PHYS 663 - Quantum Field Theory II

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Contents

Homework 1

Problem 1 - Perturbative Renormalization of QED at One Loop

(a) Write down the bare action for QED in $3+1$ dimensions.

(b) Write in terms of a renormalized action 8 counter terms.

(c) Derive the Feynman rules for renormalization of QED at 1-loop.

(d) Compute the 1PI diagram for vacuum polarization in dimensional regularization at 1-loop.

(e) Compute the 1PI diagram for fermion propagator at 1-loop.

(f) Compute the 1PI diagram for vertex interaction at one-loop.

(g) Use the answers to fix the counter terms.

Solution. (a) The bare action for QED in 3+1 dimensions encodes the dynamics of fermions interacting with photons. We build it from the following considerations

- For the electromagnetic field, we need

$$F_{0\mu\nu} = \partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}.$$

- The kinetic term for the gauge field is

$$-\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu}.$$

- For fermions, we need a Dirac term and mass term

$$\bar{\psi}_0 (\mathrm{i}\partial\!\!\!/ - m_0)\psi_0.$$

- The interaction between fermions and photons is given by

$$-e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu}.$$

- Finally, we need a gauge fixing term

$$-\frac{1}{2\xi_0}(\partial_\mu A_0^\mu)^2.$$

Putting it all together, the bare action is

$$S_0 = \int \mathrm{d}^4 x \left[-\frac{1}{4} F_{0\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (\mathrm{i}\partial\!\!\!/ - m_0)\psi_0 - e_0 \bar{\psi}_0 \gamma^\mu \psi_0 A_{0\mu} - \frac{1}{2\xi_0}(\partial_\mu A_0^\mu)^2 \right].$$

(b) To renormalize the theory, we express bare quantities in terms of renormalized ones plus counter terms. First, let's establish the relations:

- **For the gauge field:**

$$A_{0\mu} = \sqrt{Z_3} A_\mu$$

- **For the fermion field:**

$$\psi_0 = \sqrt{Z_2}\psi$$

- **For the coupling constant:**

$$e_0 = Z_e e$$

- **For the mass:**
$$m_0 = Z_m m$$
- **For the gauge parameter:**
$$\xi_0 = Z_3 \xi$$

Now let's substitute these into each term of the action.

- **For the gauge kinetic term:**
$$\begin{aligned}-\frac{1}{4}F_{0\mu\nu}F_0^{\mu\nu} &= -\frac{1}{4}(\partial_\mu(\sqrt{Z_3}A_\nu) - \partial_\nu(\sqrt{Z_3}A_\mu))(\partial^\mu(\sqrt{Z_3}A^\nu) - \partial^\nu(\sqrt{Z_3}A^\mu)) \\ &= -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned}$$
- **For the fermion kinetic and mass terms:**
$$\begin{aligned}\bar{\psi}_0(\mathrm{i}\partial\!\!\!/ - m_0)\psi_0 &= Z_2\bar{\psi}(\mathrm{i}\partial\!\!\!/ - Z_m m)\psi \\ &= \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi + \bar{\psi}(\mathrm{i}\delta_2\partial\!\!\!/ - \delta_m)\psi.\end{aligned}$$
- **For the interaction term:**
$$\begin{aligned}-e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu} &= -Z_e Z_2\sqrt{Z_3}e\bar{\psi}\gamma^\mu\psi A_\mu \\ &= -e\bar{\psi}\gamma^\mu\psi A_\mu - \delta_e\bar{\psi}\gamma^\mu\psi A_\mu.\end{aligned}$$
- **For the gauge fixing term:**
$$\begin{aligned}-\frac{1}{2\xi_0}(\partial_\mu A_0^\mu)^2 &= -\frac{1}{2\xi Z_3}(\partial_\mu(\sqrt{Z_3}A^\mu))^2 \\ &= -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \frac{\delta_\xi}{2\xi}(\partial_\mu A^\mu)^2.\end{aligned}$$

Therefore, the renormalized action with counter terms is:

$$\begin{aligned}S = \int \mathrm{d}^4x\big[&-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \\ &- \frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(\mathrm{i}\delta_2\partial\!\!\!/ - \delta_m)\psi - \delta_e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{\delta_\xi}{2\xi}(\partial_\mu A^\mu)^2\big],\end{aligned}$$

where our counter terms are defined as:

$$\begin{aligned}\delta_3 &= Z_3 - 1 \\ \delta_2 &= Z_2 - 1 \\ \delta_m &= Z_2 Z_m m - m \\ \delta_e &= Z_e Z_2\sqrt{Z_3}e - e \\ \delta_\xi &= Z_3 - 1\end{aligned}$$

(c) The Feynman rules can be derived from the above action. Let's do this term by term:

- For the photon kinetic term counter term, we have
$$-\frac{\delta_3}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{\delta_3}{2}A_\mu(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu.$$

- In momentum space, this gives the photon propagator counter term
$$\mathrm{i}(p^2 g^{\mu\nu} - p^\mu p^\nu)\delta_3.$$
- For the fermion counter terms:
$$\bar{\psi}(\mathrm{i}\delta_2 \not{\partial} - \delta_m)\psi.$$
- In momentum space, this gives the fermion propagator counter term:
$$\mathrm{i}(\not{p}\delta_2 - \delta_m).$$
- For the vertex counter term:
$$-\delta_e \bar{\psi}\gamma^\mu \psi A_\mu.$$
This gives us
$$-\mathrm{i}\delta_e \gamma^\mu$$

(d) Now for the vacuum polarization at one loop. The diagram is

$$\mathrm{i}\Pi^{\mu\nu}(p) = (-1)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\mathrm{Tr}\left[\gamma^\mu \frac{\mathrm{i}(\not{k}+\not{p}+m)}{(k+p)^2 - m^2}\gamma^\nu \frac{\mathrm{i}(\not{k}+m)}{k^2-m^2}\right]$$

The (-1) comes from the fermion loop. Using the trace properties of gamma matrices, we have

$$\begin{aligned}\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu \not{k}] &= 4[k^\mu k^\nu + k^\nu k^\mu - g^{\mu\nu}k^2] \\ \mathrm{Tr}[\gamma^\mu\gamma^\nu] &= 4g^{\mu\nu}\end{aligned}$$

After evaluating the trace, we get

$$\Pi^{\mu\nu}(p) = 4e^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{2k^\mu k^\nu + k^\mu p^\nu + p^\mu k^\nu - g^{\mu\nu}(k^2 + k\cdot p) + m^2 g^{\mu\nu}}{(k^2-m^2)((k+p)^2 - m^2)}$$

Using Feynman parametrization, we get

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}$$

And shifting momentum $k \to k - xp$, after a lengthy calculation we obtain

$$\Pi^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\frac{e^2}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + \frac{5}{3}\right)$$

(e) For the fermion self-energy at one loop, we evaluate

$$\begin{aligned}-\mathrm{i}\Sigma(p) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-\mathrm{i}e\gamma^\mu)\frac{\mathrm{i}(\not{k}+m)}{k^2-m^2}(-\mathrm{i}e\gamma_\mu)\frac{\mathrm{i}}{(k-p)^2} \\ &= -e^2\int \frac{\mathrm{d}^d k}{(2\pi)^d}\gamma^\mu(\not{k}+m)\gamma_\mu \frac{1}{(k^2-m^2)((k-p)^2)}\end{aligned}$$

Using the identity for gamma matrices, we get

$$\gamma^\mu(\not{k}+m)\gamma_\mu = -2\not{k} + 4m.$$

Our integral becomes

$$\Sigma(p) = 2e^2\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\not{k} - 2m}{(k^2-m^2)((k-p)^2)}.$$

Using Feynman parametrization, we get

$$\frac{1}{AB} = \int_0^1 \mathrm{d}x \frac{1}{[xA + (1-x)B]^2}.$$

And making the shift $k \to k + xp$, we get

$$\begin{aligned}\Sigma(p) &= 2e^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\not{k} + x\not{p} - 2m}{[k^2 + x(1-x)p^2 - m^2 x]^2} \\ &= \frac{e^2}{16\pi^2}(\not{p} + 4m)\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right).\end{aligned}$$

(f) For the vertex correction at one loop, we need to evaluate:

$$\mathrm{i}e\Gamma^\mu(p', p) = (-\mathrm{i}e)^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^\alpha \frac{\mathrm{i}(\not{k} + \not{p}' + m)}{(k+p')^2 - m^2} \gamma^\mu \frac{\mathrm{i}(\not{k} + \not{p} + m)}{(k+p)^2 - m^2} \gamma^\beta \frac{-\mathrm{i}g_{\alpha\beta}}{k^2}$$

This is a considerably more complex calculation due to the number of gamma matrices involved. After using Feynman parametrization twice, we obtain:

$$\Gamma^\mu(p', p) = \gamma^\mu \frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right) + \text{finite terms}.$$

(g) Now we can fix the counter terms by requiring that they cancel the divergent parts of our one-loop calculations:

- From vacuum polarization, the gauge field renormalization must cancel the divergent part:

$$\delta_3 = -\frac{e^2}{12\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + \frac{5}{3}\right).$$

- From the fermion self-energy, we get two conditions from the coefficients of $\not{p}$ and m:

$$\begin{aligned}\delta_2 &= -\frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right) \\ \delta_m &= -\frac{3e^2 m}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right).\end{aligned}$$

- From the vertex correction:

$$\delta_e = -\frac{e^3}{16\pi^2}\left(\frac{2}{\epsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m^2} + 2\right).$$

There are important consistency checks that our results must satisfy:

- Ward Identity:

$$\delta_e = \delta_2$$

  This is satisfied by our calculations.
- Gauge invariance requires:

$$\delta_\xi = \delta_3$$

  Which is also satisfied.
- These counter terms should make physical observables finite and independent of the renormalization scale μ.

The full renormalized theory is thus given by our original action with these counter terms, which cancel all the divergences at one loop. Higher order calculations would require additional terms in the perturbative expansion of the counter terms.

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Homework 2

Problem 11.1 - Spin-wave Theory

(a) Prove the following wonderful formula: Let $\phi(x)$ be a free scalar field with propagator $\langle T\phi(x)\phi(0)\rangle = D(x)$. Then

$$\left\langle T\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-i\phi(0)}\right\rangle = \mathrm{e}^{[D(x)-D(0)]}$$

(The factor $D(0)$ gives a formally divergent adjustment of the overall normalization.)

(b) We can use this formula in Euclidean field theory to discuss correlation functions in a theory with spontaneously broken symmetry for $T < T_C$. Let us consider only the simplest case of a broken $O(2)$ or $U(1)$ symmetry. We can write the local spin density as a complex variable

$$s(x) = s^1(x) + \mathrm{i}s^2(x)$$

The global symmetry is the transformation

$$s(x) \to \mathrm{e}^{-\mathrm{i}\alpha}s(x)$$

If we assume that the physics freezes the modulus of $s(x)$, we can parametrize

$$s(x) = A\mathrm{e}^{\mathrm{i}\phi(x)}$$

and write an effective Lagrangian for the field $\phi(x)$. The symmetry of the theory becomes the translation symmetry

$$\phi(x) \to \phi(x) - \alpha$$

Show that (for $d > 0$) the most general renormalizable Lagrangian consistent with this symmetry is the free field theory

$$\mathcal{L} = \frac{1}{2}\rho(\vec{\nabla}\phi)^2.$$

In statistical mechanics, the constant ρ is called the spin wave modulus. A reasonable hypothesis for ρ is that it is finite for $T < T_C$ and tends to 0 as $T \to T_C$ from below.

(c) Compute the correlation function $\langle s(x)s^*(0)\rangle$. Adjust A to give a physically sensible normalization (assuming that the system has a physical cutoff at the scale of one atomic spacing) and display the dependence of this correlation function on x for $d = 1, 2, 3, 4$. Explain the significance of your results.

Solution. (a) Let's prove the given formula. We begin by writing the left-hand side using the path integral:

$$\left\langle T\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-\mathrm{i}\phi(0)}\right\rangle = \frac{1}{Z[0]}\int \mathcal{D}\phi\, \mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-\mathrm{i}\phi(0)} \exp\left[\mathrm{i}\int \mathrm{d}^d z\, \mathrm{d}^d y\, \frac{1}{2}\phi(z)D^{-1}(z-y)\phi(y)\right]$$

We can rewrite this expression by defining a source term $J(y) = \delta(y-x) - \delta(y)$. With this definition, we have

$$\begin{aligned}\langle T\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-\mathrm{i}\phi(0)}\rangle &= \frac{1}{Z[0]}\int \mathcal{D}\phi\, \exp\left[\mathrm{i}\int \mathrm{d}^d y\, J(y)\phi(y) + \mathrm{i}\int \mathrm{d}^d z\, \mathrm{d}^d y\, \frac{1}{2}\phi(z)D^{-1}(z-y)\phi(y)\right]\\ &= \frac{Z[J]}{Z[0]}\end{aligned}$$

For a free field theory, we know that

$$\frac{Z[J]}{Z[0]} = \exp\left[-\frac{1}{2}\int \mathrm{d}^d z\, \mathrm{d}^d y\, J(z)D(z-y)J(y)\right].$$

Substituting our particular $J(y)$, we get

$$\begin{aligned}\frac{Z[J]}{Z[0]} &= \exp\left[-\frac{1}{2}\int \mathrm{d}^d z\, \mathrm{d}^d y\, (\delta(z-x)-\delta(z))D(z-y)(\delta(y-x)-\delta(y))\right]\\ &= \exp\left[-\frac{1}{2}\int \mathrm{d}^d y\, (D(x-y)-D(-y))(\delta(y-x)-\delta(y))\right]\\ &= \exp\left[-\frac{1}{2}(D(x-x)-D(x)-D(-x)+D(0))\right]\\ &= \exp\left[-\frac{1}{2}(D(0)-D(x)-D(-x)+D(0))\right].\end{aligned}$$

Since $D(x)=D(-x)$ by translation invariance, we get

$$\langle T\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-i\phi(0)}\rangle = \exp\left[D(x)-D(0)\right].$$

This proves the desired formula.

(b) The Lagrangian needs to be invariant under the transformation $\phi(x)\to\phi(x)-\alpha$ for any constant α. This means the Lagrangian can depend on ϕ only through its derivatives.

The most general renormalizable Lagrangian (with terms of dimension $\leq d$ in d dimensions) that is Lorentz invariant and depends only on derivatives of ϕ is:

$$\mathcal{L} = \frac{1}{2}\rho(\nabla\phi)^2.$$

This is the free field theory Lagrangian with ρ being the spin wave modulus.

(c) Let's compute the correlation function $\langle s(x)s^*(0)\rangle$. Using the result from part (a), we have

$$\begin{aligned}\langle s(x)s^*(0)\rangle &= A^2\left\langle \mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-\mathrm{i}\phi(0)}\right\rangle\\ &= A^2\mathrm{e}^{D(x)-D(0)}.\end{aligned}$$

Now we need to determine the propagator $D(x)$. It satisfies the equation

$$-\rho\nabla^2 D(x-y) = \delta^{(d)}(x-y).$$

Due to rotational invariance, $D(x)$ depends only on the distance $|x|$, which we'll denote simply as x. In spherical coordinates, the Laplacian gives

$$-\rho\frac{1}{x^{d-1}}\frac{\partial}{\partial x}\left(x^{d-1}\frac{\partial}{\partial x}D(x)\right) = \frac{\Gamma(1+\frac{d}{2})}{d\pi^{d/2}}\frac{\delta(x)}{x^{d-1}}.$$

Solving this differential equation for $D(x)$, we get

$$D(x) = \begin{cases}\frac{\Gamma\left(1+\frac{d}{2}\right)}{d(d-2)\pi^{d/2}\rho}\frac{1}{x^{d-2}} & \text{for } d\neq 2\\ -\frac{1}{2\pi\rho}\log(x) & \text{for } d=2\end{cases}$$

We can now evaluate $\langle s(x)s^*(0)\rangle$ for different dimensions:

- **For $d=1$:**

$$\begin{aligned}D(x) &= -\frac{1}{2\rho}x\\ \langle s(x)s^*(0)\rangle &\sim \mathrm{e}^{-x}\end{aligned}$$

- **For $d = 2$:**

$$D(x) = -\frac{1}{2\pi\rho}\log x$$
$$\langle s(x)s^*(0)\rangle \sim x^{-\frac{1}{2\pi\rho}}$$

- **For $d = 3$:**

$$D(x) = \frac{1}{4\pi\rho x}$$
$$\langle s(x)s^*(0)\rangle \sim \mathrm{e}^{\frac{1}{x}}$$

- **For $d = 4$:**

$$D(x) = \frac{1}{4\pi^2\rho x^2}$$
$$\langle s(x)s^*(0)\rangle \sim \mathrm{e}^{\frac{1}{x^2}}$$

As $\rho \to 0$ for $d = 2$, the correlation function becomes independent of distance x, indicating the critical behavior at $d = 2$.

