PHYS 630 - Advanced Electricity and Magnetism

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Contents

Homework 1

Problem 1
Find the charge of the Earth (positive), in terms of elementary charges (the answer is just a number N of elementary charges), so that the gravity force on a proton at the surface of the Earth is counteracted by the Coulomb repulsion. Compare with Avogadro number N_A.

Solution. For the force of gravity on a proton at the surface of Earth to be counteracted by the Coulomb repulsion, the forces must have the same magnitude. Then

$$\begin{aligned}
F_g &= F_e \\
G\frac{m_E m_p}{r^2} &= k\frac{q_E q_p}{r^2} \\
G m_E m_p &= k q_E q_p \\
q_E &= \frac{G m_E m_p}{k q_p} \\
q_E &= \frac{\left(6.67 \times 10^{-11}\right)\left(5.97 \times 10^{24}\right)\left(1.67 \times 10^{-27}\right)}{\left(9 \times 10^{9}\right)\left(1.6 \times 10^{-19}\right)} \\
q_E &= \frac{\left(6.67 \times 10^{-11}\right)\left(5.97 \times 10^{24}\right)\left(1.67 \times 10^{-27}\right)}{\left(9 \times 10^{9}\right)\left(1.6 \times 10^{-19}\right)} \\
q_E &= 4.618 \times 10^{-4}\,\mathrm{C} \\
q_E &= 2.886 \times 10^{15}\,\mathrm{e}.
\end{aligned}$$

This number of elementary charges needed is much smaller than Avogadro's number, so it would take practically very little charged particles to counteract the gravitational force induced by Earth on a proton at surface level. ■

Homework 2

Problem 1
A spherical, static charge distribution has a charge density

$$\rho(r) = \begin{cases} \left(1 - \frac{r}{r_0}\right)\rho_0, & r \le r_0 \\ 0, & r \ge r_0. \end{cases}$$

(a) Find the electric potential. (Normalize the potential so that $\phi(r_0) = 0$.) By differentiating the electric potential, find the electric field.

(b) Find the total charge $q(r)$ (charge within r), and find the electric field using Gauss' theorem. Compare the results.

(c) Find the total electromagnetic energy $\varepsilon = \int \frac{E^2}{8\pi}\,\mathrm{d}V$. Also, verify $\varepsilon = \frac{1}{2}\int \rho\phi\,\mathrm{d}V$.

Hint: Do not forget about the surface term.

Solution. (a) Since our charge distribution is spherically symmetric and has no dependence on θ and φ, then our Laplacian operator ∇^2 can be written as

$$\nabla^2\Box = \frac{1}{r^2}\partial_r\left(r^2\partial_r\Box\right).$$

Using that, the electric potential is given by

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) = -4\pi\rho.$$

Now, we will solve the differential equation for both cases: when $r \le r_0$ and when $r \ge r_0$, and make sure both solutions are equivalent at $r = r_0$.

$\boxed{\textbf{For } \mathbf{r \le r_0}\textbf{:}}$

$$\begin{aligned}
\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) &= -4\pi\rho \\
\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) &= -4\pi\rho_0\left(1 - \frac{r}{r_0}\right) \\
\partial_r\left(r^2\partial_r\phi\right) &= -4\pi\rho_0\left(r^2 - \frac{r^3}{r_0}\right) \\
r^2\partial_r\phi &= -4\pi\rho_0\left(\frac{r^3}{3} - \frac{r^4}{4r_0}\right) + c_1 \\
\partial_r\phi &= -4\pi\rho_0\left(\frac{r}{3} - \frac{r^2}{4r_0}\right) + \frac{c_1}{r^2} \\
\phi(r) &= -4\pi\rho_0\left(\frac{r^2}{6} - \frac{r^3}{12r_0}\right) - \frac{c_1}{r} + c_2 \\
\phi(r) &= -\frac{2\pi\rho_0}{3}\left(r^2 - \frac{r^3}{2r_0}\right) - \frac{c_1}{r} + c_2,
\end{aligned}$$

where c_1 and c_2 are arbitrary constants. Since there is no point charge in the center, then $c_1 = 0$. Additionally, we want to have $\phi(r_0) = 0$, which means $c_2 = \frac{\pi\rho_0}{3}r_0^2$. Thus, we have

$$\phi_{r\le r_0}(r) = \frac{\pi\rho_0}{3}\left(\frac{r^3}{r_0} - 2r^2 + r_0^2\right).$$

$\boxed{\textbf{For } \mathbf{r \geq r_0}\textbf{:}}$

$$
\begin{aligned}
\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) &= -4\pi\rho \\
\partial_r\left(r^2\partial_r\phi\right) &= 0 \\
r^2\partial_r\phi &= c_1 \\
\partial_r\phi &= \frac{c_1}{r^2} \\
\phi(r) &= -\frac{c_1}{r} + c_2,
\end{aligned}
$$

where c_1 and c_2 are arbitrary constants. Since we are outside the charged sphere, c_1 will end up being the total charge density of the charged sphere, *i.e.*

$$
-c_1 = \int_{\mathcal{V}} \rho_0 \,\mathrm{d}V = 4\pi\rho_0 \int_0^{r_0} r^2 \,\mathrm{d}r = \frac{4\pi\rho_0}{3} r_0^3.
$$

Additionally, we set c_2 so that $\phi(r_0) = 0$, which gives us $c_2 = \frac{4\pi\rho_0}{3} r_0^2$. Thus, we have

$$
\phi_{r\geq r_0}(r) = \frac{4\pi\rho_0}{3}\left(r_0^2 + \frac{r_0^3}{r}\right).
$$

Thus,

$$
\phi(r) = \begin{cases} \frac{\pi\rho_0}{3}\left(\frac{r^3}{r_0} - 2r^2 + r_0^2\right), & \text{for } r \leq r_0 \\ \frac{4\pi\rho_0}{3}\left(r_0^2 + \frac{r_0^3}{r}\right), & \text{for } r \geq r_0 \end{cases}
$$

We now differentiate with respect to r using $E = -\nabla\phi$, then the electric field is

$$
E(r) = \begin{cases} \frac{4\pi\rho_0}{3}\left(r - \frac{3r^2}{4r_0}\right), & \text{for } r \leq r_0 \\ \frac{4\pi\rho_0}{3}\frac{r_0^3}{r^2}, & \text{for } r \geq r_0 \end{cases}
$$

(b) The total charge is given by

$$
q(r) = \int_{\mathcal{V}} \rho(r') \,\mathrm{d}V'.
$$

$\boxed{\textbf{For } \mathbf{r \leq r_0}\textbf{:}}$

$$
\begin{aligned}
q(r) &= \int_{\mathcal{V}} \rho(r') \,\mathrm{d}V' \\
&= 4\pi \int_0^r \rho(r') r'^2 \,\mathrm{d}r' \\
&= 4\pi\rho_0 \int_0^r \left(1 - \frac{r'}{r_0}\right) r'^2 \,\mathrm{d}r' \\
&= 4\pi\rho_0 \int_0^r \left(r'^2 - \frac{r'^3}{r_0}\right) \mathrm{d}r' \\
&= 4\pi\rho_0 \left[\frac{r'^3}{3} - \frac{r'^4}{4r_0}\right]_0^r \\
&= 4\pi\rho_0 \left(\frac{r^3}{3} - \frac{r^4}{4r_0}\right) \\
&= 4\pi\rho_0 r^3 \left(\frac{1}{3} - \frac{r}{4r_0}\right).
\end{aligned}
$$

$\boxed{\textbf{For r} \geq \mathbf{r_0}\textbf{:}}$

$$\begin{aligned}
q(r) \equiv Q &= \int_{\mathcal{V}} \rho(r')\,\mathrm{d}V' \\
&= 4\pi \int_0^{r_0} \rho(r') r'^2\,\mathrm{d}r' \\
&= 4\pi\rho_0 \int_0^{r_0} \left(1 - \frac{r'}{r_0}\right) r'^2\,\mathrm{d}r' \\
&= 4\pi\rho_0 \int_0^{r_0} \left(r'^2 - \frac{r'^3}{r_0}\right)\mathrm{d}r' \\
&= 4\pi\rho_0 \left[\frac{r'^3}{3} - \frac{r'^4}{4r_0}\right]_0^{r_0} \\
&= 4\pi\rho_0 \left(\frac{r_0^3}{3} - \frac{r_0^3}{4}\right) \\
&= \frac{\pi\rho_0}{3} r_0^3.
\end{aligned}$$

Thus,

$$q(r) = \begin{cases} 4\pi\rho_0 r^3 \left(\frac{1}{3} - \frac{r}{4r_0}\right), & \text{for } r \leq r_0 \\ \frac{\pi\rho_0}{3} r_0^3, & \text{for } r \geq r_0 \end{cases}$$

Gauss' theorem for electrostatics states

$$\oint \mathbf{E}\cdot\mathrm{d}\mathbf{s} = 4\pi q(r)$$

Finding the electric field using Gauss' theorem, we have

$\boxed{\textbf{For r} \leq \mathbf{r_0}\textbf{:}}$

$$\begin{aligned}
\oint \mathbf{E}\cdot\mathrm{d}\mathbf{s} &= 4\pi q(r) \\
E(4\pi r^2) &= 4\pi \left[4\pi\rho_0 r^3 \left(\frac{1}{3} - \frac{r}{4r_0}\right)\right] \\
E &= 4\pi\rho_0 \left(\frac{r}{3} - \frac{r^2}{4r_0}\right).
\end{aligned}$$

$\boxed{\textbf{For r} \geq \mathbf{r_0}\textbf{:}}$

$$\begin{aligned}
\oint \mathbf{E}\cdot\mathrm{d}\mathbf{s} &= 4\pi q(r) \\
E(4\pi r^2) &= 4\pi \left(\frac{\pi\rho_0}{3} r_0^3\right) \\
E &= \frac{\pi\rho_0}{3}\frac{r_0^3}{r^2}.
\end{aligned}$$

Thus,

$$E(r) = \begin{cases} \frac{4\pi\rho_0}{3}\left(r - \frac{3r^2}{4r_0}\right), & \text{for } r \leq r_0 \\ \frac{\pi\rho_0}{3}\frac{r_0^3}{r^2}, & \text{for } r \geq r_0 \end{cases}$$

which matches what we found in (a).

(c) The electrostatic energy is given by

$$\begin{aligned}
\varepsilon &= \int_{\mathcal{V}} \frac{E^2}{8\pi}\,\mathrm{d}V \\
&= \frac{1}{8\pi}\int_0^\infty\int_0^\pi\int_0^{2\pi} E^2 r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{2}\int_0^\infty E^2 r^2\,\mathrm{d}r \\
&= \frac{1}{2}\left[\int_0^{r_0}\left(\frac{4\pi\rho_0}{3}\left(r-\frac{3r^2}{4r_0}\right)\right)^2 r^2\,\mathrm{d}r + \int_{r_0}^\infty\left(\frac{4\pi\rho_0}{3}\frac{r_0^3}{r^2}\right)^2 r^2\,\mathrm{d}r\right] \\
&= \frac{1}{2}\left[\frac{16\pi^2\rho_0^2}{9}\int_0^{r_0}\left(r-\frac{3r^2}{4r_0}\right)^2 r^2\,\mathrm{d}r + \frac{16\pi^2\rho_0^2}{9}\int_{r_0}^\infty\left(\frac{r_0^3}{r^2}\right)^2 r^2\,\mathrm{d}r\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\int_0^{r_0}\left(r-\frac{3r^2}{4r_0}\right)^2 r^2\,\mathrm{d}r + \int_{r_0}^\infty\left(\frac{r_0^3}{r^2}\right)^2 r^2\,\mathrm{d}r\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\int_0^{r_0}\left(r^2-\frac{6r^3}{4r_0}+\frac{9r^4}{16r_0^2}\right) r^2\,\mathrm{d}r + \int_{r_0}^\infty\left(\frac{r_0^6}{r^4}\right) r^2\,\mathrm{d}r\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\int_0^{r_0}\left(r^4-\frac{6r^5}{4r_0}+\frac{9r^6}{16r_0^2}\right)\mathrm{d}r + \int_{r_0}^\infty\left(\frac{r_0^6}{r^2}\right)\mathrm{d}r\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\left[\frac{r^5}{5}-\frac{r^6}{4r_0}+\frac{9r^7}{112r_0^2}\right]_0^{r_0} + \left[-\frac{r_0^6}{r}\right]_{r_0}^\infty\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left[\left(\frac{r_0^5}{5}-\frac{r_0^6}{4r_0}+\frac{9r_0^7}{112r_0^2}\right) + \left(0+\frac{r_0^6}{r_0}\right)\right] \\
&= \frac{8\pi^2\rho_0^2}{9}\left(\frac{r_0^5}{5}-\frac{r_0^5}{4}+\frac{9r_0^5}{112}+r_0^5\right) \\
&= \frac{8\pi^2\rho_0^2}{9}\left(\frac{577r_0^5}{560}\right) \\
&= \frac{577\pi^2\rho_0^2}{630}r_0^5.
\end{aligned}$$

