Homework 9 – Boson Week
Due Thursday, April 4th

Problem 1

Consider a monatomic nonrelativistic gas of bosons.

a) First, show the number density and energy density are described as follows:

$$n(\mu) = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{e^{\beta(\epsilon_p - \mu)} - 1} \tag{1}$$

$$\mathcal{E}(\mu) = E/V = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{\epsilon_p}{e^{\beta(\epsilon_p - \mu)} - 1} \tag{2}$$

with $\epsilon_p = p^2/2m$, where m is the boson mass.

b) Show in the limiting case $\mu \to -\infty$ you recover the classical result for a monatomic ideal gas:

$$\mathcal{E}_{\text{classical}} = \frac{3}{2} nkT \tag{3}$$

c) In the limit $\mu \to 0$, $n(\mu) = n_{\text{crit}}(T)$. Compute this explicitly. You might have to look up an integral to do this problem; feel free to either leave your result in terms of a Riemann zeta function, or compute the decimal coefficient explicitly.

Problem 2

For μ in-between these limits, there is an analytic expression in terms of special functions, but we might as well do the integral numerically at that point. You are free to do the integrals either way (analytically or numerically). The goal will be to eliminate μ from the equations and make a plot of $\mathcal{E}(n, T)$.

a) For $T > T_{\text{crit}}$, first fix the temperature ($T = \text{const.}$) and plot $\mathcal{E}/\mathcal{E}_{\text{classical}}$ as a function of n. If you're having trouble figuring out how to do this, first compute $\mathcal{E}(\mu)$ and $n(\mu)$ and you should be able to make a parametric plot that puts $n(\mu)$ on the x-axis and $\mathcal{E}(\mu)$ on the y-axis.

You should be able to show with this plot that $\mathcal{E}/\mathcal{E}_{\text{classical}}$ is well-approximated by a linear function of n. Describe this function, using the limits $n = 0$ and $n = n_{\text{crit}}$ from Problem 1; this gives us a decent analytical description of $\mathcal{E}(n, T)$ in this regime with no free parameters.

b) For $T < T_{\text{crit}}$, the result is no longer dependent on n, and only on the temperature; explain why this is, and use this to compute $\mathcal{E}(T)$ for all $T < T_{\text{crit}}$.

c) Make a plot of $\mathcal{E}(T)/\mathcal{E}_{\text{classical}}(T)$, as a function of all T, both above and below the critical temperature. Is your result continuous and differentiable everywhere? (If you want, you can fix $n = N_A/\text{cm}^3$).

d) Finally, since you have an analytical expression everywhere, make a plot of the heat capacity, compared with the classical result $C_V/C_{V,\text{classical}}$. Is this function continuous and differentiable everywhere?

Problem 3

Estimate the critical condensation temperature T_c of a gas of massless particles where the number density n has been fixed (If you don't like the concept that we could fix the number of particles when they're massless, then just imagine they are ultra-relativistic particles with E = pc). Do photons have a critical condensation temperature T_c?

Problem 4

We showed in class that below T_{crit}, the fraction of bosons in the ground state goes like

$$N_0/N = 1 - (T/T_{\text{crit}})^{3/2}. \tag{4}$$

How does this behavior depend on the number of dimensions? What happens when we consider massless bosons (or equivalently, ultra-relativistic bosons)?