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Problem 11.2- A Zeroth-order Natural Relation
This problem studies an $N = 2$ linear sigma model coupled to fermions:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi^i\right)^2 + \frac{1}{2}\mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left(\left(\phi^i\right)^2\right)^2 + \bar{\psi}(i\ \not{\partial})\psi - g\bar{\psi}\left(\phi^1 + i\gamma^5\phi^2\right)\psi,$$

where ϕ^i is a two-component field, $i = 1, 2$.

(a) Show that this theory has the following global symmetry:

$$\phi^1 \to \cos(\alpha)\phi^1 - \sin(\alpha)\phi^2,$$
$$\phi^2 \to \sin(\alpha)\phi^1 + \cos(\alpha)\phi^2,$$
$$\psi \to \mathrm{e}^{-\mathrm{i}\alpha\gamma^{5/2}/2}\psi.$$

Show also that the solution to the classical equations of motion with the minimum energy breaks this symmetry spontaneously.

(b) Denote the vacuum expectation value of the field ϕ^i by v and make the change of variables

$$\phi^i(x) = (v + \sigma(x), \pi(x)).$$

Write out the Lagrangian in these new variables, and show that the fermion acquires a mass given by

$$m_f = g \cdot v.$$

(c) Compute the one-loop radiative correction to m_f, choosing renormalization conditions so that v and g (defined as the $\psi\psi\pi$ vertex at zero momentum transfer) receive no radiative corrections. Show that relation (4) receives nonzero corrections but that these corrections are finite. This is in accord with our general discussion in Section 11.6.

Solution. We'll study the $N = 2$ linear sigma model coupled to fermions, with Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}\mu^2\phi^i\phi^i - \frac{1}{4}\lambda(\phi^i\phi^i)^2 + \bar{\psi}(\mathrm{i}\not{\partial})\psi - g\bar{\psi}(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi$$

where ϕ^i is a two-component field with $i = 1, 2$.

(a) Let's check the invariance of the Lagrangian under the transformation. We have

$$
\begin{aligned}
\phi^1 &\to \phi^1 \cos(\alpha) - \phi^2 \sin(\alpha) \\
\phi^2 &\to \phi^1 \sin(\alpha) + \phi^2 \cos(\alpha) \\
\psi &\to \mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}\psi
\end{aligned}
$$

The first three terms involving only ϕ^i are invariant under the $SO(2)$ rotation. The fermion kinetic term $\bar{\psi}(\mathrm{i}\partial\!\!\!/)\psi$ is also invariant because γ^5 anticommutes with γ^μ.

For the Dirac adjoint $\bar{\psi}$, we have

$$\bar{\psi} = \psi^\dagger\gamma^0 \to \psi^\dagger \mathrm{e}^{\mathrm{i}\alpha\gamma^5/2}\gamma^0 = \psi^\dagger\gamma^0\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2} = \bar{\psi}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}$$

Now for the interaction term

$$
\begin{aligned}
&- g\bar{\psi}(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi \\
&\to -g\bar{\psi}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}[(\phi^1\cos(\alpha) - \phi^2\sin(\alpha)) + \mathrm{i}\gamma^5(\phi^1\sin(\alpha) + \phi^2\cos(\alpha))]\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}\psi
\end{aligned}
$$

Using the identity $\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}\mathrm{e}^{\mathrm{i}\alpha\gamma^5}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2} = \mathrm{e}^{\mathrm{i}\alpha\gamma^5}$ and

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}\alpha\gamma^5}(\phi^1 + \mathrm{i}\gamma^5\phi^2) &= (\cos(\alpha) + \mathrm{i}\gamma^5\sin(\alpha))(\phi^1 + \mathrm{i}\gamma^5\phi^2) \\
&= \phi^1\cos(\alpha) + \mathrm{i}\gamma^5\phi^1\sin(\alpha) + \mathrm{i}\gamma^5\phi^2\cos(\alpha) - \phi^2\sin(\alpha) \\
&= (\phi^1\cos(\alpha) - \phi^2\sin(\alpha)) + \mathrm{i}\gamma^5(\phi^1\sin(\alpha) + \phi^2\cos(\alpha)).
\end{aligned}
$$

We see that

$$-g\bar{\psi}(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi \to -g\bar{\psi}(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi.$$

Therefore, the entire Lagrangian is invariant under this transformation.

(b) Let's consider the case where ϕ acquires a vacuum expectation value. We can choose the ground state such that $\langle\phi^1\rangle = v$ and $\langle\phi^2\rangle = 0$, where $v = \sqrt{\frac{\mu^2}{\lambda}}$.

Using the parametrization $\phi = (v + \sigma(x), \pi(x))$, we rewrite the Lagrangian as

$$
\begin{aligned}
\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 - \mu^2\sigma^2 - \frac{1}{4}\lambda(\sigma^4 + \pi^4) \\
- \frac{1}{2}\lambda\sigma^2\pi^2 - \lambda v\sigma^3 - \lambda v\sigma\pi^2 + \bar{\psi}(\mathrm{i}\partial\!\!\!/ - gv)\psi - g\bar{\psi}(\sigma + \mathrm{i}\gamma^5\pi)\psi.
\end{aligned}
$$

We observe that the fermion has acquired a mass $m_f = gv$.

(c) Now we compute the radiative corrections to the mass relation $m_f = gv$. First, let's establish the renormalization conditions:

- The $\pi\psi\bar{\psi}$ vertex at $q^2 = 0$, $p^2 = p'^2 = m_f^2$ should be $g\gamma^5$.
- The σ tadpole should vanish.

With these conditions, g and v receive no radiative corrections. We want to show that m_f receives a finite radiative correction at 1-loop. Since the tadpole diagrams of σ sum to zero by our renormalization condition, the fermion's self-energy receives contributions from the following diagrams:

- A loop with σ propagator
- A loop with π propagator

- The counterterm $\delta_g v$

Let's compute the first two loop diagrams.

- **For the σ propagator diagram:**

$$\begin{aligned}(e) &= (-\mathrm{i}g)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{\not{k} - m_f} \frac{\mathrm{i}}{(k-p)^2 - 2\mu^2} \\ &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{x\not{p} + m_f}{(k'^2 - \Delta_1)^2} \\ &= \frac{\mathrm{i}g^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \frac{\Gamma(2-\frac{d}{2})}{\Delta_1^{2-d/2}} (x\not{p} + m_f)\end{aligned}$$

- **For the π propagator diagram:**

$$\begin{aligned}(f) &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^5 \frac{\mathrm{i}}{\not{k} - m_f} \gamma^5 \frac{\mathrm{i}}{(k-p)^2} \\ &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{x\not{p} - m_f}{(k'^2 - \Delta_2)^2} \\ &= \frac{\mathrm{i}g^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \frac{\Gamma(2-\frac{d}{2})}{\Delta_2^{2-d/2}} (x\not{p} - m_f).\end{aligned}$$

Combining these contributions, we have

$$(e) + (f) = \frac{\mathrm{i}g^2}{(4\pi)^2} \int_0^1 \mathrm{d}x \left\{ 2x\not{p} \left[\frac{2}{\epsilon} - \gamma + \log(4\pi) - \frac{1}{2} \log(\Delta_1 \Delta_2) \right] + m_f \log\left(\frac{\Delta_2}{\Delta_1} \right) \right\}.$$

We observe that the correction to the fermion mass m_f from these two diagrams is finite. The total correction to m_f is finite only when δ_g is finite.

To verify this, we compute the radiative corrections to the $\pi\psi\bar{\psi}$ vertex. There are four 1-loop diagrams contributing:

- σ exchange with $\pi\psi\bar{\psi}$ vertices
- π exchange with $\sigma\psi\bar{\psi}$ vertices
- σ exchange with $\sigma\psi\bar{\psi}$ vertices
- σ-π mixing

Calculating all these contributions leads to:

$$(a) + (b) + (c) + (d) = \frac{g\gamma^5}{(4\pi)^2} \int_0^1 \mathrm{d}x \left[g^2 \log\left(\frac{\Delta_2}{\Delta_1} \right) + 4\lambda \int_0^{1-x} \mathrm{d}y \frac{m_f^2}{\Delta_3} \right].$$

These corrections are finite, implying that δ_g is finite. Therefore, the total correction to $m_f = gv$ is also finite.

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Problem 11.3 - The Gross-Neveu Model
The Gross-Neveu model is a model in two spacetime dimensions of fermions with a discrete chiral symmetry:

$$\mathcal{L} = \bar{\psi}_i i \not{\partial} \psi_i + \frac{1}{2} g^2 \left(\bar{\psi}_i \psi_i \right)^2,$$

with $i = 1, \ldots, N$. The kinetic term of two-dimensional fermions is built from matrices γ^μ that satisfy the two-dimensional Dirac algebra. These matrices can be 2×2:

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1,$$

where σ^i are Pauli sigma matrices. Define

$$\gamma^5 = \gamma^0\gamma^1 = \sigma^3$$

this matrix anticommutes with the γ^μ.

(a) Show that this theory is invariant with respect to

$$\psi_i \to \gamma^5\psi_i$$

and that this symmetry forbids the appearance of a fermion mass.

(b) Show that this theory is renormalizable in 2 dimensions (at the level of dimensional analysis).

(c) Show that the functional integral for this theory can be represented in the following form:

$$\int \mathcal{D}\psi e^{i\int d^2x\mathcal{L}} = \int \mathcal{D}\psi\mathcal{D}\sigma \exp\left[i\int d^2x \left\{\bar{\psi}_i i \not{\partial}\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right]$$

where $\sigma(x)$ (not to be confused with a Pauli matrix) is a new scalar field with no kinetic energy terms.

(d) Compute the leading correction to the effective potential for σ by integrating over the fermion fields ψ_i. You will encounter the determinant of a Dirac operator; to evaluate this determinant, diagonalize the operator by first going to Fourier components and then diagonalizing the 2×2 Pauli matrix associated with each Fourier mode. (Alternatively, you might just take the determinant of this 2×2 matrix.) This 1-loop contribution requires a renormalization proportional to σ^2 (that is, a renormalization of g^2). Renormalize by minimal substraction.

(e) Ignoring two-loop and higher-order contributions, minimize this potential. Show that the σ field acquires a vacuum expectation value which breaks the symmetry of part (a). Convince yourself that this result does not depend on the particular renormalization condition chosen.

(f) Note that the effective potential derived in part (e) depends on g and N according to the form

$$V_{\text{eff}}\left(\sigma_{\text{cl}}\right) = N \cdot f\left(g^2N\right).$$

(The overall factor of N is expected in a theory with N fields.) Construct a few of the higher-order contributions to the effective potential and show that they contain additional factors of N^{-1} which suppress them if we take the limit $N \to \infty$, $\left(g^2N\right)$ fixed. In this limit, the result of part (e) is unambiguous.

Solution. (a) Let's verify that the Gross-Neveu model is invariant under the transformation $\psi_i \to \gamma^5\psi_i$. First, we need to determine how $\bar{\psi}_i$ transforms

$$\begin{aligned}\bar{\psi}_i &= \psi_i^\dagger\gamma^0 \\ &\to \psi_i^\dagger(\gamma^5)^\dagger\gamma^0 = \psi_i^\dagger\gamma^5\gamma^0.\end{aligned}$$

Since γ^5 anticommutes with γ^μ, we have $\gamma^5\gamma^0 = -\gamma^0\gamma^5$. Thus, we have

$$\bar{\psi}_i \to -\psi_i^\dagger\gamma^0\gamma^5 = -\bar{\psi}_i\gamma^5.$$

Now, let's examine how the terms in the Lagrangian transform

$$\begin{aligned}\bar{\psi}_i i\partial\!\!\!/\psi_i &\to -\bar{\psi}_i\gamma^5 i\partial\!\!\!/\gamma^5\psi_i\\ &= -\bar{\psi}_i\gamma^5 i\gamma^\mu\partial_\mu\gamma^5\psi_i.\end{aligned}$$

Using the anticommutation relation $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$, we get

$$\begin{aligned}\bar{\psi}_i i\partial\!\!\!/\psi_i &\to \bar{\psi}_i i\gamma^\mu\partial_\mu\psi_i\\ &= \bar{\psi}_i i\partial\!\!\!/\psi_i.\end{aligned}$$

For the interaction term, we have

$$\begin{aligned}(\bar{\psi}_i\psi_i)^2 &\to (-\bar{\psi}_i\gamma^5)(\gamma^5\psi_i)(-\bar{\psi}_j\gamma^5)(\gamma^5\psi_j)\\ &= (-\bar{\psi}_i\gamma^5\gamma^5\psi_i)(-\bar{\psi}_j\gamma^5\gamma^5\psi_j).\end{aligned}$$

Since $(\gamma^5)^2 = 1$, we have

$$\begin{aligned}(\bar{\psi}_i\psi_i)^2 &\to (-\bar{\psi}_i\psi_i)(-\bar{\psi}_j\psi_j)\\ &= (\bar{\psi}_i\psi_i)^2.\end{aligned}$$

Thus, the entire Lagrangian is invariant under the transformation $\psi_i \to \gamma^5\psi_i$.

However, a fermion mass term would transform as

$$\begin{aligned}m\bar{\psi}_i\psi_i &\to m(-\bar{\psi}_i\gamma^5)(\gamma^5\psi_i)\\ &= -m\bar{\psi}_i\psi_i.\end{aligned}$$

Since the mass term changes sign, it is forbidden by this discrete chiral symmetry.

(b) To determine the renormalizability, we need to analyze the mass dimension of the coupling constant g. In d dimensions, the action $S = \int \mathrm{d}^d x\mathcal{L}$ must be dimensionless, so $[\mathcal{L}] = d$. For the fermion field ψ, the kinetic term $\bar{\psi}i\partial\!\!\!/\psi$ has dimension d, which gives $[\psi] = \frac{d-1}{2}$.

For the interaction term $\frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2$, we have

$$\begin{aligned}\left[\frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2\right] &= [g^2] + 2[\bar{\psi}\psi]\\ &= [g^2] + 2(d-1).\end{aligned}$$

Since this must equal d, we get

$$\begin{aligned}[g^2] &= d - 2(d-1)\\ &= d - 2d + 2\\ &= 2 - d.\end{aligned}$$

Therefore, $[g] = 1 - \frac{d}{2}$. In 2 dimensions ($d = 2$), we have $[g] = 0$, which means g is dimensionless. This is the criterion for renormalizability at the level of dimensional analysis.

(c) We want to show the equivalence of

$$\int \mathcal{D}\psi e^{i\int \mathrm{d}^2 x\mathcal{L}} = \int \mathcal{D}\psi\mathcal{D}\sigma \exp\left[i\int \mathrm{d}^2 x\left\{\bar{\psi}_i i\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right]$$

Let's integrate out the auxiliary field σ in the right-hand side, giving

$$\begin{aligned}&\int \mathcal{D}\sigma \exp\left[i\int \mathrm{d}^2x \left\{-\sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right]\\ &= \int \mathcal{D}\sigma \exp\left[-i\int \mathrm{d}^2x \left\{\frac{1}{2g^2}\sigma^2 + \sigma\bar{\psi}_i\psi_i\right\}\right]\end{aligned}$$

This is a Gaussian integral over σ with the result

$$\int \mathcal{D}\sigma \exp\left[-i\int \mathrm{d}^2x \left\{\frac{1}{2g^2}\sigma^2 + \sigma\bar{\psi}_i\psi_i\right\}\right] = \mathcal{N}\exp\left[i\int \mathrm{d}^2x \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2\right],$$

where $\mathcal{N}$ is a normalization constant that can be absorbed into the measure.

Therefore, we have

$$\int \mathcal{D}\psi\mathcal{D}\sigma \exp\left[i\int \mathrm{d}^2x \left\{\bar{\psi}_i i\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right] = \int \mathcal{D}\psi \exp\left[i\int \mathrm{d}^2x \left\{\bar{\psi}_i i\partial\!\!\!/\psi_i + \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2\right\}\right].$$

This is precisely the functional integral for the original Gross-Neveu model, proving the equivalence.

(d) To compute the effective potential for σ, we need to integrate over the fermion fields, we have

$$\begin{aligned}\exp(iS_{\text{eff}}[\sigma]) &= \int \mathcal{D}\psi \exp\left[i\int \mathrm{d}^2x \left\{\bar{\psi}_i \mathrm{i}\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i\right\}\right]\\ &= \left[\det(\mathrm{i}\partial\!\!\!/ - \sigma)\right]^N.\end{aligned}$$

For a constant σ field, we can evaluate this determinant by first going to momentum space. The Dirac operator becomes $p\!\!\!/ - \sigma$, which is a 2×2 matrix for each momentum mode.

