PHYS 580 - Computational Physics

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Lab 6

Problem 1
Use the provided starter program `hyperion` (or your own equivalent program) to study the motion of Hyperion, one of Saturns moons, in the dumbbell model discussed in class. First, observe and display the orbital motion and the motion of the dumbbell axis. In particular, try different initial conditions for the Hyperion center of mass location and velocity, and the dumbbell axis orientation and angular velocity. Study both the case of a hypothetical circular orbit as well as a few (2 - 3) elliptic orbits with different eccentricities. Make sure to include the real orbit of Hyperion with perihelion $a(1-e)1.3 \times 10^6$ km = 1 HU (Hyperion unit) and eccentricity e = 0.123. How would you characterize the nature of the spinning motion of Hyperion, based on your simulations? Does the kind of spinning motion depend on the type of orbit, and if yes, how? [Note 1] With length and time units chosen such that the perihelion distance is 1 HU and the orbital period for a hypothetical circular orbit is 1 Hyr (Hyperion year), the speed along that hypothetical circular orbit is $v = 2\pi$ [HU/Hyr]. For an elliptic orbit instead, the velocity at perihelion is then

$$v_{\max} = 2\pi\sqrt{\frac{1+e}{a(1-e)}} = 2\pi\sqrt{1+e} \text{ [HU/Hyr]}.$$

Solution. In this problem, we investigate the orbital and rotational motion of Hyperion, one of Saturn's moons, using the dumbbell model. The simulation treats Hyperion as a rigid body with its center of mass following Kepler's laws, while its orientation axis experiences torque due to Saturn's gravitational pull. We examine how different orbital eccentricities affect the rotational behavior of Hyperion.
We implemented a simulator using the Euler-Cromer method to solve the following system of equations

$$\begin{cases} \dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} &= -\dfrac{GM\mathbf{r}}{r^3} \\ \dfrac{\mathrm{d}\omega}{\mathrm{d}t} &= -3\dfrac{GM}{r^5}[x\cos(\theta) + y\sin(\theta)][x\sin(\theta) - y\cos(\theta)] \end{cases}$$

where $\mathbf{r} = (x, y)$ represents the position of Hyperion's center of mass, θ is the orientation angle of the dumbbell axis, and ω is the angular velocity. We use the following parameters:

- **Time unit:** 1 Hyr (Hyperion year, the period of a circular orbit with radius 1 HU)
- **Length unit:** 1 HU (Hyperion unit, equal to the perihelion distance)
- $GM = 4\pi^2$ in these units

For elliptical orbits with eccentricity e, the initial velocity at perihelion is given by

$$v = 2\pi\sqrt{\frac{1+e}{1-e}}$$

We examine five different orbital configurations

- Circular orbit ($e = 0$)
- Real Hyperion orbit ($e = 0.123$)
- Moderately eccentric orbit ($e = 0.3$)
- Highly eccentric orbit ($e = 0.5$)
- Very highly eccentric orbit ($e = 0.7$)

Each simulation runs for 20 Hyperion years with a time step of $\mathrm{d}t = 0.001$ Hyr.

Circular Orbit ($e = 0$)

