Master Problem Set, Math 572

February 11, 2025

Problem 1 (Not trivial)**.** Let S_g be a genus g surface. We say a set X of simple (no self intersections) closed (loops) curves on S_g is *nice* if no two pairs of distinct curves are homotopic, every curve is non-trivial (i.e. not homotopic to the constant curve), and no two pairs of distinct curves intersect. We say X is *awesome* if X is maximal with respect to these properties.

(1) Prove that $|X| = 3g-3$. [Hint: Cut the surface along the curves in X into pieces. Use maximality to say something about the topology of the pieces. This should do all of the problem.]

(2) Take $Z = S_g - \cup_{\gamma \in X} \gamma$ and let $\{P_1, \dots, P_r\}$ be the path components of Z. Prove that $r = 2g-2$.

(3) Prove that each P_i is homeomorphic to $S^2 - \{\text{three points}\}$. These are called pairs of pants.

Pants play an important role in the study of surfaces.

Problem 2 (Not easy)**.** Let X be a topological space. We say that an open covering $\mathscr{C}$ of X is *good* if

(a) $\mathscr{C}$ is locally finite.

(b) For any finite collection $V_1, \dots, V_n \in \mathscr{C}$ with non-trivial intersection, we have that $V_1 \cap \dots \cap V_n$ is contractible.

Let $\mathrm{N}(\mathscr{C})$ be the nerve associated to the covering $\mathscr{C}$. This is the simplical complex built by taking the elements of $\mathscr{C}$ to be the vertices. We connected U, V by an edge if and only if $U \cap V \neq \emptyset$, and add a 2–simplex if and only if $U \cap V \cap W \neq \emptyset$, .etc. Prove that X and $\mathrm{N}(\mathscr{C})$ are homotopy equivalent.

Problem 3. Let M be a n–manifold.

(1) Prove that every open covering has a good refinement. [Hint: Prove it for an open subset of $\mathbf{R}^n$]

(2) Prove that M is homotopy equivalent to a simplicial complex. [Hint: Use previous problem]

(3) Prove that if M is compact, then M is homotopy equivalent to a finite simplicial complex.

Problem 4 (Easy)**.** Prove that $\mathbf{R}^2 - \{(0,0)\}$ deformation retracts the unit circle centered at $(0,0)$.

Problem 5 (Not trivial)**.** View S^1 as the modulus 1 complex numbers and let $X = S^1 \times S^1 - (1,1)$. Let A be the union of the two circles $\left\{(t,i) \,:\, t \in S^1\right\}$ and $\left\{(i,t) \,:\, t \in S^1\right\}$. Prove that X deformation retracts A. [Hint: Think of the torus as coming from the square with opposite sides identified and view $(1,1)$ as being the center of the square. Then push out towards the boundary of the square]

Problem 6 (Fun but long)**.** Let X be a topological space and let $G = \mathrm{Homeo}(X)$, the group of homeomorphisms $\psi \colon X \to X$. We can equip G with the compact-open topology: This is the topology generated by all of the subsets

$$W(K,V) = \{\psi \in G \,:\, \psi(K) \subset V\}$$

where $K \subset X$ varies over all compact subsets of X and $V \subset X$ varies over all open subsets of X.

(1) Prove or disprove: the action map

$$G \times X \to X$$

given by $(\psi, x) \mapsto \psi(x)$ is continuous.

(2) Prove that if X is regular and locally compact, then the group multiplication map

$$G \times G \longrightarrow G, \quad (g,h) \mapsto gh$$

is continuous in the compact-open topology on G; the domain is given the product topology with each factor having the compact-open topology.

(3) Prove that if X is compact and Hausdorff, then G is a topological group. Specifically, prove that the inverse map
$$G \longrightarrow G, \quad g \mapsto g^{-1}$$
is continuous.

(4) Prove that for $\psi_1, \psi_2 \in G$ that ψ_1, ψ_2 are isotopic (a homotopy through homeomorphisms) if and only if there exists a continuous path $c\colon [0,1] \to G$ such that $c(0) = \psi_1$ and $c(1) = \psi_2$.

