MA 562 - Introduction to Differential Geometry and Topology

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Contents

Homework 1

Recommended practice/review problems (ungraded, please do not submit): Exercises A.1, A.2, A.3, A.10, A.11, A.15, A.21, A.46, 1.18, 1.20.

Problem 1.6 - 1.7

- Show that $\mathbb{RP}^n$ is Hausdorff and second-countable, and is therefore a topological n-manifold.
- Show that $\mathbb{RP}^n$ is compact. [Hint: show that the restriction of π to $\mathbb{S}^n$ is surjective.]

Solution. • **Hausdorff** Let $[x]$ and $[y]$ be two distinct points of $\mathbb{RP}^n$, *i.e.* two distinct 1-dimensional subspaces of $\mathbb{R}^{n+1}$, that are spanned by the unit vectors x and y, respectively. Since $\mathbb{S}^n$ is Hausdorff, it is not hard to see that we can find pairwise disjoint open sets $U, \bar{U}, V, \bar{V} \subsetneq \mathbb{S}^n$, such that $x \in U$, $-x \in \bar{U}$, $y \in V$, and $-y \in \bar{V}$, where $\bar{A} = -a$ such that $a \in A$. Let $\hat{U} = U \cup \bar{U}$ and $\hat{V} = V \cup \bar{V}$. We define

$$\begin{aligned} \varphi : \mathbb{R}^n - \{0\} &\to \mathbb{S}^n \\ p &\mapsto \frac{p}{||p||}. \end{aligned}$$

We claim that $U = \pi\left(\varphi^{-1}\left(\hat{U}\right)\right)$ and $V = \pi\left(\varphi^{-1}\left(\hat{V}\right)\right)$ are open, disjoint, and contain $[x]$ and $[y]$, respectively. It is clear that $[x] \in U$ and $[y] \in V$. Let $[p] \in U \cap V$, then $[p] = \pi(u) = \pi(v)$ for some $u \in \varphi^{-1}\left(\hat{U}\right)$ and $v \in \varphi^{-1}\left(\hat{V}\right)$. Then $u = \lambda v$ for some $\lambda \in \mathbb{R} - \{0\}$. Hence, $\varphi(u) = \pm\varphi(v)$. However, this implies that $\varphi(u) \in \hat{U} \cap \hat{V}$, which is a contradiction. We conclude that $\mathbb{RP}^n$ is Hausdorff.

Second-countable Let $\mathcal{B}$ be a countable basis for $\mathbb{R}^n$. We claim that $\pi(\mathcal{B}) = \{\pi(B) \mid B \in \mathcal{B}\}$ is a basis for $\mathbb{RP}^n$. Define

$$\begin{aligned} f_t : \mathbb{R}^n - \{0\} &\to \mathbb{R}^n - \{0\} \\ p &\mapsto tp, \end{aligned}$$

for every $t \in \mathbb{R}^n - \{0\}$. Note that f_t is continuous and has a continuous inverse $f_{\frac{1}{t}}$. Hence, if U is open, then $f_t(U)$ is also open. We claim that $\pi^{-1}(\pi(U)) = \bigcup_{t\in\mathbb{R}-\{0\}} f_t(U)$ for every open set $U \in \mathbb{R}^n - \{0\}$. Let $p \in \pi^{-1}(\pi(U))$. Then $\pi(p) \in \pi(U)$, implying that there is a $u \in U$ such that u spans the same vector space as p. Hence, $p = \lambda u$, for some non-zero λ, and therefore $p \in \bigcup_{t\in\mathbb{R}-\{0\}} f_t(U)$.

Conversely, suppose that $p \in \bigcup_{t\in\mathbb{R}-\{0\}} f_t(U)$. Then $p = f_\lambda(u) = \lambda u$, for some $u \in U$ and some non-zero λ. Hence, $\pi(p) = \pi(\lambda u) = \pi(u)$, so $p \in \pi^{-1}(\pi(U))$, proving the claim.

