PHYS 660 - Quantum Mechanics I

Modern Quantum Mechanics by *J. J. Sakurai*

Student: **Ralph Razzouk**

Homework 2

Problem 1

A spin $\frac{1}{2}$ system is in the state $|\uparrow\rangle_{\hat{n}}$, namely in a spin up eigenstate in an arbitrary direction defined by the unit vector $\hat{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$.

(a) If the component S_z is measured, find the possible results of the measurement and their probabilities.

(b) Evaluate the dispersion σ_x, given by

$$\sigma_x^2 = {}_{\hat{n}}\left\langle\uparrow\middle|\left(S_x - \bar{S}_x\right)^2\middle|\uparrow\right\rangle_{\hat{n}},$$

where $\bar{S}_x = {}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}$.

(c) Check your answers for the cases $\theta = 0, \pi$ and $\theta = \pi/2$, $\phi = 0$.

Proof. (a) Given an arbitrary direction defined by the unit vector $\hat{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, we can represent $|\uparrow\rangle_{\hat{n}}$ as

$$|\uparrow\rangle_{\hat{n}} = \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle.$$

Measuring S_x will give us the eigenvalues of S_x which are $\pm\frac{\hbar}{2}$. The probabilities are as follows

- **Probability of getting $+\frac{\hbar}{2}$:**

$$\begin{aligned}
|\langle S_x|\uparrow\rangle_{\hat{n}}|^2 &= \left|\left(\frac{1}{\sqrt{2}}\langle\uparrow| + \frac{1}{\sqrt{2}}\langle\downarrow|\right)\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle\right)\right|^2 \\
&= \frac{1}{2}\left|(\langle\uparrow| + \langle\downarrow|)\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle\right)\right|^2 \\
&= \frac{1}{2}\left|\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}\right|^2 \\
&= \frac{1}{2}\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}\right)\left(\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\right) \\
&= \frac{1}{2}\left[\cos^2\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(e^{i\varphi} + e^{-i\varphi}\right) + \sin^2\left(\frac{\theta}{2}\right)\right] \\
&= \frac{1}{2}\left[1 + \sin(\theta)\cos(\varphi)\right].
\end{aligned}$$

- **Probability of getting $-\frac{\hbar}{2}$:**
  Since this is a two-state system, then

$$|\langle S_x|\downarrow\rangle_{\hat{n}}|^2 = 1 - |\langle S_x|\uparrow\rangle_{\hat{n}}|^2 = \frac{1}{2}\left[1 - \sin(\theta)\cos(\varphi)\right].$$

(b) We have that

$$
\begin{aligned}
\sigma_x^2 &= {}_{\hat{n}}\Big\langle\uparrow\Big|\left(S_x-\bar{S}_x\right)^2\Big|\uparrow\Big\rangle_{\hat{n}}\\
&= {}_{\hat{n}}\langle\uparrow|S_x^2-2S_x\bar{S}_x+\bar{S}_x^2|\uparrow\rangle_{\hat{n}}\\
&= {}_{\hat{n}}\langle\uparrow|S_x^2|\uparrow\rangle_{\hat{n}} - 2\,{}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}\,{}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}} + {}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}^2\,{}_{\hat{n}}\langle\uparrow|\uparrow\rangle_{\hat{n}}\\
&= {}_{\hat{n}}\langle\uparrow|S_x^2|\uparrow\rangle_{\hat{n}} - 2\,{}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}^2 + {}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}^2\\
&= {}_{\hat{n}}\langle\uparrow|S_x^2|\uparrow\rangle_{\hat{n}} - {}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}^2\\
&= \left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\langle\downarrow|\right)S_x^2\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle\right)\\
&\quad - \left[\left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\langle\downarrow|\right)S_x\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle\right)\right]^2\\
&= \frac{\hbar^2}{4}\left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\langle\downarrow|\right)\mathbb{I}\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle\right)\\
&\quad - \frac{\hbar^2}{4}\left[\left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\langle\downarrow|\right)\sigma_x\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle\right)\right]^2\\
&= \frac{\hbar^2}{4}\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right)\\
&\quad - \frac{\hbar^2}{4}\left[\left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\langle\downarrow|\right)\left(\sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\uparrow\rangle + \cos\left(\frac{\theta}{2}\right)|\downarrow\rangle\right)\right]^2\\
&= \frac{\hbar^2}{4}\left(\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right)\right) - \frac{\hbar^2}{4}\left[\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(e^{i\varphi}+e^{-i\varphi}\right)\right]^2\\
&= \frac{\hbar^2}{4}\left[1-\left[\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)\left(e^{i\varphi}+e^{-i\varphi}\right)\right]^2\right]\\
&= \frac{\hbar^2}{4}\left[1-\left(\sin(\theta)\cos(\varphi)\right)^2\right]\\
&= \frac{\hbar^2}{4}\left[1-\sin^2(\theta)\cos^2(\varphi)\right].
\end{aligned}
$$

