PHYS 601 - Methods of Theoretical Physics II

Mathematical Methods for Physicists by *Arfken, Weber, Harris*

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Homework 3

Problem 1

For each differential equation below, find all the singularities (including those at infinity) and state whether each is regular or irregular.

NAME	EXPRESSION
Hypergeometric	$x(x-1)y'' + [(1+a+b)x - c]\,y' + aby = 0$
Legendre	$(1-x^2)y'' - 2xy' + \ell(\ell+1)y = 0$
Chebyshev	$(1-x^2)y'' - xy' + n^2y = 0$
Confluent Hypergeometric	$xy'' + (c-x)y' - ay = 0$
Laguerre	$xy'' + (1-x)y' + ay = 0$
Bessel	$x^2y'' + xy' + (x^2-n^2)y = 0$
Simple Harmonic Oscillator	$y'' + \omega^2 y = 0$
Hermite	$y'' - 2xy' + 2\alpha y = 0$

Solution. Consider the general form of a second-order ordinary differential equation

$$y'' + P(x)y' + Q(x)y = 0.$$

An ordinary differential equation is said to have singularities when the highest order term has zeroes or when the lower order terms have poles.

- A point x_0 is said to be an **ordinary point** if $P(x)$ and $Q(x)$ are analytic at $x = x_0$.
- A point x_0 is said to be a **singular point** if $P(x)$ and $Q(x)$ are not analytic at $x = x_0$.
  - A point x_0 is said to be a **regular singular point** if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are analytic at x_0.
  - Otherwise, x_0 is said to be an **irregular singular point**.

There remains one region of interest, which is as $x \to \infty$. To study the ODE at infinity, we make a variable change of $z = \frac{1}{x}$. Now, we study what happens at $z = 0$. Under such a transformation, the transformed forms of $P(x)$ and $Q(x)$ will have to be studied using the analysis above. Accordingly, we have

$$\tilde{P}(z) = \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \quad \text{and} \quad \tilde{Q}(z) = \frac{Q\left(\frac{1}{z}\right)}{z^4}$$

- **Hypergeometric:**

$$x(x-1)y'' + [(1+a+b)x - c]\,y' + aby = 0$$

The above ODE reduces to

$$y'' + \frac{(1+a+b)x - c}{x(x-1)}y' + \frac{ab}{x(x-1)}y = 0,$$

where

$$P(x) = \frac{(1+a+b)x - c}{x(x-1)} \qquad \text{and} \qquad Q(x) = \frac{ab}{x(x-1)}.$$

The hypergeometric ODE has interesting points at $x_0 = 0, 1, \infty$.

For $x_0 = 0$**:**

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.
- The quantity $(x-x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0} \\ &= \left.\frac{(1+a+b)x-c}{x-1}\right|_{x=0} \\ &= c \sim \text{finite.}\end{aligned}$$

- The quantity $(x-x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= x^2Q(x)\big|_{x=0} \\ &= \left.\frac{abx}{x-1}\right|_{x=0} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 = 1$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.
- The quantity $(x-x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1} \\ &= \left.\frac{(1+a+b)x-c}{x}\right|_{x=1} \\ &= 1+a+b-c \sim \text{finite.}\end{aligned}$$

- The quantity $(x-x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= (x-1)^2Q(x)\big|_{x=1} \\ &= \left.\frac{ab(x-1)}{x}\right|_{x=1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{\frac{(1+a+b)\frac{1}{z}-c}{\frac{1}{z}\left(\frac{1}{z}-1\right)}}{z^2} \\ &= \frac{2}{z} - \frac{1+a+b-cz}{z\left(1-z\right)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{ab}{\frac{1}{z}\left(\frac{1}{z}-1\right)}}{z^4} \\ &= \frac{ab}{z^2\left(1-z\right)}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned} (z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - \frac{1 + a + b - cz}{1 - z}\Big|_{z=0} \\ &= 1 - a - b \sim \text{finite.} \end{aligned}$$

- The quantity $(z - z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned} (z - z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{ab}{(1 - z)}\Big|_{z=0} \\ &= ab \sim \text{finite.} \end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Legendre:**

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0$$

The above ODE reduces to

$$y'' - \frac{2x}{1 - x^2}y' + \frac{\ell(\ell + 1)}{1 - x^2}y = 0,$$

where

$$P(x) = -\frac{2x}{1 - x^2} = -\frac{2x}{(1 - x)(1 + x)} \qquad \text{and} \qquad Q(x) = \frac{\ell(\ell + 1)}{1 - x^2} = \frac{\ell(\ell + 1)}{(1 - x)(1 + x)}.$$

The Legendre ODE has interesting points at $x_0 = \pm 1, \infty$.

For $x_0 = -1$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned} (x - x_0)P(x)\big|_{x_0} &= (x + 1)P(x)\big|_{x=-1} \\ &= -\frac{2x}{1 - x}\Big|_{x=-1} \\ &= 1 \sim \text{finite.} \end{aligned}$$

- The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned} (x - x_0)^2Q(x)\big|_{x_0} &= (x + 1)^2Q(x)\big|_{x=-1} \\ &= \frac{\ell(\ell + 1)(x + 1)}{1 - x}\Big|_{x=-1} \\ &= 0 \sim \text{finite.} \end{aligned}$$

Thus, $x_0 = -1$ is a regular singular point.

For $x_0 = 1$:

– The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.

