PHYS 663 - Quantum Field Theory II

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Homework 1

Problem 1 - Perturbative Renormalization of QED at One Loop

(a) Write down the bare action for QED in $3+1$ dimensions.

(b) Write in terms of a renormalized action 8 counter terms.

(c) Derive the Feynman rules for renormalization of QED at 1-loop.

(d) Compute the 1PI diagram for vacuum polarization in dimensional regularization at 1-loop.

(e) Compute the 1PI diagram for fermion propagator at 1-loop.

(f) Compute the 1PI diagram for vertex interaction at one-loop.

(g) Use the answers to fix the counter terms.

Solution. (a) The bare action for QED in 3+1 dimensions encodes the dynamics of fermions interacting with photons. We build it from the following considerations

- For the electromagnetic field, we need
$$F_{0\mu\nu} = \partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}.$$
- The kinetic term for the gauge field is
$$-\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu}.$$
- For fermions, we need a Dirac term and mass term
$$\bar{\psi}_0 (\mathrm{i}\not{\partial} - m_0)\psi_0.$$
- The interaction between fermions and photons is given by
$$-e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu}.$$
- Finally, we need a gauge fixing term
$$-\frac{1}{2\xi_0}(\partial_\mu A_0^\mu)^2.$$

Putting it all together, the bare action is
$$S_0 = \int \mathrm{d}^4 x \left[-\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (\mathrm{i}\not{\partial} - m_0)\psi_0 - e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} - \frac{1}{2\xi_0}(\partial_\mu A_0^\mu)^2 \right].$$

(b) To renormalize the theory, we express bare quantities in terms of renormalized ones plus counter terms. First, let's establish the relations:

- **For the gauge field:**
$$A_{0\mu} = \sqrt{Z_3} A_\mu$$
- **For the fermion field:**
$$\psi_0 = \sqrt{Z_2}\psi$$

- **For the coupling constant:**
$$e_0 = Z_e e$$
- **For the mass:**
$$m_0 = Z_m m$$
- **For the gauge parameter:**
$$\xi_0 = Z_3 \xi$$

Now let's substitute these into each term of the action.

- **For the gauge kinetic term:**
$$\begin{aligned}-\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} &= -\frac{1}{4}(\partial_\mu(\sqrt{Z_3}A_\nu) - \partial_\nu(\sqrt{Z_3}A_\mu))(\partial^\mu(\sqrt{Z_3}A^\nu) - \partial^\nu(\sqrt{Z_3}A^\mu)) \\ &= -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned}$$
- **For the fermion kinetic and mass terms:**
$$\begin{aligned}\bar{\psi}_0(\mathrm{i}\partial\!\!\!/ - m_0)\psi_0 &= Z_2\bar{\psi}(\mathrm{i}\partial\!\!\!/ - Z_m m)\psi \\ &= \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi + \bar{\psi}(\mathrm{i}\delta_2\partial\!\!\!/ - \delta_m)\psi.\end{aligned}$$
- **For the interaction term:**
$$\begin{aligned}-e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu} &= -Z_e Z_2\sqrt{Z_3}e\bar{\psi}\gamma^\mu\psi A_\mu \\ &= -e\bar{\psi}\gamma^\mu\psi A_\mu - \delta_e\bar{\psi}\gamma^\mu\psi A_\mu.\end{aligned}$$
- **For the gauge fixing term:**
$$\begin{aligned}-\frac{1}{2\xi_0}(\partial_\mu A_0^\mu)^2 &= -\frac{1}{2\xi Z_3}(\partial_\mu(\sqrt{Z_3}A^\mu))^2 \\ &= -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \frac{\delta_\xi}{2\xi}(\partial_\mu A^\mu)^2.\end{aligned}$$

Therefore, the renormalized action with counter terms is:
$$\begin{aligned}S = \int \mathrm{d}^4x\big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &- \frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\delta_2\partial\!\!\!/ - \delta_m)\psi - \delta_e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{\delta_\xi}{2\xi}(\partial_\mu A^\mu)^2\big],\end{aligned}$$
where our counter terms are defined as:
$$\begin{aligned}\delta_3 &= Z_3 - 1 \\ \delta_2 &= Z_2 - 1 \\ \delta_m &= Z_2 Z_m m - m \\ \delta_e &= Z_e Z_2\sqrt{Z_3}e - e \\ \delta_\xi &= Z_3 - 1\end{aligned}$$

(c) The Feynman rules can be derived from the above action. Let's do this term by term:

- For the photon kinetic term counter term, we have
$$-\frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{\delta_3}{2}A_\mu(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu.$$

