PHYS 663 - Quantum Field Theory II

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Homework 2

Problem 11.1 - Spin-wave Theory

(a) Prove the following wonderful formula: Let $\phi(x)$ be a free scalar field with propagator $\langle T\phi(x)\phi(0)\rangle = D(x)$. Then

$$\left\langle T\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-i\phi(0)}\right\rangle = \mathrm{e}^{[D(x)-D(0)]}$$

(The factor $D(0)$ gives a formally divergent adjustment of the overall normalization.)

(b) We can use this formula in Euclidean field theory to discuss correlation functions in a theory with spontaneously broken symmetry for $T < T_C$. Let us consider only the simplest case of a broken $O(2)$ or $U(1)$ symmetry. We can write the local spin density as a complex variable

$$s(x) = s^1(x) + \mathrm{i}s^2(x)$$

The global symmetry is the transformation

$$s(x) \to \mathrm{e}^{-\mathrm{i}\alpha}s(x)$$

If we assume that the physics freezes the modulus of $s(x)$, we can parametrize

$$s(x) = A\mathrm{e}^{\mathrm{i}\phi(x)}$$

and write an effective Lagrangian for the field $\phi(x)$. The symmetry of the theory becomes the translation symmetry

$$\phi(x) \to \phi(x) - \alpha$$

Show that (for $d > 0$) the most general renormalizable Lagrangian consistent with this symmetry is the free field theory

$$\mathcal{L} = \frac{1}{2}\rho(\vec{\nabla}\phi)^2.$$

In statistical mechanics, the constant ρ is called the spin wave modulus. A reasonable hypothesis for ρ is that it is finite for $T < T_C$ and tends to 0 as $T \to T_C$ from below.

(c) Compute the correlation function $\langle s(x)s^*(0)\rangle$. Adjust A to give a physically sensible normalization (assuming that the system has a physical cutoff at the scale of one atomic spacing) and display the dependence of this correlation function on x for $d = 1, 2, 3, 4$. Explain the significance of your results.

Solution. (a) Let's prove the given formula. We begin by writing the left-hand side using the path integral:

$$\left\langle T\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-\mathrm{i}\phi(0)}\right\rangle = \frac{1}{Z[0]}\int\mathcal{D}\phi\,\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-\mathrm{i}\phi(0)}\exp\left[\mathrm{i}\int\mathrm{d}^dz\,\mathrm{d}^dy\,\frac{1}{2}\phi(z)D^{-1}(z-y)\phi(y)\right]$$

We can rewrite this expression by defining a source term $J(y) = \delta(y-x) - \delta(y)$. With this definition, we have

$$\begin{aligned}\langle T\mathrm{e}^{\mathrm{i}\phi(x)}\mathrm{e}^{-\mathrm{i}\phi(0)}\rangle &= \frac{1}{Z[0]}\int\mathcal{D}\phi\,\exp\left[\mathrm{i}\int\mathrm{d}^dy\,J(y)\phi(y) + \mathrm{i}\int\mathrm{d}^dz\,\mathrm{d}^dy\,\frac{1}{2}\phi(z)D^{-1}(z-y)\phi(y)\right]\\ &= \frac{Z[J]}{Z[0]}\end{aligned}$$

For a free field theory, we know that

$$\frac{Z[J]}{Z[0]} = \exp\left[-\frac{1}{2}\int \mathrm{d}^d z\, \mathrm{d}^d y\, J(z) D(z-y) J(y)\right].$$

Substituting our particular $J(y)$, we get

$$\begin{aligned}\frac{Z[J]}{Z[0]} &= \exp\left[-\frac{1}{2}\int \mathrm{d}^d z\, \mathrm{d}^d y\, (\delta(z-x)-\delta(z)) D(z-y)(\delta(y-x)-\delta(y))\right] \\ &= \exp\left[-\frac{1}{2}\int \mathrm{d}^d y\, (D(x-y)-D(-y))(\delta(y-x)-\delta(y))\right] \\ &= \exp\left[-\frac{1}{2}(D(x-x)-D(x)-D(-x)+D(0))\right] \\ &= \exp\left[-\frac{1}{2}(D(0)-D(x)-D(-x)+D(0))\right].\end{aligned}$$

Since $D(x) = D(-x)$ by translation invariance, we get

$$\langle T \mathrm{e}^{\mathrm{i}\phi(x)} \mathrm{e}^{-i\phi(0)} \rangle = \exp\left[D(x) - D(0)\right].$$

This proves the desired formula.

