PHYS 662 - Quantum Field Theory I

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Homework 1

Problem 1 - Translation Invariance on a Lattice

1. Consider the following generalization of the model we discussed in class

$$\mathcal{H} = \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} \lambda_{ij} q_i q_j,$$

where λ_{ij} depends only on $i - j$.

   (a) Show that the model is translationally invariant.

   (b) Find the normal modes.

2. Consider the case where the index i runs over a two-dimensional lattice and λ_{ij} is non-zero only for nearest neighbors and is independent of location

$$\lambda_{ij} = \delta_{j,j+1}\lambda.$$

   (a) Find the normal modes.

   (b) Find the dispersion relation.

Solution. 1. (a) The model is translationally invariant if the map

$$T : \begin{pmatrix} x_i \\ p_i \end{pmatrix} \to \begin{pmatrix} x_{i+1} \\ p_{i+1} \end{pmatrix}$$

yields the same Hamiltonian. We have

$$\begin{aligned} T(\mathcal{H}) &= T\left(\sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} \lambda_{ij} q_i q_j\right) \\ &= \sum_i \frac{p_{i+1}^2}{2m} + \frac{1}{2}\sum_{i,j} \lambda_{i+1,j+1} q_{i+1} q_{j+1} \\ &= \sum_i \frac{p_i^2}{2m} + \frac{1}{2}\sum_{i,j} \lambda_{i,j} q_i q_j, \end{aligned}$$

where $\lambda_{i+1,j+1}$ is only dependent on $(i+1) - (j+1) = i - j$.

Thus, our model is translationally invariant.

(b) Consider the Fourier modes

$$q_i = \frac{1}{\sqrt{N}} \sum_k \mathrm{e}^{\mathrm{i}kx_k} q_k \qquad \Longleftrightarrow \qquad q_k = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}kx_i} q_i$$

$$p_i = \frac{1}{\sqrt{N}} \sum_k \mathrm{e}^{\mathrm{i}kx_k} p_k \qquad \Longleftrightarrow \qquad p_k = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathrm{e}^{-\mathrm{i}kx_i} p_i$$

We will find the Fourier modes of the momentum part first

$$
\begin{aligned}
\sum_i \frac{p_i^2}{2m} &= \frac{1}{2m}\sum_{i=1}^N \left(\frac{1}{\sqrt{N}}\sum_k \mathrm{e}^{\mathrm{i}kx_i}p_k\right)\left(\frac{1}{\sqrt{N}}\sum_{k'} \mathrm{e}^{\mathrm{i}k'x_i}p_{k'}\right) \\
&= \frac{1}{2mN}\sum_{i=1}^N\sum_k\sum_{k'} \mathrm{e}^{\mathrm{i}(k+k')x_i}p_kp_{k'} \\
&= \frac{N}{2mN}\sum_k\sum_{k'}\delta(k+k')p_kp_{k'} \\
&= \frac{1}{2m}\sum_k p_kp_{-k}.
\end{aligned}
$$

Now, we will find the Fourier modes of the position part. We have

$$
\begin{aligned}
\frac{1}{2}\sum_{i,j}\lambda_{ij}q_iq_j &= \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N\lambda_{ij}q_iq_j \\
&= \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N\lambda_{ij}\left(\frac{1}{\sqrt{N}}\sum_k \mathrm{e}^{\mathrm{i}kx_i}q_k\right)\left(\frac{1}{\sqrt{N}}\sum_{k'} \mathrm{e}^{\mathrm{i}k'x_j}q_{k'}\right) \\
&= \frac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N\sum_k\sum_{k'}\lambda_{ij}\mathrm{e}^{\mathrm{i}kx_i}\mathrm{e}^{\mathrm{i}k'x_j}q_kq_{k'} \\
&= \frac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N\sum_k\sum_{k'}\lambda_{ij}\mathrm{e}^{\mathrm{i}kx_i}\mathrm{e}^{\mathrm{i}k'x_i}\mathrm{e}^{-\mathrm{i}k'x_i}\mathrm{e}^{\mathrm{i}k'x_j}q_kq_{k'} \\
&= \frac{1}{2N}\sum_{i=1}^N\sum_k\sum_{k'}\mathrm{e}^{\mathrm{i}(k+k')x_i}q_kq_{k'}\sum_{j=1}^N\lambda(j-i)\mathrm{e}^{\mathrm{i}k'(x_j-x_i)} \\
&= \frac{1}{2N}\sum_{i=1}^N\sum_k\sum_{k'}\mathrm{e}^{\mathrm{i}(k+k')x_i}q_kq_{k'}\tilde{\lambda}(-k') \\
&= \frac{N}{2N}\sum_k\sum_{k'}\delta(k+k')q_kq_{k'}\tilde{\lambda}(-k') \\
&= \frac{1}{2}\sum_k\tilde{\lambda}(k)q_kq_{-k}.
\end{aligned}
$$

