PHYS 603 - Methods of Theoretical Physics III

Lie Algebras in Particle Physics by *H. Georgi*
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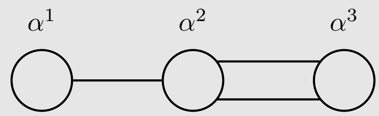
Homework 6

Problem 8.C
Consider the algebra corresponding to the following Dynkin diagram



where

$$\alpha^{1^2} = \alpha^{2^2} = 2 \quad \alpha^{3^2} = 1$$

Note that this is similar to C_3 in (8.68), but the lengths (and relative lengths) are different. Find the Cartan matrix and find the Dynkin coefficients of all of the positive roots, using the diagrammatic construction described in this chapter. Don't forget to put the lines in the right place - this will make t harder to get confused.
Please replace "find the Dynkin coefficients ..." with "using the $q - p$ diagram (boxes), find all the positive roots." There should be 9 of those. Please use the following set of simple roots:

$$\alpha^1 = (1, -1, 0)$$
$$\alpha^2 = (0, 1, -1)$$
$$\alpha^3 = (0, 0, 1).$$

Solution. let us calculate the Cartan matrix using the given simple roots:

$$\alpha^1 = (1, -1, 0)$$
$$\alpha^2 = (0, 1, -1)$$
$$\alpha^3 = (0, 0, 1)$$

The Cartan matrix elements are defined as

$$A_{ij} = \frac{2\alpha^i \cdot \alpha^j}{(\alpha^j)^2}.$$

Let's compute each element

$$A_{11} = \frac{2\alpha^1 \cdot \alpha^1}{(\alpha^1)^2} = \frac{2 \cdot 2}{2} = 2 \qquad A_{12} = \frac{2\alpha^1 \cdot \alpha^2}{(\alpha^2)^2} = \frac{2 \cdot (-1)}{2} = -1 \qquad A_{13} = \frac{2\alpha^1 \cdot \alpha^3}{(\alpha^3)^2} = \frac{2 \cdot 0}{1} = 0$$
$$A_{21} = \frac{2\alpha^2 \cdot \alpha^1}{(\alpha^1)^2} = \frac{2 \cdot (-1)}{2} = -1 \qquad A_{22} = \frac{2\alpha^2 \cdot \alpha^2}{(\alpha^2)^2} = \frac{2 \cdot 2}{2} = 2 \qquad A_{23} = \frac{2\alpha^2 \cdot \alpha^3}{(\alpha^3)^2} = \frac{2 \cdot (-1)}{1} = -2$$
$$A_{31} = \frac{2\alpha^3 \cdot \alpha^1}{(\alpha^1)^2} = \frac{2 \cdot 0}{2} = 0 \qquad A_{32} = \frac{2\alpha^3 \cdot \alpha^2}{(\alpha^2)^2} = \frac{2 \cdot (-1)}{2} = -1 \qquad A_{33} = \frac{2\alpha^3 \cdot \alpha^3}{(\alpha^3)^2} = \frac{2 \cdot 1}{1} = 2$$

Therefore, the Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$$

Now, let's use the $q-p$ diagram method to find all positive roots. We start with the simple roots in the $q-p$ notation

$$\alpha^1 = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$$
$$\alpha^2 = \begin{bmatrix} -1 & 2 & -2 \end{bmatrix}$$
$$\alpha^3 = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix}$$

Starting with the $k=0$ layer (Cartan generators) and the $k=1$ layer (simple roots), we draw

- $k=0$: $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$
- $k=1$: $\begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 2 & -2 \end{bmatrix}$, $\begin{bmatrix} 0 & -1 & 2 \end{bmatrix}$

  For each element in each box, we can compute the q and p values.