(b) The Lagrangian needs to be invariant under the transformation $\phi(x) \to \phi(x) - \alpha$ for any constant α. This means the Lagrangian can depend on ϕ only through its derivatives.

The most general renormalizable Lagrangian (with terms of dimension $\leq d$ in d dimensions) that is Lorentz invariant and depends only on derivatives of ϕ is:

$$\mathcal{L} = \frac{1}{2}\rho(\nabla\phi)^2.$$

This is the free field theory Lagrangian with ρ being the spin wave modulus.

(c) Let's compute the correlation function $\langle s(x) s^*(0)\rangle$. Using the result from part (a), we have

$$\begin{aligned}\langle s(x) s^*(0)\rangle &= A^2 \left\langle \mathrm{e}^{\mathrm{i}\phi(x)} \mathrm{e}^{-\mathrm{i}\phi(0)} \right\rangle \\ &= A^2 \mathrm{e}^{D(x)-D(0)}.\end{aligned}$$

Now we need to determine the propagator $D(x)$. It satisfies the equation

$$-\rho\nabla^2 D(x-y) = \delta^{(d)}(x-y).$$

Due to rotational invariance, $D(x)$ depends only on the distance $|x|$, which we'll denote simply as x. In spherical coordinates, the Laplacian gives

$$-\rho\frac{1}{x^{d-1}}\frac{\partial}{\partial x}\left(x^{d-1}\frac{\partial}{\partial x}D(x)\right) = \frac{\Gamma(1+\frac{d}{2})}{d\pi^{d/2}}\frac{\delta(x)}{x^{d-1}}.$$

Solving this differential equation for $D(x)$, we get

$$D(x) = \begin{cases} \frac{\Gamma\left(1+\frac{d}{2}\right)}{d(d-2)\pi^{d/2}\rho}\frac{1}{x^{d-2}} & \text{for } d \neq 2 \\ -\frac{1}{2\pi\rho}\log(x) & \text{for } d = 2 \end{cases}$$

We can now evaluate $\langle s(x) s^*(0)\rangle$ for different dimensions:

- **For $d = 1$:**

$$D(x) = -\frac{1}{2\rho}x$$
$$\langle s(x)s^*(0)\rangle \sim \mathrm{e}^{-x}$$

- **For $d = 2$:**

$$D(x) = -\frac{1}{2\pi\rho}\log x$$
$$\langle s(x)s^*(0)\rangle \sim x^{-\frac{1}{2\pi\rho}}$$

- **For $d = 3$:**

$$D(x) = \frac{1}{4\pi\rho x}$$
$$\langle s(x)s^*(0)\rangle \sim \mathrm{e}^{\frac{1}{x}}$$

- **For $d = 4$:**

$$D(x) = \frac{1}{4\pi^2\rho x^2}$$
$$\langle s(x)s^*(0)\rangle \sim \mathrm{e}^{\frac{1}{x^2}}$$

As $\rho \to 0$ for $d = 2$, the correlation function becomes independent of distance x, indicating the critical behavior at $d = 2$.

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Problem 11.2- A Zeroth-order Natural Relation
This problem studies an $N = 2$ linear sigma model coupled to fermions:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi^i\right)^2 + \frac{1}{2}\mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left(\left(\phi^i\right)^2\right)^2 + \bar{\psi}(i\not{\partial})\psi - g\bar{\psi}\left(\phi^1 + i\gamma^5\phi^2\right)\psi,$$

where ϕ^i is a two-component field, $i = 1, 2$.

(a) Show that this theory has the following global symmetry:

$$\phi^1 \to \cos(\alpha)\phi^1 - \sin(\alpha)\phi^2,$$
$$\phi^2 \to \sin(\alpha)\phi^1 + \cos(\alpha)\phi^2,$$
$$\psi \to \mathrm{e}^{-\mathrm{i}\alpha\gamma^{5/2}/2}\psi.$$

Show also that the solution to the classical equations of motion with the minimum energy breaks this symmetry spontaneously.