Showing that $\mathbb{RP}^n$ is second-countable is equivalent to proving that $\pi(\mathcal{B})$ is a basis. Let $[p] \in \pi(B_1) \cap \pi(B_2)$ for two basis sets $B_1, B_2 \in \mathcal{B}$. Then $p \in \pi^{-1}(\pi(B_1)) \cap \pi^{-1}(\pi(B_2))$, which is open by our previous work. Since this set is non-empty, there is a basis set B_3 contained in $\pi^{-1}(\pi(B_1)) \cap \pi^{-1}(\pi(B_2))$. Then $\pi(B_3) \subseteq \pi(B_1) \cap \pi(B_2)$, showing that $\pi(\mathcal{B})$ is a basis.

Since the condition of locally Euclidean was shown in the textbook, all three conditions are met, and therefore $\mathbb{RP}^n$ is a topological n-manifold.

- **Compactness** We have to notice that any element $[x] \in \mathbb{RP}^n$, with an arbitrary representation $x \in \mathbb{R}^{n+1}$, has another normalized representation, given by

$$\tilde{x} \equiv \frac{x}{||x||},$$

since

$$[x] = \pi(x) = \pi\left(\frac{x}{||x||}\right) = [\tilde{x}],$$

which lies on the unit sphere $\mathbb{S}^n$.

Now, consider the restriction $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ of $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$ to the unit sphere $\mathbb{S}^n$. By the previous argument, $\pi|_{\mathbb{S}^n}$ must be surjective, *i.e.* $\pi(\mathbb{S}^n) = \mathbb{RP}^n$.

Since $\mathbb{S}^n$ is compact, and the image of a compact set under a continuous function is compact, then $\mathbb{RP}^n$ is compact.

■

Problem 1-1
Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second-countable, but not Hausdorff. (This space is called the ***line with two origins***.)

Solution. Let

$$\begin{aligned} \pi : X &\to M \\ x &\mapsto [x] \end{aligned}$$

be the natural projection, sending each point to its equivalence class. This map defines the quotient topology of M. Let p_1 and p_2 denote the upper origin $[(0, 1)]$ and the lower origin $[(0, -1)]$ of M, respectively.
$\boxed{\pi \textbf{ is an open map}}$ A result from general topology states

Lemma . A quotient map π is an open map if and only if

$$U \subset X \text{ is open } \implies \pi^{-1}(\pi(U)) \subset X \text{ is open.}$$

This property holds in the case of the line with two origins: fix an open subset $U \subset X$. Then it is easy to see that $\pi^{-1}(\pi(U)) = U \cup U^{\mathrm{R}}$ where U^{R} denotes the reflection of U about the x-axis. Both sets are open in X, making π an open map.
$\boxed{M \textbf{ is second countable}}$ Define an "open interval" in M to be the image of the corresponding open interval in $\mathbb{R}$ under the quotient map π as

$$\begin{aligned} ([a], 0) &\equiv \pi \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| \; a < x < 0 \right\} = \pi \left\{ \begin{pmatrix} x \\ -1 \end{pmatrix} \middle| \; a < x < 0 \right\} \\ (0, [b]) &\equiv \pi \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} \middle| \; 0 < x < b \right\} = \pi \left\{ \begin{pmatrix} x \\ -1 \end{pmatrix} \middle| \; 0 < x < b \right\}. \end{aligned}$$

These "open intervals" are open in M since π is an open map. Imitating the basis $\{(-q, q) \mid q \in \mathbb{Q}_+\}$ of the real line, we can define a countable basis $\mathcal{B}$ of M as

$$\mathcal{B} \equiv \left\{ \begin{array}{c} ([-q], 0) \cup \{p_1\} \cup (0, [q]) \\ ([-q], 0) \cup \{p_2\} \cup (0, [q]) \end{array} \middle| \; q \in \mathbb{Q}_+ \right\}$$

