MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*
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Homework 4

Problem 4-4
Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$. (*Used on pp. 502, 542.*)

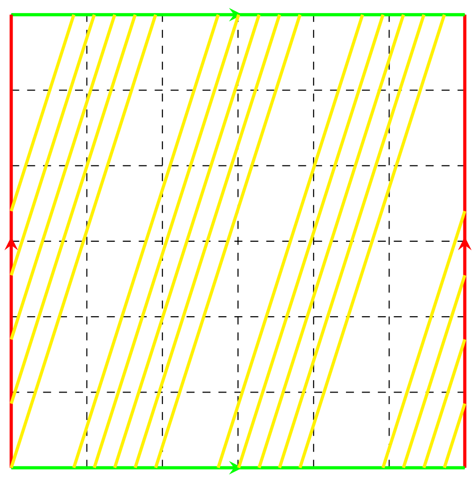
Solution. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$ denote the torus, and let α be any irrational number. The curve γ defined in Example 4.20 is

$$\begin{aligned} \gamma : \mathbb{R} &\to \mathbb{T}^2 \\ t &\mapsto \left(\mathrm{e}^{2\pi \mathrm{i} t}, \mathrm{e}^{2\pi \mathrm{i}\alpha t}\right). \end{aligned}$$

To show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$, we have two ways:

- Show that $\gamma(\mathbb{R})$ intersects every open set in $\mathbb{T}^2$, or
- Show that the closure of $\gamma(\mathbb{R})$ is equal to $\mathbb{T}^2$ itself, *i.e.* every point of $\mathbb{T}^2$ is a limit point of $\gamma(\mathbb{R})$.

We will proceed with the first option.



As a hand-wavy argument, it feels intuitive that, given any open set in $\mathbb{T}^2$, no matter how small, there will always be a line from $\gamma(\mathbb{R})$ that intersects it, since the irrationality of α allows $\gamma(\mathbb{R})$ to fully cover $\mathbb{T}^2$. If that was not the case, then that means there would be lines that coincide and areas left unoccupied, which contradicts the irrationality of α.
We use Dirichlet's Approximation Theorem (Lemma 4.21) to do so.

Lemma 1. Given $\alpha \in \mathbb{R}$ and any positive integer N, there exists integers n and m with $1 \leq n \leq N$, such that $|n\alpha - m| \leq \frac{1}{N}$.

This means we are guaranteed the existence of $n, m \in \mathbb{Z}$ such that $|n\alpha - m| < \epsilon$, for some $\epsilon > 0$. Let $\beta = n\alpha - m$. Note that

$$\mathrm{e}^{2\pi \mathrm{i} n\alpha} = \mathrm{e}^{2\pi \mathrm{i}(n\alpha - m)} = \mathrm{e}^{2\pi \mathrm{i}\beta}.$$

We have that

$$\{U \times V \mid U, V \subseteq \mathbb{S}^1 \text{ are open}\}$$

is a basis for the topology on $\mathbb{T}^2$.

Consider an open set $V \subseteq \mathbb{S}^1$, then we can find an open interval $(a, b) \subset \mathbb{R}$ such that

$$\{e^{2\pi i x} \mid x \in (a, b)\} \subseteq V.$$

From $|\beta| < \epsilon = b - a$, we can see that there exists an integer $k \in \mathbb{Z}$ such that $k\beta \in (a, b)$. In fact,

$$e^{2\pi i n\alpha k} = e^{2\pi i k\beta} \in V.$$

We will use the above argument now on the basis of $\mathbb{T}^2$. Given any open sets $U, V \in \mathbb{S}^1$, we can always choose $x \in \mathbb{R}$ such that $e^{2\pi i x} \in U$ and then use the above to find an integer k such that $e^{2\pi i(x+kn)\alpha} \in V$, where $e^{2\pi i kn} = 1$.
We have

$$\gamma(x + kn) = \left(e^{2\pi i(x+kn)}, e^{2\pi i(x+kn)\alpha}\right) = \left(e^{2\pi i x}, e^{2\pi i(x+kn)\alpha}\right) \in U \times V.$$

Since this is true for arbitrary U and V, then $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.

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Problem 4-7
Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29; in other words, assuming that $\tilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P with or without boundary, a map $F : \tilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\tilde{N}$. [Remark: this shows that the property described in Theorem 4.29 is "characteristic" in the same sense as that in which Theorem A.27(a) is characteristic of the quotient topology.]