– The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1} \\ &= -\frac{2x}{1+x}\bigg|_{x=1} \\ &= -1 \sim \text{finite.}\end{aligned}$$

– The quantity $(x - x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2Q(x)\big|_{x_0} &= (x-1)^2Q(x)\big|_{x=1} \\ &= \frac{\ell(\ell+1)(x-1)}{1+x}\bigg|_{x=1} \\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-\frac{2\left(\frac{1}{z}\right)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^2} \\ &= \frac{2}{z} + \frac{2}{z(z-1)(z+1)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{\ell(\ell+1)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^4} \\ &= \frac{\ell(\ell+1)}{z^2(z-1)(z+1)}.\end{aligned}$$

– The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.

– The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{2}{(z-1)(z+1)}\bigg|_{z=0} \\ & 0 \sim \text{finite.}\end{aligned}$$

– The quantity $(z - z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z - z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{\ell(\ell+1)}{(z-1)(z+1)}\bigg|_{z=0} \\ &= -\ell(\ell+1) \sim \text{finite.}\end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Chebyshev:**

$$(1-x^2)y'' - xy' + n^2 y = 0$$

The above ODE reduces to

$$y'' - \frac{x}{1-x^2}y' + \frac{n^2}{1-x^2}y = 0,$$

where

$$P(x) = -\frac{x}{1-x^2} = -\frac{x}{(1-x)(1+x)} \qquad \text{and} \qquad Q(x) = \frac{n^2}{1-x^2} = \frac{n^2}{(1-x)(1+x)}.$$

The Chebyshev ODE has interesting points at $x_0 = \pm 1, \infty$.

For $x_0 = -1$**:**

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned} (x-x_0)P(x)\big|_{x_0} &= (x+1)P(x)\big|_{x=-1} \\ &= -\frac{x}{1-x}\bigg|_{x=-1} \\ &= \frac{1}{2} \sim \text{finite.} \end{aligned}$$

- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is

$$\begin{aligned} (x-x_0)^2 Q(x)\big|_{x_0} &= (x+1)^2 Q(x)\big|_{x=-1} \\ &= \frac{(x+1)n^2}{1-x}\bigg|_{x=-1} \\ &= 0 \sim \text{finite.} \end{aligned}$$

Thus, $x_0 = -1$ is a regular singular point.

For $x_0 = 1$**:**

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned} (x-x_0)P(x)\big|_{x_0} &= (x-1)P(x)\big|_{x=1} \\ &= -\frac{x}{1+x}\bigg|_{x=1} \\ &= -\frac{1}{2} \sim \text{finite.} \end{aligned}$$

- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is

$$\begin{aligned} (x-x_0)^2 Q(x)\big|_{x_0} &= (x-1)^2 Q(x)\big|_{x=1} \\ &= \frac{(x-1)n^2}{1+x}\bigg|_{x=1} \\ &= 0 \sim \text{finite.} \end{aligned}$$

Thus, $x_0 = 1$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-\frac{\left(\frac{1}{z}\right)}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^2} \\ &= \frac{2}{z} + \frac{1}{z(z-1)(z+1)},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{n^2}{\left(1-\frac{1}{z}\right)\left(1+\frac{1}{z}\right)}}{z^4} \\ &= \frac{n^2}{z^2(z-1)(z+1)}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{1}{(z-1)(z+1)}\Big|_{z=0} \\ &= 1 \sim \text{finite}.\end{aligned}$$

- The quantity $(z - z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{n^2}{(z-1)(z+1)}\Big|_{z=0} \\ &= -n^2 \sim \text{finite}.\end{aligned}$$

Hence, $z_0 = 0$ is a regular singular point, and thus, $x_0 \to \infty$ is a regular singular point.

- **Confluent Hypergeometric:**

$$xy'' + (c - x)y' - ay = 0$$

The above ODE reduces to

$$y'' + \frac{c-x}{x}y' - \frac{a}{x}y = 0,$$

where

$$P(x) = \frac{c-x}{x} \qquad \text{and} \qquad Q(x) = -\frac{a}{x}.$$

The Confluent Hypergeometric ODE has interesting points at $x_0 = 0, \infty$.
For $x_0 = 0$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.

- The quantity $(x-x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)|_{x_0} &= xP(x)|_{x=0}\\ &= c-x|_{x=0}\\ &= c \sim \text{finite.}\end{aligned}$$

- The quantity $(x-x_0)^2Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= x^2Q(x)\big|_{x=0}\\ &= -ax|_{x=0}\\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2}\\ &= \frac{2}{z} - \frac{\frac{c-\frac{1}{z}}{\frac{1}{z}}}{z^2}\\ &= \frac{2}{z} - \frac{cz-1}{z^2},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4}\\ &= \frac{-\frac{a}{\frac{1}{z}}}{z^4}\\ &= -\frac{a}{z^3}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z-z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0}\\ &= 2 - \frac{cz-1}{z}\Bigg|_{z=0}\\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

- **Laguerre:**

$$xy'' + (1-x)y' + ay = 0$$

The above ODE reduces to

$$y'' + \frac{1-x}{x}y' + \frac{a}{x}y = 0,$$

where

$$P(x) = \frac{1-x}{x} \qquad \text{and} \qquad Q(x) = \frac{a}{x}.$$

The Laguerre ODE has interesting points at $x_0 = 0, \infty$.
For $x_0 = 0$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = -1$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0}\\ &= 1 - x\big|_{x=0}\\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x-x_0)^2Q(x)\big|_{x_0} &= x^2Q(x)\big|_{x=0}\\ &= ax\big|_{x=0}\\ &= 0 \sim \text{finite.}\end{aligned}$$

Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:
Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2}\\ &= \frac{2}{z} - \frac{\frac{1-\frac{1}{z}}{\frac{1}{z}}}{z^2}\\ &= \frac{2}{z} - \frac{z-1}{z^2},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4}\\ &= \frac{\frac{a}{\frac{1}{z}}}{z^4}\\ &= \frac{a}{z^3}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0}\\ &= 2 - \frac{z-1}{z}\Big|_{z=0}\\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

- **Bessel:**

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

The above ODE reduces to

$$y'' + \frac{1}{x}y' + \frac{x^2-n^2}{x^2}y = 0,$$

where

$$P(x) = \frac{1}{x} \qquad \text{and} \qquad Q(x) = \frac{x^2 - n^2}{x^2}.$$

The Bessel ODE has interesting points at $x_0 = 0, \infty$.

For $x_0 = 0$:

- The analyticity of $P(x)$ and $Q(x)$, evaluated at x_0, is not satisfied. Hence, $x_0 = 0$ is a singular point.
- The quantity $(x - x_0)P(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)P(x)\big|_{x_0} &= xP(x)\big|_{x=0} \\ &= 1\big|_{x=0} \\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(x - x_0)^2 Q(x)$, evaluated at x_0, is

$$\begin{aligned}(x - x_0)^2 Q(x)\big|_{x_0} &= x^2 Q(x)\big|_{x=0} \\ &= x^2 - n^2\big|_{x=0} \\ &= -n^2 \sim \text{finite.}\end{aligned}$$

  Thus, $x_0 = 0$ is a regular singular point.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{z}{z^2} \\ &= \frac{2}{z} - \frac{1}{z},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\frac{\left(\frac{1}{z}\right)^2 - zn^2}{\left(\frac{1}{z}\right)^2}}{z^4} \\ &= \frac{1 - n^2 z^2}{z^4}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 - 1\big|_{z=0} \\ &= 1 \sim \text{finite.}\end{aligned}$$

- The quantity $(z-z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{1-n^2z^2}{z^2}\Big|_{z=0} \\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

Simple Harmonic Oscillator:

$$y'' + \omega^2 y = 0$$

The above ODE is already in a reduced for where

$$P(x) = 0 \qquad \text{and} \qquad Q(x) = \omega^2.$$

The simple harmonic oscillator ODE has interesting points at $x_0 = \infty$.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{\omega^2}{z^4}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z-z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z-z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2\big|_{z=0} \\ &= 2 \sim \text{finite}.\end{aligned}$$

- The quantity $(z-z_0)^2\tilde{Q}(z)$, evaluated at x_0, is

$$\begin{aligned}(z-z_0)^2\tilde{Q}(z)\Big|_{z_0} &= z^2\tilde{Q}(z)\Big|_{z=0} \\ &= \frac{\omega^2}{z^2}\Big|_{z=0} \\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

Hermite:

$$y'' - 2xy' + 2\alpha y = 0$$

The above ODE is already in a reduced for where

$$P(x) = -2x \qquad \text{and} \qquad Q(x) = 2\alpha.$$

The Hermite ODE has interesting points at $x_0 = \infty$.

For $x_0 \to \infty$:

Letting $z = \frac{1}{x}$, then we study $z_0 = 0$. Accordingly, we have

$$\begin{aligned}\tilde{P}(z) &= \frac{2}{z} - \frac{P\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} - \frac{-2\left(\frac{1}{z}\right)}{z^2} \\ &= \frac{2}{z} + \frac{2}{z^3},\end{aligned}$$

and

$$\begin{aligned}\tilde{Q}(z) &= \frac{Q\left(\frac{1}{z}\right)}{z^4} \\ &= \frac{2\alpha}{z^4}.\end{aligned}$$

- The analyticity of $\tilde{P}(z)$ and $\tilde{Q}(z)$, evaluated at z_0, is not satisfied. Hence, $z_0 = 0$ is a singular point.
- The quantity $(z - z_0)\tilde{P}(z)$, evaluated at z_0, is

$$\begin{aligned}(z - z_0)\tilde{P}(z)\Big|_{z_0} &= z\tilde{P}(z)\Big|_{z=0} \\ &= 2 + \frac{2}{z^2}\Big|_{z=0} \\ &\to \infty.\end{aligned}$$

Hence, $z_0 = 0$ is an irregular singular point, and thus, $x_0 \to \infty$ is an irregular singular point.

To summarize, we have

NAME	SINGULARITIES
Hypergeometric	Regular at $x = 0, 1, \infty$
Legendre	Regular at $x = \pm 1, \infty$
Chebyshev	Regular at $x = \pm 1, \infty$
Confluent Hypergeometric	Regular at $x = 0$ and irregular at $x = \infty$
Laguerre	Regular at $x = 0$ and irregular at $x = \infty$
Bessel	Regular at $x = 0$ and irregular at $x = \infty$
Simple Harmonic Oscillator	Irregular at $x = \infty$
Hermite	Irregular at $x = \infty$

■

Problem 2
For some of the above equations, $q(x) = 0$ when expressed in Sturm-Liouiville form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right]0 - \left[q(x) - \lambda w(x)\right]y = 0.$$

When $\lambda = 0$ also, the Sturm-Liouiville equation has a solution $y(x)$ determined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p(x)},$$

(a) Show this.