It is easy to verify that $\mathcal{B}$ indeed constitutes a basis for M.
$\boxed{M \textbf{ is locally Euclidean}}$ For each point $[p] \in M$, we should find an open neighbourhood $U \subset M$ containing $[p]$ for which there is a homeomorphism $\varphi : U \to \mathbb{R}$.
The trick is to notice that the upper part of X can generate the quotient line with the upper origin p_1 and the lower part of X can generate the quotient line with the lower origin p_2 (i.e., any point $[p] \in M$ is contained in $\pi(\mathbb{R} \times \{1\}) = M - \{p_2\}$ or in $\pi(\mathbb{R} \times \{-1\}) = M - \{p_1\}$). In other words, two large sets already satisfy the desired open neighbourhood condition: $M - \{p_2\}$ and $M - \{p_1\}$. Both sets are open since π is open.
Such separation results in additional properties. By construction, we can find at least one x such that $p \sim (x, 1)$ for all $p \in M - \{p_2\}$ and $p \sim (x, -1)$ for all $p \in M - \{p_1\}$. Conversely, if there is one more x' with $p \sim (x', 1)$ or $p \sim (x', -1)$, respectively, we must necessarily have $x = x'$. This allows us to write our

equivalence classes $[p]$ in either $M - \{p_2\}$ or $M - \{p_1\}$ as $[x]$ for some scalars $x \in \mathbb{R}$ without any problems (which is the first hint that our manifold is one-dimensional). Using this property, we define

$$\begin{aligned} \varphi_1 : M - \{p_2\} &\to \mathbb{R} \\ [x] &\mapsto x \end{aligned}$$

and

$$\begin{aligned} \varphi_2 : M - \{p_1\} &\to \mathbb{R} \\ [x] &\mapsto x. \end{aligned}$$

These functions are

- **(Bijective)** φ_1 is obviously surjective. To show injectivity, suppose that $\varphi_1([x]) = \varphi_1([x'])$ for some $[x], [x'] \in M - \{p_2\}$. Then $x = x'$ and so $[(x, 1)] = [(x', 1)]$. A similar argument works for φ_2.
- **(Continuous)** Let $\widetilde{U}$ be open in $\mathbb{R}$. The $\phi_1^{-1}(U)$ maps to $\pi(U \times \{1\})$ and $\phi_2^{-1}(U)$ to $\pi(U \times \{-1\})$, which are both open since π is an open map.

$\boxed{M \textbf{ is not Hausdorff}}$ We argue that the points $[(0, 1)]$ and $[(0, -1)]$ are not separable in M. Suppose, for the sake of contradiction, that we can find two disjoint, open sets U and U' containing $[(0, 1)]$ and $[(0, -1)]$, respectively. Since π is continuous, the sets $\pi^{-1}(U)$ and $\pi^{-1}(U')$ are two open subsets of X containing $(0, 1)$ and $(0, -1)$, respectively. We can then find $\varepsilon > 0$ small enough so that

$$\begin{aligned} (-\varepsilon, \varepsilon) \times \{1\} &\subseteq \pi^{-1}(U) \\ (-\varepsilon, \varepsilon) \times \{-1\} &\subseteq \pi^{-1}(U'). \end{aligned}$$

In other words, since the image of the preimage is a subset of the original set,

$$\begin{aligned} \pi((-\varepsilon, \varepsilon) \times \{1\}) &\subseteq U \\ \pi((-\varepsilon, \varepsilon) \times \{-1\}) &\subseteq U', \end{aligned}$$

which is a contradiction to the fact that $U \cap U' = \emptyset$.

