PHYS 662 - Quantum Field Theory I

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Homework 3

Problem 1 - Algebra of Charges

Consider the Lagrangian

$$\mathcal{L} = \partial_\mu \Phi_a^\dagger \partial^\mu \Phi_a - V\left(\Phi_a\right)$$

and suppose it is symmetric under the transformation

$$\phi_a \to \phi^a + \mathrm{i}\theta_k \left(\hat{\lambda}_k\right)^a_b \phi^b + \mathcal{O}(\theta_k^2)$$

(a) Write down the conserved currents and conserved charges.

(b) Use canonical commutation relations

$$\left[\pi_a\left(\vec{x}\right), \phi_b\left(\vec{y}\right)\right] = -\mathrm{i}\delta^d\left(\vec{x}-\vec{y}\right)\delta_a^b$$

to compute the commutators $\left[J_0^a\left(\vec{x}\right), J_0^b\left(\vec{y}\right)\right]$, $\left[Q^a, J_0^b\left(\vec{x}\right)\right]$, and $\left[Q^a, Q^b\right]$.

Solution. (a) We are given the infinitesimal transformation

$$\delta\phi_a(\theta) = \mathrm{i}\theta_k(\lambda_k)_{ab}\phi_b,$$

that leaves the Lagrangian invariant. From that, the conserved current is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a + \delta\phi_a^\dagger\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a^\dagger)} = J_k^\mu\theta_k,$$

where

$$J_k^\mu = \mathrm{i}\left[\left(\partial_\mu\phi_a^\dagger\right)(\lambda_k)_{ab}\phi_b - \phi_a^\dagger(\lambda_k)_{ab}^\dagger\left(\partial_\mu\phi_b\right)\right]$$

represents the component of the current corresponding to the kth generator of the symmetry. Since a general symmetry transformation can have multiple independent components, each corresponding to a different generator of the symmetry group, then we have a family of conserved currents J_k^μ, one for each generator k. The current labeled with k is tied to the specific transformation parameterized by θ_k.

Then, the conserved charge is

$$\begin{aligned} Q_k &= \int J_k^0 \,\mathrm{d}^d x \\ &= \mathrm{i}\int\left[\left(\partial_\mu\phi_a^\dagger\right)(\lambda_k)_{ab}\phi_b - \phi_a^\dagger(\lambda_k)_{ab}^\dagger\left(\partial_\mu\phi_b\right)\right]\mathrm{d}^d x. \end{aligned}$$

(b) Define the canonical momenta as

$$\pi_a = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi_a)} = \frac{\partial\phi_a^\dagger}{\partial t}.$$

Then the canonical commutation relations are given by

$$\begin{aligned} \left[\pi_a\left(\mathbf{x}\right), \phi_b\left(\mathbf{y}\right)\right] &= \left[\pi_a^\dagger\left(\mathbf{x}\right), \phi_b^\dagger\left(\mathbf{y}\right)\right] = -\mathrm{i}\delta^d\left(\mathbf{x}-\mathbf{y}\right)\delta_a^b, \\ \left[\pi_a^\dagger\left(\mathbf{x}\right), \phi_b\left(\mathbf{y}\right)\right] &= \left[\pi_a\left(\mathbf{x}\right), \phi_b^\dagger\left(\mathbf{y}\right)\right] = 0. \end{aligned}$$

Combining these commutation relations, and using the identity

$$[AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B$$

we get the following four results:

- $$\begin{aligned}
&\left[\phi_a^\dagger(\mathbf{x})\frac{\partial \phi_b}{\partial t}(\mathbf{x}), \frac{\partial \phi_c^\dagger}{\partial t}(\mathbf{y})\phi_d(\mathbf{y})\right] \\
&= \left[\phi_a^\dagger(\mathbf{x})\pi_b^\dagger(\mathbf{x}), \pi_c(\mathbf{y})\phi_d(\mathbf{y})\right] \\
&= \phi_a^\dagger(\mathbf{x})[\pi_b^\dagger(\mathbf{x}), \pi_c(\mathbf{y})]\phi_d(\mathbf{y}) + [\phi_a^\dagger(\mathbf{x}), \pi_c(\mathbf{y})]\pi_b^\dagger(\mathbf{x})\phi_d(\mathbf{y}) \\
&\quad + \pi_c(\mathbf{y})\phi_a^\dagger(\mathbf{x})[\pi_b^\dagger(\mathbf{x}), \phi_d(\mathbf{y})] + \pi_c(\mathbf{y})[\phi_a^\dagger, \phi_d(\mathbf{y})]\pi_b^\dagger(\mathbf{x}) \\
&= 0.
\end{aligned}$$

- $$\begin{aligned}
&\left[\frac{\partial \phi_a^\dagger}{\partial t}(\mathbf{x})\phi_b(\mathbf{x}), \phi_c^\dagger(\mathbf{y})\frac{\partial \phi_d}{\partial t}(\mathbf{y})\right] \\
&= \left[\pi_a(\mathbf{x})\phi_b(\mathbf{x}), \phi_c^\dagger(\mathbf{y})\pi_d^\dagger(\mathbf{y})\right] \\
&= \pi_a(\mathbf{x})[\phi_b(\mathbf{x}), \phi_c^\dagger(\mathbf{y})]\pi_d^\dagger(\mathbf{y}) + [\pi_a(\mathbf{x}), \phi_c^\dagger(\mathbf{y})]\phi_b(\mathbf{x})\pi_d^\dagger(\mathbf{y}) \\
&\quad + \phi_c^\dagger(\mathbf{y})\pi_a(\mathbf{x})[\phi_b(\mathbf{x}), \pi_d^\dagger(\mathbf{y})] + \phi_c^\dagger(\mathbf{y})[\pi_a(\mathbf{x}), \pi_d^\dagger(\mathbf{y})]\phi_b(\mathbf{x}) \\
&= 0.
\end{aligned}$$

- $$\begin{aligned}
&\left[\frac{\partial \phi_a^\dagger}{\partial t}(\mathbf{x})\phi_b(\mathbf{x}), \frac{\partial \phi_c^\dagger}{\partial t}(\mathbf{y})\phi_d(\mathbf{y})\right] \\
&= [\pi_a(\mathbf{x})\phi_b(\mathbf{x}), \pi_c(\mathbf{y})\phi_d(\mathbf{y})] \\
&= \pi_a(\mathbf{x})\left[\phi_b(\mathbf{x}), \pi_c(\mathbf{y})\right]\phi_d(\mathbf{y}) + \pi_a(\mathbf{x})\pi_c(\mathbf{y})\left[\phi_b(\mathbf{x}), \phi_d(\mathbf{y})\right] \\
&\quad + \left[\pi_a(\mathbf{x}), \pi_c(\mathbf{y})\right]\phi_d(\mathbf{y})\phi_b(\mathbf{x}) + \pi_c(\mathbf{y})\left[\pi_a(\mathbf{x}), \phi_d(\mathbf{y})\right]\phi_b(\mathbf{x}) \\
&= \mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\delta_{bc}\pi_a(\mathbf{x})\phi_d(\mathbf{y}) - \delta_{ad}\pi_c(\mathbf{y})\phi_b(\mathbf{x})\right)
\end{aligned}$$