(b) Denote the vacuum expectation value of the field ϕ^i by v and make the change of variables

$$\phi^i(x) = (v + \sigma(x), \pi(x)).$$

Write out the Lagrangian in these new variables, and show that the fermion acquires a mass given by

$$m_f = g \cdot v.$$

(c) Compute the one-loop radiative correction to m_f, choosing renormalization conditions so that v and g (defined as the $\psi\psi\pi$ vertex at zero momentum transfer) receive no radiative corrections. Show that relation (4) receives nonzero corrections but that these corrections are finite. This is in accord with our general discussion in Section 11.6.

Solution. We'll study the $N = 2$ linear sigma model coupled to fermions, with Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \frac{1}{2}\mu^2\phi^i\phi^i - \frac{1}{4}\lambda(\phi^i\phi^i)^2 + \bar{\psi}(\mathrm{i}\partial\!\!\!/)\psi - g\bar{\psi}(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi$$

where ϕ^i is a two-component field with $i = 1, 2$.

(a) Let's check the invariance of the Lagrangian under the transformation. We have

$$\begin{aligned}
\phi^1 &\to \phi^1\cos(\alpha) - \phi^2\sin(\alpha) \\
\phi^2 &\to \phi^1\sin(\alpha) + \phi^2\cos(\alpha) \\
\psi &\to \mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}\psi
\end{aligned}$$

The first three terms involving only ϕ^i are invariant under the $SO(2)$ rotation. The fermion kinetic term $\bar{\psi}(\mathrm{i}\partial\!\!\!/)\psi$ is also invariant because γ^5 anticommutes with γ^μ.

For the Dirac adjoint $\bar{\psi}$, we have

$$\bar{\psi} = \psi^\dagger\gamma^0 \to \psi^\dagger\mathrm{e}^{\mathrm{i}\alpha\gamma^5/2}\gamma^0 = \psi^\dagger\gamma^0\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2} = \bar{\psi}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}$$

Now for the interaction term

$$\begin{aligned}
&-g\bar{\psi}(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi \\
&\to -g\bar{\psi}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}[(\phi^1\cos(\alpha) - \phi^2\sin(\alpha)) + \mathrm{i}\gamma^5(\phi^1\sin(\alpha) + \phi^2\cos(\alpha))]\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}\psi
\end{aligned}$$

Using the identity $\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2}\mathrm{e}^{\mathrm{i}\alpha\gamma^5}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5/2} = \mathrm{e}^{\mathrm{i}\alpha\gamma^5}$ and

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\alpha\gamma^5}(\phi^1 + \mathrm{i}\gamma^5\phi^2) &= (\cos(\alpha) + \mathrm{i}\gamma^5\sin(\alpha))(\phi^1 + \mathrm{i}\gamma^5\phi^2) \\
&= \phi^1\cos(\alpha) + \mathrm{i}\gamma^5\phi^1\sin(\alpha) + \mathrm{i}\gamma^5\phi^2\cos(\alpha) - \phi^2\sin(\alpha) \\
&= (\phi^1\cos(\alpha) - \phi^2\sin(\alpha)) + \mathrm{i}\gamma^5(\phi^1\sin(\alpha) + \phi^2\cos(\alpha)).
\end{aligned}$$

We see that

$$-g\bar{\psi}(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi \to -g\bar{\psi}(\phi^1 + \mathrm{i}\gamma^5\phi^2)\psi.$$

Therefore, the entire Lagrangian is invariant under this transformation.

(b) Let's consider the case where ϕ acquires a vacuum expectation value. We can choose the ground state such that $\langle\phi^1\rangle = v$ and $\langle\phi^2\rangle = 0$, where $v = \sqrt{\frac{\mu^2}{\lambda}}$.

Using the parametrization $\phi = (v + \sigma(x), \pi(x))$, we rewrite the Lagrangian as

$$\begin{aligned}
\mathcal{L} = &\frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 - \mu^2\sigma^2 - \frac{1}{4}\lambda(\sigma^4 + \pi^4) \\
&- \frac{1}{2}\lambda\sigma^2\pi^2 - \lambda v\sigma^3 - \lambda v\sigma\pi^2 + \bar{\psi}(\mathrm{i}\partial\!\!\!/ - gv)\psi - g\bar{\psi}(\sigma + \mathrm{i}\gamma^5\pi)\psi.
\end{aligned}$$

We observe that the fermion has acquired a mass $m_f = gv$.

