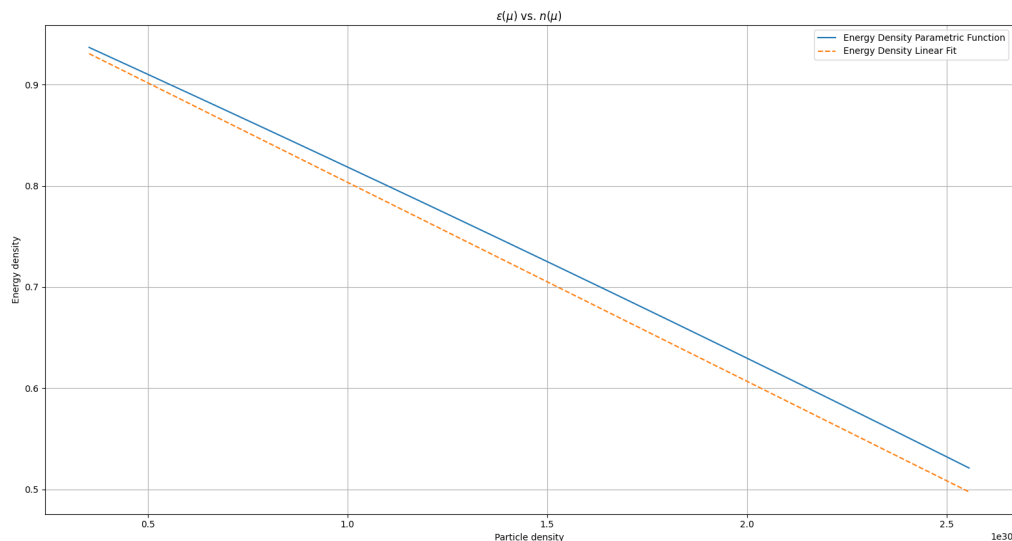


Figure 12: Plot of $\mathcal{E}(\mu)$ vs. $n(\mu)$.

Solution

```

import numpy as np
import scipy.integrate as int
import matplotlib.pyplot as plt

##### CONSTANTS #####
c = 3e8 # Speed of light (m/s)
hbar = 6.582e-16 # Planck's constant (eV/Hz) | 1.05e-34 (J*s)
k_B = 8.617e-5 # Boltzmann constant (eV/K) | 1.38e-23 (J/K)
# m_e = 9.11e-31 # Electron mass (GeV/c^2)
# e = 1.6e-19 # Elementary charge (C)
N_A = 6.022e23 # Avogadro's number (cm^-3)

T = 1e5 # Temperature (K)
beta = 1 / (k_B*T) # Beta (1/eV)

m = 2829570 / c**2 # Mass of Helium-4 Atom (eV/c)
zeta_1 = 2.612 # Riemann Zeta Function of 3/2
zeta_2 = 1.341 # Riemann Zeta Function of 5/2
C = 1/2 * np.pi**2 * hbar**3

##### PROBLEM 2 #####
N = 10000
mu = np.linspace(-10, 0, N) # Chemical Potential (eV)
p = np.linspace(0, 1e5 / c, N) # Momentum (eV/c)
epsilon = p**2 / (2*m) # Energy of Particle (J)

n = np.zeros_like(mu) # Number Density
varepsilon = np.zeros_like(mu) # Energy density
varepsilon_classical = np.zeros_like(mu) # Classical energy density

##### PART (a) #####
for i, u in enumerate(mu):
    # Integrands for n(mu) and varepsilon(mu)
    n_i = 1/(np.exp(beta*(epsilon - u)) - 1)
    varepsilon_i = epsilon/(np.exp(beta*(epsilon - u)) - 1)

    # Integrating over momentum
    n[i] = C * np.trapz(p**2 * n_i, p)
    varepsilon[i] = C * np.trapz(p**2 * varepsilon_i, p)

varepsilon_classical = 3/2 * n * k_B * T

n_crit = (m / (2 * np.pi * hbar**2 * beta))**(3/2) * zeta_1
varepsilon_a = 1 - (0.51 / n_crit) * n
T_crit = (N_A * T**(3/2) / n_crit)**(2/3)

plt.plot(n, varepsilon/varepsilon_classical, label='Energy Density Parametric Function')
plt.plot(n, varepsilon_a, linestyle='--', label='Energy Density Linear Fit')

plt.title('$\varepsilon(\mu)$ vs. $n(\mu)$')
plt.xlabel('Particle density')
plt.ylabel('Energy density')
plt.grid()
plt.legend()
plt.show()

```

Listing 8: Code for Problem 2 (a)