(b) Use this result to produce a second solution [in addition to those given on the sheet distributed in class] to the Legendre, Laguerre, and Hermite equations.

Solution. (a) Given the Sturm-Liouiville form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right]0 - \left[q(x) - \lambda w(x)\right]y = 0,$$

where $q(x) = \lambda = 0$, then we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)y'\right] = 0 \qquad \Longrightarrow \qquad p(x)y' &= \text{constant} \\ \Longrightarrow \qquad y' &= \frac{\text{constant}}{p(x)} \\ &= \frac{1}{p(x)}, \end{aligned}$$

where the last step was done by absorbing the constant into $p(x)$.

(b)

- **Legendre:** The Legendre ODE is given by

$$(1 - x^2)y'' - 2xy' + \ell(\ell + 1)y = 0,$$

which can be rewritten in the Sturm-Liouiville form as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)y'\right] + \ell(\ell + 1)y = 0,$$

where $p(x) = (1 - x^2)$, $q(x) = 0$, and $\lambda = \ell(\ell + 1)$.
By setting $\ell = 0$ and using the property in part (a), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p(x)} = \frac{1}{1 - x^2}$$

$$y = \int \frac{1}{1 - x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) + C,$$

which is the second solution of the Legendre ODE.

- **Laguerre:** The Laguerre ODE is given by

$$xy'' + (1 - x)y' + ay = 0.$$

To rewrite this in the Sturm-Liouiville form, we need to use an integrating factor $I(x)$, which would have initially been in the equation but was cancelled for its non-zero value. Let

$$xIy'' + (1 - x)Iy' + aIy = 0$$

be our new ODE. We require that $\frac{\mathrm{d}}{\mathrm{d}x}(xI) = (1-x)I$, and so we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(xI) = I + x\frac{\mathrm{d}I}{\mathrm{d}x} = (1-x)I$$
$$\implies \frac{\mathrm{d}I}{\mathrm{d}x} = -I$$
$$\implies I = e^{-x}.$$

Thus, our Laguerre ODE becomes

$$xe^{-x}y'' + (1-x)e^{-x}y' + ae^{-x}y = 0.$$

Rewriting in the Sturm-Liouiville form, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[xe^{-x}y'\right] + ae^{-x}y = 0,$$

where $p(x) = xe^{-x}$, $q(x) = 0$, and ae^{-x}.
By setting $ae^{-x} = 0 \implies a = 0$, since $e^{-x} \neq 0$, and using the property in part (a), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p(x)} = \frac{1}{xe^{-x}}$$
$$y = \int \frac{1}{xe^{-x}}\,\mathrm{d}x = Ei(x) + C,$$

which is the second solution of the Laguerre ODE, where $Ei(x)$ is defined to be the exponential integral.

- **Hermite:** The Hermite ODE is given by

$$y'' - 2xy' + 2\alpha y = 0.$$

To rewrite this in the Sturm-Liouiville form, we need to use an integrating factor $I(x)$, which would have initially been in the equation but was cancelled for its non-zero value. Let

$$Iy'' - 2xIy' + 2\alpha Iy = 0$$

be our new ODE. We require that $\frac{\mathrm{d}}{\mathrm{d}x}(I) = -2xI$, and so we have

$$\frac{\mathrm{d}I}{\mathrm{d}x} = -2xI$$
$$\implies I = e^{-x^2}.$$

Thus, our Hermite ODE becomes

$$e^{-x^2}y'' - 2xe^{-x^2}y' + 2\alpha e^{-x^2}y = 0$$

Rewriting in the Sturm-Liouiville form, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-x^2}y'\right] + 2\alpha e^{-x^2}y = 0,$$

where $p(x) = e^{-x^2}$, $q(x) = 0$, and $2\alpha e^{-x^2}$.
By setting $2\alpha e^{-x^2} = 0 \implies \alpha = 0$, since $e^{-x^2} \neq 0$, and using the property in part (a), we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{p(x)} = \frac{1}{e^{-x^2}}$$
$$y = \int e^{x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}\,\mathrm{erfi}(x)}{2} + C,$$

which is the second solution of the Hermite ODE, where $\mathrm{erfi}(x)$ is defined to be the imaginary error function.

■

Homework 4

Problem 1
The first four Legendre polynomials are

$$P_0(x) = 1, \quad P_2 = \frac{1}{2}(3x^2 - 1),$$
$$P_1(x) = x, \quad P_3 = \frac{1}{2}(5x^3 - 3x).$$

Obtain these four polynomials by each of the following methods:

(a) Generating function,

(b) Rodrigues' formula,

(c) Schmidt orthogonalization,

(d) Series solution.