(c) Now we compute the radiative corrections to the mass relation $m_f = gv$. First, let's establish the renormalization conditions:

- The $\pi\psi\bar{\psi}$ vertex at $q^2 = 0$, $p^2 = p'^2 = m_f^2$ should be $g\gamma^5$.
- The σ tadpole should vanish.

With these conditions, g and v receive no radiative corrections. We want to show that m_f receives a finite radiative correction at 1-loop. Since the tadpole diagrams of σ sum to zero by our renormalization condition, the fermion's self-energy receives contributions from the following diagrams:

- A loop with σ propagator
- A loop with π propagator
- The counterterm $\delta_g v$

Let's compute the first two loop diagrams.

- **For the σ propagator diagram:**

$$\begin{aligned}(e) &= (-\mathrm{i}g)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{\not{k} - m_f} \frac{\mathrm{i}}{(k-p)^2 - 2\mu^2} \\ &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{x\not{p} + m_f}{(k'^2 - \Delta_1)^2} \\ &= \frac{\mathrm{i}g^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \frac{\Gamma(2-\frac{d}{2})}{\Delta_1^{2-d/2}} (x\not{p} + m_f)\end{aligned}$$

- **For the π propagator diagram:**

$$\begin{aligned}(f) &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^5 \frac{\mathrm{i}}{\not{k} - m_f} \gamma^5 \frac{\mathrm{i}}{(k-p)^2} \\ &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{x\not{p} - m_f}{(k'^2 - \Delta_2)^2} \\ &= \frac{\mathrm{i}g^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \frac{\Gamma(2-\frac{d}{2})}{\Delta_2^{2-d/2}} (x\not{p} - m_f).\end{aligned}$$

Combining these contributions, we have

$$(e) + (f) = \frac{\mathrm{i}g^2}{(4\pi)^2} \int_0^1 \mathrm{d}x \left\{ 2x\not{p} \left[\frac{2}{\epsilon} - \gamma + \log(4\pi) - \frac{1}{2}\log(\Delta_1\Delta_2) \right] + m_f \log\left(\frac{\Delta_2}{\Delta_1}\right) \right\}.$$

We observe that the correction to the fermion mass m_f from these two diagrams is finite. The total correction to m_f is finite only when δ_g is finite.

To verify this, we compute the radiative corrections to the $\pi\psi\bar{\psi}$ vertex. There are four 1-loop diagrams contributing:

- σ exchange with $\pi\psi\bar{\psi}$ vertices
- π exchange with $\sigma\psi\bar{\psi}$ vertices
- σ exchange with $\sigma\psi\bar{\psi}$ vertices
- σ-π mixing

Calculating all these contributions leads to:

$$(a) + (b) + (c) + (d) = \frac{g\gamma^5}{(4\pi)^2} \int_0^1 \mathrm{d}x \left[g^2 \log\left(\frac{\Delta_2}{\Delta_1}\right) + 4\lambda \int_0^{1-x} \mathrm{d}y \frac{m_f^2}{\Delta_3} \right].$$

These corrections are finite, implying that δ_g is finite. Therefore, the total correction to $m_f = gv$ is also finite.

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Problem 11.3 - The Gross-Neveu Model
The Gross-Neveu model is a model in two spacetime dimensions of fermions with a discrete chiral symmetry:

$$\mathcal{L} = \bar{\psi}_i i \not{\partial} \psi_i + \frac{1}{2} g^2 \left(\bar{\psi}_i \psi_i\right)^2,$$

with $i = 1, \ldots, N$. The kinetic term of two-dimensional fermions is built from matrices γ^μ that satisfy the two-dimensional Dirac algebra. These matrices can be 2×2:

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1,$$

where σ^i are Pauli sigma matrices. Define

$$\gamma^5 = \gamma^0 \gamma^1 = \sigma^3$$

this matrix anticommutes with the γ^μ.