(c)
- **For $\theta = 0$ or $\theta = \pi$:** We measure $+\frac{\hbar}{2}$ with probability $\frac{1}{2}$ and $-\frac{\hbar}{2}$ with probability $\frac{1}{2}$ and the dispersion has a value of $\frac{\hbar^2}{4}$.
- **For $\theta = \frac{\pi}{2}$ and $\phi = 0$:** We measure $+\frac{\hbar}{2}$ with probability 1 and $-\frac{\hbar}{2}$ with probability 0 and the dispersion has a value of 0, which implies that there are no other possible values to measure.

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Problem 2
Continuing from Problem 1, a series of Stern-Gerlach experiments are done to measure different components of the spin in succession. The beams are directed along direction $\hat{x}$ and the experiments are done as follows:

- The first device accepts only $s_z = \hbar/2$ states, (i.e. those with $s_z = -\hbar/2$ are blocked) thus creating a polarized beam for the next devices.
- The second device accepts only states with $s_{\hat{n}} = \hbar/2$, where $\hat{n}$ is a unit vector perpendicular to $\hat{x}$.
- The third device accepts only $s_z = -\hbar/2$.

(a) What is the ratio of the intensities of the final $s_z = -\hbar/2$ beam and the initial $s_z = \hbar/2$ polarized beam?

(b) How should the orientation of the second device, namely the vector $\hat{n}$, should be chosen to maximize the ratio computed in (a).

Proof. (a) Denote the first device by s_{z+}, the second device by $s_{\hat{n}}$, and the third device by s_{z-}. The ratio of intensities of the final beam and the initial polarized beam is given by

$$\frac{I_f}{I_i} = \left|\langle s_{\hat{n}}|s_{z-}\rangle\right|^2 \left|\langle s_{z+}|s_{\hat{n}}\rangle\right|^2 .$$

In other words, the ratio of intensities will be given by the amplitude remaining after the second and third devices.

We have

$$\begin{aligned}
|s_{z+}\rangle &= |\uparrow\rangle , \\
|s_{z-}\rangle &= |\downarrow\rangle , \\
|s_{\hat{n}}\rangle &= \cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle ,
\end{aligned}$$

then, replacing in the ratio of intensities, we get

$$\begin{aligned}
\frac{I_f}{I_i} &= \left|\langle s_{\hat{n}}|s_{z-}\rangle\right|^2 \left|\langle s_{z+}|s_{\hat{n}}\rangle\right|^2 \\
&= \langle s_{\hat{n}}|s_{z-}\rangle \langle s_{z-}|s_{\hat{n}}\rangle \langle s_{z+}|s_{\hat{n}}\rangle \langle s_{\hat{n}}|s_{z+}\rangle \\
&= \left[\left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\langle\downarrow|\right)|\downarrow\rangle\right]\left[\langle\downarrow|\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle\right)\right] \\
&\quad \left[\langle\uparrow|\left(\cos\left(\frac{\theta}{2}\right)|\uparrow\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|\downarrow\rangle\right)\right]\left[\left(\cos\left(\frac{\theta}{2}\right)\langle\uparrow| + \sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\langle\downarrow|\right)|\uparrow\rangle\right] \\
&= \left[\sin\left(\frac{\theta}{2}\right)e^{-i\varphi}\right]\left[\sin\left(\frac{\theta}{2}\right)e^{i\varphi}\right]\left[\cos\left(\frac{\theta}{2}\right)\right]\left[\cos\left(\frac{\theta}{2}\right)\right] \\
&= \sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \\
&= \frac{\sin^2(\theta)}{4}.
\end{aligned}$$