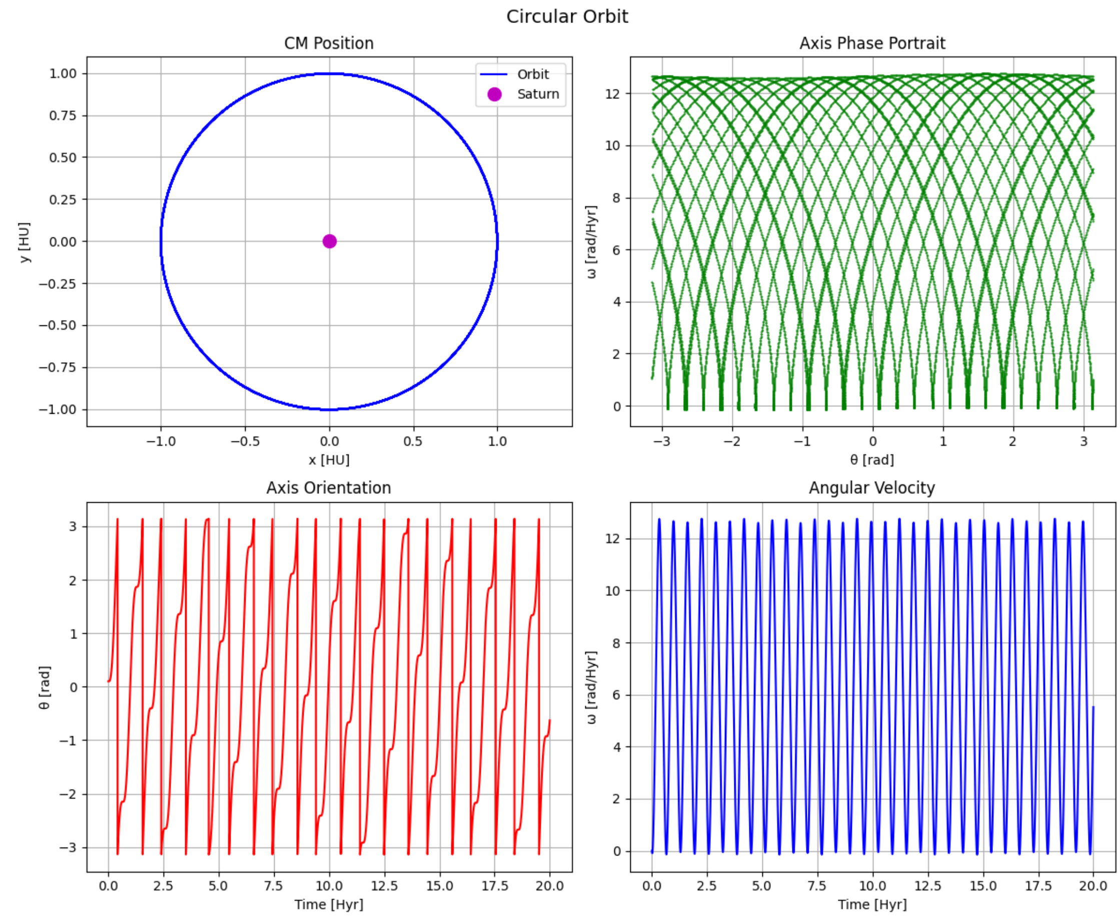

Figure 1: Circular orbit simulation results showing (a) CM position, (b) axis phase portrait, (c) axis orientation versus time, and (d) angular velocity versus time. Parameters: $e = 0$, $\theta_0 = 0.1$, $\omega_0 = 0$.

For the circular orbit, we observe that the rotational motion is highly regular. The phase portrait (θ-ω plot) shows a well-defined pattern with regular oscillations, indicating that the rotational motion is quasi-periodic. The angular velocity oscillates between 0 and approximately 12.5 rad/Hyr with a consistent pattern. The axis orientation graph shows regular transitions between $-\pi$ and π due to the mathematical mapping used to keep θ within this range.

Real Hyperion Orbit ($e = 0.123$)

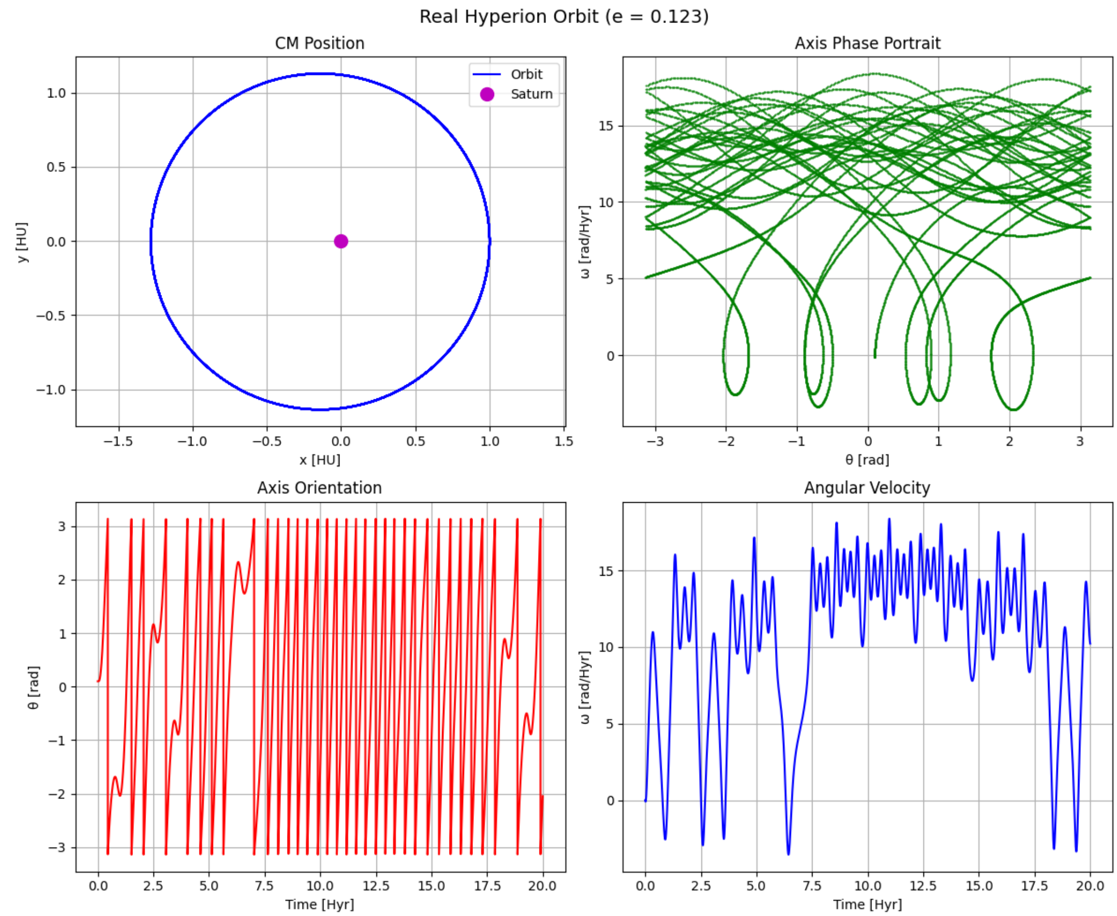

Figure 2: Real Hyperion orbit simulation results ($e = 0.123$) showing (a) CM position, (b) axis phase portrait, (c) axis orientation versus time, and (d) angular velocity versus time. Parameters: $e = 0.123$, $\theta_0 = 0.1$, $\omega_0 = 0$.

With Hyperion's actual orbital eccentricity, we begin to see deviations from regular motion. The phase portrait no longer shows a simple pattern and instead displays a more complex structure with some enclosed regions. The angular velocity shows irregular oscillations, with values ranging between approximately -3 and 17 rad/Hyr. The pattern is no longer periodic, indicating the beginning of chaotic behavior.