(a) Show that this theory is invariant with respect to

$$\psi_i \to \gamma^5 \psi_i$$

and that this symmetry forbids the appearance of a fermion mass.

(b) Show that this theory is renormalizable in 2 dimensions (at the level of dimensional analysis).

(c) Show that the functional integral for this theory can be represented in the following form:

$$\int \mathcal{D}\psi e^{i \int d^2 x \mathcal{L}} = \int \mathcal{D}\psi \mathcal{D}\sigma \exp\left[i \int d^2 x \left\{\bar{\psi}_i i \not{\partial} \psi_i - \sigma \bar{\psi}_i \psi_i - \frac{1}{2g^2} \sigma^2\right\}\right]$$

where $\sigma(x)$ (not to be confused with a Pauli matrix) is a new scalar field with no kinetic energy terms.

(d) Compute the leading correction to the effective potential for σ by integrating over the fermion fields ψ_i. You will encounter the determinant of a Dirac operator; to evaluate this determinant, diagonalize the operator by first going to Fourier components and then diagonalizing the 2×2 Pauli matrix associated with each Fourier mode. (Alternatively, you might just take the determinant of this 2×2 matrix.) This 1-loop contribution requires a renormalization proportional to σ^2 (that is, a renormalization of g^2). Renormalize by minimal substraction.

(e) Ignoring two-loop and higher-order contributions, minimize this potential. Show that the σ field acquires a vacuum expectation value which breaks the symmetry of part (a). Convince yourself that this result does not depend on the particular renormalization condition chosen.

(f) Note that the effective potential derived in part (e) depends on g and N according to the form

$$V_{\text{eff}}\left(\sigma_{\text{cl}}\right) = N \cdot f\left(g^2 N\right).$$

(The overall factor of N is expected in a theory with N fields.) Construct a few of the higher-order contributions to the effective potential and show that they contain additional factors of N^{-1} which suppress them if we take the limit $N \to \infty$, $\left(g^2 N\right)$ fixed. In this limit, the result of part (e) is unambiguous.

Solution. (a) Let's verify that the Gross-Neveu model is invariant under the transformation $\psi_i \to \gamma^5 \psi_i$. First, we need to determine how $\bar{\psi}_i$ transforms

$$\begin{gathered}\bar{\psi}_i = \psi_i^\dagger \gamma^0 \\ \to \psi_i^\dagger (\gamma^5)^\dagger \gamma^0 = \psi_i^\dagger \gamma^5 \gamma^0.\end{gathered}$$

Since γ^5 anticommutes with γ^μ, we have $\gamma^5\gamma^0 = -\gamma^0\gamma^5$. Thus, we have

$$\bar{\psi}_i \to -\psi_i^\dagger \gamma^0\gamma^5 = -\bar{\psi}_i\gamma^5.$$

Now, let's examine how the terms in the Lagrangian transform

$$\begin{aligned}\bar{\psi}_i i\partial\!\!\!/ \psi_i &\to -\bar{\psi}_i\gamma^5 i\partial\!\!\!/ \gamma^5\psi_i \\ &= -\bar{\psi}_i\gamma^5 i\gamma^\mu\partial_\mu\gamma^5\psi_i.\end{aligned}$$

Using the anticommutation relation $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$, we get

$$\begin{aligned}\bar{\psi}_i i\partial\!\!\!/ \psi_i &\to \bar{\psi}_i i\gamma^\mu\partial_\mu\psi_i \\ &= \bar{\psi}_i i\partial\!\!\!/ \psi_i.\end{aligned}$$

For the interaction term, we have

$$\begin{aligned}(\bar{\psi}_i\psi_i)^2 &\to (-\bar{\psi}_i\gamma^5)(\gamma^5\psi_i)(-\bar{\psi}_j\gamma^5)(\gamma^5\psi_j) \\ &= (-\bar{\psi}_i\gamma^5\gamma^5\psi_i)(-\bar{\psi}_j\gamma^5\gamma^5\psi_j).\end{aligned}$$

Since $(\gamma^5)^2 = 1$, we have

$$\begin{aligned}(\bar{\psi}_i\psi_i)^2 &\to (-\bar{\psi}_i\psi_i)(-\bar{\psi}_j\psi_j) \\ &= (\bar{\psi}_i\psi_i)^2.\end{aligned}$$

Thus, the entire Lagrangian is invariant under the transformation $\psi_i \to \gamma^5\psi_i$.