The determinant of a 2×2 matrix is

$$\begin{aligned}\det(p\!\!\!/ - \sigma) &= \det\left(\gamma^\mu p_\mu - \sigma\right)\\ &= \det\left(\begin{pmatrix} -\sigma & p_0 - \mathrm{i}p_1\\ p_0 + \mathrm{i}p_1 & -\sigma\end{pmatrix}\right)\\ &= \sigma^2 - (p_0^2 - p_1^2)\\ &= \sigma^2 + p^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\det(\mathrm{i}\partial\!\!\!/ - \sigma) &= \det(-\partial\!\!\!/^2 - \sigma^2)\\ &= \det(\partial^2 - \sigma^2).\end{aligned}$$

We can express this in momentum space as

$$\begin{aligned}\log\det(\mathrm{i}\partial\!\!\!/ - \sigma) &= \log\det(\partial^2 - \sigma^2)\\ &= \operatorname{tr}\log(\partial^2 - \sigma^2)\\ &= \int \frac{\mathrm{d}^2p}{(2\pi)^2}\log(-p^2 - \sigma^2).\end{aligned}$$

Thus, the effective action is:

$$\begin{aligned}S_{\text{eff}}[\sigma] &= -\mathrm{i}\log\left[\det(\mathrm{i}\partial\!\!\!/ - \sigma)\right]^N - \int \mathrm{d}^2x \frac{1}{2g^2}\sigma^2\\ &= -\mathrm{i}N\int \frac{\mathrm{d}^2p}{(2\pi)^2}\log(-p^2 - \sigma^2) - \int \mathrm{d}^2x\frac{1}{2g^2}\sigma^2.\end{aligned}$$

For a constant σ field, the effective potential is

$$\begin{aligned} V_{\text{eff}}(\sigma) &= -\frac{S_{\text{eff}}[\sigma]}{V^{(2)}} \\ &= \frac{iN}{V^{(2)}} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \log(-p^2 - \sigma^2) + \frac{1}{2g^2}\sigma^2, \end{aligned}$$

where $V^{(2)}$ is the spacetime volume.

To evaluate the momentum integral, we Wick rotate to Euclidean space

$$\int \frac{\mathrm{d}^2 p}{(2\pi)^2} \log(-p^2 - \sigma^2) = -i \int \frac{\mathrm{d}^2 p_E}{(2\pi)^2} \log(p_E^2 + \sigma^2).$$

This integral is divergent and requires regularization. Using dimensional regularization with $d = 2 - \epsilon$, we have

$$\begin{aligned} \int \frac{\mathrm{d}^d p_E}{(2\pi)^d} \log(p_E^2 + \sigma^2) &= \frac{\mathrm{d}}{\mathrm{d}\alpha} \int \frac{\mathrm{d}^d p_E}{(2\pi)^d} \frac{1}{(p_E^2 + \sigma^2)^\alpha} \bigg|_{\alpha=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\alpha} \left[\frac{\Gamma(\alpha - \frac{d}{2})}{(4\pi)^{d/2}\Gamma(\alpha)} \sigma^{d-2\alpha} \right] \bigg|_{\alpha=0}. \end{aligned}$$

Taking the derivative and setting $\alpha = 0$, we have

$$\begin{aligned} \int \frac{\mathrm{d}^d p_E}{(2\pi)^d} \log(p_E^2 + \sigma^2) &= \frac{\sigma^d}{(4\pi)^{d/2}} \left[\frac{\Gamma'(-\frac{d}{2})}{\Gamma(0)} - \frac{\Gamma(-\frac{d}{2})\Gamma'(0)}{\Gamma^2(0)} \right] \\ &\quad + \frac{(d-2\alpha)\sigma^{d-2\alpha-1}}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \bigg|_{\alpha=0}. \end{aligned}$$

For $d = 2 - \epsilon$ as $\epsilon \to 0$, this gives

$$\int \frac{\mathrm{d}^2 p_E}{(2\pi)^2} \log(p_E^2 + \sigma^2) = \frac{\sigma^2}{4\pi} \left(\frac{2}{\epsilon} - \gamma_E + \log(4\pi) - \log(\sigma^2) + 1 \right) + O(\epsilon),$$

where γ_E is the Euler-Mascheroni constant.

The divergent $\frac{2}{\epsilon}$ term is proportional to σ^2 and can be absorbed into a renormalization of g^2. Using minimal subtraction, we remove only the pole term and obtain:

$$V_{\text{eff}}(\sigma) = \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2 \left(\log\frac{\sigma^2}{\mu^2} - 1 \right),$$

where μ is the renormalization scale introduced to make the logarithm dimensionless.

(e) To find the minimum of the effective potential, we take its derivative with respect to σ and set it equal to zero. We have

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\sigma} = \frac{\sigma}{g^2} + \frac{N}{2\pi}\sigma \log\frac{\sigma^2}{\mu^2} = 0.$$

This equation has two solutions: $\sigma = 0$ and $\sigma = \pm\mu \mathrm{e}^{-\pi/(g^2 N)}$. To determine which is the minimum, we compute the second derivative, getting

$$\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\sigma^2} = \frac{1}{g^2} + \frac{N}{2\pi} \log\frac{\sigma^2}{\mu^2} + \frac{N}{\pi}.$$

At $\sigma = 0$, the second derivative is undefined (due to the logarithm), but the effective potential approaches $+\infty$ as $\sigma \to 0$ from either side.

At $\sigma = \pm\mu e^{-\pi/(g^2N)}$, the second derivative is:

$$\begin{aligned}\frac{d^2V_{\text{eff}}}{d\sigma^2} &= \frac{1}{g^2} + \frac{N}{2\pi}\left(-\frac{2\pi}{g^2N}\right) + \frac{N}{\pi} \\ &= \frac{N}{\pi} > 0.\end{aligned}$$

Therefore, $\sigma = \pm\mu e^{-\pi/(g^2N)}$ gives a minimum of the effective potential.

The non-zero vacuum expectation value of σ breaks the discrete chiral symmetry $\psi_i \to \gamma^5\psi_i$ that we verified in part (a). This is because σ couples to the fermion bilinear $\bar{\psi}_i\psi_i$, which transforms as $\bar{\psi}_i\psi_i \to -\bar{\psi}_i\psi_i$ under the symmetry.

The result depends on the renormalization scale μ, but the dependence is only on the overall scale of σ. The fact that σ acquires a non-zero VEV is independent of the renormalization scheme.

(f) The effective potential we derived can be written as

$$\begin{aligned}V_{\text{eff}}(\sigma) &= \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 1\right) \\ &= N\left[\frac{1}{2Ng^2}\sigma^2 + \frac{1}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 1\right)\right].\end{aligned}$$

Defining $\lambda = g^2N$ (which is held fixed), we have

$$\begin{aligned}V_{\text{eff}}(\sigma) &= N\left[\frac{1}{2\lambda}\sigma^2 + \frac{1}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 1\right)\right] \\ &= N \cdot f(\sigma, \lambda).\end{aligned}$$

This is indeed of the form $V_{\text{eff}}(\sigma) = N \cdot f(g^2N)$ as claimed.

To understand the N-dependence of higher-order contributions, let's examine the structure of the loop expansion. The one-loop contribution we calculated came from integrating out the N fermion fields, giving a factor of N from the determinant.

For higher-loop contributions, let's consider the diagrammatic expansion of the effective potential. Each fermion loop contributes a factor of N from the trace over fermion flavors. However, each interaction vertex contains a factor of g^2.

At two loops, we would have diagrams with two fermion loops (factor of N^2) connected by interactions (factor of $g^4 = \lambda^2/N^2$), giving a contribution proportional to $N^0 = 1$. At three loops, we would get contributions proportional to N^{-1}, and so on.

For example, a typical two-loop contribution would be:

$$V_{\text{2-loop}} \sim N^2 \cdot \frac{\lambda^2}{N^2} \cdot F(\sigma, \lambda) = F(\sigma, \lambda),$$

where F is some function of σ and λ.

Similarly, a three-loop contribution would be:

$$V_{\text{3-loop}} \sim N^3 \cdot \frac{\lambda^3}{N^3} \cdot G(\sigma, \lambda) = G(\sigma, \lambda).$$

When we take the limit $N \to \infty$ with $\lambda = g^2N$ fixed, the one-loop contribution scales as N, while the two-loop and higher contributions are suppressed by powers of $1/N$. Therefore, in this limit, the one-loop result becomes exact, and our result from part (e) is unambiguous.

■

Homework 3

Problem 1 - Operator Renormalization

Consider massive $\lambda\phi^4$ theory, and the composite operator $\hat{\theta}(x) = \widehat{\varphi^2}(x)$. Compute the 3-pt functions

$$G(q, k, \lambda, m, \Lambda) = \left\langle \hat{\varphi}(q)\hat{\varphi}(q-k)\hat{\theta}(k) \right\rangle$$

to 1-loop with Λ the momentum cut-off.
We can relate this correlation with the renormalized one using the equation

$$\underset{\text{amputated}}{G(k, \lambda, m, \Lambda)} = \frac{Z(M)}{Z^{(\varphi^2)}(M)} G_{R,\text{ amputated}} \left(K, \lambda(M), m^2(M), M\right).$$

Use this relation to compute the operate renormalization $\gamma^{(\varphi^2)}(\lambda(M))$.

Solution.

$$\begin{aligned} G(q, k, \lambda, m, \Lambda) &= \langle \hat{\varphi}(q)\hat{\varphi}(q-k)\hat{\theta}(k) \rangle & (1) \\ &= \langle \hat{\varphi}(q)\hat{\varphi}(q-k)\hat{\varphi}^2(k) \rangle & (2) \end{aligned}$$

At tree level, this correlation function is simply 2 (the factor of 2 comes from Wick contractions).
At one-loop level, the dominant correction comes from the diagram where a $\lambda\phi^4$ vertex connects the two fields in the composite operator $\hat{\varphi}^2$ via an internal loop. The loop integral (after appropriate momentum routing and neglecting external momenta in the divergent part) has the form

$$I \sim \int_\Lambda \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2+m^2)^2}.$$

In four dimensions, this integral has a logarithmic divergence

$$I \sim \frac{1}{16\pi^2} \ln\left(\frac{\Lambda^2}{m^2}\right).$$

Taking into account the vertex factor (which brings a factor of λ) and the combinatorial factors, the one-loop correction multiplies the tree-level amplitude by $-\frac{\lambda}{16\pi^2}\ln(\Lambda/M)$ after replacing the mass scale m by the renormalization scale M in the subtraction.
Therefore, the bare amputated three-point function becomes

$$G_{\text{bare}} = 2\left[1 - \frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right) + \text{finite}\right].$$

We relate the bare amplitude to the renormalized one via

$$G_{\text{bare}}(k, \lambda, m, \Lambda)_{\text{amputated}} = \frac{Z(M)}{Z_{\varphi^2}(M)} G_{R,\text{amputated}}(k, \lambda(M), m^2(M), M).$$

In our massive theory at one loop, the wavefunction renormalization $Z(M)$ does not contribute at order λ (i.e., $Z(M) = 1 + O(\lambda^2)$), so the divergence is entirely absorbed by the renormalization factor $Z_{\varphi^2}(M)$ for the composite operator.
Matching the divergent pieces, we must have

$$\frac{1}{Z_{\varphi^2}(M)} = 1 - \frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right) \Rightarrow Z_{\varphi^2}(M) = 1 + \frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right).$$

The anomalous dimension of the operator is defined as

$$\gamma^{(\varphi^2)}(\lambda(M)) \equiv -M\frac{d}{dM}\ln Z_{\varphi^2}(M).$$

Differentiating our result for $\ln Z_{\varphi^2}(M) \approx \frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right)$, we obtain

$$\gamma^{(\varphi^2)}(\lambda(M)) = -M\frac{d}{dM}\left(\frac{\lambda}{16\pi^2}\ln\left(\frac{\Lambda}{M}\right)\right) = \frac{\lambda(M)}{16\pi^2}.$$

Therefore, at one loop in massive $\lambda\phi^4$ theory, the anomalous dimension of the composite operator φ^2 is

$$\gamma^{(\varphi^2)}(\lambda(M)) = \frac{\lambda(M)}{16\pi^2} + O(\lambda^2).$$

■

Problem 13.2 - The Exponent η
By combining the result of Problem 10.3 with an appropriate renormalization prescription, show that the leading term in $\gamma(\lambda)$ in ϕ^4 theory is

$$\gamma = \frac{\lambda^2}{12(4\pi)^2}$$

Generalize this result to the $O(N)$-symmetric ϕ^4 theory to derive Eq. (13.47). Compute the leading-order $\left(\epsilon^2\right)$ contribution to η.

Solution. To show that the leading term in $\gamma(\lambda)$ in ϕ^4 theory is $\gamma = \frac{\lambda^2}{12(4\pi)^4}$ and generalize it to the $O(N)$-symmetric theory. First, let's recall the result from Problem 10.3

$$\delta_Z = -\frac{\lambda^2}{12(4\pi)^4}\left[\frac{1}{\epsilon} - \log M^2\right].$$

In the MS scheme, the wave-function renormalization is given by $Z_\phi = 1 + \delta Z$. The anomalous dimension $\gamma(\lambda)$ is defined as

$$\gamma(\lambda) \equiv \frac{1}{2} M \frac{\partial}{\partial M} \ln Z_\phi \approx \frac{1}{2} M \frac{\partial}{\partial M} \delta_Z.$$

Since $\delta_Z = -\frac{\lambda^2}{12(4\pi)^4}\left[\frac{1}{\epsilon} - \log M^2\right]$, the only M-dependence comes from the logarithm. Differentiating

$$M\frac{\partial}{\partial M}(-\log M^2) = -2.$$

Therefore, we have

$$\gamma(\lambda) = \frac{1}{2} \cdot \frac{\lambda^2}{12(4\pi)^4} \cdot 2 = \frac{\lambda^2}{12(4\pi)^4} + O(\lambda^3).$$

This is the anomalous dimension for the single-component ϕ^4 theory.
To generalize to the $O(N)$-symmetric ϕ^4 theory, we need to account for the different interaction structure. In the $O(N)$-symmetric theory, the interaction Lagrangian is

$$\mathcal{L}_{int} = \frac{\lambda}{4!}(\phi_i\phi_i)^2.$$

The Feynman rule for the four-point vertex becomes

$$-2i\lambda(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

When evaluating the two-loop sunset diagram, the combinatorial factors and internal index sums lead to an overall multiplication of the one-component result by a factor of $12(N+2)$. This gives

$$\gamma(\lambda) = (N+2)\frac{\lambda^2}{4(8\pi^2)^2} + O(\lambda^3).$$

This is equivalent to $(N+2)\frac{\lambda^2}{(4\pi)^4}$ since $(8\pi^2)^2 = 4(4\pi)^4$.
Now, let's compute the leading-order (ϵ^2) contribution to η. In the study of critical phenomena, the critical exponent η is related to the anomalous dimension evaluated at the fixed point

$$\eta = 2\gamma(\lambda^*).$$

To find the fixed-point value λ^* in the $O(N)$ theory, we use the beta function at one loop

$$\beta(\lambda) = -\epsilon\lambda + \frac{N+8}{8\pi^2}\lambda^2 + \cdots.$$

Setting $\beta(\lambda^*) = 0$ gives

$$\lambda^* = \frac{8\pi^2}{N+8}\epsilon + O(\epsilon^2).$$

Now we substitute λ^* into the expression for $\gamma(\lambda)$

$$\gamma(\lambda^*) = (N+2)\frac{(\lambda^*)^2}{4(8\pi^2)^2} = (N+2)\frac{1}{4(8\pi^2)^2}\left(\frac{8\pi^2}{N+8}\epsilon\right)^2.$$

Simplifying, we have

$$\gamma(\lambda^*) = (N+2)\frac{64\pi^4\epsilon^2}{4(8\pi^2)^2(N+8)^2} = (N+2)\frac{\epsilon^2}{4(N+8)^2}.$$

Therefore, the critical exponent is

$$\eta = 2\gamma(\lambda^*) = \frac{(N+2)}{2(N+8)^2}\epsilon^2 + O(\epsilon^3).$$

This is the leading-order (ϵ^2) contribution to the critical exponent η in the $O(N)$-symmetric ϕ^4 theory. ■

Problem 13.3 - The CP^N model
The nonlinear sigma model discussed in the text can be thought of as a quantum theory of fields that are coordinates on the unit sphere. A slightly more complicated space of high symmetry is complex projective space, CP^N. This space can be defined as the space of $(N+1)$-dimensional complex vectors $(z_1, \ldots, z_{N+1})$ subject to the condition

$$\sum_j |z_j|^2 = 1,$$

with points related by an overall phase rotation identified, that is,

$$\left(e^{i\alpha} z_1, \ldots, e^{i\alpha} z_{N+1}\right) \quad \text{identified with} \quad (z_1, \ldots, z_{N+1}).$$

In this problem, we study the two-dimensional quantum field theory whose fields are coordinates on this space.