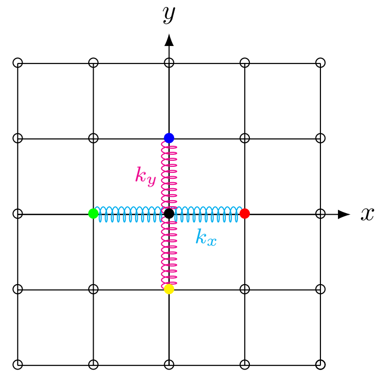
As a worded explanation of what happened: we Fourier transformed our coordinates to momentum space, inserted an identity $1 = \mathrm{e}^{\mathrm{i}k'x_i}\mathrm{e}^{-\mathrm{i}k'x_i}$, factored x_i's in the exponents in one exponential and took the difference of x_j and x_i in another. Then, the term $\sum_{j=1}^N\lambda(j-i)\mathrm{e}^{\mathrm{i}k'(x_j-x_i)}$ is the Fourier transform of the λ_{ij} we started with. Finally, the Fourier expansion of a δ-function is defined by $\sum_{i=1}^N\mathrm{e}^{\mathrm{i}(k+k')x_i} = N\delta(k+k')$, which leads us to the final result.
Finally, our Fourier transformed Hamiltonian is given by

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2m}\sum_k p_kp_{-k} + \frac{1}{2}\sum_k\tilde{\lambda}(k)q_kq_{-k} \\
&= \frac{1}{2m}\sum_k p_kp_{-k} + \frac{m}{2}\sum_k\omega_k^2q_kq_{-k},
\end{aligned}
$$

where

$$
\omega_k^2 = \frac{1}{m}\tilde{\lambda}(k) = \frac{1}{m}\sum_{j=1}^N\lambda(j-i)\mathrm{e}^{-\mathrm{i}k(x_j-x_i)}.
$$

2. (a) The index i running over a two-dimensional lattice is the same as two indices i and j running each on a one-dimensional line. We have that $\lambda_{i,j}$ is non-zero only for nearest neighbor interactions, which means it is interacting either with the particle to its right, to its left, to the top, or to the bottom, and not multiple at the same time. The figure below should help illustrate that.



From these facts, we can simplify our Hamiltonian to

$$
\begin{aligned}
\mathcal{H} &= \sum_{i,j} \frac{p_{i,j}^2}{2m} + \frac{1}{2} \sum_{i,j,k,\ell} \lambda_{k,\ell} q_{i,j} q_{k,\ell} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{p_{i,j}^2}{2m} + \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} \sum_{k=i-1}^{i+1} \sum_{\ell=j-1}^{j+1} \lambda_{k,\ell} q_{i,j} q_{k,\ell} \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{p_{i,j}^2}{2m} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda \left(q_{i,j} q_{i+1,j} + q_{i,j} q_{i-1,j} + q_{i,j} q_{i,j+1} + q_{i,j} q_{i,j-1} \right),
\end{aligned}
$$

where $\lambda_{k,\ell}$ refers to the spring constant **between** $q_{i,j}$ and $q_{k,\ell}$, since we will always center our point at (i,j), so we relieve ourselves from this redundancy in subscript notation. Additionally, we note that when $i=k$ and $j=\ell$, then $\lambda_{i,j}=0$ since that refers to the constant between a point and itself, and that $\lambda_{i\pm1,j\pm1}=0$.

Consider the Fourier modes

$$
\begin{aligned}
q_{i,j} &= \frac{1}{N} \sum_{k_x} \sum_{k_y} \mathrm{e}^{\mathrm{i}(k_x x_i + k_y y_j)} q_{k_x,k_y} \qquad \Longleftrightarrow \qquad q_{k_x,k_y} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}(k_x x_i + k_y y_j)} q_{i,j} \\
p_{i,j} &= \frac{1}{N} \sum_{k_x} \sum_{k_y} \mathrm{e}^{\mathrm{i}(k_x x_i + k_y y_j)} p_{k_x,k_y} \qquad \Longleftrightarrow \qquad p_{k_x,k_y} = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}(k_x x_i + k_y y_j)} p_{i,j}
\end{aligned}
$$

where we assume that $k_x = k_y$, but we will keep the subscripts there for now.