Additionally, we can write $E^2 = E\cdot E = E\cdot(-\nabla\phi)$. By replacing in the equation for the electrostatic energy, we get

$$\begin{aligned}
\varepsilon &= \int_{\mathcal{V}} \frac{E^2}{8\pi}\,\mathrm{d}V \\
&= -\frac{1}{8\pi}\int_{\mathcal{V}} \mathbf{E}\cdot\nabla\phi\,\mathrm{d}V \\
&= -\frac{1}{8\pi}\int_{\mathcal{V}} \nabla\cdot(\phi\mathbf{E})\,\mathrm{d}V + \frac{1}{8\pi}\int_{\mathcal{V}}\phi(\nabla\cdot\mathbf{E})\,\mathrm{d}V \\
&= \frac{1}{8\pi}\oint_{\mathcal{S}}\phi\mathbf{E}\cdot\mathrm{d}\mathbf{S} + \frac{1}{2}\int_{\mathcal{V}}\phi\rho\,\mathrm{d}V.
\end{aligned}$$

We aim to show that $\varepsilon = \frac{1}{2}\int_{\mathcal{V}}\phi\rho\,\mathrm{d}V$, *i.e.* that $\frac{1}{8\pi}\oint_{\mathcal{S}}\phi\mathbf{E}\cdot\mathrm{d}\mathbf{S} = 0$.

We have

$$\begin{aligned}
\frac{1}{2}\int_{\mathcal{V}} \phi\rho \,\mathrm{d}V &= \frac{4\pi}{2}\left[\int_0^{r_0} \phi\rho r^2 \,\mathrm{d}r + \int_{r_0}^{\infty} \phi\rho r^2 \,\mathrm{d}r\right] \\
&= 2\pi\left[\int_0^{r_0} \frac{\pi\rho_0}{3}\left(\frac{r^3}{r_0} - 2r^2 + r_0^2\right)\left(1 - \frac{r}{r_0}\right)\rho_0 r^2 \,\mathrm{d}r + 0\right] \\
&= \frac{2\pi^2\rho_0^2}{3}\int_0^{r_0}\left(\frac{r^3}{r_0} - 2r^2 + r_0^2\right)\left(1 - \frac{r}{r_0}\right) r^2 \,\mathrm{d}r \\
&= \frac{2\pi^2\rho_0^2}{3}\int_0^{r_0}\left(\frac{r^5}{r_0} - 2r^4 + r_0^2 r^2 - \frac{r^6}{r_0^2} + \frac{2r^5}{r_0} - r_0 r^3\right)\mathrm{d}r \\
&= \frac{2\pi^2\rho_0^2}{3}\int_0^{r_0}\left(\frac{3r^5}{r_0} - 2r^4 + r_0^2 r^2 - \frac{r^6}{r_0^2} - r_0 r^3\right)\mathrm{d}r \\
&= \frac{2\pi^2\rho_0^2}{3}\left[\left(\frac{3r^6}{6r_0} - \frac{2r^5}{5} + \frac{r_0^2 r^3}{3} - \frac{r^7}{7r_0^2} - \frac{r_0 r^4}{4}\right)\right]_0^{r_0} \\
&= \frac{2\pi^2\rho_0^2}{3}\left(\frac{3}{6} - \frac{2}{5} + \frac{1}{3} - \frac{1}{7} - \frac{1}{4}\right) r_0^5 \\
&= \frac{2\pi^2\rho_0^2}{3}\left(\frac{17}{420} r_0^5\right) \\
&= \frac{17\pi^2\rho_0^2}{630} r_0^5,
\end{aligned}$$

so $\epsilon \neq \frac{1}{2}\int_{\mathcal{V}} \phi\rho \,\mathrm{d}V$. This is because the potential at infinity is not zero. The surface term is

$$\frac{4\pi}{8\pi} Q\phi_\infty = \frac{\pi^2\rho_0^2}{18} r_0^5.$$

If we normalize the potential at infinity to zero or $\phi(r_0) = \frac{Q}{r_0}$, then

$$\phi(r) = \frac{\pi\rho_0}{3}\left(\frac{r^3}{r_0} - 2r^2 + 2r_0^2\right), \quad r \leq r_0,$$

and, from that, we have

$$\frac{1}{2}\int_{\mathcal{V}} \phi\rho \,\mathrm{d}V = \frac{26\pi^2\rho_0^2}{315} r_0^5 = \epsilon$$

which matches what we found earlier.

■

Problem 2
Consider a charge q_0 surrounded by a cloud with charge density

$$\rho_e = -\frac{q_0}{\pi a^3}\mathrm{e}^{-\frac{2r}{a}}$$

Find the total charge of the system, the total potential $\phi(r)$, and the total electric field $\mathbf{E}(r)$. Plot the total potential $\phi(r)$ and compare with Coulomb.
Hint:

$$\int x^2 \mathrm{e}^{-\frac{x}{a}}\,\mathrm{d}x = -a\mathrm{e}^{-\frac{x}{a}}\left(2x^2 + 2ax + a^2\right).$$

Solution. The total charge Q enclosed is given by

$$\begin{aligned}
Q &= q_0 + \int_{\mathcal{V}} \rho_e \,\mathrm{d}V \\
&= q_0 + \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho_e r^2 \sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= q_0 + 4\pi \int_0^{\infty}\left(-\frac{q_0}{\pi a^3}\mathrm{e}^{-\frac{2r}{a}}\right) r^2\,\mathrm{d}r \\
&= q_0 - \frac{q_0}{a^3}\int_0^{\infty}(2r)^2\mathrm{e}^{-\frac{2r}{a}}\,\mathrm{d}r \\
&= q_0 - \frac{q_0}{a^3}\left|-a\left(2x^2 + 2ax + a^2\right)\mathrm{e}^{-\frac{2r}{a}}\right|_0^{\infty} \\
&= q_0 - \frac{q_0}{a^2}(a^2) \\
&= q_0 - q_0 \\
&= 0.
\end{aligned}$$

The total electric potential $\phi(r)$ of the system is given by

$$\begin{aligned}
\phi(r) &= \frac{q_0}{r} + \int_V \frac{\rho_c(r')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' \\
&= \frac{q_0}{r} + \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\left(-\frac{q_0}{\pi a^3}\mathrm{e}^{-\frac{2r'}{a}}\right)}{\sqrt{r^2 + r'^2 - 2rr'\cos(\theta)}} r'^2 \sin(\theta)\,\mathrm{d}r'\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{q_0}{r} - 2\pi \int_0^{\pi}\int_0^{\infty} \frac{q_0}{\pi a^3}\frac{\mathrm{e}^{-\frac{2r'}{a}}}{\sqrt{r^2 + r'^2 - 2rr'\cos(\theta)}} r'^2 \sin(\theta)\,\mathrm{d}r'\,\mathrm{d}\theta \\
&= \frac{q_0}{r} - \frac{2q_0}{a^3}\int_0^{\pi}\int_0^{\infty} \frac{r'^2\sin(\theta)\mathrm{e}^{-\frac{2r'}{a}}}{\sqrt{r^2 + r'^2 - 2rr'\cos(\theta)}}\,\mathrm{d}r'\,\mathrm{d}\theta \\
&= \frac{q_0}{r} - \frac{2q_0}{a^3}\left(\frac{a^2}{2}\right) \\
&= \frac{q_0}{r} - \frac{q_0}{a} \\
&= q_0\left(\frac{1}{r} - \frac{1}{a}\right).
\end{aligned}$$

The total electric field $\mathbf{E}(r)$ is given by

$$\begin{aligned}
E(r) &= -\nabla\phi(r) \\
&= -q_0\nabla\left(\frac{1}{r} - \frac{1}{a}\right) \\
&= \frac{q_0}{r^2}.
\end{aligned}$$

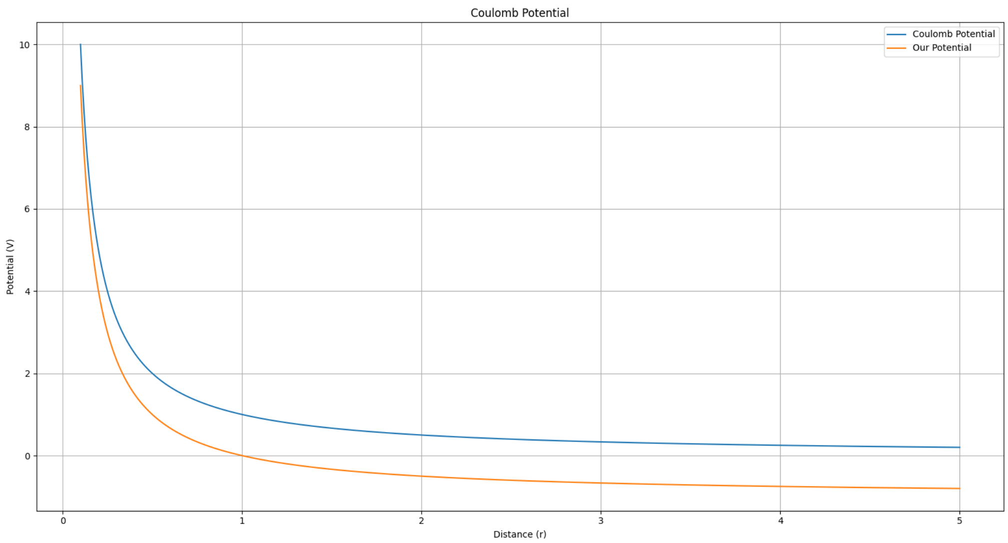

Figure 1: Plot of Our Potential vs. the Coulomb Potential

■

Homework 3

Problem 1
A sphere is charged with a surface charge $\sigma = \sigma_0 \cos(\theta)$. Find electric potential inside and outside of the sphere. Hint: since both inside and outside are vacuum, use an expansion in spherical harmonics plus a jump at the surface.

Solution. Let R be the radius of the charged sphere with surface charge $\sigma = \sigma_0 \cos(\theta)$. Given $\sigma(\theta)$, we could solve this using direct integration

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\theta)}{r} \, \mathrm{d}a,$$

but solving this using separation of variables is simpler.
The spherical harmonics expansion of the electric potential is given by

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos(\theta)).$$

For $r \leq R$

The term inversely proportional to $r^{\ell+1}$ diverges when $r = 0$, so we set $B_\ell = 0$. Then, we have

$$V_{\text{in}}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos(\theta)).$$

For $r \geq R$

The term proportional to r^ℓ diverges when $r \to \infty$, so we set $A_\ell = 0$. Then, we have

$$V_{\text{out}}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos(\theta)).$$

The piece-wise functions must be joined together using proper boundary conditions, which is that the functions V_{in} and V_{out} are continuous at the surface when $r = R$ and their radial derivatives are discontinuous at $r = R$.

