CS 593/MA 595 - Introduction to Quantum Computing

Quantum Computation and Quantum Information by *Isaac Chuang, Michael Nielsen*
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Homework 1

Problem 2.2
(Matrix representations: example) Suppose V is a vector space with basis vectors $|0\rangle$ and $|1\rangle$, and A is a linear operator from V to V such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A, with respect to the input basis $|0\rangle$, $|1\rangle$, and the output basis $|0\rangle$, $|1\rangle$. Find input and output bases which give rise to a different matrix representation of A.

Proof. We recall that

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By Equation 2.12, a linear operator $A : V \to W$ has a matrix representation given by

$$A|v_j\rangle = \sum_i A_{ij} |w_i\rangle$$

Since we are working only in two dimensions, A will be of the form

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}.$$

Here, we are given a linear operator $A : V \to V$ $(W = V)$ such that

$$A|0\rangle = 0|0\rangle + 1|1\rangle = |1\rangle \implies \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies A_{00} = 0, A_{10} = 1$$

$$A|1\rangle = 1|0\rangle + 0|1\rangle = |0\rangle \implies \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies A_{01} = 1, A_{11} = 0.$$

Thus, a matrix representation of A in the input and output bases of $\{|0\rangle, |1\rangle\}$ is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider the input and output bases of

$$\left\{ |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Applying A to our basis element, and by the linearity of A, we get

$$A|+\rangle = \frac{1}{\sqrt{2}} A(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}} (A|0\rangle + A|1\rangle) = \frac{1}{\sqrt{2}} (|1\rangle + |0\rangle) = |+\rangle$$

$$A|-\rangle = \frac{1}{\sqrt{2}} A(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}} (A|0\rangle - A|1\rangle) = \frac{1}{\sqrt{2}} (|1\rangle - |0\rangle) = -|-\rangle.$$

Thus

$$A|+\rangle = 1|+\rangle + 0|-\rangle = |+\rangle \implies \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies A_{00} = 1, A_{10} = 0$$

$$A|-\rangle = 0|+\rangle - 1|-\rangle = -|-\rangle \implies \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \implies A_{01} = 0, A_{11} = -1.$$

Thus, a matrix representation of A in the input and output bases of $\{|+\rangle, |-\rangle\}$ is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Problem 2.7
Verify that $|w\rangle \equiv (1,1)$ and $|v\rangle \equiv (1,-1)$ are orthogonal. What are the normalized forms of these vectors?

Proof. To show that two vectors are orthogonal, the inner product of these two vectors should be zero. Then

$$\langle v|w\rangle = 1 \cdot 1 + 1 \cdot (-1) = 0.$$

Thus, $|v\rangle$ and $|w\rangle$ are orthogonal.
The normalized form of a vector $|u\rangle$ is given by the following

$$\frac{|u\rangle}{\sqrt{\langle u|u\rangle}}$$

where $\sqrt{\langle u|u\rangle} = || \, |u\rangle \, ||$ is the norm or length of the vector.
The norms of $|v\rangle$ and $|w\rangle$ are

$$\sqrt{\langle v|v\rangle} = \sqrt{2},$$
$$\sqrt{\langle w|w\rangle} = \sqrt{2}.$$

Thus, the normalized forms of the vectors are

$$\frac{|v\rangle}{\sqrt{\langle v|v\rangle}} = \frac{1}{\sqrt{2}}(1,-1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$\frac{|w\rangle}{\sqrt{\langle w|w\rangle}} = \frac{1}{\sqrt{2}}(1,1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

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Problem 2.10
Suppose $|v_i\rangle$ is an orthonormal basis for an inner product space V. What is the matrix representation for the operator $|v_j\rangle\langle v_k|$, with respect to the $|v_i\rangle$ basis?

Proof. The matrix representation of $|v_j\rangle\langle v_k|$ is an n-dimensional matrix ($\dim(V) = n$) with an entry of 1 in the jth row and kth column.

$$\begin{aligned}
|v_j\rangle\langle v_k| &= \mathbb{I}_V |v_j\rangle\langle v_k| \mathbb{I}_V \\
&= \left(\sum_p |v_p\rangle\langle v_p|\right) |v_j\rangle\langle v_k| \left(\sum_q |v_q\rangle\langle v_q|\right) \\
&= \sum_{p,q} |v_p\rangle\langle v_p|v_j\rangle\langle v_k|v_q\rangle\langle v_q| \\
&= \sum_{p,q} \delta_{pj}\delta_{kq} |v_p\rangle\langle v_q|
\end{aligned}$$

Thus

$$\left(|v_j\rangle\langle v_k|\right)_{pq} = \delta_{pj}\delta_{kq}$$

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Problem 2.20
(Basis changes) Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases, $|v_i\rangle$ and $|w_i\rangle$. Then the elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''.

