PHYS 663 - Quantum Field Theory II

An Introduction to Quantum Fied Theory by *Peskin and Schroeder*
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Homework 6

Problem 1

(a) Compute the homotopy groups of $\mathbb{S}^2$.

(b) Compute the homotopy groups of a torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Solution. (a) The homotopy groups of $\mathbb{S}^2$ are

$$\begin{aligned}
\pi_0(\mathbb{S}^2) &= \{e\} \\
\pi_1(\mathbb{S}^2) &= 0 \\
\pi_2(\mathbb{S}^2) &= \mathbb{Z} \\
\pi_3(\mathbb{S}^2) &= \mathbb{Z} \\
\pi_4(\mathbb{S}^2) &= \mathbb{Z}_2 \\
\pi_5(\mathbb{S}^2) &= \mathbb{Z}_2 \\
\pi_6(\mathbb{S}^2) &= \mathbb{Z}_{12}
\end{aligned}$$

(1) For $\pi_0(\mathbb{S}^2)$: Since $\mathbb{S}^2$ is connected, $\pi_0(\mathbb{S}^2)$ consists of a single element.

(2) For $\pi_1(\mathbb{S}^2)$: Any loop on the 2-sphere can be continuously deformed to a point, so the fundamental group is trivial.

(3) For $\pi_2(\mathbb{S}^2)$: By the Hurewicz theorem, since $\pi_k(\mathbb{S}^2) = 0$ for all $k < 2$, we have $\pi_2(\mathbb{S}^2) \cong H_2(\mathbb{S}^2) \cong \mathbb{Z}$.

(4) For $\pi_3(\mathbb{S}^2)$: This can be computed using the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$. The associated long exact sequence gives:

$$\pi_3(\mathbb{S}^1) \to \pi_3(\mathbb{S}^3) \to \pi_3(\mathbb{S}^2) \to \pi_2(\mathbb{S}^1)$$

Since $\pi_3(\mathbb{S}^1) = 0$ and $\pi_2(\mathbb{S}^1) = 0$, while $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$, we conclude $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$.

(5) For $\pi_4(\mathbb{S}^2)$ and higher: The calculation becomes more complex and relies on advanced techniques. The results are $\pi_4(\mathbb{S}^2) \cong \mathbb{Z}_2$, $\pi_5(\mathbb{S}^2) \cong \mathbb{Z}_2$, and $\pi_6(\mathbb{S}^2) \cong \mathbb{Z}_{12}$.

(b) For the torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$, we can use the product formula for homotopy groups

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y) \quad \text{for } n \geq 1.$$

The homotopy groups of $\mathbb{S}^1$ are

$$\begin{aligned}
\pi_1(\mathbb{S}^1) &= \mathbb{Z} \\
\pi_n(\mathbb{S}^1) &= 0 \quad \text{for } n \geq 2
\end{aligned}$$

Therefore, the homotopy groups of the torus are

$$\begin{aligned}
\pi_0(T^2) &= \{e\} \quad \text{(since } T^2 \text{ is connected)} \\
\pi_1(T^2) &= \pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z} \\
\pi_n(T^2) &= \pi_n(\mathbb{S}^1) \times \pi_n(\mathbb{S}^1) = 0 \times 0 = 0 \quad \text{for } n \geq 2
\end{aligned}$$

The fundamental group $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ corresponds to the two independent ways to wind around the torus - one for each circle factor.

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Problem 2
Show that the homotopy group $\pi_r(M)$ is abelian for $r > 1$.

Solution. Let (M, m_0) be a pointed topological space, and consider the homotopy group

$$\pi_r(M, m_0) = [S^r, (M, m_0)].$$

which consists of homotopy classes of pointed maps from the r-sphere to M.
We shall employ the Eckmann-Hilton argument. For $r > 1$, we can define two binary operations on $\pi_r(M, m_0)$. To construct these operations, we represent each element of $\pi_r(M, m_0)$ as a map $f : I^r \to M$ from the unit cube $I^r = [0, 1]^r$ to M, where f maps the boundary ∂I^r to the basepoint m_0.
First, define the operation $+_1$ by concatenating maps along the first coordinate

$$(f +_1 g)(t_1, t_2, \dots, t_r) = \begin{cases} f(2t_1, t_2, \dots, t_r), & \text{if } 0 \le t_1 \le \frac{1}{2} \\ g(2t_1 - 1, t_2, \dots, t_r), & \text{if } \frac{1}{2} \le t_1 \le 1 \end{cases}$$

Similarly, define $+_2$ by concatenating along the second coordinate

$$(f +_2 g)(t_1, t_2, \dots, t_r) = \begin{cases} f(t_1, 2t_2, \dots, t_r), & \text{if } 0 \le t_2 \le \frac{1}{2} \\ g(t_1, 2t_2 - 1, \dots, t_r), & \text{if } \frac{1}{2} \le t_2 \le 1 \end{cases}$$

Both operations are well-defined on homotopy classes and give $\pi_r(M, m_0)$ a group structure. The constant map to m_0 serves as the identity element for both operations.
Now, the crucial observation is that these operations satisfy the interchange law

$$(f +_1 g) +_2 (h +_1 k) = (f +_2 h) +_1 (g +_2 k)$$

This can be verified directly by examining the definitions: both sides correspond to placing f, g, h, and k in the same positions in a 2×2 grid in the t_1-t_2 plane.
The Eckmann-Hilton argument then proceeds as follows. Let e denote the identity element for both operations. For any $a, b \in \pi_r(M, m_0)$, we have

$$\begin{aligned} a +_1 b &= (a +_1 b) +_2 (e +_1 e) \\ &= (a +_2 e) +_1 (b +_2 e) \\ &= a +_1 b \end{aligned}$$

Similarly

$$\begin{aligned} a +_2 b &= (e +_1 e) +_2 (a +_1 b) \\ &= (e +_2 a) +_1 (e +_2 b) \\ &= a +_1 b \end{aligned}$$

Thus, $+_1 = +_2$. Let us denote this operation as $+$.
Now, for any $a, b \in \pi_r(M, m_0)$

$$\begin{aligned} a + b &= (a +_1 e) +_2 (e +_1 b) \\ &= (a +_2 e) +_1 (e +_2 b) \\ &= (e +_2 a) +_1 (b +_2 e) \\ &= (e +_1 b) +_2 (a +_1 e) \\ &= b + a \end{aligned}$$

Therefore, the operation $+$ is commutative, which means $\pi_r(M, m_0)$ is an abelian group for $r > 1$. ■