PHYS 617 - Statistical Mechanics

A Modern Course in Statistical Physics by *Linda E. Reichl*
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Homework 1

Problem 1
Estimate the number of electrons in the sun.

Note: When I ask you to "estimate" a number, I want zero significant digits. Your answer should just be 10 to some power (and you get full credit even if you're one power off). Accordingly, your derivation should avoid any complexity at a higher level of detail than the order-of-magnitude estimate you are looking for.

Proof. The mass of the Sun $m_\odot \approx 10^{30}$ kg and the mass of a proton $m_p \approx 10^{-27}$ kg. Since the Sun is primarily composed of hydrogen (1 proton, 1 electron), and since the mass of a proton is much larger than that of an electron, then the number of electrons in the Sun can be estimated as

$$\# \, e^- \in \text{Sun} \approx \frac{m_\odot}{m_p} = \frac{10^30}{10^{-27}} = 10^{57} \text{ kg}.$$

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Problem 2
Show that the diffusion equation

$$\dot{u} - \nu u'' = 0$$

is a conservation law. Define the total "charge" between two points a and b:

$$Q \equiv \int_a^b u \, \mathrm{d}x$$

and show that the rate of change of Q in time is determined only by the flux of charge through the points a and b. What is this flux (rate of flow of u) and can it be interpreted physically?

Proof. We have defined

$$Q = \int_a^b u(x,t) \, \mathrm{d}x.$$

Calculating the rate of change of Q in time, we have

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\int_a^b u(x,t) \, \mathrm{d}x \right) \\
&= \int_a^b \frac{\partial u(x,t)}{\partial t} \, \mathrm{d}x \\
&= \nu \int_a^b \frac{\partial^2 u(x,t)}{\partial x^2} \, \mathrm{d}x \\
&= \nu \left. \frac{\partial u(x,t)}{\partial x} \right|_a^b \\
&= \nu \left[\frac{\partial u(b,t)}{\partial x} - \frac{\partial u(a,t)}{\partial x} \right] \\
&= \nu \left[u'(b,t) - u'(a,t) \right].
\end{aligned}$$

The flux of charge through the boundaries at a and b can be defined as

$$J_a \equiv \nu \frac{\partial u(a,t)}{\partial x}, \quad J_b \equiv \nu \frac{\partial u(b,t)}{\partial x}.$$

Thus, we have

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = J_b - J_a.$$

The flux is the rate of flow of u and can be interpreted physically as the rate of flow of the material through the boundaries of the system. The flux is positive when the material is flowing into the system and negative when the material is flowing out of the system. Thus, the rate of change of Q is determined only by the flux of charge through the boundaries.

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Problem 3
Estimate how long it takes a photon to escape the sun. Assume a photon mean free path given by λ and the radius of the sun is given by R.

Note: when I ask you to "estimate" a formula (rather than a number) the result should not include any dimensionless coefficients like π or 2. I am only looking for how the answer scales as a function of the different variables in the system.

Proof. The problem of finding out the time it takes for a photon to escape the sun (from the center) is equivalent to a random walk of that photon. In other words, the time τ taken by the photon to reach the surface is the time t it takes in each step multiplied by the number of those steps N taken, given by

$$\tau = tN.$$

The time t for each step is given by

$$t = \frac{\lambda}{c}$$

and the number of steps N taken is given by the linear distance, which is the radius of the Sun R, covered by random walk, given by

$$N = \frac{R^2}{\lambda^2}.$$

Putting everything together, we get

$$\tau = tN = \frac{\lambda}{c}\frac{R^2}{\lambda^2} = \frac{R^2}{\lambda c}.$$

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Problem 4
Solve the diffusion equation for an initial condition given by a single Fourier mode:

$$\rho(x,0) = e^{ikx}$$

What is $\rho(x,t)$?