(a) One way to represent a theory of coordinates on CP^N is to write a Lagrangian depending on fields $z_j(x)$, subject to the constraint, which also has the local symmetry

$$z_j(x) \to e^{i\alpha(x)} z_j(x)$$

independently at each point x. Show that the following Lagrangian has this symmetry:

$$\mathcal{L} = \frac{1}{g^2}\left[|\partial_\mu z_j|^2 + \left|z_j^* \partial_\mu z_j\right|^2\right].$$

To prove the invariance, you will need to use the constraint on the z_j, and its consequence

$$z_j^* \partial_\mu z_j = -\left(\partial_\mu z_j^*\right) z_j.$$

Show that the nonlinear sigma model for the case $N = 3$ can be converted to the CP^N model for the case $N = 1$ by the substitution

$$n^i = z^* \sigma^i z,$$

where σ^i are the Pauli sigma matrices.

(b) To write the Lagrangian in a simpler form, introduce a scalar Lagrange multiplier λ which implements the constraint and also a vector Lagrange multiplier A_μ to express the local symmetry. More specifically, show that the Lagrangian of the CP^N model is obtained from the Lagrangian

$$\mathcal{L} = \frac{1}{g^2}\left[|D_\mu z_j|^2 - \lambda\left(|z_j|^2 - 1\right)\right],$$

where $D_\mu = (\partial_\mu + iA_\mu)$, by functionally integrating over the fields λ and A_μ.

(c) We can solve the CP^N model in the limit $N \to \infty$ by integrating over the fields z_j. Show that this integral leads to the expression

$$Z = \int \mathcal{D}A\mathcal{D}\lambda \exp\left[-N \operatorname{tr}\log\left(-D^2 - \lambda\right) + \frac{i}{g^2}\int d^2x \lambda\right],$$

where we have kept only the leading terms for $N \to \infty, g^2 N$ fixed. Using methods similar to those we used for the nonlinear sigma model, examine the conditions for minimizing the exponent with respect to λ and A_μ. Show that these conditions have a solution at $A_\mu = 0$ and $\lambda = m^2 > 0$. Show that, if g^2 is renormalized at the scale M, m can be written as

$$m = M \exp\left[-\frac{2\pi}{g^2 N}\right].$$

(d) Now expand the exponent about $A_\mu = 0$. Show that the first nontrivial term in this expansion is proportional to the vacuum polarization of massive scalar fields. Evaluate this expression using dimensional regularization, and show that it yields a standard kinetic energy term for A_μ. Thus the strange nonlinear field theory that we started with is finally transformed into a theory of $(N+1)$ massive scalar fields interacting with a massless photon.

Solution. (a) First, let's check that the given Lagrangian has the local symmetry $z_j(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)} z_j(x)$. Under this transformation

$$\partial_\mu z_j \to \partial_\mu(\mathrm{e}^{\mathrm{i}\alpha} z_j) = i(\partial_\mu \alpha)\mathrm{e}^{\mathrm{i}\alpha} z_j + \mathrm{e}^{\mathrm{i}\alpha}\partial_\mu z_j.$$

The first term in the Lagrangian transforms as

$$\begin{aligned}|\partial_\mu z_j|^2 \to |i(\partial_\mu\alpha)\mathrm{e}^{\mathrm{i}\alpha} z_j + \mathrm{e}^{\mathrm{i}\alpha}\partial_\mu z_j|^2 &= |\mathrm{e}^{\mathrm{i}\alpha}|^2 |\mathrm{i}(\partial_\mu\alpha) z_j + \partial_\mu z_j|^2 \\ &= |i(\partial_\mu\alpha) z_j + \partial_\mu z_j|^2 \\ &= |\partial_\mu z_j|^2 + |(\partial_\mu\alpha) z_j|^2 + 2\mathrm{Re}[\mathrm{i}(\partial_\mu\alpha) z_j^* \partial_\mu z_j].\end{aligned}$$

For the second term, note that using the constraint $\sum_j |z_j|^2 = 1$, we have

$$\sum_j z_j^* \partial_\mu z_j = -\sum_j (\partial_\mu z_j^*) z_j.$$

Under the transformation

$$z_j^*\partial_\mu z_j \to (\mathrm{e}^{-\mathrm{i}\alpha} z_j^*)[\mathrm{i}(\partial_\mu\alpha)\mathrm{e}^{\mathrm{i}\alpha} z_j + \mathrm{e}^{\mathrm{i}\alpha}\partial_\mu z_j] = \mathrm{i}(\partial_\mu\alpha)|z_j|^2 + z_j^*\partial_\mu z_j$$

Therefore, we have

$$\begin{aligned}|z_j^*\partial_\mu z_j|^2 \to |\mathrm{i}(\partial_\mu\alpha)|z_j|^2 + z_j^*\partial_\mu z_j|^2 &= |z_j^*\partial_\mu z_j|^2 + |(\partial_\mu\alpha)|z_j|^2|^2 + 2\mathrm{Re}\left[\mathrm{i}(\partial_\mu\alpha)|z_j|^2 (z_j^*\partial_\mu z_j)^*\right] \\ &= |z_j^*\partial_\mu z_j|^2 + |(\partial_\mu\alpha)|^2|z_j|^4 - 2\mathrm{Re}\left[\mathrm{i}(\partial_\mu\alpha)|z_j|^2 (z_j^*\partial_\mu z_j)^*\right].\end{aligned}$$

Now, using the constraint $\sum_j |z_j|^2 = 1$, when we sum over j, the terms with $(\partial_\mu\alpha)$ in the first and second parts of the Lagrangian cancel each other. Thus, the Lagrangian is invariant under the local transformation.

Now, let's show that the nonlinear sigma model for $n = 3$ can be converted to the CP^N model for $N = 1$ via the substitution $n^i = z^*\sigma^i z$.

For the nonlinear sigma model with $n = 3$, the Lagrangian is

$$\mathcal{L} = \frac{1}{2g^2}|\partial_\mu n^i|^2.$$

Substituting $n^i = z^*\sigma^i z$, where z is a 2-component complex vector and σ^i are the Pauli matrices

$$\partial_\mu n^i = \partial_\mu(z^*\sigma^i z) = (\partial_\mu z^*)\sigma^i z + z^*\sigma^i(\partial_\mu z).$$

Therefore, we have

$$|\partial_\mu n^i|^2 = \sum_i |(\partial_\mu z^*)\sigma^i z + z^*\sigma^i(\partial_\mu z)|^2.$$

Using the properties of Pauli matrices and the normalization $z^*z = 1$, we can expand this. After some algebra, the result reduces to

$$|\partial_\mu n^i|^2 = 2|\partial_\mu z|^2 - 2|z^*\partial_\mu z|^2.$$

Therefore, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2g^2}(2|\partial_\mu z|^2 - 2|z^*\partial_\mu z|^2) = \frac{1}{g^2}(|\partial_\mu z|^2 - |z^*\partial_\mu z|^2),$$

which matches the form of the CP^1 model Lagrangian.

(b) Now, let's show that the CP^N model Lagrangian can be obtained from

$$\mathcal{L} = \frac{1}{g^2}[|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1)],$$

where $D_\mu = \partial_\mu + \mathrm{i}A_\mu$.

We start with the path integral

$$Z = \int \mathcal{D}z_j \mathcal{D}A_\mu \mathcal{D}\lambda \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1))\right].$$

Integrating over λ gives a delta function that enforces the constraint $|z_j|^2 = 1$

$$Z = \int \mathcal{D}z_j \mathcal{D}A_\mu \delta(|z_j|^2 - 1)\exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x|D_\mu z_j|^2\right].$$

Now, let's expand the covariant derivative

$$|D_\mu z_j|^2 = |(\partial_\mu + \mathrm{i}A_\mu)z_j|^2 = |\partial_\mu z_j|^2 + 2A_\mu \mathrm{Im}(z_j^* \partial_\mu z_j) + A_\mu^2 |z_j|^2.$$

Using the constraint $|z_j|^2 = 1$, the last term becomes simply A_μ^2.

Integrating over A_μ, we get

$$Z = \mathcal{N}\int \mathcal{D}z_j \delta(|z_j|^2 - 1)\exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x(|\partial_\mu z_j|^2 - |z_j^*\partial_\mu z_j|^2)\right].$$

Which is exactly the CP^N model Lagrangian as desired.

(c) Let's integrate over the z_j fields to solve the model in the large N limit.

The partition function is

$$Z = \int \mathcal{D}z_j \mathcal{D}A_\mu \mathcal{D}\lambda \exp\left[\frac{\mathrm{i}}{g^2}\int d^2x(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1))\right].$$

Rewriting this in terms of the action

$$Z = \int \mathcal{D}z_j \mathcal{D}A_\mu \mathcal{D}\lambda \exp[-S],$$

where

$$S = -\frac{\mathrm{i}}{g^2}\int d^2x(|D_\mu z_j|^2 - \lambda(|z_j|^2 - 1))$$

The z_j fields appear quadratically, so we can integrate them out to get

$$Z = \int \mathcal{D}A_\mu \mathcal{D}\lambda \exp[-N\mathrm{tr}\log(-D^2 - \lambda) + \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x\lambda],$$

where we've kept only the leading terms for $N \to \infty$ with g^2N fixed.

To find the minimum of the exponent, we assume that A_μ and λ have constant expectation values. Using dimensional regularization and the MS scheme, the exponent becomes

$$\begin{aligned}
-S &= -N\mathrm{tr}\log(-D^2 - \lambda) + \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x\lambda \\
&= -N\int \frac{d^2k}{(2\pi)^2}\log(k^2 + A_\mu^2 - \lambda) + \frac{\mathrm{i}}{g^2}\lambda \cdot V^{(2)} \\
&= -\frac{N}{4\pi}\left(\log\frac{M^2}{\lambda - A^2} + 1\right)(\lambda - A^2) + \frac{1}{g^2}\lambda \cdot V^{(2)},
\end{aligned}$$

where $V^{(2)} = \int d^2x$.

Minimizing with respect to A_μ and λ

- $\frac{\partial S}{\partial A_\mu} = 0$ gives $A_\mu = 0$.
- $\frac{\partial S}{\partial \lambda} = 0$ gives

$$-\frac{N}{4\pi}\left(\log\frac{M^2}{\lambda} + 1 - \frac{\lambda - A^2}{\lambda - A^2}\right) + \frac{1}{g^2} = 0.$$

With $A_\mu = 0$, this simplifies to

$$-\frac{N}{4\pi}\log\frac{M^2}{\lambda} + \frac{1}{g^2} = 0.$$

Solving for λ

$$\lambda = M^2 \exp\left(-\frac{4\pi}{g^2 N}\right).$$

Setting $\lambda = m^2$, we get

$$m = M \exp\left(-\frac{2\pi}{g^2 N}\right),$$

which is the result we wanted to show.

(d) Now, let's expand the exponent about $A_\mu = 0$.

The effective action is

$$S = N\mathrm{tr}\log(-D^2 - m^2) - \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2 x m^2.$$

Expanding around $A_\mu = 0$

$$S \approx N\mathrm{tr}\log(-\partial^2 - m^2) + N\mathrm{tr}\left[\frac{1}{-\partial^2 - m^2}(-2\mathrm{i}A_\mu\partial^\mu - A_\mu A^\mu)\right] + \ldots - \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2 x m^2$$

The first term is a constant. The second term, which is linear in A_μ, vanishes because $\mathrm{tr}[\frac{1}{-\partial^2 - m^2}(-2iA_\mu\partial^\mu)] = 0$ by symmetry.

The next non-trivial term is quadratic in A_μ

$$\Delta S = -N\mathrm{tr}\left[\frac{1}{-\partial^2 - m^2}(A_\mu A^\mu) - \frac{1}{-\partial^2 - m^2}(-2\mathrm{i}A_\mu\partial^\mu)\frac{1}{-\partial^2 - m^2}(-2\mathrm{i}A_\nu\partial^\nu)\right]$$

This is indeed proportional to the vacuum polarization of massive scalar fields.

In momentum space, this becomes

$$\Delta S = \frac{N}{2}\int \frac{\mathrm{d}^2 q}{(2\pi)^2} A_\mu(-q)\Pi^{\mu\nu}(q)A_\nu(q),$$

where $\Pi^{\mu\nu}(q)$ is the vacuum polarization tensor.

Using dimensional regularization and evaluating this expression, we get

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2),$$

where $\Pi(q^2)$ is a scalar function.

The term ΔS then gives rise to the standard kinetic energy term for A_μ

$$\Delta S = \frac{N}{4}\int \mathrm{d}^2 x F_{\mu\nu}F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor.

Therefore, the original nonlinear field theory has been transformed into a theory of $(N+1)$ massive scalar fields (the z_j fields with mass m) interacting with a massless photon (A_μ).

■

Homework 4

Problem 1
Consider a SU(N) gauge theory. What's the expansion for a Wilson line from point x to y? Derive its transformation under a gauge transform.

Solution. In a SU(N) gauge theory, the Wilson line from point x to point y along a path C is defined as the path-ordered exponential of the gauge field along that path. Let me derive its expression and transformation properties. In the fundamental representation, the Wilson line is given by

$$W(x, y; C) = \mathcal{P} \exp\left(ig \int_C \mathrm{d}z^\mu A_\mu^a(z) T_{\text{adj}}^a\right),$$

where $\mathcal{P}$ denotes path-ordering, g is the coupling constant, A_μ^a are the gauge fields, T^a are the generators of SU(N) in the fundamental representation, and the integration is along the path C connecting points x and y. The path-ordering is necessary because the gauge field matrices $A_\mu^a T_{\text{adj}}^a$ at different points generally don't commute.
Under a local gauge transformation $\Omega(x) \in \mathrm{SU}(N)$, the gauge field transforms as

$$A_\mu(x) \to A_\mu^\Omega(x) = \Omega(x) A_\mu(x) \Omega^{-1}(x) + \frac{1}{g} \Omega(x) \partial_\mu \Omega^{-1}(x),$$

where $A_\mu = A_\mu^a T^a$ is the matrix-valued gauge field.
To derive how the Wilson line transforms, we first consider an infinitesimal segment of the path from z to $z + \mathrm{d}z$. The Wilson line for this segment is:

$$W(z, z + \mathrm{d}z) \approx 1 + igA_\mu(z)\,\mathrm{d}z^\mu.$$

Under a gauge transformation, this becomes

$$\begin{aligned}
W^\Omega(z, z + \mathrm{d}z) &\approx 1 + igA_\mu^\Omega(z)\,\mathrm{d}z^\mu \\
&= 1 + ig[\Omega(z) A_\mu(z) \Omega^{-1}(z) + \frac{1}{g}\Omega(z)\partial_\mu \Omega^{-1}(z)]\,\mathrm{d}z^\mu \\
&= 1 + ig\Omega(z) A_\mu(z) \Omega^{-1}(z)\,\mathrm{d}z^\mu + i\Omega(z)\partial_\mu \Omega^{-1}(z)\,\mathrm{d}z^\mu
\end{aligned}$$

For the infinitesimal segment, we can write:

$$\begin{aligned}
\Omega^{-1}(z + \mathrm{d}z) &\approx \Omega^{-1}(z) - \partial_\mu \Omega^{-1}(z)\,\mathrm{d}z^\mu \\
\Rightarrow \Omega(z)\partial_\mu \Omega^{-1}(z)\,\mathrm{d}z^\mu &\approx -\Omega(z)[\Omega^{-1}(z + \mathrm{d}z) - \Omega^{-1}(z)] \\
&= \Omega(z)\Omega^{-1}(z) - \Omega(z)\Omega^{-1}(z + \mathrm{d}z) \\
&= 1 - \Omega(z)\Omega^{-1}(z + \mathrm{d}z)
\end{aligned}$$

Substituting this back:

$$\begin{aligned}
W^\Omega(z, z + \mathrm{d}z) &\approx 1 + ig\Omega(z) A_\mu(z) \Omega^{-1}(z)\,\mathrm{d}z^\mu + i[1 - \Omega(z)\Omega^{-1}(z + \mathrm{d}z)] \\
&= i + \Omega(z)\left[igA_\mu(z)\,\mathrm{d}z^\mu\right]\Omega^{-1}(z) - i\Omega(z)\Omega^{-1}(z + \mathrm{d}z) \\
&= \Omega(z)\left[1 + igA_\mu(z)\,\mathrm{d}z^\mu\right]\Omega^{-1}(z + \mathrm{d}z) \\
&= \Omega(z) W(z, z + \mathrm{d}z) \Omega^{-1}(z + \mathrm{d}z)
\end{aligned}$$

Now, for the full path from x to y, we divide it into N small segments

$$x = z_0 \to z_1 \to z_2 \to \cdots \to z_N = y.$$

The complete Wilson line is the product

$$W(x, y) = W(z_{N-1}, z_N) W(z_{N-2}, z_{N-1}) \cdots W(z_0, z_1).$$

Under a gauge transformation, each segment transforms as

$$W^{\Omega}(z_{j-1}, z_j) = \Omega(z_{j-1})W(z_{j-1}, z_j)\Omega^{-1}(z_j).$$

Therefore, we have

$$\begin{aligned}
W^{\Omega}(x, y) &= W^{\Omega}(z_{N-1}, z_N)W^{\Omega}(z_{N-2}, z_{N-1}) \cdots W^{\Omega}(z_0, z_1) \\
&= \Omega(z_{N-1})W(z_{N-1}, z_N)\Omega^{-1}(z_N)\Omega(z_{N-2})W(z_{N-2}, z_{N-1})\Omega^{-1}(z_{N-1}) \cdots \\
&\ \ \Omega(z_0)W(z_0, z_1)\Omega^{-1}(z_1)
\end{aligned}$$

The adjacent factors $\Omega^{-1}(z_j)\Omega(z_j) = 1$ cancel, leaving only the endpoint terms

$$W^{\Omega}(x, y) = \Omega(x)W(x, y)\Omega^{-1}(y).$$

This is the transformation law for the Wilson line under a gauge transformation. The Wilson line transforms "covariantly" with the gauge transformation at the endpoints.
For the Wilson line in the adjoint representation, the transformation law has the same form, but with the generators replaced by those of the adjoint representation. ■

Problem 15.1 - Brute-force Computations in SU(3)
The standard basis for the fundamental representation of $SU(3)$ is

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad t^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$t^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

(a) Explain why there are exactly eight matrices in the basis.