(5) Prove that there exists a bijection between $G/\sim$ and the path components of G in the compact-open topology where $\sim$ is the equivalence relation on G given by isotopy.

(6) Prove that the path component G_0 that contains the identity element is an open normal subgroup of G.

(7) Prove that the quotient $\mathrm{Map}(X) = G/G_0$ is a discrete group. This is called the mapping class group of X.

Problem 7. Let $\Gamma = \mathrm{PSL}(2,\mathbf{Z}) = \mathrm{SL}(2,\mathbf{Z})/\pm \mathrm{I}_2$ and let Γ act on complex upper half space $\mathbf{H} = \{z \in \mathbf{C} \,:\, \mathrm{Im}(z) > 0\}$ by Möbius transformations:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$
Define
$$V = \{z \in \mathbf{H} \,:\, \mathrm{Stab}(z) = \{1\}\}.$$
Prove that V is an open, dense subset of $\mathbf{H}$. [Hint: One should be able to succeed with a brute force approach since the action is explicit. Write out the equations for the stabilizer of a general point and show it does not have integral solutions often]

Problem 8 (Easy)**.** Let X be a compact, connected graph (i.e. a compact CW complex with only 0 and 1-cells). Prove that if Y is a subgraph, then X retracts Y. Deduce that the inclusion map $Y \to X$ induces an injective homomorphism on fundamental groups. [Hint: Collapse stuff not in Y]

Problem 9 (Easy)**.** Prove that every path connected graph deformation retracts to a wedge of circles. [Hint: Collapse any edge that is not a loop]

Problem 10 (Massey)**.** Let $F\colon X \times [0,1] \to X$ be continuous and define $f_t\colon X \to X$ via $f_t(x) = F(x,t)$. Assume that $f_0 = f_1 = \mathrm{Id}_X$.

(a) For each $x_0 \in X$, prove that $c_{x_0}\colon [0,1] \to X$ given by $c_{x_0}(t) = F(x_0,t)$ is a loop at x_0.

(b) Prove that $[c_{x_0}] \in \pi_1(X,x_0)$ is central in $\pi_1(X,x_0)$. That is, $[c_{x_0}]$ commutes with every $[c] \in \pi_1(X,x_0)$.

Problem 11. Let $\mathscr{L}(X,x)$ be the space of loops based at x; these are continuous maps $(S^1,1) \to (X,x)$ with the compact open topology. Prove that there is a bijection between the free homotopy classes $[c]$ for $c \in \mathscr{L}(X,x)$ are in bijection with the conjugacy classes of $\pi_1(X,x)$. [Hint: Think about the difference between (free) homotopy and pointed homotopy]

Problem 12 (Easy)**.** Let X,Y be path connected. Prove that if X,Y are homotopy equivalent, then $\pi_1(X,x) \cong \pi_1(Y,y)$ for any $x \in X$ and $y \in Y$. [Hint: Pick homotopy equivalent maps and use functorality]

Problem 13. Let G be a Lie group and let $H \leq G$ be a closed subgroup. Prove that G/H is a manifold and that $\dim(G/H) = \dim(G) - \dim(H)$. [Hint: Local submersion theorem]

Problem 14. Let $\mathrm{SO}(n)$ be the group of rotations about the origin in $\mathbf{R}^n$. This gives us an action of $\mathrm{SO}(n)$ on S^{n-1}, the unit sphere in $\mathbf{R}^n$ centered about the origin.

(a) Prove that this action is transitive.

(b) Let $H = \mathrm{Stab}(x)$ where $x = (0,0,\dots,0,1)$. Prove that the map

$$\psi_x\colon\ \mathrm{SO}(n)/H \to S^{n-1}$$

given by $\psi_x(gH) = gx$ is a homeomorphism where $\mathrm{SO}(n)/H$ is given the quotient topology ($\mathrm{SO}(n)$ is given the subspace topology from $\mathrm{M}(n,\mathbf{R}) = \mathbf{R}^{n^2}$ and S^{n-1} is given the subspace topology from $\mathbf{R}^n$.