Proof. The diffusion equation is given by

$$\dot{\rho}(x,t) - \nu\rho''(x,t) = 0.$$

Let

$$r(k,t) = \mathcal{F}\left[\rho(x,t)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \rho(x,t)e^{ikx}\,\mathrm{d}x$$

be the Fourier transform of $\rho(x,t)$. Then, our PDE becomes

$$\dot{r}(k,t) + \nu k^2 r(k,t) = 0.$$

The solution to the ODE above is given by

$$r(k,t) = r(k,0)e^{-\nu k^2 t}$$

where

$$r(k,0) = \mathcal{F}\left[\rho(x,0)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \rho(x,0)e^{ikx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}e^{ikx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\left.\frac{e^{2ikx}}{2ik}\right|_{-\infty}^{0} = \frac{1}{2ik\sqrt{2\pi}}$$

We now have

$$r(k,t) = \frac{1}{2ik\sqrt{2\pi}}e^{-\nu k^2 t}.$$

Taking the inverse Fourier transform, we get

$$\begin{aligned}\rho(x,t) = \left[r(k,t)\right]^{-T} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r(k,t)e^{-ikx}\,\mathrm{d}k \\ &= \frac{1}{4ik\pi}\int_{-\infty}^{\infty} e^{-\nu k^2 t}e^{-ikx}\,\mathrm{d}k.\end{aligned}$$

Using the following property

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}e^{c}$$

with $a = \nu t$, $b = -ix$, and $c = 0$, we get

$$\rho(x,t) = \frac{1}{4ik\pi}\left(\sqrt{\frac{\pi}{\nu t}}e^{\frac{-x^2}{4\nu t}}\right) = \frac{1}{2ik\sqrt{4\pi\nu t}}e^{\frac{-x^2}{4\nu t}}.$$

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Problem 5
Derive the next-order correction to Stirling's approximation:

$$\ln(N!) = N\ln(N) - N + f(N)$$

where $f(N)$ is the correction. Do this by finding a relationship between $f(N)$ and $f(N+1)$ and setting that (approximately) to

$$f(N+1) - f(N) \approx \frac{\mathrm{d}f}{\mathrm{d}N}$$

Make a plot of the ratio of $N!$ to its approximation and show that it asymptotes to a constant:

$$\frac{N!}{\text{Appx.}} = e^{\ln(N!) - N\ln(N) + N - f(N)}$$

Make a plot of the above formula for the $f(N)$ you computed and $f(N) = 0$ with no correction, to show the difference. I highly recommend keeping it in the exponential form, as $N!$ and N^N are both very large and will not be representable as standard floating-point numbers for $N >\sim 100$; so take the difference $\ln(N!) - N\ln(N)$ first before exponentiating.

Depending on the detail of your calculation, your ratio should asymptote to a constant which may not be equal to unity. Double-check the formula by looking up Stirling's Approximation on google (assuming you haven't already) and check that the constant term you get is consistent with the overall dimensionless constant in the most detailed formula you can find on the internet.

Proof. Computing the derivative's approximation

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}N} &\approx f(N+1) - f(N) \\
&= \ln((N+1)!) - (N+1)\ln(N+1) + (N+1) - \ln(N!) + N\ln(N) - N \\
&= \ln\left(\frac{(N+1)!}{N!}\right) + \ln\left(\frac{N^N}{(N+1)^{N+1}}\right) + 1 \\
&= \ln(N+1) + \ln\left(\frac{N^N}{(N+1)^{N+1}}\right) + 1 \\
&= N\ln\left(\frac{N}{N+1}\right) + 1
\end{aligned}$$

Taking the integral, we get

$$\begin{aligned}
f(N) &= \int (N+1)\ln\left(\frac{N}{N+1}\right) + 1\,\mathrm{d}N \\
&= \frac{\ln(|N+1|) + N^2\ln\left(\frac{N}{N+1}\right) + N}{2}
\end{aligned}$$