(b)

```
##### PART (b) #####
T_n = np.linspace(0, T_crit, N) # Temperature (K)
beta_1 = 1 / (k_B*T_n) # 1/eV
n_c = (m / (2 * np.pi * hbar**2 * beta_1))**(3/2) * zeta_1 # Critical density

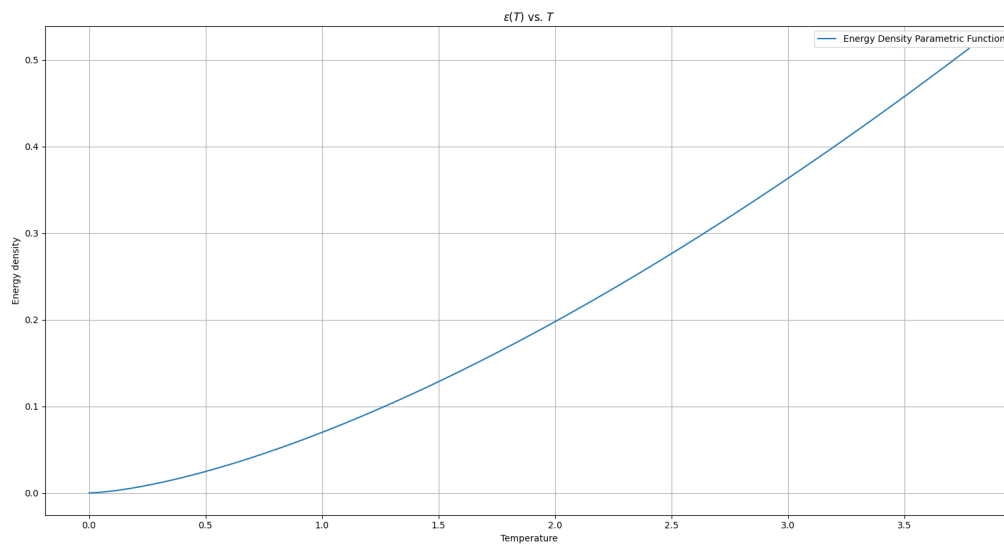
varepsilon_classical = np.zeros_like(T) # Classical energy density
varepsilon_classical = 3/2 * N_A * k_B * T_n

varepsilon_l = 3/2 * n_c * beta_1**-1 * zeta_2 / zeta_1

plt.plot(T_n, varepsilon_l/varepsilon_classical, label='Energy Density Parametric Function')

plt.title('\varepsilon(T) vs. T')
plt.xlabel('Temperature')
plt.ylabel('Energy density')
plt.grid()
plt.legend()
plt.show()
```

Listing 9: Code for Problem 2 (b)

Figure 13: Plot of $\mathcal{E}(T)$ vs. T .

(c)

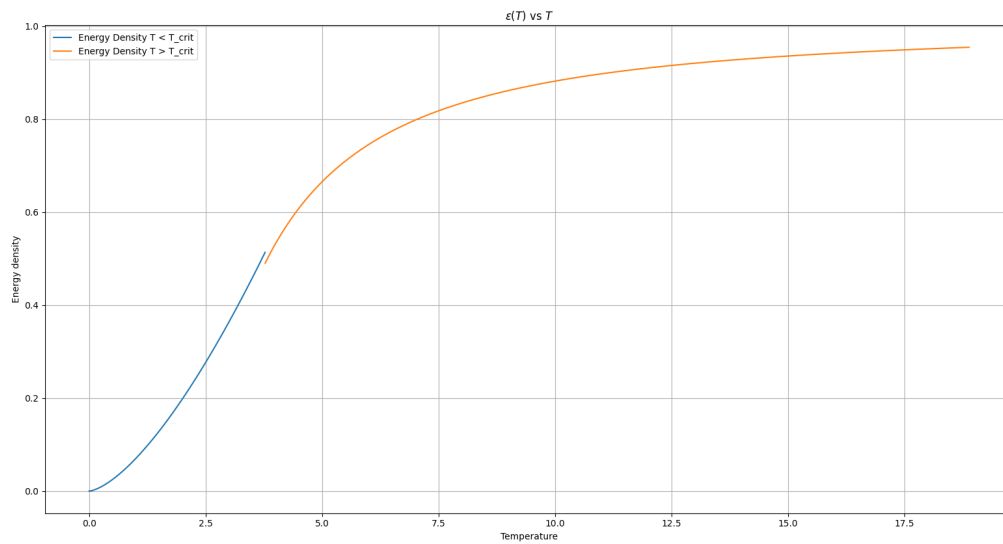
```
##### PART (c) #####
T = np.linspace(T_crit, 5*T_crit, N) # K
beta = 1 / (k_B*T) # 1/eV

varepsilon_h = 1 - 0.51*(T_crit / T)**(3/2)

plt.plot(T_n, varepsilon_l/varepsilon_classical, label='Energy Density T < T_crit')
plt.plot(T, varepsilon_h, label='Energy Density T > T_crit')

plt.title('$\\varepsilon(T)$ vs $T$')
plt.xlabel('Temperature')
plt.ylabel('Energy density')
plt.grid()
plt.legend()
plt.show()
```

Listing 10: Code for Problem 2 (c)

Figure 14: Plot of $\mathcal{E}(T)$ vs. T .

