MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*
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Homework 7

Problem 7-2
Let G be a Lie group.

(a) Let $m : G \times G \to G$ denote the multiplication map. Using Proposition 3.14 to identify $T_{(e,e)}(G \times G)$ with $T_eG \oplus T_eG$, show that the differential $\mathrm{d}m_{(e,e)} : T_eG \oplus T_eG \to T_eG$ is given by

$$\mathrm{d}m_{(e,e)}(X, Y) = X + Y.$$

[Hint: compute $\mathrm{d}m_{(e,e)}(X, 0)$ and $\mathrm{d}m_{(e,e)}(0, Y)$ separately.]

(b) Let $i : G \to G$ denote the inversion map. Show that $\mathrm{d}i_e : T_eG \to T_eG$ is given by $\mathrm{d}i_e(X) = -X$. (Used on pp. 203, 522.)

Solution. (a) We have

$$\begin{aligned} \mathrm{d}m_{(e,e)}(X, Y) &= \mathrm{d}m_{(e,e)}(X, 0) + \mathrm{d}m_{(e,e)}(0, Y) \\ &= \mathrm{d}(m^1)_e(X) + \mathrm{d}(m^2)_e(Y), \end{aligned}$$

where

$$\begin{aligned} m^1 : G &\to G, \\ x &\mapsto m(x, e), \end{aligned} \qquad \text{and} \qquad \begin{aligned} m^2 : G &\to G, \\ y &\mapsto m(e, y). \end{aligned}$$

We have that $m^1 = m^2 = \mathrm{Id}_G$, thus, $\mathrm{d}m_{(e,e)}(X, Y) = X + Y$.

(b) We construct a constant map c by defining it to be the composition of three other maps m, n, and q, defined as

$$\begin{aligned} n : G &\to G \times G, \\ x &\mapsto (x, x), \end{aligned} \qquad \begin{aligned} q : G \times G &\to G \times G, \\ (x, y) &\mapsto (x, i(y)), \end{aligned}$$

and m defined as stated in the problem. Then $c = m \circ n \circ q$ is a constant map. Thus $\mathrm{d}c_e(X) = 0$. From that, we have

$$\begin{aligned} 0 &= \mathrm{d}c_e(X) \\ &= \mathrm{d}m_{(e,e)}\left(\mathrm{d}n_{(e,e)}\left(\mathrm{d}q_e(X)\right)\right) \\ &= \mathrm{d}m_{(e,e)}\left(\mathrm{d}n_{(e,e)}(X, X)\right) \\ &= \mathrm{d}m_{(e,e)}\left(X, \mathrm{d}i_e(X)\right) \\ &= X + \mathrm{d}i_e(X), \end{aligned}$$

where the last step follows from part (a). Therefore,

$$\mathrm{d}i_e(X) = -X.$$

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Problem 7-4
Let $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ denote the determinant function. Use Corollary 3.25 to compute the differential of det, as follows.

(a) For any $A \in \mathrm{M}(n,\mathbb{R})$, show that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det\left(I_n + tA\right) = \operatorname{tr}(A)$$

where $\operatorname{tr}\left(A^i_j\right) = \sum_i A^i_i$ is the trace of A. [Hint: the defining equation (B.3) expresses $\det\left(I_n + tA\right)$ as a polynomial in t. What is the linear term?]

(b) For $X \in \mathrm{GL}(n,\mathbb{R})$ and $B \in T_X\mathrm{GL}(n,\mathbb{R}) \cong \mathrm{M}(n,\mathbb{R})$, show that

$$\mathrm{d}(\det)_X(B) = (\det(X))\operatorname{tr}\left(X^{-1}B\right)$$

[Hint: $\det(X + tB) = \det(X)\det\left(I_n + tX^{-1}B\right)$.] (Used on p. 203)

Solution. (a) We aim to compute the derivative of $\det(\mathbb{I}_n + tA)$ with respect to t at $t = 0$. Using the hint, we can express $\det(\mathbb{I}_n + tA)$ as a polynomial in t as follows

$$\det(\mathbb{I}_n + tA) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (\mathbb{I}_n + tA)_{i,\sigma(i)}.$$

The terms in the product are

$$(\mathbb{I}_n + tA)_{i,\sigma(i)} = \delta_{i,\sigma(i)} + tA_{i,\sigma(i)},$$

where $\delta_{i,\sigma(i)}$ is the Kronecker delta. Plugging this back in the determinant expansion, we get

$$\det(\mathbb{I}_n + tA) \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(\delta_{i,\sigma(i)} + tA_{i,\sigma(i)}\right).$$

Expanding the product using the binomial theorem, we have

$$\prod_{i=1}^{n} \left(\delta_{i,\sigma(i)} + tA_{i,\sigma(i)}\right) = \sum_{k=0}^{n} t^k \sum_{1\le i_1\le\cdots\le i_k\le n} A_{i_1,\sigma(i_1)}\delta_{i_1,\sigma(i_1)} \cdots A_{i_k,\sigma(i_k)}\delta_{i_k,\sigma(i_k)}.$$

