PHYS 603 - Methods of Theoretical Physics III

Lie Algebras in Particle Physics by *H. Georgi*
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Homework 4

The components of an angular momentum $\mathbf{J}$ (which can be orbital, spin, or total) can be organized into a spin-1 tensor operator as J_{+1}, J_0, J_{-1}, where $J_0 = J_z$, and

$$J_{\pm 1} = \mp \frac{1}{\sqrt{2}} (J_x \pm i J_y) = \mp J^{\pm}.$$

Note in particular that $J_{+1} = -J^{+}$. For this homework, please assume $\hbar = 1$.

Problem 1
Problem 4.B from the textbook. You do not have to answer the last question of the problem.
The operator $(r_{+1})^2$ satisfies

$$\left[L^{+}, (r_{+1})^2\right] = 0.$$

It is therefore the O_{+2} component of a spin 2 tensor operator. Construct the other components, O_m. Note that the product of tensor operators transforms like the tensor product of their representations. What is the connection of this with the spherical harmonics, $Y_{l,m}(\theta, \phi)$? Hint: let $r_1 = \sin(\theta)\cos(\phi)$, $r_2 = \sin(\theta)\sin(\phi)$, and $r_3 = \cos(\theta)$. Can you generalize this construction to arbitrary ℓ and explain what is going on?

Solution. We're asked to construct the components of a spin-2 tensor operator, starting from the fact that $(r_{+1})^2$ satisfies

$$\left[J_{+}, (r_{+1})^2\right] = 0,$$

making it the O_{+2} component of a spin-2 tensor operator.
First, let's recall that the position vector $\mathbf{r}$ transforms as a spin-1 tensor under rotations with components:

$$\begin{aligned} r_{+1} &= -\frac{1}{\sqrt{2}}(r_x + \mathrm{i} r_y) \\ r_0 &= r_z \\ r_{-1} &= \frac{1}{\sqrt{2}}(r_x - \mathrm{i} r_y) \end{aligned}$$

For a spin-1 tensor, the commutation relations with angular momentum operators are:

$$\begin{cases} [J_{+}, r_{+1}] = 0 \\ [J_{+}, r_0] = \sqrt{2} r_{+1} \\ [J_{+}, r_{-1}] = \sqrt{2} r_0 \\ [J_{-}, r_{+1}] = \sqrt{2} r_0 \\ [J_{-}, r_0] = \sqrt{2} r_{-1} \\ [J_{-}, r_{-1}] = 0 \end{cases}$$

To find all components of our spin-2 tensor operator, we can use the lowering operator J_{-} systematically:

- **For O_{+2}:**

$$\begin{aligned}
O_{+2} &= (r_{+1})^2 \\
&= \left(-\frac{1}{\sqrt{2}}(r_x + \mathrm{i}r_y)\right)^2 \\
&= \frac{1}{2}(r_x + \mathrm{i}r_y)^2 \\
&= \frac{1}{2}(r_x^2 - r_y^2 + 2\mathrm{i}r_x r_y).
\end{aligned}$$

- **For O_{+1}:**

$$\begin{aligned}
O_{+1} &= \frac{1}{\sqrt{2}}\left[J_-, O_{+2}\right] \\
&= \frac{1}{\sqrt{2}}\left[J_-, (r_{+1})^2\right] \\
&= \frac{1}{\sqrt{2}}\left(2r_{+1}\left[J_-, r_{+1}\right]\right) \\
&= \frac{1}{\sqrt{2}}(2r_{+1}\sqrt{2}r_0) \\
&= 2r_{+1}r_0 \\
&= -\sqrt{2}(r_x + \mathrm{i}r_y)r_z
\end{aligned}$$

