PHYS 630 - Advanced Electricity and Magnetism
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Homework 4

Problem 1
A metal sphere of radius r has dipolar magnetic field, shown in the figure below. Imagine a cylindrical surface of radius $R > r$ (R is measured from the center of the sphere) aligned with the dipole. Some magnetic field lines close within the cylinder, some intersect the cylinder. Find the polar angle θ that the last field line that closes inside the cylinder makes on the surface of the sphere.

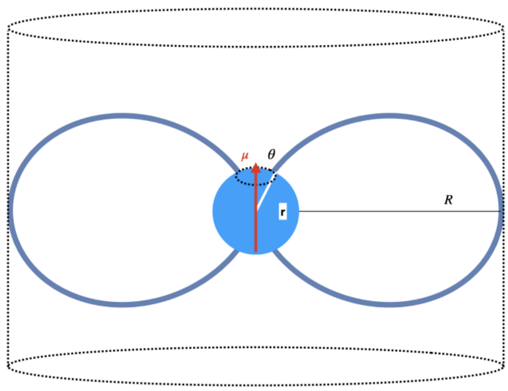

Figure 1: Dipole within a cylinder

Solution. We have that

$$\frac{1}{r}\frac{\mathrm{d}r}{B_r} = \frac{\mathrm{d}\theta}{B_\theta},$$

where $\mathbf{B} = (B_r, B_\theta, B_\varphi) = (2\cos(\theta), \sin(\theta), 0)$. We will now solve for r. We have

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{B_r}{B_\theta} r \\
\frac{\mathrm{d}r}{r} &= \frac{B_r}{B_\theta}\,\mathrm{d}\theta \\
\frac{\mathrm{d}r}{r} &= \frac{2\cos(\theta)}{\sin(\theta)}\,\mathrm{d}\theta \\
\ln(r) &= 2\ln(\sin(\theta)) + c \\
r &= \mathrm{e}^c \sin^2(\theta) \\
r &= c\sin^2(\theta).
\end{aligned}$$

To determine c, we have that when $\theta = \frac{\pi}{2}$, $r\left(\frac{\pi}{2}\right) = R$, thus $c = R$, so that

$$r = R\sin^2(\theta).$$

Thus, the polar angle θ on the sphere at which the last magnetic field line closes inside the cylinder, denoted θ_ℓ, is

$$\theta_\ell = \sin^{-1}\left(\sqrt{\frac{r}{R}}\right),$$

where r here is the fixed radius of the sphere.

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Problem 2
A metal sphere of radius R carries a surface current $g_\varphi = g_0 \sin(\theta)$. Find magnetic field inside and outside. Hint: look for $A_\varphi \propto f(r)\sin(\theta)$, find solutions of $\nabla^2 \mathbf{A} = 0$ inside and outside, then use the fact that the radial component of the magnetic field should be continuous, while the tangential component experiences a jump, given by

$$B_\theta^{(+)} - B_\theta^{(-)} = \frac{4\pi}{c} g.$$

Solution. Let us consider $A_\varphi = f(r)\sin(\theta)$. In spherical coordinates, the Laplacian is given by

$$\nabla^2 \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \mathbf{A}}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \mathbf{A}}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 \mathbf{A}}{\partial \varphi^2}.$$

In our problem, we have a that the radial and polar components of $\mathbf{A}$ are zero, so we consider the Laplacian of the azimuthal component. Then

