PHYS 662 - Quantum Field Theory I

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Homework 7

Problem 1
Consider a theory of real scalar fields A, B, and C with Lagrangian density

$$\begin{aligned}\delta = &-\frac{1}{2}\partial^\mu A\partial_\mu A - \frac{1}{2}m_A^2 A^2 \\ &-\frac{1}{2}\partial^\mu B\partial_\mu B - \frac{1}{2}m_B^2 B^2 \\ &-\frac{1}{2}\partial^\mu C\partial_\mu C - \frac{1}{2}m_C^2 C^2 \\ &+ gABC\end{aligned}$$

Write down the tree level amplitudes for these processes

$$\begin{array}{ll} AA \to AA & AA \to BC \\ AA \to AB & AB \to AB \\ AA \to BB & AB \to AC \end{array}$$

Solution. Let us start by analyzing the propagators and vertices from the given Lagrangian density. We can then use these to determine the tree-level amplitudes for the given scattering processes. Looking at the Lagrangian density, we can see that it consists of three Klein-Gordon terms and an interaction term. The Klein-Gordon terms lead to the propagators

$$\begin{aligned}\langle AA\rangle &= \frac{\mathrm{i}}{p^2 - m_A^2}, \\ \langle BB\rangle &= \frac{\mathrm{i}}{p^2 - m_B^2}, \\ \langle CC\rangle &= \frac{\mathrm{i}}{p^2 - m_C^2}.\end{aligned}$$

The interaction term $gABC$ leads to a single type of vertex where all three fields meet, with vertex factor $\mathrm{i}g$. Importantly, this means no two fields of the same type can interact directly at a vertex.
For tree-level amplitudes in $\alpha\beta \to \gamma\delta$ scattering, we must have external legs connected by vertices and propagators. Due to the form of our interaction vertex, we can have three possible diagram topologies:

1. s-channel: Initial particles combine at a vertex, intermediate particle propagates, then splits at second vertex
2. t-channel: One initial particle exchanges with one final particle through an intermediate propagator
3. u-channel: Similar to t-channel but with different final state pairing

Let us now analyze each given process:

1. $AA \to AA$: Since our vertex requires three different fields, there are no allowed tree-level diagrams. Therefore
$$\mathcal{M}(AA \to AA) = 0.$$
2. $AA \to AB$: Again, no diagrams are possible since each vertex needs three different fields. Therefore
$$\mathcal{M}(AA \to AB) = 0.$$

3. $AA \to BB$: Here we can have t-channel and u-channel diagrams with a C propagator. Therefore
$$\mathcal{M}(AA \to BB) = -g^2\left(\frac{1}{(p_1-k_1)^2-m_C^2}+\frac{1}{(p_1-k_2)^2-m_C^2}\right).$$
where p_i are initial momenta and k_i are final momenta.

4. $AA \to BC$: No allowed diagrams since each vertex requires exactly three different fields. Therefore
$$\mathcal{M}(AA \to BC) = 0.$$

5. $AB \to AB$: Here we can have s-channel and u-channel diagrams with a C propagator. Therefore
$$\mathcal{M}(AB \to AB) = -g^2\left(\frac{1}{(p_1+p_2)^2-m_C^2}+\frac{1}{(p_1-k_2)^2-m_C^2}\right).$$

6. $AB \to AC$: No allowed diagrams are possible. Therefore
$$\mathcal{M}(AB \to AC) = 0.$$

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Problem 2 - Rutherford Scattering
The cross section for scattering an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. We treat the field $A_\mu(x)$ as a classical potential and consider the interaction
$$\hat{H}_I = \int \mathrm{d}^3x\ e\hat{\bar{\psi}}\gamma^\mu\hat{\psi}A_\mu,$$
where $\psi(x)$ is a quantized Dirac field.

(a) Compute the T-matrix for electron scattering off a localized classical potential to the lowest order.

(b) If $A_\mu(x)$ is time-independent, it is natural to define
$$\langle p'|\mathrm{i}T|p\rangle \equiv \mathrm{i}M(2\pi)\delta(E_f-E_i),$$
where E_i and E_f are the initial and final energies respectively.
Show that the cross section for scattering off a time-independent localized potential is
$$\mathrm{d}\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{\mathrm{d}^3p_f}{(2\pi)^3}\frac{1}{2E_f}\left|M\left(p_i-p_f\right)\right|^2 (2\pi)\,\delta(E_f-E_i),$$
where v_i is the initial velocity.
Integrate over $|p_f|$ to find $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$.