(b) To maximize the ratio of intensities, the argument of the sine function should be $\frac{\pi}{2}$. The second device should be chosen to be at a 90° angle with respect to the first device, which also ends up being the same angle from the third device. In other words, the second device should be angles right in between the $|\uparrow\rangle$ and $|\downarrow\rangle$ states, in order to get the maximum intensity of $\frac{1}{4}$.

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Problem 3
Consider a particle in a state with a Gaussian wave-function

$$\langle x|\psi\rangle = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{ikx - \frac{1}{4\sigma^2}(x-x_0)^2}. \tag{1}$$

(a) Compute $\langle\psi|\hat{x}|\psi\rangle$, $\langle\psi|\hat{p}|\psi\rangle$, $\left\langle\psi\middle|(\Delta x)^2\middle|\psi\right\rangle$, and $\left\langle\psi\middle|(\Delta p)^2\middle|\psi\right\rangle$.

(b) Check that such state has minimal uncertainty, namely

$$\sqrt{\langle\psi|(\Delta x)^2|\psi\rangle}\sqrt{\langle\psi|(\Delta p)^2|\psi\rangle} = \frac{\hbar}{2}.$$

(c) Show that for this state

$$\langle x|\Delta x|\psi\rangle = i\lambda \langle x|\Delta p|\psi\rangle$$

where $\lambda \in \mathbb{R}$. How does this relate to the minimal uncertainty property? **Hint:** Recall the proof of the uncertainty principle based on defining an operator $\mathcal{O} = \Delta x + i\mu\Delta p$ and computing $\left\langle\psi\middle|\mathcal{O}^\dagger\mathcal{O}\middle|\psi\right\rangle \geq 0$.

Proof. (a) In the position basis, $\hat{x} = x$, and we have

$$\begin{aligned}
\langle\psi|\hat{x}|\psi\rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-ikx} e^{ikx} x e^{-\frac{(x-x_0)^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-x_0)^2}{2\sigma^2}} \,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \left(\sqrt{2\sigma^2}t + x_0\right) e^{-t^2}\sqrt{2\sigma^2} \,\mathrm{d}t \\
&= \frac{1}{\sqrt{\pi}} \left[\sqrt{2\sigma^2}\int_{-\infty}^{\infty} te^{-t^2} \,\mathrm{d}t + x_0 \int_{-\infty}^{\infty} e^{-t^2} \,\mathrm{d}t\right] \\
&= \frac{1}{\sqrt{\pi}} \left[0 + x_0\sqrt{\pi}\,\mathrm{d}t\right] \\
&= x_0,
\end{aligned}$$

where the final jump was due to two things: the first integral is odd over an even domain and so evaluates to zero, and the second integral is equal to $\sqrt{\pi}$. Our solution makes sense as the Gaussian wave-function is centered at the mean $\mu = x_0$.

In the momentum basis, $\hat{p} = -i\hbar\frac{\partial}{\partial x}$, and we have

$$
\begin{aligned}
\langle\psi|\hat{p}|\psi\rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \left(-i\hbar\frac{\partial}{\partial x}\right)\left(e^{ikx}e^{-\frac{(x-x_0)^2}{4\sigma^2}}\right) \mathrm{d}x \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \frac{\partial}{\partial x}\left(e^{ikx}e^{-\frac{(x-x_0)^2}{4\sigma^2}}\right) \mathrm{d}x \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \left(ik - \frac{2(x-x_0)}{4\sigma^2}\right) e^{ikx}e^{-\frac{(x-x_0)^2}{4\sigma^2}} \mathrm{d}x \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \left(ik - \frac{2(x-x_0)}{4\sigma^2}\right) e^{-\frac{(x-x_0)^2}{2\sigma^2}} \mathrm{d}x \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \left(ik - \frac{2t}{4\sigma^2}\right) e^{-\frac{t^2}{2\sigma^2}} \mathrm{d}t \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \left[ik\int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} \mathrm{d}t - \frac{1}{2\sigma^2}\int_{-\infty}^{\infty} te^{-\frac{t^2}{2\sigma^2}} \mathrm{d}t\right] \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \left[ik\sqrt{2\pi\sigma^2} - 0\right] \\
&= \hbar k.
\end{aligned}
$$