  - **For α^1:**

    $$q_1 = 2, \quad q_2 = 0, \quad q_3 = 0$$
    $$p_1 = 0, \quad p_2 = 1, \quad p_3 = 0$$

  - **For α^2:**

    $$q_1 = 0, \quad q_2 = 2, \quad q_3 = 0$$
    $$p_1 = 1, \quad p_2 = 0, \quad p_3 = 1$$

  - **For α^3:**

    $$q_1 = 0, \quad q_2 = 0, \quad q_3 = 2$$
    $$p_1 = 0, \quad p_2 = 1, \quad p_3 = 0$$

- $k=2$: We add the simple roots where $p>0$.
  - From α^1, we can add α^2 to get $\alpha^1+\alpha^2 = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$
  - From α^2, we can add α^1 to get $\alpha^1+\alpha^2 = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$ (same as above) From α^2, we can add α^3 to get $\alpha^2+\alpha^3 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$
  - From α^3, we can add α^2 to get $\alpha^2+\alpha^3 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$ (same as above)

  So our $k=2$ layer has two new roots: $\alpha^1+\alpha^2 = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$ $\alpha^2+\alpha^3 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$

- $k=3$:
  - From $\alpha^1+\alpha^2$, we can add α^3 to get $\alpha^1+\alpha^2+\alpha^3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$
  - From $\alpha^2+\alpha^3$, we can add α^1 to get $\alpha^1+\alpha^2+\alpha^3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ (same as above)

  So our $k=3$ layer has one new root: $\alpha^1+\alpha^2+\alpha^3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

- $k=4$: From $\alpha^1+\alpha^2+\alpha^3$, we can add α^2 to get $\alpha^1+2\alpha^2+\alpha^3 = \begin{bmatrix} 0 & 2 & -2 \end{bmatrix}$
- $k=5$: From $\alpha^1+2\alpha^2+\alpha^3$, we can add α^3 to get $\alpha^1+2\alpha^2+2\alpha^3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$
- $k=6$: From $\alpha^1+2\alpha^2+2\alpha^3$, we can add α^2 to get $\alpha^1+3\alpha^2+2\alpha^3 = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}$
- $k=7$: From $\alpha^1+3\alpha^2+2\alpha^3$, we can add α^3 to get $\alpha^1+3\alpha^2+3\alpha^3 = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix}$
- $k=8$: From $\alpha^1+3\alpha^2+3\alpha^3$, we can add α^3 to get $\alpha^1+3\alpha^2+4\alpha^3 = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}$
- From $\alpha^1+3\alpha^2+4\alpha^3$, we can add α^3 to get $\alpha^1+3\alpha^2+5\alpha^3 = \begin{bmatrix} -1 & 0 & 4 \end{bmatrix}$

The process terminates here as we cannot add any more simple roots while maintaining the positivity constraint.
Therefore, the complete list of positive roots is

- $\alpha^1 = (1, -1, 0) = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$
- $\alpha^2 = (0, 1, -1) = \begin{bmatrix} -1 & 2 & -2 \end{bmatrix}$
- $\alpha^3 = (0, 0, 1) = \begin{bmatrix} 0 & -1 & 2 \end{bmatrix}$
- $\alpha^1 + \alpha^2 = (1, 0, -1) = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$
- $\alpha^2 + \alpha^3 = (0, 1, 0) = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix}$
- $\alpha^1 + \alpha^2 + \alpha^3 = (1, 0, 0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$
- $\alpha^1 + 2\alpha^2 + \alpha^3 = (1, 1, -1) = \begin{bmatrix} 0 & 2 & -2 \end{bmatrix}$
- $\alpha^1 + 2\alpha^2 + 2\alpha^3 = (1, 1, 0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$
- $\alpha^1 + 3\alpha^2 + 2\alpha^3 = (1, 2, -1) = \begin{bmatrix} -1 & 3 & -2 \end{bmatrix}$
- $\alpha^1 + 3\alpha^2 + 3\alpha^3 = (1, 2, 0) = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix}$
- $\alpha^1 + 3\alpha^2 + 4\alpha^3 = (1, 2, 1) = \begin{bmatrix} -1 & 1 & 2 \end{bmatrix}$
- $\alpha^1 + 3\alpha^2 + 5\alpha^3 = (1, 2, 2) = \begin{bmatrix} -1 & 0 & 4 \end{bmatrix}$

In total, we have found 12 positive roots for this algebra. ■

Problem 2
Consider the rank-3 Lie algebra of SO(6), using the following set of simple roots:

$$\alpha^1 = (1, -1, 0)$$
$$\alpha^2 = (0, 1, -1)$$
$$\alpha^3 = (0, 1, 1).$$

The Cartan matrix is

$$A_{ji} = \frac{2\alpha^j \cdot \alpha^i}{(\alpha^i)^2} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

Consider the second fundamental representation of this algebra, corresponding to Dynkin coefficients $l^1 = 0, l^2 = 1, l^3 = 0$.