We now apply the Fourier modes to $p_{i,j}p_{m,n}$, getting

$$\begin{aligned}
\sum_{i,j}\frac{p_{i,j}^2}{2m} &= \frac{1}{2m}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\frac{1}{N}\sum_{k_x}\sum_{k_y}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}p_{k_x,k_y}\right)\left(\frac{1}{N}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_x'x_i+k_y'y_j)}p_{k_x',k_y'}\right)\\
&= \frac{1}{2mN^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i+k_y'y_j)}p_{k_x,k_y}p_{k_x',k_y'}\\
&= \frac{1}{2mN^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_x+k_x')x_i}\mathrm{e}^{\mathrm{i}(k_y+k_y')y_j}p_{k_x,k_y}p_{k_x',k_y'}\\
&= \frac{1}{2mN^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\left(\sum_{i=1}^{N}\mathrm{e}^{\mathrm{i}(k_x+k_x')x_i}\right)\left(\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}(k_y+k_y')y_j}\right)p_{k_x,k_y}p_{k_x',k_y'}\\
&= \frac{1}{2mN^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\left(N\delta(k_x+k_x')\right)\left(N\delta(k_y+k_y')\right)p_{k_x,k_y}p_{k_x',k_y'}\\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y}p_{k_x,k_y}p_{-k_x,-k_y}.
\end{aligned}$$

After applying the Fourier modes to $q_{i,j}q_{m,n}$, we get

$$\begin{aligned}
q_{i,j}q_{m,n} &= \left(\frac{1}{N}\sum_{k_x}\sum_{k_y}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}q_{k_x,k_y}\right)\left(\frac{1}{N}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_x'x_i'+k_y'y_j')}q_{k_x',k_y'}\right)\\
&= \frac{1}{N^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i'+k_y'y_j')}q_{k_x,k_y}q_{k_x',k_y'}.
\end{aligned}$$

We will now insert an identity, defined by

$$\mathrm{e}^{\mathrm{i}(k_x'x_i+k_y'y_j)}\mathrm{e}^{-\mathrm{i}(k_x'x_i+k_y'y_j)} = 1.$$

$$\begin{aligned}
q_{i,j}q_{m,n} &= \frac{1}{N^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i'+k_y'y_j')}q_{k_x,k_y}q_{k_x',k_y'}\\
&= \frac{1}{N^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}(k_xx_i+k_yy_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i+k_y'y_j)}\mathrm{e}^{-\mathrm{i}(k_x'x_i+k_y'y_j)}\mathrm{e}^{\mathrm{i}(k_x'x_i'+k_y'y_j')}q_{k_x,k_y}q_{k_x',k_y'}\\
&= \frac{1}{N^2}\sum_{k_x}\sum_{k_y}\sum_{k_x'}\sum_{k_y'}\mathrm{e}^{\mathrm{i}\left[(k_x+k_x')x_i+(k_y+k_y')y_j\right]}\mathrm{e}^{\mathrm{i}(k_x'(x_i'-x_i)+k_y'(y_j'-y_j))}q_{k_x,k_y}q_{k_x',k_y'}.
\end{aligned}$$

We can notice that, if and when this is done for the entire Hamiltonian, the addition summations over i and j will make the first two exponential terms turn in δ-functions for k_x and k_y, which will end up giving us $k_x' = -k_x$ and $k_y' = -k_y$.

For the second exponential term, the x_i' will, in reality, be either x_{i+1} pr x_{i-1}. The points (i,j) and $(i\pm1,j)$ or $(i,j\pm1)$ are at a distance of ϵ away from each other. This means, for $q_{i,j}q_{i\pm1,j}$, we can write $x_{i\pm1} = x_i \pm \epsilon$, and for $q_{i,j}q_{i,j\pm1}$, we can write $y_{j\pm1} = y_j \pm \epsilon$.