$$\begin{aligned}
\left(\nabla^2 \mathbf{A}\right)_\varphi &= \nabla^2 A_\varphi - \frac{1}{r^2 \sin^2(\theta)} A_\varphi \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_\varphi}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial A_\varphi}{\partial \theta}\right) + \cancel{\frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 A_\varphi}{\partial \varphi^2}} - \frac{1}{r^2 \sin^2(\theta)} A_\varphi \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_\varphi}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial A_\varphi}{\partial \theta}\right) - \frac{1}{r^2 \sin^2(\theta)} A_\varphi \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f(r)}{\partial r}\sin(\theta)\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \sin(\theta)}{\partial \theta} f(r)\right) - \frac{f(r)}{r^2 \sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\frac{\partial}{\partial r}\left(r^2 f'(r)\right) + \frac{f(r)}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\cos(\theta)\right) - \frac{f(r)}{r^2 \sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\left[2rf'(r) + r^2 f''(r)\right] + \frac{f(r)}{2r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\sin(2\theta) - \frac{f(r)}{r^2 \sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\left[2rf'(r) + r^2 f''(r)\right] + \frac{f(r)}{r^2 \sin(\theta)}\cos(2\theta) - \frac{f(r)}{r^2 \sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\left[2rf'(r) + r^2 f''(r)\right] + \frac{f(r)}{r^2 \sin(\theta)}(1 - 2\sin^2(\theta)) - \frac{f(r)}{r^2 \sin(\theta)} \\
&= \frac{\sin(\theta)}{r^2}\left[2rf'(r) + r^2 f''(r)\right] + \frac{f(r)}{r^2 \sin(\theta)} - \frac{2f(r)\sin(\theta)}{r^2} - \frac{f(r)}{r^2 \sin(\theta)} \\
&= \sin(\theta)\left[f''(r) + \frac{2}{r}f'(r) - \frac{2}{r^2}f(r)\right] \\
&= 0 \\
&\implies f''(r) + \frac{2}{r}f'(r) - \frac{2}{r^2}f(r) = 0
\end{aligned}$$

Assume a solution form of $f(r) = cr^n$, then $f'(r) = cnr^{n-1}$ and $f''(r) = cn(n-1)r^{n-2}$. Replacing in the differential equation above, we have

$$\begin{aligned}
cn(n-1)r^{n-2} + \frac{2}{r}cnr^{n-1} - \frac{2}{r^2}cr^n &= 0 \\
cn(n-1)r^{n-2} + 2cnr^{n-2} - 2cr^{n-2} &= 0 \\
\left(n(n-1) + 2n - 2\right) r^{n-2} &= 0 \\
\left(n^2 + n - 2\right) r^{n-2} &= 0 \\
(n-1)(n+2)r^{n-2} &= 0,
\end{aligned}$$

and thus $n = 1$ or $n = -2$. We consider a superposition solution of both powers, and we get

$$f(r) = \frac{c_1}{r^2} + c_2 r.$$

Thus,

$$\begin{aligned}
A_\varphi &= \left(\frac{c_1}{r^2} + c_2 r\right)\sin(\theta), \\
\mathbf{A} &= \left(\frac{c_1}{r^2} + c_2 r\right)\sin(\theta)\hat{\boldsymbol{\varphi}}.
\end{aligned}$$

Because of this, we can divide $\mathbf{A}$ into two components:

- **Internal**: $\mathbf{A}_{\text{in}} = c_2 r \sin(\theta)\hat{\boldsymbol{\varphi}}$.
- **External**: $\mathbf{A}_{\text{ext}} = \frac{c_1}{r^2}\sin(\theta)\hat{\boldsymbol{\varphi}}$.

To find the magnetic field $\mathbf{B}$, we take the curl of $\mathbf{A}$, getting

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} \\
&= \begin{vmatrix} \frac{1}{r^2\sin(\theta)}\hat{\mathbf{r}} & \frac{1}{r\sin(\theta)}\hat{\boldsymbol{\theta}} & \frac{1}{r}\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ A_r & rA_\theta & r\sin(\theta)A_\varphi \end{vmatrix} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(r\sin(\theta)A_\varphi) - \frac{\partial}{\partial\varphi}(rA_\theta)\right]\hat{\mathbf{r}} \\
&\quad - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)A_\varphi) - \frac{\partial}{\partial\varphi}(A_r)\right]\hat{\boldsymbol{\theta}} \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}(A_r)\right]\hat{\boldsymbol{\varphi}} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(r\sin(\theta)A_\varphi)\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)A_\varphi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)\right]\hat{\boldsymbol{\varphi}}.
\end{aligned}$$

We know that $A_r = A_\theta = 0$. Additionally, we divide the magnetic field just like we did with the magnetic potential. We have