(d)

```
##### PART (d) #####
C_l = np.diff(varepsilon_l)/np.diff(T_n) # heat capacity
C_h = np.diff(varepsilon_h)/np.diff(T) # heat capacity

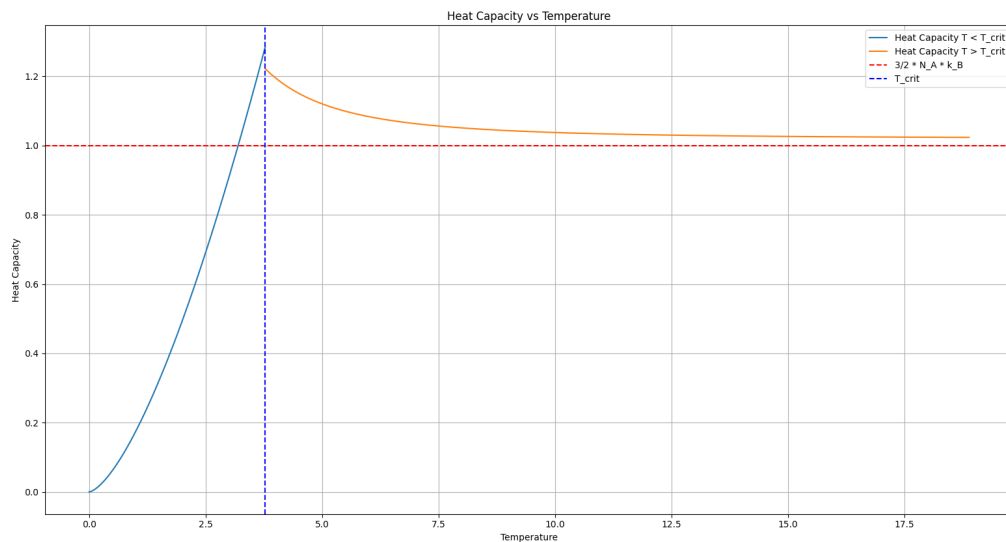
varepsilon_classical = np.zeros_like(T_n) # Classical energy density
varepsilon_classical = 3/2 * N_A * k_B * T_n
C_classical = np.diff(varepsilon_classical)/np.diff(T_n) # Classical heat capacity

plt.plot(T_n[:-1], C_l/C_classical, label='Heat Capacity T < T_crit')
plt.plot(T[:-1], C_h + 1.02, label='Heat Capacity T > T_crit')

plt.axhline(y=1, color='r', linestyle='--', label='3/2 * N_A * k_B')
plt.axvline(x=T_crit, color='b', linestyle='--', label = 'T_crit')

plt.title('Heat Capacity vs Temperature')
plt.xlabel('Temperature')
plt.ylabel('Heat Capacity')
plt.grid()
plt.legend()
plt.show()
```

Listing 11: Code for Problem 2 (d)

Figure 15: Plot of C_V vs. T .



Problem 3

Estimate the critical condensation temperature T_c of a gas of massless particles where the number density n has been fixed (If you don't like the concept that we could fix the number of particles when they're massless, then just imagine they are ultra-relativistic particles with $E = pc$). Do photons have a critical condensation temperature T_c ?

Solution. (a) From Problem 1, we have

$$\ln(\mathbb{Q}) = - \sum_q \ln \left(1 - e^{-\beta(\epsilon_q - \mu)} \right),$$

$$n(\mu) = \frac{N}{V} = \frac{1}{(2\pi\hbar)^3} \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} d^3p.$$

Here, we consider the number of particles $N(\mu)$, or the number density $n(\mu)$ equivalently, to be fixed. Since we are looking for a condensation temperature, then we are looking for the case when $\mu \rightarrow 0$.