After doing that to all the pairs of points we have, we get

$$\begin{aligned}
H &= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} \\
&\quad + \frac{\lambda}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_x}\sum_{k_y}\sum_{k'_x}\sum_{k'_y} \mathrm{e}^{\mathrm{i}\left[(k_x+k'_x)x_i+(k_y+k'_y)y_j\right]}\Big[\mathrm{e}^{\mathrm{i}\left[k'_x(x_{i+1}-x_i)+k'_y(y_j-y_j)\right]}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad + \mathrm{e}^{\mathrm{i}\left[k'_x(x_{i-1}-x_i)+k'_y(y_j-y_j)\right]}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad + \mathrm{e}^{\mathrm{i}\left[k'_x(x_i-x_i)+k'_y(y_{j+1}-y_j)\right]}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad + \mathrm{e}^{\mathrm{i}\left[k'_x(x_i-x_i)+k'_y(y_{j-1}-y_j)\right]}q_{k_x,k_y}q_{k'_x,k'_y}\Big] \\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} \\
&\quad + \frac{\lambda}{2N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k_x}\sum_{k_y}\sum_{k'_x}\sum_{k'_y} \mathrm{e}^{\mathrm{i}\left[(k_x+k'_x)x_i+(k_y+k'_y)y_j\right]}\Big[\mathrm{e}^{\mathrm{i}k'_x\epsilon}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad + \mathrm{e}^{-\mathrm{i}k'_x\epsilon}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad + \mathrm{e}^{\mathrm{i}k'_y\epsilon}q_{k_x,k_y}q_{k'_x,k'_y} \\
&\qquad + \mathrm{e}^{-\mathrm{i}k'_y\epsilon}q_{k_x,k_y}q_{k'_x,k'_y}\Big] \\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} \\
&\quad + \frac{\lambda}{2N^2}\sum_{k_x}\sum_{k_y}\sum_{k'_x}\sum_{k'_y}\left(\sum_{i=1}^{N}\mathrm{e}^{\mathrm{i}(k_x+k'_x)x_i}\right)\left(\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}(k_y+k'_y)y_j}\right) \\
&\qquad \times q_{k_x,k_y}q_{k'_x,k'_y}\left[\mathrm{e}^{\mathrm{i}k'_x\epsilon}+\mathrm{e}^{-\mathrm{i}k'_x\epsilon}+\mathrm{e}^{\mathrm{i}k'_y\epsilon}+\mathrm{e}^{-\mathrm{i}k'_y\epsilon}\right] \\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} \\
&\quad + \frac{\lambda}{2N^2}\sum_{k_x}\sum_{k_y}\sum_{k'_x}\sum_{k'_y}\left(N\delta(k_x+k'_x)\right)\left(N\delta(k_y+k'_y)\right)q_{k_x,k_y}q_{k'_x,k'_y}\left[2\cos(k'_x\epsilon)+2\cos(k'_y\epsilon)\right] \\
&= \frac{1}{2m}\sum_{k_x}\sum_{k_y} p_{k_x,k_y}p_{-k_x,-k_y} + \frac{\lambda}{2}\sum_{k_x}\sum_{k_y} q_{k_x,k_y}q_{-k_x,-k_y}\left[2\cos(k_x\epsilon)+2\cos(k_y\epsilon)\right] \\
&= \sum_{k_x}\sum_{k_y}\left[\frac{1}{2m}p_{k_x,k_y}p_{-k_x,-k_y} + \frac{1}{2}2\lambda\left(\cos(k_x\epsilon)+\cos(k_y\epsilon)\right)q_{k_x,k_y}q_{-k_x,-k_y}\right] \\
&= \sum_{k_x}\sum_{k_y}\left[\frac{1}{2m}p_{k_x,k_y}p_{-k_x,-k_y} + \frac{1}{2}m\omega_k^2 q_{k_x,k_y}q_{-k_x,-k_y}\right]
\end{aligned}$$

From this, we can say that

$$\omega_k^2 = \frac{2\lambda}{m}\left(\cos(k_x\epsilon)+\cos(k_y\epsilon)\right) = \omega_{k_x}^2 + \omega_{k_y}^2,$$

where

$$\omega_{k_x}^2 = \frac{2\lambda}{m}\cos(k_x\epsilon) \quad \text{and} \quad \omega_{k_y}^2 = \frac{2\lambda}{m}\cos(k_y\epsilon).$$

(b) To find the dispersion relation, we must plug in the normal modes in the equation of motion of the system. To do that, we first have to find the Hamilton-Jacobi equations by

$$\begin{aligned}\dot{q}_k &= \{q_k, \mathcal{H}\},\\ \dot{p}_k &= \{p_k, \mathcal{H}\}.\end{aligned}$$