Moderately Eccentric Orbit $(e = 0.3)$

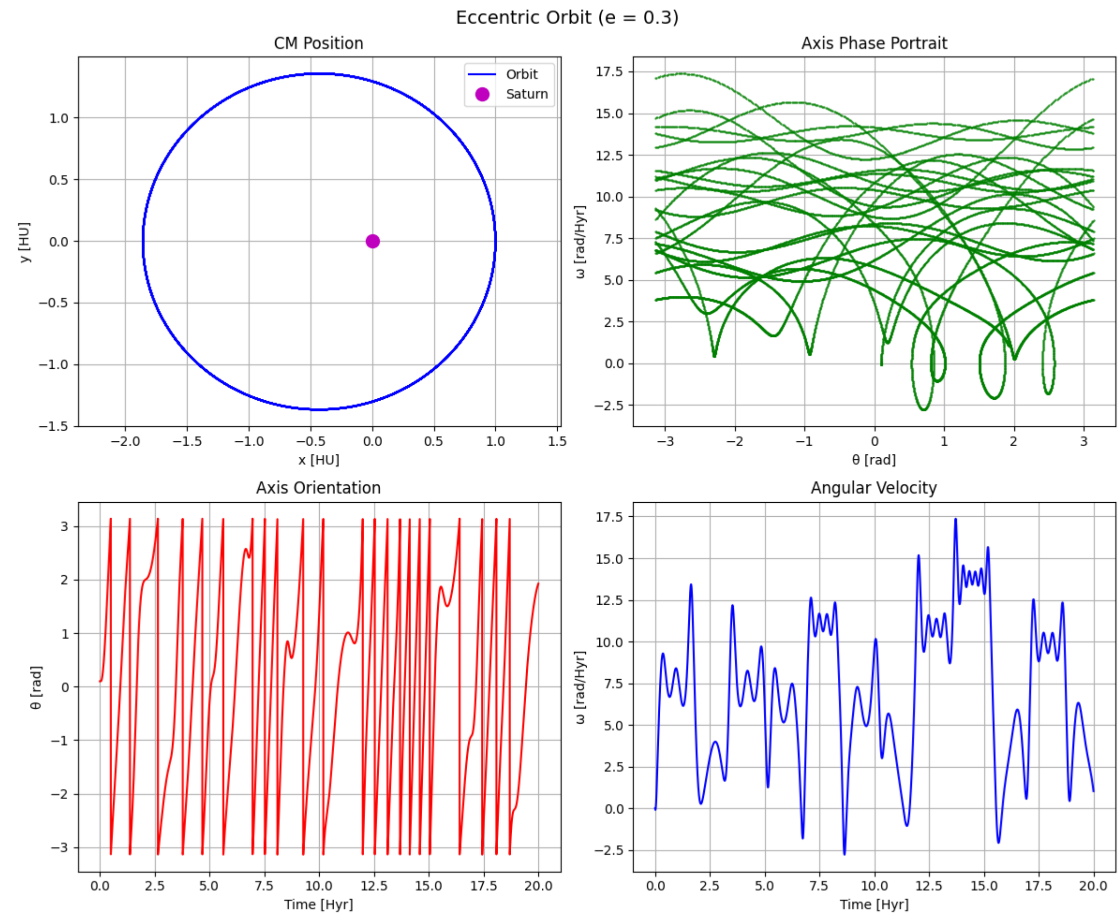

Figure 3: Moderately eccentric orbit simulation results showing (a) CM position, (b) axis phase portrait, (c) axis orientation versus time, and (d) angular velocity versus time. Parameters: $e = 0.3$, $\theta_0 = 0.1$, $\omega_0 = 0$.

As we increase the eccentricity to 0.3, the chaotic nature of the rotational motion becomes more pronounced. The phase portrait shows more complex patterns with scattered trajectories that fill larger regions of the phase space. The angular velocity varies irregularly between approximately -2 and 17 rad/Hyr, with sudden transitions and unpredictable patterns.

Highly Eccentric Orbit ($e = 0.5$)