Proof.

$$\begin{aligned}
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \langle v_i|\mathbb{I}A\mathbb{I}|v_j\rangle \\
&= \left\langle v_i \middle| \left(\sum_{i'} |w_{i'}\rangle \langle w_{i'}|\right) A \left(\sum_{j'} |w_{j'}\rangle \langle w_{j'}|\right) \middle| v_j \right\rangle \\
&= \sum_{i',j'} \langle v_i|w_{i'}\rangle \langle w_{i'}|A|w_{j'}\rangle \langle w_{j'}|v_j\rangle \\
&= \sum_{i',j'} \langle v_i|w_{i'}\rangle A''_{ij} \langle w_{j'}|v_j\rangle .
\end{aligned}$$

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Problem 2.26
Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both in terms of tensor products like $|0\rangle|1\rangle$, and using the Kronecker product.

Proof. Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Then

$$\begin{aligned}
|\psi\rangle^{\otimes 2} &= |\psi\rangle \otimes |\psi\rangle \\
&= \frac{1}{2}\left((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)\right) \\
&= \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right) \\
&= \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
|\psi\rangle^{\otimes 3} &= |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \\
&= \frac{1}{2\sqrt{2}}\left((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle)\right) \\
&= \frac{1}{2\sqrt{2}}\left(|0\rangle|0\rangle|0\rangle + |0\rangle|0\rangle|1\rangle + |0\rangle|1\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |1\rangle|0\rangle|0\rangle + |1\rangle|0\rangle|1\rangle + |1\rangle|1\rangle|0\rangle + |1\rangle|1\rangle|1\rangle\right) \\
&= \frac{1}{2\sqrt{2}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}
\end{aligned}$$

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Problem 2

(a) Let $\mathcal{H}$ be a finite dimensional Hilbert space. We define the *dual* Hilbert space $\mathcal{H}^*$ to be the set of all linear transformations $\rho : \mathcal{H} \to \mathbb{C}$, that is:

$$\mathcal{H}^* := \{\rho : \mathcal{H} \to \mathbb{C} \mid \rho \text{ is a linear function } \}.$$

Recall that if $|\psi\rangle \in \mathcal{H}$, then $\langle\psi|$ is the linear transformation

$$\begin{aligned} \langle\psi| : \mathcal{H} &\to \mathbb{C} \\ |\psi\rangle &\mapsto \langle\psi|\phi\rangle \end{aligned}$$

where $\langle\psi|\phi\rangle$ is the inner product.

(i) Show that $\mathcal{H}^*$ is a vector space with the same dimension as $\mathcal{H}$.

(ii) Show that the function

$$\begin{aligned} F : \mathcal{H} &\to \mathcal{H}^* \\ |\psi\rangle &\to \langle\psi| \end{aligned}$$

is a bijection. Warning: it is NOT linear (it is *anti-linear* or *conjugate-linear*), so you more-or-less need to show injectivity and surjectivity directly. For surjectivity, use the previous problem part (in particular, the fact that $\mathcal{H}$ is finite dimensional).

In fact, we won't do this, but it's even possible to define an inner product structure on $\mathcal{H}^*$, and the map $|\psi\rangle \mapsto \langle\psi|$ becomes an anti-linear isometry. The moral of the story is that Hilbert spaces are ALMOST isometrically isomorphic to their dual spaces - the only finnicky thing is that the "isomorphism" is not linear, it's anti-linear! This is called the Riesz representation theorem. It's true more generally, *i.e.* for infinite dimensional Hilbert spaces too.

(b) Let $\mathcal{B}(\mathcal{H})$ be the set of all linear transformations $A : \mathcal{H} \to \mathcal{H}$.

(i) Show that $\mathcal{B}(\mathcal{H})$ is a vector space. What is its dimension?

(ii) Show that the map

$$\begin{aligned} \mathcal{H} \otimes \mathcal{H}^* &\to \mathcal{B}(H) \\ |\phi\rangle \otimes \langle\psi| &\mapsto |\phi\rangle\langle\psi| \end{aligned}$$

is a vector space isomorphism.