Solution. (a) The generating function of the Legendre polynomials is given by

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

The Taylor expansion of the left hand-side has the form of

$$g(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t)\bigg|_{t=0} t^n.$$

By comparison, we have

$$P_n(x) = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t)\bigg|_{t=0}.$$

- **For $n = 0$:**

$$\begin{aligned} P_0(x) &= \frac{1}{0!} \frac{\mathrm{d}^0}{\mathrm{d}t^0} g(x,t)\bigg|_{t=0} \\ &= g(x,t)|_{t=0} \\ &= \frac{1}{\sqrt{1-2xt+t^2}}\bigg|_{t=0} \\ &= 1. \end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned} P_1(x) &= \frac{1}{1!} \frac{\mathrm{d}^1}{\mathrm{d}t^1} g(x,t)\bigg|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} g(x,t)\bigg|_{t=0} \\ &= \frac{x-t}{(1-2tx+t^2)^{\frac{3}{2}}}\bigg|_{t=0} \\ &= x. \end{aligned}$$

- **For** $n = 2$**:**

$$\begin{aligned}
P_2(x) &= \frac{1}{2!}\,\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(x,t)\bigg|_{t=0} \\
&= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(x,t)\bigg|_{t=0} \\
&= \frac{1}{2}\frac{2t^2 - 4xt + 3x^2 - 1}{(1 - 2xt + t^2)^{\frac{5}{2}}}\bigg|_{t=0} \\
&= \frac{1}{2}(3x^2 - 1).
\end{aligned}$$

- **For** $n = 3$**:**

$$\begin{aligned}
P_3(x) &= \frac{1}{3!}\,\frac{\mathrm{d}^3}{\mathrm{d}t^3} g(x,t)\bigg|_{t=0} \\
&= \frac{1}{6}\frac{\mathrm{d}^3}{\mathrm{d}t^3} g(x,t)\bigg|_{t=0} \\
&= \frac{1}{6}\frac{3\,(x - t)\left(2t^2 - 4xt + 5x^2 - 3\right)}{(1 - 2xt + t^2)^{\frac{7}{2}}}\bigg|_{t=0} \\
&= \frac{1}{2}(5x^3 - 3x).
\end{aligned}$$

(b) The Rodrigues' formula for the Legendre polynomials is given by

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2 - 1)^n.$$

Calculating, we have

- **For** $n = 0$**:**

$$\begin{aligned}
P_0(x) &= \frac{1}{2^0 0!}\frac{\mathrm{d}^0}{\mathrm{d}x^0}(x^2 - 1)^0 \\
&= 1.
\end{aligned}$$

- **For** $n = 1$**:**

$$\begin{aligned}
P_1(x) &= \frac{1}{2^1 1!}\frac{\mathrm{d}^1}{\mathrm{d}x^1}(x^2 - 1)^1 \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 1) \\
&= x.
\end{aligned}$$

- **For** $n = 2$**:**

$$\begin{aligned}
P_2(x) &= \frac{1}{2^2 2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2 - 1)^2 \\
&= \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^4 - 2x^2 + 1) \\
&= \frac{1}{8}(12x^2 - 4) \\
&= \frac{1}{2}(3x^2 - 1).
\end{aligned}$$

- **For $n = 3$:**

$$\begin{aligned}
P_3(x) &= \frac{1}{2^3 3!}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^2-1)^3 \\
&= \frac{1}{48}\frac{\mathrm{d}^3}{\mathrm{d}x^3}(x^6 - 3x^4 + 3x^2 - 1) \\
&= \frac{1}{48}(120x^3 - 72x) \\
&= \frac{1}{2}(5x^3 - 3x).
\end{aligned}$$

(c) We define an inner product

$$\langle f|g\rangle = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x$$

and a set of functions

$$u_n(x) = x^n,$$

where n is a non-negative integer.

We now generate a set of orthonormal functions $\phi_n(x)$ using the Gram-Schmidt orthogonalization process. We have

- **For $n = 0$:**

$$\begin{aligned}
\phi_0(x) &= \frac{u_0(x)}{\sqrt{\langle u_0|u_0\rangle}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned}
\psi_1(x) &= u_1(x) - \langle \phi_0|u_1\rangle\,\phi_0(x) \\
&= x - \frac{1}{2}\int_{-1}^{1} x\,\mathrm{d}x \\
&= x.
\end{aligned}$$

Normalizing, we have

$$\phi_1(x) = \frac{\psi_1(x)}{\sqrt{\langle \psi_1|\psi_1\rangle}} = \frac{\psi_1(x)}{\sqrt{\int_{-1}^{1} x^2\,\mathrm{d}x}} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}x$$

- **For $n = 2$:**

$$\begin{aligned}
\psi_2(x) &= u_2(x) - \langle \phi_0|u_2\rangle\,\phi_0(x) - \langle \phi_1|u_2\rangle\,\phi_1(x) \\
&= x^2 - \frac{1}{2}\int_{-1}^{1} x^2\,\mathrm{d}x - \frac{3}{2}x\int_{-1}^{1} x^3\,\mathrm{d}x \\
&= x^2 - \frac{1}{3}.
\end{aligned}$$

Normalizing, we have

$$\phi_2(x) = \frac{\psi_2(x)}{\sqrt{\langle \psi_2|\psi_2\rangle}} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2\mathrm{d}x}} = \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1).$$

- **For $n = 3$:**

$$\begin{aligned}
\psi_3(x) &= u_3(x) - \langle \phi_0 | u_3 \rangle \, \phi_0(x) - \langle \phi_1 | u_3 \rangle \, \phi_1(x) - \langle \phi_2 | u_3 \rangle \, \phi_2(x) \\
&= x^3 - \frac{1}{2} \int_{-1}^{1} x^3 \, \mathrm{d}x - \frac{3}{2} x \int_{-1}^{1} x^4 \, \mathrm{d}x - \frac{5}{8}(3x^2 - 1) \int_{-1}^{1} (3x^2 - 1) x^3 \, \mathrm{d}x \\
&= x^3 - \frac{3}{2} x \int_{-1}^{1} x^4 \, \mathrm{d}x \\
&= x^3 - \frac{3}{5} x.
\end{aligned}$$