- **Continuous**: We have that

$$\begin{gathered} V_{\text{in}}(R,\theta) = V_{\text{out}}(R,\theta) \\ \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos(\theta)) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos(\theta)). \end{gathered}$$

We will now invoke the orthogonality property for the Legendre functions to simplify the summation and equate the coefficients, specifically, we have

$$\int_1^{-1} P_\ell(x) P_{\ell'}(x) \, \mathrm{d}x = \int_0^{\pi} P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta)) \sin(\theta) \, \mathrm{d}\theta = \begin{cases} 0, & \text{if } \ell \neq \ell', \\ \frac{2}{2\ell+1}, & \text{if } \ell = \ell'. \end{cases}$$

Thus, by multiplying both sides by $P_{\ell'}(\cos(\theta))\sin(\theta)$, and then integrating over θ from 0 to π, we have

$$\begin{aligned}
\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos(\theta)) \times P_{\ell'}(\cos(\theta))\sin(\theta) &= \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos(\theta)) \times P_{\ell'}(\cos(\theta))\sin(\theta) \\
\int_0^\pi \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos(\theta)) \times P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta &= \int_0^\pi \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos(\theta)) \times P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
\sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta &= \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} \int_0^\pi P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
A_\ell R^\ell \left(\frac{2}{2\ell+1}\right) &= \frac{B_\ell}{R^{\ell+1}} \left(\frac{2}{2\ell+1}\right) \\
A_\ell R^\ell &= \frac{B_\ell}{R^{\ell+1}} \\
\implies B_\ell &= A_\ell R^{2\ell+1}.
\end{aligned}$$

We used the orthogonality property for the Legendre functions to get rid of the summation and equate the coefficients.

- **Not Differentiable**: We have that

$$\begin{aligned}
\left.\left(\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r}\right)\right|_{r=R} &= -\frac{\sigma(\theta)}{\epsilon_0} \\
\left.\left(-\sum_{\ell=0}^{\infty} (\ell+1)\frac{B_\ell}{r^{\ell+2}} P_\ell(\cos(\theta)) - \sum_{\ell=0}^{\infty} \ell A_\ell r^{\ell-1} P_\ell(\cos(\theta))\right)\right|_{r=R} &= -\frac{\sigma(\theta)}{\epsilon_0} \\
-\sum_{\ell=0}^{\infty} (\ell+1)\frac{B_\ell}{R^{\ell+2}} P_\ell(\cos(\theta)) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= -\frac{\sigma(\theta)}{\epsilon_0} \\
-\sum_{\ell=0}^{\infty} (\ell+1)\frac{A_\ell R^{2\ell+1}}{R^{\ell+2}} P_\ell(\cos(\theta)) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= -\frac{\sigma(\theta)}{\epsilon_0} \\
-\sum_{\ell=0}^{\infty} (\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) - \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= -\frac{\sigma(\theta)}{\epsilon_0} \\
\sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= \frac{\sigma(\theta)}{\epsilon_0}.
\end{aligned}$$

Similar to the case of continuity, we can determine the coefficients using the orthogonality of the Legendre polynomials. We have

$$\begin{aligned}
\sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) &= \frac{\sigma(\theta)}{\epsilon_0} \\
\sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta))\sin(\theta) &= \frac{\sigma(\theta)}{\epsilon_0} P_{\ell'}(\cos(\theta))\sin(\theta) \\
\int_0^\pi \sum_{\ell=0}^{\infty} (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos(\theta)) P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta &= \frac{1}{\epsilon_0}\int_0^\pi \sigma(\theta) P_{\ell'}(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
(2\ell+1) 2 A_\ell R^{\ell-1} \left(\frac{2}{2\ell+1}\right) &= \frac{1}{\epsilon_0}\int_0^\pi \sigma(\theta) P_\ell(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
A_\ell &= \frac{1}{2R^{\ell-1}\epsilon_0}\int_0^\pi \sigma(\theta) P_\ell(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta.
\end{aligned}$$

Now, considering the given conditions of our problem, we have $\sigma(\theta) = \sigma_0 \cos(\theta)$, for some constant σ_0, then notice that our surface charge density is proportional to $P_1(\cos(\theta))$, *i.e* $\sigma(\theta) = \sigma_0 P_1(\cos(\theta))$, which means that all A_ℓ's are null except for $\ell = 1$, which gives

$$\begin{aligned}
A_1 &= \frac{1}{2R^0\epsilon_0}\int_0^\pi \sigma(\theta)P_1(\cos(\theta))\sin(\theta)\,\mathrm{d}\theta \\
A_1 &= \frac{\sigma_0}{2\epsilon_0}\int_0^\pi \cos^2(\theta)\sin(\theta)\,\mathrm{d}\theta \\
A_1 &= \frac{\sigma_0}{2\epsilon_0}\left[-\frac{\cos^3(\theta)}{3}\right]\Bigg|_0^\pi \\
A_1 &= \frac{\sigma_0}{2\epsilon_0}\left(\frac{2}{3}\right) \\
A_1 &= \frac{\sigma_0}{3\epsilon_0}.
\end{aligned}$$

From this, we have that

$$B_1 = A_1 R^3 = \frac{\sigma_0 R^3}{3\epsilon_0}.$$

The potential inside the sphere is

$$V_{\text{in}}(r,\theta) = A_1 r P_1(\cos(\theta)) = \frac{\sigma_0}{3\epsilon_0} r\cos(\theta).$$

The potential outside the sphere is

$$V_{\text{out}}(r,\theta) = \frac{B_1}{r^2}P_1(\cos(\theta)) = \frac{\sigma_0 R^3}{3\epsilon_0}\frac{1}{r^2}\cos(\theta).$$

Thus, the total electric potential is

$$V(r,\theta) = \begin{cases} \frac{\sigma_0}{3\epsilon_0} r\cos(\theta), & \text{if } r \leq R, \\ \frac{\sigma_0 R^3}{3\epsilon_0}\frac{1}{r^2}\cos(\theta), & \text{if } r \geq R. \end{cases}$$

$\blacksquare$

Problem 2
A straight current wire has current distribution $j_z = j_0 (1 - \varpi/\varpi_0)$ where $\varpi \le \varpi_0$ is a cylindrical coordinate. Find the total current, the vector potential, and the magnetic field.

Solution. We have a current distribution of

$$j_x = j_y = 0, \quad j_z = j_0 \left(1 - \frac{\varpi}{\varpi_0}\right), \quad \varpi \le \varpi_0.$$

The total current I is given by

$$\begin{aligned} I &= \int_{\mathcal{A}} \mathbf{j} \cdot \mathrm{d}\mathbf{A} \\ &= \int_0^{\varpi_0} j_0 \left(1 - \frac{\varpi}{\varpi_0}\right) 2\pi\varpi \, \mathrm{d}\varpi \\ &= 2\pi j_0 \int_0^{\varpi_0} \left(\varpi - \frac{\varpi^2}{\varpi_0}\right) \mathrm{d}\varpi \\ &= 2\pi j_0 \left(\frac{\varpi^2}{2} - \frac{\varpi^3}{3\varpi_0}\right)\Bigg|_0^{\varpi_0} \\ &= 2\pi j_0 \left(\frac{\varpi_0^2}{2} - \frac{\varpi_0^3}{3\varpi_0}\right) \\ &= \frac{\pi j_0 \varpi_0^2}{3}. \end{aligned}$$

The vector potential $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{c} \int_{\mathcal{V}} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, \mathrm{d}V'.$$

Equivalently, we can solve the differential equation

$$\nabla^2 \mathbf{A} = \frac{1}{\rho} \partial_\rho \left(\rho \partial_\rho \mathbf{A}\right) = -\mu_0 \mathbf{j}(\rho) = -\frac{4\pi}{c} \mathbf{j}(\rho)$$

to get the vector potential, noting that there is no ϕ or z dependence since the vector potential does not rely on them. Additionally, since $j_x = j_y = 0$, then $A_x = A_y = 0$. We have

$$\begin{aligned} \nabla^2 A_z &= -\frac{4\pi}{c} j_z \\ \frac{1}{\varpi} \partial_\varpi \left(\varpi \partial_\varpi A_z\right) &= -\frac{4\pi j_0}{c} \left(1 - \frac{\varpi}{\varpi_0}\right) \\ \varpi \partial_\varpi A_z &= -\frac{4\pi j_0}{c} \left(\frac{\varpi^2}{2} - \frac{\varpi^3}{3\varpi_0}\right) + c_1 \\ \partial_\varpi A_z &= -\frac{4\pi j_0}{c} \left(\frac{\varpi}{2} - \frac{\varpi^2}{3\varpi_0}\right) + \frac{c_1}{\varpi} \\ A_z &= -\frac{4\pi j_0}{c} \left(\frac{\varpi^2}{4} - \frac{\varpi^3}{9\varpi_0}\right) + c_1 \ln(\varpi) + c_2. \end{aligned}$$

Thus

$$\mathbf{A} = (0, 0, A_z) = \left[-\frac{4\pi j_0}{c} \left(\frac{\varpi^2}{4} - \frac{\varpi^3}{9\varpi_0}\right) + c_1 \ln(\varpi) + c_2\right] \mathbf{z}.$$

The magnetic field $\mathbf{B}$ can be found by

$$\begin{aligned}
\mathbf{B} = \nabla \times \mathbf{A} &= \frac{1}{c}\int_{\mathcal{V}} \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\,\mathrm{d}V' \\
&= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)\hat{\mathbf{z}} \\
&= \frac{1}{\rho}\frac{\partial A_z}{\partial \varphi}\hat{\boldsymbol{\rho}} - \frac{\partial A_z}{\partial \rho}\hat{\boldsymbol{\varphi}} \\
&= -\frac{\partial A_z}{\partial \rho}\hat{\boldsymbol{\varphi}} \\
&\equiv -\frac{\partial A_z}{\partial \varpi}\hat{\boldsymbol{\varphi}} \\
&= \left[\frac{4\pi j_0}{c}\left(\frac{\varpi}{8} - \frac{\varpi^2}{27\varpi_0}\right) - \frac{c_1}{\varpi}\right]\hat{\boldsymbol{\varphi}}.
\end{aligned}$$

■

Problem 3
Current density is given by

$$j_\phi = C_1 r^3 \mathrm{e}^{-\frac{2r}{3a}} \sin(\theta)\cos^2(\theta)$$

where C_1 is some constant and we are in spherical coordinates $r - \theta - \phi$. Find the magnetic moment.

Solution. The magnetic moment is given by

$$\boldsymbol{\mu} = \frac{1}{2}\int_{\mathcal{V}} \mathbf{r}\times\mathbf{j}\,\mathrm{d}V.$$

Using (r,θ,ϕ) coordinates, we have

$$\begin{aligned}\mathbf{r} &= (r\sin(\theta)\cos(\phi), r\sin(\theta)\sin(\phi), r\cos(\theta)),\\ \mathbf{j} &= (j_r, j_\theta, j_\phi),\end{aligned}$$

where

$$\begin{aligned}j_r = j_\theta &= 0,\\ j_\phi &= C_1 r^3 \mathrm{e}^{-\frac{2r}{3a}} \sin(\theta)\cos^2(\theta).\end{aligned}$$

The cross product term is

$$\begin{aligned}\mathbf{r}\times\mathbf{j} &= \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z}\\ r\sin(\theta)\cos(\phi) & r\sin(\theta)\sin(\phi) & r\cos(\theta)\\ -j_\phi\sin(\phi) & j_\phi\cos(\phi) & 0\end{vmatrix}\\ &= -j_\phi r\cos(\theta)\cos(\phi)\mathbf{x} - j_\phi r\cos(\theta)\sin(\phi)\mathbf{y} + \left[j_\phi r\sin(\theta)\cos^2(\phi) + j_\phi\sin(\theta)\sin^2(\theta)\right]\mathbf{z}\\ &= -j_\phi r\cos(\theta)\cos(\phi)\mathbf{x} - j_\phi r\cos(\theta)\sin(\phi)\mathbf{y} + j_\phi r\sin(\theta)\mathbf{z}\\ &= j_\phi r\left[-\cos(\theta)\cos(\phi)\mathbf{x} - \cos(\theta)\sin(\phi)\mathbf{y} + \sin(\theta)\mathbf{z}\right].\end{aligned}$$

We will integrate coordinate-wise to solve for the magnetic moment $\boldsymbol{\mu}$.

- **Along x**:

$$\begin{aligned}\mu_x &= \frac{1}{2}\int_{\mathcal{V}} (\mathbf{r}\times\mathbf{j})_x\,\mathrm{d}V\\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(-j_\phi r\cos(\theta)\cos(\phi)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(-\left(C_1 r^3\mathrm{e}^{-\frac{2r}{3a}}\sin(\theta)\cos^2(\theta)\right) r\cos(\theta)\cos(\phi)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= -\frac{C_1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left(r^6\mathrm{e}^{-\frac{2r}{3a}}\sin^2(\theta)\cos^3(\theta)\cos(\phi)\right)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= -\frac{C_1}{2}\left(\int_0^{\infty} r^6\mathrm{e}^{-\frac{2r}{3a}}\,\mathrm{d}r\right)\left(\int_0^{\pi}\sin^2(\theta)\cos^3(\theta)\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\cos(\phi)\,\mathrm{d}\phi\right)\\ &= 0,\end{aligned}$$

  since $\int_0^{2\pi}\cos(\phi)\,\mathrm{d}\phi = 0$.