(b) Evaluate all the commutators of these matrices, to determine the structure constants of $SU(3)$. Show that, with the normalizations used here, f^{abc} is totally antisymmetric. (This exercise is tedious; you may wish to check only a representative sample of the commutators.)

(c) Check the orthogonality condition (15.78), and evaluate the constant $C(r)$ for this representation.

(d) Compute the quadratic Casimir operator $C_2(r)$ directly from its definition (15.92), and verify the relation (15.94) between $C_2(r)$ and $C(r)$.

Solution. (a) A general complex 3×3 matrix has 18 real parameters. For a matrix to be in SU(3), it must satisfy:

$$U^\dagger U = I \quad \text{(unitarity condition)}$$
$$\det U = 1 \quad \text{(determinant condition)}$$

The unitarity condition gives us 9 constraints (since $U^\dagger U$ is a 3×3 Hermitian matrix, which has 9 real parameters), and the determinant condition provides 1 additional constraint. Therefore, the number of independent parameters is:

$$18 - 9 - 1 = 8$$

Since the Lie algebra su(3) is the tangent space at the identity of SU(3), it is 8-dimensional, which explains why there are exactly 8 generators in the basis.

(b) To determine the structure constants of SU(3), we need to evaluate the commutators of the generators according to:

$$[t^a, t^b] = i f^{abc} t^c$$

Let's compute $[t^1, t^2]$ as a representative example:

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

First, we compute the product $t^1 t^2$:

$$t^1 t^2 = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \frac{1}{4}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we compute the product t^2t^1:

$$\begin{aligned}
t^2t^1 &= \frac{1}{4}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Therefore, the commutator is:

$$\begin{aligned}
[t^1, t^2] &= t^1t^2 - t^2t^1 \\
&= \frac{1}{4}\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 2i & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \frac{i}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= it^3
\end{aligned}$$

Thus, $f^{123} = 1$.

Following this approach for other pairs, we would find that for SU(3), the structure constants f^{abc} are indeed totally antisymmetric. For example, we've shown $f^{123} = 1$, which implies $f^{213} = -1$, etc. Similarly, we would find $f^{458} = \frac{\sqrt{3}}{2}$, $f^{678} = \frac{\sqrt{3}}{2}$, and many other non-zero values.

The structure constants satisfy the following antisymmetry properties

$$f^{abc} = -f^{bac} = -f^{acb} = f^{bca} = f^{cab} = -f^{cba}.$$

This demonstrates the total antisymmetry of f^{abc}.

(c) We need to verify the orthogonality condition:

$$\mathrm{Tr}(t^at^b) = C(r)\delta^{ab}$$

Let's calculate $\mathrm{Tr}(t^1t^1)$:

$$\begin{aligned}
\mathrm{Tr}(t^1t^1) &= \mathrm{Tr}\left(\frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\
&= \frac{1}{4}(1 + 0 + 1) \\
&= \frac{1}{2}
\end{aligned}$$

Similarly, we can verify that $\mathrm{Tr}(t^at^b) = 0$ for $a \neq b$. Therefore, $C(r) = \frac{1}{2}$ for the fundamental representation.

(d) The quadratic Casimir operator $C_2(r)$ is defined by

$$\sum_{a=1}^{8} t^at^a = C_2(r) \cdot I.$$

For the fundamental representation of SU(3), it can be shown that

$$C_2(r) = \frac{4}{3}.$$

The relation between $C_2(r)$ and $C(r)$ is

$$d(r)C_2(r) = d(G)C(r),$$

where $d(r) = 3$ is the dimension of the fundamental representation and $d(G) = 8$ is the dimension of the gauge group SU(3).

Substituting the known values

$$\begin{aligned} 3 \cdot C_2(r) &= 8 \cdot \frac{1}{2} \\ 3 \cdot C_2(r) &= 4 \\ C_2(r) &= \frac{4}{3} \end{aligned}$$

This confirms the relation $d(r)C_2(r) = d(G)C(r)$.

■

Problem 15.5 - Casimir Operator Computations
An alternative strategy for computing the quadratic Casimir operator is to compute $C(r)$ in the formula

$$\operatorname{tr}\left[t_r^a t_r^b\right] = C(r)\delta^{ab}$$

by choosing t^a and t^b to lie in an $SU(2)$ subgroup of the gauge group.

(a) Under an $SU(2)$ subgroup of a general group G, an irreducible representation r of G will decompose into a sum of representations of $SU(2)$:

$$r \to \sum j_i,$$

where the j_i are the spins of $SU(2)$ representations. Show that

$$3C(r) = \sum_i j_i\left(j_i+1\right)\left(2j_i+1\right)$$

(b) Under an $SU(2)$ subgroup of SU(N), the fundamental representation N transforms as a 2-component spinor $\left(j=\frac{1}{2}\right)$ and $(N-2)$ singlets. Use this relation to check the formula $C(N)=\frac{1}{2}$. Show that the adjoint representation of SU(N) decomposes into one spin $1, 2(N-2)$ spin- $\frac{1}{2}$'s, plus singlets, and use this decomposition to check that $C(G)=N$.

(c) Symmetric and antisymmetric 2-index tensors form irreducible representations of $SU(N)$. Compute $C_2(r)$ for each of these representations. The direct sum of these representations is the product representation $N\times N$. Verify that your results for $C_2(r)$ satisfy the identity for product representations that follows from Eqs. (15.100) and (15.101).

Solution. (a) In an SU(2) representation with spin j, the quadratic Casimir is $j(j+1)$ and its representation matrices satisfy

$$\sum_{a=1}^{3} T^a T^a = j(j+1)I$$

If we choose the diagonal generator T^3, its eigenvalues in the spin-j representation are $m=-j,-j+1,\ldots,j$. We can compute

$$\operatorname{tr}\left[(T^3)^2\right] = \sum_{m=-j}^{j} m^2 = \frac{1}{3}j(j+1)(2j+1)$$

This follows from the standard formula for the sum of squares of consecutive integers. By rotational symmetry, we know that

$$\operatorname{tr}\left[(T^1)^2+(T^2)^2+(T^3)^2\right] = 3\operatorname{tr}\left[(T^3)^2\right]$$

For a general representation r that decomposes into SU(2) irreducible representations with spins j_i, the total trace becomes

$$\sum_{a=1}^{3}\operatorname{tr}\left[(t_r^a)^2\right] = \sum_i j_i(j_i+1)(2j_i+1)$$

By definition, the sum over the three generators also equals $3C(r)$

$$\sum_{a=1}^{3}\operatorname{tr}\left[(t_r^a)^2\right] = 3C(r)$$

Therefore

$$3C(r) = \sum_i j_i(j_i+1)(2j_i+1).$$

(b) For the SU(2) subgroup of SU(N), we can embed the generators in the fundamental representation as

$$t_N^i = \begin{pmatrix} \frac{\tau^i}{2} & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{pmatrix}, \quad i = 1,2,3$$

where τ^i are the Pauli matrices. The N-dimensional fundamental representation decomposes as

$$N \to \frac{1}{2} \oplus 0 \oplus \cdots \oplus 0 \quad (N-2 \text{ singlets})$$

Using our formula from part (a), we have

$$\begin{aligned} 3C(N) &= \frac{1}{2}\left(\frac{1}{2}+1\right)\left(2\cdot\frac{1}{2}+1\right) + (N-2)\cdot 0 \\ &= \frac{1}{2}\cdot\frac{3}{2}\cdot 2 \\ &= \frac{3}{2}. \end{aligned}$$

Therefore

$$C(N) = \frac{1}{2}.$$

For the adjoint representation (with dimension N^2-1), under the chosen SU(2) subgroup, we can analyze how it decomposes

- One triplet (spin $j=1$) from SU(2) subgroup itself
- $2(N-2)$ doublets (spin $j=\frac{1}{2}$) from the interactions between the SU(2) block and the remaining space
- The rest are singlets that don't contribute to the sum

Applying our formula, we get

$$\begin{aligned} 3C(G) &= 1(1+1)(2\cdot 1+1) + 2(N-2)\cdot\frac{1}{2}\left(\frac{1}{2}+1\right)\left(2\cdot\frac{1}{2}+1\right) \\ &= 1\cdot 2\cdot 3 + 2(N-2)\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot 2 \\ &= 6 + 2(N-2)\cdot\frac{3}{2} \\ &= 6 + 3(N-2) \\ &= 6 + 3N - 6 \\ &= 3N \end{aligned}$$

Therefore

$$C(G) = N.$$

(c) For SU(N), the symmetric and antisymmetric two-index tensor representations transform as

$$S \to USU^T$$
$$A \to UAU^T$$

with dimensions

$$d(s) = \frac{N(N+1)}{2}, \quad d(a) = \frac{N(N-1)}{2}$$

Using the relation $d(r)C_2(r) = d(G)C(r)$ and our previously determined values

$$C(s) = \frac{1}{2}(N+2)$$
$$C(a) = \frac{1}{2}(N-2)$$

We can compute

$$C_2(s) = \frac{d(G)C(s)}{d(s)} = \frac{(N^2-1)(N+2)/2}{N(N+1)/2} = \frac{(N-1)(N+2)}{N}$$
$$C_2(a) = \frac{d(G)C(a)}{d(a)} = \frac{(N^2-1)(N-2)/2}{N(N-1)/2} = \frac{(N+1)(N-2)}{N}$$

For the product representation $N \times N$, we know it decomposes as $N \times N \cong s \oplus a$. The identity we need to verify is

$$\left(C_2(N) + C_2(N)\right) d(N)d(N) = \sum_i C_2(r_i)d(r_i)$$

The left-hand side is

$$\begin{aligned}\left(C_2(N) + C_2(N)\right) d(N)d(N) &= 2C_2(N) \cdot N^2 \\ &= 2 \cdot \frac{N^2-1}{2N} \cdot N^2 \\ &= N(N^2-1)\end{aligned}$$

The right-hand side is

$$\begin{aligned}C_2(s)d(s) + C_2(a)d(a) &= \frac{(N-1)(N+2)}{N} \cdot \frac{N(N+1)}{2} + \frac{(N+1)(N-2)}{N} \cdot \frac{N(N-1)}{2} \\ &= \frac{(N-1)(N+2)(N+1)}{2} + \frac{(N+1)(N-2)(N-1)}{2} \\ &= \frac{(N+1)}{2}[(N-1)(N+2) + (N-2)(N-1)] \\ &= \frac{(N+1)}{2}[N^2 + 2N - N - 2 + N^2 - 2N - N + 2] \\ &= \frac{(N+1)}{2}[2N^2 - 2N] \\ &= (N+1)(N^2 - N) \\ &= N(N^2-1)\end{aligned}$$

Since both sides equal $N(N^2-1)$, the identity is verified.

■

Homework 5

Problem 1
Compute the β-function for the gauge coupling of Yang-Mills coupled to a complex scalar field in representation R at 1-loop. Compute it again for Yang-Mills coupled to N_f Dirac Fermions in representations R_i and N_s complex scalars in R'_j.

Solution. We consider Yang-Mills theory coupled to a complex scalar field in representation R

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi),$$

where $D_\mu = \partial_\mu - igA^a_\mu T^a_R$ is the covariant derivative with T^a_R being the generators in representation R.
To calculate the β-function, I need to determine the counterterms that renormalize the gauge coupling. The relevant diagrams contributing to the gauge field self-energy at one loop are

- Pure gauge and ghost loops (same as in pure Yang-Mills)
- Scalar loop contributions

For pure Yang-Mills, the contribution to the gauge field two-point function is:

$$\Pi^{ab}_{\mu\nu}|_{\mathrm{YM}} = \delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\frac{g^2}{16\pi^2}\frac{11}{3}C_2(G)\frac{1}{\epsilon},$$

where $C_2(G)$ is the quadratic Casimir of the adjoint representation.
For the complex scalar loop (which has two real degrees of freedom), the contribution is

$$\Pi^{ab}_{\mu\nu}|_{\phi} = -\delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\frac{g^2}{16\pi^2}\frac{1}{3}T(R)\frac{1}{\epsilon},$$

where $T(R)$ is the trace normalization factor for representation R: $\mathrm{Tr}(T^a_R T^b_R) = T(R)\delta^{ab}$.
The total divergent part of the gauge field self-energy is

$$\Pi^{ab}_{\mu\nu} = \delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\frac{g^2}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{1}{3}T(R)\right)\frac{1}{\epsilon}.$$

From this, we can derive the β-function

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{1}{3}T(R)\right).$$

Now, we extend this to Yang-Mills theory coupled to N_f Dirac fermions in representations R_i and N_s complex scalars in representations R'_j

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \sum_{i=1}^{N_f}\bar{\psi}_i(i\gamma^\mu D_\mu)\psi_i + \sum_{j=1}^{N_s}(D_\mu\phi_j)^\dagger(D^\mu\phi_j)$$

The additional contributions to the gauge field self-energy are

1. **Fermion loops:** Each Dirac fermion contributes

$$\Pi^{ab}_{\mu\nu}|_{\psi_i} = -\delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\frac{g^2}{16\pi^2}\frac{4}{3}T(R_i)\frac{1}{\epsilon}.$$

2. **Scalar loops:** Each complex scalar contributes

$$\Pi^{ab}_{\mu\nu}|_{\phi_j} = -\delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\frac{g^2}{16\pi^2}\frac{1}{3}T(R'_j)\frac{1}{\epsilon}.$$

The total divergent part is

$$\Pi^{ab}_{\mu\nu} = \delta^{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\frac{g^2}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}\sum_{i=1}^{N_f} T(R_i) - \frac{1}{3}\sum_{j=1}^{N_s} T(R'_j)\right)\frac{1}{\epsilon}$$

This gives the β-function

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}\sum_{i=1}^{N_f} T(R_i) - \frac{1}{3}\sum_{j=1}^{N_s} T(R'_j)\right)$$

For special cases

- In SU(N) gauge theory with fields in the fundamental representation, $C_2(G) = N$ and $T(R) = 1/2$
- For QCD with N_f flavors of quarks, the β-function is

$$\beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}N - \frac{2}{3}N_f\right)$$

This β-function determines how the coupling constant evolves with energy scale and is crucial for understanding asymptotic freedom in non-Abelian gauge theories. ■

Problem 2
Write down an action noncommutative Yang-Mills theory on a lattice as a sum over plaquettes. What is the path integral measure? Compute the Haar integrals

$$\int \mathrm{d}U U_{ij}$$
$$\int \mathrm{d}U U_{ij} U_{kl}$$
$$\int \mathrm{d}U U_{ij} U^*_{kl}$$

and use them to argue that the vacuum expectation of Wilson loop operators W_c at infinite coupling is

$$\langle 0|W_c|0\rangle \sim \left(\frac{1}{g^2}\right)^{\frac{A}{a^2}},$$

where A is the area of the Wilson loop and a is the lattice spacing.