- **For O_0:**

$$\begin{aligned}
O_0 &= \frac{1}{\sqrt{6}}\left[J_-, O_{+1}\right] \\
&= \frac{1}{\sqrt{6}}\left[J_-, 2r_{+1}r_0\right] \\
&= \frac{2}{\sqrt{6}}\left(\left[J_-, r_{+1}\right]r_0 + r_{+1}\left[J_-, r_0\right]\right) \\
&= \frac{2}{\sqrt{6}}(\sqrt{2}r_0r_0 + r_{+1}\sqrt{2}r_{-1}) \\
&= \frac{2\sqrt{2}}{\sqrt{6}}(r_0^2 + r_{+1}r_{-1}).
\end{aligned}$$

In spherical coordinates, this becomes:

$$\begin{aligned}
O_0 &= \frac{2\sqrt{2}}{\sqrt{6}}\left(\cos^2(\theta) + \left(-\frac{1}{\sqrt{2}}\sin(\theta)\mathrm{e}^{\mathrm{i}\phi}\right)\left(\frac{1}{\sqrt{2}}\sin(\theta)\mathrm{e}^{-\mathrm{i}\phi}\right)\right) \\
&= \frac{2\sqrt{2}}{\sqrt{6}}\left(\cos^2(\theta) - \frac{1}{2}\sin^2(\theta)\right) \\
&= \frac{2\sqrt{2}}{\sqrt{6}}\left(\cos^2(\theta) - \frac{1}{2}(1 - \cos^2(\theta))\right) \\
&= \frac{2\sqrt{2}}{\sqrt{6}}\left(\frac{3\cos^2(\theta) - 1}{2}\right) \\
&= \frac{1}{\sqrt{3}}(3\cos^2(\theta) - 1).
\end{aligned}$$

- **For O_{-1}:**

$$
\begin{aligned}
O_{-1} &= \frac{1}{\sqrt{2}}\left[J_-, O_0\right] \\
&= \frac{2\sqrt{2}}{\sqrt{6}\sqrt{2}}\left[J_-, r_0^2 + r_{+1}r_{-1}\right] \\
&= \frac{2}{\sqrt{6}}\left(2r_0\left[J_-, r_0\right] + \left[J_-, r_{+1}\right]r_{-1} + r_{+1}\left[J_-, r_{-1}\right]\right) \\
&= \frac{2}{\sqrt{6}}(2r_0\sqrt{2}r_{-1} + \sqrt{2}r_0 r_{-1} + 0) \\
&= \frac{2\sqrt{2}}{\sqrt{6}}(3r_0 r_{-1}) \\
&= 2\sqrt{3}r_0 r_{-1} \\
&= \sqrt{6}\cos(\theta)\sin(\theta)\mathrm{e}^{-\mathrm{i}\phi}.
\end{aligned}
$$

- **For O_{-2}:**

$$
\begin{aligned}
O_{-2} &= \frac{1}{\sqrt{1}}\left[J_-, O_{-1}\right] \\
&= \left[J_-, 2\sqrt{3}r_0 r_{-1}\right] \\
&= 2\sqrt{3}(\left[J_-, r_0\right]r_{-1} + r_0\left[J_-, r_{-1}\right]) \\
&= 2\sqrt{3}(\sqrt{2}r_{-1}r_{-1} + 0) \\
&= 2\sqrt{6}r_{-1}^2 \\
&= \sqrt{6}\sin^2(\theta)\mathrm{e}^{-2\mathrm{i}\phi}.
\end{aligned}
$$

To summarize, our tensor operator components are:

$$
\begin{cases}
O_{+2} = \frac{1}{2}(r_x + \mathrm{i}r_y)^2 = \frac{1}{2}\sin^2\theta\mathrm{e}^{2\mathrm{i}\phi} \\
O_{+1} = -\sqrt{2}(r_x + \mathrm{i}r_y)r_z = -\sqrt{2}\sin(\theta)\cos(\theta)\mathrm{e}^{\mathrm{i}\phi} \\
O_0 = \frac{1}{\sqrt{3}}(3\cos^2(\theta) - 1) \\
O_{-1} = \sqrt{6}\cos(\theta)\sin(\theta)\mathrm{e}^{-\mathrm{i}\phi} \\
O_{-2} = \sqrt{6}\sin^2(\theta)\mathrm{e}^{-2\mathrm{i}\phi}.
\end{cases}
$$