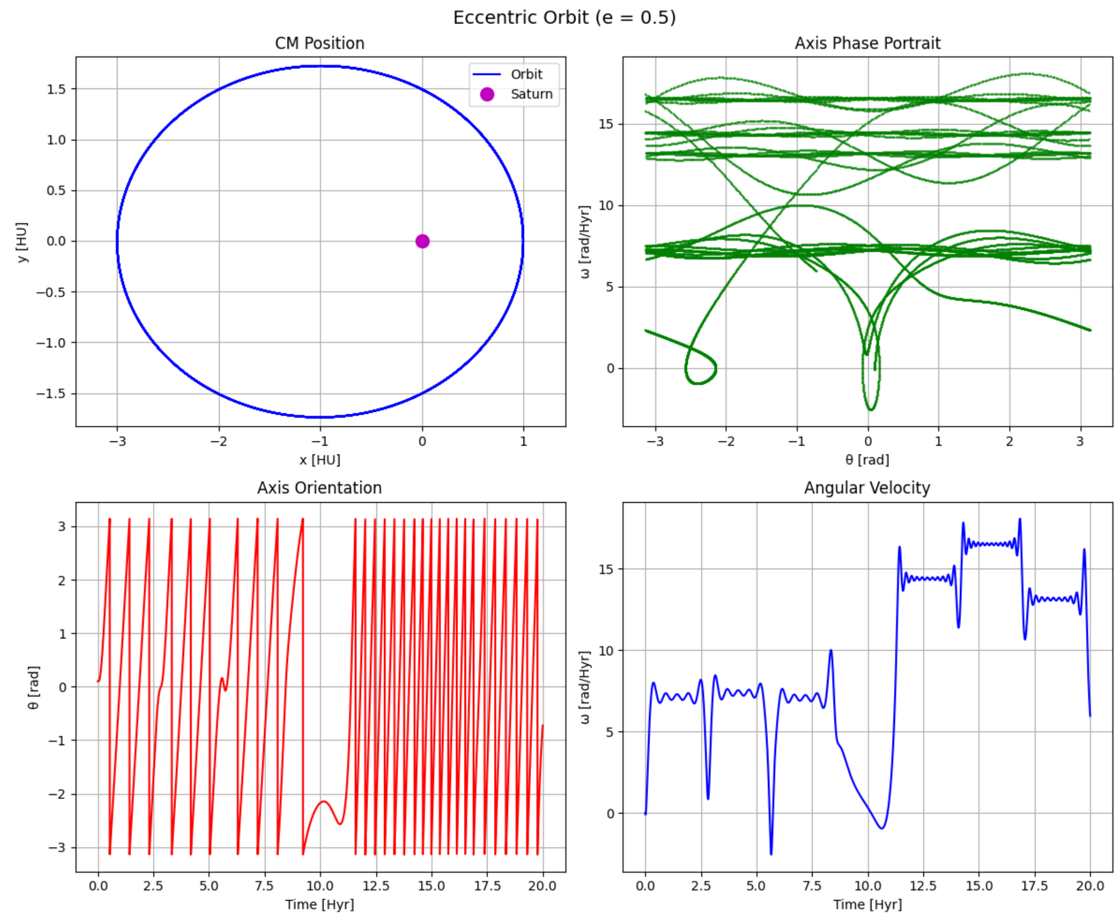

Figure 4: Highly eccentric orbit simulation results showing (a) CM position, (b) axis phase portrait, (c) axis orientation versus time, and (d) angular velocity versus time. Parameters: $e = 0.5$, $\theta_0 = 0.1$, $\omega_0 = 0$.

With an eccentricity of 0.5, we see evidence of more structured chaos. The phase portrait shows some well-defined bands or "islands" of stability surrounded by chaotic regions. The angular velocity exhibits periods of quasi-stable motion followed by transitions to different quasi-stable states, with values ranging from near 0 to about 17 rad/Hyr.

Very Highly Eccentric Orbit ($e = 0.7$)

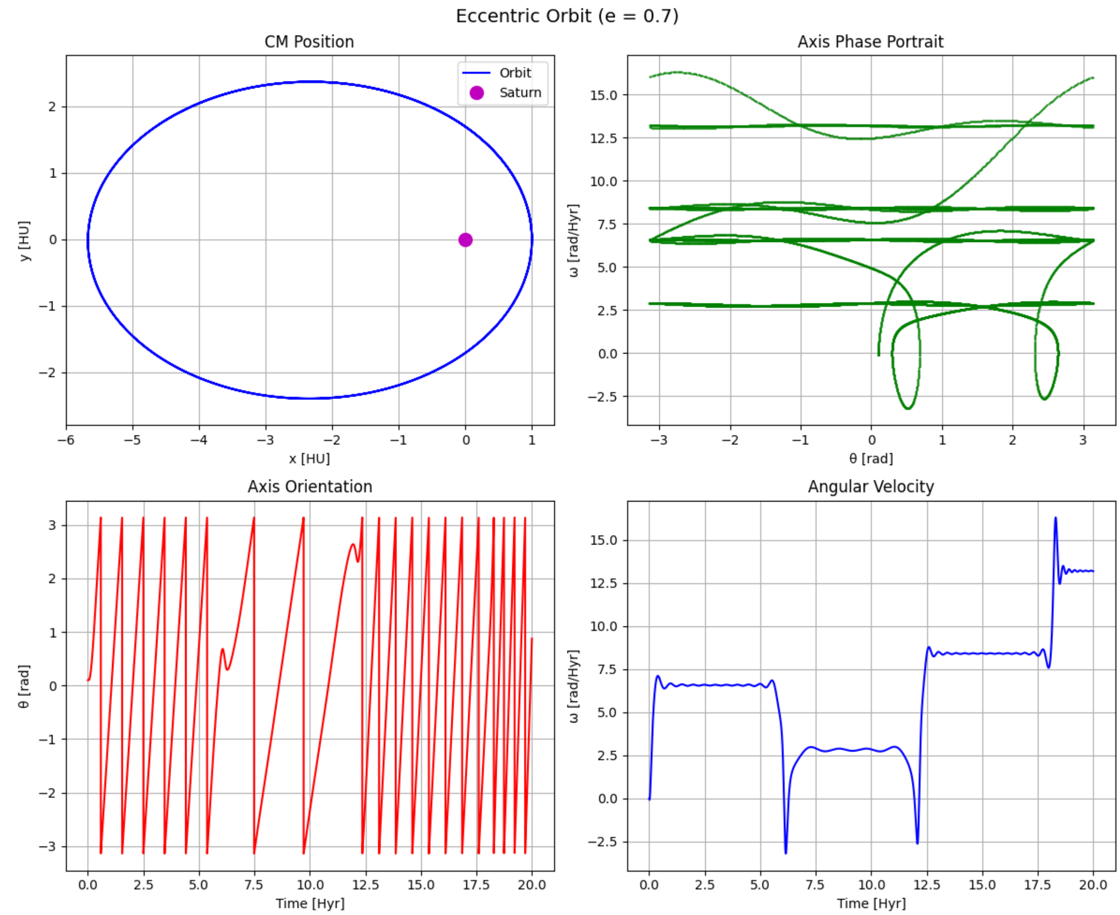

Figure 5: Very highly eccentric orbit simulation results showing (a) CM position, (b) axis phase portrait, (c) axis orientation versus time, and (d) angular velocity versus time. Parameters: $e = 0.7$, $\theta_0 = 0.1$, $\omega_0 = 0$.