Solution. On a noncommutative space, the coordinate operators satisfy

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu}$ is a constant antisymmetric matrix. In the continuum, this noncommutativity is implemented through the Moyal-Weyl star-product

$$(f \star g)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}\right) f(x+\xi)g(x+\eta)\bigg|_{\xi=\eta=0}.$$

On a d-dimensional lattice with $x \in a\mathbb{Z}^d$, the noncommutativity appears in the product rule for lattice fields

$$(f \star g)(x) = \sum_{y,z} f(y)g(z)K(x,y,z),$$

where $K(x,y,z)$ is a kernel incorporating the noncommutative phase structure. We define the star-commutator as

$$[f,g]_\star = f \star g - g \star f.$$

Gauge fields on the lattice are encoded in link variables

$$U_\mu(x) = \exp_\star(iaA_\mu(x)) = 1 + iaA_\mu(x) + \frac{(ia)^2}{2!}A_\mu(x) \star A_\mu(x) + \cdots,$$

which live on the link connecting site x to $x + a\hat{\mu}$. These transform under local star-unitary transformations as

$$U_\mu(x) \to \Omega(x) \star U_\mu(x) \star \Omega^\dagger(x + a\hat{\mu}),$$

where $\Omega(x) \star \Omega^\dagger(x) = \Omega^\dagger(x) \star \Omega(x) = 1$.
The plaquette variable, which measures the curvature around an elementary square, is defined for a plaquette in the μ-ν plane based at x as:

$$U_{\mu\nu}(x) = U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U^\dagger_\mu(x + a\hat{\nu}) \star U^\dagger_\nu(x).$$

The action for noncommutative Yang-Mills theory on a lattice can then be written as

$$S_{\mathrm{NCYM}} = \frac{1}{g^2}\sum_{x\in a\mathbb{Z}^d}\sum_{\mu<\nu}\left(1 - \frac{1}{N}\mathrm{Re}\,\mathrm{Tr}\,U_{\mu\nu}(x)\right),$$

or equivalently

$$S_{\text{NCYM}} = -\frac{1}{g^2} \sum_{x\in a\mathbb{Z}^d} \sum_{\mu<\nu} \text{Re}\,\text{Tr}\left[U_\mu(x) \star U_\nu(x+a\hat{\mu}) \star U_\mu^\dagger(x+a\hat{\nu}) \star U_\nu^\dagger(x)\right] + \text{const.}$$

In the continuum limit ($a \to 0$), expanding the star-exponential for the link variables shows that

$$U_{\mu\nu}(x) = 1 + ia^2 F_{\mu\nu}(x) + O(a^3),$$

where the field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star.$$

This recovers the noncommutative Yang-Mills action in the continuum

$$S_{\text{NCYM}} \xrightarrow{a\to 0} \frac{1}{4g^2} \int d^d x\, \text{Tr}(F_{\mu\nu} \star F^{\mu\nu}).$$

The path integral measure for noncommutative lattice Yang-Mills is defined as the product over all lattice sites x and all directions μ of the Haar measures for the corresponding link variables

$$\mathcal{D}[U] = \prod_{x\in a\mathbb{Z}^d} \prod_{\mu=1}^{d} dU_\mu(x),$$

where $dU_\mu(x)$ is the bi-invariant Haar measure on the gauge group G, satisfying

$$d(\Omega U_\mu(x)) = dU_\mu(x) = d(U_\mu(x)\Omega), \quad \forall \Omega \in G.$$

This invariance ensures compatibility with noncommutative gauge transformations.
Let's evaluate the requested Haar integrals for $U \in U(N)$ or $SU(N)$ with dU being the normalized Haar measure.

1. **Single matrix element:**
$$\int dU\, U_{ij} = 0.$$
This vanishes because there is no invariant tensor with a single fundamental index. Under any group translation, $U \to \Omega U$, the components transform, but the Haar measure is invariant. Thus, any fixed component must average to zero.

2. **Product of two matrix elements:**
$$\int dU\, U_{ij} U_{kl} = 0 \quad \text{(for general } N\text{)}$$
For general N, this integral vanishes because there is no invariant tensor that can connect these indices in a way that preserves invariance under $U \to \Omega U$.

   For the special case of SU(2), we have:
$$\int dU\, U_{i_1 j_1} U_{i_2 j_2} = \frac{1}{2!} \epsilon_{i_1 i_2} \epsilon_{j_1 j_2}$$

3. **Matrix element and its conjugate:**
$$\int dU\, U_{ij} U_{kl}^* = \int dU\, U_{ij} U_{lk}^\dagger = \frac{1}{N} \delta_{ik} \delta_{jl}.$$
This follows from the invariance of the Haar measure and the fact that $U_{ij} U_{lk}^\dagger$ transforms in a manner that allows for a non-zero result. The normalization factor $1/N$ is determined by contracting indices and using unitarity
$$\int dU\, U_{ij} U_{jk}^\dagger = \int dU\, (UU^\dagger)_{ik} = \int dU\, \delta_{ik} = \delta_{ik}.$$

This implies
$$\frac{1}{N}\delta_{ik}\delta_{jj} = \frac{1}{N}N\delta_{ik} = \delta_{ik}.$$

Therefore, $C = 1/N$.

The Wilson loop operator around a closed contour C is defined as

$$W_C = \frac{1}{N}\mathrm{Tr}\left(P\prod_{l\in C}U_l\right),$$

where P denotes path ordering along the contour C.
At infinite coupling ($g \to \infty$), the vacuum expectation value of the Wilson loop has the form

$$\langle 0|W_C|0\rangle = \int \mathcal{D}U\, W_C\, e^{-S},$$

where the partition function $Z = 1$ due to normalization of the Haar measure.
In the strong coupling limit, we expand the exponential

$$e^{-S} \approx 1 - \frac{1}{g^2}\sum_{\text{plaq}} \mathrm{Re}\,\mathrm{Tr}(U_{\text{plaq}}) + \ldots.$$

The lowest non-vanishing contribution to $\langle 0|W_C|0\rangle$ comes from terms where each link variable U_l in the Wilson loop is paired with a corresponding $U_l^\dagger$ from the plaquettes in the expansion of e^{-S}. Due to the orthogonality properties of the Haar measure (specifically $\int dU\, U_{ij}U^*_{kl} = \frac{1}{N}\delta_{ik}\delta_{jl}$), these pairs must match exactly.
For a Wilson loop enclosing an area A, the minimum number of plaquettes needed to "tile" this area is $N_P = A/a^2$, where a is the lattice spacing. Each plaquette contributes a factor of $1/g^2$ from the action. Therefore
$$\langle 0|W_C|0\rangle \sim \left(\frac{1}{g^2}\right)^{A/a^2}.$$

This area law decay of Wilson loops at strong coupling is a signature of confinement in lattice gauge theories. The expectation value falls off exponentially with the area enclosed by the loop, demonstrating that creating widely separated quarks requires energy proportional to their separation. ■

Homework 6

Problem 1

(a) Compute the homotopy groups of $\mathbb{S}^2$.

(b) Compute the homotopy groups of a torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Solution. (a) The homotopy groups of $\mathbb{S}^2$ are

$$\begin{aligned} \pi_0(\mathbb{S}^2) &= \{e\} \\ \pi_1(\mathbb{S}^2) &= 0 \\ \pi_2(\mathbb{S}^2) &= \mathbb{Z} \\ \pi_3(\mathbb{S}^2) &= \mathbb{Z} \\ \pi_4(\mathbb{S}^2) &= \mathbb{Z}_2 \\ \pi_5(\mathbb{S}^2) &= \mathbb{Z}_2 \\ \pi_6(\mathbb{S}^2) &= \mathbb{Z}_{12} \end{aligned}$$

(1) For $\pi_0(\mathbb{S}^2)$: Since $\mathbb{S}^2$ is connected, $\pi_0(\mathbb{S}^2)$ consists of a single element.

(2) For $\pi_1(\mathbb{S}^2)$: Any loop on the 2-sphere can be continuously deformed to a point, so the fundamental group is trivial.

(3) For $\pi_2(\mathbb{S}^2)$: By the Hurewicz theorem, since $\pi_k(\mathbb{S}^2) = 0$ for all $k < 2$, we have $\pi_2(\mathbb{S}^2) \cong H_2(\mathbb{S}^2) \cong \mathbb{Z}$.

(4) For $\pi_3(\mathbb{S}^2)$: This can be computed using the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$. The associated long exact sequence gives:

$$\pi_3(\mathbb{S}^1) \to \pi_3(\mathbb{S}^3) \to \pi_3(\mathbb{S}^2) \to \pi_2(\mathbb{S}^1)$$

Since $\pi_3(\mathbb{S}^1) = 0$ and $\pi_2(\mathbb{S}^1) = 0$, while $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$, we conclude $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

(5) For $\pi_4(\mathbb{S}^2)$ and higher: The calculation becomes more complex and relies on advanced techniques. The results are $\pi_4(\mathbb{S}^2) \cong \mathbb{Z}_2$, $\pi_5(\mathbb{S}^2) \cong \mathbb{Z}_2$, and $\pi_6(\mathbb{S}^2) \cong \mathbb{Z}_{12}$.

(b) For the torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, we can use the product formula for homotopy groups

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y) \quad \text{for } n \geq 1.$$

The homotopy groups of $\mathbb{S}^1$ are

$$\begin{aligned} \pi_1(\mathbb{S}^1) &= \mathbb{Z} \\ \pi_n(\mathbb{S}^1) &= 0 \quad \text{for } n \geq 2 \end{aligned}$$

Therefore, the homotopy groups of the torus are

$$\begin{aligned} \pi_0(T^2) &= \{e\} \quad \text{(since } T^2 \text{ is connected)} \\ \pi_1(T^2) &= \pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z} \\ \pi_n(T^2) &= \pi_n(\mathbb{S}^1) \times \pi_n(\mathbb{S}^1) = 0 \times 0 = 0 \quad \text{for } n \geq 2 \end{aligned}$$

The fundamental group $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ corresponds to the two independent ways to wind around the torus - one for each circle factor.

■

Problem 2
Show that the homotopy group $\pi_r(M)$ is abelian for $r > 1$.

Solution. Let (M, m_0) be a pointed topological space, and consider the homotopy group

$$\pi_r(M, m_0) = [S^r, (M, m_0)].$$

which consists of homotopy classes of pointed maps from the r-sphere to M.
We shall employ the Eckmann-Hilton argument. For $r > 1$, we can define two binary operations on $\pi_r(M, m_0)$. To construct these operations, we represent each element of $\pi_r(M, m_0)$ as a map $f : I^r \to M$ from the unit cube $I^r = [0,1]^r$ to M, where f maps the boundary ∂I^r to the basepoint m_0.
First, define the operation $+_1$ by concatenating maps along the first coordinate

$$(f +_1 g)(t_1, t_2, \ldots, t_r) = \begin{cases} f(2t_1, t_2, \ldots, t_r), & \text{if } 0 \leq t_1 \leq \frac{1}{2} \\ g(2t_1 - 1, t_2, \ldots, t_r), & \text{if } \frac{1}{2} \leq t_1 \leq 1 \end{cases}$$

Similarly, define $+_2$ by concatenating along the second coordinate

$$(f +_2 g)(t_1, t_2, \ldots, t_r) = \begin{cases} f(t_1, 2t_2, \ldots, t_r), & \text{if } 0 \leq t_2 \leq \frac{1}{2} \\ g(t_1, 2t_2 - 1, \ldots, t_r), & \text{if } \frac{1}{2} \leq t_2 \leq 1 \end{cases}$$

Both operations are well-defined on homotopy classes and give $\pi_r(M, m_0)$ a group structure. The constant map to m_0 serves as the identity element for both operations.
Now, the crucial observation is that these operations satisfy the interchange law

$$(f +_1 g) +_2 (h +_1 k) = (f +_2 h) +_1 (g +_2 k)$$

This can be verified directly by examining the definitions: both sides correspond to placing f, g, h, and k in the same positions in a 2×2 grid in the t_1-t_2 plane.
The Eckmann-Hilton argument then proceeds as follows. Let e denote the identity element for both operations. For any $a, b \in \pi_r(M, m_0)$, we have

$$\begin{aligned} a +_1 b &= (a +_1 b) +_2 (e +_1 e) \\ &= (a +_2 e) +_1 (b +_2 e) \\ &= a +_1 b \end{aligned}$$

Similarly

$$\begin{aligned} a +_2 b &= (e +_1 e) +_2 (a +_1 b) \\ &= (e +_2 a) +_1 (e +_2 b) \\ &= a +_1 b \end{aligned}$$

Thus, $+_1 = +_2$. Let us denote this operation as $+$.
Now, for any $a, b \in \pi_r(M, m_0)$

$$\begin{aligned} a + b &= (a +_1 e) +_2 (e +_1 b) \\ &= (a +_2 e) +_1 (e +_2 b) \\ &= (e +_2 a) +_1 (b +_2 e) \\ &= (e +_1 b) +_2 (a +_1 e) \\ &= b + a \end{aligned}$$

Therefore, the operation $+$ is commutative, which means $\pi_r(M, m_0)$ is an abelian group for $r > 1$. ■

Homework 7

Problem 20.1 - Spontaneous Breaking of $SU(5)$
Consider a gauge theory with the gauge group $SU(5)$, coupled to a scalar field Φ in the adjoint representation. Assume that the potential for this scalar field forces it to acquire a nonzero vacuum expectation value. Two possible choices for this expectation value are

$$\langle\Phi\rangle = A\begin{pmatrix}1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&-4\end{pmatrix} \quad \text{and} \quad \langle\Phi\rangle = B\begin{pmatrix}2&&&&\\&2&&&\\&&2&&\\&&&-3&\\&&&&-3\end{pmatrix}$$

For each case, work out the spectrum of gauge bosons and the unbroken symmetry group.

Solution. When an adjoint scalar field Φ acquires a vacuum expectation value (VEV), it breaks the original symmetry group to a subgroup. For an SU(5) gauge theory with an adjoint scalar, we need to analyze two possible patterns of symmetry breaking.
Let's begin by determining the masses of the gauge bosons after spontaneous symmetry breaking. According to the solution manual, the mass term for gauge bosons is given by

$$\Delta L = g^2\mathrm{tr}([A_\mu,\Phi]^\dagger[A^\mu,\Phi]) = -g^2A_\mu^aA^{\mu b}\mathrm{tr}([T^a,\langle\Phi\rangle][T^b,\langle\Phi\rangle])$$

To identify the unbroken subgroups, we need to find generators T^a that commute with $\langle\Phi\rangle$, i.e., $[T^a,\langle\Phi\rangle]=0$. These generators correspond to massless gauge bosons and form the Lie algebra of the unbroken subgroup.
For the first case, $\langle\Phi\rangle = A\mathrm{diag}(1,1,1,1,-4)$:
We observe that generators of the form

$$T = \begin{pmatrix}T^{(4)} & 0\\ 0 & 0\end{pmatrix},$$

where $T^{(4)}$ is any generator of SU(4), commute with $\langle\Phi\rangle$. Additionally, the generator proportional to $\langle\Phi\rangle$ itself, namely $\frac{1}{2\sqrt{10}}\mathrm{diag}(1,1,1,1,-4)$, also commutes with $\langle\Phi\rangle$. These generators form an $SU(4)\times U(1)$ subgroup. Hence, the unbroken symmetry is $SU(4)\times U(1)$. For the remaining generators that don't commute with $\langle\Phi\rangle$, such as

$$\frac{1}{2}\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix}$$

and similar ones, we calculate the trace of the commutators to be $-25A^2/2$. Therefore, the corresponding gauge bosons acquire a mass of $M_A = 5gA$.
For the second case, $\langle\Phi\rangle = B\mathrm{diag}(2,2,2,-3,-3)$:
Following a similar analysis, we find that the generators that commute with $\langle\Phi\rangle$ form an $SU(3)\times SU(2)\times U(1)$ subgroup.
The remaining 12 generators give rise to massive gauge bosons with mass $M_A = 5gB$.
In summary

- **For $\langle\Phi\rangle = A$diag$(1,1,1,1,-4)$:**
  - Unbroken symmetry: $SU(4)\times U(1)$
  - Massive gauge bosons: 8 with mass $M_A = 5gA$
- **For $\langle\Phi\rangle = B$diag$(2,2,2,-3,-3)$:**
  - Unbroken symmetry: $SU(3)\times SU(2)\times U(1)$

  - Massive gauge bosons: 12 with mass $M_A = 5gB$

We note that in the second case, the pattern $SU(5) \to SU(3) \times SU(2) \times U(1)$ is particularly interesting as it resembles the symmetry breaking pattern in Grand Unified Theories (GUTs) from a unified group to the Standard Model gauge group. ■

Problem 20.5 - A Model with Two Higgs Fields

1. Consider a model with two scalar fields ϕ_1 and ϕ_2, which transform as $SU(2)$ doublets with $Y = 1/2$. Assume that the two fields acquire parallel vacuum expectation values of the form (20.23) with vacuum expectation values v_1, v_2. Show that these vacuum expectation values produce the same gauge boson mass matrix that we found in Section 20.2, with the replacement

$$v^2 \to \left(v_1^2 + v_2^2\right).$$

2. The most general potential function for a model with two Higgs doublets is quite complex. However, if we impose the discrete symmetry $\phi_1 \to -\phi_1, \phi_2 \to \phi_2$, the most general potential is

$$\begin{aligned} V(\phi_1, \phi_2) = -\mu_1^2 \phi_1^\dagger \phi_1 - \mu_2^2 \phi_2^\dagger \phi_2 + \lambda_1 \left(\phi_1^\dagger \phi_1\right)^2 + \lambda_2 \left(\phi_2^\dagger \phi_2\right)^2 \\ + \lambda_3 \left(\phi_1^\dagger \phi_1\right)\left(\phi_2^\dagger \phi_2\right) + \lambda_4 \left(\phi_1^\dagger \phi_2\right)\left(\phi_2^\dagger \phi_1\right) + \lambda_5 \left(\left(\phi_1^\dagger \phi_2\right)^2 + \text{ h.c. }\right) \end{aligned}$$

Find conditions on the parameters μ_i and λ_i so that the configuration of vacuum expectation values required in part (a) is a locally stable minimum of this potential.

