PHYS 617 - Statistical Mechanics

A Modern Course in Statistical Physics by *Linda E. Reichl*

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Homework 6

Problem 1

(a) Estimate the kinematic viscosity ν of a plasma made out of electrons and protons. You may assume the plasma has mass density ρ and temperature T.

(b) Show for a plasma with characteristic mass M, size L, temperature T and velocity V, the Reynolds number is given by

$$Re \sim LV/\nu = C\left(\frac{M}{M_\odot}\right)\left(\frac{L}{R_\odot}\right)^{-2}\left(\frac{T}{1\,\mathrm{K}}\right)^{-\frac{5}{2}}\left(\frac{V}{1\,\mathrm{km/s}}\right).$$

What is your estimate for the coefficient C? (this might help to explain why astrophysical flows are often treated as having zero viscosity).

Solution. (a) Considering a plasma, the mass of the protons will dominate the mass of electrons, so we can say $m \sim m_p$. We know that the diffusion constant ν of a particle is given by $\nu \sim \lambda v_{th}$. Additionally, for a plasma, the collision cross-section σ_c is given by

$$\sigma_c \sim b^2 \sim \left(\frac{e^2}{4\pi\epsilon_0}\frac{1}{k_B T}\right)^2,$$

and the mean free path of a particle is given by

$$\lambda \sim \frac{1}{n\sigma_c}.$$

Then,

$$\begin{aligned}
\nu &\sim \lambda v_{th} \\
&\sim \frac{1}{n\sigma_c}\sqrt{\frac{k_B T}{m}} \\
&\sim \frac{V}{N}\left(\frac{4\pi\epsilon_0}{e^2}k_B T\right)^2\sqrt{\frac{k_B T}{m}} \\
&\sim \frac{M}{N\rho}\left(\frac{\epsilon_0}{e^2}\right)^2\frac{(k_B T)^{\frac{5}{2}}}{\sqrt{m}} \\
&\sim \frac{m}{\rho}\left(\frac{\epsilon_0}{e^2}\right)^2\frac{(k_B T)^{\frac{5}{2}}}{\sqrt{m}} \\
&\sim \left(\frac{\epsilon_0}{e^2}\right)^2\frac{(k_B T)^{\frac{5}{2}}\sqrt{m}}{\rho}.
\end{aligned}$$

(b) The Reynolds number Re is given by

$$\begin{aligned}
Re &\sim \frac{LV}{\nu} \\
&\sim LV\left(\frac{e^2}{\epsilon_0}\right)^2 \frac{\rho}{(k_BT)^{\frac{5}{2}}\sqrt{m}} \\
&\sim LV\left(\frac{e^2}{\epsilon_0}\right)^2 \frac{\frac{m}{L^3}}{(k_BT)^{\frac{5}{2}}\sqrt{m}} \\
&\sim \left(\frac{e^2}{\epsilon_0}\right)^2 \frac{\sqrt{m}V}{L^2(k_BT)^{\frac{5}{2}}} \\
&\sim \frac{1}{\sqrt{m}}\left(\frac{e^2}{\epsilon_0}\right)^2 \frac{mV}{L^2(k_BT)^{\frac{5}{2}}} \\
&\sim \frac{1}{\sqrt{m}k_B^{\frac{5}{2}}}\left(\frac{e^2}{\epsilon_0}\right)^2 \frac{mV}{L^2(T)^{\frac{5}{2}}} \\
&\equiv CML^{-2}T^{-\frac{5}{2}}V.
\end{aligned}$$

We can see that the constant has to be on the order of $\frac{e^4}{\sqrt{m}k_B^{\frac{5}{2}}\epsilon_0^2} \sim 10^{17}$, which is huge. From that, we can say that ν has to be extremely small, and hence, the astrophysical plasma can be assumed to have zero viscosity.

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Problem 2
Compute the electric susceptibility χ_E of water vapor. Assume that a water molecule has a given electric dipole moment p and it can have any random orientation in the presence of an electric field. The susceptibility should be some function of number density n and temperature T. You may find it useful to know the following expansion of $\coth(x)$ for small x:

$$\coth(x) \approx \frac{1}{x} + \frac{1}{3}x + \cdots .$$

If it's been a while since your last E&M course, the electric susceptibility of a material is determined by its affinity for inducing polarization in the presence of an electric field

$$\vec{P} = \epsilon_0 \chi_E \vec{E},$$

where $\vec{P}$ is the dipole moments per unit volume. Don't worry about the electric field produced by the dipoles themselves; $\vec{E}$ is meant to represent the total electric field, so it is already taken into account.