If $\mu \rightarrow 0$, then

$$\begin{aligned} n(0) &= \frac{1}{(2\pi\hbar)^3} \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} d^3p \\ &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{e^{\beta\epsilon_q} - 1} d^3p \\ &= \frac{1}{(2\pi\hbar)^3} \int_0^\infty \frac{1}{e^{\beta\epsilon_q} - 1} 4\pi p^2 dp \\ &= \frac{4\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2}{e^{\beta pc} - 1} dp \\ &= \frac{1}{2\pi^2\hbar^3} \int_0^\infty \frac{p^2}{e^{\beta pc} - 1} dp. \end{aligned}$$

Now, let $x = \beta pc \implies dx = \beta c dp$. Then

$$\begin{aligned} n(0) &= \frac{1}{2\pi^2\hbar^3} \frac{1}{(\beta c)^3} \int_0^\infty \frac{x^2}{e^x - 1} dx \\ &= \frac{1}{2\pi^2\hbar^3} \frac{1}{(\beta c)^3} \Gamma(3)\zeta(3) \\ &= \frac{1}{\pi^2\hbar^3} \frac{1}{(\beta c)^3} \zeta(3) \\ &= \frac{1}{\pi^2\hbar^3} \frac{(k_B T)^3}{c^3} \zeta(3) \\ \implies T_c &= \left(\frac{\pi^2\hbar^3 c^3 n(0)}{k_B^3 \zeta(3)} \right)^{\frac{1}{3}} \\ &= \frac{\hbar c}{k_B} \left(\frac{\pi^2 n(0)}{\zeta(3)} \right)^{\frac{1}{3}}. \end{aligned}$$

■

Problem 4

We showed in class that below T_{crit} , the fraction of bosons in the ground state goes like

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{crit}}} \right)^{\frac{3}{2}}.$$

How does this behavior depend on the number of dimensions? What happens when we consider massless bosons (or equivalently, ultra-relativistic bosons)?

Solution. In n dimensions, we will now have to integrate over $d^n x$ and $d^n p$. Let S_{n-1} be the surface area of the $(n-1)$ -sphere embedded in n -dimensional Euclidean space. The integration over $d^n x$ will give us V and we will convert the integral over $d^n p$ by using the surface area of a $(n-1)$ -sphere with radius p . Then our number of particles $N(\mu)$ will be

$$\begin{aligned} N(\mu) &= \frac{1}{(2\pi\hbar)^n} \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} d^n x d^n p \\ &= \frac{V}{(2\pi\hbar)^n} \int \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} d^n p \\ &= \frac{V}{(2\pi\hbar)^n} \int_0^\infty \frac{1}{e^{\beta(\epsilon_q - \mu)} - 1} \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} p^{n-1} dp \\ &= \frac{V}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{p^{n-1}}{e^{\beta(\epsilon_q - \mu)} - 1} dp. \end{aligned}$$

Taking the $\mu \rightarrow 0$ limit, we have

$$\begin{aligned} n(0) \equiv \frac{N(\mu)}{V} &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{p^{n-1}}{e^{\beta(\epsilon_q - \mu)} - 1} dp \\ &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{p^{n-1}}{e^{\beta\epsilon_q} - 1} dp \\ &= \frac{1}{2^{n-1}\pi^{\frac{n}{2}}\hbar^n\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{p^{n-1}}{e^{\beta\frac{p^2}{2m}} - 1} dp. \end{aligned}$$

Letting $x = \beta \frac{p^2}{2m} \implies dx = \beta \frac{p}{m} dp$, we have

$$\begin{aligned}
 n(0) &= \frac{1}{2^{n-1} \pi^{\frac{n}{2}} \hbar^n \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{\left(\frac{2mx}{\beta}\right)^{\frac{n-1}{2}}}{e^x - 1} \frac{mx}{\beta} \sqrt{\frac{\beta}{2mx}} dx \\
 &= \frac{2^{\frac{n-1}{2}}}{2^{n-1} \sqrt{2} \pi^{\frac{n}{2}} \hbar^n \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{\left(\frac{mx}{\beta}\right)^{\frac{n-1}{2}}}{e^x - 1} \sqrt{\frac{mx}{\beta}} dx \\
 &= \frac{2^{\frac{n-1}{2}}}{2^{n-\frac{1}{2}} \pi^{\frac{n}{2}} \hbar^n \Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{\beta}\right)^{\frac{n}{2}} \int_0^\infty \frac{x^{\frac{n}{2}}}{e^x - 1} dx \\
 &= \frac{1}{2^{\frac{n}{2}} \pi^{\frac{n}{2}} \hbar^n \Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{\beta}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} + 1\right) \zeta\left(\frac{n}{2} + 1\right) \\
 &= \frac{1}{2^{\frac{n}{2}} \pi^{\frac{n}{2}} \hbar^n} \left(\frac{m}{\beta}\right)^{\frac{n}{2}} \left(\frac{n}{2}\right) \zeta\left(\frac{n}{2} + 1\right) \\
 &= \frac{n}{2^{\frac{n}{2}+1} \pi^{\frac{n}{2}} \hbar^n} \left(\frac{m}{\beta}\right)^{\frac{n}{2}} \zeta\left(\frac{n}{2} + 1\right) \\
 &= \frac{n}{2^{\frac{n}{2}+1} \pi^{\frac{n}{2}} \hbar^n} (mk_B T)^{\frac{n}{2}} \zeta\left(\frac{n}{2} + 1\right) \\
 &= \left(\frac{mk_B T}{2\pi \hbar^2}\right)^{\frac{n}{2}} \frac{n}{2} \zeta\left(\frac{n}{2} + 1\right).
 \end{aligned}$$