■

Problem 1-7
Let N denote the ***north pole*** $(0, \ldots, 0, 1) \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1}$, and let S denote the ***south pole*** $(0, \ldots, 0, -1)$. Define the ***stereographic projection*** $\sigma : \mathbb{S}^n \backslash \{N\} \to \mathbb{R}^n$ by

$$\sigma\left(x^1, \ldots, x^{n+1}\right) = \frac{\left(x^1, \ldots, x^n\right)}{1 - x^{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \backslash \{S\}$.

(a) For any $x \in \mathbb{S}^n \backslash \{N\}$, show that $\sigma(x) = u$, where $(u, 0)$ is the point where the line through N and x intersects the linear subspace where $x^{n+1} = 0$ (Fig. 1.13). Similarly, show that $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace. (For this reason, $\tilde{\sigma}$ is called ***stereographic projection from the south pole***.)

(b) Show that σ is bijective, and

$$\sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}$$

(c) Compute the transition map $\tilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ or $\tilde{\sigma}$ are called ***stereographic coordinates***.)

(d) Show that this smooth structure is the same as the one defined in Example 1.31.

Solution. (a) Fix an arbitrary $x \in \mathbb{S}^n \backslash \{N\}$. A line passing through N and x is given by

$$\begin{aligned} L_{(N,x)} : \mathbb{R} &\to \mathbb{R}^{n+1} \\ t &\mapsto N + t(N - x) \end{aligned}$$

or in vector notation,

$$L_{(N,x)}(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} + t\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} x^1 \\ \vdots \\ x^n \\ x^{n+1} \end{bmatrix}\right) = \begin{bmatrix} -tx^1 \\ \vdots \\ -tx^n \\ 1 + t\left(1 - x^{n+1}\right) \end{bmatrix}$$

Figure 1: Stereographic projection from the north pole in case of $\mathbb{S}^1$.

Where does L exactly cross the x^{n+1}-axis? This happens at the value t_0 for which the $(n+1)$th component of $L(t_0)$ is zero. To derive it, we set

$$1 + t_0\left(1 - x^{n+1}\right) \overset{!}{=} 0 \implies t_0 = -\frac{1}{1 - x^{n+1}},$$

which results in

$$L(t_0) = \left[\frac{x^1}{1 - x^{n+1}}, \ldots, \frac{x^n}{1 - x^{n+1}}\right] = [\sigma(x), 0].$$

Therefore, the intersection point is $(\sigma(x), 0)$.

(b) We compute the formula for $\sigma \circ \sigma^{-1}$ and $\sigma^{-1} \circ \sigma$. First, we derive a formula for σ^2 by

$$
\begin{aligned}
|\sigma(x)|^2 &= \frac{\left(x^1\right)^2 + \ldots + \left(x^n\right)^2}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{\left(x^1\right)^2 + \ldots + \left(x^n\right)^2 + \left(x^{n+1}\right)^2 - \left(x^{n+1}\right)^2}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{|x| - \left(x^{n+1}\right)^2}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{1 - \left(x^{n+1}\right)^2}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{\left(1 - x^{n+1}\right)\left(1 + x^{n+1}\right)}{\left(1 - x^{n+1}\right)^2} \\
&= \frac{1 + x^{n+1}}{1 - x^{n+1}}.
\end{aligned}
$$

Using the above, we compute $\sigma^{-1} \circ \sigma(x)$

$$
\begin{aligned}
\sigma^{-1} \circ \sigma(x) &= \sigma^{-1}\left(\frac{\left(x^1, \ldots x^n\right)}{1 - x^{n+1}}\right) \\
&= \frac{\left(2x^1, \ldots, 2x^n, \left(|\sigma(x)|^2 - 1\right)\left(1 - x^{n+1}\right)\right)}{\left(|\sigma(x)|^2 + 1\right)\left(1 - x^{n+1}\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n, \left(\frac{1+x^{n+1}}{1-x^{n+1}} - 1\right)\left(1 - x^{n+1}\right)\right)}{\left(\frac{1+x^{n+1}}{1-x^{n+1}} + 1\right)\left(1 - x^{n+1}\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n, \left(\frac{2x^{n+1}}{1-x^{n+1}}\right)\left(1 - x^{n+1}\right)\right)}{\left(\frac{2}{1-x^{n+1}}\right)\left(1 - x^{n+1}\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n, 2x^{n+1}\right)}{2} \\
&= \left(x^1, \ldots, x^{n+1}\right) \\
&= x.
\end{aligned}
$$

