PHYS 630 - Advanced Electricity and Magnetism

Student: **Ralph Razzouk**

Homework 7

Problem 1
Estimate the deflection angle for an electron of energy $\epsilon = 1$ keV passing at a distance $\ell = 1$ μm from the proton.

Solution. The deflection angle θ for a charged particle in a Coulomb field can be estimated using the formula:

$$\theta \approx \frac{2b}{\ell}.$$

where b is the impact parameter given by $b = \frac{k|q_1 q_2|}{mv^2}$.
For the velocity v, we use the energy which is

$$\epsilon = \frac{1}{2}mv^2 \implies v = \sqrt{\frac{2\epsilon}{m}} = \sqrt{\frac{2 \times 1.60 \times 10^{-16}}{9.11 \times 10^{-31}}} \approx 1.87 \times 10^7 \text{ m/s}.$$

Calculating b, we have

$$b = \frac{ke^2}{mv^2} = \frac{(8.99 \times 10^9)\left(1.60 \times 10^{-19}\right)^2}{(9.11 \times 10^{-31})(1.87 \times 10^7)^2} \approx 7.224 \times 10^{-13}\,\text{m}.$$

Finally, calculating θ, we have

$$\theta \approx \frac{2b}{\ell} = \frac{2(7.224 \times 10^{-13})}{10^{-6}} \approx 1.44 \times 10^{-6}\,\text{radians}.$$

Therefore, the deflection angle is approximately 1.44×10^{-6} radians or about 0.297 arcseconds.
This small angle makes sense because the electron's energy is relatively high (1 keV) and the distance of closest approach is large (1 μm).

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Problem 2
It's a bright sunny day. The Sun is aligned with the zenith. You are holding a parasol of radius $r = 50$ cm. Find the force from the Sun (assume complete absorption). Look up luminosity of the Sun and distance to the Sun.

Solution. The luminosity of the sun is $L_\odot = 3.828 \times 10^{26}$ W and the distance to the sun is 1 AU $= 1.496 \times 10^8$ km.
The solar irradiance at Earth's distance is

$$I = \frac{L_\odot}{4\pi d^2} = \frac{3.828 \times 10^{26}}{4\pi(1.496 \times 10^{11})^2} \approx 1361\,\text{W/m}^2.$$

This value is known as the solar constant.
The area of the parasol is

$$A = \pi r^2 = \pi(0.5)^2 = 0.785\,\text{m}^2.$$

The power received by the parasol is

$$P = IA = (1361)(0.785) = 1068.4\,\text{W}.$$

Assuming complete absorption, the force from the Sun on the parasol is

$$F = \frac{P}{c} = \frac{1068.4}{2.998 \times 10^8} = 3.56 \times 10^{-6}\,\text{N}.$$

Therefore, the Sun exerts a force of approximately $3.56\,\mu$N on the parasol.
This is a very small force, roughly equivalent to the weight of a few micrograms on Earth, but it is responsible for important effects like radiation pressure in space and comet tail formation. ■

Problem 3
An electromagnetic wave is propagating along z and is polarized along x, so that its vector potential is

$$\mathbf{A} = A_0 \left(\cos\left(\omega t - k_z z\right), 0, 0\right),$$

where $\omega = k_z c$.
Calculate the averaged Poynting flux (over period of oscillation). Represent this linearly polarized wave as a sum of two circularly polarized waves and write the vector potential for $\mathbf{A}_R$ and $\mathbf{A}_L$, the right and left-polarized components. There are media where right and left circular polarizations propagate with different speeds (called gyrotropic). After passing through such a media one of the waves (does not matter which, right or left) is delayed in phase by π (so that it becomes $\propto \omega t - k_z z + \pi$). Find the new polarization after the wave leaves that gyrotropic medium.

Solution. We first calculate the fields and Poynting flux for the initial wave. From $\mathbf{A}$ we can find $\mathbf{B}$ and $\mathbf{E}$, as follows

$$\mathbf{B} = \nabla \times \mathbf{A} = A_0 \left(0, -k_z \cos(\omega t - k_z z), 0\right),$$
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = A_0 \omega \left(\sin(\omega t - k_z z), 0, 0\right).$$

The Poynting vector is given by

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

Substituting, we get

$$\mathbf{S} = \frac{A_0^2 \omega k_z}{\mu_0} \left(0, 0, \sin(\omega t - k_z z)\cos(\omega t - k_z z)\right).$$