Problem 15 (Massey)**.** Let X be compact and let $f\colon X \to Y$ is a local homeomorphism.

(1) Prove $f^{-1}(y)$ is finite for all $y \in Y$.

(2) Prove that if Y is connected and Hausdorff, then f is surjective.

Problem 16 (Massey)**.** Let X be locally compact and Hausdorff and assume that G acts by homeomorphisms on X. Prove that G acts freely and for every compact subset $K \subset X$ that

$$\{g \in G \ :\ gK \cap K \neq \emptyset\}$$

is finite (we will say G acts properly discontinuously when this holds). Prove that X/G is locally compact and Hausdorff.

Problem 17 (Massey)**.** Let G be a topological group and let $\Gamma \leq G$ be a discrete subgroup. Prove that there exists a neighborhood V of the identity element such that $\gamma V \cap V = \emptyset$ for all $\gamma \in \Gamma$, $\gamma \neq 1$.

Problem 18 (Massey)**.** Let X,Y be path connected and locally path connected. Assume further that X is compact and Hausdorff and Y is Hausdorff. Prove that if $f\colon X \to Y$ is a local homeomorphism, then f is a covering map.

Problem 19 (Easy)**.** Prove that $\pi_1(X \times Y, (x,y)) \cong \pi_1(X,x) \times \pi_1(Y,y)$.

Problem 20 (Massey)**.** Let G,H be path connected, locally path connected topological groups and let $\psi\colon G \to H$ be a covering map. Prove that $\ker(\psi)$ is discrete and central.

Problem 21 (Massey)**.** Let X,Y be path connected, locally path connected spaces and assume that G acts freely and properly discontinuously on both X,Y. Let $f\colon X \to Y$ be continuous and G–equivariant. Let $q_X, q_Y : X,Y \to X/G, Y/G$ be the associated quotient maps.

(1) Prove that there exists a function $\overline{f}\colon X/G \to Y/G$ such that

$$\overline{f} \circ q_X = q_Y \circ f.$$

This yields the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q_X"'] & Y \arrow[d, "q_Y"] \\ X/G \arrow[r, "\overline{f}"'] & Y/G \end{tikzcd}$$

(2) Prove that

$$\pi_1(X/G)/(q_X)_\star(\pi_1(X)) \cong \pi_1(Y/G)/(q_Y)_\star(\pi_1(Y)).$$

Problem 22. Prove that any continuous function $f\colon S^n \to S^1$ is homotopic to the constant map if $n > 1$. [Hint: this might be hard?]

Problem 23. Let X be a set and let G be a group that acts on X. Assume that $\gamma,\eta \in G$ such that there exists disjoint subsets $\mathrm{Ping}, \mathrm{Pong} \subset X$ such that

$$\gamma^n(\mathrm{Ping}) \subset \mathrm{Pong}, \qquad \eta^m(\mathrm{Pong}) \subset \mathrm{Ping}$$

for all $m \neq 0$. Prove that the subgroup $\langle \gamma, \eta \rangle$ generated by γ and η is isomorphic to $\langle \gamma \rangle * \langle \eta \rangle$.

Problem 24. Let X be a set and let G be a group that acts on X. Assume that $H, K \leq G$ are subgroups of G such that there exists disjoint subsets $\mathrm{Ping}, \mathrm{Pong} \subset X$ such that

$$h(\mathrm{Ping}) \subset \mathrm{Pong}, \quad k(\mathrm{Pong}) \subset \mathrm{Ping}$$

for all $h \in H$, $k \in K$, and $n, k \neq 1$. Prove that the subgroup $\langle H, K \rangle$ generated by H and K is isomorphic to $H * K$.

Problem 25. Prove that $S^1 - x \cong \mathbf{R}$ for any $x \in S^1$. In particular, we can view S^1 as the 1–point compactification of $\mathbf{R}$.