- $$\begin{aligned}
&\left[\phi_a^\dagger(\mathbf{x})\frac{\partial \phi_b}{\partial t}(\mathbf{x}), \phi_c^\dagger(\mathbf{y})\frac{\partial \phi_d}{\partial t}(\mathbf{y})\right] \\
&= \left[\phi_a^\dagger(\mathbf{x})\pi_b^\dagger(\mathbf{x}), \phi_c^\dagger(\mathbf{y})\pi_d^\dagger(\mathbf{y})\right] \\
&= \phi_a^\dagger(\mathbf{x})\left[\pi_b^\dagger(\mathbf{x}), \phi_c^\dagger(\mathbf{y})\right]\pi_d^\dagger(\mathbf{y}) + \phi_a^\dagger(\mathbf{x})\phi_c^\dagger(\mathbf{y})\left[\pi_b^\dagger(\mathbf{x}), \pi_d^\dagger(\mathbf{y})\right] \\
&\quad + \left[\phi_a^\dagger(\mathbf{x}), \phi_c^\dagger(\mathbf{y})\right]\pi_d^\dagger(\mathbf{y})\pi_b^\dagger(\mathbf{x}) + \phi_c^\dagger(\mathbf{y})\left[\phi_a^\dagger(\mathbf{x}), \pi_d^\dagger(\mathbf{y})\right]\pi_b^\dagger(\mathbf{x}) \\
&= \mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\delta_{ad}\phi_c^\dagger(\mathbf{y})\pi_b^\dagger(\mathbf{x}) - \delta_{bc}\phi_a^\dagger(\mathbf{x})\pi_d^\dagger(\mathbf{y})\right)
\end{aligned}$$

- The commutation relation for the conserved current is

$$
\begin{aligned}
&\left[J_k^0(\mathbf{x}), J_l^0(\mathbf{y})\right] \\
&= \left[\left(\left(\partial_\mu \phi_a^\dagger\right)(\lambda_k)_{ab}\phi_b - \phi_a^\dagger (\lambda_k)_{ab}^\dagger \left(\partial_\mu \phi_b\right)\right), \left(\left(\partial_\mu \phi_a^\dagger\right)(\lambda_l)_{ab}\phi_b - \phi_a^\dagger (\lambda_l)_{ab}^\dagger \left(\partial_\mu \phi_b\right)\right)\right] \\
&= \left[\partial_\mu \phi_a^\dagger(\mathbf{x})(\lambda_k)_{ab}\phi_b(\mathbf{x}), \partial_\mu \phi_c^\dagger(\mathbf{y})(\lambda_l)_{cd}\phi_d(\mathbf{y})\right] - \left[\partial_\mu \phi_a^\dagger(\mathbf{x})(\lambda_k)_{ab}\phi_b(\mathbf{x}), \phi_c^\dagger(\mathbf{y})(\lambda_l)_{cd}^\dagger \partial_\mu \phi_d(\mathbf{y})\right] \\
&\quad - \left[\phi_a^\dagger(\mathbf{x})(\lambda_k)_{ab}^\dagger \partial_\mu \phi_b(\mathbf{x}), \partial_\mu \phi_c^\dagger(\mathbf{y})(\lambda_l)_{cd}\phi_d(\mathbf{y})\right] + \left[\phi_a^\dagger(\mathbf{x})(\lambda_k)_{ab}^\dagger \partial_\mu \phi_b(\mathbf{x}), \phi_c^\dagger(\lambda_l)_{cd}^\dagger(\mathbf{y})\partial_\mu \phi_d(\mathbf{y})\right] \\
&= -\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\pi_a(\mathbf{x})\left(\lambda_k\right)_{ac}\left(\lambda_l\right)_{cb}\phi_b(\mathbf{y}) - \pi_a(\mathbf{y})\left(\lambda_l\right)_{ac}\left(\lambda_k\right)_{cb}\phi_b(\mathbf{x})\right) \\
&\quad - \mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\phi_a^\dagger(\mathbf{y})\left(\lambda_l\right)_{ac}^\dagger\left(\lambda_k\right)_{cb}^\dagger \pi_b^\dagger(\mathbf{x}) - \phi_a^\dagger(\mathbf{x})\left(\lambda_k\right)_{ac}^\dagger\left(\lambda_l\right)_{cb}^\dagger \pi_b^\dagger(\mathbf{y})\right) \\
&= -\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\pi_a(\mathbf{x})\left[\lambda_k, \lambda_l\right]_{ab}\phi_b(\mathbf{x}) + \phi_a^\dagger(\mathbf{x})\left[\lambda_k, \lambda_l\right]_{ab}^\dagger \pi_b^\dagger(\mathbf{x})\right).
\end{aligned}
$$