Then, we define

$$T_{\text{crit}} \equiv \left(\frac{2\pi \hbar^2}{mk_B}\right) \left(\frac{2}{n \zeta\left(\frac{n}{2} + 1\right)}\right)^{\frac{2}{n}}$$

Thus,

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{crit}}}\right)^{\frac{n}{2}}.$$

When we consider massless bosons, *i.e.* ultra-relativistic bosons, then we have

$$\begin{aligned}
 n(0) \equiv \frac{N(\mu)}{V} &= \frac{1}{2^{n-1} \pi^{\frac{n}{2}} \hbar^n \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{p^{n-1}}{e^{\beta \epsilon_q} - 1} dp \\
 &= \frac{1}{2^{n-1} \pi^{\frac{n}{2}} \hbar^n \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{p^{n-1}}{e^{\beta pc} - 1} dp.
 \end{aligned}$$

Letting $x = \beta pc \implies dx = \beta c dp$, then

$$\begin{aligned}
 n(0) &\equiv \frac{N(\mu)}{V} = \frac{1}{2^{n-1} \pi^{\frac{n}{2}} \hbar^n \Gamma(\frac{n}{2})} \int_0^\infty \frac{\left(\frac{x}{\beta c}\right)^{n-1}}{e^x - 1} \frac{1}{\beta c} dx \\
 &= \frac{1}{2^{n-1} \pi^{\frac{n}{2}} \hbar^n \Gamma(\frac{n}{2})} \left(\frac{1}{\beta c}\right)^n \int_0^\infty \frac{x^{n-1}}{e^x - 1} dx \\
 &= \frac{1}{2^{n-1} \pi^{\frac{n}{2}} \hbar^n \Gamma(\frac{n}{2})} \left(\frac{1}{\beta c}\right)^n \Gamma(n) \zeta(n) \\
 &= \frac{\Gamma(n)}{2^{n-1} \Gamma(\frac{n}{2})} \left(\frac{1}{\pi \hbar^2} \left(\frac{1}{\beta c}\right)^2\right)^{\frac{n}{2}} \zeta(n) \\
 &= \frac{\Gamma(n)}{2^{n-1} \Gamma(\frac{n}{2})} \left(\frac{(k_B T)^2}{\pi c^2 \hbar^2}\right)^{\frac{n}{2}} \zeta(n) \\
 &= \frac{2^{n-1} \Gamma(\frac{n+1}{2})}{2^{n-1} \sqrt{\pi}} \left(\frac{(k_B T)^2}{\pi c^2 \hbar^2}\right)^{\frac{n}{2}} \zeta(n) \\
 &= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \left(\frac{(k_B T)^2}{\pi c^2 \hbar^2}\right)^{\frac{n}{2}} \zeta(n) \\
 &= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}} \left(\frac{k_B T}{\sqrt{\pi} c \hbar}\right)^n \zeta(n).
 \end{aligned}$$

Then, we define

$$T_{\text{crit}} \equiv \left(\frac{\sqrt{\pi} c \hbar}{k_B}\right) \left(\frac{\sqrt{\pi}}{\Gamma(\frac{n+1}{2}) \zeta(n)}\right)^{\frac{1}{n}}.$$

Thus,

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_{\text{crit}}}\right)^n.$$

■

Homework 10

Problem 1

- (a) Show that in Debye theory, the energy as a function of temperature can be expressed in terms of the following integral (ignoring the ground state energy):

$$E = 9N \frac{(k_B T)^4}{(\hbar \omega_D)^3} \int_0^{\beta \hbar \omega_D} \frac{x^3}{e^x - 1} dx$$

- (b) Show that this gives the correct limiting behavior in the high-temperature limit and low-temperature limit.
- (c) Compute this integral explicitly, either numerically or analytically, and compare with the Einstein formula:

$$E = \frac{3N\hbar\omega_0}{e^{\beta\hbar\omega_0} - 1}$$

where ω_0 here is some characteristic oscillation frequency of the solid. Can we relate ω_D to ω_0 ? How do their heat capacities compare? Note: if you are computing the integral numerically, you want to be careful near $x = 0$ where the denominator blows up but the integrand $x^3/(e^x - 1)$ is finite. Just make sure you're doing the calculation to high accuracy and always use double-precision floating point numbers.

- (d) Attached is that same experimental data from that diamond we looked at a few weeks ago. Compare the data using both Einstein and Debye's formulas (you will have to take the derivative dE/dT). Which gives the better fit? Use this data to find the value of ω_D for diamond.
- (e) Compute ω_D as a theoretical value based on the material properties of diamond and compare to the value you found in part (d). You will have to look up some of the properties of diamond online. Keep about two significant figures (depending on how precisely you've worked out ω_D from part (d)).