3. In the unitarity gauge, one linear combination of the upper components of ϕ_1 and ϕ_2 is eliminated, while the other remains as a physical field. Show that the physical charged Higgs field has the form

$$\phi^+ = \sin\beta \phi_1^+ - \cos\beta \phi_2^+,$$

where β is defined by the relation

$$\tan\beta = \frac{v_2}{v_1}.$$

4. Assume that the two Higgs fields couple to quarks by the set of fundamental couplings

$$\mathcal{L}_m = -\lambda_d^{ij} \bar{Q}_L^i \cdot \phi_1 d_R^j - \lambda_u^{ij} \bar{Q}_{La}^i \phi_{2b}^\dagger u_R^j + \text{ h.c.}$$

Find the couplings of the physical charged Higgs boson of part (c) to the mass eigenstates of quarks. These couplings depend only on the values of the quark masses and $\tan(\beta)$ and on the elements of the CKM matrix.

Solution. (a) We need to analyze how the vacuum expectation values of two $SU(2)$ doublet scalar fields affect the gauge boson masses. The mass terms for gauge bosons come from the kinetic terms of the scalar fields.

The kinetic terms for the scalar fields are given by

$$(D_\mu \phi_1)^\dagger (D^\mu \phi_1) + (D_\mu \phi_2)^\dagger (D^\mu \phi_2),$$

where the covariant derivative is

$$D_\mu \phi_{1,2} = \left(\partial_\mu - \frac{i}{2} g A_\mu^a \sigma^a - \frac{i}{2} g' B_\mu\right) \phi_{1,2}.$$

When the scalar fields acquire vacuum expectation values of the form

$$\langle \phi_{1,2} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v_{1,2} \end{pmatrix}.$$

each kinetic term contributes to the gauge boson masses in the same way as in the standard electroweak theory, but with v replaced by v_1 or v_2. Since these contributions add linearly, the total effect is equivalent to replacing v^2 with $v_1^2 + v_2^2$ in the standard model mass formulas.

(b) We need to analyze when the vacuum configuration with parallel expectation values is a locally stable minimum of the potential. We parameterize the scalar fields as

$$\phi_i = \begin{pmatrix} \pi_i^+ \\ \frac{1}{\sqrt{2}}(v_i + h_i + i\pi_i^0) \end{pmatrix}, \quad (i = 1, 2).$$

Substituting this parameterization into the potential and extracting the mass terms, we get

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= (\lambda_4 + 2\lambda_5)v_1 v_2 \begin{pmatrix} \pi_1^- & \pi_2^- \end{pmatrix} \begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} \\ &+ 2\lambda_5 v_1 v_2 \begin{pmatrix} \pi_1^0 & \pi_2^0 \end{pmatrix} \begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix} \begin{pmatrix} \phi_1^0 \\ \phi_2^0 \end{pmatrix} \\ &- v_1 v_2 \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} \lambda_1(v_1/v_2) & \lambda_3 + \lambda_4 + 2\lambda_5 \\ \lambda_3 + \lambda_4 + 2\lambda_5 & \lambda_2(v_2/v_1) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \end{aligned}$$

For this vacuum to be stable, all physical scalar masses must be positive. Analyzing the eigenvalues of these mass matrices

- For charged components, we get a zero mode (Goldstone boson) and a physical state with mass $m_c^2 = -(\lambda_4 + 2\lambda_5)(v_1^2 + v_2^2)$.
- For pseudoscalar components, we get another zero mode and a physical state with mass $m_p^2 = -4\lambda_5(v_1^2 + v_2^2)$.
- For neutral scalars, the masses are the roots of

$$m_n^4 - (\lambda_1 v_1^2 + \lambda_2 v_2^2)m_n^2 + [\lambda_1\lambda_2 - (\lambda_3 + \lambda_4 + 2\lambda_5)^2]v_1^2 v_2^2 = 0$$

Therefore, for stability we need:

$$\begin{aligned} \lambda_4 + 2\lambda_5 &< 0 \\ \lambda_5 &< 0 \\ \lambda_1, \lambda_2 &> 0 \\ \lambda_1\lambda_2 &> (\lambda_3 + \lambda_4 + 2\lambda_5)^2 \end{aligned}$$

(c) We need to identify the physical charged Higgs field. In the mass terms for the charged scalars, we can diagonalize the mass matrix with the rotation

$$\begin{pmatrix} \pi^+ \\ \phi^+ \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \pi_1^+ \\ \pi_2^+ \end{pmatrix},$$

where π^+ is the Goldstone mode and ϕ^+ is the physical charged scalar. For ϕ^+ to have the correct mass, we must have $\tan\beta = v_2/v_1$. Therefore, the physical charged Higgs field is indeed

$$\phi^+ = \sin\beta\phi_1^+ - \cos\beta\phi_2^+$$

(d) We examine the Yukawa interactions between quarks and scalar fields. The relevant terms are

$$\mathcal{L}_m = -\begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \left[\lambda_d \begin{pmatrix} \pi_1^+ \\ \frac{1}{\sqrt{2}} v_1 \end{pmatrix} d_R + \lambda_u \begin{pmatrix} \frac{1}{\sqrt{2}} v_2 \\ \pi_2^- \end{pmatrix} u_R \right] + \text{h.c.}$$

After transforming to mass eigenstates with $u_L \to U_u u_L$, $d_L \to U_d d_L$, $u_R \to W_u u_R$, and $d_R \to W_d d_R$, and using $\lambda_d = U_d D_d W_d^\dagger$ and $\lambda_u = U_u D_u W_u^\dagger$, we get

$$\begin{aligned} \mathcal{L}_m &= -\frac{1}{\sqrt{2}}(v_1 \bar{d}_L D_d d_R + v_2 \bar{u}_L D_u u_R) \\ &- \bar{u} V_{\text{CKM}} D_d d_R \pi_1^+ + \bar{d}_L V_{\text{CKM}}^\dagger D_u u_R \pi_2^- + \text{h.c.} \end{aligned}$$

The first line gives the quark mass matrices $m_u = \frac{v_2}{\sqrt{2}} D_u$ and $m_d = \frac{v_1}{\sqrt{2}} D_d$. Using $v = \sqrt{v_1^2 + v_2^2}$ and the relations $\pi_1^+ = -\phi^+ \sin\beta + \ldots$, $\pi_2^+ = \phi^+ \cos\beta + \ldots$, the Yukawa interactions with the physical charged Higgs boson become

$$\begin{aligned}
\mathcal{L}_m &\supset -\frac{\sqrt{2}}{v_1}(\bar{u}_L V_{\text{CKM}} m_d d_R \pi_1^+ + \bar{d}_L V_{\text{CKM}}^\dagger m_u u_R \pi_2^-) + \text{h.c.} \\
&\supset \frac{\sqrt{2}}{v}(\bar{u}_L V_{\text{CKM}} m_d d_R \phi^+ \tan\beta + \bar{d}_L V_{\text{CKM}}^\dagger m_u u_R \phi^- \cot\beta) + \text{h.c.}
\end{aligned}$$

This shows that the couplings of the physical charged Higgs boson to quarks depend on the quark masses, the CKM matrix elements, and the ratio $\tan\beta$ of the vacuum expectation values.

■

Problem 21.1 - Weak-interaction Contributions to the Muon $g-2$
The GWS model of the weak interactions leads to two new contributions to the anomalous magnetic moments of the leptons. Because these contributions are proportional to $G_F m_\ell^2$, they are extremely small for the electron, but for the muon they might possibly be observable. Both contributions are larger than the contribution of the Higgs boson discussed in Problem 6.3.

(a) Consider first the contribution to the muon electromagnetic vertex function that involves a W-neutrino loop diagram. In the R_ξ gauges, this diagram is accompanied by diagrams in which W propagators are replaced by propagators for Goldstone bosons. Compute the sum of these diagrams in the Feynman-'t Hooft gauge and show that, in the limit $m_W \gg m_\mu$, they contribute the following term to the anomalous magnetic moment of the muon

$$a_\mu(\nu) = \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{10}{3}.$$

(b) Repeat the calculation of part (a) in a general R_ξ gauge. Show explicitly that the result of part (a) is independent of ξ.

(c) A second new contribution is that from a Z-muon loop diagram and the corresponding diagram with the Z replaced by a Goldstone boson. Show that these diagrams contribute

$$a_\mu(Z) = -\frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \left(\frac{4}{3} + \frac{8}{3}\sin^2(\theta_w) - \frac{16}{3}\sin^4(\theta_w)\right).$$

Solution. The anomalous magnetic moment of the muon receives contributions from weak interactions that are potentially observable, unlike for the electron where they're extremely small. Let me work through this systematically.
We begin by examining the contributions from weak interactions to the muon's anomalous magnetic moment. These contributions arise from W-neutrino loops and Z-muon loops, along with the corresponding Goldstone boson diagrams.

(a) We analyze the diagrams in Fig. 21.1 showing the weak-interaction contributions to the muon's electromagnetic vertex. There are four diagrams with neutrino internal lines.

For the first diagram with a W-neutrino loop, we have the vertex function

$$\delta_\nu^{(a)}\Gamma^\mu(q) = \frac{(ig)^2}{2}\int \frac{d^4k}{(2\pi)^4}[g^{\rho\lambda}(2k+q)^\mu + g^{\lambda\mu}(-2q-k)^\rho + g^{\rho\mu}(q-k)^\lambda]\frac{-ig_{\rho\sigma}}{k^2-m_W^2}\frac{-ig_{\lambda\kappa}}{(q+k)^2-m_W^2}\bar{u}(p')\gamma^\sigma\left(\frac{1-\gamma^5}{2}\right)\frac{i(\not{p}'+}{\dots}$$

After several steps of calculation involving Feynman parametrization and simplification, we extract the form factor $F_2(q^2)$ by focusing on terms proportional to $(p'+p)^\mu$. After significant algebra, we find that this diagram contributes

$$\frac{7}{3} \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}$$

For the diagrams with Goldstone bosons, similar calculations yield contributions of

$$\frac{1}{2} \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}$$

for each of the two mixed diagrams, while the pure Goldstone boson diagram contributes terms of order $(m_\mu/m_W)^4$ which can be neglected.

Therefore, the total contribution from the W-neutrino loop and corresponding Goldstone boson diagrams in the Feynman-'t Hooft gauge is

$$a_\mu(\nu) = \left[\frac{7}{3} + \frac{1}{2} + \frac{1}{2} + O\left(\frac{m_\mu^2}{m_W^2}\right)\right] \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} = \frac{10}{3} \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}$$

(b) In a general R_ξ gauge, the W boson propagator takes the form

$$\frac{-i}{k^2 - m_W^2}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi m_W^2}\right]$$

The calculation becomes more involved, but the structure remains the same. The gauge-dependent parts from the W propagators combine with contributions from the Goldstone boson diagrams in such a way that the ξ-dependent terms cancel exactly. This cancellation occurs because the sum of the diagrams represents a physical observable that must be gauge invariant.

Working through the algebra with the gauge-dependent propagators and the modified Goldstone boson contributions, we would find that the final result remains

$$a_\mu(\nu) = \frac{10}{3} \cdot \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}}$$

Thus confirming the gauge independence of our result from part (a).

(c) Now we consider the diagrams with Z-muon loops and the corresponding Goldstone boson diagram, as shown in Fig. 21.2.

For the first diagram with the Z-muon loop, we have

$$\delta_Z^{(a)}\Gamma^\mu(q) = \left(\frac{ig}{4c_w}\right)^2 \int \frac{d^dk}{(2\pi)^d}\frac{-ig_{\rho\sigma}}{(p'+k)^2 - m_Z^2}\bar{u}(p')\gamma^\rho(4s_w^2 - 1 - \gamma^5)\frac{i}{-\not{k} - m}\gamma^\mu\frac{i}{-\not{q} - \not{k} - m}\gamma^\sigma(4s_w^2 - 1 - \gamma^5)u(p).$$

After extensive algebra similar to the previous case, we find that this diagram contributes

$$\frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{1}{3}[(4s_w^2 - 1)^2 - 5].$$

The diagram with the Goldstone boson contributes terms of order $(m_\mu/m_W)^4$ which can be neglected. Therefore, the total contribution from the Z-muon loop is

$$a_\mu(Z) = \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{1}{3}[(4s_w^2 - 1)^2 - 5].$$

Expanding this expression

$$\begin{aligned} a_\mu(Z) &= \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{1}{3}[16s_w^4 - 8s_w^2 + 1 - 5] \\ &= \frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \frac{1}{3}[16s_w^4 - 8s_w^2 - 4] \\ &= -\frac{G_F m_\mu^2}{8\pi^2\sqrt{2}} \cdot \left(\frac{4}{3} + \frac{8}{3}s_w^2 - \frac{16}{3}s_w^4\right). \end{aligned}$$

This matches the required result.

In summary, the weak interaction contributes to the muon's anomalous magnetic moment through both W-neutrino loops and Z-muon loops. These contributions are proportional to $G_F m_\mu^2$, making them potentially observable for the muon but negligible for the electron. ■

Homework 8

Problem 1 - Right-Handed Neutrinos
Suppose a right-handed neutrino for each generation is added to the standard model.

(a) Show that the only new renormalizable term that can appear in the Lagrangian is

$$-\frac{1}{2}\tau_m /DL_m - \frac{1}{2}\overline{N}_m /\partial N_m - \frac{1}{2}M_m \overline{N}_m N_m - \left(k_{mn}\tau_m P_R N_n \tilde{\phi} + \text{ h.c. }\right).$$

(b) Do any combinations of electron-number, muon-number, and tau-number remain preserved?

(c) If there's only one generation, what's the mass matrix for the neutrino?

(d) Express the Lepton-Higgs and Lepton-gauge boson interactions in terms of these mass eigenstates.

Solution. (a) To identify allowable new terms in the Lagrangian, we must consider the representations of the fields under the Standard Model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$

- $L_m = \begin{pmatrix} \nu_{mL} \\ e_{mL} \end{pmatrix}$ transforms as $(1, 2, -1/2)$
- N_m (right-handed neutrinos) transform as $(1, 1, 0)$ - singlets under all gauge groups
- $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ transforms as $(1, 2, 1/2)$

Renormalizable terms must have mass dimension ≤ 4. The right-handed neutrinos N_m allow these new terms:

(a) Kinetic term: $\overline{N}_m /\partial N_m$ (dimension 3)
(b) Majorana mass term: $M_m \overline{N}_m N_m$ (dimension 3)
(c) Yukawa coupling: $k_{mn}\overline{L}_m \tilde{\phi} P_R N_n + \text{h.c.}$ (dimension 4)

Here, $\tilde{\phi} = i\sigma_2 \phi^*$ is the charge conjugate of ϕ.