Computing the variance of position, we get

$$
\begin{aligned}
\langle\psi|(\Delta x)^2|\psi\rangle &= \langle\psi|(\hat{x} - \langle x\rangle)^2|\psi\rangle \\
&= \langle\psi|x^2|\psi\rangle - \langle x\rangle^2 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}} \mathrm{d}x - x_0^2 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (t+x_0)^2 e^{-\frac{t^2}{2\sigma^2}} \mathrm{d}t - x_0^2 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \left[-\frac{t}{\sigma^2}(t+x_0)^2 e^{-\frac{t^2}{2\sigma^2}}\Big|_{-\infty}^{\infty} - \frac{2t^2}{\sigma^4}(t+x_0)e^{-\frac{t^2}{2\sigma^2}}\Big|_{-\infty}^{\infty} + \frac{2t^3}{\sigma^6}e^{-\frac{t^2}{2\sigma^2}}\Big|_{-\infty}^{\infty}\right] - x_0^2 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\sqrt{2\pi\sigma^2}\left(x_0^2 + \sigma^2\right) - x_0^2 \\
&= \left(x_0^2 + \sigma^2\right) - x_0^2 \\
&= \sigma^2.
\end{aligned}
$$

Computing the variance of momentum, we get

$$\begin{aligned}
\langle\psi|(\Delta p)^2|\psi\rangle &= \langle\psi|(\hat{p}-\langle p\rangle)^2|\psi\rangle \\
&= \langle\psi|p^2|\psi\rangle - \langle p\rangle^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(e^{-ikx-\frac{(x-x_0)^2}{4\sigma^2}}\right)\frac{\partial^2}{\partial x^2}\left(e^{ikx-\frac{(x-x_0)^2}{4\sigma^2}}\right)\mathrm{d}x - (\hbar k)^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(e^{-ikx-\frac{(x-x_0)^2}{4\sigma^2}}\right)\left(ik-\frac{2(x-x_0)}{4\sigma^2}\right)^2 e^{ikx-\frac{(x-x_0)^2}{4\sigma^2}}\mathrm{d}x - (\hbar k)^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(ik-\frac{2(x-x_0)}{4\sigma^2}\right)^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}}\mathrm{d}x - (\hbar k)^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(-k^2-\frac{ik(x-x_0)}{\sigma^2}-\frac{(x-x_0)^2}{4\sigma^4}\right) e^{-\frac{(x-x_0)^2}{2\sigma^2}}\mathrm{d}x - (\hbar k)^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\left(-\frac{\sqrt{\pi}\left(4k^2\sigma^2+1\right)}{2\sqrt{2\sigma^2}}\right) - (\hbar k)^2 \\
&= \frac{\hbar^2\left(4k^2\sigma^2+1\right)}{4\sigma^2} - (\hbar k)^2 \\
&= \hbar^2k^2 + \frac{\hbar^2}{4\sigma^2} - (\hbar k)^2 \\
&= \frac{\hbar^2}{4\sigma^2}.
\end{aligned}$$

(b) The uncertainty is

$$\sqrt{\langle\psi|(\Delta x)^2|\psi\rangle}\sqrt{\langle\psi|(\Delta p)^2|\psi\rangle} = \sqrt{\sigma^2}\sqrt{\frac{\hbar^2}{4\sigma^2}} = \frac{\hbar}{2},$$

and it fulfils the minimal uncertainty, as needed. This was expected as the condition of a Gaussian wave-functions for position and momentum creates the minimum uncertainty state.