- **Along y**:

$$\begin{aligned}
\mu_y &= \frac{1}{2}\int_{\mathcal{V}} (\mathbf{r}\times\mathbf{j})_y \,\mathrm{d}V \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(-j_\phi r\cos(\theta)\sin(\phi)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(-\left(C_1 r^3 \mathrm{e}^{-\frac{2r}{3a}}\sin(\theta)\cos^2(\theta)\right) r\cos(\theta)\sin(\phi)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= -\frac{C_1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(r^6 \mathrm{e}^{-\frac{2r}{3a}}\sin^2(\theta)\cos^3(\theta)\sin(\phi)\right)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= -\frac{C_1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} r^6 \mathrm{e}^{-\frac{2r}{3a}}\sin^3(\theta)\cos^2(\theta)\cos(\phi)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= -\frac{C_1}{2}\left(\int_0^{\infty} r^6 \mathrm{e}^{-\frac{2r}{3a}}\,\mathrm{d}r\right)\left(\int_0^{\pi}\sin^2(\theta)\cos^3(\theta)\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\sin(\phi)\,\mathrm{d}\phi\right) \\
&= 0,
\end{aligned}$$

since $\int_0^{2\pi}\sin(\phi)\,\mathrm{d}\phi = 0$.

- **Along z**:

$$\begin{aligned}
\mu_z &= \frac{1}{2}\int_{\mathcal{V}} (\mathbf{r}\times\mathbf{j})_z \,\mathrm{d}V \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(j_\phi r\sin(\theta)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(\left(C_1 r^3 \mathrm{e}^{-\frac{2r}{3a}}\sin(\theta)\cos^2(\theta)\right) r\sin(\theta)\right) r^2\sin(\theta)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{C_1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \left(r^6 \mathrm{e}^{-\frac{2r}{3a}}\sin^3(\theta)\cos^2(\theta)\right)\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{C_1}{2}\left(\int_0^{\infty} r^6 \mathrm{e}^{-\frac{2r}{3a}}\,\mathrm{d}r\right)\left(\int_0^{\pi}\sin^3(\theta)\cos^2(\theta)\,\mathrm{d}\theta\right)\left(\int_0^{2\pi}\mathrm{d}\phi\right) \\
&= \frac{C_1}{2}\left(\frac{98415a^7}{8}\right)\left(\frac{4}{15}\right)(2\pi) \\
&= \frac{6561\pi a^7}{2}C_1.
\end{aligned}$$

Thus, we have

$$\boldsymbol{\mu} = \left(0, 0, \frac{6561\pi a^7}{2}C_1\right) = \frac{6561\pi a^7}{2}C_1\mathbf{z}.$$

■

Homework 4

Problem 1
A metal sphere of radius r has dipolar magnetic field, shown in the figure below. Imagine a cylindrical surface of radius $R > r$ (R is measured from the center of the sphere) aligned with the dipole. Some magnetic field lines close within the cylinder, some intersect the cylinder. Find the polar angle θ that the last field line that closes inside the cylinder makes on the surface of the sphere.

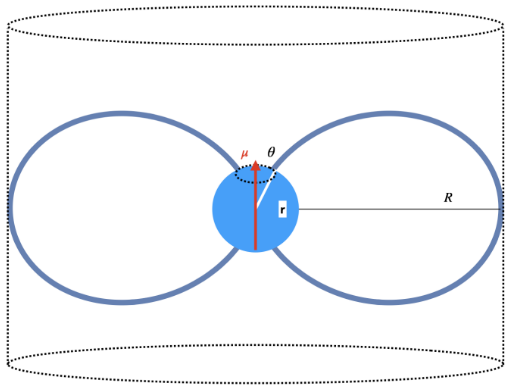

Figure 2: Dipole within a cylinder

Solution. We have that

$$\frac{1}{r}\frac{\mathrm{d}r}{B_r} = \frac{\mathrm{d}\theta}{B_\theta},$$

where $\mathbf{B} = (B_r, B_\theta, B_\varphi) = (2\cos(\theta), \sin(\theta), 0)$. We will now solve for r. We have

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{B_r}{B_\theta} r \\
\frac{\mathrm{d}r}{r} &= \frac{B_r}{B_\theta}\,\mathrm{d}\theta \\
\frac{\mathrm{d}r}{r} &= \frac{2\cos(\theta)}{\sin(\theta)}\,\mathrm{d}\theta \\
\ln(r) &= 2\ln(\sin(\theta)) + c \\
r &= \mathrm{e}^c \sin^2(\theta) \\
r &= c\sin^2(\theta).
\end{aligned}$$

To determine c, we have that when $\theta = \frac{\pi}{2}$, $r\left(\frac{\pi}{2}\right) = R$, thus $c = R$, so that

$$r = R\sin^2(\theta).$$

Thus, the polar angle θ on the sphere at which the last magnetic field line closes inside the cylinder, denoted θ_ℓ, is

$$\theta_\ell = \sin^{-1}\left(\sqrt{\frac{r}{R}}\right),$$

where r here is the fixed radius of the sphere.

■

Problem 2
A metal sphere of radius R carries a surface current $g_\varphi = g_0 \sin(\theta)$. Find magnetic field inside and outside. Hint: look for $A_\varphi \propto f(r)\sin(\theta)$, find solutions of $\nabla^2 \mathbf{A} = 0$ inside and outside, then use the fact that the radial component of the magnetic field should be continuous, while the tangential component experiences a jump, given by

$$B_\theta^{(+)} - B_\theta^{(-)} = \frac{4\pi}{c} g.$$

Solution. Let us consider $A_\varphi = f(r)\sin(\theta)$. In spherical coordinates, the Laplacian is given by

$$\nabla^2 \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \mathbf{A}}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \mathbf{A}}{\partial \theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 \mathbf{A}}{\partial \varphi^2}.$$

In our problem, we have a that the radial and polar components of $\mathbf{A}$ are zero, so we consider the Laplacian of the azimuthal component. Then

$$\begin{aligned}
\left(\nabla^2 \mathbf{A}\right)_\varphi &= \nabla^2 A_\varphi - \frac{1}{r^2\sin^2(\theta)}A_\varphi \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_\varphi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial A_\varphi}{\partial \theta}\right) + \cancel{\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 A_\varphi}{\partial \varphi^2}} - \frac{1}{r^2\sin^2(\theta)}A_\varphi \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_\varphi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial A_\varphi}{\partial \theta}\right) - \frac{1}{r^2\sin^2(\theta)}A_\varphi \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f(r)}{\partial r}\sin(\theta)\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \sin(\theta)}{\partial \theta}f(r)\right) - \frac{f(r)}{r^2\sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\frac{\partial}{\partial r}\left(r^2 f'(r)\right) + \frac{f(r)}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\cos(\theta)\right) - \frac{f(r)}{r^2\sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\left[2rf'(r) + r^2 f''(r)\right] + \frac{f(r)}{2r^2\sin(\theta)}\frac{\partial}{\partial \theta}\sin(2\theta) - \frac{f(r)}{r^2\sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\left[2rf'(r) + r^2 f''(r)\right] + \frac{f(r)}{r^2\sin(\theta)}\cos(2\theta) - \frac{f(r)}{r^2\sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\left[2rf'(r) + r^2 f''(r)\right] + \frac{f(r)}{r^2\sin(\theta)}(1 - 2\sin^2(\theta)) - \frac{f(r)}{r^2\sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\left[2rf'(r) + r^2 f''(r)\right] + \frac{f(r)}{r^2\sin(\theta)} - \frac{2f(r)\sin(\theta)}{r^2} - \frac{f(r)}{r^2\sin(\theta)} \\
&= \sin(\theta)\left[f''(r) + \frac{2}{r}f'(r) - \frac{2}{r^2}f(r)\right] \\
&= 0 \\
&\implies f''(r) + \frac{2}{r}f'(r) - \frac{2}{r^2}f(r) = 0
\end{aligned}$$

Assume a solution form of $f(r) = cr^n$, then $f'(r) = cnr^{n-1}$ and $f''(r) = cn(n-1)r^{n-2}$. Replacing in the differential equation above, we have

$$\begin{aligned}
cn(n-1)r^{n-2} + \frac{2}{r}cnr^{n-1} - \frac{2}{r^2}cr^n &= 0 \\
cn(n-1)r^{n-2} + 2cnr^{n-2} - 2cr^{n-2} &= 0 \\
\left(n(n-1) + 2n - 2\right)r^{n-2} &= 0 \\
\left(n^2 + n - 2\right)r^{n-2} &= 0 \\
(n-1)(n+2)r^{n-2} &= 0,
\end{aligned}$$

and thus $n = 1$ or $n = -2$. We consider a superposition solution of both powers, and we get

$$f(r) = \frac{c_1}{r^2} + c_2 r.$$

Thus,

$$\begin{aligned}
A_\varphi &= \left(\frac{c_1}{r^2} + c_2 r\right)\sin(\theta), \\
\mathbf{A} &= \left(\frac{c_1}{r^2} + c_2 r\right)\sin(\theta)\hat{\boldsymbol{\varphi}}.
\end{aligned}$$

Because of this, we can divide $\mathbf{A}$ into two components:

- **Internal**: $\mathbf{A}_{\text{in}} = c_2 r \sin(\theta)\hat{\boldsymbol{\varphi}}$.

- **External**: $\mathbf{A}_{\text{ext}} = \frac{c_1}{r^2}\sin(\theta)\hat{\boldsymbol{\varphi}}$.

To find the magnetic field $\mathbf{B}$, we take the curl of $\mathbf{A}$, getting

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
&= \begin{vmatrix} \frac{1}{r^2\sin(\theta)}\hat{\mathbf{r}} & \frac{1}{r\sin(\theta)}\hat{\boldsymbol{\theta}} & \frac{1}{r}\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin(\theta)A_\varphi \end{vmatrix} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(r\sin(\theta)A_\varphi) - \frac{\partial}{\partial\varphi}(rA_\theta)\right]\hat{\mathbf{r}} \\
&\quad - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)A_\varphi) - \frac{\partial}{\partial\varphi}(A_r)\right]\hat{\boldsymbol{\theta}} \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}(A_r)\right]\hat{\boldsymbol{\varphi}} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(r\sin(\theta)A_\varphi)\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)A_\varphi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)\right]\hat{\boldsymbol{\varphi}}.
\end{aligned}$$

We know that $A_r = A_\theta = 0$. Additionally, we divide the magnetic field just like we did with the magnetic potential. We have

$$\begin{aligned}
\mathbf{B}_{\text{in}} &= \nabla \times \mathbf{A}_{\text{in}} \\
&= \begin{vmatrix} \frac{1}{r^2\sin(\theta)}\hat{\mathbf{r}} & \frac{1}{r\sin(\theta)}\hat{\boldsymbol{\theta}} & \frac{1}{r}\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & r\sin(\theta)A_\varphi \end{vmatrix} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(r\sin(\theta)A_\varphi)\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)A_\varphi)\right]\hat{\boldsymbol{\theta}} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(c_2r^2\sin^2(\theta))\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(c_2r^2\sin^2(\theta))\right]\hat{\boldsymbol{\theta}} \\
&= \frac{c_2}{\sin(\theta)}\left[2\sin(\theta)\cos(\theta)\right]\hat{\mathbf{r}} - \frac{c_2\sin(\theta)}{r}\left[2r\right]\hat{\boldsymbol{\theta}} \\
&= 2c_2\cos(\theta)\hat{\mathbf{r}} - 2c_2\sin(\theta)\hat{\boldsymbol{\theta}}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_{\text{ext}} &= \nabla \times \mathbf{A}_{\text{ext}} \\
&= \begin{vmatrix} \frac{1}{r^2\sin(\theta)}\hat{\mathbf{r}} & \frac{1}{r\sin(\theta)}\hat{\boldsymbol{\theta}} & \frac{1}{r}\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & r\sin(\theta)A_\varphi \end{vmatrix} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(r\sin(\theta)A_\varphi)\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)A_\varphi)\right]\hat{\boldsymbol{\theta}} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}\left(\frac{c_1}{r}\sin^2(\theta)\right)\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}\left(\frac{c_1}{r}\sin^2(\theta)\right)\right]\hat{\boldsymbol{\theta}} \\
&= \frac{c_1}{r^3\sin(\theta)}\left[2\sin(\theta)\cos(\theta)\right]\hat{\mathbf{r}} - \frac{c_1\sin(\theta)}{r}\left[-\frac{1}{r^2}\right]\hat{\boldsymbol{\theta}} \\
&= \frac{2c_1\cos(\theta)}{r^3}\hat{\mathbf{r}} + \frac{c_1\sin(\theta)}{r^3}\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Thus,

$$\begin{cases} \mathbf{B}_{\text{in}} &= 2c_2\cos(\theta)\hat{\mathbf{r}} - 2c_2\sin(\theta)\hat{\boldsymbol{\theta}}, \\ \mathbf{B}_{\text{ext}} &= \frac{2c_1\cos(\theta)}{r^3}\hat{\mathbf{r}} + \frac{c_1\sin(\theta)}{r^3}\hat{\boldsymbol{\theta}}. \end{cases}$$

We will now use the boundary conditions.