Solution. (a) We start with the interaction Hamiltonian
$$H_I = e\int \mathrm{d}^3x\ \bar{\psi}\gamma^\mu\psi A_\mu.$$
To find the T-matrix element to lowest order, we need to evaluate
$$\langle p'|\mathrm{i}T|p\rangle = \left\langle p'\middle|-\mathrm{i}\int \mathrm{d}^3x\ H_I(x)\middle|p\right\rangle.$$
For single electron states we have
$$\langle p'|\bar{\psi}\gamma^\mu\psi|p\rangle = \bar{u}(p')\gamma^\mu u(p)\mathrm{e}^{\mathrm{i}(p'-p)x}.$$

Therefore,

$$\langle p'|\mathrm{i}T|p\rangle = -\mathrm{i}e\int \mathrm{d}^3x\ A_\mu(x)\bar{u}(p')\gamma^\mu u(p)\mathrm{e}^{\mathrm{i}(p'-p)x} = -\mathrm{i}e\tilde{A}_\mu(p'-p)\bar{u}(p')\gamma^\mu u(p),$$

where $\tilde{A}_\mu(q)$ is the Fourier transform of $A_\mu(x)$.

(b) For a time-independent potential, we define

$$\langle p'|\mathrm{i}T|p\rangle = \mathrm{i}M(2\pi)\delta(E_f - E_i).$$

The differential cross section is given by

$$\mathrm{d}\sigma = \frac{1}{v_i}\frac{\mathrm{d}^3p_f}{(2\pi)^3}\frac{1}{2E_f}\frac{1}{2E_i}|M|^2(2\pi)\delta(E_f - E_i).$$

To integrate over $|p_f|$, we use the energy delta function. First, we write

$$\mathrm{d}^3p_f = p_f^2\,\mathrm{d}p_f\,\mathrm{d}\Omega.$$

The delta function enforces $E_f = E_i$, which means $|p_f| = |p_i|$ for elastic scattering. Thus,

$$\delta(E_f - E_i) = \delta\left(\sqrt{p_f^2+m^2} - \sqrt{p_i^2+m^2}\right) = \frac{E_f}{p_f}\delta(p_f - p_i).$$

Integrating over p_f we have

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{v_i}\frac{p_i^2}{(2\pi)^3}\frac{1}{2E_i}\frac{1}{2E_i}|M|^2(2\pi)\frac{E_i}{p_i}\\ &= \frac{1}{(4\pi)^2}\frac{p_i}{v_i}|M|^2.\end{aligned}$$

Using $v_i = p_i/E_i$, we get our final result

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{E_i}{(4\pi)^2}|M|^2.$$

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Problem 3
Any particle that couples to electrons can produce a correction to $\mathrm{g}-2$. Because the $\mathrm{g}-2$ agrees with QED to high accuracy, the corrections to $\mathrm{g}-2$ constrain properties of hypothetical particles.

(a) Consider the Higgs boson h which couples to electron as

$$H_{\text{int}} = \int \mathrm{d}^3x\ \frac{\lambda}{\sqrt{2}}h\bar{\psi}\psi.$$

Compute the contribution of a virtual Higgs boson to election's $\mathrm{g}-2$, in terms of λ and the mass of Higgs m_h.

(b) If the experimental value of $a = \mathrm{g}-2$ agrees with a_{QED} up to 10^{-10}, what limit does this place on λ and m_h?

Solution. (a) The one-loop vertex correction from the Higgs boson is given by

$$-\mathrm{i}e\Gamma^\mu = \left(-\mathrm{i}\frac{\lambda}{\sqrt{2}}\right)^2 \mathrm{i}^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{k^2 - m_h^2} \frac{(\not{p}' - \not{k} + m)}{(p'-k)^2 - m^2} \gamma^\mu \frac{(\not{p} - \not{k} + m)}{(p-k)^2 - m^2}.$$

Using Feynman parameters we can write

$$\frac{1}{ABC} = 2\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\; \delta(x+y+z-1)\frac{1}{(Ax+By+Cz)^3}.$$

After the substitution $l = k - yp' - zp$, and keeping only terms relevant for g-2, we get

$$\Gamma^\mu = \frac{\lambda^2}{16\pi^2}\int_0^1 \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\; \frac{\delta(x+y+z-1)}{xm_h^2 + (y+z)^2m^2 - yzq^2}\left[\gamma^\mu - \frac{2m^2}{q^2}((2-y-z)^2-2)\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\right].$$

The g-2 factor is related to the form factor $F_2(0)$ by

$$\frac{g-2}{2} = F_2(0).$$

Therefore

$$a = \frac{g-2}{2} = \frac{\lambda^2}{8\pi^2}\int_0^1 \mathrm{d}Y\; \frac{2-(2-Y)^2}{(1-Y)m_h^2 + Y^2m^2},$$

where we changed variables to $Y = y + z$.
Evaluating this integral we get

$$a = \frac{\lambda^2}{8\pi^2}\frac{m^2}{m_h^2}.$$

(b) Given that the experimental value agrees with QED up to 10^{-10}, we must have

$$\frac{\lambda^2}{8\pi^2}\frac{m^2}{m_h^2} < 10^{-10}.$$

Solving for λ

$$\lambda < 10^{-5}\frac{m_h}{m}\sqrt{8\pi^2} \approx 10^{-4}\frac{m_h}{m}.$$

For a Higgs mass of $m_h \approx 125$ GeV and electron mass $m \approx 0.5$ MeV, this gives

$$\lambda < 2.5 \times 10^{-2}.$$

This puts a fairly stringent constraint on the coupling between the Higgs boson and electrons. ■