The Poisson bracket is given by

$$\{f, g\} = \sum_{\ell=1}^{N} \left(\frac{\partial f}{\partial q_\ell}\frac{\partial g}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell}\frac{\partial g}{\partial q_\ell} \right).$$

Finding $\dot{q}_k$: We have

$$\begin{aligned}\dot{q}_k &= \{q_k, \mathcal{H}\}\\ &= \sum_{\ell=1}^{N} \left(\frac{\partial q_k}{\partial q_\ell}\frac{\partial \mathcal{H}}{\partial p_\ell} - \frac{\partial q_k}{\partial p_\ell}\frac{\partial \mathcal{H}}{\partial q_\ell} \right)\\ &= \sum_{\ell=1}^{N} \left(\delta_{k\ell}\frac{p_{-\ell}}{m} - 0 \right)\\ &= \frac{p_{-k}}{m}.\end{aligned}$$

Finding $\dot{p}_k$: We have

$$\begin{aligned}\dot{p}_k &= \{p_k, \mathcal{H}\}\\ &= \sum_{\ell=1}^{N} \left(\frac{\partial p_k}{\partial q_\ell}\frac{\partial \mathcal{H}}{\partial p_\ell} - \frac{\partial p_k}{\partial p_\ell}\frac{\partial \mathcal{H}}{\partial q_\ell} \right)\\ &= \sum_{\ell=1}^{N} (0 - \delta_{k\ell}\lambda(\cos(k_x\epsilon) + \cos(k_y\epsilon))q_{-\ell})\\ &= -\lambda(\cos(k_x\epsilon) + \cos(k_y\epsilon))q_{-k}.\end{aligned}$$

Now that we have these expressions, we can use them to reach a second-order differential equation and solve it. From the equation of $\dot{p}_k$, we have

$$\dot{p}_{-k} = -\lambda(\cos(k_x\epsilon) + \cos(k_y\epsilon))q_k,$$

and then, using a rearrangement of the equation for $\dot{q}_k$, we have

$$\ddot{q}_k = -\frac{k}{m}(\cos(k_x\epsilon) + \cos(k_y\epsilon))q_k.$$

Plugging in a solution of the form

$$\begin{aligned} q_k &= \sum_\omega \mathrm{e}^{-\mathrm{i}\omega t} q_{k,\omega}\\ \implies \dot{q}_k &= \sum_\omega (-\mathrm{i}\omega)\mathrm{e}^{-\mathrm{i}\omega t} q_{k,\omega}\\ \implies \ddot{q}_k &= -\sum_\omega \omega^2 \mathrm{e}^{-\mathrm{i}\omega t} q_{k,\omega}\\ &= -\sum_\omega \frac{\lambda}{m}(\cos(k_x\epsilon) + \cos(k_y\epsilon))\mathrm{e}^{-\mathrm{i}\omega t} q_{k,\omega}.\end{aligned}$$

Therefore, the dispersion relation is given by

$$\omega^2 - \frac{\lambda}{m}(\cos(k_x\epsilon) + \cos(k_y\epsilon)) = 0.$$

■

Problem 2 - One-dimensional Heisenberg Model
Consider a one-dimensional array of ferromagnets with $\vec{S}_i$ a vector degree of freedom on each site. Take the Hamiltonian

$$\mathcal{H} = -J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$

with $J > 0$. The operators $\vec{S}_i$ satisfy the $SU(2)$ algebra

$$\left[\left(\vec{S}_i\right)_\alpha, \left(\vec{S}_j\right)_\beta\right] = \mathrm{i}\delta_{i,j}\epsilon^{\alpha\beta\gamma}\left(\vec{S}_j\right)^\gamma.$$

(a) Write down the classical equations of motion for $\vec{S}_i$ variables as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{S}_i = \cdots .$$

(b) Treat spins as classical variables and take the continuum limit to find the equations of motion of the low energy field.