- **Radial Components**: The radial components are continuous at the surface of the sphere ($r = R$). Specifically

$$\begin{aligned}
B_r^{(+)}\Big|_{r=R} &= B_r^{(-)}\Big|_{r=R} \\
\frac{2c_1\cos(\theta)}{R^3} &= 2c_2\cos(\theta) \\
c_2 &= \frac{c_1}{R^3}
\end{aligned}$$

- **Polar Components**: The polar components are discontinuous at the surface of the sphere ($r = R$). Specifically

$$\begin{aligned}
B_\theta^{(+)}\Big|_{r=R} - B_\theta^{(-)}\Big|_{r=R} &= \frac{4\pi}{c}g \\
\frac{c_1\sin(\theta)}{R^3} + 2c_2\sin(\theta) &= \frac{4\pi}{c}g_0\sin(\theta) \\
\frac{c_1\sin(\theta)}{R^3} + \frac{2c_1\sin(\theta)}{R^3} &= \frac{4\pi}{c}g_0\sin(\theta) \\
\frac{3c_1\sin(\theta)}{R^3} &= \frac{4\pi}{c}g_0\sin(\theta) \\
c_1 &= \frac{4\pi R^3}{3c}g_0 \\
\implies c_2 &= \frac{4\pi}{3c}g_0.
\end{aligned}$$

Therefore, the internal and external magnetic fields are

$$\begin{cases} \mathbf{B}_{\text{in}} &= \frac{8\pi g_0}{3c}\cos(\theta)\hat{\mathbf{r}} - \frac{8\pi g_0}{3c}\sin(\theta)\hat{\boldsymbol{\theta}}, \\ \mathbf{B}_{\text{ext}} &= \frac{8\pi g_0}{3c}\frac{R^3}{r^3}\cos(\theta)\hat{\mathbf{r}} + \frac{4\pi g_0}{3c}\frac{R^3}{r^3}\sin(\theta)\hat{\boldsymbol{\theta}}. \end{cases}$$

■

Homework 5

Problem 1
At time $t = 0$, an electron has a velocity v_0 directed along the y-axis. There is an electric field E_0 along the z-axis. Find $y(t)$ and $z(t)$ (fully relativistic).

Solution. When our system is fully relativistic, we have that the momentum is given by $p = \gamma m v$, where γ is the Lorentz factor given by

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{1}{\sqrt{1 - \beta^2}}.$$

The Lorentz force is given by

$$\mathbf{F} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right),$$

where we can also right, using Newton's second law, that

$$\dot{\mathbf{p}} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F},$$

assuming that the mass is constant.
At $t = 0$, the magnetic field has no effect, so $\dot{\mathbf{p}} = e\mathbf{E}$. Additionally, there is no force along the y-axis. From these, we have

$$\begin{cases} \dot{p}_x = F_x = 0 \\ \dot{p}_y = F_y = 0 \\ \dot{p}_z = F_z = eE_0 \end{cases} \implies \begin{cases} p_x = p_{0x} = 0 \\ p_y = p_{0y} = p_0 \\ p_z = eE_0 t \end{cases}$$

The kinetic energy of the particle is

$$\begin{aligned} \varepsilon &= \sqrt{m^2c^4 + (\mathbf{p}c)^2} \\ &= \sqrt{m^2c^4 + p_x^2c^2 + p_y^2c^2 + p_z^2c^2} \\ &= \sqrt{m^2c^4 + p_0^2c^2 + (eE_0t)^2c^2} \\ &= \sqrt{\varepsilon_0^2 + (eE_0tc)^2}. \end{aligned}$$

Since this is a fully relativistic system, we know that $\varepsilon = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$, then by taking the ratio between them, we have

$$\frac{\mathbf{p}}{\varepsilon} = \frac{\mathbf{v}}{c^2} \implies \mathbf{v} = \frac{\mathbf{p}c^2}{\varepsilon}.$$

Solving for the components of $\mathbf{v}$ component-wise, we have

$$\begin{cases} v_y = \dot{y}(t) = \dfrac{p_y c^2}{\varepsilon} = \dfrac{p_0 c^2}{\sqrt{\varepsilon_0^2 + (eE_0tc)^2}}, \\ v_z = \dot{z}(t) = \dfrac{p_z c^2}{\varepsilon} = \dfrac{eE_0tc^2}{\sqrt{\varepsilon_0^2 + (eE_0tc)^2}}, \end{cases} \implies \begin{cases} y(t) = p_0c^2 \displaystyle\int \frac{1}{\sqrt{\varepsilon_0^2 + (eE_0tc)^2}}\,\mathrm{d}t, \\ z(t) = eE_0c^2 \displaystyle\int \frac{t}{\sqrt{\varepsilon_0^2 + (eE_0tc)^2}}\,\mathrm{d}t, \end{cases}$$

$$\implies \begin{cases} y(t) = \dfrac{p_0 c}{eE_0}\sinh^{-1}\left(\dfrac{eE_0tc}{\varepsilon_0}\right), \\ z(t) = \dfrac{1}{eE_0}\sqrt{\varepsilon_0^2 + (eE_0tc)^2}. \end{cases}$$

■

Problem 2
Earth's magnetic field on the surface is approximately 1 Gauss. Earth magnetosphere extends approximately to 10 Earth radii, where it interacts with the solar wind, as shown in the figure below. The solar wind has a velocity $v_w \approx 500\,\mathrm{km/s}$. Find the Larmor radii of protons from the Solar wind as they penetrate in the outer parts of the Earth magnetosphere.

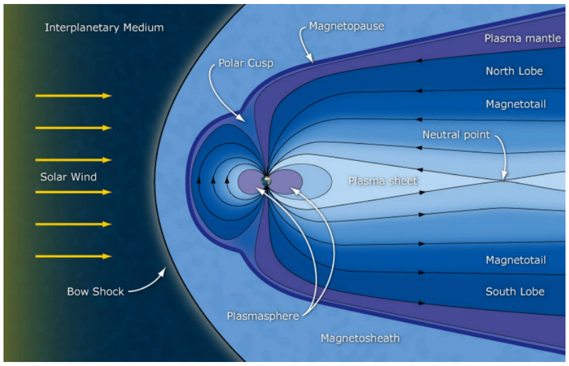

Figure 3: Magnetosphere of Earth

Solution. The Larmor radius is given by

$$r_L = \frac{\gamma v_\perp}{\omega_B},$$

where $v_\perp = v_0 \cos(\omega_c t)$, $\omega_B = \frac{eB}{mc}$, and $\omega_c = \frac{\omega_B}{\gamma}$ is the cyclotron frequency.
Since $v_w \ll c$, then $\gamma \approx 1$. Additionally, $\cos(\omega_c t) \approx 1$ by the small-angle formula since $\omega_c \approx \omega_B$ is very small. The Larmor radius is then

$$\begin{aligned} r_L &= \frac{\gamma\left(v_0 \cos(\omega_c t)\right)}{\left(\frac{eB}{mc}\right)} \\ &\approx \frac{mcv_0}{eB}\cos(\omega_c t) \quad (\text{take } c\cos(\omega_c t) \approx 1) \\ &= \frac{mv_w}{eB}. \end{aligned}$$

We are given that the magnetic field of Earth at the surface is $B_0 = 1\,\mathrm{Gs}$. The magnetic field of Earth at a distance of $10R_\oplus$ will be given by

$$\begin{aligned} \mathbf{B}(r) &= B_0\left(\frac{R^3}{r^3} - 1\right)\cos(\theta) \\ &\approx B_0\left(\frac{R_\oplus}{r}\right)^3 \\ &= B_0\left(\frac{R_\oplus}{10R_\oplus}\right)^3 \\ &= B_0\left(\frac{1}{10}\right)^3 \\ &= \frac{1}{1000B_0} \\ &= 10^{-3}\,\mathrm{Gs} \\ &= 10^{-7}\,\mathrm{T}. \end{aligned}$$

Thus, by replacing our values into the equation for the Larmor radius, we have that

$$\begin{aligned} r_L &= \frac{mv_w}{eB} \\ &= \frac{(1.67\times10^{-27})(5\times10^5)}{(1.602\times10^{-19})(10^{-7})} \\ &= 52.122\,\text{km}. \end{aligned}$$

■

Problem 3
Some laboratory device produces horizontal magnetic field of 10^3 G. A proton happens to be inside this device, as shown in the figure below. Find the drift velocity of a proton induced by the gravity of Earth.

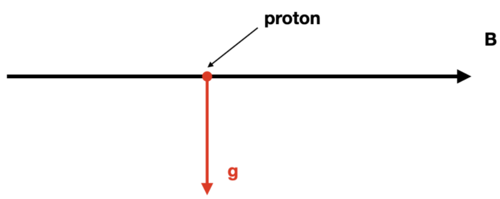

Figure 4: Free-body Diagram of a Proton

Solution. Assuming a slow drift, *i.e.* that the magnetic field $\mathbf{B}$ does not change fast enough on the Larmor scale, then the drift velocity can be approximated by

$$\begin{aligned} \frac{v_d}{c} &= \frac{\mathbf{F}\times\mathbf{B}}{qB^2} \\ &= \frac{m\mathbf{g}\times\mathbf{B}}{qB^2} \\ &= \frac{mgB_x}{eB_x^2} \\ &= \frac{(1.67\times10^{-27})(9.81)}{(1.602\times10^{-19})(10^{-1})} \\ &= 1.02\times10^{-6}\,\text{m/s}, \end{aligned}$$

where v_d is in the $\hat{\mathbf{z}}$ direction (out of the page). ■

Homework 6

Problem 1
A particle moves in the dipolar Earth magnetosphere. It is reflected at a polar angle $\theta_r = \frac{\pi}{4}$. Find its pitch angle α at the equator, as shown in the figure below.
Hint: The particle moves along a given dipolar field line parameterized by $r = R\sin^2(\theta)$. Using the expression for the strength of the dipolar magnetic field at a given radius r and angle θ, use $r \to R\sin^2(\theta)$. This will give the magnetic field along a given field line. Use conservation of the adiabatic invariant to find $\alpha(\theta)$ (reflection point corresponds to $\alpha = \frac{\pi}{2}$).

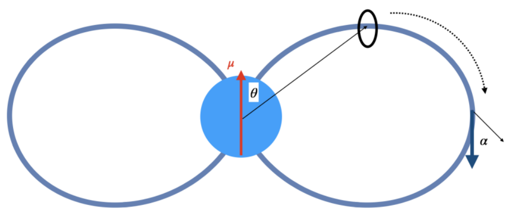

Figure 5: Bouncing particle in the dipolar magnetosphere. At the reflection point (a ring), the pitch angle is $\alpha = \frac{\pi}{2}$.