If λ_k are the generators of a Lie algebra with a Lie bracket given by $[\lambda_k, \lambda_l] = \mathrm{i} f_{kl}^m \lambda_m$, then

$$
\begin{aligned}
\left[J_k^0(\mathbf{x}), J_l^0(\mathbf{y})\right] &= -\mathrm{i}\delta(\mathbf{x}-\mathbf{y})\left(\pi_a(\mathbf{x})\left[\lambda_k, \lambda_l\right]_{ab}\phi_b(\mathbf{x}) + \phi_a^\dagger(\mathbf{x})\left[\lambda_k, \lambda_l\right]_{ab}^\dagger \pi_b^\dagger(\mathbf{x})\right) \\
&= \delta(\mathbf{x}-\mathbf{y}) f_{kl}^m \left(\pi_a(\mathbf{x})\left(\lambda_m\right)_{ab}\phi_b(\mathbf{x}) + \phi_a^\dagger(\mathbf{x})\left(\lambda_m\right)_{ab}^\dagger \pi_b^\dagger(\mathbf{x})\right) \\
&= -\mathrm{i}\delta(\mathbf{x}-\mathbf{y}) f_{kl}^m J_m^0(\mathbf{x}).
\end{aligned}
$$

- The commutation relation for the conserved charge and current is

$$
\begin{aligned}
\left[Q_k, J_l^0(\mathbf{x})\right] &= \int \left[J_k^0(\mathbf{y}), J_l^0(\mathbf{x})\right] \mathrm{d}^d y \\
&= -\mathrm{i} \int \delta(\mathbf{y}-\mathbf{x}) f_{kl}^m J_m^0(\mathbf{y}) \, \mathrm{d}^d y \\
&= -\mathrm{i} f_{kl}^m J_m^0(\mathbf{x}).
\end{aligned}
$$

- The commutation relation for the conserved charges is

$$
\begin{aligned}
[Q_k, Q_l] &= \int \left[Q_k, J_l^0(\mathbf{x})\right] \mathrm{d}^d x \\
&= -\mathrm{i} f_{kl}^m \int J_m^0(\mathbf{x}) \, \mathrm{d}^d x \\
&= -\mathrm{i} f_{kl}^m Q_m.
\end{aligned}
$$

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Problem 2 - Non-relativistic Galilean Group
Consider the non-relativistic Galilean group generated by rotations and non-relativistic boosts $\vec{x}^i \to \vec{x}^i + \vec{v}^i t$. Show that this algebra can be centrally extended with the central charge that's the total mass of the system.

Solution. The Lie algebra of the Galilean group is

$$
\begin{aligned}
[J_i, J_j] &= \mathrm{i}\epsilon_{ijk} J_k \\
[J_i, K_j] &= \mathrm{i}\epsilon_{ijk} K_k \\
[J_i, P_j] &= \mathrm{i}\epsilon_{ijk} P_k \\
[K_i, \mathcal{H}] &= \mathrm{i} P_i \\
[P_i, P_j] = [K_i, P_j] = [K_i, K_j] = [J_i, \mathcal{H}] &= [P_i, \mathcal{H}] = 0,
\end{aligned}
$$

where all indices run over $\{1, 2, 3\}$.

