PHYS 617 - Statistical Mechanics

A Modern Course in Statistical Physics by *Linda E. Reichl*
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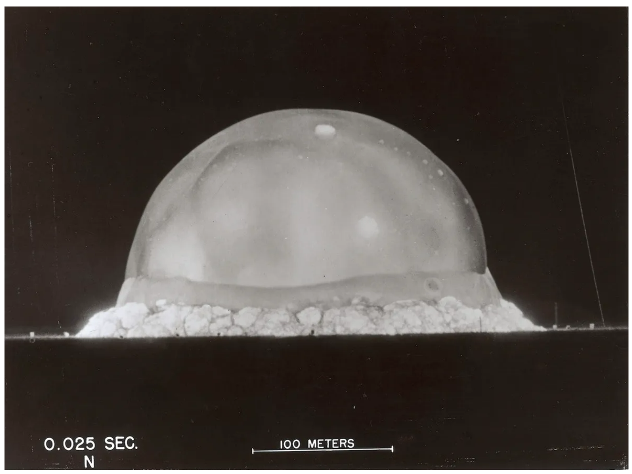
Homework 5

Problem 1
The figure below was published in Time magazine in September of 1945. It is an image of an atomic test. At the time of publication, this was extremely new military technology and the explosion energy was classified by the US government.
Estimate the explosion energy. Express your answer in tons of TNT, as (for some reason) those are the units the military preferred. 1 ton of TNT is about 4×10^9 Joules or 4×10^{16} erg.



Solution. The **Taylor–von Neumann–Sedov** blast wave describes the shock wave generated by a powerful explosion. Consider an explosion releasing a significant amount of energy E within a confined space and short time frame. This results in a robust spherical shock wave expanding outward from the explosion's center. When the shock wave has traversed a large distance relative to the explosion's size, a self-similar solution is sought to characterize the flow. Here, details regarding the explosion's size and duration become negligible, with only the energy released E dictating the shock wave's behavior. With high precision, it can be assumed that the explosion occurred instantaneously at a single point (e.g., the origin $r = 0$) at time $t = 0$. Within this self-similar region, the shock wave remains immensely strong, such that the pressure P_1 behind it significantly surpasses the pressure ahead of it (typically atmospheric pressure), denoted as P_2, which can be disregarded in analysis. While the pressure of the undisturbed gas may be insignificant, the density of this gas, denoted as ρ_0, cannot be ignored due to finite density differences across strong shock waves, as per the Rankine–Hugoniot conditions. This approximation entails setting $p_0 = 0$ and the corresponding sound speed $c_0 = 0$, yet retaining a non-zero density, i.e., $\rho_0 \neq 0$.
The only parameters available at our disposal are the energy E and the undisturbed gas density ρ_0. The only non-dimensional combination available from r, t, ρ_0, and E is

$$R = \beta \left(\frac{Et^2}{\rho_0} \right)^{\frac{1}{5}}.$$

We have that

$$R \sim \left(\frac{Et^2}{\rho_0} \right)^{\frac{1}{5}} \implies E \sim \frac{\rho_0 R^5}{t^2} = \frac{(1.293\,\mathrm{kg\,m^{-3}})(160\,\mathrm{m})^5}{(0.025\,\mathrm{s})^2} = 2.169 \times 10^{14}\,\mathrm{J} = 54,232 \text{ tons of TNT}.$$

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Problem 2

(a) In class, we discussed the condition for convective instability, being that the entropy has a negative vertical gradient. Please explain this reasoning again in your own words.

(b) Assuming hydrostatic equilibrium, show that this implies a maximally steep temperature gradient in the atmosphere, beyond which instability occurs. What is the steepest negative temperature gradient allowed in Earth's atmosphere, in °C/km? Don't forget the Earth's atmosphere is a diatomic gas, made mostly out of nitrogen molecules, with $m = 28m_p$.