$$\begin{aligned}
\mathbf{B}_{\text{in}} &= \nabla \times \mathbf{A}_{\text{in}} \\
&= \begin{vmatrix} \frac{1}{r^2\sin(\theta)}\hat{\mathbf{r}} & \frac{1}{r\sin(\theta)}\hat{\boldsymbol{\theta}} & \frac{1}{r}\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & r\sin(\theta)A_\varphi \end{vmatrix} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(r\sin(\theta)A_\varphi)\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)A_\varphi)\right]\hat{\boldsymbol{\theta}} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(c_2 r^2\sin^2(\theta))\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(c_2 r^2\sin^2(\theta))\right]\hat{\boldsymbol{\theta}} \\
&= \frac{c_2}{\sin(\theta)}\left[2\sin(\theta)\cos(\theta)\right]\hat{\mathbf{r}} - \frac{c_2\sin(\theta)}{r}\left[2r\right]\hat{\boldsymbol{\theta}} \\
&= 2c_2\cos(\theta)\hat{\mathbf{r}} - 2c_2\sin(\theta)\hat{\boldsymbol{\theta}}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}_{\text{ext}} &= \nabla \times \mathbf{A}_{\text{ext}} \\
&= \begin{vmatrix} \frac{1}{r^2\sin(\theta)}\hat{\mathbf{r}} & \frac{1}{r\sin(\theta)}\hat{\boldsymbol{\theta}} & \frac{1}{r}\hat{\boldsymbol{\varphi}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & r\sin(\theta)A_\varphi \end{vmatrix} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}(r\sin(\theta)A_\varphi)\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}(r\sin(\theta)A_\varphi)\right]\hat{\boldsymbol{\theta}} \\
&= \frac{1}{r^2\sin(\theta)}\left[\frac{\partial}{\partial\theta}\left(\frac{c_1}{r}\sin^2(\theta)\right)\right]\hat{\mathbf{r}} - \frac{1}{r\sin(\theta)}\left[\frac{\partial}{\partial r}\left(\frac{c_1}{r}\sin^2(\theta)\right)\right]\hat{\boldsymbol{\theta}} \\
&= \frac{c_1}{r^3\sin(\theta)}\left[2\sin(\theta)\cos(\theta)\right]\hat{\mathbf{r}} - \frac{c_1\sin(\theta)}{r}\left[-\frac{1}{r^2}\right]\hat{\boldsymbol{\theta}} \\
&= \frac{2c_1\cos(\theta)}{r^3}\hat{\mathbf{r}} + \frac{c_1\sin(\theta)}{r^3}\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Thus,

$$\begin{cases} \mathbf{B}_{\text{in}} &= 2c_2\cos(\theta)\hat{\mathbf{r}} - 2c_2\sin(\theta)\hat{\boldsymbol{\theta}}, \\ \mathbf{B}_{\text{ext}} &= \frac{2c_1\cos(\theta)}{r^3}\hat{\mathbf{r}} + \frac{c_1\sin(\theta)}{r^3}\hat{\boldsymbol{\theta}}. \end{cases}$$

We will now use the boundary conditions.

- **Radial Components**: The radial components are continuous at the surface of the sphere ($r = R$). Specifically

$$\begin{aligned}
B_r^{(+)}\Big|_{r=R} &= B_r^{(-)}\Big|_{r=R} \\
\frac{2c_1\cos(\theta)}{R^3} &= 2c_2\cos(\theta) \\
c_2 &= \frac{c_1}{R^3}
\end{aligned}$$

- **Polar Components**: The polar components are discontinuous at the surface of the sphere ($r = R$). Specifically

$$\begin{aligned}
B_\theta^{(+)}\Big|_{r=R} - B_\theta^{(-)}\Big|_{r=R} &= \frac{4\pi}{c}g \\
\frac{c_1\sin(\theta)}{R^3} + 2c_2\sin(\theta) &= \frac{4\pi}{c}g_0\sin(\theta) \\
\frac{c_1\sin(\theta)}{R^3} + \frac{2c_1\sin(\theta)}{R^3} &= \frac{4\pi}{c}g_0\sin(\theta) \\
\frac{3c_1\sin(\theta)}{R^3} &= \frac{4\pi}{c}g_0\sin(\theta) \\
c_1 &= \frac{4\pi R^3}{3c}g_0 \\
\implies c_2 &= \frac{4\pi}{3c}g_0.
\end{aligned}$$

Therefore, the internal and external magnetic fields are

$$\begin{cases} \mathbf{B}_{\text{in}} &= \frac{8\pi g_0}{3c}\cos(\theta)\hat{\mathbf{r}} - \frac{8\pi g_0}{3c}\sin(\theta)\hat{\boldsymbol{\theta}}, \\ \mathbf{B}_{\text{ext}} &= \frac{8\pi g_0}{3c}\frac{R^3}{r^3}\cos(\theta)\hat{\mathbf{r}} + \frac{4\pi g_0}{3c}\frac{R^3}{r^3}\sin(\theta)\hat{\boldsymbol{\theta}}. \end{cases}$$

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