Problem 26. $\mathrm{SL}(2, \mathbf{R})$ acts on $\mathbf{R} \cup \{\infty\}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} t = \frac{at + b}{ct + d}$$

where if the denominator is zero, t is sent to ∞. Prove that this action is continuous.

Problem 27. Prove that the matrices

$$\begin{pmatrix} 1 & 168 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 168 & 1 \end{pmatrix}$$

generate a free group in $\mathrm{SL}(2, \mathbf{Z})$. [Hint: Use the Ping Pong Lemma and the action from the previous problem]

Problem 28. Prove that $\mathrm{PSL}(2, \mathbf{Z}) \cong (\mathbf{Z}/2\mathbf{Z}) * (\mathbf{Z}/3\mathbf{Z})$. [Hint: Same as last problem]

Problem 29. Prove that if G is a group and $H \leq G$ is a subgroup such that the index $[G : H] = 2$. Prove that H is normal in G. Deduce that any 2–fold covering must be regular.

Problem 30. Let Γ be a finitely generated group with finite generating set $X = \{x_1, \dots, x_n\}$. Let $F(X)$ be the free group on X and let $\psi \colon F(X) \to \Gamma$ be the homomorphism given by the universal mapping property for free groups.

(1) Prove ψ is surjective.

(2) Given a group G, let

$$R(\star, G) = \{\rho : \star \to G \ : \ \rho \text{ is a homomorphism}\}.$$

Define

$$\psi^\star \colon R(\Gamma, G) \to R(F(X), G)$$

by $\psi^\star(\rho) = \rho \circ \psi$. Prove $\psi^\star$ is injective.

(3) Define $W \colon G^n \to R(F(X), G)$ where $W(g_1, \dots, g_n) = \varphi$ is the unique homomorphism determined by $\varphi(x_j) = g_j$. Prove that W is a bijection.

(4) Prove that if G is finite, then $R(\Gamma, G)$ is finite.

(5) Prove that there are only finitely subgroups of index at most n for all $n \in \mathbf{N}$.

Problem 31. Let G be a group and let $H \leq G$ be a subgroup of G. Define $\mathrm{Core}_G(H)$ to be the largest normal subgroup of G that is contained in H.

(1) Prove that

$$\mathrm{Core}_G(H) = \bigcap_{g \in G} gHg^{-1}.$$

(2) Prove that if $[G : H] < \infty$, then $[G : \mathrm{Core}_G(H)] < \infty$.

(3) Deduce that every finite cover $Y \to X$, there exists a finite cover $Z \to Y$ such that $Z \to Y \to X$ is a regular cover of X.

Problem 32. Draw all the 3–fold covers of the wedge of 2 circles and describe the associated homomorphisms to $\mathrm{Sym}(3)$. [Hint: Write down all of the homomorphisms from $F_2 \to \mathrm{Sym}(3)$ that have transitive image]

Problem 33. Draw all the 3–fold covers of the 2–torus and describe the associated homomorphisms to $\mathrm{Sym}(3)$. [Hint: Write down all of the homomorphisms from $\pi_1(T^2) \to \mathrm{Sym}(3)$ that have transitive image]

Problem 34. Draw all the 3–fold covers of the genus 2 surface and describe the associated homomorphisms to $\mathrm{Sym}(3)$. [Hint: Write down all of the homomorphisms from $\pi_1(S_2) \to \mathrm{Sym}(3)$ that have transitive image]

Problem 35. Let S_g denote the closed genus $g \geq 2$ surface and let G be a finite group that acts freely on S_g by orientation preserving homeomorphisms.

(1) Prove that G acts properly discontinuously on S_g. [Hint: easy]

(2) Prove that S_g/G is a closed surface and $S_g \to S_g/G$ is a covering map [Hint: Use (1)].