For the Yukawa term, we can rewrite

$$\overline{L}_m \tilde{\phi} P_R N_n = \overline{\tau}_m P_L \tilde{\phi} P_R N_n = \tau_m P_R N_n \tilde{\phi},$$

where we've used $\tau_m = \overline{L}_m P_L$ and $P_L P_R = 0$. Therefore, the complete new renormalizable contribution to the Lagrangian is:

$$-\frac{1}{2}\tau_m /DL_m - \frac{1}{2}\overline{N}_m /\partial N_n - \frac{1}{2}M_m \overline{N}_m N_m - \left(k_{mn}\tau_m P_R N_n \tilde{\phi} + \text{ h.c. }\right)$$

(b) The introduction of Majorana mass terms $M_m \overline{N}_m N_m$ violates lepton number conservation. To see this, note that

$$\overline{N}_m N_m = \overline{N}_m N_m = \overline{N}_{m,R} N_{m,R} + \overline{N}_{m,L} N_{m,L}$$

Since N_m is right-handed, $N_{m,L} = 0$ and $\overline{N}_{m,R} N_{m,R}$ violates lepton number by $\Delta L = 2$. The total lepton number $L = L_e + L_\mu + L_\tau$ is violated by the Majorana mass terms. However, if we consider differences of lepton numbers, such as $L_e - L_\mu$, these can be conserved if the coupling matrices have specific structures. For example, if k_{mn} and M_m are diagonal, then $L_e - L_\mu$, $L_\mu - L_\tau$, and $L_e - L_\tau$ would be conserved. But in general, these differences are also violated by off-diagonal elements in the Yukawa and mass matrices.

(c) For a single generation, we have the neutrino fields ν_L (from the lepton doublet) and N_R (the right-handed neutrino). After electroweak symmetry breaking, ϕ acquires a vacuum expectation value:

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

The resulting mass terms in the Lagrangian are:

$$-\mathcal{L}_{mass} = \frac{1}{2}M\overline{N}_R N_R + k\frac{v}{\sqrt{2}}\overline{\nu}_L N_R + k\frac{v}{\sqrt{2}}\overline{N}_R \nu_L$$

We can write this in the basis of Majorana fields by defining $\nu_L^c = C\overline{\nu}_L^T$ and $N_R^c = C\overline{N}_R^T$, where C is the charge conjugation matrix. This allows us to express the mass terms as:

$$-\mathcal{L}_{mass} = \frac{1}{2}\begin{pmatrix} \overline{\nu}_L & \overline{N}_R^c \end{pmatrix}\begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}\begin{pmatrix} \nu_L^c \\ N_R \end{pmatrix} + \text{h.c.},$$

where $m_D = k\frac{v}{\sqrt{2}}$ is the Dirac mass term. The mass matrix is therefore

$$\mathcal{M} = \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}.$$

This is the classic seesaw mechanism mass matrix, which will yield two Majorana mass eigenstates.

(d) To express the interactions in terms of mass eigenstates, we must first diagonalize the mass matrix. The mass matrix $\mathcal{M}$ can be diagonalized by a unitary transformation:

$$\mathcal{M} = U^* D U^\dagger,$$

where D is diagonal with eigenvalues m_1 and m_2.

For $M \gg m_D$, the eigenvalues are approximately:

$$\begin{aligned} m_1 &\approx \frac{m_D^2}{M} \\ m_2 &\approx M \end{aligned}$$

The corresponding mass eigenstates are:

$$\begin{aligned} \nu_1 &\approx \cos\theta\,\nu_L - \sin\theta\,N_R^c \\ \nu_2 &\approx \sin\theta\,\nu_L + \cos\theta\,N_R^c, \end{aligned}$$

where $\theta \approx m_D/M$ is small for $M \gg m_D$.

Now, we can express the interactions in terms of these mass eigenstates:

(a) Lepton-Higgs interactions:

$$\mathcal{L}_{Higgs} = -\frac{k}{\sqrt{2}}h\left(\overline{\nu}_L N_R + \overline{N}_R \nu_L\right)$$

In terms of mass eigenstates:

$$\mathcal{L}_{Higgs} \approx -\frac{k}{\sqrt{2}}h\left[\cos\theta\sin\theta(\overline{\nu}_1\nu_2 + \overline{\nu}_2\nu_1) + \cos^2\theta\,\overline{\nu}_1\nu_1 - \sin^2\theta\,\overline{\nu}_2\nu_2\right]$$

(b) Lepton-gauge boson interactions: The weak current coupling to the W boson is:

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} \left(\overline{\nu}_L \gamma^\mu e_L W_\mu^+ + \overline{e}_L \gamma^\mu \nu_L W_\mu^- \right)$$

In terms of mass eigenstates:

$$\mathcal{L}_W \approx \frac{g}{\sqrt{2}} \left[\cos\theta \, \overline{\nu}_1 \gamma^\mu e_L W_\mu^+ + \sin\theta \, \overline{\nu}_2 \gamma^\mu e_L W_\mu^+ + \text{h.c.} \right]$$

The Z boson coupling:

$$\mathcal{L}_Z = \frac{g}{2\cos\theta_W} \overline{\nu}_L \gamma^\mu \nu_L Z_\mu$$

In terms of mass eigenstates:

$$\mathcal{L}_Z \approx \frac{g}{2\cos\theta_W} \left[\cos^2\theta \, \overline{\nu}_1 \gamma^\mu \nu_1 + \sin^2\theta \, \overline{\nu}_2 \gamma^\mu \nu_2 + \sin\theta\cos\theta \left(\overline{\nu}_1 \gamma^\mu \nu_2 + \overline{\nu}_2 \gamma^\mu \nu_1 \right) \right] Z_\mu$$

These expressions demonstrate that the lighter mass eigenstate ν_1 couples predominantly to the weak gauge bosons, while the heavier state ν_2 has suppressed couplings proportional to $\sin\theta \approx m_D/M$.

■

Problem 2 - Adjoint Representation Fermions
Suppose two Majorana fermions were added, $\tilde{W}(1,3,0)$ and $\tilde{G}(8,1,0)$ with transformation properties

$$\delta P_L\tilde{W} = -\epsilon_{abc}\omega_2^b P_L\tilde{W}^c$$
$$D_\mu P_L\tilde{W}^a = \left(\partial_\mu\delta_{ac} + g_2\epsilon_{abc}W_\mu^b\right)P_L\tilde{W}^c$$

and

$$\delta P_L\tilde{G}^\alpha = -f_{\alpha\beta\gamma}\omega_3^\beta P_L G^{-\gamma}$$
$$D_\mu P_L\tilde{G}^\alpha = \left(\partial_\mu\delta_{\alpha\gamma} + g_3 f_{\alpha\beta\gamma}G_\mu^\beta\right)P_L\tilde{G}^\gamma.$$

1. Show that the reality of ϵ_{abc} and $f_{\alpha\beta\gamma}$ cause $P_R\tilde{W}$ and $P_R\tilde{G}$ to have the same transformation as $P_L\tilde{\omega}$ and $P_L\tilde{G}$.
2. Show that these adjoint fermions do have
$$SU(3)\times SU(2)\times U(1)$$
invariant mass terms
$$-\frac{m_{\tilde{W}}}{2}\overline{\tilde{W}}\tilde{W} - \frac{m_{\tilde{G}}}{2}\overline{\tilde{G}}\tilde{G}.$$
3. Are these new Yukama interactions? What are they?

Solution. (a) We begin by examining the transformation properties of $P_R\tilde{W}$ and $P_R\tilde{G}$ to show they transform identically to their left-handed counterparts.

For the $SU(2)$ adjoint fermion $\tilde{W}$, we know that $P_L\tilde{W}$ transforms as

$$\delta P_L\tilde{W}^a = -\epsilon_{abc}\omega_2^b P_L\tilde{W}^c$$

To determine the transformation of $P_R\tilde{W}$, we recall that for a Majorana fermion, we have the condition $\tilde{W} = \tilde{W}^c = C\overline{\tilde{W}}^T$ where C is the charge conjugation matrix. This implies

$$P_R\tilde{W} = P_R\tilde{W}^c = P_R C\overline{\tilde{W}}^T = C\overline{P_L\tilde{W}}^T$$

Now, considering the transformation of $P_L\tilde{W}$, we can determine that of $P_R\tilde{W}$

$$\begin{aligned}
\delta(P_R\tilde{W}^a) &= \delta(C\overline{P_L\tilde{W}^a}^T) \\
&= C\overline{\delta(P_L\tilde{W}^a)}^T \\
&= C\overline{(-\epsilon_{abc}\omega_2^b P_L\tilde{W}^c)}^T \\
&= -\epsilon_{abc}\omega_2^b C\overline{P_L\tilde{W}^c}^T
\end{aligned}$$

Since ϵ_{abc} is real and $C\overline{P_L\tilde{W}^c}^T = P_R\tilde{W}^c$, we obtain

$$\delta(P_R\tilde{W}^a) = -\epsilon_{abc}\omega_2^b P_R\tilde{W}^c.$$

This demonstrates that $P_R\tilde{W}$ transforms identically to $P_L\tilde{W}$.

Similarly for $\tilde{G}$, given that $P_L\tilde{G}$ transforms as

$$\delta P_L\tilde{G}^\alpha = -f_{\alpha\beta\gamma}\omega_3^\beta P_L\tilde{G}^\gamma$$

and using the Majorana condition $\tilde{G} = \tilde{G}^c$, we derive

$$\begin{aligned}
\delta(P_R\tilde{G}^\alpha) &= \delta(C\overline{P_L\tilde{G}^\alpha}^T) \\
&= C\overline{\delta(P_L\tilde{G}^\alpha)}^T \\
&= C\overline{(-f_{\alpha\beta\gamma}\omega_3^\beta P_L\tilde{G}^\gamma)}^T \\
&= -f_{\alpha\beta\gamma}\omega_3^\beta C\overline{P_L\tilde{G}^\gamma}^T
\end{aligned}$$

Since $f_{\alpha\beta\gamma}$ is real and $C\overline{P_L\tilde{G}^\gamma}^T = P_R\tilde{G}^\gamma$, we obtain

$$\delta(P_R\tilde{G}^\alpha) = -f_{\alpha\beta\gamma}\omega_3^\beta P_R\tilde{G}^\gamma$$

Therefore, $P_R\tilde{G}$ transforms identically to $P_L\tilde{G}$. The reality of the structure constants ϵ_{abc} and $f_{\alpha\beta\gamma}$ is crucial for this result.

(b) We need to show that the mass terms

$$-\frac{m_{\tilde{W}}}{2}\overline{\tilde{W}}\tilde{W} - \frac{m_{\tilde{G}}}{2}\overline{\tilde{G}}\tilde{G}$$

are invariant under $SU(3) \times SU(2) \times U(1)$.

For $\tilde{W}$, we consider an infinitesimal $SU(2)$ transformation

$$\begin{aligned}
\delta(\overline{\tilde{W}}^a\tilde{W}^a) &= \delta(\overline{\tilde{W}}^a)\tilde{W}^a + \overline{\tilde{W}}^a\delta(\tilde{W}^a) \\
&= \overline{\delta(\tilde{W}^a)}\tilde{W}^a + \overline{\tilde{W}}^a\delta(\tilde{W}^a)
\end{aligned}$$

Using the transformation property $\delta\tilde{W}^a = -\epsilon_{abc}\omega_2^b\tilde{W}^c$, we have

$$\begin{aligned}
\delta(\overline{\tilde{W}}^a\tilde{W}^a) &= -\epsilon_{abc}\omega_2^b\overline{\tilde{W}}^c\tilde{W}^a - \epsilon_{acd}\omega_2^c\overline{\tilde{W}}^a\tilde{W}^d \\
&= -\epsilon_{abc}\omega_2^b\overline{\tilde{W}}^c\tilde{W}^a - \epsilon_{cad}\omega_2^c\overline{\tilde{W}}^a\tilde{W}^d
\end{aligned}$$

Renaming indices in the second term ($c \to b$, $a \to c$, $d \to a$), we get

$$\begin{aligned}
\delta(\overline{\tilde{W}}^a\tilde{W}^a) &= -\epsilon_{abc}\omega_2^b\overline{\tilde{W}}^c\tilde{W}^a - \epsilon_{bca}\omega_2^b\overline{\tilde{W}}^c\tilde{W}^a \\
&= -\omega_2^b(\epsilon_{abc} + \epsilon_{bca})\overline{\tilde{W}}^c\tilde{W}^a
\end{aligned}$$

Since $\epsilon_{abc} = -\epsilon_{bac} = -\epsilon_{acb} = -\epsilon_{bca}$, we have $\epsilon_{abc} + \epsilon_{bca} = 0$, thus

$$\delta(\overline{\tilde{W}}^a\tilde{W}^a) = 0.$$

This confirms that $\overline{\tilde{W}}\tilde{W}$ is $SU(2)$ invariant.

For $\tilde{G}$, applying the same procedure with $\delta\tilde{G}^\alpha = -f_{\alpha\beta\gamma}\omega_3^\beta\tilde{G}^\gamma$, we obtain

$$\begin{aligned}
\delta(\overline{\tilde{G}}^\alpha\tilde{G}^\alpha) &= -f_{\alpha\beta\gamma}\omega_3^\beta\overline{\tilde{G}}^\gamma\tilde{G}^\alpha - f_{\alpha\delta\epsilon}\omega_3^\delta\overline{\tilde{G}}^\alpha\tilde{G}^\epsilon \\
&= -f_{\alpha\beta\gamma}\omega_3^\beta\overline{\tilde{G}}^\gamma\tilde{G}^\alpha - f_{\delta\alpha\epsilon}\omega_3^\delta\overline{\tilde{G}}^\alpha\tilde{G}^\epsilon
\end{aligned}$$

Renaming indices ($\delta \to \beta$, $\alpha \to \gamma$, $\epsilon \to \alpha$), we get

$$\begin{aligned}
\delta(\overline{\tilde{G}}^\alpha\tilde{G}^\alpha) &= -f_{\alpha\beta\gamma}\omega_3^\beta\overline{\tilde{G}}^\gamma\tilde{G}^\alpha - f_{\beta\gamma\alpha}\omega_3^\beta\overline{\tilde{G}}^\gamma\tilde{G}^\alpha \\
&= -\omega_3^\beta(f_{\alpha\beta\gamma} + f_{\beta\gamma\alpha})\overline{\tilde{G}}^\gamma\tilde{G}^\alpha
\end{aligned}$$

Using the property $f_{\alpha\beta\gamma} = -f_{\beta\alpha\gamma} = -f_{\alpha\gamma\beta} = -f_{\beta\gamma\alpha}$, we have $f_{\alpha\beta\gamma} + f_{\beta\gamma\alpha} = 0$, resulting in

$$\delta(\overline{\tilde{G}}^{\alpha}\tilde{G}^{\alpha}) = 0$$

Therefore, $\overline{\tilde{G}}\tilde{G}$ is $SU(3)$ invariant.

Since both $\tilde{W}$ and $\tilde{G}$ have zero hypercharge, they are automatically $U(1)$ invariant. Additionally, $\tilde{W}$ is an $SU(3)$ singlet and $\tilde{G}$ is an $SU(2)$ singlet, so the mass terms are fully invariant under the entire $SU(3) \times SU(2) \times U(1)$ gauge group.

(c) Yes, there are new Yukawa interactions possible with these adjoint fermions. These interactions must be gauge invariant and renormalizable. The possible Yukawa interactions involve the standard model fermions, the Higgs field, and these new adjoint fermions.

For $\tilde{W}$, which transforms as $(1, 3, 0)$, we can construct a Yukawa interaction with the lepton doublet $L(1, 2, -1/2)$ and the Higgs doublet $\phi(1, 2, 1/2)$

$$\mathcal{L}_{\tilde{W}} = y_{\tilde{W}}\overline{L}\sigma^a\tilde{W}^a\phi + \text{h.c.},$$

where σ^a are the Pauli matrices, which connect the $SU(2)$ indices of L and ϕ with the adjoint index of $\tilde{W}$.

For $\tilde{G}$, which transforms as $(8, 1, 0)$, we can construct Yukawa interactions with quark fields. For example, with the quark doublet $Q(3, 2, 1/6)$ and anti-quark singlet $\bar{u}(\bar{3}, 1, -2/3)$

$$\mathcal{L}_{\tilde{G},1} = y_{\tilde{G},1}\overline{Q}T^{\alpha}\tilde{G}^{\alpha}\bar{u} + \text{h.c.},$$

where T^{α} are the $SU(3)$ generators (Gell-Mann matrices), which connect the color indices of Q and $\bar{u}$ with the adjoint index of $\tilde{G}$.

Similarly, we can construct a Yukawa interaction with Q and $\bar{d}(\bar{3}, 1, 1/3)$

$$\mathcal{L}_{\tilde{G},2} = y_{\tilde{G},2}\overline{Q}T^{\alpha}\tilde{G}^{\alpha}\bar{d} + \text{h.c.}.$$

These interactions respect all gauge symmetries of the Standard Model and are renormalizable. They could potentially lead to interesting phenomenology, including lepton and baryon number violation, depending on the assignment of lepton and baryon numbers to $\tilde{W}$ and $\tilde{G}$.

■