Solution. Suppose M and N are smooth manifolds and let $\tilde{N}$ be the same set as N but with a possibly different topology and smooth structure. Let $\pi : M \to N$ and $\tilde{\pi} : M \to \tilde{N}$ be surjective smooth submersions. Consider $\mathrm{Id}_N : N \to \tilde{N}$ and $\mathrm{Id}_{\tilde{N}} : \tilde{N} \to \tilde{N}$. Clearly, Id_N is a homeomorphism. We need to show smoothness of the map and its inverse.
We have that $\mathrm{Id}_{\tilde{N}}$ is smooth, then $\mathrm{Id}_{\tilde{N}} \circ \tilde{\pi} = \tilde{\pi}$ is also smooth, since $\tilde{\pi}$ is also smooth. From that, $\mathrm{Id}_N \circ \pi = \tilde{\pi}$ is smooth, and thus, so is Id_N.
Now to show that the inverse is also smooth, notice that $\mathrm{Id}_N^{-1} \circ \tilde{\pi} = \pi$ is smooth, and then so is Id_N^{-1}.
Therefore, Id_N is a diffeomorphism. ■

Problem 4-8
This problem shows that the converse of Theorem 4.29 is false. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) = xy$. Show that π is surjective and smooth, and for each smooth manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Solution. Let

$$\begin{aligned} \pi : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto xy. \end{aligned}$$

- **Surjective**: For all $z \in \mathbb{R}$, there exists $(x, y) \in \mathbb{R}^2$ such that $\pi(x, y) = z$. In fact, let $(x, y) = (z, 1)$ or $(x, y) = (1, z)$, then $\pi(x, y) = z$.

  Thus, π is surjective.

- **Smoothness**: π is clearly smooth since it a polynomial

- $\boxed{\Longrightarrow}$ Suppose $F : \mathbb{R} \to P$ is smooth, then $F \circ \pi$ is smooth since it is a composition of smooth maps.

$\boxed{\Longleftarrow}$ Suppose $F \circ \pi$ is smooth, then if we compose with a smooth function, say f, that acts "like" the inverse of π, then we get that $F \circ \pi \circ f = F$ is smooth. In that case, take

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^2 \\ x &\mapsto (x, 1) \end{aligned}$$

and thus, F is smooth.

Thus, for each manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth.

- **Not a Smooth Submersion**: We compute

$$\mathrm{d}\pi(x, y) = \begin{pmatrix} \partial_x(xy) \\ \partial_y(xy) \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix},$$

which has a rank 0 at $(0, 0)$. Thus, π is not a smooth submersion.

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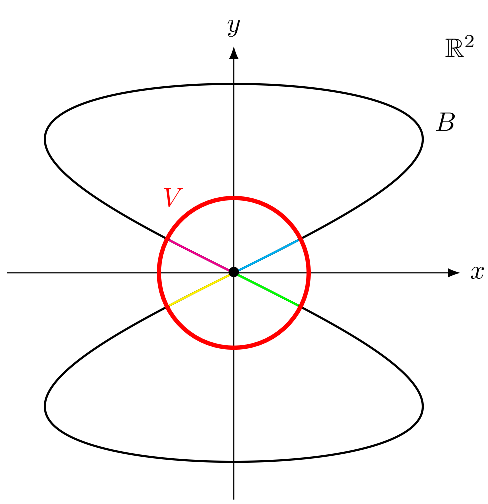
Problem 5-4
Show that the image of the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: this is not the same as showing that β is not an embedding.]

Solution. Consider the curve

$$\begin{aligned} \beta : (-\pi, \pi) &\to \mathbb{R}^2 \\ t &\mapsto (\sin(2t), \sin(t)). \end{aligned}$$

Let $B \equiv \beta(-\pi, \pi)$ be the image of the curve. Let V be a neighborhood of 0 in $\mathbb{R}^2$, then $B \cap V$ is open in B. For small enough V, the intersection of B and V excluding the origin, *i.e.* $(B \cap V) \setminus 0$, will have four connected components.



This means that $B \cap V$ cannot be homeomorphic to any open ball in $\mathbb{R}^n$ since removing a point from $B \cap V$ created four connected components while removing a point from $\mathbb{R}^n$ creates either two connected components if $n = 1$, or one connected component otherwise. This helps us conclude that B, with the subspace topology, is not a topological manifold.
Therefore, the image of β is not an embedded submanifold of $\mathbb{R}^2$.

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