Similarly, $\sigma \circ \sigma^{-1}$ is given by

$$
\begin{aligned}
\sigma \circ \sigma^{-1}(x) &= \sigma\left(\frac{\left(2x^1, \ldots, 2x^n, |x|^2 - 1\right)}{|x|^2 + 1}\right) \\
&= \frac{\left(2x^1, \ldots, 2x^n\right)}{\left(|x|^2 + 1\right)\left(1 - \frac{|x|^2 - 1}{|x|^2 + 1}\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n\right)}{|x|^2 + 1 - \left(|x|^2 - 1\right)} \\
&= \frac{\left(2x^1, \ldots, 2x^n\right)}{2} \\
&= \left(x^1, \ldots, x^n\right) \\
&= x.
\end{aligned}
$$

Thus, σ is a bijection.

(c) The transition map is given by

$$\begin{aligned}\tilde{\sigma} \circ \sigma^{-1} : \sigma(\mathbb{S}^n \backslash \{N, S\} &\to \tilde{\sigma}(\mathbb{S}^n \backslash \{N, S\}) \\ (u^1, \ldots, u^n) &\mapsto \frac{(u^1, \ldots, u^n)}{|u|}.\end{aligned}$$

To see why, we compute

$$\begin{aligned}\tilde{\sigma} \circ \sigma^{-1}(u^1, \ldots, u^n) &= \tilde{\sigma}\left(\frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}\right) \\ &= -\sigma\left((-1)\frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}\right) \\ &= -\sigma\left(\frac{\left(2u^1, \ldots, 2u^n, |u|^2 - 1\right)}{-|u|^2 - 1}\right) \\ &= -\frac{\left(2u^1, \ldots, 2u^n\right)}{(-|u|^2 - 1)\left(1 - \frac{|u|^2-1}{-|u|^2-1}\right)} \\ &= -\frac{\left(2u^1, \ldots, 2u^n\right)}{(-|u|^2 - 1)\left(1 + \frac{|u|^2-1}{|u|^2+1}\right)} \\ &= -\frac{\left(2u^1, \ldots, 2u^n\right)}{(-|u|^2 - 1)\left(\frac{2|u|^2}{|u|^2+1}\right)} \\ &= -\frac{\left(2u^1, \ldots, 2u^n\right)}{-2|u|^2} \\ &= \frac{\left(u^1, \ldots, u^n\right)}{|u|^2} \\ &= \frac{u}{|u|^2}.\end{aligned}$$

We can skip a similar computation for its inverse by noticing that it is its own inverse

$$\begin{aligned}(\tilde{\sigma} \circ \sigma^{-1}) \circ (\tilde{\sigma} \circ \sigma^{-1}) &= \tilde{\sigma} \circ \sigma^{-1}\left(\frac{u}{|u|^2}\right) \\ &= \frac{\frac{u}{|u|^2}}{\left|\frac{u}{|u|^2}\right|^2} \\ &= \frac{\frac{u}{|u|^2}}{\frac{u^2}{|u|^4}} \\ &= \frac{\frac{u}{|u|^2}}{\frac{1}{|u|^2}} \\ &= u.\end{aligned}$$

In other words, the transition map is a diffeomorphism. Same goes for $\sigma \circ \tilde{\sigma}^{-1}$. Since $\mathbb{S}^n \backslash \{N\}$ and $\mathbb{S}^n \backslash \{S\}$ form an open cover of $\mathbb{S}^n$, we conclude that $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ define a smooth structure on $\mathbb{S}^n$.