Solution. The magnetic field of a dipole is given by

$$\mathbf{B} = \{2\cos(\theta), \sin(\theta), 0\} \left(\frac{R}{r}\right)^3 B_0,$$

where $r = R\sin^2(\theta)$, giving us

$$\mathbf{B} = \{2\cos(\theta), \sin(\theta), 0\} \frac{B_0}{\sin^6(\theta)}.$$

Finding the magnitude of $\mathbf{B}$, we have

$$\begin{aligned}
|\mathbf{B}| &= \sqrt{\mathbf{B}_r^2 + \mathbf{B}_\theta^2 + \mathbf{B}_\varphi^2} \\
&= \sqrt{\left(\frac{2\cos(\theta)B_0}{\sin^6(\theta)}\right)^2 + \left(\frac{\sin(\theta)B_0}{\sin^6(\theta)}\right)^2 + 0} \\
&= \frac{B_0}{\sin^6(\theta)} \sqrt{4\cos^2(\theta) + \sin^2(\theta)}.
\end{aligned}$$

From the conservation of adiabatic invariant, we know that

$$\frac{\sin^2(\alpha)}{B_\alpha} = \frac{\sin^2(\alpha_r)}{B_r},$$

where α is the pitch angle at the equator, $B_\alpha = B_0$ is the magnetic field at the equator, and the subscript r

refers to the reflection point, *i.e.* when $\alpha_r = \frac{\pi}{2}$. We know that $\theta_r = \frac{\pi}{4}$, then

$$\begin{aligned}
\frac{\sin^2(\alpha)}{B_\alpha} &= \frac{1}{B_r} \\
\sin^2(\alpha) &= \frac{B_\alpha}{B_r} \\
\sin(\alpha) &= \sqrt{\frac{B_\alpha}{B_r}} \\
\alpha &= \sin^{-1}\left(\sqrt{\frac{B_\alpha}{B_r}}\right) \\
\alpha &= \sin^{-1}\left(\sqrt{\frac{B_0}{\frac{B_0}{\sin^6(\theta_r)}\sqrt{4\cos^2(\theta_r)+\sin^2(\theta_r)}}}\right) \\
\alpha &= \sin^{-1}\left(\sqrt{\frac{\sin^6(\theta_r)}{\sqrt{4\cos^2(\theta_r)+\sin^2(\theta_r)}}}\right) \\
\alpha &= \sin^{-1}\left(\frac{\sin^3(\theta_r)}{\left(4\cos^2(\theta_r)+\sin^2(\theta_r)\right)^{\frac{1}{4}}}\right) \\
\alpha &= \sin^{-1}\left(\frac{\sin^3\left(\frac{\pi}{4}\right)}{\left(4\cos^2\left(\frac{\pi}{4}\right)+\sin^2\left(\frac{\pi}{4}\right)\right)^{\frac{1}{4}}}\right) \\
\alpha &= 0.285 \text{ rad.}
\end{aligned}$$

■

Problem 2 - Lorentz Transformations for a Particle in a Magnetic Field
Consider a particle moving in a magnetic field along a helical trajectory with a velocity β and a pitch angle α, so that

$$\begin{aligned}
\mu &= \cos(\alpha), \\
\beta_\parallel &= \beta\mu, \\
\beta_\perp &= \beta\sqrt{1-\mu^2}.
\end{aligned}$$

Make a boost with $\beta_\parallel$ to the frame K', where the parallel velocity is zero, $\beta'_\parallel = 0$. (Prime denotes that frame K'). In the frame K', find the velocity $\beta'_\perp$, momentum $p'_\perp$, and Lorentz factor $\gamma'_\perp$.

Solution. We choose the axes such that

$$\frac{v_x}{c} = \beta_\parallel, \quad \frac{v_y}{c} = \beta_\perp, \quad v_z = 0,$$

$$\implies v_x = c\beta_\parallel, \quad v_y = c\beta_\perp, \quad v_z = 0.$$

In the frame K', we have $\beta'_\parallel = 0$, then

$$v'_x = c\beta'_\parallel = 0, \quad v'_y = c\beta'_\perp, \quad v'_z = 0.$$

The Lorentz boosts are then given by

$$\begin{aligned}
v_x &= \frac{v'_x + V}{1 + \frac{V}{c^2} v'_x}, \\
v_y &= \frac{v'_y \sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V}{c^2} v'_x}, \\
v_z &= \frac{v'_z \sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V}{c^2} v'_x}.
\end{aligned}$$

We have that $v'_x = 0$, then

$$\begin{aligned}
&v_x = V = c\beta_\parallel = c\beta\mu, \\
&v_y = v'_y \sqrt{1 - \frac{V^2}{c^2}} \implies v'_y = \frac{v_y}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{c\beta\sqrt{1-\mu^2}}{\sqrt{1-(\beta\mu)^2}} \equiv c\beta'_\perp,
\end{aligned}$$

where

$$\begin{aligned}
\beta'_\perp &\equiv \frac{\beta\sqrt{1-\mu^2}}{\sqrt{1-(\beta\mu)^2}} \\
&= \frac{\beta\sin(\alpha)}{\sqrt{1-(\beta\mu)^2}} \\
&= \gamma_\parallel \beta \sin(\alpha).
\end{aligned}$$

The relativistic momentum in the frame K' is given by $\mathbf{p} = \gamma m \mathbf{v}$, divided in to parallel and perpendicular components

$$\begin{aligned}
p'_\parallel &= \gamma' m \beta'_\parallel, \\
p'_\perp &= \gamma' m \beta'_\perp,
\end{aligned}$$

where γ' is the Lorentz factor in the frame K'. Since we only have the perpendicular component of the velocity, we have that $\gamma'(\beta'_\perp) = \gamma'_\perp$, which gives

$$\begin{aligned}
\gamma'_\perp &= \frac{1}{\sqrt{1 - \left(\frac{\beta'_\perp}{c}\right)^2}} \\
&= \frac{1}{\sqrt{1 - \left(\frac{\beta\sin(\alpha)}{\sqrt{1-(\beta\mu)^2}}\right)^2}} \\
&= \frac{1}{\sqrt{1 - \left(\frac{\beta^2\sin^2(\alpha)}{1-(\beta\mu)^2}\right)}} \\
&= \frac{\sqrt{1-(\beta\mu)^2}}{\sqrt{1-(\beta\mu)^2 - \beta^2\sin^2(\alpha)}} \\
&= \frac{\sqrt{1-(\beta\mu)^2}}{\sqrt{1-\beta^2\cos^2(\alpha) - \beta^2\sin^2(\alpha)}} \\
&= \frac{1}{\gamma_\parallel\sqrt{1-\beta^2}} \\
&= \frac{\gamma}{\gamma_\parallel}.
\end{aligned}$$

Thus, the perpendicular momentum component in the frame K' is

$$\begin{aligned} p'_\perp &= \gamma'_\perp m\beta'_\perp \\ &= \gamma m\beta \sin(\alpha) \\ &= p_\perp. \end{aligned}$$

■

Problem 3 - Lorentz Transformations for Fields
A magnetic dipole μ is aligned with z-axis and is moving with a velocity βc along the x direction. For small $\beta \ll 1$, find the electric field in the lab frame.

Solution. The electromagnetic field strength tensor, considering $c = 1$, is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix},$$

and a Lorentz transformation for a boost along the x direction with velocity βc can be written as

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

such that

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \Lambda^\mu{}_\nu \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}.$$

For $\beta \ll 1$, then $\gamma = 1$ and relativistic effects are ignored. Additionally, we will use the Minkowski metric given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} F^{\mu\nu} &= \Lambda^\mu{}_\lambda F_{\lambda\rho} \Lambda_\rho{}^\nu \\ &= \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\beta E_x & E_x & E_y & E_z \\ -E_x & \beta E_x & -B_z & B_y \\ -E_y - \beta B_z & \beta E_y + B_z & 0 & -B_x \\ -E_z - \beta B_y & \beta E_z - B_y & B_x & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & E_x(1-\beta^2) & E_y + \beta B_z & E_z - \beta B_y \\ E_x(\beta^2 - 1) & \beta E_x & -\beta E_y - B_z & -\beta E_z + B_y \\ -E_y - \beta B_z & \beta E_y + B_z & 0 & -B_x \\ -E_z + \beta B_y & \beta E_z - B_y & B_x & 0 \end{pmatrix}. \end{aligned}$$

Thus, the electric field in the lab frame is

$$\mathbf{E} = E_x(1-\beta^2)\hat{\mathbf{x}} + (E_y + \beta B_z)\hat{\mathbf{y}} + (E_z - \beta B_y)\hat{\mathbf{z}}.$$

■

Homework 7

Problem 1
Estimate the deflection angle for an electron of energy $\epsilon = 1$ keV passing at a distance $\ell = 1$ μm from the proton.

Solution. The deflection angle θ for a charged particle in a Coulomb field can be estimated using the formula:

$$\theta \approx \frac{2b}{\ell}.$$

where b is the impact parameter given by $b = \frac{k|q_1 q_2|}{mv^2}$.
For the velocity v, we use the energy which is

$$\epsilon = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{2\epsilon}{m}} = \sqrt{\frac{2 \times 1.60 \times 10^{-16}}{9.11 \times 10^{-31}}} \approx 1.87 \times 10^7 \,\mathrm{m/s}.$$

Calculating b, we have

$$b = \frac{ke^2}{mv^2} = \frac{(8.99 \times 10^9)\left(1.60 \times 10^{-19}\right)^2}{(9.11 \times 10^{-31})(1.87 \times 10^7)^2} \approx 7.224 \times 10^{-13} \,\mathrm{m}.$$

Finally, calculating θ, we have

$$\theta \approx \frac{2b}{\ell} = \frac{2(7.224 \times 10^{-13})}{10^{-6}} \approx 1.44 \times 10^{-6} \,\text{radians}.$$

Therefore, the deflection angle is approximately 1.44×10^{-6} radians or about 0.297 arcseconds.
This small angle makes sense because the electron's energy is relatively high (1 keV) and the distance of closest approach is large (1 μm).

■

Problem 2
It's a bright sunny day. The Sun is aligned with the zenith. You are holding a parasol of radius $r = 50$ cm. Find the force from the Sun (assume complete absorption). Look up luminosity of the Sun and distance to the Sun.

Solution. The luminosity of the sun is $L_\odot = 3.828 \times 10^{26}$ W and the distance to the sun is 1 AU $= 1.496 \times 10^8$ km.
The solar irradiance at Earth's distance is

$$I = \frac{L_\odot}{4\pi d^2} = \frac{3.828 \times 10^{26}}{4\pi(1.496 \times 10^{11})^2} \approx 1361 \,\mathrm{W/m^2}.$$

This value is known as the solar constant.
The area of the parasol is

$$A = \pi r^2 = \pi(0.5)^2 = 0.785 \,\mathrm{m}^2.$$

The power received by the parasol is

$$P = IA = (1361)(0.785) = 1068.4 \,\mathrm{W}.$$

Assuming complete absorption, the force from the Sun on the parasol is

$$F = \frac{P}{c} = \frac{1068.4}{2.998 \times 10^8} = 3.56 \times 10^{-6} \,\mathrm{N}.$$

Therefore, the Sun exerts a force of approximately $3.56 \,\mu$N on the parasol.
This is a very small force, roughly equivalent to the weight of a few micrograms on Earth, but it is responsible for important effects like radiation pressure in space and comet tail formation. ■

Problem 3
An electromagnetic wave is propagating along z and is polarized along x, so that its vector potential is

$$\mathbf{A} = A_0 \left(\cos\left(\omega t - k_z z\right), 0, 0 \right),$$

where $\omega = k_z c$.
Calculate the averaged Poynting flux (over period of oscillation). Represent this linearly polarized wave as a sum of two circularly polarized waves and write the vector potential for $\mathbf{A}_R$ and $\mathbf{A}_L$, the right and left-polarized components. There are media where right and left circular polarizations propagate with different speeds (called gyrotropic). After passing through such a media one of the waves (does not matter which, right or left) is delayed in phase by π (so that it becomes $\propto \omega t - k_z z + \pi$). Find the new polarization after the wave leaves that gyrotropic medium.