Solution. (a) We have that

$$\begin{aligned} E &= \sum_{\text{modes } \omega} \hbar \omega n_B(\beta) \\ &= \sum_{\text{modes } \omega} \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1} \\ &= 3 \iint \frac{\hbar \omega}{e^{\beta \hbar \omega} - 1} \frac{d^3 x d^3 p}{(2\pi \hbar)^3} \\ &= \frac{3V\hbar}{(2\pi \hbar)^3} \int \frac{\omega}{e^{\beta \hbar \omega} - 1} d^3 p. \end{aligned}$$

We know that $\omega = kc$ and $p = \hbar k = \frac{\hbar \omega}{c} \implies dp = \frac{\hbar}{c} d\omega$. Then

$$\begin{aligned} E &= \frac{3V\hbar}{(2\pi \hbar)^3} \int \frac{\omega}{e^{\beta \hbar \omega} - 1} \left(\frac{\hbar}{c}\right)^3 d^3 \omega \\ &= \frac{3V\hbar}{(2\pi c)^3} \int_0^{\omega_D} \frac{\omega}{e^{\beta \hbar \omega} - 1} 4\pi \omega^2 d\omega \\ &= \frac{3V\hbar}{2\pi^2 c^3} \int_0^{\omega_D} \frac{\omega^3}{e^{\beta \hbar \omega} - 1} d\omega. \end{aligned}$$

Let $x = \beta\hbar\omega \rightarrow dx = \beta\hbar d\omega$, then

$$\begin{aligned} E &= \frac{3V\hbar}{2\pi^2c^3} \int_0^{\beta\hbar\omega_D} \frac{\left(\frac{x}{\beta\hbar}\right)^3}{e^x - 1} \frac{1}{\beta\hbar} dx \\ &= \frac{3V}{2\pi^2(\hbar c)^3\beta^4} \int_0^{\beta\hbar\omega_D} \frac{x^3}{e^x - 1} dx. \end{aligned}$$

The Debye frequency ω_D is defined as $\omega_D^3 = 6\pi^2nc^3 \Rightarrow \frac{1}{c^3} = \frac{6\pi^2N}{\omega_D^3V}$. Replacing, we have

$$\begin{aligned} E &= \frac{3V}{2\pi^2(\hbar c)^3\beta^4} \int_0^{\beta\hbar\omega_D} \frac{x^3}{e^x - 1} dx \\ &= \frac{3V}{2\pi^2\hbar^3\beta^4} \frac{6\pi^2N}{\omega_D^3V} \int_0^{\beta\hbar\omega_D} \frac{x^3}{e^x - 1} dx \\ &= \frac{9N}{(\omega_D\hbar)^3\beta^4} \int_0^{\beta\hbar\omega_D} \frac{x^3}{e^x - 1} dx \\ &= 9N \frac{(k_BT)^4}{(\omega_D\hbar)^3} \int_0^{\beta\hbar\omega_D} \frac{x^3}{e^x - 1} dx. \end{aligned}$$

- (b) • For high temperatures, we have $\beta\hbar\omega_D \ll 1$, then $e^x \sim 1 + x$, and then

$$\begin{aligned} E &\sim 9N \frac{(k_BT)^4}{(\omega_D\hbar)^3} \int_0^{\beta\hbar\omega_D} x^2 dx \\ &= 9N \frac{(k_BT)^4}{(\omega_D\hbar)^3} \frac{(\beta\hbar\omega_D)^3}{3} \\ &= 3Nk_BT. \end{aligned}$$

- For low temperatures, we have $\beta\hbar\omega_D \gg 1$, then

$$\begin{aligned} E &\sim 9N \frac{(k_BT)^4}{(\omega_D\hbar)^3} \int_0^\infty \frac{x^3}{e^x - 1} dx \\ &= 9N \frac{(k_BT)^4}{(\omega_D\hbar)^3} \Gamma(4)\zeta(4) \\ &= 9N \frac{(k_BT)^4}{(\omega_D\hbar)^3} \frac{\pi^4}{15} \\ &= \frac{3N(\pi k_BT)^4}{5(\omega_D\hbar)^3}. \end{aligned}$$

(c) The energy per particle of each mostly coincide.