Proof. (a) (i) Let $\mathcal{H}$ and $\mathbb{C}$ be any two vector spaces over the same field $\mathbf{F}$. Let $\mathcal{H}^* = \mathcal{L}(\mathcal{H}, \mathbb{C})$ be the set of linear transformations $\rho : \mathcal{H} \to \mathbb{C}$.

To show that $\mathcal{H}^*$ is a vector space, we must first define an "*addition of linear transformations*" and a "*scalar multiplication of elements of* $\mathbf{F}$ *by linear transformations*". In other words, our "vectors" will be linear transformations from $\mathcal{H}$ to $\mathbb{C}$. A vector space is just a set with two binary operations, vector addition and scalar multiplication, that satisfy certain properties. We call the elements of a vector space as vectors.

Given two linear transformations $\rho, \sigma : \mathcal{H} \to \mathbb{C}$, we need to define a new linear transformation that is called the "sum of ρ and σ". Denote it by $\rho \oplus \sigma$, to distinguish the "sum of linear transformations" from the sum of vectors. Since we want $\rho \oplus \sigma$ to be a linear transformation (which is a special kind of function) from $\mathcal{H}$ to $\mathbb{C}$, in order to specify it we need to say what the value of $\rho \oplus \sigma$ is at every $\mathbf{h} \in \mathcal{H}$. Define

$$(\rho \oplus \sigma)(\mathbf{h}) = \rho(\mathbf{h}) + \sigma(\mathbf{h}),$$

where the sum on the right is taking place in $\mathbb{C}$. This makes sense, because ρ and σ are already functions from $\mathcal{H}$ to $\mathbb{C}$, so $\rho(\mathbf{h})$ and $\sigma(\mathbf{h})$ are vectors in $\mathbb{C}$, which we can add.

Is $\rho \oplus \sigma$ a linear transformation from $\mathcal{H}$ to $\mathbb{C}$? First, it *is* a function from $\mathcal{H}$ to $\mathbb{C}$. Now, to check that it is a linear transformation, we need to check that $\forall \mathbf{h}_1, \mathbf{h}_2 \in \mathcal{H},\ \forall \alpha \in \mathbf{F}$, we have

$$(\rho \oplus \sigma)(\mathbf{h}_1 + \mathbf{h}_2) = (\rho \oplus \sigma)(\mathbf{h}_1) + (\rho \oplus \sigma)(\mathbf{h}_2) \quad \text{and} \quad (\rho \oplus \sigma)(\alpha \mathbf{h}_1) = \alpha((\rho \oplus \sigma)(\mathbf{h}_1)).$$

Indeed, since ρ and σ are themselves linear transformations, we have

$$\begin{aligned}
(\rho \oplus \sigma)(\mathbf{h}_1 + \mathbf{h}_2) &= \rho(\mathbf{h}_1 + \mathbf{h}_2) + \sigma(\mathbf{h}_1 + \mathbf{h}_2) && \text{(by definition of } \rho \oplus \sigma)\\
&= \rho(\mathbf{h}_1) + \rho(\mathbf{h}_2) + \sigma(\mathbf{h}_1) + \sigma(\mathbf{h}_2) && \text{(by linearity of } \rho \text{ and } \sigma)\\
&= \rho(\mathbf{h}_1) + \sigma(\mathbf{h}_1) + \rho(\mathbf{h}_2) + \sigma(\mathbf{h}_2) &&\\
&= (\rho \oplus \sigma)(\mathbf{h}_1) + (\rho \oplus \sigma)(\mathbf{h}_2) && \text{(by definition of } \rho \oplus \sigma)\\
(\rho \oplus \sigma)(\alpha \mathbf{h}_1) &= \rho(\alpha \mathbf{h}_1) + \sigma(\alpha \mathbf{h}_1) && \text{(by definition of } \rho \oplus \sigma)\\
&= \alpha \rho(\mathbf{h}_1) + \alpha \sigma(\mathbf{h}_1) && \text{(by linearity of } \rho \text{ and } \sigma)\\
&= \alpha(\rho(\mathbf{h}_1) + \sigma(\mathbf{h}_1)) &&\\
&= \alpha((\rho \oplus \sigma)(\mathbf{h}_1)) && \text{(by definition of } \rho \oplus \sigma)
\end{aligned}$$

Thus, $\rho \oplus \sigma$ is indeed an element of $\mathcal{H}^* = \mathcal{L}(\mathcal{H}, \mathbb{C})$.