  Normalizing, we have

$$\phi_3(x) = \frac{\psi_3(x)}{\sqrt{\langle \psi_3 | \psi_3 \rangle}} = \frac{x^3 - \frac{3}{5}x}{\sqrt{\int_{-1}^{1} \left(x^3 - \frac{3}{5}x\right)^2 \mathrm{d}x}} = \frac{1}{2} \sqrt{\frac{7}{2}} (5x^3 - 3x).$$

The Legendre polynomials $P_n(x)$ would then be

$$P_n(x) = \sqrt{\frac{2}{2n+1}} \phi_n(x)$$

(d) We seek solutions of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_\lambda \neq 0.$$

Deriving and replacing back into the Legendre ODE, we get

$$(1 - x^2) \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - 2x \sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda) x^{k+\lambda-1} + n(n+1) \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1) x^{k+\lambda-2} - \sum_{\lambda=0}^{\infty} a_\lambda \left[(k+\lambda)(k+\lambda+1) - n(n+1)\right] x^{k+\lambda} = 0$$

Setting $\lambda = 0$, we get

- **Lowest order x^{k-2}:** This gives us the indicial equation

$$a_0 k(k-1) = 0 \implies k = 0, 1.$$

- **First order x^{k-1}:**

$$a_1 (k+1) k = 0 \implies \begin{cases} a_1 \text{ arbitrary}, & \text{if } k = 0, \\ a_1 = 0, & \text{if } k = 1, \end{cases}$$

- **General order x^{k+j}:**

$$a_{j+2}(k+j+2)(k+j+1) - a_j \left[(k+j)(k+j+1) - n(n+1)\right] = 0$$

$$a_{j+2} = a_j \frac{(k+j)(k+j+1) - n(n+1)}{(k+j+1)(k+j+2)}$$

  - **For $k = 0$:** We have

$$a_{j+2} = a_j \frac{j(j+1) - n(n+1)}{(j+1)(j+2)}.$$

    Since a_1 is arbitrary for $k = 0$, then we can set it to zero. Then we get a solution explicitly for even powers and

$$y_{2n} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda}$$

- **For $k = 1$:** We have
$$a_{j+2} = a_j \frac{(j+1)(j+2) - n(n+1)}{(j+2)(j+3)}.$$
Since $a_1 = 0$ for $k = 1$, then we get a solution explicitly for odd powers and
$$y_{2n+1} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda+1}$$
- **For $n = 0$:** We have
$$y_0 = P_0(x) = a_0.$$
Set $a_0 = 1$, then
$$P_0(x) = 1.$$
- **For $n = 1$:** We have
$$y_1 = P_1(x) = a_0 x.$$
Set $a_0 = 1$, then
$$P_1(x) = x.$$
- **For $n = 2$:** We have
$$y_2 = P_2(x) = a_0 - 3a_0 x^2.$$
Set $a_0 = -\frac{1}{2}$, then
$$P_2(x) = \frac{1}{2}(3x^2 - 1).$$
- **For $n = 3$:** We have
$$y_3 = P_3(x) = a_0 x - \frac{5}{3} a_0 x^3.$$
Set $a_0 = -\frac{3}{2}$, then
$$P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

$\blacksquare$

Problem 2
The Hermite differential equation is $H_n'' - 2xH_n' + 2nH_n = 0$.

(a) Solve this equation by series solution and show that it terminates for integral values of n.

(b) Use the series solution to generate the first four Hermite polynomials which are

$$H_0(x) = 1, \qquad H_2 = 4x^2 - 2,$$
$$H_1(x) = 2x, \quad H_3 = 8x^3 - 12x.$$

(c) Obtain the first four Hermite polynomials using the generating function which is

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

(d) Using the generating function, derive the recurrence relations

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,$$

$$H_n'(x) - 2nH_{n-1}(x) = 0.$$

(e) Using the result of part (d), verify that the $H_n(x)$ defined by the generating function obeys the Hermite differential equation.

Solution. (a) We seek solutions of the form

$$y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}, \quad a_\lambda \neq 0.$$

Deriving and replacing back into the Legendre ODE, we get

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} - 2x\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)x^{k+\lambda-1} + 2n\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0$$

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)x^{k+\lambda-2} - \sum_{\lambda=0}^{\infty} 2a_\lambda (k+\lambda-n)x^{k+\lambda} = 0$$

Setting $\lambda = 0$, we get

- **Lowest order** x^{k-2}: This gives us the indicial equation

$$a_0 k(k-1) = 0 \implies k = 0, 1.$$

- **First order** x^{k-1}:

$$a_1(k+1)k = 0 \implies \begin{cases} a_1 \text{ arbitrary}, & \text{if } k=0, \\ a_1 = 0, & \text{if } k = 1, \end{cases}$$

- **General order** x^{k+j}:

$$a_{j+2}(k+j+2)(k+j+1) - 2a_j(k+j-n) = 0$$

$$a_{j+2} = a_j \frac{2(k+j-n)}{(k+j+1)(k+j+2)}$$

– **For $k = 0$:** We have

$$a_{j+2} = a_j \frac{2(j-n)}{(j+1)(j+2)}.$$

Since a_1 is arbitrary for $k = 0$, then we can set it to zero. Then we get a solution explicitly for even powers and

$$y_{2n} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda}$$

Notice, since j is even from the recurrence relation, that if n is also even, then there will be some term that is zero which terminates the series.