(c) We have

$$\begin{aligned}
\langle x|\Delta p|\psi\rangle &= \langle x|\hat{p}-\langle p\rangle|\psi\rangle \\
&= \langle x|\hat{p}|\psi\rangle - \langle x|\langle p\rangle|\psi\rangle \\
&= \hat{p}\,\langle x|\psi\rangle - \langle p\rangle\,\langle x|\psi\rangle \\
&= [\hat{p}-\langle p\rangle]\,\langle x|\psi\rangle \\
&= \left[-i\hbar\frac{\partial}{\partial x}-\langle p\rangle\right]\langle x|\psi\rangle \\
&= \left[-i\hbar\left(ik-\frac{2(x-x_0)}{4\sigma^2}\right)-\hbar k\right]\langle x|\psi\rangle \\
&= \left[\hbar k+\frac{i\hbar(x-x_0)}{2\sigma^2}-\hbar k\right]\langle x|\psi\rangle \\
&= \left[\frac{i\hbar(x-x_0)}{2\sigma^2}\right]\langle x|\psi\rangle \\
&= \left[\frac{i\hbar}{2\sigma^2}\right](\hat{x}-\langle x\rangle)\,\langle x|\psi\rangle \\
&= \left[\frac{i\hbar}{2\sigma^2}\right]\langle x|\hat{x}-\langle x\rangle|\psi\rangle \\
&= \left[\frac{i\hbar}{2\sigma^2}\right]\langle x|\Delta x|\psi\rangle\,.
\end{aligned}$$

Thus,

$$\langle x|\Delta x|\psi\rangle = i\lambda \langle x|\Delta p|\psi\rangle = -\frac{2i\sigma^2}{\hbar} \langle x|\Delta p|\psi\rangle \implies \lambda = -\frac{2\sigma^2}{\hbar}.$$

If we rearrange the terms, we get

$$\sqrt{\langle\psi|(\Delta x)^2|\psi\rangle}\sqrt{\langle\psi|(\Delta p)^2|\psi\rangle} = \frac{\hbar}{2} = -\frac{\sigma^2}{\lambda}.$$

∎

Problem 4
Continuing from Problem 3,

(a) Check the momentum wave function $\tilde{\psi}(p) = \langle p|\psi\rangle$ for the state $|\psi\rangle$.

(b) Using $\tilde{\psi}(p) = \langle p|\psi\rangle$, compute $\langle\psi|\hat{p}|\psi\rangle$, $\left\langle\psi\middle|(\Delta p)^2\middle|\psi\right\rangle$ and check that you obtain the same results as in Problem 3.

Proof. (a) We have that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}} = \langle p|x\rangle^* .$$

Checking the momentum wave function, we get

$$\begin{aligned}
\tilde{\psi}(p) &= \langle p|\psi\rangle \\
&= \int_{-\infty}^{\infty} \langle p|x\rangle \langle x|\psi\rangle \, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ipx}{\hbar}}\right)\left(\frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{ikx - \frac{1}{4\sigma^2}(x-x_0)^2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} \left(e^{-\frac{ipx}{\hbar}}\right)\left(e^{ikx - \frac{1}{4\sigma^2}(x-x_0)^2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} \left(e^{ix\left(k-\frac{p}{\hbar}\right) - \frac{1}{4\sigma^2}(x-x_0)^2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{ix_0\left(k-\frac{p}{\hbar}\right)} \int_{-\infty}^{\infty} \left(e^{i(x-x_0)\left(k-\frac{p}{\hbar}\right) - \frac{1}{4\sigma^2}(x-x_0)^2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{ix_0\left(k-\frac{p}{\hbar}\right)} 2\sqrt{\pi\sigma^2} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2} \\
&= \sqrt{\frac{2\sigma}{\hbar}} \frac{1}{(2\pi)^{\frac{1}{4}}} e^{ix_0\left(k-\frac{p}{\hbar}\right)} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2} \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} e^{ix_0\left(k-\frac{p}{\hbar}\right)} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\langle\psi|\hat{p}|\psi\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\psi|p\rangle\,\langle p|\hat{p}|p'\rangle\,\langle p'|\psi\rangle\,\mathrm{d}p\,\mathrm{d}p' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\,p\delta(p-p')\,\tilde{\psi}(p')\,\mathrm{d}p\,\mathrm{d}p' \\
&= \int_{-\infty}^{\infty} \tilde{\psi}^*(p)p\tilde{\psi}(p)\,\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(e^{-ix_0\left(k-\frac{p}{\hbar}\right)}e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) p \left(e^{ix_0\left(k-\frac{p}{\hbar}\right)}e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right)\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) p \left(e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right)\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} pe^{-2\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\,\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}\hbar^2 k}{\sqrt{2\sigma^2}} \\
&= \hbar k.
\end{aligned}$$