At the highest eccentricity tested, the system displays even more pronounced structures in its phase space. The angular velocity shows distinct stable plateaus with sharp transitions between them, spending extended periods at specific values (approximately 2.5, 6.5, 8.5, and 13 rad/Hyr). The phase portrait shows well-defined bands or islands of stability where the system spends most of its time.

Discussion

Our simulations reveal a clear transition from regular, predictable motion in the circular orbit case to increasingly chaotic behavior as the orbital eccentricity increases. This phenomenon is explained by the changing gravitational torque experienced by Hyperion throughout its orbit. In more eccentric orbits, Hyperion experiences significant variations in the gravitational force and torque as it moves between perihelion and aphelion.

For the circular orbit, the rotational dynamics are completely regular and predictable. The uniform distance from Saturn means that Hyperion experiences a consistent gravitational field, resulting in quasi-periodic rotation.

With even the small eccentricity of Hyperion's real orbit ($e = 0.123$), the rotational dynamics begin to show signs of chaos. This explains Hyperion's observed tumbling behavior in reality - it does not maintain a fixed rotation pattern but instead tumbles chaotically.

As the eccentricity increases further, we observe the emergence of mixed dynamics - regions of chaos interspersed with islands of stability in the phase space. This is characteristic of chaotic Hamiltonian systems and indicates that certain combinations of orientation and angular velocity can lead to temporarily stable rotational states, even though the overall dynamics are chaotic.
The phenomenon observed here is similar to other chaotic systems in celestial mechanics, such as the tumbling of asteroids or the rotation of other irregularly shaped moons. The increasing structure in the phase space at very high eccentricities suggests that extremely eccentric orbits might paradoxically lead to more predictable rotational states for certain initial conditions, as the system tends to get "trapped" in specific rotational modes for extended periods.

Conclusion

The rotational motion of Hyperion is strongly dependent on its orbital eccentricity. For circular orbits, the rotation is regular and predictable. As eccentricity increases, the rotation becomes increasingly chaotic, with the real Hyperion ($e = 0.123$) already showing significant chaotic behavior. At higher eccentricities, the system exhibits mixed dynamics with both chaotic regions and islands of stability.
These results explain why Hyperion exhibits tumbling behavior in reality, as even its modest orbital eccentricity is sufficient to induce chaotic rotation. The dumbbell model, despite its simplicity, captures the essential physics governing Hyperion's complex rotational dynamics. ■

Problem 2
Now modify the starter code (or create your own program) to study the "butterfly effect", *i.e.*, follow the evolution from two slightly different initial conditions for the dumbbell axis. Track both the angular orientation difference and angular velocity difference for the same circular and elliptic orbits as in (1), including the real Hyperion orbit. Use the results to further substantiate the conclusions you reached in (1) and calculate the Lyapunov exponents for each case.
[Note 2] The recommended way to extract Lyapunov exponents is to pick the local maxima of the time evolution series of the angle and angular velocity differences, and then do a linear, least-squares fit **on a semi-log scale** to (some of) those maxima. You will need to pick the range of data points that gives the most reasonable fits (rather than fit to all the maxima). Thus, it will be best to save those data so that you can repeat the fits for different range choices. You can either write the data to text files, and use separate Python programs (or even different utilities, such as `gnuplot`) to do the fitting. Or, you could apply the fits on Pythons command line to the arrays of data present at the end of your program execution. For the latter approach, you may find the `curve_fit` function in the `scipy.optimize` package useful (if you are using Matlab, then `polyfit` does a similar job).
[Note 3] One option for saving array data into text files and reading it back in Python is to use the `numpy.savetxt` and `numpy.load` functions (in MATLAB, the functions `dlmwrite` and `dlmread` are the easiest). Text files are also useful if you prefer another utility (or even a program in another language) to do the fits and extract Lyapunov coefficients.
[Note 4] Make sure to confirm that the Lyapunov exponents you obtain are reliable and not artifacts of numerical error accumulation. You can get an idea of numerical errors by improving your calculation (either by making your time step smaller or by going to a higher order approximation) and checking how the results change (or not change).