Solution. The potential energy of the water vapor molecule due to an electric field is $\epsilon(\theta) = -\mathbf{P}\cdot\mathbf{E} = -PE\cos(\theta)$. Then the probability of having a given potential energy V for some angle θ is

$$\mathbb{P}(\theta) = \frac{1}{N}\mathrm{e}^{-\frac{V}{k_BT}} = \frac{1}{N}\mathrm{e}^{PE\cos(\theta)}.$$

We know that $\cos(\theta) \in [-1, 1]$, which means that the energy can vary from $-PE$ to $+PE$. Then the expected value (average value) of the energy of the system is given by

$$\begin{aligned}
\langle \epsilon \rangle &= \frac{\int_{-PE}^{PE} \epsilon e^{-\frac{\epsilon}{k_B T}}\,\mathrm{d}\epsilon}{\int_{-PE}^{PE} e^{-\frac{\epsilon}{k_B T}}\,\mathrm{d}\epsilon} \\
&= \frac{-k_B T \epsilon e^{-\frac{\epsilon}{k_B T}}\Big|_{-PE}^{PE} + k_B T \int_{-PE}^{PE} e^{-\frac{\epsilon}{k_B T}}\,\mathrm{d}\epsilon}{\int_{-PE}^{PE} e^{-\frac{\epsilon}{k_B T}}\,\mathrm{d}\epsilon} \\
&= \frac{-k_B T PE \left(e^{-\frac{PE}{k_B T}} + e^{\frac{PE}{k_B T}}\right) - (k_B T)^2 \left(e^{-\frac{PE}{k_B T}} - e^{\frac{PE}{k_B T}}\right)}{-k_B T \left(e^{-\frac{PE}{k_B T}} - e^{\frac{PE}{k_B T}}\right)} \\
&= k_B T - \frac{e^{\frac{PE}{k_B T}} + e^{-\frac{PE}{k_B T}}}{e^{\frac{PE}{k_B T}} - e^{-\frac{PE}{k_B T}}} PE \\
&= k_B T - PE \coth\left(\frac{PE}{k_B T}\right).
\end{aligned}$$

Since $PE \ll k_B T$, then we can use the small-angle approximation for coth to get

$$\begin{aligned}
\langle \epsilon \rangle &= k_B T - PE \coth\left(\frac{PE}{k_B T}\right) \\
&\approx k_B T - PE \left(\frac{1}{\frac{PE}{k_B T}} + \frac{1}{3}\frac{PE}{k_B T}\right) \\
&= -\frac{1}{3}\frac{(PE)^2}{k_B T}.
\end{aligned}$$

For the polarization vector, we can write it as the average polarization of each particle multiplied by the number of particles, *i.e.*

$$\mathbf{P} = n \langle p \rangle .$$

We can now take $\langle P \rangle$ to be parallel to the electric field as

$$\langle \epsilon \rangle = - \langle p \rangle E \implies \langle p \rangle = -\frac{\langle \epsilon \rangle}{E}$$

$$-\frac{n \langle \epsilon \rangle}{E} = \epsilon_0 \chi_E E \implies \chi_E = -\frac{n \langle \epsilon \rangle}{E^2 \epsilon_0} = \frac{np^2}{3\epsilon_0 k_B T}.$$

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Problem 3

(a) Compute the single-particle partition function

$$z = \int \frac{\mathrm{d}^3x\,\mathrm{d}^3p}{h^3} \mathrm{e}^{-\beta\epsilon_p}$$

for a single particle in a classical ideal monatomic gas.

(b) Let's look at the quantum version. Imagine an infinite square well with sides of length L. Show that the partition function for this system, can be expressed as

$$z = \left(\sum_{n=1}^{\infty} \mathrm{e}^{-\beta\epsilon_0 n^2}\right)^3$$

for some energy scale ϵ_0.

Write a computer program to compute this explicitly as a function of $\beta\epsilon_0$ by approximating the infinite sum up to very large N. Make a plot of $z(\beta\epsilon_0)$, where

$$z(x) = \left(\sum_{n=1}^{\infty} \mathrm{e}^{-xn^2}\right)^3$$

and examine the limiting cases $\beta\epsilon_0 \ll 1$ and $\beta\epsilon_0 \gg 1$. Does your solution agree with the classical result in the approximate limit? Plot the classical result alongside the quantum result. Make sure the plot is clear enough that I can actually see where these curves agree and where they disagree.