Now,

$$\begin{aligned}
\left\langle\psi\middle|\hat{p}^2\middle|\psi\right\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\psi|p\rangle\,\left\langle p\middle|\hat{p}^2\middle|p'\right\rangle\,\langle p'|\psi\rangle\,\mathrm{d}p\,\mathrm{d}p' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\,p^2\delta(p-p')\,\tilde{\psi}(p')\,\mathrm{d}p\,\mathrm{d}p' \\
&= \int_{-\infty}^{\infty} \tilde{\psi}^*(p)p^2\tilde{\psi}(p)\,\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(e^{-ix_0\left(k-\frac{p}{\hbar}\right)}e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) p^2 \left(e^{ix_0\left(k-\frac{p}{\hbar}\right)}e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right)\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) p^2 \left(e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right)\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} p^2e^{-2\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\,\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}\hbar^3\left(4k^2\sigma^2+1\right)}{2^{\frac{5}{2}}\sigma^3} \\
&= \frac{\hbar^2\left(4k^2\sigma^2+1\right)}{4\sigma^2} \\
&= \hbar^2k^2 + \frac{\hbar^2}{4\sigma^2}.
\end{aligned}$$

Checking, we have

$$\langle\psi|\Delta p|\psi\rangle = \left\langle\psi\middle|\hat{p}^2\middle|\psi\right\rangle - \langle\psi|\hat{p}|\psi\rangle^2 = \hbar^2k^2 + \frac{\hbar^2}{4\sigma^2} - \hbar^2k^2 = \frac{\hbar^2}{4\sigma^2}.$$

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Problem 5
Given the translation operator
$$U(a) = e^{-i\frac{\hat{p}a}{\hbar}}.$$

(a) Use the fundamental commutation relation
$$[\hat{p}, \hat{x}] = -i\hbar$$
to compute the commutator $[\hat{x}, U(a)]$.

(b) Given a state $|\psi\rangle$ such that $\langle\psi|\hat{x}|\psi\rangle = \bar{x}$, what is the mean value of $\hat{x}$ in the state $|\phi\rangle = U(a)|\psi\rangle$?

Proof. (a) We are given
$$[\hat{p}, \hat{x}] = -i\hbar,$$
which implies, by the properties of the commutator, that
$$[\hat{x}, \hat{p}] = i\hbar.$$

Notice that
$$\begin{aligned} U(a) &= e^{-i\frac{\hat{p}a}{\hbar}} \\ &= \sum_{k=0}^{\infty} \frac{\left(-i\frac{\hat{p}a}{\hbar}\right)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{\left(-i\frac{a}{\hbar}\right)^k}{k!}\hat{p}^k. \end{aligned}$$

If we can find a relation for the commutator between the position operator and the kth degree of the momentum operator, we can then solve what is needed. In other words, we need to find a general formula for $[\hat{x}, \hat{p}^k]$.

Consider the following argument

- **For** $k = 1$**:**
$$[\hat{x}, \hat{p}^1] = i\hbar,$$
which is given to us.
- **For** $k = 2$**:**
$$\begin{aligned} [\hat{x}, \hat{p}^2] &= [\hat{x}, \hat{p}\hat{p}] \\ &= \hat{p}[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}]\hat{p} \\ &= i\hbar\hat{p} + i\hbar\hat{p} \\ &= 2i\hbar\hat{p}. \end{aligned}$$
- **For** $k = 3$**:**
$$\begin{aligned} [\hat{x}, \hat{p}^3] &= [\hat{x}, (\hat{p})^2\hat{p}] \\ &= \hat{p}^2[\hat{x}, \hat{p}] + [\hat{x}, \hat{p}^2]\hat{p} \\ &= i\hbar\hat{p}^2 + 2i\hbar\hat{p}^2 \\ &= 3i\hbar\hat{p}^2. \end{aligned}$$
- **For** $k = n$**:** We will claim that it is represented by the following formula
$$[\hat{x}, \hat{p}^n] = i\hbar n\hat{p}^{n-1}.$$