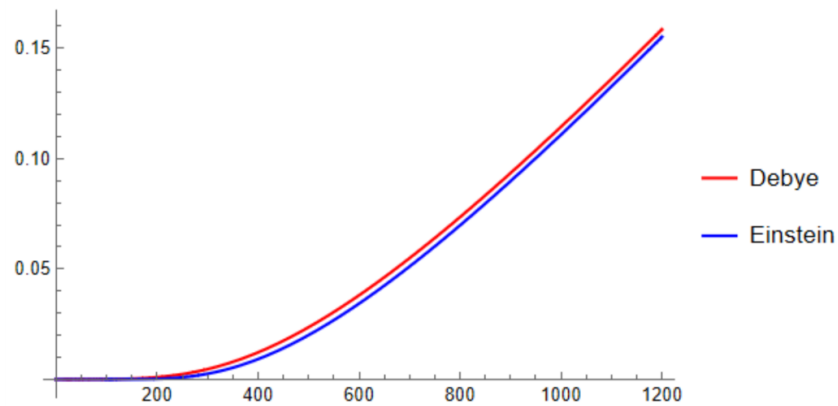


Figure 16: Comparison Plot of Debye vs. Einstein.

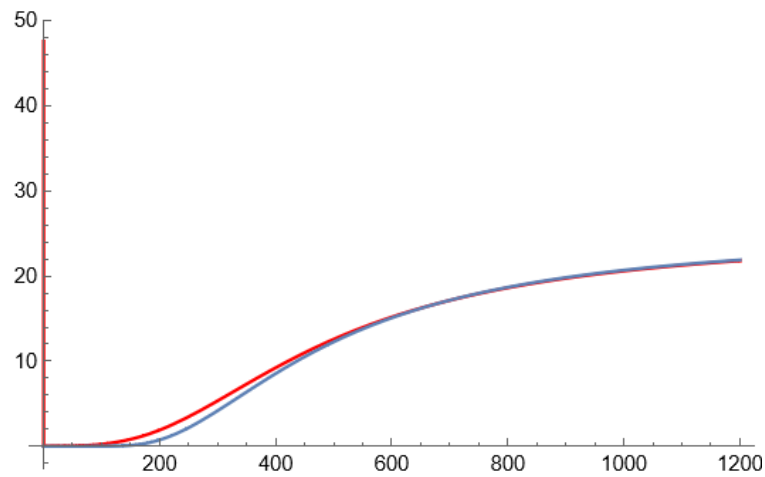


Figure 17: Comparison Plot of Debye vs. Einstein.

- (d) For high temperatures, they both coincide. At lower temperatures, the Debye equations better fits the data.

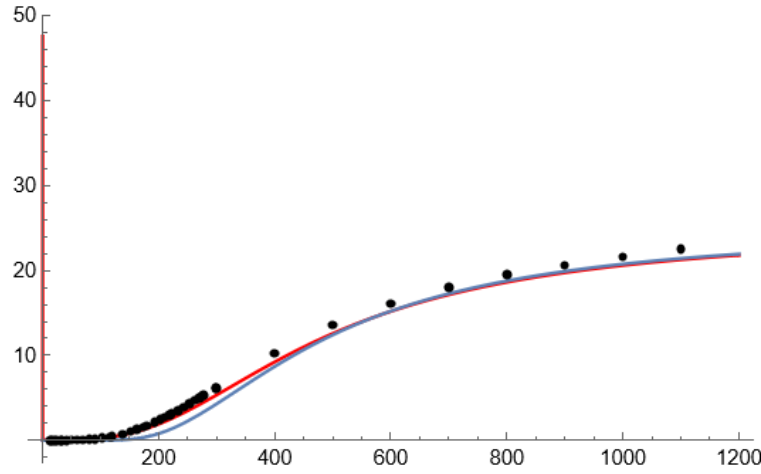


Figure 18: Comparison Plot of Debye vs. Einstein vs. Data.

- (e) For a diamond, $c_s = 12,000 \text{ m/s}$, $a = 3.57 \times 10^{10} \text{ m}$, $n = 176.2 \times 10^{27} \text{ m}^{-3}$, and $w_D^3 = 6\pi^2 n c_s^3$. Then, we have

$$\varepsilon_1 = \frac{E_1}{N} = 9 \frac{(k_B T)^4}{(\hbar \omega_D)^3} \int_0^{\beta \hbar \omega_D} \frac{x^3}{e^x - 1} dx$$

$$\varepsilon_2 = \frac{E_2}{N} = \frac{3 \hbar \omega_0}{e^{\beta \hbar \omega_0} - 1}$$

Additionally, $\omega_D = (6\pi^2 n c_s^3)^{\frac{1}{3}} = 2.6 \times 10^{14} \text{ s}^{-1}$. Then

$$\frac{dE_2}{dT} = - \frac{3(\hbar \omega_D)^2 e^{\beta \hbar \omega_D}}{(e^{\beta \hbar \omega_D} - 1)^2} \frac{d\beta}{dT} N_A$$

and

$$C_{V,2} = \frac{3(\hbar \omega_D)^2 e^{\beta \hbar \omega_D} N_A}{k_B T^2 (e^{\beta \hbar \omega_D} - 1)^2}.$$

The curve fits give us

$$\omega'_d = 0.996 \omega_D$$

$$\omega'_0 = 0.757 \omega_D,$$

where ω'_d and ω'_0 are the fitted values and ω_D is the theoretical value based on material properties. ■

Problem 2

Estimate the formula for sound speed in a material assuming a set of masses on springs characterized by mass m , spring constant κ , and an atomic spacing a . Given this estimate, show the Debye frequency ω_D is of the same order as ω_0 , the characteristic oscillation frequency of a single spring-mass system.