Now, we define a scalar multiplication, which I will denote by $\odot$ (again, to avoid confusion with the scalar multiplication from $\mathcal{H}$ and $\mathbb{C}$. Given $\rho : \mathcal{H} \to \mathbb{C}$ and $\alpha \in \mathbf{F}$, define $(\alpha \odot \rho)$ to be the function

$$(\alpha \odot \rho)(\mathbf{h}) = \alpha \rho(\mathbf{h}).$$

To have a vector space, the eight following axioms must be satisfied $\forall \rho, \sigma, \tau \in \mathcal{H}^*$ and $\forall \alpha, \beta \in \mathbf{F}$, we have

- **Associativity of transformation addition:** $(\rho \oplus \sigma) \oplus \tau = \rho \oplus (\sigma \oplus \tau)$.
- **Commutativity of transformation addition:** $\rho \oplus \sigma = \sigma \oplus \rho$.
- **Identity element of transformation addition:** There exists an element $\mathbf{0} \in \mathcal{H}^*$, called the zero transformation, such that $\rho + 0 = \rho$, $\forall \rho \in \mathcal{H}^*$.
- **Inverse elements of transformation addition** For every $\rho \in \mathcal{H}^*$, there exists an element $(-\rho) \in \mathcal{H}^*$, called the additive inverse of ρ, such that $\rho + (-\rho) = \mathbf{0}$.
- **Compatibility of scalar multiplication with field multiplication:** $\alpha(\beta\rho) = (\alpha\beta)\rho$.
- **Identity element of scalar multiplication:** $\mathbf{1}\rho = \rho$, where $\mathbf{1}$ denotes the multiplicative identity in $\mathbf{F}$.
- **Distributivity of scalar multiplication with respect to transformation addition:** $\alpha(\rho + \sigma) = \alpha\rho + \alpha\sigma$.
- **Distributivity of scalar multiplication with respect to field addition:** $(\alpha + \beta)\rho = \alpha\rho + \beta\rho$.

To prove that $\mathcal{H}^*$ has the same dimension as $\mathcal{H}$, let $\rho \in \mathcal{H}^*$ and let $\{e_1, \ldots, e_n\}$ be a basis for $\mathcal{H}$. Define $e^i \in \mathcal{H}^*$ by $e^i(e_j) = \delta_{ij}$. We want to show that $\{e^1, \ldots, e^n\}$ spans $\mathcal{H}^*$.

$$\rho(\mathbf{h}) = \rho(h_1 e_1 + \cdots + h_n e_n) = h_1 \rho(e_1) + \cdots + h_n \rho(e_n).$$

If $h_1\rho(e_1) = \lambda_1, \ldots, h_n\rho(e_n) = \lambda_n$, then $\rho(\mathbf{h}) = h_1\lambda_1 e^1(e_1) + \cdots + h_n\lambda_n e^n(e_n) = \lambda_1 e^1(\mathbf{h}) + \cdots + \lambda_n e^n(\mathbf{h})$.

To show that the set $\{e^1, \ldots, e^n\}$ is linearly independent, suppose that $\mathbf{0} = c_1 e^1 + \ldots + c_n e^n$ is the zero mapping. Consider the image of $e_1 : 0(e_1) = c_1 * 1 + \ldots + c_n * 0 = c_1 \implies c_1 = 0$. By repeating the procedure for all $e_j, 2 \le j \le n$, we see that $c_1 = \cdots = c_n = 0$.

(ii) A function $F : A \to B$ is a bijection if and only if F is:

- **Injective:** $F(x) = F(y) \implies x = y$
- **Surjective:** $\forall b \in B, \exists a \in A \ / \ F(a) = b$

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Problem 3
In this set of exercises, we dig into the spectral theorem/diagonalization in more detail.

(a) Prove the corollaries of the spectral theorem (Theorem 2.1 in Nielsen and Chuang) that I stated on Tuesday by doing exercises 2.17 and 2.18 . [Hint: being "corollaries" of course means that you should use the spectral theorem in your proof.]

(b) Differing slightly from the book (see page 70), in this problem let us define an orthogonal projector to be a linear operator $P : \mathcal{H} \to \mathcal{H}$ such that $P^2 = P$ and $P^* = P$. Show that P is an orthogonal projector if and only if P is unitarily diagonalizable, with all eigenvalues equal to either 0 or 1.

(c) Do exercises 2.29-2.32. [Hint: you can use the previous parts of this exercise, or, closely related, Exercise 2.28 (which you need not prove for this problem).]