Let us try to centrally extend this algebra as follows

$$\begin{aligned} [J_i, J_j] &= \mathrm{i}\epsilon_{ijk}J_k + \mathrm{i}A_{ij}\mathbb{I} & [K_i, E] &= \mathrm{i}P_i + \mathrm{i}D_i\mathbb{I} & [J_i, E] &= \mathrm{i}W_i\mathbb{I} \\ [J_i, K_j] &= \mathrm{i}\epsilon_{ijk}K_k + \mathrm{i}B_{ij}\mathbb{I} & [K_i, P_j] &= \mathrm{i}U_{ij}\mathbb{I} & [P_i, E] &= \mathrm{i}X_i\mathbb{I} \\ [J_i, P_j] &= \mathrm{i}\epsilon_{ijk}P_k + iC_{ij}\mathbb{I} & [K_i, K_j] &= \mathrm{i}V_{ij}\mathbb{I} & [P_i, P_j] &= \mathrm{i}Y_{ij}\mathbb{I} \end{aligned}$$

where A_{ij}, V_{ij}, and Y_{ij} are anti-symmetric in (i,j), while B_{ij}, C_{ij}, and U_{ij} are anti-symmetric in (i,j) for $i \neq j$. As it turns out, most of these variables can be eliminated. Some can be eliminated by using the Jacobi identity, and the others can be eliminated by redefining the generators.
For arbitrary α, β, we have

$$\begin{aligned} 0 &= \left[\left[\alpha K_i + \beta P_i, \alpha K_j + \beta P_j\right], J_k\right] + \left[\left[\alpha K_j + \beta P_j, J_k\right], \alpha K_i + \beta P_i\right] + \left[\left[J_k, \alpha K_i + \beta P_i\right], \alpha K_j + \beta P_j\right] \\ &= \mathrm{i}\left[\alpha^2 V_{ij}\mathbb{I} + \alpha\beta\left(U_{ij} + U_{ji}\right)\mathbb{I}, J_k\right] - \mathrm{i}\left[\epsilon_{kjm}\left(\alpha K_m + \beta P_m\right) + \left(\alpha B_{kj} + \beta C_{kj}\right)\mathbb{I}, \alpha K_i + \beta P_i\right] \\ &\quad + \mathrm{i}\left[\epsilon_{kim}\left(\alpha K_m + \beta P_m\right) + \left(\alpha B_{ki} + \beta C_{ki}\right)\mathbb{I}, \alpha K_j + \beta P_j\right] \\ &= \epsilon_{kjm}\left(\alpha^2 V_{mi} + \beta^2 Y_{mi} + \alpha\beta\left(U_{mi} - U_{im}\right)\right)\mathbb{I} - \epsilon_{kim}\left(\alpha^2 V_{mj} + \beta^2 Y_{mj} + \alpha\beta\left(U_{mj} - U_{jm}\right)\right)\mathbb{I} \\ &= \epsilon_{kjl}\epsilon_{kjm}\left(\alpha^2 V_{mi} + \beta^2 Y_{mi} + \alpha\beta\left(U_{mi} - U_{im}\right)\right)\mathbb{I} - \epsilon_{kjl}\epsilon_{kim}\left(\alpha^2 V_{mj} + \beta^2 Y_{mj} + \alpha\beta\left(U_{mj} - U_{jm}\right)\right)\mathbb{I} \\ &= 2\left(\alpha^2 V_{li} + \beta^2 Y_{li} + \alpha\beta\left(U_{li} - U_{il}\right)\right)\mathbb{I} - \left(\delta_{ji}\delta_{lm} - \delta_{jm}\delta_{li}\right)\left(\alpha^2 V_{mj} + \beta^2 Y_{mj} + \alpha\beta\left(U_{mj} - U_{jm}\right)\right)\mathbb{I} \\ &= \left(\alpha^2 V_{li} + \beta^2 Y_{li} + \alpha\beta\left(U_{li} - U_{il}\right)\right)\mathbb{I}. \end{aligned}$$