(3) Prove that $G = \mathbf{Z}/(g-1)\mathbf{Z}$ acts freely on S_g by orientation preserving homeomorphisms and $S_g/G \cong S_2$. [Hint: Draw a picture]

(4) Prove that S_g cannot cover $S_1 = T^2$, the 2–torus. [Hint: use π_1]

(5) Prove that if G acts freely on S_g by orientation preserving homeomorphisms, then $|G| \leq g-1$.

Problem 36. Let X be a path connected, locally path connected space and let $f\colon X \to X$ be a homeomorphism. Define $M_f = X \times [0,1]/\sim$ where $(x,0) \sim (f(x),1)$ for all $x \in X$. Let $\pi_1(X) = \langle Y \parallel R \rangle$.

(1) Prove that $\pi_1(M_f) \cong \left\langle Y, t \parallel R,\ tyt^{-1} = f_\star(y) \text{ for all } y \in Y \right\rangle$.

(2) Prove that if X is locally compact, then M_f is locally compact.

(3) Prove that if X is Hausdorff, then M_f is Hausdorff.

(4) Prove if X is compact, then M_f is compact.

(5) Prove that if X is a topological n–manifold, then M_f is a topology $(n+1)$–manifold.

(6) Prove that if $f_1, f_2\colon X \to X$ are homotopic, then M_{f_1} and M_{f_2} are homotopy equivalent.

Problem 37. Let X and Y be path connected, locally path connected topological spaces and let $Y \to X$ be a covering map.

(1) Prove that if X admits a CW structure, then Y admits a CW structure.

(2) Prove if X is a n–manifold, then Y is n–manifold.

(3) Assume X is compact. Prove Y is compact if and only if $Y \to X$ is a finite cover.

(4) Prove that if X is locally homogenous, then Y is locally homogenous.

(5) Assume X admits a path metric. Prove that Y can be given a path metric such that the covering map $Y \to X$ is a local isometry.

Problem 38. Let M, N be two closed, connected n–manifolds with $n > 2$ and let $M\#N$ be the connect sum. Prove that $\pi_1(M\#N) \cong \pi_1(M) * \pi_1(N)$.

Problem 39. Let X and Y be path connected, locally path connected spaces with a common subspace Z. Specifically, let $i_X, i_Y\colon Z \to X, Y$ be such that Z is homeomorphic with its image under both. Prove that if Z is simply connected, then $\pi_1(W) = \pi_1(X) * \pi_1(Y)$ where $W = X \cup_Z Y$ is the space given by gluing X and Y along Z.

Problem 40. Prove the following:

(1) $\mathrm{SL}(2,\mathbf{R})$ is homotopy equivalent to S^1. [Hint: QR factorization/Iwasawa decomposition]

(2) $\mathrm{SL}(2,\mathbf{C})$ is homotopy equivalent to S^3. [Hint: Same as for (1)]

Problem 41. Let $g \geq 0$ and take a wedge of g circles inside of $\mathbf{R}^3$ and take a closed tubular neighborhood H_g such that the boundary is a genus g surface S_g. H_g is called genus g handle body. When $g = 0$, this is just a closed 3–ball.

Let $f\colon S \to S$ be an orientation reversing homeomorphism. Define $H_g \cup_f H_g$ to be two copies of H_g glued along the boundary surface S_g via the mapping f.

(1) For $f = \mathrm{Id}$ and for all g, prove that $H_g \cup_{\mathrm{Id}} H_g \cong S^3$.

(2) Prove that if f_1, f_2 are homotopic, then $H_g \cup_{f_1} H_g$ and $H_g \cup_{f_2} H_g$ are homotopic.

(3) Let $T^3 = (S^1)^3$ be the 3–torus. Prove that $T_3 \cong H_3 \cup_f H_3$ for some f.

These decompositions are called Heegard decompositions. It turns out that every closed, orientable 3–manifold admits such a decomposition.

Problem 42. Compute $\pi_1(\mathbf{R}^3 - X)$ where X is:

(1) The union of the x, y and z axes.

(2) 2 disjoint 2–spheres.

(3) A genus g handle body.

(4) 3 parallel lines.

(5) 2 linked circles.