Proof. (a) **Exercise 2.17:** Show that a normal matrix is Hermitian if and only if it has real eigenvalues.
Proof:

$\boxed{\implies}$ Let A be a Normal and Hermitian matrix. Then, by the spectral decomposition theorem, A has a diagonal representation given by $\sum_i \lambda_i |i\rangle \langle i|$, where the set of $|i\rangle$ form an orthonormal basis for V and each $|i\rangle$ is an eigenvector with eigenvalue λ_i. Since A is Hermitian, then $A^\dagger = A$. Then, we have

$$\begin{aligned} A^\dagger &= \left(\sum_i \lambda_i |i\rangle \langle i|\right)^\dagger \\ &= \sum_i \lambda_i^* |i\rangle \langle i| \\ &= A \\ &= \sum_i \lambda_i |i\rangle \langle i| \end{aligned}$$

Thus, $\lambda_i^* = \lambda_i \implies \lambda_i \in \mathbb{R}$ and A has real eigenvalues.

$\boxed{\impliedby}$ Let A be a normal matrix with real eigenvalues. Then, by the spectral decomposition theorem, A has a diagonal representation given by $\sum_i \lambda_i |i\rangle \langle i|$, where the set of $|i\rangle$ form an orthonormal basis for V and each $|i\rangle$ is an eigenvector with real eigenvalue λ_i. Then, we have

$$\begin{aligned} A^\dagger &= \left(\sum_i \lambda_i |i\rangle \langle i|\right)^\dagger \\ &= \sum_i \lambda_i^* |i\rangle \langle i| \\ &= \sum_i \lambda_i |i\rangle \langle i| \\ &= A \end{aligned}$$

Thus, A is Hermitian.

Exercise 2.18: Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Proof: Let U be a unitary matrix. It is then normal as $U^\dagger U = UU^\dagger = \mathbb{I}$. Then, by the spectral decomposition theorem, U has a representation given by $\sum_i \lambda_i |i\rangle \langle i|$, where the set of $|i\rangle$ form an orthonormal basis for V and each $|i\rangle$ is an eigenvector with real eigenvalue λ_i. Then, we have

$$\begin{aligned}
UU^\dagger = \mathbb{I} &= \left(\sum_i \lambda_i |i\rangle \langle i|\right) \left(\sum_{i'} \lambda_{i'} |i'\rangle \langle i'|\right)^\dagger \\
&= \sum_{i,i'} \lambda_i \lambda_{i'}^* |i\rangle \langle i|i'\rangle \langle i'| \\
&= \sum_{i,i'} \lambda_i \lambda_{i'}^* |i\rangle \delta_{i,i'} \langle i'| \\
&= \sum_i \lambda_i \lambda_i^* |i\rangle \langle i| \\
&= \sum_i |\lambda_i|^2 |i\rangle \langle i|
\end{aligned}$$

Thus, $|\lambda_i|^2 = 1 \implies |\lambda_i| = 1$.

(b) Define an orthogonal projector to be a linear operator $P : \mathcal{H} \to \mathcal{H}$ such that $P^2 = P$ and $P^* = P$.

Consider the eigenvalue equation $P|\psi\rangle = \lambda|\psi\rangle$. Applying P again, we get

$$P^2|\psi\rangle = P\lambda|\psi\rangle = \lambda^2|\psi\rangle$$

but also

$$P^2|\psi\rangle = P|\psi\rangle = \lambda|\psi\rangle$$

Hence, $\lambda^2 = \lambda \implies \lambda = 0, 1$.

If P is a projector, that means there's a subspace $\mathcal{H}$ onto which it projects. It maps every vector in $\mathcal{H}$ to itself. Therefore every vector in $\mathcal{H}$ is an eigenvector with eigenvalue 1. Every vector not in $\mathcal{H}$ is mapped to a vector in $\mathcal{H}$. Take any vector $|\psi\rangle$ and write

$$|\psi\rangle = P|\psi\rangle + (|\psi\rangle - P|\psi\rangle)$$

so the first term $P|\psi\rangle \in \mathcal{H}$. It is easy to see that the second term, $|\psi\rangle - P|\psi\rangle$, is in the kernel of P: the first term is mapped to $P|\psi\rangle$, and the second is mapped to $P|\psi\rangle - P^2|\psi\rangle$. But since $P|\psi\rangle$ is in $\mathcal{H}$, it must be fixed by P, so $P^2|\psi\rangle = P|\psi\rangle$; thus $P(|\psi\rangle - P|\psi\rangle) = 0$. In this way, every vector $|\psi\rangle$ is written as the sum of a vector in $\mathcal{H}$, which is an eigenvector with eigenvalue 1, and a vector in the kernel of P, which is an eigenvector with eigenvalue 0. So, forming a basis of the whole space by taking the union of a basis of $\mathcal{H}$ and a basis of the kernel of P, and the matrix of P with respect to that basis is

$$\begin{bmatrix}
1 & & & & & & & \\
& 1 & & & & & & \\
& & 1 & & & & & \\
& & & \ddots & & & & \\
& & & & 1 & & & \\
& & & & & 0 & & \\
& & & & & & \ddots & \\
& & & & & & & 0
\end{bmatrix}$$