Solution. (a) Convective instability refers to a situation where a fluid (such as air or water) becomes unstable and starts to convect or circulate due to changes in temperature or density. The condition for convective instability is closely related to the concept of entropy, which is a measure of disorder or randomness in a system. In the context of convective instability, when we say that entropy has a negative vertical gradient, we're essentially saying that entropy decreases as you move upward in the fluid. This means that the fluid becomes more ordered or less random as you go up. Now, imagine a scenario where you have a parcel of fluid that's slightly warmer (and thus less dense) than the surrounding fluid. If you lift this parcel upwards, it will be in an environment where entropy is decreasing with height (negative gradient). As the parcel rises, it finds itself in a region where the fluid around it is becoming more ordered (lower entropy). Since nature tends towards disorder (higher entropy), the parcel is going to have a tendency to keep rising, leading to convective motion. This process is often observed in the atmosphere, where warm air rising from the Earth's surface encounters cooler air aloft. The warm air parcel, being less dense, continues to rise due to convective instability until it reaches a level where its temperature is similar to the surrounding air, at which point it may stop rising or spread out horizontally.

(b) Assuming hydrostatic equilibrium, we have that $P = K\rho^\gamma$. We also know that convective instability occurs when

$$\frac{\partial}{\partial z}\left(\frac{P}{\rho^\gamma}\right) > 0.$$

Under the assumption that we are dealing with an ideal gas, then $PV = nk_BT \implies P = \frac{\rho k_B}{m}T$, and we have

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\begin{aligned}
\frac{\partial T}{\partial z} &= \frac{\partial T}{\partial P}\frac{\partial P}{\partial z} + \frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial z} \\
&= \left(\frac{m}{\rho k_B}\right)\frac{\partial P}{\partial z} + \left(-\frac{mP}{\rho^2 k_B}\right)\frac{\partial \rho}{\partial z} \\
&= \left(\frac{T}{P}\right)\frac{\partial P}{\partial z} + \left(-\frac{T}{\rho}\right)\frac{\partial \rho}{\partial z} \\
&= \frac{T}{P}\left[\frac{\partial P}{\partial z} - \frac{P}{\rho}\frac{\partial \rho}{\partial z}\right] \\
&= \frac{T}{P}\left[\frac{\partial P}{\partial z} - \frac{P}{\rho}\frac{\partial \rho}{\partial P}\frac{\partial P}{\partial z}\right] \\
&= \frac{T}{P}\left[\frac{\partial P}{\partial z} - \frac{P}{\rho}\left(\frac{1}{K\gamma}\left(\frac{P}{K}\right)^{\frac{1}{\gamma}-1}\right)\frac{\partial P}{\partial z}\right] \\
&= \frac{T}{P}\left[\frac{\partial P}{\partial z} - \frac{1}{\rho\gamma}\left(\frac{P}{K}\right)^{\frac{1}{\gamma}}\frac{\partial P}{\partial z}\right] \\
&= \frac{T}{P}\left[\frac{\partial P}{\partial z} - \frac{1}{\gamma}\frac{\partial P}{\partial z}\right] \\
&= \frac{T}{P}\frac{\partial P}{\partial z}\left(\frac{\gamma-1}{\gamma}\right) \\
&= \frac{T}{P}(-g\rho)\left(\frac{\gamma-1}{\gamma}\right) \\
&= \frac{T}{P}\left(-g\frac{Pm}{k_B T}\right)\left(\frac{\gamma-1}{\gamma}\right) \\
&= -\frac{mg}{k_B}\left(\frac{\gamma-1}{\gamma}\right).
\end{aligned}
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In Earth's atmosphere, assuming abundance in Nitrogen, the steepest negative temperature gradient is

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\begin{aligned}
\frac{\partial T}{\partial z} &= -\frac{mg}{k_B}\left(\frac{\gamma-1}{\gamma}\right) \\
&= -\frac{28m_p(9.81)}{1.38\times 10^{-23}}\left(\frac{\frac{7}{5}-1}{\frac{7}{5}}\right) \\
&= -9.49\times 10^{-3}\,{}^\circ\mathrm{C}\,\mathrm{m}^{-1} \\
&= -9.49\,{}^\circ\mathrm{C}\,\mathrm{km}^{-1}.
\end{aligned}
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Problem 3

Imagine a black hole with mass M embedded in a uniform gas. Far from this mass, as $r \to \infty$, the density $\rho \to \rho_0$ and sound speed $c_s \to c_0$ are constants. Near the hole, gas will accrete until it reaches a steady-state, spherically symmetric solution as a function of r. For this problem, consider a "black hole" just to be an accreting point mass; interestingly, relativity never enters into this calculation.