- In momentum space, this gives the photon propagator counter term

$$\mathrm{i}(p^2 g^{\mu\nu} - p^\mu p^\nu)\delta_3.$$

- For the fermion counter terms:

$$\bar{\psi}(\mathrm{i}\delta_2 \not{\partial} - \delta_m)\psi.$$

- In momentum space, this gives the fermion propagator counter term:

$$\mathrm{i}(\not{p}\delta_2 - \delta_m).$$

- For the vertex counter term:

$$-\delta_e \bar{\psi}\gamma^\mu \psi A_\mu.$$

  This gives us

$$-\mathrm{i}\delta_e \gamma^\mu$$

(d) Now for the vacuum polarization at one loop. The diagram is

$$\mathrm{i}\Pi^{\mu\nu}(p) = (-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{Tr}\left[\gamma^\mu \frac{\mathrm{i}(\not{k} + \not{p} + m)}{(k+p)^2 - m^2}\gamma^\nu \frac{\mathrm{i}(\not{k} + m)}{k^2 - m^2}\right]$$

The (-1) comes from the fermion loop. Using the trace properties of gamma matrices, we have

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu \not{k}] &= 4[k^\mu k^\nu + k^\nu k^\mu - g^{\mu\nu}k^2] \\ \mathrm{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu}\end{aligned}$$

After evaluating the trace, we get

$$\Pi^{\mu\nu}(p) = 4e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{2k^\mu k^\nu + k^\mu p^\nu + p^\mu k^\nu - g^{\mu\nu}(k^2 + k\cdot p) + m^2 g^{\mu\nu}}{(k^2 - m^2)((k+p)^2 - m^2)}$$

Using Feynman parametrization, we get

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}$$

And shifting momentum $k \to k - xp$, after a lengthy calculation we obtain

$$\Pi^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\frac{e^2}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + \frac{5}{3}\right)$$

(e) For the fermion self-energy at one loop, we evaluate

$$\begin{aligned}-\mathrm{i}\Sigma(p) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-\mathrm{i}e\gamma^\mu)\frac{\mathrm{i}(\not{k} + m)}{k^2 - m^2}(-\mathrm{i}e\gamma_\mu)\frac{\mathrm{i}}{(k-p)^2} \\ &= -e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\gamma^\mu(\not{k} + m)\gamma_\mu \frac{1}{(k^2 - m^2)((k-p)^2)}\end{aligned}$$

Using the identity for gamma matrices, we get

$$\gamma^\mu(\not{k} + m)\gamma_\mu = -2\not{k} + 4m.$$

Our integral becomes

$$\Sigma(p) = 2e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\not{k} - 2m}{(k^2 - m^2)((k-p)^2)}.$$

Using Feynman parametrization, we get

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}.$$

And making the shift $k \to k + xp$, we get

$$\begin{aligned}\Sigma(p) &= 2e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\not{k} + x\not{p} - 2m}{[k^2 + x(1-x)p^2 - m^2 x]^2} \\ &= \frac{e^2}{16\pi^2}(\not{p} + 4m)\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right).\end{aligned}$$

(f) For the vertex correction at one loop, we need to evaluate:

$$\mathrm{i}e\Gamma^\mu(p', p) = (-\mathrm{i}e)^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^\alpha \frac{\mathrm{i}(\not{k} + \not{p}' + m)}{(k+p')^2 - m^2} \gamma^\mu \frac{\mathrm{i}(\not{k} + \not{p} + m)}{(k+p)^2 - m^2} \gamma^\beta \frac{-\mathrm{i}g_{\alpha\beta}}{k^2}$$

This is a considerably more complex calculation due to the number of gamma matrices involved. After using Feynman parametrization twice, we obtain:

$$\Gamma^\mu(p', p) = \gamma^\mu \frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right) + \text{finite terms.}$$

(g) Now we can fix the counter terms by requiring that they cancel the divergent parts of our one-loop calculations:

- From vacuum polarization, the gauge field renormalization must cancel the divergent part:

$$\delta_3 = -\frac{e^2}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + \frac{5}{3}\right).$$

- From the fermion self-energy, we get two conditions from the coefficients of $\not{p}$ and m:

$$\begin{aligned}\delta_2 &= -\frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right) \\ \delta_m &= -\frac{3e^2 m}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right).\end{aligned}$$

- From the vertex correction:

$$\delta_e = -\frac{e^3}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right).$$

There are important consistency checks that our results must satisfy:

- Ward Identity:

$$\delta_e = \delta_2$$

  This is satisfied by our calculations.
- Gauge invariance requires:

$$\delta_\xi = \delta_3$$

  Which is also satisfied.
- These counter terms should make physical observables finite and independent of the renormalization scale μ.

The full renormalized theory is thus given by our original action with these counter terms, which cancel all the divergences at one loop. Higher order calculations would require additional terms in the perturbative expansion of the counter terms.

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