(d) The charts from Example 1.31 can be grouped into three categories:

- $\left(U^+_{n+1}, \phi^+_{n+1}\right)$ which contain N.

- $\left(U_{n+1}^{-}, \phi_{n+1}^{-}\right)$ which contain S.
- $\left(U_i^{\pm}, \phi_i^{\pm}\right)$ for $i = 1, \ldots, n$ which do not contain N and S.

We argue that each type of chart is smoothly compatible with σ; the case for $\tilde{\sigma}$ is similar. In the first and in the second case, we have

$$\phi_{n+1}^{\pm} \circ \sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, 2u^n\right)}{|u|^2 + 1}$$

Both of these functions are smooth. The inverses are given by

$$\sigma \circ \left(\phi_{n+1}^{\pm}\right)^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(u^1, \ldots, u^n\right)}{1 \mp \sqrt{1 - |u|^2}}$$

Thus, $\sigma \circ \left(\phi_{n+1}^{-}\right)^{-1}$ is smooth. Since the domain $\left(\phi_{n+1}^{+}\right)\left(U_{n+1}^{+} \backslash \{N\}\right)$ does not include $\mathbf{0}$, $\sigma \circ \left(\phi_{n+1}^{+}\right)^{-1}$ is smooth.

For the third case, we have

$$\phi_i^{\pm} \circ \sigma^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(2u^1, \ldots, \widehat{2u^i}, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}$$

which is smooth, and

$$\sigma \circ \left(\phi_i^{\pm}\right)^{-1}\left(u^1, \ldots, u^n\right) = \frac{\left(u^1, \ldots, u^{i-1}, \sqrt{1 - |u|^2}, u^i, \ldots, u^{n-1}\right)}{1 - u^n}$$

which is also smooth, since $u^n \neq 1$ in $U_i^{\pm}$. In other words, the smooth atlas from Example 1.31 is smoothly compatible with the smooth atlas of stereographic projection.

■

Problem 1-12
Prove Proposition 1.45 (a product of smooth manifolds together with one smooth manifold with boundary is a smooth manifold with boundary).

Solution. We will show that if M is a smooth m-manifold and N is a smooth n-manifold with boundary, then $M \times N$ is a smooth manifold with boundary, and $\partial(M \times N) = M \times \partial N$.
Using the simpler result that finite products of smooth (boundary-less) manifolds are smooth manifolds, the problem can be reduced to the case where $k = 1$. The general case follows by induction.
First, $M \times N$ is Hausdorff and second-countable, since both M and N are. Given charts (U, ϕ) and (V, ψ) for M and N, respectively, we let $(U \times V, \phi \times \psi)$ be a chart for $M \times N$. The collection of all such charts then gives $M \times N$ the structure of a smooth manifold with boundary, as we now show.
Let $(U_1 \times V_1, \phi_1 \times \psi_1)$ and $(U_2 \times V_2, \phi_2 \times \psi_2)$ be two intersecting charts. Note that $(\phi_2 \times \psi_2) \circ (\phi_1 \times \psi_1)^{-1} = \left(\phi_2 \circ \phi_1^{-1}\right) \times \left(\psi_2 \times \psi_1^{-1}\right)$ has domain $(\phi_1 \circ \psi_1)([U_1 \times V_1] \cap [U_2 \times V_2]) = \phi_1(U_1 \cap U_2) \times \psi_1(V_1 \cap V_2)$. Since $\psi_1(V_1 \cap V_2)$ can be extended to an open set such that $\psi_2 \circ \psi_1^{-1}$ is smooth, we have that the product charts are smoothly compatible. Thus, $M \times N$ is a smooth manifold with boundary.
Let $(x, y) \in \partial(M \times N)$. Then (x, y) is in the domain of some boundary chart $(U \times V, \phi \times \psi)$. Since $\phi(U)$ is open in $\mathbb{R}^m$, it follows that (V, ψ) must be a boundary chart. Moreover, $\psi(y)$ should lie on the boundary of $\mathbb{H}^n$, since otherwise there is an interior chart (V', ψ') whose domain contains y', and thus $(U \times V', \phi \times \psi')$ is an interior chart containing (x, y'). Thus, $y \in \partial N$, so $(x, y) \in M \times \partial N$. Conversely, let $(x, y) \in M \times \partial N$. Then there is a boundary chart (V, ψ) such that $y \in V$ and $\psi(y)$ is in $\partial\mathbb{H}^n$. Then, if u is in the domain of a chart (U, ϕ), then $(\phi \times \psi)(x, y) = (\phi(x), \psi(y)) \in \partial\mathbb{H}^{m+n}$, so $(x, y) \in \partial(M \times N)$. Hence, $\partial(M \times N) = M \times \partial N$. ■