(a) Estimate the location of the "sonic point"; i.e. the distance from the black hole where the accretion flow becomes supersonic, and sound waves can no longer travel upstream.

(b) Estimate the accretion rate $\dot{M}$ by determining its value at the sonic point.

Solution. (a) The speed of sound is very important in fluids as it is the speed at which waves and force propagate. In case of the Bondi flow, when the speed of the accreting gas is lower than v_c, pressure waves can travel "counter-current" and influence the external flow. Past v_c, inside the radius r_c, the inward flow is so fast that no fluid pressure wave/force manage to travel outside, since it travels slower than the medium itself. So all fluid dynamics of the star are confined inside this radius now. It is a similar principle as the black hole, its event horizon for light waves being equivalent to the Bondi critical point for sound waves.

The location of the sonic point for the black hole would be a Schwarzchild radius away from the center. The Schwarzchild radius is given by

$$R_s = \frac{2GM}{c^2} \sim \frac{GM}{c^2}.$$

(b) Since we have mass density conservation, then $\partial_t \rho + \nabla \cdot (\rho v) = 0$. Taking the volume integral of this quantity (assuming it is constant in time), we have

$$\begin{aligned}\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v)\right] \mathrm{d}V &= \frac{\partial}{\partial t}\int_V \rho \,\mathrm{d}V + \oint_A (\rho \mathbf{v}) \cdot \mathrm{d}\mathbf{A} \\ &= \frac{\partial}{\partial t} M - \rho v \oint_A \mathrm{d}A \\ &= \dot{M} - 4\pi r^2 \rho v \\ &= 0.\end{aligned}$$

Then

$$\dot{M} = 4\pi R_s^2 \rho v \sim R_s^2 \rho v \sim \frac{(GM)^2 \rho}{c^3}.$$

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Problem 4

2D turbulence behaves fundamentally differently from 3D turbulence. The reason is that in 2D the "vorticity" $\omega = \nabla \times v$ obeys a conservation law (maybe I can show this at some point; it's not too hard to derive). Formulate an analogous vorticity cascade argument (analogous to the energy cascade we derived in class) and derive the following:

(a) How does velocity v of an eddy scale with its size ℓ? (in 3D we saw $v \propto \ell^{\frac{1}{3}}$)

(b) Given this, what is the slope of the power spectrum $\mathrm{d}E/\,\mathrm{d}k$ of 2D turbulence (instead of the $k^{-\frac{5}{3}}$ law we found in 3D)?

Note: the most familiar example you might have of 2D turbulence is weather patterns on the surface of the earth (on scales much larger than the height of the atmosphere, so the flow is effectively 2D). Another example is convective zones on planets like Jupiter. The steeper power-law in 2D turbulence puts more energy in long-lived large-scale structures like hurricanes or Jupiter's big red dot.

Solution. (a) Writing the velocity vector $\mathbf{v}$ in cylindrical coordinates, we have $\mathbf{v} = v(r)\hat{\phi}$, then the vorticity ω is

$$\begin{aligned}\omega &= \nabla \times v \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rv)\,\hat{\phi} \\ &= \frac{1}{r}\left(v + r\frac{\partial v}{\partial r}\right)\hat{\phi} \\ &= \left(\frac{v}{r} + \frac{\partial v}{\partial r}\right)\hat{\phi}.\end{aligned}$$

In analogy to the energy cascade, we have that the vorticity flow (enstrophy) from scale i to $i+1$ will be constant. Hence

$$\omega = \nabla \times v = \text{constant} \implies \omega \sim \frac{v}{\ell} \implies v \propto \ell.$$

(b) We have that the wave-number $k = \frac{1}{\ell} \implies \ell \propto k^{-1}$. Then, the slope of the power spectrum is

$$\frac{\mathrm{d}E}{\mathrm{d}k} \sim \frac{v^2}{k} \sim \frac{\ell^2}{k} \propto \ell^3.$$

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