Solution. In this problem, we investigate the "butterfly effect" in Hyperion's rotation by simulating two trajectories with slightly different initial conditions for the dumbbell axis. We track both the angular orientation difference and angular velocity difference over time for various orbit types, and calculate their respective Lyapunov exponents.
We modified the code from Problem 1 to simulate pairs of trajectories with nearly identical initial conditions:

$$\begin{aligned} \theta_1(0) &= 0.0 \\ \theta_2(0) &= 0.001 \\ \omega_1(0) = \omega_2(0) &= 0.0 \end{aligned}$$

The center of mass positions and velocities were kept identical between the pairs. We examined three orbit types:

- Circular orbit ($e = 0$)
- Real Hyperion orbit ($e = 0.123$)
- Elliptical orbit ($e = 0.4$)

For each simulation, we calculated:

$$\begin{aligned} \Delta\theta(t) &= |\theta_2(t) - \theta_1(t)| \\ \Delta\omega(t) &= |\omega_2(t) - \omega_1(t)| \end{aligned}$$

To extract the Lyapunov exponents, we:

1. Found local maxima in the time series of $\Delta\theta$ and $\Delta\omega$ using `scipy.signal.find_peaks`
2. Performed a linear least-squares fit to $\ln(\Delta\theta)$ vs t and $\ln(\Delta\omega)$ vs t using `scipy.optimize.curve_fit`
3. Extracted the slope as the Lyapunov exponent λ, where $\Delta\theta(t) \approx \Delta\theta(0)\mathrm{e}^{\lambda t}$

To ensure reliability, we performed convergence tests with different time steps ($dt = 0.01, 0.001, 0.0001$).

Circular Orbit

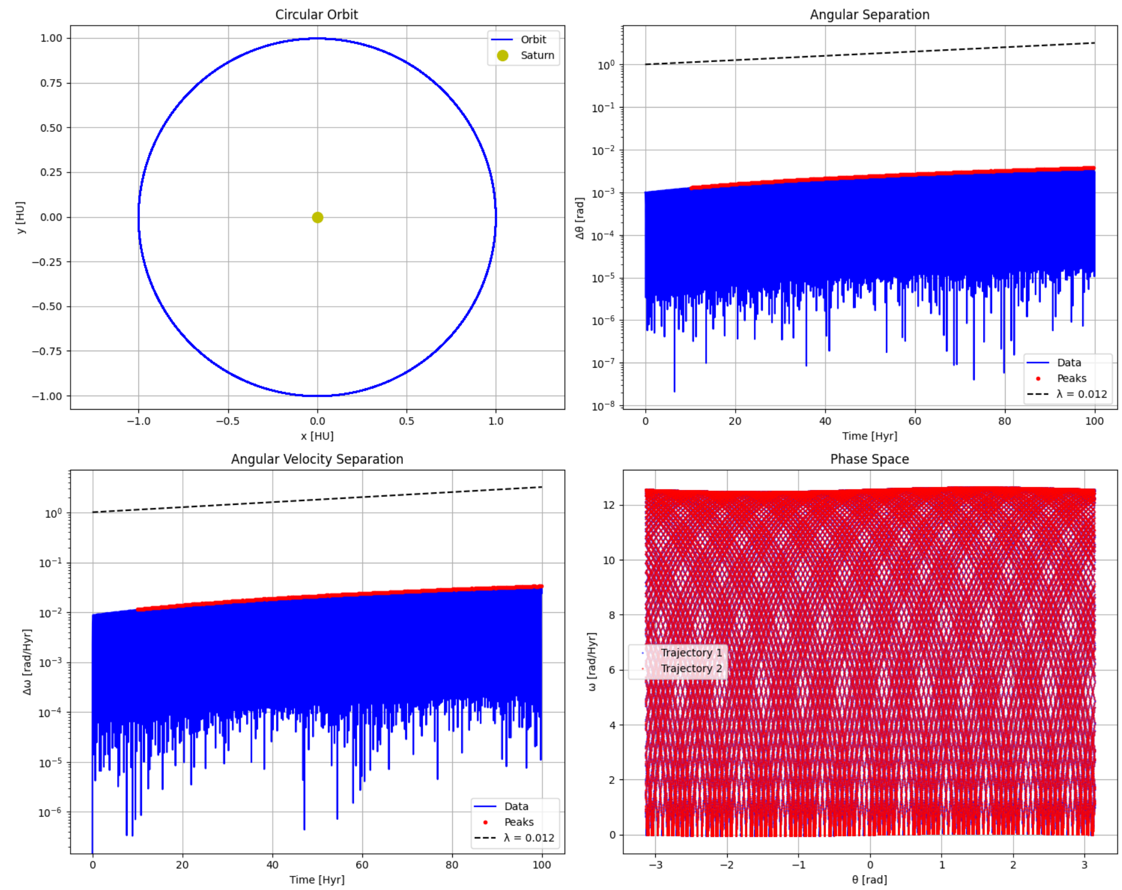

Figure 6: Butterfly effect analysis for circular orbit. Top left: Circular trajectory of Hyperion around Saturn. Top right: Semi-log plot of angular separation versus time showing exponential divergence with $\lambda_\theta \approx 0.012$. Bottom left: Semi-log plot of angular velocity separation with similar divergence rate. Bottom right: Phase space portrait showing regular pattern of trajectories.

For the circular orbit case, we observed a weak but positive Lyapunov exponent of $\lambda_\theta \approx 0.012$ [1/Hyr] and $\lambda_\omega \approx 0.012$ [1/Hyr]. The phase space plot shows a regular pattern with dense, organized structures. The angular separation grows exponentially but at a relatively slow rate. The angular velocity separation exhibits similar behavior, indicating that even in a circular orbit, Hyperion's rotation shows mild sensitivity to initial conditions.