However, a fermion mass term would transform as

$$\begin{aligned}m\bar{\psi}_i\psi_i &\to m(-\bar{\psi}_i\gamma^5)(\gamma^5\psi_i) \\ &= -m\bar{\psi}_i\psi_i.\end{aligned}$$

Since the mass term changes sign, it is forbidden by this discrete chiral symmetry.

(b) To determine the renormalizability, we need to analyze the mass dimension of the coupling constant g. In d dimensions, the action $S = \int \mathrm{d}^d x\mathcal{L}$ must be dimensionless, so $[\mathcal{L}] = d$. For the fermion field ψ, the kinetic term $\bar{\psi}i\partial\!\!\!/\psi$ has dimension d, which gives $[\psi] = \frac{d-1}{2}$.

For the interaction term $\frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2$, we have

$$\begin{aligned}\left[\frac{1}{2}g^2(\bar{\psi}_i\psi_i)^2\right] &= [g^2] + 2[\bar{\psi}\psi] \\ &= [g^2] + 2(d-1).\end{aligned}$$

Since this must equal d, we get

$$\begin{aligned}[g^2] &= d - 2(d-1) \\ &= d - 2d + 2 \\ &= 2 - d.\end{aligned}$$

Therefore, $[g] = 1 - \frac{d}{2}$. In 2 dimensions ($d = 2$), we have $[g] = 0$, which means g is dimensionless. This is the criterion for renormalizability at the level of dimensional analysis.

(c) We want to show the equivalence of

$$\int \mathcal{D}\psi e^{i\int \mathrm{d}^2x\mathcal{L}} = \int \mathcal{D}\psi\mathcal{D}\sigma \exp\left[i\int \mathrm{d}^2x \left\{\bar{\psi}_i i\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right]$$

Let's integrate out the auxiliary field σ in the right-hand side, giving

$$\begin{aligned}&\int \mathcal{D}\sigma \exp\left[i\int \mathrm{d}^2x \left\{-\sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right]\\ &= \int \mathcal{D}\sigma \exp\left[-i\int \mathrm{d}^2x \left\{\frac{1}{2g^2}\sigma^2 + \sigma\bar{\psi}_i\psi_i\right\}\right]\end{aligned}$$

This is a Gaussian integral over σ with the result

$$\int \mathcal{D}\sigma \exp\left[-i\int \mathrm{d}^2x \left\{\frac{1}{2g^2}\sigma^2 + \sigma\bar{\psi}_i\psi_i\right\}\right] = \mathcal{N}\exp\left[i\int \mathrm{d}^2x \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2\right],$$

where $\mathcal{N}$ is a normalization constant that can be absorbed into the measure.
Therefore, we have

$$\int \mathcal{D}\psi\mathcal{D}\sigma \exp\left[i\int \mathrm{d}^2x \left\{\bar{\psi}_i i\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i - \frac{1}{2g^2}\sigma^2\right\}\right] = \int \mathcal{D}\psi \exp\left[i\int \mathrm{d}^2x \left\{\bar{\psi}_i i\partial\!\!\!/\psi_i + \frac{g^2}{2}(\bar{\psi}_i\psi_i)^2\right\}\right].$$

This is precisely the functional integral for the original Gross-Neveu model, proving the equivalence.

(d) To compute the effective potential for σ, we need to integrate over the fermion fields, we have

$$\begin{aligned}\exp(iS_{\text{eff}}[\sigma]) &= \int \mathcal{D}\psi \exp\left[i\int \mathrm{d}^2x \left\{\bar{\psi}_i \mathrm{i}\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i\right\}\right]\\ &= \left[\det(\mathrm{i}\partial\!\!\!/ - \sigma)\right]^N.\end{aligned}$$