With appropriate normalization factors, these components are proportional to the spherical harmonics $Y_{2,m}(\theta,\phi)$. Specifically, they correspond to the solid spherical harmonics $r^2 Y_{2,m}(\theta,\phi)$.
This construction generalizes to arbitrary ℓ. Starting with $(r_{+1})^\ell$, we can generate all components of a tensor that transforms like the spherical harmonics $Y_{\ell,m}(\theta,\phi)$. The connection exists because both the tensor operators and spherical harmonics transform in the same way under rotations - they form irreducible representations of angular momentum ℓ. ■

Problem 2
Let $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{L}$ is the orbital angular momentum, and $\mathbf{S}$ is spin. Denote the standard basis states of the spin- J irrep of $\mathbf{J}$ as $|J, M\rangle$. Please assume $\hbar = 1$.

(a) Suppose that
$$\langle 1, 1|S_z|1, 1\rangle = A$$
is known. Compute
$$\langle 1, M|S_z|1, M\rangle$$
for the two remaining values of M.

(b) Compute
$$\left\langle 1, M\middle|S^-\middle|1, M+1\right\rangle$$
for all possible M, in terms of the same constant A as in (a).

(c) Same for
$$\left\langle 1, M\middle|S^+\middle|1, M-1\right\rangle$$

(d) Because J_z and $J^{\pm}$acting on $|1, M\rangle$ each produce another state of the same irrep (and we know what state that is), one can use the results of (b) and (c) to compute, in terms of A, the matrix element
$$\langle 1, M|\mathbf{S} \cdot \mathbf{J}|1, M\rangle\,.$$
Compute it (in terms of A) for all M, using the observation that
$$\mathbf{S} \cdot \mathbf{J} = S^+J^- + S^-J^+ + S_zJ_z.$$

Solution. Let's solve this problem carefully, working through each part step by step.

(a) We have to find $\langle 1, M|S_z|1, M\rangle$ for $M = 0, -1$. We know that $\langle 1, 1|S_z|1, 1\rangle = A$. To find the matrix elements for other values of M, we can use the fact that S_z is a component of a tensor operator. We know $\mathbf{J} = \mathbf{L} + \mathbf{S}$, so for any state $|1, M\rangle$, we have
$$\begin{cases} J_z\,|1, M\rangle = M\,|1, M\rangle \\ (L_z + S_z)\,|1, M\rangle = M\,|1, M\rangle \end{cases}$$
The states $|1, M\rangle$ are eigenstates of J_z with eigenvalue M, but they are not generally eigenstates of L_z or S_z individually.

Using the Wigner-Eckart theorem, we know that the matrix elements of S_z within a given irreducible representation of $\mathbf{J}$ are related by Clebsch-Gordan coefficients. Since S_z is the $q = 0$ component of a rank-1 tensor operator, we have
$$\langle 1, M|S_z|1, M\rangle = \langle 1||S||1\rangle\,\frac{\langle 1, M|1, 0; 1, M\rangle}{\sqrt{3}}.$$
Here, $\langle 1||S||1\rangle$ is the reduced matrix element, and the Clebsch-Gordan coefficient $\langle 1, M|1, 0; 1, M\rangle = \frac{M}{\sqrt{1(1+1)}}$ for the $q = 0$ component.