We will prove it by mathematical induction. The case for $k=1$ is given to us. Assume this is true for all k up to n. Let us show it is indeed true for $k=n+1$ and that we get $[\hat{x},\hat{p}^{n+1}]=i\hbar(n+1)\hat{p}^n$. Consider

$$\begin{aligned}[\hat{x},\hat{p}^{n+1}] &= [\hat{x},\hat{p}^n\hat{p}] \\ &= \hat{p}^n[\hat{x},\hat{p}]+[\hat{x},\hat{p}^n]\hat{p} \\ &= i\hbar\hat{p}^n + i\hbar n\hat{p}^{n-1}\hat{p} \qquad (\text{since } [\hat{x},\hat{p}^n]=i\hbar n\hat{p}^{n-1}) \\ &= i\hbar(n+1)\hat{p}^n,\end{aligned}$$

and is, thus, proven.

Computing the needed commutator, we have

$$\begin{aligned}[\hat{x},U(a)] &= \left[\hat{x}, e^{-i\frac{\hat{p}a}{\hbar}}\right] \\ &= \left[\hat{x}, \sum_{k=0}^{\infty}\frac{\left(-i\frac{a}{\hbar}\right)^k}{k!}\hat{p}^k\right] \\ &= \sum_{k=0}^{\infty}\frac{\left(-i\frac{a}{\hbar}\right)^k}{k!}[\hat{x},\hat{p}^k] \\ &= \sum_{k=0}^{\infty}\frac{\left(-i\frac{a}{\hbar}\right)^k}{k!}\left[i\hbar k\hat{p}^{k-1}\right] \\ &= \left(-i\frac{a}{\hbar}\right)(i\hbar)\sum_{k=0}^{\infty}\frac{\left(-i\frac{\hat{p}a}{\hbar}\right)^{k-1}}{(k-1)!} \\ &= a\sum_{k=0}^{\infty}\frac{\left(-i\frac{\hat{p}a}{\hbar}\right)^{k-1}}{(k-1)!} \\ &= aU(a).\end{aligned}$$

(b) From part (a), we have

$$[\hat{x},U(a)] = \hat{x}U(a)-U(a)\hat{x} = aU(a) \qquad \Longrightarrow \qquad \hat{x}U(a) = aU(a)+U(a)\hat{x}.$$

Also note that $U(a)$ represents the translation operator, which is unitary.

Given the state $|\phi\rangle = U(a)|\psi\rangle$, then $\langle\phi| = U^\dagger(a)\langle\psi|$. Taking the inner product of $\hat{x}$ in the $|\phi\rangle$ basis, we get

$$\begin{aligned}\langle\phi|\hat{x}|\phi\rangle &= \left\langle\psi\middle|U^\dagger(a)\hat{x}U(a)\middle|\psi\right\rangle \\ &= \left\langle\psi\middle|U^\dagger(a)\left(aU(a)+U(a)\hat{x}\right)\middle|\psi\right\rangle \\ &= \left\langle\psi\middle|U^\dagger(a)aU(a)\middle|\psi\right\rangle + \left\langle\psi\middle|U^\dagger(a)U(a)\hat{x}\middle|\psi\right\rangle \\ &= a\left\langle\psi\middle|U^\dagger(a)U(a)\middle|\psi\right\rangle + \left\langle\psi\middle|U^\dagger(a)U(a)\hat{x}\middle|\psi\right\rangle \\ &= a\langle\psi|\psi\rangle + \langle\psi|\hat{x}|\psi\rangle \\ &= a+\bar{x}.\end{aligned}$$

Thus, the mean value of $\hat{x}$ in the state $|\phi\rangle$ is the same as it is in the state $|\psi\rangle$ but shifted by a units.

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