– **For $k = 1$:** We have

$$a_{j+2} = a_j \frac{2(j+1-n)}{(j+2)(j+3)}.$$

Since $a_1 = 0$ for $k = 1$, then we get a solution explicitly for odd powers and

$$y_{2n+1} = \sum_{\lambda=0}^{\infty} a_{2\lambda} x^{2\lambda+1}$$

Notice, since j is odd from the recurrence relation, that if n is also odd, then there will be some term that is zero which terminates the series.

(b)

- **For $n = 0$:** We have

$$y_0 = H_0(x) = a_0.$$

Set $a_0 = 1$, then

$$H_0(x) = 1.$$

- **For $n = 1$:** We have

$$y_1 = H_1(x) = a_0 x.$$

Set $a_0 = 2$, then

$$H_1(x) = 2x.$$

- **For $n = 2$:** We have

$$y_2 = H_2(x) = a_0 - 2a_0 x^2.$$

Set $a_0 = -2$, then

$$H_2(x) = 4x^2 - 2.$$

- **For $n = 3$:** We have

$$y_3 = H_3(x) = a_0 x - \frac{2}{3} a_0 x^3.$$

Set $a_0 = -12$, then

$$H_3(x) = 8x^3 - 12x.$$

(c) The generating function of the Hermite polynomials is given by

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}.$$

The Taylor expansion of g has the form of

$$g(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t) \right|_{t=0} t^n.$$

By comparison, we have

$$H_n(x) = \left. \frac{\mathrm{d}^n}{\mathrm{d}t^n} g(x,t) \right|_{t=0}.$$

- **For $n = 0$:**

$$\begin{aligned} H_0(x) &= \frac{\mathrm{d}^0}{\mathrm{d}t^0} g(x,t)\Big|_{t=0} \\ &= g(x,t)|_{t=0} \\ &= e^{-t^2+2tx}\Big|_{t=0} \\ &= 1. \end{aligned}$$

- **For $n = 1$:**

$$\begin{aligned} H_1(x) &= \frac{\mathrm{d}^1}{\mathrm{d}t^1} g(x,t)\Big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} g(x,t)\Big|_{t=0} \\ &= 2(x-t)e^{-t^2+2tx}\Big|_{t=0} \\ &= 2x. \end{aligned}$$

- **For $n = 2$:**

$$\begin{aligned} H_2(x) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} g(x,t)\Big|_{t=0} \\ &= \frac{\mathrm{d}^2}{\mathrm{d}t^2} e^{-t^2+2tx}\Big|_{t=0} \\ &= (2(x-t))^2 e^{-t^2+2tx} - 2e^{-t^2+2tx}\Big|_{t=0} \\ &= 4x^2 - 2. \end{aligned}$$

- **For $n = 3$:**

$$\begin{aligned} H_3(x) &= \frac{\mathrm{d}^3}{\mathrm{d}t^3} g(x,t)\Big|_{t=0} \\ &= \frac{\mathrm{d}^3}{\mathrm{d}t^3} e^{-t^2+2tx}\Big|_{t=0} \\ &= (2(x-t))^3 e^{-t^2+2tx} - 8(x-t)e^{-t^2+2tx} - 4(x-t)e^{-t^2+2tx}\Big|_{t=0} \\ &= 8x^3 - 12x. \end{aligned}$$

(d) The generating function of the Hermite polynomials is given by

$$g(x,t) = e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

- Deriving both sides with respect to t, we get

$$\frac{\partial g(x,t)}{\partial t} = 2(x-t)e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!},$$

which implies that

$$\begin{aligned} 2(x-t)\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} \\ \sum_{n=0}^{\infty} 2xH_n(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} &= \sum_{n=0}^{\infty} H_n(x)\frac{t^{n-1}}{(n-1)!} \end{aligned}$$

$$\sum_{m=1}^{\infty} 2H_{m-1}(x)\frac{t^m}{(m-1)!} - \sum_{m=0}^{\infty} 2xH_m(x)\frac{t^m}{m!} + \sum_{m=-1}^{\infty} H_m(x)\frac{t^m}{m!} = 0$$
$$\sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!} - \sum_{m=0}^{\infty} 2xH_m(x)\frac{t^m}{m!} + \sum_{m=-1}^{\infty} H_{m+1}(x)\frac{t^m}{m!} = 0$$
$$\implies H_{n+1} - 2xH_n + 2nH_{n-1} = 0.$$

- Deriving both sides with respect to x, we get

$$\frac{\partial g(x,t)}{\partial x} = 2te^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!},$$

which implies that

$$\sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!}$$
$$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} - \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!} = 0$$

$$\sum_{m=0}^{\infty} H_m'(x)\frac{t^m}{m!} - \sum_{m=1}^{\infty} 2H_{m-1}(x)\frac{t^m}{(m-1)!} = 0$$
$$\sum_{m=0}^{\infty} H_m'(x)\frac{t^m}{m!} - \sum_{m=1}^{\infty} 2mH_{m-1}(x)\frac{t^m}{m!} = 0$$
$$\implies H_n' - 2nH_{n-1} = 0.$$

(e) Replacing in the Hermite differential equation, we have

$$\begin{aligned} H_n'' - 2xH_n' + 2nH_n & \\ &= 2nH_{n-1}' - 4nxH_{n-1} + 2nH_n \\ &= 4n^2H_{n-2}' - 4nxH_{n-1} + 2nH_n \\ &= 0. \end{aligned}$$

■

Problem 3
Use the generating function for the Bessel functions,

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n,$$

to obtain the following recurrence relations

(a) $J_{n-1} + J_{n+1} = \frac{2n}{x} J_n$,

(b) $J_{n-1} - J_{n+1} = 2J_n'$,

(c) $J_{n-1} - \frac{n}{x} J_n = J_n'$,

(d) $J_{n+1} - \frac{n}{x} J_n = -J_n'$.