Time averaging, using $\langle \sin(\alpha)\cos(\alpha)\rangle = \frac{1}{2}$, we have

$$\langle \mathbf{S} \rangle = \frac{A_0^2 \omega k_z}{2\mu_0} (0, 0, 1)$$

Now for circular polarization decomposition: A linearly polarized wave can be represented as a sum of right and left circular polarizations:

$$\mathbf{A}_R = \frac{A_0}{2} \left(\cos(\omega t - k_z z), -\mathrm{i}\cos(\omega t - k_z z), 0\right),$$
$$\mathbf{A}_L = \frac{A_0}{2} \left(\cos(\omega t - k_z z), \mathrm{i}\cos(\omega t - k_z z), 0\right).$$

We can check that $\mathbf{A}_R + \mathbf{A}_L = \mathbf{A}$, since the imaginary parts cancel.
After passing through the gyrotropic medium, one component gets phase shifted by π: If right component is shifted:

$$\mathbf{A}'_R = \frac{A_0}{2} \left(\cos(\omega t - k_z z + \pi), -\mathrm{i}\cos(\omega t - k_z z + \pi), 0\right),$$
$$\mathbf{A}'_L = \frac{A_0}{2} \left(\cos(\omega t - k_z z), \mathrm{i}\cos(\omega t - k_z z), 0\right).$$

The resulting wave is

$$\mathbf{A}' = \mathbf{A}'_R + \mathbf{A}'_L = A_0 \left(0, \cos(\omega t - k_z z), 0\right).$$

The resulting wave is linearly polarized along y axis. This is an interesting result. The gyrotropic medium has rotated the polarization by 90 degrees. This is the basic principle behind optical isolators and other devices that manipulate polarization states. ■

Problem 4
An electromagnetic wave $\mathbf{A}_1$ is propagating along z and is polarized along x, so that its vector potential is
$$\mathbf{A}_1 = A_0 \left(\cos\left(\omega t - k_z z\right), 0, 0\right),$$
where $\omega = k_z c$.
Another identical electromagnetic wave $\mathbf{A}_2$ propagating along z and is polarized along x, is added. Calculate averaged Poynting flux (over period of oscillation) of the two waves together.

Solution. Let's calculate the total fields and Poynting flux for two identical waves. Starting with the vector potentials, we have

$$\mathbf{A}_1 = A_0 \left(\cos(\omega t - k_z z), 0, 0\right),$$
$$\mathbf{A}_2 = A_0 \left(\cos(\omega t - k_z z), 0, 0\right).$$

The total vector potential is

$$\mathbf{A}_{tot} = \mathbf{A}_1 + \mathbf{A}_2 = 2A_0 \left(\cos(\omega t - k_z z), 0, 0\right).$$

From this, we can find $\mathbf{B}$ and $\mathbf{E}$, given by

$$\mathbf{B}_{tot} = \nabla \times \mathbf{A}_{tot} = 2A_0 \left(0, -k_z \cos(\omega t - k_z z), 0\right),$$
$$\mathbf{E}_{tot} = -\frac{\partial \mathbf{A}_{tot}}{\partial t} = 2A_0 \omega \left(\sin(\omega t - k_z z), 0, 0\right).$$

The Poynting vector is given by
$$\mathbf{S}_{tot} = \frac{1}{\mu_0} \mathbf{E}_{tot} \times \mathbf{B}_{tot}.$$

Substituting, we have
$$\mathbf{S}_{tot} = \frac{4A_0^2 \omega k_z}{\mu_0} \left(0, 0, \sin(\omega t - k_z z) \cos(\omega t - k_z z)\right).$$

Time averaging, using $\langle \sin(\alpha) \cos(\alpha) \rangle = \frac{1}{2}$, we have

$$\langle \mathbf{S}_{tot} \rangle = \frac{2A_0^2 \omega k_z}{\mu_0} (0, 0, 1).$$

Note that
$$\langle \mathbf{S}_{tot} \rangle = 4 \langle \mathbf{S}_1 \rangle = 4 \langle \mathbf{S}_2 \rangle.$$

The total Poynting flux is four times larger than the flux of each individual wave. This is because the amplitudes of $\mathbf{E}$ and $\mathbf{B}$ fields doubled and because the Poynting flux is proportional to the product of $\mathbf{E}$ and $\mathbf{B}$, hence the factor of 4. This demonstrates constructive interference of two identical waves leading to intensity enhancement. ■