Therefore, we have
$$\langle 1, M|S_z|1, M\rangle = \langle 1||S||1\rangle \cdot \frac{M}{\sqrt{2}}$$
Since we know $\langle 1, 1|S_z|1, 1\rangle = A$, we can determine that
$$A = \langle 1||S||1\rangle \cdot \frac{1}{\sqrt{2}}$$
$$\langle 1||S||1\rangle = A\sqrt{2}$$

Now we can calculate the other matrix elements. We get

$$\langle 1,0|S_z|1,0\rangle = A\sqrt{2}\cdot\frac{0}{\sqrt{2}} = 0,$$
$$\langle 1,-1|S_z|1,-1\rangle = A\sqrt{2}\cdot\frac{-1}{\sqrt{2}} = -A.$$

(b) **Computing** $\langle 1,M|S^-|1,M+1\rangle$**:**
S^- is the lowering operator for spin, and it corresponds to the $q=-1$ component of the tensor operator $\mathbf{S}$.
Using the Wigner-Eckart theorem, we have

$$\left\langle 1,M\middle|S^-\middle|1,M+1\right\rangle = \langle 1||S||1\rangle\cdot\frac{\langle 1,M|1,-1;1,M+1\rangle}{\sqrt{3}}$$

The relevant Clebsch-Gordan coefficient is

$$\langle 1,M|1,-1;1,M+1\rangle = \sqrt{\frac{(1-M)(1+M+1)}{2}}$$

Substituting, we have

$$\begin{aligned}\left\langle 1,M\middle|S^-\middle|1,M+1\right\rangle &= A\sqrt{2}\cdot\frac{1}{\sqrt{3}}\cdot\sqrt{\frac{(1-M)(1+M+1)}{2}}\\ &= \frac{A\sqrt{(1-M)(2+M)}}{\sqrt{3}}\end{aligned}$$

- **For** $M=0$**:**
$$\left\langle 1,0\middle|S^-\middle|1,1\right\rangle = \frac{A\sqrt{(1-0)(2+0)}}{\sqrt{3}} = \frac{A\sqrt{2}}{\sqrt{3}}$$
- **For** $M=-1$**:**
$$\left\langle 1,-1\middle|S^-\middle|1,0\right\rangle = \frac{A\sqrt{(1-(-1))(2+(-1))}}{\sqrt{3}} = \frac{A\sqrt{2}}{\sqrt{3}}$$

(c) **Computing** $\langle 1,M|S^+|1,M-1\rangle$**:**
The calculation for S^+ follows similarly. S^+ corresponds to the $q=+1$ component of $\mathbf{S}$. We have

$$\left\langle 1,M\middle|S^+\middle|1,M-1\right\rangle = \langle 1||S||1\rangle\cdot\frac{\langle 1,M|1,1;1,M-1\rangle}{\sqrt{3}}.$$

The Clebsch-Gordan coefficient is

$$\langle 1,M|1,1;1,M-1\rangle = \sqrt{\frac{(1+M)(1-M+1)}{2}}.$$

Substituting, we have

$$\begin{aligned}\left\langle 1,M\middle|S^+\middle|1,M-1\right\rangle &= A\sqrt{2}\cdot\frac{1}{\sqrt{3}}\cdot\sqrt{\frac{(1+M)(1-M+1)}{2}}\\ &= \frac{A\sqrt{(1+M)(2-M)}}{\sqrt{3}}\end{aligned}$$

- **For** $M=0$**:**
$$\left\langle 1,0\middle|S^+\middle|1,-1\right\rangle = \frac{A\sqrt{(1+0)(2-0)}}{\sqrt{3}} = \frac{A\sqrt{2}}{\sqrt{3}}$$

- **For $M = 1$:**

$$\left\langle 1,1\middle|S^+\middle|1,0\right\rangle = \frac{A\sqrt{(1+1)(2-1)}}{\sqrt{3}} = \frac{A\sqrt{2}}{\sqrt{3}}$$

(d) **Computing $\langle 1, M|\mathbf{S}\cdot\mathbf{J}|1, M\rangle$:**

We need to compute $\langle 1, M|\mathbf{S}\cdot\mathbf{J}|1, M\rangle$ using

$$\mathbf{S}\cdot\mathbf{J} = S^+J^- + S^-J^+ + S_zJ_z.$$

Let's calculate each term. We have

(i) $\langle 1, M|S_zJ_z|1, M\rangle = M \cdot \langle 1, M|S_z|1, M\rangle$

(ii) For $\langle 1, M|S^+J^-|1, M\rangle$:
  - $J^-|1, M\rangle = \sqrt{(1+M)(1-M+1)}\,|1, M-1\rangle$.
  - Then we apply S^+ using our result from part (c).