Solution. We have

$$\begin{aligned} m\ddot{x}_i &= \kappa [(x_{i+1} - x_i) - a] + \kappa [a - (x_i - x_{i-1})] \\ &= \kappa (x_{i+1} - 2x_i + x_{i-1}). \end{aligned}$$

Consider

$$x_n = e^{i(\omega t - kna)} \implies x_n = -\omega^2 e^{i(\omega t - kna)},$$

then

$$\begin{aligned} -m\omega^2 &= \kappa (e^{ika} - 2 + e^{-ika}) \\ \implies \omega^2 &= \frac{\kappa}{m} (2 - (e^{ika} + e^{-ika})) \\ &= \frac{\kappa}{m} (2 - 2\cos(ka)) \\ &= \frac{2\kappa}{m} (1 - \cos(ka)) \\ &= \frac{4\kappa}{m} \sin^2\left(\frac{ka}{2}\right) \\ \implies \omega &= 2\sqrt{\frac{\kappa}{m}} \left| \sin\left(\frac{ka}{2}\right) \right|. \end{aligned}$$

The speed of sound is given by taking the limit $k \rightarrow 0$ of the derivative of ω with respect to k . Thus,

$$\begin{aligned} v_{\text{sound}} &= \lim_{k \rightarrow 0} \frac{d\omega}{dk} \\ &= \lim_{k \rightarrow 0} 2\sqrt{\frac{\kappa}{m}} \frac{a}{2} \left| \cos\left(\frac{ka}{2}\right) \right| \\ &= a\sqrt{\frac{\kappa}{m}} \\ &= a\omega_0. \end{aligned}$$

Since $\omega_D^3 = 6\pi^2 n v_s^3$, then

$$\omega_D^3 = 6\pi^2 n (a\omega_0)^3 \sim na^3 \omega_0^3.$$

Additionally, $n \sim \frac{1}{a^3}$, which gives us

$$\omega_D \sim \omega_0.$$

■

Problem 3

(Optional) Our model for solids seems pretty good at this point! But how about we get rid of the ad-hoc cutoff and instead sum over modes using the dispersion relation $\omega(k_x, k_y, k_z)$ we found in class (assuming a square lattice even though we know diamonds are more complicated). This will amount to a 3D integral that must be computed numerically for a given choice of temperature (there's not a lot of analytical work we can do to simplify things). Perform the integral for the energy numerically for a range of temperatures, and compute the derivative with temperature numerically as well to get the heat capacity. Check against our other methods and the diamond dataset. At this point the results will be very similar to the Debye model, but you should be able to see subtle distinctions that give a better fit to the data.

Solution.



Practice Exam

Closed book, no notes. If you need a formula, please just ask for it. I am happy to provide a range of formulas within reason.

Your grade will be based on your process to the solution, not on the answer itself. If you just write down an answer with no explanation, that gets zero credit.

Problem 1

Estimate the critical temperature T_{BE} of transition to a Bose-Einstein condensate, as a function of the number density n and the particle mass m (assume massive bosons, like ^4He).

Solution. ■

Problem 2

Imagine a set of non-interacting particles that can each be in one of two quantum states, separated by energy ΔE (if you want, you can imagine them as particles with spins in a magnetic field). Compute the total energy of the system as a function of the number of particles and the temperature $E(N, T)$.

Solution. ■

Problem 3

Recall for Fermions we could write the total number of particles as a sum over eigenstates:

$$N = \sum_{\text{eigenstates}} (\text{occupation } \#)$$

Use this fact to derive the Fermi energy for a cold (zero temperature) gas of electrons as a function of the number density n_e and electron mass m_e . Actually, if you have some other method of deriving the Fermi energy, that's fine too. If you're having trouble you may estimate the Fermi energy instead for half credit.

Solution. ■

Problem 4

Recall in class we discussed the heat capacity of solids using (at least) three different models. Derive the energy $E(T)$ for a solid using any one of these models (in principle you could take the derivative to get the heat capacity but I won't make you do that). When you are done, you must also write out in words the next correction you would need to make to improve the model.

Solution. ■

Problem 5

Derive the ideal gas law $PV = Nk_B T$ by any means.

Solution. ■