Replacing this in the determinant expression, we have

$$\det(\mathbb{I}_n + tA) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{k=0}^{n} t^k \sum_{1\le i_1\le\cdots\le i_k\le n} A_{i_1,\sigma(i_1)}\delta_{i_1,\sigma(i_1)} \cdots A_{i_k,\sigma(i_k)}\delta_{i_k,\sigma(i_k)}.$$

The linear term in t is given by

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} A_{i,\sigma(i)}\delta_{i,\sigma(i)},$$

which is just the trace of A, $\operatorname{tr}(A)$.
Thus,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det\left(\mathbb{I}_n + tA\right) = \operatorname{tr}(A).$$

(b) Let us view the expression $X + tB$ as the image, at time t, of a curve γ on $M(n,\mathbb{R})$, namely

$$\begin{aligned} \gamma : \mathbb{R} &\to M(n,\mathbb{R}), \\ t &\mapsto X + tB. \end{aligned}$$

The expression $\det(X+tB)$ can thus be seen as the composition of γ with the determinant function, given by

$$t \overset{\gamma}{\longmapsto} X+tB \overset{\det}{\longmapsto} \det(X+tB).$$

Computing the derivative of this composition with respect to t by an application of the chain rule yields

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(X+tB) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det\circ\gamma(t) \\
&= \mathrm{d}(\det)_{\gamma(0)}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\gamma(t)\right) \\
&= \mathrm{d}(\det)_{X+0B}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(X+tB)\right) \\
&= \mathrm{d}(\det)_X B.
\end{aligned}$$

On the other hand, using the hint and the result from part (a), we repeat the computation of this derivative as follows

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(X+tB) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(X)\,\det(\mathbb{I}_n + tX^{-1}B) && \text{(by the hint)} \\
&= \det(X)\,\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(\mathbb{I}_n + tX^{-1}B) && \text{(Leibniz product rule)} \\
&= (\det(X))\,\mathrm{tr}(X^{-1}B). && \text{(by part (a))}
\end{aligned}$$

Equating them, we obtain $\mathrm{d}(\det)_X(B) = (\det(X))\,\mathrm{tr}\left(X^{-1}B\right)$.

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Problem 8-1
Prove Lemma 8.6 (the extension lemma for vector fields).

Lemma 1. Extension Lemma for Vector Fields Let M be a smooth manifold with or without boundary, and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a smooth global vector field $\tilde{X}$ on M such that $\left.\tilde{X}\right|_A = X$ and $\mathrm{supp}(\tilde{X}) \subseteq U$.

Solution. We first construct a smooth function $\phi : U \to \mathbb{R}$ with $0 \le \phi(x) \le 1$ for all $x \in U$, such that $\phi(x) = 1$ for $x \in A$ and $\phi(x) = 0$ for x in a neighborhood of the boundary of U. This can be done using a smooth bump function. Notice that ϕ is smooth on U.
We will now extend the vector field X to a vector field $\tilde{X}$ on U.
For each $x \in \partial A$, take V_x to be a coordinate neighborhood containing x and let $e_1, \dots, e_n$ be the standard coordinate vector fields with respect to the coordinates on V_x. In that neighborhood, define a vector field $\tilde{X}_x$ given by

$$\tilde{X}_x(y) = \sum_{i=1}^n \phi(y) X_x^i(y) e_i,$$

where X_x^i are extensions of the coordinate components of X (restricted to V_x), satisfying the condition that $\tilde{X}$ agrees with X on $V_x \cap A$ by the extension lemma for smooth functions (Lemma 2.26). Replacing V_x by $V_x \cap U$, we may assume that $V_x \subseteq U$.
The family of sets

$$\big(A \setminus \bigcup_{x\in\partial A} V_x\big) \cup \{V_x \mid x \in \partial A\} \cup \{U \setminus A\}$$

is an open cover of U. Let $\{\theta : U \to \mathbb{R}\} \cup \{\varphi_x : U \to \mathbb{R} \mid x \in \partial A\} \cup \{\psi : U \to \mathbb{R}\}$ be a smooth partition of unity subordinate to this open cover, with $\operatorname{supp}(\theta) \subseteq A \setminus \bigcup_{x\in\partial A} V_x$, $\operatorname{supp}(\varphi_x) \subseteq V_x$, and $\operatorname{supp}(\psi) \subseteq U \setminus A$. Our construction is complete by defining the vector field $\tilde{X}$ on U by

$$\tilde{X}(y) := \sum_{x\in\partial A} \varphi_x(y)\tilde{X}_x(y) + \theta(y)X(y)$$

for all $y \in U$.
Therefore, by definition of the partition of unity, this agrees with X on A, and is smooth on all of U.

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Problem 8-12
Let $F : \mathbb{R}^2 \to \mathbb{RP}^2$ be the smooth map $F(x, y) = [x, y, 1]$, and let $X \in \mathfrak{X}\left(\mathbb{R}^2\right)$ be defined by $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Prove that there is a vector field $Y \in \mathfrak{X}\left(\mathbb{RP}^2\right)$ that is F-related to X, and compute its coordinate representation in terms of each of the charts defined in Example 1.5.