Plotting the two functions, with the computed $f(N)$ and with $f(N) = 0$, we get

```
import matplotlib.pyplot as plt
import numpy as np
from scipy.special import factorial

N = np.arange(0, 170, 1)
f = ( np.log( np.abs(N+1)) + N**2 * np.log( N / (N+1) ) + N) /2

asymp = np.exp( np.log(factorial(N)) - N*np.log(N) + N - f)
asymp_0 = np.exp(np.log(factorial(N)) - N*np.log(N) + N)
```

```
plt.plot(N, asymp, 'b', N, asymp_value, 'r--')
plt.plot(N, asymp_0, 'g')
plt.show()
```

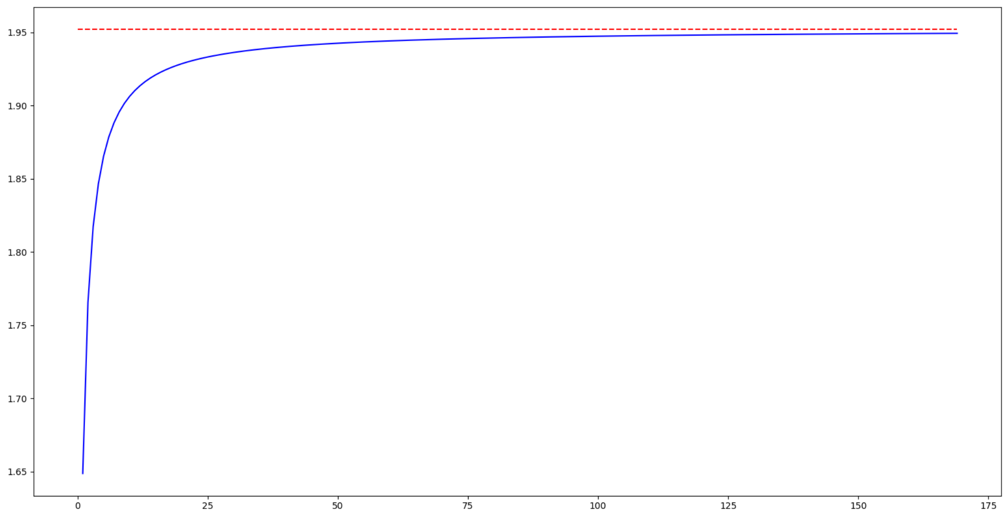

Figure 1: Plot of the ratio with the computed $f(N)$

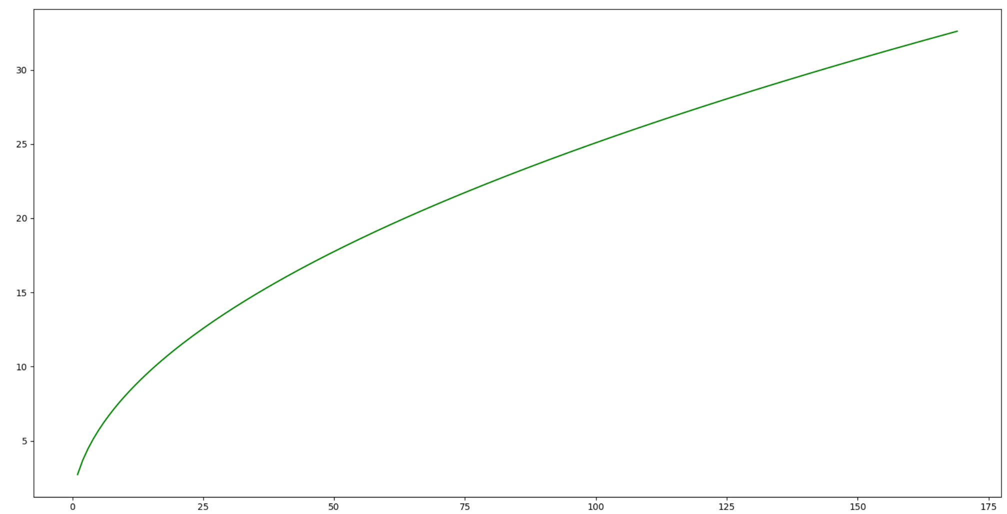

Figure 2: Plot of the ratio with $f(N) = 0$

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