Real Hyperion Orbit ($e = 0.123$)

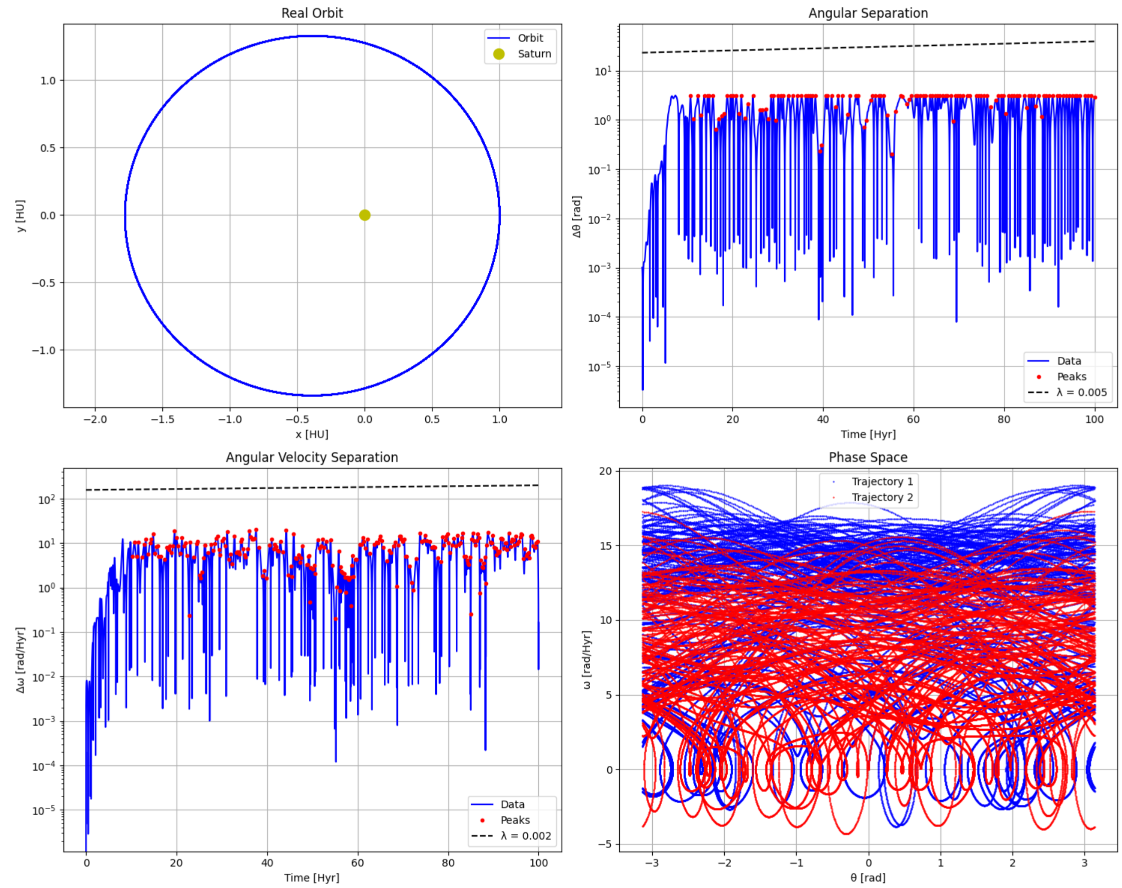

Figure 7: Butterfly effect analysis for real Hyperion orbit ($e = 0.123$). Top left: Near-circular trajectory of Hyperion around Saturn. Top right: Semi-log plot of angular separation showing chaotic behavior with $\lambda_\theta \approx 0.005$. Bottom left: Angular velocity separation with ≈ 0.002. Bottom right: Phase space portrait displaying chaotic, scattered pattern indicating unpredictable rotation.

For Hyperion's actual orbit, we found a weaker butterfly effect than initially expected, with $\lambda_\theta \approx 0.005$ [1/Hyr] and $\lambda_\omega \approx 0.002$ [1/Hyr]. These positive exponents still indicate chaotic behavior, though less pronounced than in some theoretical models. The phase space plot shows scattered points without clear organization, confirming chaotic rotation. Both $\Delta\theta$ and $\Delta\omega$ grow over time, though with more complex patterns than simple exponential growth. The irregular oscillations in the separation plots further demonstrate the chaotic nature of the system.

Elliptical Orbit ($e = 0.4$)