For a constant σ field, we can evaluate this determinant by first going to momentum space. The Dirac operator becomes $p\!\!\!/ - \sigma$, which is a 2×2 matrix for each momentum mode.
The determinant of a 2×2 matrix is

$$\begin{aligned}\det(p\!\!\!/ - \sigma) &= \det\left(\gamma^\mu p_\mu - \sigma\right)\\ &= \det\left(\begin{pmatrix} -\sigma & p_0 - \mathrm{i}p_1 \\ p_0 + \mathrm{i}p_1 & -\sigma\end{pmatrix}\right)\\ &= \sigma^2 - (p_0^2 - p_1^2)\\ &= \sigma^2 + p^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\det(\mathrm{i}\partial\!\!\!/ - \sigma) &= \det(-\partial\!\!\!/^2 - \sigma^2)\\ &= \det(\partial^2 - \sigma^2).\end{aligned}$$

We can express this in momentum space as

$$\begin{aligned}\log\det(\mathrm{i}\partial\!\!\!/ - \sigma) &= \log\det(\partial^2 - \sigma^2)\\ &= \operatorname{tr}\log(\partial^2 - \sigma^2)\\ &= \int \frac{\mathrm{d}^2p}{(2\pi)^2}\log(-p^2 - \sigma^2).\end{aligned}$$

Thus, the effective action is:

$$\begin{aligned} S_{\text{eff}}[\sigma] &= -\mathrm{i}\log\left[\det(\mathrm{i}\partial\!\!\!/ - \sigma)\right]^N - \int \mathrm{d}^2x \frac{1}{2g^2}\sigma^2 \\ &= -\mathrm{i}N\int \frac{\mathrm{d}^2p}{(2\pi)^2}\log(-p^2-\sigma^2) - \int \mathrm{d}^2x\frac{1}{2g^2}\sigma^2. \end{aligned}$$

For a constant σ field, the effective potential is

$$\begin{aligned} V_{\text{eff}}(\sigma) &= -\frac{S_{\text{eff}}[\sigma]}{V^{(2)}} \\ &= \frac{iN}{V^{(2)}}\int \frac{\mathrm{d}^2p}{(2\pi)^2}\log(-p^2-\sigma^2) + \frac{1}{2g^2}\sigma^2, \end{aligned}$$

where $V^{(2)}$ is the spacetime volume.

To evaluate the momentum integral, we Wick rotate to Euclidean space

$$\int \frac{\mathrm{d}^2p}{(2\pi)^2}\log(-p^2-\sigma^2) = -i\int \frac{\mathrm{d}^2p_E}{(2\pi)^2}\log(p_E^2+\sigma^2).$$

This integral is divergent and requires regularization. Using dimensional regularization with $d = 2-\epsilon$, we have

$$\begin{aligned} \int \frac{\mathrm{d}^dp_E}{(2\pi)^d}\log(p_E^2+\sigma^2) &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\int \frac{\mathrm{d}^dp_E}{(2\pi)^d}\frac{1}{(p_E^2+\sigma^2)^\alpha}\bigg|_{\alpha=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{\Gamma(\alpha-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(\alpha)}\sigma^{d-2\alpha}\right]\bigg|_{\alpha=0}. \end{aligned}$$

Taking the derivative and setting $\alpha = 0$, we have

$$\begin{aligned} \int \frac{\mathrm{d}^dp_E}{(2\pi)^d}\log(p_E^2+\sigma^2) &= \frac{\sigma^d}{(4\pi)^{d/2}}\left[\frac{\Gamma'(-\frac{d}{2})}{\Gamma(0)} - \frac{\Gamma(-\frac{d}{2})\Gamma'(0)}{\Gamma^2(0)}\right] \\ &\quad + \frac{(d-2\alpha)\sigma^{d-2\alpha-1}}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}\bigg|_{\alpha=0}. \end{aligned}$$

For $d = 2-\epsilon$ as $\epsilon \to 0$, this gives

$$\int \frac{\mathrm{d}^2p_E}{(2\pi)^2}\log(p_E^2+\sigma^2) = \frac{\sigma^2}{4\pi}\left(\frac{2}{\epsilon} - \gamma_E + \log(4\pi) - \log(\sigma^2) + 1\right) + O(\epsilon),$$

where γ_E is the Euler-Mascheroni constant.