Solution. We first calculate the fields and Poynting flux for the initial wave. From $\mathbf{A}$ we can find $\mathbf{B}$ and $\mathbf{E}$, as follows

$$\mathbf{B} = \nabla \times \mathbf{A} = A_0 \left(0, -k_z \cos(\omega t - k_z z), 0\right),$$
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = A_0 \omega \left(\sin(\omega t - k_z z), 0, 0\right).$$

The Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

Substituting, we get

$$\mathbf{S} = \frac{A_0^2 \omega k_z}{\mu_0} \left(0, 0, \sin(\omega t - k_z z) \cos(\omega t - k_z z)\right).$$

Time averaging, using $\langle \sin(\alpha) \cos(\alpha) \rangle = \frac{1}{2}$, we have

$$\langle \mathbf{S} \rangle = \frac{A_0^2 \omega k_z}{2\mu_0} \left(0, 0, 1\right)$$

Now for circular polarization decomposition: A linearly polarized wave can be represented as a sum of right and left circular polarizations:

$$\mathbf{A}_R = \frac{A_0}{2} \left(\cos(\omega t - k_z z), -\mathrm{i} \cos(\omega t - k_z z), 0\right),$$
$$\mathbf{A}_L = \frac{A_0}{2} \left(\cos(\omega t - k_z z), \mathrm{i} \cos(\omega t - k_z z), 0\right).$$

We can check that $\mathbf{A}_R + \mathbf{A}_L = \mathbf{A}$, since the imaginary parts cancel.
After passing through the gyrotropic medium, one component gets phase shifted by π: If right component is shifted:

$$\mathbf{A}'_R = \frac{A_0}{2} \left(\cos(\omega t - k_z z + \pi), -\mathrm{i} \cos(\omega t - k_z z + \pi), 0\right),$$
$$\mathbf{A}'_L = \frac{A_0}{2} \left(\cos(\omega t - k_z z), \mathrm{i} \cos(\omega t - k_z z), 0\right).$$

The resulting wave is

$$\mathbf{A}' = \mathbf{A}'_R + \mathbf{A}'_L = A_0 \left(0, \cos(\omega t - k_z z), 0\right).$$

The resulting wave is linearly polarized along y axis. This is an interesting result. The gyrotropic medium has rotated the polarization by 90 degrees. This is the basic principle behind optical isolators and other devices that manipulate polarization states. ■

Problem 4
An electromagnetic wave $\mathbf{A}_1$ is propagating along z and is polarized along x, so that its vector potential is

$$\mathbf{A}_1 = A_0 \left(\cos\left(\omega t - k_z z\right), 0, 0\right),$$

where $\omega = k_z c$.
Another identical electromagnetic wave $\mathbf{A}_2$ propagating along z and is polarized along x, is added. Calculate averaged Poynting flux (over period of oscillation) of the two waves together.

Solution. Let's calculate the total fields and Poynting flux for two identical waves. Starting with the vector potentials, we have

$$\mathbf{A}_1 = A_0 \left(\cos(\omega t - k_z z), 0, 0\right),$$
$$\mathbf{A}_2 = A_0 \left(\cos(\omega t - k_z z), 0, 0\right).$$

The total vector potential is

$$\mathbf{A}_{tot} = \mathbf{A}_1 + \mathbf{A}_2 = 2A_0 \left(\cos(\omega t - k_z z), 0, 0\right).$$

From this, we can find $\mathbf{B}$ and $\mathbf{E}$, given by

$$\mathbf{B}_{tot} = \nabla \times \mathbf{A}_{tot} = 2A_0 \left(0, -k_z \cos(\omega t - k_z z), 0\right),$$
$$\mathbf{E}_{tot} = -\frac{\partial \mathbf{A}_{tot}}{\partial t} = 2A_0\omega \left(\sin(\omega t - k_z z), 0, 0\right).$$

The Poynting vector is given by

$$\mathbf{S}_{tot} = \frac{1}{\mu_0}\mathbf{E}_{tot} \times \mathbf{B}_{tot}.$$

Substituting, we have

$$\mathbf{S}_{tot} = \frac{4A_0^2\omega k_z}{\mu_0} \left(0, 0, \sin(\omega t - k_z z)\cos(\omega t - k_z z)\right).$$

Time averaging, using $\langle \sin(\alpha)\cos(\alpha)\rangle = \frac{1}{2}$, we have

$$\langle \mathbf{S}_{tot}\rangle = \frac{2A_0^2\omega k_z}{\mu_0} (0, 0, 1).$$

Note that

$$\langle \mathbf{S}_{tot}\rangle = 4\langle \mathbf{S}_1\rangle = 4\langle \mathbf{S}_2\rangle.$$

The total Poynting flux is four times larger than the flux of each individual wave. This is because the amplitudes of $\mathbf{E}$ and $\mathbf{B}$ fields doubled and because the Poynting flux is proportional to the product of $\mathbf{E}$ and $\mathbf{B}$, hence the factor of 4. This demonstrates constructive interference of two identical waves leading to intensity enhancement. ■

Homework 8

Problem 1
A charged particle (charge e and mass m) is in motion in circularly polarized electromagnetic wave propagating along magnetic field. A circularly wave of amplitude $E_w = B_w$ propagates along the guiding magnetic field B_0. Find the velocity of the particle v (non-relativistic). Keep in mind that there are two possible circular polarizations, two possible signs of charge, and two possible directions of wave propagating (along and against the guide field).
[Hint: Let the guide field be along z. Write electromagnetic field of the wave (check that Maxwell's equations are satisfied). Assume that at each moment $v_z = 0$. Let a particle be located at $z = 0$. Assume that at each moment velocity is aligned or counter-aligned with the waves magnetic field, and hence perpendicular to the electric field (energy and value of velocity are conserved).]

Solution. We need to find the velocity of the particle $\mathbf{v}$ (non-relativistic). We make the following assumptions inspired by the hint:

- The guide field is along the z-axis.
- At each moment, $v_z = 0$.
- The particle is located at $z = 0$.
- The particle's velocity is counter-aligned with the wave's magnetic field, and hence perpendicular to the electric field (energy and value of velocity are conserved).

The electromagnetic field of the circularly polarized wave propagating along the z-axis and the guiding magnetic field $\mathbf{B}_0$ is given by:

$$\begin{aligned}
\mathbf{B}_0 &= (0,\, 0,\, B_0)\,, \\
\mathbf{E} &= E_\omega\left(\sin(\omega t - k_z z),\, -\cos(\omega t - k_z z),\, 0\right), \\
\mathbf{B} &= B_\omega\left(\cos(\omega t - k_z z),\, \sin(\omega t - k_z z),\, 0\right).
\end{aligned}$$

The force equation for a charged particle in this electromagnetic field is given by

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times(\mathbf{B}+\mathbf{B}_0)\right).$$

Assuming the particle's velocity is counter-aligned with the wave's magnetic field, we have

$$\mathbf{v} = -\left(\cos(\omega t),\, \sin(\omega t),\, 0\right),$$

and then $\mathbf{v}\times\mathbf{B} = 0$. Thus

$$\begin{aligned}
m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}_0\right) &\implies \begin{cases} m\dfrac{\mathrm{d}v_x}{\mathrm{d}t} = e\left(E_x + \dfrac{v_y}{c}B_0\right) \\ m\dfrac{\mathrm{d}v_y}{\mathrm{d}t} = e\left(E_y - \dfrac{v_x}{c}B_0\right) \end{cases} \\
&\implies \begin{cases} \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \dfrac{e}{m}\left(E_w\cos(\omega t) + \dfrac{v_y}{c}B_0\right) \\ \dfrac{\mathrm{d}v_y}{\mathrm{d}t} = \dfrac{e}{m}\left(E_w\sin(\omega t) - \dfrac{v_x}{c}B_0\right) \end{cases} \\
&\implies \begin{cases} \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \dfrac{eE_w}{m}\cos(\omega t) + \dfrac{eB_0}{mc}v_y \\ \dfrac{\mathrm{d}v_y}{\mathrm{d}t} = \dfrac{eE_w}{m}\sin(\omega t) - \dfrac{eB_0}{mc}v_x \end{cases}
\end{aligned}$$

The first equation rewritten gives

$$v_y = \frac{mc}{eB_0}\frac{\mathrm{d}v_x}{\mathrm{d}t} - \frac{cE_w}{B_0}\cos(\omega t),$$

and now taking the derivative, we get

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{mc}{eB_0}\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} + \frac{c\omega E_w}{B_0}\sin(\omega t).$$

Replacing in the second equation gives

$$\begin{aligned}
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_w}{m}\sin(\omega t) - \frac{eB_0}{mc}v_x \\
\frac{mc}{eB_0}\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} + \frac{c\omega E_w}{B_0}\sin(\omega t) &= \frac{eE_w}{m}\sin(\omega t) - \frac{eB_0}{mc}v_x \\
\frac{mc}{eB_0}\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} + \frac{eB_0}{mc}v_x + \left(\frac{c\omega E_w}{B_0} - \frac{eE_w}{m}\right)\sin(\omega t) = 0 \\
\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} + \left(\frac{eB_0}{mc}\right)^2 v_x + \frac{eE_w}{m}\left(\omega - \frac{eB_0}{mc}\right)\sin(\omega t) = 0 \\
\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} + \omega_0^2 v_x &= \frac{eE_w}{m}\left(\omega_0 - \omega\right)\sin(\omega t),
\end{aligned}$$

where

$$\omega_0 = \frac{eB_0}{mc}.$$

The solution to the second-order differential equation above is

$$\begin{aligned}
v_x(t) &= \frac{\frac{eE_\omega}{m}(\omega_0 - \omega)}{\omega_0^2 - \omega^2}\sin(\omega t) + c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t) \\
v_x(t) &= \frac{eE_\omega}{m(\omega_0 + \omega)}\sin(\omega t) + c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t).
\end{aligned}$$

Using v_x, we compute v_y by

$$\begin{aligned}
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_\omega}{m}\sin(\omega t) - \frac{eB_0}{mc}v_x \\
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_\omega}{m}\sin(\omega t) - \frac{eB_0}{mc}\left(\frac{eE_\omega}{m(\omega_0 + \omega)}\sin(\omega t) + c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t)\right) \\
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_\omega}{m}\sin(\omega t) - w_0\frac{eE_\omega}{m(\omega_0 + \omega)}\sin(\omega t) - w_0 c_1\sin(\omega_0 t) - w_0 c_2\cos(\omega_0 t) \\
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= \frac{eE_\omega}{m}\left(1 - \frac{\omega_0}{\omega_0 + \omega}\right)\sin(\omega t) - w_0 c_1\sin(\omega_0 t) - w_0 c_2\cos(\omega_0 t) \\
v_y(t) &= -\frac{eE_\omega}{m\omega}\left(\frac{\omega}{\omega_0 + \omega}\right)\cos(\omega t) + c_1\cos(\omega_0 t) - c_2\sin(\omega_0 t) + c_3 \\
v_y(t) &= -\frac{eE_\omega}{m\left(\omega_0 + \omega\right)}\cos(\omega t) + c_1\cos(\omega_0 t) - c_2\sin(\omega_0 t) + c_3.
\end{aligned}$$

For completion, we note again that

$$v_z(t) = 0.$$

The velocity v will be the magnitude, which will be

$$v = \frac{eE_\omega}{m(\omega_0 + \omega)}.$$

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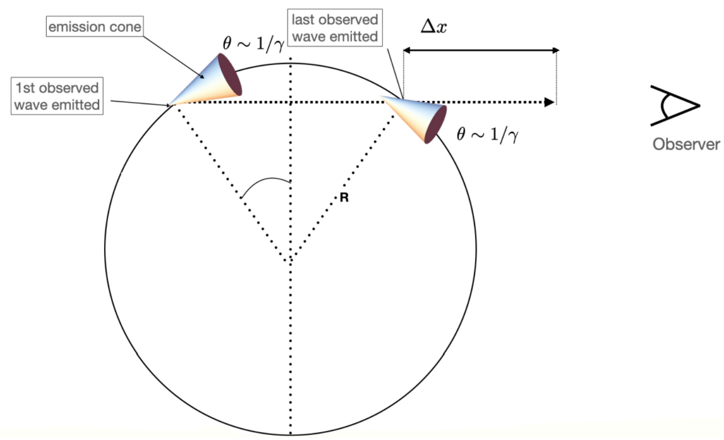
Homework 9

Problem 1
A particle is moving relativistically with Lorentz factor $\gamma \gg 1$ along a circle of radius R, as shown in the figure below. By aberration, a particle emits mostly within an angle $\sim 1/\gamma$ with respect to its instantaneous velocity. Estimate a distance Δx and corresponding time $\Delta x/c$ between first and last moment that the observer sees the particle. (An electromagnetic signal is first emitted at the "first observed wave emitted", and propagates with speed c. The particle nearly catches up with its own radiation, and so at the point "last observed wave emitted", it is behind the first emitted wave by Δx.)