Solution. (a) The energy of a single particle is given by $\epsilon_p = \frac{p^2}{2m}$. Then the single-particle partition function is

$$\begin{aligned}
z &= \int \frac{\mathrm{d}^3x\,\mathrm{d}^3p}{h^3} \mathrm{e}^{-\beta\epsilon_p} \\
&= \frac{1}{h^3} \int_0^L \mathrm{d}^3x \int_0^{\infty} \mathrm{e}^{-\frac{\beta p^2}{2m}}\,\mathrm{d}^3p \\
&= \frac{1}{h^3} \left(L^3\right) \int_0^{\infty} \mathrm{e}^{-\frac{\beta p^2}{2m}}\,4\pi p^2\,\mathrm{d}p \\
&= \frac{L^3}{h^3} \left(\frac{2\pi m}{\beta}\right)^{\frac{3}{2}} \\
&= \left(\frac{2\pi m L^2}{\beta h^2}\right)^{\frac{3}{2}}.
\end{aligned}$$

(b) The energy in a quantum well is given by

$$\epsilon_n = \frac{n^2\hbar^2}{8mL^2} = \epsilon_0 n^2.$$

In three dimensions, we can write $n^2 = n_x^2 + n_y^2 + n_z^2$, and we get

$$\begin{aligned}
z &= \sum_{n=1}^{\infty} \mathrm{e}^{-\beta\epsilon_n} \\
&= \sum_{n=1}^{\infty} \mathrm{e}^{-\beta\epsilon_0 n^2} \\
&= \sum_{n_x, n_y, n_z}^{\infty} \mathrm{e}^{-\beta\epsilon_0 (n_x^2 + n_y^2 + n_z^2)} \\
&= \left(\sum_{n_x=1}^{\infty} \mathrm{e}^{-\beta\epsilon_0 n_x^2}\right)\left(\sum_{n_y=1}^{\infty} \mathrm{e}^{-\beta\epsilon_0 n_y^2}\right)\left(\sum_{n_z=1}^{\infty} \mathrm{e}^{-\beta\epsilon_0 n_z^2}\right) \\
&= \left(\sum_{n=1}^{\infty} \mathrm{e}^{-\beta\epsilon_0 n^2}\right)^3 .
\end{aligned}$$

Setting $x = \beta\epsilon_0$, we have

$$z(x) = \left(\sum_{n=1}^{\infty} \mathrm{e}^{-xn^2}\right)^3 .$$

We can rewrite the partition function from part (a) as

$$z_1 = \left(\frac{2\pi m L^2}{\beta h^2}\right)^{\frac{3}{2}} = \left(\frac{\pi}{4\beta\epsilon_0}\right)^{\frac{3}{2}} = \left(\frac{\pi}{4x}\right)^{\frac{3}{2}} .$$

For large N (~ 1000), we have

```python
import numpy as np
import matplotlib.pyplot as plt

N = 1000
x = np.linspace(0.005, 10, N)

z_classical = (np.pi / (4 * x))**(3/2)

n = 1
summand = 0
while (n < N):
    summand += np.exp(-x*n**2)
    n += 1
z_quantum = summand**3


plt.plot(x, z_classical, 'r')
plt.plot(x, z_quantum, 'b')
plt.show()
```

Listing 1: Code block of the plot shown

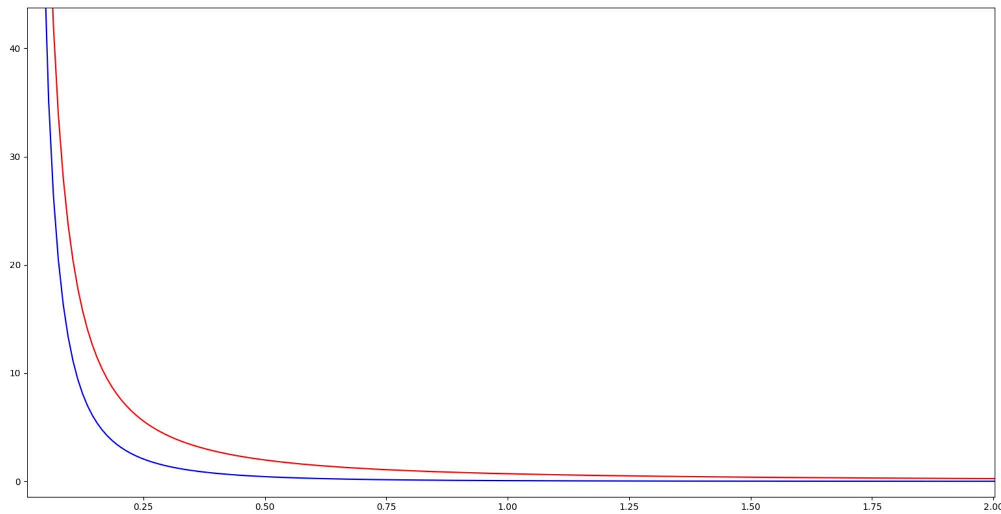

Figure 1: Plot showing the classical partition function (red) vs. the quantum partition function (blue)

For both functions, we have

- For $\beta\epsilon_0 \ll 1$, $z \to \infty$.
- For $\beta\epsilon_0 \gg 1$, $z \to 0$.

The main difference is that the quantum partition function decays asymptotically to zero faster than the classical partition function, and the latter is less singular at the origin than the former.

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