Solution. Recall that two vector fields X and Y are F-related if $F_*X = Y$, where F_* is the pushforward of F. In other words, we need to find a vector Y on $\mathbb{RP}^2$ such that $F_*X_p = Y_{F(p)}$, for all $p \in \mathbb{R}^2$.
To compute the pushforward F_*X_p, for some point p, we can use the chain rule. Let $f : \mathbb{RP}^2 \to \mathbb{R}$ be a smooth function. Then

$$\begin{aligned}(F_*X_p)(f) &= X_p(f \circ F)\\ &= \frac{\partial(f\circ F)}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial(f\circ F)}{\partial y}\frac{\partial y}{\partial x} - \frac{\partial(f\circ F)}{\partial x}\frac{\partial x}{\partial y} - \frac{\partial(f\circ F)}{\partial y}\frac{\partial y}{\partial y}.\end{aligned}$$

Computing the partial derivatives of $f \circ F$ with respect to x and y, we have

$$\begin{aligned}\frac{\partial(f\circ F)}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x},\\ \frac{\partial(f\circ F)}{\partial y} &= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}.\end{aligned}$$

Substituting these expressions back, we get

$$(F_*X_p)(f) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}.$$

We can now define the vector field Y on $\mathbb{RP}^2$ by $Y = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$. Then, for any smooth function $f : \mathbb{RP}^2 \to \mathbb{R}$, we have

$$Y_{F(p)}(f) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = (F_*X_p)(f),$$

which shows that Y is F-related to X.
The charts defined in Example 1.5 are

$$\begin{aligned}\varphi_1 : U_1 &\to \mathbb{R}^2\\ [1, a, b] &\mapsto (a, b).\end{aligned}$$

$$\begin{aligned}\varphi_2 : U_2 &\to \mathbb{R}^2\\ [c, 1, d] &\mapsto (c, d).\end{aligned}$$

$$\begin{aligned}\varphi_3 : U_3 &\to \mathbb{R}^2\\ [x, y, 1] &\mapsto (x, y).\end{aligned}$$

Notice that $\varphi_3^{-1} = F$. Computing the coordinate representation of Y in terms of the charts, we have

$$X = \begin{cases} (1+a^2)\dfrac{\partial}{\partial a} + ab\dfrac{\partial}{\partial b}, & \text{for } U_1 \cap U_3 \\ -(1+c^2)\dfrac{\partial}{\partial c} - cd\dfrac{\partial}{\partial d}, & \text{for } U_2 \cap U_3 \\ x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}, & \text{for } U_3. \end{cases}$$

Therefore, there is a vector field $Y \in \mathfrak{X}\left(\mathbb{RP}^2\right)$ that is F-related to X.

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Problem 8-13
Show that there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point. [Hint: try using stereographic projection; see Problem 1-7.]

Solution. Let (u, v) and (w, t) be the stereographic coordinates relative to the projection from the north pole and from the south pole, respectively. The maps are then

$$\varphi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2,$$
$$\varphi_S : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2.$$

The change of coordinates map and its inverse are

$$(u, v) = \varphi_N \circ \varphi_S^{-1}(w, t) = \frac{(w, t)}{w^2 + t^2},$$
$$(w, t) = \varphi_S \circ \varphi_N^{-1}(u, v) = \frac{(u, v)}{u^2 + v^2}.$$

Consider the vector field $\frac{\partial}{\partial u} = \partial_u$ in coordinates (we could also do ∂_v), defined on the domain of φ_N.
For some point p in the intersection of the two coordinate charts, $p \in \mathbb{S}^2 \setminus \{N, S\}$, we can compute ∂_u in the stereographic coordinates relative to the projection from the south pole, $(w, t) = \varphi_S \circ \varphi_N^{-1}(u, v)$, given by

$$\begin{aligned} \partial_u &= \left(t^2 - w^2\right)\frac{\partial}{\partial w} - 2wt\frac{\partial}{\partial t} \\ &= \left(t^2 - w^2\right)\partial_w - 2wt\partial_t, \qquad p \in \mathbb{S}^2 \setminus \{N, S\}. \end{aligned}$$

This vector field can be extended at the north pole. Thus,

$$X_p = \begin{cases} \left(\varphi_N^{-1}\right)_* (\partial_u), & p \in \mathbb{S}^2 \setminus \{N\} \\ \left(\varphi_S^{-1}\right)_* \left(t^2 - w^2\right)\partial_w - 2wt\partial_t, & p \in \mathbb{S}^2 \setminus \{S\} \end{cases}$$

X_p is a well-defined vector field on all of $\mathbb{S}^2$ and is smooth. By construction, we have that $X_N = 0$ and $X_p = \partial_u \neq 0$ on $\mathbb{S} \setminus \{N\}$.
Therefore, there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point. ■