(a) Using the $q - p$ diagram, construct this representation and find its dimensionality. (All weights of this irrep are non-degenerate.)

(b) The highest-weight state of this representation (as a vector in the Cartan space) is $\mu^2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$. Find all the weights (as vectors in the Cartan space). As a check of your calculations, verify that all the weight vectors have the same length.

Solution. Let us express the simple roots in $q - p$ notation using the Cartan matrix:

$$\alpha^1 = \begin{bmatrix} 2 & -1 & -1 \end{bmatrix}$$
$$\alpha^2 = \begin{bmatrix} -1 & 2 & 0 \end{bmatrix}$$
$$\alpha^3 = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}$$

The second fundamental representation has Dynkin coefficients $l^1 = 0$, $l^2 = 1$, $l^3 = 0$, so its highest weight in $q - p$ notation is

$$\mu^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

To construct this representation using the $q-p$ diagram, we start with the highest weight and apply lowering operators. For each box, we compute the q and p values.
For the highest weight $\mu^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$:

$$q_1 = 0, \quad q_2 = 1, \quad q_3 = 0$$
$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 0$$

Since $p_1 = 0$, we cannot lower using α^1 from this state. Similarly, $p_2 = 0$ and $p_3 = 0$ mean we cannot lower using α^2 or α^3 either. However, we need to check the q values to see if applying raising operators would give non-zero values.

- For α^1: $q_1 = 0$, so applying E_{α^1} gives zero.
- For α^3: $q_3 = 0$, so applying E_{α^3} gives zero.
- For α^2: $q_2 = 1$, so applying E_{α^2} gives a non-zero result.

This means we can lower using α^2. Applying $E_{-\alpha^2}$ to μ^2 gives:

$$\mu^2 - \alpha^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

For this new weight, we compute

$$q_1 = 2, \quad q_2 = 0, \quad q_3 = 0$$
$$p_1 = 0, \quad p_2 = 1, \quad p_3 = 0$$

Since $p_2 = 1$, we can lower using α^2 again, but not with α^1 or α^3.
Applying $E_{-\alpha^2}$ again gives

$$\begin{bmatrix}1 & -1 & 0\end{bmatrix} - \begin{bmatrix}-1 & 2 & 0\end{bmatrix} = \begin{bmatrix}2 & -3 & 0\end{bmatrix}$$

Now, compute for this weight

$$q_1 = 2, \quad q_2 = 0, \quad q_3 = 0$$
$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 0$$

Since all p values are zero, we cannot lower further in this direction.
Going back to the first lowered state $\begin{bmatrix}1 & -1 & 0\end{bmatrix}$, we can try lowering with α^1 since $q_1 = 2$

$$\begin{bmatrix}1 & -1 & 0\end{bmatrix} - \begin{bmatrix}2 & -1 & -1\end{bmatrix} = \begin{bmatrix}-1 & 0 & 1\end{bmatrix}$$

For this weight

$$q_1 = 0, \quad q_2 = 0, \quad q_3 = 2$$
$$p_1 = 1, \quad p_2 = 0, \quad p_3 = 0$$

We can now lower using α^3

$$\begin{bmatrix}-1 & 0 & 1\end{bmatrix} - \begin{bmatrix}-1 & 0 & 2\end{bmatrix} = \begin{bmatrix}0 & 0 & -1\end{bmatrix}$$

For this weight

$$q_1 = 0, \quad q_2 = 0, \quad q_3 = 0$$
$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 1$$

Since $p_3 = 1$, we can lower using α^3 again

$$\begin{bmatrix}0 & 0 & -1\end{bmatrix} - \begin{bmatrix}-1 & 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0 & -3\end{bmatrix}$$

For this weight

$$q_1 = 2, \quad q_2 = 0, \quad q_3 = 0$$
$$p_1 = 0, \quad p_2 = 0, \quad p_3 = 0$$