The divergent $\frac{2}{\epsilon}$ term is proportional to σ^2 and can be absorbed into a renormalization of g^2. Using minimal subtraction, we remove only the pole term and obtain:

$$V_{\text{eff}}(\sigma) = \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 1\right),$$

where μ is the renormalization scale introduced to make the logarithm dimensionless.

(e) To find the minimum of the effective potential, we take its derivative with respect to σ and set it equal to zero. We have

$$\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}\sigma} = \frac{\sigma}{g^2} + \frac{N}{2\pi}\sigma\log\frac{\sigma^2}{\mu^2} = 0.$$

This equation has two solutions: $\sigma = 0$ and $\sigma = \pm\mu e^{-\pi/(g^2 N)}$. To determine which is the minimum, we compute the second derivative, getting

$$\frac{d^2 V_{\text{eff}}}{d\sigma^2} = \frac{1}{g^2} + \frac{N}{2\pi}\log\frac{\sigma^2}{\mu^2} + \frac{N}{\pi}.$$

At $\sigma = 0$, the second derivative is undefined (due to the logarithm), but the effective potential approaches $+\infty$ as $\sigma \to 0$ from either side.

At $\sigma = \pm\mu e^{-\pi/(g^2 N)}$, the second derivative is:

$$\begin{aligned}\frac{d^2 V_{\text{eff}}}{d\sigma^2} &= \frac{1}{g^2} + \frac{N}{2\pi}\left(-\frac{2\pi}{g^2 N}\right) + \frac{N}{\pi} \\ &= \frac{N}{\pi} > 0.\end{aligned}$$

Therefore, $\sigma = \pm\mu e^{-\pi/(g^2 N)}$ gives a minimum of the effective potential.

The non-zero vacuum expectation value of σ breaks the discrete chiral symmetry $\psi_i \to \gamma^5\psi_i$ that we verified in part (a). This is because σ couples to the fermion bilinear $\bar{\psi}_i\psi_i$, which transforms as $\bar{\psi}_i\psi_i \to -\bar{\psi}_i\psi_i$ under the symmetry.

The result depends on the renormalization scale μ, but the dependence is only on the overall scale of σ. The fact that σ acquires a non-zero VEV is independent of the renormalization scheme.

(f) The effective potential we derived can be written as

$$\begin{aligned}V_{\text{eff}}(\sigma) &= \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 1\right) \\ &= N\left[\frac{1}{2Ng^2}\sigma^2 + \frac{1}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 1\right)\right].\end{aligned}$$

Defining $\lambda = g^2 N$ (which is held fixed), we have

$$\begin{aligned}V_{\text{eff}}(\sigma) &= N\left[\frac{1}{2\lambda}\sigma^2 + \frac{1}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 1\right)\right] \\ &= N \cdot f(\sigma, \lambda).\end{aligned}$$

This is indeed of the form $V_{\text{eff}}(\sigma) = N \cdot f(g^2 N)$ as claimed.

To understand the N-dependence of higher-order contributions, let's examine the structure of the loop expansion. The one-loop contribution we calculated came from integrating out the N fermion fields, giving a factor of N from the determinant.

For higher-loop contributions, let's consider the diagrammatic expansion of the effective potential. Each fermion loop contributes a factor of N from the trace over fermion flavors. However, each interaction vertex contains a factor of g^2.

At two loops, we would have diagrams with two fermion loops (factor of N^2) connected by interactions (factor of $g^4 = \lambda^2/N^2$), giving a contribution proportional to $N^0 = 1$. At three loops, we would get contributions proportional to N^{-1}, and so on.

For example, a typical two-loop contribution would be:

$$V_{\text{2-loop}} \sim N^2 \cdot \frac{\lambda^2}{N^2} \cdot F(\sigma, \lambda) = F(\sigma, \lambda),$$

where F is some function of σ and λ.

Similarly, a three-loop contribution would be:

$$V_{\text{3-loop}} \sim N^3 \cdot \frac{\lambda^3}{N^3} \cdot G(\sigma, \lambda) = G(\sigma, \lambda).$$

When we take the limit $N \to \infty$ with $\lambda = g^2 N$ fixed, the one-loop contribution scales as N, while the two-loop and higher contributions are suppressed by powers of $1/N$. Therefore, in this limit, the one-loop result becomes exact, and our result from part (e) is unambiguous.

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