Homework 2

Recommended practice/review problems (ungraded, please do not submit): Problem 1-6, Exercises 2.2, 2.3, 2.9, 2.11, 2.16, 2.27

Problem 2-1
Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi\left(U \cap f^{-1}(V)\right)$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

Solution. Since f is not continuous, it is not smooth in the sense that we have defined in this chapter. Moreover, f is smooth far from $x = 0$ since it is constant there.
Let $\epsilon > 0$, $U = (-\epsilon, \epsilon)$, $\varphi = \mathrm{id}$, $V = \left(\frac{1}{2}, \frac{3}{2}\right)$, and $\psi = \mathrm{id}$. Then U contains $x = 0$ and V contains $f(x) = 1$.
Let (U, id) and (V, id) be coordinate charts for $\mathbb{R}$, then $\mathrm{id} \circ f \circ \mathrm{id}^{-1} = f$ is the constant map on $f\left(U \cap f^{-1}(V)\right) = [0, \epsilon)$, and is therefore smooth. ■

Problem 2-3
For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the **nth power map for** $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.

(b) $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is the **antipodal map** $\alpha(x) = -x$.

(c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, where we think of $\mathbb{S}^3$ as the subset $\left\{(w, z) : |w|^2 + |z|^2 = 1\right\}$ of $\mathbb{C}^2$.

Solution. (a) Let $z \in \mathbb{S}^1$ and let (U, θ) be an angle coordinate chart containing z, and let (V, ϕ) be an angle coordinate chart containing z^n. Then $\phi \circ p_n \circ \theta^{-1}(x) = \phi \circ p_n(\mathrm{e}^{ix}) = \phi(\mathrm{e}^{inx}) = nx + 2k\pi$, for some k, which is constant on each component of $\theta(U \cap p_n^{-1}(V))$. Note that $U \cap p_n^{-1}(V)$ is open, since p_n is continuous. Thus, p_n is smooth.

(b) Let $p \in \mathbb{S}^n$ and assume that $(\mathbb{S}^n \setminus \{N\}, \sigma)$ is the stereographic chart from the north and it contains p. Then $(\mathbb{S}^n \setminus S, \tilde{\sigma})$ contains $\alpha(p)$, where $\tilde{\sigma}$ is the stereographic projection from the south. A computation shows $\tilde{\alpha} \circ \alpha \circ \sigma^{-1}(u) = -u$, which is smooth. Thus, α is smooth.