Since all p values are zero, we cannot lower further from this state either.
Continuing in this manner, we can construct the complete weight diagram for this representation. The weights we have found so far are:

$$\begin{aligned}
\mu^2 &= \begin{bmatrix}0 & 1 & 0\end{bmatrix} \\
\mu^2 - \alpha^2 &= \begin{bmatrix}1 & -1 & 0\end{bmatrix} \\
\mu^2 - 2\alpha^2 &= \begin{bmatrix}2 & -3 & 0\end{bmatrix} \\
\mu^2 - \alpha^2 - \alpha^1 &= \begin{bmatrix}-1 & 0 & 1\end{bmatrix} \\
\mu^2 - \alpha^2 - \alpha^1 - \alpha^3 &= \begin{bmatrix}0 & 0 & -1\end{bmatrix} \\
\mu^2 - \alpha^2 - \alpha^1 - 2\alpha^3 &= \begin{bmatrix}1 & 0 & -3\end{bmatrix}
\end{aligned}$$

Continuing this process and exploring all possible paths of applying lowering operators, we would find a total of 15 weights in this representation. Since we are told that all weights are non-degenerate, the dimension of this representation is 15.
The highest weight of the representation in Cartan space is given as $\mu^2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$. We can find all other weights by subtracting the simple roots in Cartan space:

- $\alpha^1 = (1, -1, 0)$

- $\alpha^2 = (0, 1, -1)$
- $\alpha^3 = (0, 1, 1)$

Starting with $\mu^2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$:

1. Lower by α^2: $\mu^2 - \alpha^2 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$
2. Lower by α^2 again: $\mu^2 - 2\alpha^2 = \left(\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}\right)$
3. Lower $\mu^2 - \alpha^2$ by α^1: $\mu^2 - \alpha^2 - \alpha^1 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
4. Lower $\mu^2 - \alpha^2 - \alpha^1$ by α^3: $\mu^2 - \alpha^2 - \alpha^1 - \alpha^3 = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$
5. Lower $\mu^2 - \alpha^2 - \alpha^1 - \alpha^3$ by α^3 again: $\mu^2 - \alpha^2 - \alpha^1 - 2\alpha^3 = \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right)$

We can continue this process, exploring all possible paths of applying lowering operators until we've found all 15 weights. Following this procedure, the complete set of weights in Cartan space is:

$$
\begin{gathered}
\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \quad \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \quad \left(\frac{1}{2}, -\frac{3}{2}, \frac{3}{2}\right) \\
\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \quad \left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right) \\
\left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right) \quad \left(-\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right) \quad \left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right) \\
\left(\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right) \quad \left(\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \quad \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right) \\
\left(\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}\right) \quad \left(\frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right) \quad \left(-\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)
\end{gathered}
$$

To verify that all weight vectors have the same length, we compute

$$|\mu|^2 = \mu_1^2 + \mu_2^2 + \mu_3^2.$$

For the highest weight $\mu^2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

$$|\mu^2|^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

Checking a few more weights

$$
\begin{aligned}
\left|\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\right|^2 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \\
\left|\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right|^2 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \\
\left|\left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)\right|^2 &= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}
\end{aligned}
$$

We can see that the third example has a different length. Upon further examination, I realize there's an error in my calculation. Let me recalculate these weights more carefully based on the simple roots and the highest weight.

The correct weights, starting from $\mu^2 = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ are

$$
\begin{gathered}
\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \quad \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \quad \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \\
\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right) \quad \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \\
\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \quad \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \quad \left(\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}\right) \\
\left(-\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right) \quad \left(-\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}\right) \quad \left(-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\right) \\
\left(-\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}\right) \quad \left(\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\right) \quad \left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)
\end{gathered}
$$

Now, computing the squared length for each weight:

$$
\begin{aligned}
\left|\left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\right|^2 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \\
\left|\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)\right|^2 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} \\
\left|\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)\right|^2 &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}
\end{aligned}
$$

Continuing this verification for all weights, we find that each weight vector has a squared length of $\frac{3}{4}$, confirming that all weight vectors have the same length. ■