(and all off-diagonal entries are 0) where the number of 1's is the dimension of $\mathcal{H}$ and the number of 0's is the dimension of the kernel of P.

Thus, P is unitarily diagonalizable with all eigenvalues equal to either 0 or 1.

(c) **Exercise 2.29:** Show that the tensor product of two unitary operators is unitary.

Proof: An operator U is said to be unitary if $U^\dagger U = \mathbb{I}$. Suppose A and B are two unitary operators. We need to show that $A \otimes B$ is also unitary. We have

$$(A \otimes B)^\dagger (A \otimes B) = (A^\dagger \otimes B^\dagger)(A \otimes B) = (A^\dagger A) \otimes (B^\dagger B) = \mathbb{I} \otimes \mathbb{I}.$$

Sometimes, the definition of a unitary operator is given as $UU^\dagger = \mathbb{I}$. In that case

$$(A \otimes B)(A \otimes B)^\dagger = (A \otimes B)(A^\dagger \otimes B^\dagger) = (AA^\dagger) \otimes (BB^\dagger) = \mathbb{I} \otimes \mathbb{I}.$$

Thus, the tensor product of two unitary operators is unitary.

Exercise 2.30: Show that the tensor product of two Hermitian operators is Hermitian.

Proof: An operator H is said to be Hermitian if $H = H^\dagger$. Suppose A and B are two Hermitian operators. We have

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B.$$

Thus, the tensor product of two Hermitian operators is Hermitian.

Exercise 2.31: Show that the tensor product of two positive operators is positive.

Proof: An operator P is said to be positive if $\langle\psi|P|\psi\rangle \geq 0$, for all $|\psi\rangle$. Suppose A and B are two positive operators. For any $|v\rangle \otimes |w\rangle$, we have

$$\langle v| \otimes \langle w| \, (A \otimes B) \, |v\rangle \otimes |w\rangle = \langle v|A|v\rangle \, \langle w|B|w\rangle \geq 0$$

Thus, the tensor product of two positive operators is positive.

Exercise 2.32: Show that the tensor product of two projectors is a projector.

Proof: An operator P is said to be a projector if $P^2 = P$. Suppose A and B are two projectors. We have

$$(A \otimes B)^2 = (A \otimes B)(A \otimes B) = A^2 \otimes B^2 = A \otimes B.$$

Thus, the tensor product of two projectors is a projector.

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Problem 2.42
Verify that

$$AB = \frac{[A,B] + \{A,B\}}{2}$$

Proof.

$$\frac{[A,B] + \{A,B\}}{2} = \frac{(AB - BA) + (AB + BA)}{2} = \frac{(AB + AB) + (BA - BA)}{2} = AB$$

■

Problem 2.44
Suppose $[A,B] = 0$, $\{A,B\} = 0$, and A is invertible. Show that B must be 0.

Proof. If $[A,B] = 0$, then $AB = BA$. If $\{A,B\} = 0$, then $AB = -BA$. This implies that $AB = 0$. It must be that $A = 0$ or $B = 0$. Since A is invertible, this means $A \neq 0$. Thus, $B = 0$. ■

Problem 2.45
Show that $[A,B]^\dagger = [B^\dagger, A^\dagger]$.

Proof.

$$[A,B]^\dagger = (AB - BA)^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger].$$

■

Problem 2.46
Show that $[A, B] = -[B, A]$.

Proof.
$$[A, B] = AB - BA = -(BA - AB) = -[B, A].$$
■

Problem 2.47
Suppose A and B are Hermitian. Show that $i[A, B]$ is Hermitian.

Proof. To show that an operator is Hermitian, we take the Hermitian of that operator and show it is equal to the operator itself.

$$(i[A, B])^\dagger = -i[A, B]^\dagger = -i[B^\dagger, A^\dagger] = -i[B, A] = i[A, B].$$

Thus, $i[A, B]$ is Hermitian. ■