Solution. Consider a particle moving in a circle at relativistic speed with Lorentz factor is $\gamma \gg 1$. The radiation is beamed within angle $\theta \sim 1/\gamma$.
The arc length $\Delta\ell$ taken by the particle is given by

$$\Delta\ell = 2\theta R \sim \frac{2R}{\gamma}.$$

and the time Δt that passes for the particle after taking the arc length path is

$$\Delta t = \frac{\Delta\ell}{v} \sim \frac{2R}{\gamma v}.$$

Thus, the distance between the first and last moment the observer sees the particle is

$$\begin{aligned}\Delta x &= c\Delta t - v\Delta t \\ &= \frac{2Rc}{\gamma v} - \frac{2R}{\gamma} \\ &= \frac{2Rc}{\gamma v}\left(1 - \frac{v}{c}\right)\end{aligned}$$

and the time between the first and last moment the observer sees the particle is

$$\Delta\tau = \frac{\Delta x}{c} = \frac{2R}{\gamma v}\left(1 - \frac{v}{c}\right).$$

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Problem 2 - Magneto-dipolar Emission
A sphere of radius R carries dipolar magnetic field B_0 and rotates with spin frequency Ω. Estimate the emitted power. [Hint: Check Landau & Lifshitz, vol 2, eq. (71.5)]

Solution. Landau and Lifshitz, Volume 2, Eq. 71.5 states that the total radiation I is given by

$$I = \frac{2}{3c^3}\ddot{d}^2 + \frac{1}{180c^5}\dddot{D}^2_{\alpha\beta} + \frac{2}{3c^3}\ddot{m}^2,$$

where the terms corresponds to dipole radiation, quadrupole radiation, and magnetic dipole radiation, respectively.
To get the power, we have to integrate the radiation. Since we are considering a dipolar magnetic field, we only consider the last term. Additionally, we know that the magnetic field has the form

$$|\mathbf{B_0}| = \frac{\mu_0}{2}\frac{IR^2}{(z^2+R^2)^{\frac{3}{2}}}.$$

For $r \gg R$, we have

$$\begin{aligned}
|\mathbf{B_0}| &= \frac{\mu_0}{2}\frac{IR^2}{z^3} \\
&= \frac{\mu_0}{4\pi}\frac{2I\pi R^2}{r^3} \\
&= \frac{\mu_0}{4\pi}\frac{2IA}{r^3} \\
&= \frac{\mu_0}{4\pi}\frac{2}{r^3}m,
\end{aligned}$$

where A is the area of the loop. From the previous, we can write $\mathbf{m} \sim r^3\mathbf{B_0}$.
The magnetic dipole moment can be expressed in terms of its angular position as

$$m = m_0\cos(\Omega t)\hat{\mathbf{n}},$$

where m_0 is the magnitude of the dipole, Ω is the spin frequency, and $\hat{\mathbf{n}}$ is along $\hat{z}$. Taking the second time derivative, we have

$$\ddot{m} = -m_0\Omega^2\cos(\Omega t)\hat{z} = -\Omega^2 m.$$

To calculate power, we only need the magnitude of the magnetic dipole radiation; hence, $|\ddot{m}| = \Omega^2 m$. Thus, we have

$$\begin{aligned}
P &= \frac{2}{3c^3}|\ddot{m}^2| \\
&= \frac{2}{3c^3}\left(\Omega^2 m\right)^2 \\
&= \frac{2}{3c^3}\left(\Omega^2 r^3\mathbf{B_0}\right)^2 \\
&= \frac{2}{3c^3}\Omega^4 r^6\mathbf{B_0}^2.
\end{aligned}$$

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Problem 3
In the problem above, the mass of the sphere is M (so that its moment of inertia is $I \approx (2/5)MR^2$). An observer can measure spin Ω and the rate of change $\dot{\Omega}$. Find the magnetic field on the surface in terms of M, R, Ω and $\dot{\Omega}$.

Solution. The rotational energy is given by

$$E = \frac{1}{2} I \Omega^2.$$

We know that $\frac{\mathrm{d}E}{\mathrm{d}t} = -P$, so we calculate the time derivative of the energy. We have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} = I\Omega\dot{\Omega} &= -P \\ &= \frac{2}{3c^3}\Omega^4 r^6 \mathbf{B_0}^2.\end{aligned}$$

Thus, the magnetic field on the surface is

$$\begin{aligned}|\mathbf{B_0}| &= \sqrt{\frac{3c^3 I\Omega\dot{\Omega}}{2\Omega^4 r^6}} \\ &= \frac{c}{r^3}\sqrt{\frac{3cI\dot{\Omega}}{2\Omega^3}} \\ &= \frac{c}{\Omega r^3}\sqrt{\frac{3cI\dot{\Omega}}{2\Omega}}.\end{aligned}$$

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Homework 10

Problem 1
A relativistic electron with initial Lorentz factor $\gamma_0 \gg 1$ is moving in magnetic field of value B_0 (no parallel velocity). Estimate how the peak frequency of the synchrotron emission evolves with time (as long as $\gamma \geq 1$). You may use approximate values, neglecting factors of $\sim$ few.

Solution. Recall that, for a relativistic electron in a magnetic field, the synchrotron peak frequency is given by

$$\omega_{\text{peak}} = \frac{3}{2}\gamma^2 \omega_B,$$

where $\omega_B = \frac{eB_0}{m_e c}$ is the cyclotron frequency. Taking the time derivative of the synchroton peak frequency, we get

$$\begin{aligned}
\dot{\omega}_{\text{peak}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3}{2}\frac{eB_0}{m_e c}\gamma^2\right) \\
&= \frac{3}{2}\frac{eB_0}{m_e c}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{1-\frac{v^2}{c^2}}\right) \\
&= \frac{3}{2}\frac{eB_0}{m_e c}\frac{2v}{\left(1-\frac{v^2}{c^2}\right)^2} \\
&= \frac{3eB_0}{m_e c}v\gamma^4 \\
&= \frac{3eB_0}{m_e c}\frac{1}{\gamma}\sqrt{\gamma^2-1}\,\gamma^4 \\
&\sim \gamma^3.
\end{aligned}$$

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Problem 2
A circularly polarized electromagnetic wave propagating along $+z$ has an electric field

$$\mathbf{E}_w = \big(\sin(\omega t - kz),\, \cos(\omega t - kz),\, 0\big)E_0.$$

A conducting plate is located at $z = 0$ (so that, for $z > 0$, the total electric field is zero). Find the reflected wave, and the total electromagnetic field in the region $z < 0$.

Solution. The incident wave is

$$\mathbf{E}_i = E_0\big(\sin(\omega t - kz),\, \cos(\omega t - kz)\big).$$

For the reflected wave, we expect a similar form but with kz replaced by $-kz$ (propagating in $-z$ direction) and with possibly different amplitudes and phases, given by

$$\mathbf{E}_r = E_0\big(A\sin(\omega t + kz + \phi_1),\, B\cos(\omega t + kz + \phi_2)\big).$$

At the conducting plate ($z = 0$), the total tangential electric field must be zero. Thus,

$$\mathbf{E}_{\text{total}}(z=0) = \mathbf{E}_i(z=0) + \mathbf{E}_r(z=0) = 0.$$

This gives us the following two equations (for the x and y components)

$$\begin{aligned}
\sin(\omega t) + A\sin(\omega t + \phi_1) &= 0, \\
\cos(\omega t) + B\cos(\omega t + \phi_2) &= 0.
\end{aligned}$$

These equations must be satisfied for all t, which requires that

$$A = B = 1, \quad \phi_1 = \phi_2 = \pi.$$

Therefore, the reflected wave is

$$\mathbf{E}_r = -E_0\big(\sin(\omega t + kz),\, \cos(\omega t + kz)\big).$$

The total field in the region $z < 0$ is

$$\begin{aligned}
\mathbf{E}_{\text{total}} &= \mathbf{E}_i + \mathbf{E}_r \\
&= E_0\big(\sin(\omega t - kz),\, \cos(\omega t - kz)\big) + E_0\big(-\sin(\omega t + kz),\, -\cos(\omega t + kz)\big) \\
&= E_0\big(\sin(\omega t - kz) - \sin(\omega t + kz),\, \cos(\omega t - kz) - \cos(\omega t + kz)\big) \\
&= E_0\big(2\cos(\omega t)\sin(kz),\, 2\sin(\omega t)\sin(kz)\big) \\
&= 2E_0\big(\cos(\omega t)\sin(kz),\, -\sin(\omega t)\sin(kz)\big).
\end{aligned}$$

The magnetic field can be found using Maxwell's equations or noting that $\mathbf{B} = \frac{1}{c}\hat{k} \times \mathbf{E}$ for each wave. The total magnetic field is then

$$\mathbf{B}_{\text{total}} = -\frac{2E_0}{c}\big(\sin(\omega t)\cos(kz),\, \cos(\omega t)\cos(kz)\big).$$

Therefore, the reflected wave has the same amplitude but opposite polarization, the total field is a standing wave with amplitude modulated by $\sin(kz)$, and at the conducting plate ($z = 0$), the total electric field is zero as required. The field maintains circular polarization at each point in space, but with an amplitude that varies with position. ■

Homework 11

Problem 1
A light of intensity I_0 falls on a layer of plasma (a mix of free ions and electrons) with density n (in units cm^{-3}, assume single charged) and thickness l (cm). Find intensity of light that passed unscattered (assume heavy ions do not respond to radiation, and wave-electron interaction occurs in single-particle regime).

Solution. The intensity of unscattered light follows the Beer-Lambert law given by

$$I = I_0 \mathrm{e}^{-\sigma n l},$$

where σ is the scattering cross-section for single-particle interaction, n is plasma density (cm^{-3}), l is layer thickness (cm)
In single-particle regime, σ is the Thomson cross-section, given by

$$\sigma = \frac{8\pi}{3} r_e^2 = 6.65 \times 10^{-25} \text{cm}^2,$$

where $r_e = 2.818 \times 10^{-13}$ cm is the classical electron radius.
Thus, the final intensity is

$$I = I_0 \mathrm{e}^{-\frac{8\pi}{3} r_e^2 n l}.$$

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Problem 2
Why does a simple charge in circular motion emit radiation while a constant loop current does not?

Solution. A particle in circular motion changes its direction and and velocity at every point in time, thus making the particle have non-zero acceleration, which in turn leads to the emission of radiation. A particle in a constant loop current does not have any change in velocity, making the acceleration zero and no emission of radiation. ■

Problem 3
A dielectric sphere of radius R with dielectric constant ϵ has constant charge density ρ_e. Find electric field (both inside and outside).
Hint:

$$\begin{aligned} \operatorname{div} \mathbf{D} &= 4\pi\rho_e \\ \mathbf{E} &= -\nabla\Phi \\ \mathbf{D} &= \epsilon\mathbf{E} \end{aligned}$$

and the boundary condition is

$$\epsilon \partial_r \Phi\big|_{R_-} = \partial_r \Phi\big|_{R_+}$$

Solution.

- **Inside the sphere ($r \leq R$):** From $\operatorname{div} \mathbf{D} = 4\pi\rho_e$, we get

  $$\nabla^2 \Phi = -\frac{4\pi\rho_e}{\epsilon_0}.$$

  Solving with spherical symmetry, we get

  $$\Phi(r) = -\frac{4\pi\rho_e}{3\epsilon_0} r^2 + A.$$

  The boundary condition at $r = 0$ requires $A = 0$. Thus,

  $$\Phi(r) = -\frac{4\pi\rho_e}{3\epsilon_0} r^2.$$

- **Outside the sphere** ($r > R$)**:** The Laplace equation is $\nabla^2\Phi = 0$. The general solution is then

$$\Phi(r) = \frac{B}{r} + C.$$

The boundary condition at $r = R$ requires Φ to be continuous. Hence

$$\epsilon\partial_r\Phi|\, R- = \partial_r\Phi|\, R+.$$

These conditions give

$$\begin{cases} \mathbf{E}_{\text{inside}}(r) = -\dfrac{4\pi\rho_e r}{3\epsilon_0}, & \text{for } r < R, \\ \mathbf{E}_{\text{outside}}(r) = \dfrac{4\pi R^3\rho_e}{3\epsilon_0 r^2}, & \text{for } r > R. \end{cases}$$

■