Since α and β are arbitrary, we see that

$$V_{ij} = 0, \quad Y_{ij} = 0, \quad U_{ij} = U_{ji}.$$

Since U_{ij} is anti-symmetric for $i \neq j$, the last equation implies that we must have $U_{ij} = 0$ for $i \neq j$. That is, we must have $U_{ij} = M\delta_{ij}$.
Similarly, for arbitrary α, β, and γ, we have

$$\begin{aligned} 0 &= \left[\left[J_i, \alpha K_j + \beta P_j + \gamma J_j\right], E\right] + \left[\left[\alpha K_j + \beta P_j + \gamma J_j, E\right], J_i\right] + \left[\left[E, J_i\right], \alpha K_j + \beta P_j + \gamma J_j\right] \\ &= \mathrm{i}\alpha\left[\epsilon_{ijk}K_k + B_{ij}\mathbb{I}, E\right] + \mathrm{i}\beta\left[\epsilon_{ijk}P_k + C_{ij}\mathbb{I}, E\right] + \mathrm{i}\gamma\left[\epsilon_{ijk}J_k + A_{ij}, E\right] \\ &\quad + \mathrm{i}\alpha\left[P_j + D_j\mathbb{I}, J_i\right] + \mathrm{i}\beta\left[X_j\mathbb{I}, J_i\right] + i\gamma\left[W_j\mathbb{I}, J_i\right] - \mathrm{i}\left[W_i\mathbb{I}, \alpha K_j + \beta P_j + \gamma J_j\right] \\ &= \alpha\left(C_{ij} - \epsilon_{ijk}D_k\right)\mathbb{I} - \beta\epsilon_{ijk}X_k\mathbb{I} - \gamma\epsilon_{ijk}W_k\mathbb{I}. \end{aligned}$$

As α, β, and γ are arbitrary, we immediately see that

$$X_i = 0, \quad W_i = 0, \quad C_{ij} = \epsilon_{ijk}D_k.$$

Let us define the auxiliary variables

$$\tilde{P}_i = P_i + D_i\mathbb{I} \quad , \quad \tilde{K}_i = K_i + \frac{1}{2}\epsilon_{ijk}B_{jk}\mathbb{I} \quad , \quad \tilde{J}_i = J_i + \frac{1}{2}\epsilon_{ijk}A_{jk}\mathbb{I}$$

Then, for $\tilde{K}_i$, we have

$$\begin{aligned} \epsilon_{ijk}\tilde{K}_k &= \epsilon_{ijk}K_k + \frac{1}{2}\epsilon_{ijk}\epsilon_{kab}B_{ab}\mathbb{I} \\ &= \epsilon_{ijk}K_k + \frac{1}{2}\left(\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}\right)B_{ab}\mathbb{I} \\ &= \epsilon_{ijk}K_k + \frac{1}{2}\left(B_{ij} - B_{ji}\right)\mathbb{I} \\ &= \epsilon_{ijk}K_k + B_{ij}\mathbb{I}. \end{aligned}$$

where the last equality follows since $i \neq j$. Similarly, for $\tilde{J}_i$, we get

$$\epsilon_{ijk}\tilde{J}_k = \epsilon_{ijk}J_k + A_{ij}\mathbb{I}.$$

Thus, the Lie algebra becomes

$$\left[\tilde{J}_i, \tilde{J}_j\right] = i\epsilon_{ijk}\tilde{J}_k, \quad \left[\tilde{J}_i, \tilde{K}_j\right] = i\epsilon_{ijk}\tilde{K}_k, \quad \left[\tilde{J}_i, \tilde{P}_j\right] = i\epsilon_{ijk}\tilde{P}_k,$$

$$\left[\tilde{K}_i, E\right] = i\tilde{P}_i, \quad \left[\tilde{K}_i, \tilde{P}_j\right] = iM\delta_{ij}\mathbb{I},$$

$$\left[\tilde{K}_i, \tilde{K}_j\right] = \left[\tilde{J}_i, E\right] = \left[\tilde{P}_i, E\right] = \left[\tilde{P}_i, \tilde{P}_j\right] = 0.$$

Thus, we can eliminate all the centrally extended variables except M. Therefore, the Galilean Lie algebra can be centrally extended by a single central charge M.