Solution. (a) Using the Hamilton-Jacobi equation, we have

$$\begin{aligned}
\dot{\vec{S}}_i &= \{\vec{S}_i, \mathcal{H}\} \\
&= -\frac{\mathrm{i}}{\hbar}\left[\vec{S}_i, \mathcal{H}\right] \\
&= -\frac{\mathrm{i}}{\hbar}\left[\vec{S}_i, -J\sum_j \vec{S}_j \cdot \vec{S}_{j+1}\right] \\
&= J\frac{\mathrm{i}}{\hbar}\left[\vec{S}_i, \sum_j \vec{S}_j \cdot \vec{S}_{j+1}\right].
\end{aligned}$$

To break this down component-wise, we can write

$$\begin{aligned}
\left(\dot{\vec{S}}_i\right)_\alpha &= J\frac{\mathrm{i}}{\hbar}\left[\left(\vec{S}_i\right)_\alpha, \sum_j\sum_\beta \left(\vec{S}_j\right)_\beta\left(\vec{S}_{j+1}\right)_\beta\right] \\
&= J\frac{\mathrm{i}}{\hbar}\sum_j\sum_\beta\left[\left(\vec{S}_i\right)_\alpha, \left(\vec{S}_j\right)_\beta\left(\vec{S}_{j+1}\right)_\beta\right] \\
&= J\frac{\mathrm{i}}{\hbar}\sum_j\sum_\beta\left(\left(\vec{S}_j\right)_\beta\left[\left(\vec{S}_i\right)_\alpha, \left(\vec{S}_{j+1}\right)_\beta\right] + \left[\left(\vec{S}_i\right)_\alpha, \left(\vec{S}_j\right)_\beta\right]\left(\vec{S}_{j+1}\right)_\beta\right) \\
&= J\frac{\mathrm{i}}{\hbar}\sum_j\sum_\beta\sum_\gamma\left(\mathrm{i}\delta_{i,j+1}\epsilon^{\alpha\beta\gamma}\left(\vec{S}_j\right)_\beta\left(\vec{S}_{j+1}\right)_\gamma + \mathrm{i}\delta_{i,j}\epsilon^{\alpha\beta\gamma}\left(\vec{S}_j\right)_\gamma\left(\vec{S}_{j+1}\right)_\beta\right) \\
&= J\frac{\mathrm{i}}{\hbar}\sum_\beta\sum_\gamma \mathrm{i}\epsilon^{\alpha\beta\gamma}\left(\left(\vec{S}_{i-1}\right)_\beta\left(\vec{S}_i\right)_\gamma + \left(\vec{S}_i\right)_\gamma\left(\vec{S}_{i+1}\right)_\beta\right) \\
&= -\frac{J}{\hbar}\sum_\beta\sum_\gamma \epsilon^{\alpha\beta\gamma}\left(\left(\vec{S}_{i-1}\right)_\beta + \left(\vec{S}_{i+1}\right)_\beta\right)\left(\vec{S}_i\right)_\gamma \\
&= -\frac{J}{\hbar}\left[\left(\vec{S}_{i-1} + \vec{S}_{i+1}\right) \times \vec{S}_i\right]_\alpha.
\end{aligned}$$

Therefore,

$$\dot{\vec{S}}_i = \frac{J}{\hbar}\vec{S}_i \times \left(\vec{S}_{i+1} + \vec{S}_{i-1}\right).$$

(b) We introduce a lattice spacing a and a spin field $\vec{S}(x,t)$ at position x and time t such that $x = na$. We define

$$\delta(na,t) \equiv \delta_n(t).$$

Assume a small value of a and then we expand in a Taylor series around $x = na$. We have

$$\begin{aligned}
S_{n+1}(t) + S_{n-1}(t) &= S(x+a,t) + S(x-a,t) \\
&= \left(S(x,t) + \frac{\partial S(x,t)}{\partial x} a + \frac{\partial^2 S(x,t)}{\partial x^2} a^2 + \cdots \right) \\
&\quad + \left(S(x,t) + \frac{\partial S(x,t)}{\partial x} (-a) + \frac{\partial^2 S(x,t)}{\partial x^2} (-a)^2 + \cdots \right) \\
&= 2S(x,t) + \frac{\partial^2 S(x,t)}{\partial x^2} a^2 + \mathcal{O}(a^4).
\end{aligned}$$

We now replace the previous equation in the final result obtained in part (a), getting

$$\begin{aligned}
\dot{\vec{S}}_i &= \frac{J}{\hbar} \vec{S}_i \times \left(\vec{S}_{i+1} + \vec{S}_{i-1} \right) \\
&= \frac{J}{\hbar} \vec{S}_i \times \left(2\vec{S}_i + \frac{\partial^2 \vec{S}_i}{\partial x^2} a^2 \right) \\
&= \frac{J}{\hbar} \vec{S}_i \times \frac{\partial^2 \vec{S}_i}{\partial x^2} a^2.
\end{aligned}$$

The continuum limit is then achieved by taking $a \to 0$.

■