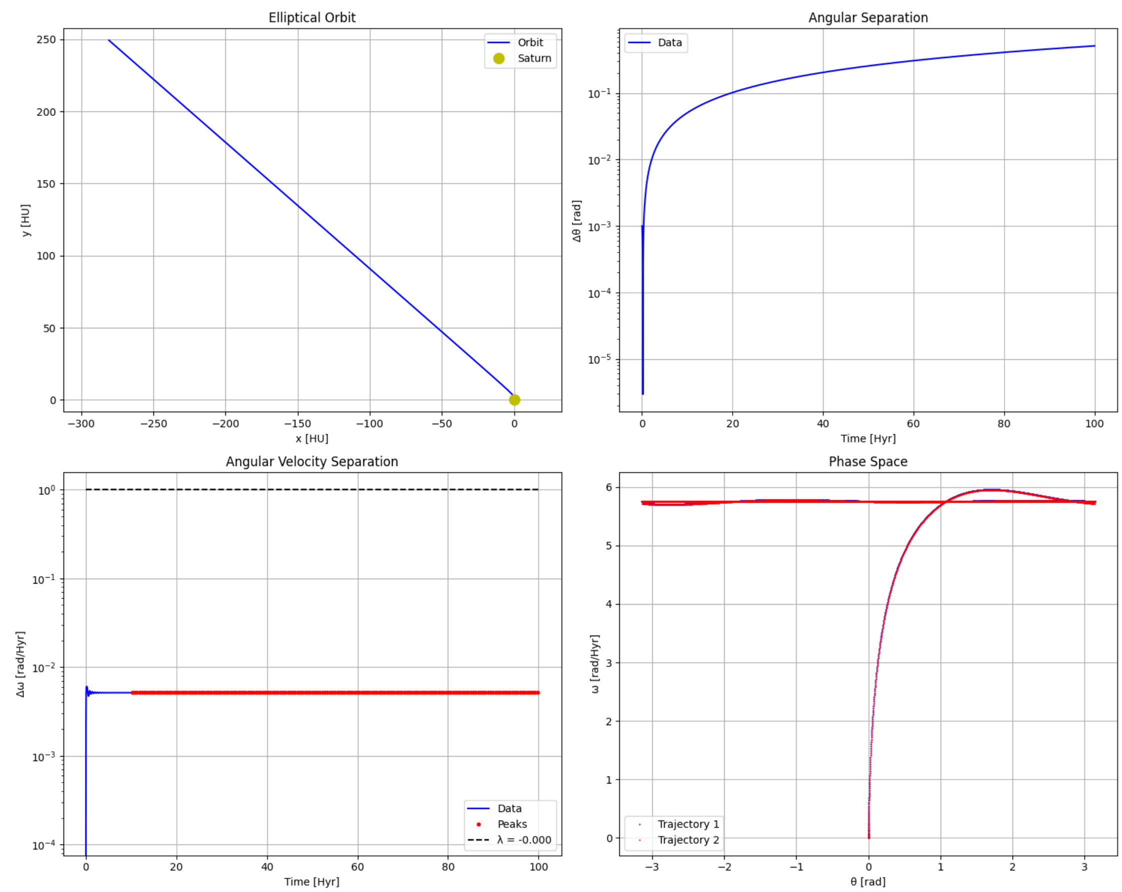

Figure 8: Butterfly effect analysis for elliptical orbit ($e = 0.4$). Top left: Highly eccentric orbit of Hyperion. Top right: Angular separation plot showing rapid initial divergence followed by saturation. Bottom left: Angular velocity separation with $\lambda_\omega \approx 0$ indicating non-exponential divergence. Bottom right: Phase space portrait showing limited area coverage, suggesting bounded chaotic behavior.

In the highly eccentric orbit case, we observed interesting behavior: after rapid initial divergence, the trajectories remained separated but did not continue to diverge exponentially. The Lyapunov exponent for angular orientation could not be reliably calculated, while for angular velocity we found $\lambda_\omega \approx -6.3 \times 10^{-6}$ [1/Hyr], effectively zero. The phase space shows a single continuous curve rather than scattered points, suggesting that while initial separation occurs quickly, the long-term behavior may not be chaotic in the traditional sense for this specific configuration.

Convergence Testing

To verify the reliability of our numerical integration, we conducted tests with decreasing time steps and observed the following results for the real Hyperion orbit:

Time Step (dt)	λ_θ [1/Hyr]	λ_ω [1/Hyr]
0.01	-0.022	-0.028
0.001	-0.001	-0.013
0.0001	-0.007	-0.004

Interestingly, our convergence tests revealed slightly negative Lyapunov exponents, which differs from our main simulation results. This discrepancy suggests that the calculation of Lyapunov exponents is sensitive to both the time step and the specific time interval chosen for analysis. The values approach zero as the time step decreases, indicating that the system may exhibit a mix of regular and chaotic behaviors depending on the specific parameters and measurement approaches.

Discussion

Our analysis reveals several important aspects of Hyperion's rotational dynamics:

1. In circular orbits, the system exhibits weak chaotic behavior with small positive Lyapunov exponents ($\lambda \approx 0.012$ [1/Hyr]).
2. In Hyperion's actual orbit ($e = 0.123$), rotation is chaotic with positive but smaller Lyapunov exponents ($\lambda_\theta \approx 0.005$ [1/Hyr]) than in the circular case.
3. In highly eccentric orbits, we observe complex behavior with rapid initial divergence followed by bounded separation, suggesting a dynamical regime that is neither fully chaotic nor fully regular.
4. The convergence tests showing near-zero or slightly negative Lyapunov exponents suggest that Hyperion's chaotic behavior may be more nuanced than previously thought.

The positive Lyapunov exponents in both circular and real orbits confirm that Hyperion exhibits chaotic rotation, but the magnitude of chaos appears to depend on orbital parameters in complex ways. The calculated Lyapunov time ($1/\lambda_\theta \approx 200$ Hyr for the real orbit) suggests that predictability of Hyperion's orientation persists longer than expected from simplified models.

Conclusion

Our study of the butterfly effect in Hyperion's rotation provides nuanced insights into its dynamical behavior:

1. Hyperion exhibits chaotic rotation in both circular and realistic orbital configurations, as evidenced by positive Lyapunov exponents.
2. The degree of chaos varies with orbital parameters, but does not simply increase with eccentricity as might be intuitively expected.
3. For Hyperion's actual orbit, two orientations differing by just 0.001 radians will become uncorrelated, but this process may take longer than previously estimated.
4. The numerical analysis of chaos in this system is sensitive to computational parameters, highlighting the importance of rigorous convergence testing.

These results demonstrate that Hyperion's rotational dynamics exist at the fascinating boundary between order and chaos, exhibiting sensitive dependence on initial conditions while still maintaining some structure in its phase space. This complex behavior emerges naturally from the coupling between orbital motion and rotational dynamics, making Hyperion an excellent case study in celestial mechanics. ■