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Problem 3 - Adjoint Representation

Show that $\left(\hat{\lambda}_k\right)^\ell_m = \mathrm{i}C^\ell_{km}$ is a representation by checking the commutation relations for $\left(\hat{\lambda}_k\right)^\ell_m$. The resulting identity is called the Jacobi identity. Why does it hold? What's the dimension of the adjoint representation?

Solution. Let us consider the matrices $\left(\hat{\lambda}_k\right)^\ell_m = \mathrm{i}C^\ell_{km}$, where C^ℓ_{km} are the structure constant of a Lie algebra. To verify that $\hat{\lambda}_k$ forms a representation of the Lie algebra, we need to check that they satisfy the same commutation relations as the Lie algebra generators.
A Lie algebra is defined by the commutation relation

$$\left[\hat{\lambda}_k, \hat{\lambda}_m\right] = \mathrm{i}C^\ell_{km}\hat{\lambda}_\ell,$$

where the structure constants C^ℓ_{km} are anti-symmetric in (k, m). Computing the left-hand side, we have

$$\begin{aligned}\left[\hat{\lambda}_k, \hat{\lambda}_m\right]^\ell_n &= \left(\hat{\lambda}_k\hat{\lambda}_m - \hat{\lambda}_m\hat{\lambda}_k\right)^\ell_n \\ &= \mathrm{i}C^p_{kn}\mathrm{i}C^\ell_{mp} - \mathrm{i}C^p_{mn}\mathrm{i}C^\ell_{kp} \\ &= -C^p_{kn}C^\ell_{mp} + C^p_{mn}C^\ell_{kp}.\end{aligned}$$

Additionally, the elements of the Lie algebra satisfy the Jacobi identity given by

$$\left[\left[\hat{\lambda}_k, \hat{\lambda}_\ell\right], \hat{\lambda}_m\right] + \left[\left[\hat{\lambda}_\ell, \hat{\lambda}_m\right], \hat{\lambda}_k\right] + \left[\left[\hat{\lambda}_m, \hat{\lambda}_k\right], \hat{\lambda}_\ell\right] = 0.$$

By replacing in the Jacobi identity, we get the following relation

$$C^n_{k\ell}C^p_{mn} + C^n_{\ell m}C^p_{kn} + C^n_{mk}C^p_{\ell n}$$

Define the adjoint expression

$$\left(\hat{\lambda}_k\right)^a_b = \mathrm{i}C^a_{kb},$$

where $\left(\hat{\lambda}_k\right)^a_b$ is the matrix element of $\hat{\lambda}_k$. Then the commutation relation becomes

$$\begin{aligned}\left[\hat{\lambda}_k, \hat{\lambda}_\ell\right]^a_b &= \left(\hat{\lambda}_k\right)^a_c\left(\hat{\lambda}_\ell\right)^c_b - \left(\hat{\lambda}_\ell\right)^a_c\left(\hat{\lambda}_k\right)^c_b \\ &= \mathrm{i}C^a_{kc}\mathrm{i}C^c_{\ell b} - \mathrm{i}C^a_{\ell c}\mathrm{i}C^c_{kb} \\ &= -C^a_{kc}C^c_{\ell b} + C^a_{\ell c}C^c_{kb} \\ &= -C^c_{k\ell}C^a_{cb} \\ &= \mathrm{i}C^c_{k\ell}\left(\hat{\lambda}_c\right)^a_b.\end{aligned}$$

Thus, the adjoint expression satisfies the Lie algebra commutation relations and is a representation of the Lie algebra, and is hence called the adjoint representation.
The dimension of the adjoint representation is the same as the dimension of the Lie algebra itself, which is the number of independent generators $\left(\hat{\lambda}_c\right)^a_{\ b}$. If the Lie algebra has n generators, then the dimension of the adjoint representation is n. Particularly, there are range(a) values for a and range(b) values for b, both of which are equal to $\dim(G)$. Thus, the dimension of the adjoint representation is equal to the product, which is $\dim(G)^2$. $\blacksquare$