(e) Using the above results, verify that J_n satisfies Bessel's equation,

$$x^2 J_n'' + xJ_n' + (x^2 - n^2)J_n = 0.$$

(e) Verify that the series solution

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

satisfies the same equation.

Solution. The generating function of the Bessel functions is given by

$$g(x,t) = e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n.$$

(a) Deriving both sides with respect to t, we get

$$\frac{\partial g(x,t)}{\partial t} = \frac{x}{2}\left(1 + \frac{1}{t^2}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1},$$

which implies that

$$\frac{x}{2}\left(1 + \frac{1}{t^2}\right) \sum_{-\infty}^{\infty} J_n(x)t^n = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1}$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^n + \frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^{n-2} = \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1}$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^{n-2} - \sum_{-\infty}^{\infty} nJ_n(x)t^{n-1} + \frac{x}{2}\sum_{-\infty}^{\infty} J_n(x)t^n = 0$$

$$\frac{x}{2}\sum_{-\infty}^{\infty} J_{m+1}(x)t^{m-1} - \sum_{-\infty}^{\infty} mJ_m(x)t^{m-1} + \frac{x}{2}\sum_{-\infty}^{\infty} J_{m-1}(x)t^{m-1} = 0$$

$$\implies J_{n+1} + J_{n-1} = \frac{2n}{x} J_n.$$

(b) Deriving both sides with respect to x, we get

$$\frac{\partial g(x,t)}{\partial x} = \frac{\left(t - \frac{1}{t}\right)}{2} e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n'(x) t^n,$$

which implies that

$$\begin{aligned}
\frac{\left(t - \frac{1}{t}\right)}{2} \sum_{-\infty}^{\infty} J_n(x) t^n &= \sum_{-\infty}^{\infty} J_n'(x) t^n \\
\frac{1}{2} \sum_{-\infty}^{\infty} J_n(x) t^{n+1} - \frac{1}{2} \sum_{-\infty}^{\infty} J_n(x) t^{n-1} &= \sum_{-\infty}^{\infty} J_n'(x) t^n
\end{aligned}$$

$$\begin{aligned}
\sum_{-\infty}^{\infty} J_n'(x) t^n - \frac{1}{2} \sum_{-\infty}^{\infty} J_n(x) t^{n+1} + \frac{1}{2} \sum_{-\infty}^{\infty} J_n(x) t^{n-1} &= 0 \\
\sum_{-\infty}^{\infty} J_m'(x) t^m - \frac{1}{2} \sum_{-\infty}^{\infty} J_{m-1}(x) t^m + \frac{1}{2} \sum_{-\infty}^{\infty} J_{m+1}(x) t^m &= 0
\end{aligned}$$

$$\implies J_{n-1} - J_{n+1} = 2J_n'.$$

(c) Adding the two equations derived in parts (a) and (b), we get

$$J_{n-1} - \frac{n}{x} J_n = J_n'.$$

(d) Subtracting the two equations derived in parts (a) and (b), we get

$$J_{n+1} - \frac{n}{x} J_n = -J_n'.$$

(e) Bessel's equation is given by

$$x^2 J_n'' + x J_n' + (x^2 - n^2) J_n = 0.$$

Deriving and replacing, we get

$$\begin{aligned}
x^2 J_n'' + x J_n' + (x^2 - n^2) J_n &= \frac{x^2}{2}(J_{n-1}' - J_{n+1}') + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\
&= \frac{x^2}{4}\left(J_{n-2} - 2J_n + J_{n+2}\right) + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\
&= \frac{x}{2}\left((n-1)J_{n-1} + (n+1)J_{n+1}\right) - x^2 J_n + \frac{x}{2}(J_{n-1} - J_{n+1}) + (x^2 - n^2) J_n \\
&= \frac{x}{2}\left((n-1)J_{n-1} + (n+1)J_{n+1} + J_{n-1} - J_{n+1}\right) - n^2 J_n \\
&= \frac{nx}{2}\left(J_{n-1} + J_{n+1}\right) - n^2 J_n \\
&= \frac{nx}{2}\frac{2n}{x} J_n - n^2 J_n \\
&= 0.
\end{aligned}$$

(f) Given the series solution

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!} \left(\frac{x}{2}\right)^{n+2s}$$

Deriving and replacing, we get

$$
\begin{aligned}
x^2 J_n'' + x J_n' + (x^2 - n^2) J_n &= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)(n+2s-1)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(x^2-n^2)\left(\frac{x}{2}\right)^{n+2s} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)(n+2s-1)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}(n+2s)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad - \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}n^2\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2\left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left[(n+2s)(n+2s-1)+(n+2s)-n^2\right]\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2\left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2 s\,(n+s)\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2\left(\frac{x}{2}\right)^{n+2s+2} \\
&= \sum_{s=0}^{\infty} \frac{(-1)^s}{(s-1)!(n+s-1)!}2^2\left(\frac{x}{2}\right)^{n+2s} \\
&\quad + \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}2^2\left(\frac{x}{2}\right)^{n+2s+2} \\
&= 0.
\end{aligned}
$$

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