(iii) For $\langle 1, M|S^-J^+|1, M\rangle$:
  - $J^+|1, M\rangle = \sqrt{(1-M)(1+M+1)}\,|1, M+1\rangle$.
  - Then we apply S^- using our result from part (b).

Let's compute these terms for each value of M:

- **For $M = 1$:**

$$\langle 1,1|S_zJ_z|1,1\rangle = 1\cdot A = A$$
$$\left\langle 1,1\middle|S^+J^-\middle|1,1\right\rangle = \left\langle 1,1\middle|S^+\middle|\sqrt{2}\,|1,0\rangle\right\rangle = \sqrt{2}\cdot\frac{A\sqrt{2}}{\sqrt{3}} = \frac{2A}{\sqrt{3}}$$
$$\left\langle 1,1\middle|S^-J^+\middle|1,1\right\rangle = 0 \text{ (since } J^+\,|1,1\rangle = 0).$$

Therefore, we have

$$\langle 1,1|\mathbf{S}\cdot\mathbf{J}|1,1\rangle = A + \frac{2A}{\sqrt{3}} + 0 = A\left(1+\frac{2}{\sqrt{3}}\right).$$

- **For $M = 0$:**

$$\langle 1,0|S_zJ_z|1,0\rangle = 0\cdot 0 = 0$$
$$\left\langle 1,0\middle|S^+J^-\middle|1,0\right\rangle = \left\langle 1,0\middle|S^+\middle|1\cdot|1,-1\rangle\right\rangle = 1\cdot\frac{A\sqrt{2}}{\sqrt{3}} = \frac{A\sqrt{2}}{\sqrt{3}}$$
$$\left\langle 1,0\middle|S^-J^+\middle|1,0\right\rangle = \left\langle 1,0\middle|S^-\middle|1\cdot|1,1\rangle\right\rangle = 1\cdot\frac{A\sqrt{2}}{\sqrt{3}} = \frac{A\sqrt{2}}{\sqrt{3}}.$$

Therefore, we have

$$\langle 1,0|\mathbf{S}\cdot\mathbf{J}|1,0\rangle = 0 + \frac{A\sqrt{2}}{\sqrt{3}} + \frac{A\sqrt{2}}{\sqrt{3}} = \frac{2A\sqrt{2}}{\sqrt{3}}.$$

- **For $M = -1$:**

$$\langle 1,-1|S_zJ_z|1,-1\rangle = (-1)\cdot(-A) = A$$
$$\left\langle 1,-1\middle|S^+J^-\middle|1,-1\right\rangle = 0 \text{ (since } J^-\,|1,-1\rangle = 0)$$
$$\left\langle 1,-1\middle|S^-J^+\middle|1,-1\right\rangle = \left\langle 1,-1\middle|S^-\middle|\sqrt{2}\,|1,0\rangle\right\rangle = \sqrt{2}\cdot\frac{A\sqrt{2}}{\sqrt{3}} = \frac{2A}{\sqrt{3}}.$$

Therefore, we have

$$\langle 1,-1|\mathbf{S}\cdot\mathbf{J}|1,-1\rangle = A + 0 + \frac{2A}{\sqrt{3}} = A\left(1+\frac{2}{\sqrt{3}}\right).$$

We can see that $\langle 1,1|\mathbf{S}\cdot\mathbf{J}|1,1\rangle = \langle 1,-1|\mathbf{S}\cdot\mathbf{J}|1,-1\rangle = A\left(1+\frac{2}{\sqrt{3}}\right)$ and $\langle 1,0|\mathbf{S}\cdot\mathbf{J}|1,0\rangle = \frac{2A\sqrt{2}}{\sqrt{3}}$.

$\blacksquare$