(c) The given function F is defined over two complex variables. We can rewrite F in terms of real coordinates, which gives us

$$F(x^1, x^2, x^3, x^4) = \left(2x^1x^3 + 2x^2x^4, 2x^1x^4 - 2x^2x^3, (x^3)^2 + (x^4)^2 - (x^1)^2 - (x^2)^2\right),$$

so F is continuous since it is the restriction of a continuous function. Additionally, we have

$$\sigma_{\mathbb{S}^2} \circ F \circ \sigma_{\mathbb{S}^3}^{-1}(u^1, u^2, u^3) = \frac{\left(8u^1u^3 + 4u^2(|u|^2 - 1), 4u^1(|u|^2 - 1) - 8u^2u^3\right)}{1 + (2u^1)^2 + (2u^2)^2 - (2u^3)^2 - (|u|^2 - 1)^2},$$

which is smooth on $\sigma_{\mathbb{S}^3}\left(\mathbb{S}^3 \setminus \{N\} \cap F^{-1}\left(\mathbb{S}^2 \setminus \{N\}\right)\right)$. Here $\sigma_{\mathbb{S}^n}$ is the stereographic projection from the north of $\mathbb{S}^n$. Similar computations for different pairs of charts show that F is indeed a smooth function.

■

Problem 2-5
Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\widetilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.

(b) Show that f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

Solution. In Example 1.23, $\psi(x) = x^3$, *i.e.* $\psi^{-1}(x) = x^{\frac{1}{3}}$.

(a) The coordinate representation $\psi \circ f \circ \mathrm{id}^{-1} = \psi \circ f$ is smooth since both $\psi(x) = x^3$ and f are smooth in the usual sense.

(b) The coordinate representation is $\mathrm{id} \circ f \circ \psi^{-1} = f \circ \psi^{-1}$ and $f \circ \psi^{-1}(x) = f(x^{\frac{1}{3}})$.

$\boxed{\Longrightarrow}$ Assume that f is a smooth map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$. Notice that $\psi^{(j)}(0) = 0$ for all $j \neq 3$. Then, if we want $f^{(n)}(0) = 0$, then there must be an n-tuple $(m_1, \ldots, m_n)$ such that $m_j = 0$ for all $j \neq 3$. Thus, $n = 3m_3$.

$\boxed{\Longleftarrow}$ Let $F = f \circ \psi^{-1}$. We aim to prove that F is smooth, but we first have to prove a little proposition.

Proposition I. f $F \in C^k(\mathbb{R})$, then so is $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}})$.

Proof. We will prove this by induction on k.

- For $k = 0$, we have $x^{\frac{1}{3}} F(x^{\frac{1}{3}})$, and the result is clear since $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}})$ is continuous.
- Inductive step: Let $F \in C^k(\mathbb{R})$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{k+\frac{1}{3}} F(x^{\frac{1}{3}})\right] = x^{(k-1)+\frac{1}{3}} F(x^{\frac{1}{3}}) + \frac{1}{3} x^{(k-1)+\frac{1}{3}} x^{\frac{1}{3}} F'(x^{\frac{1}{3}}).$$

  By induction, we have
$$x^{(k-1)+\frac{1}{3}} F(x^{\frac{1}{3}}) \in C^{k-1}(\mathbb{R}).$$
  Since $xF'(x) \in C^{k-1}(\mathbb{R})$, then $\frac{1}{3} x^{(k-1)+\frac{1}{3}} F'(x^{\frac{1}{3}}) \in C^{k-1}(\mathbb{R})$, by induction. Since its derivative is $C^{k-1}(\mathbb{R})$, then it must be that $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}}) \in C^k(\mathbb{R})$.

∎

Now, suppose that $f^{(n)}(0) = 0$ for all n not integral multiples of 3. We can use the Taylor remainder theorem to write f up to the $3m$th term as

$$f(x) = f(0) + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(6)}(0)}{6!} x^6 + \ldots + + \frac{f^{(3m)}(0)}{(3m)!} x^{3m} + x^{3m+1} F_{3m+1}(x),$$

for some smooth function F_{3m+1}, then

$$f(x^{\frac{1}{3}}) = f(0) + \frac{f^{(3)}(0)}{3!} x + \frac{f^{(6)}(0)}{6!} x^2 + \ldots + + \frac{f^{(3m)}(0)}{(3m)!} x^m + x^{m+\frac{1}{3}} F_{3m+1}(x^{\frac{1}{3}}).$$

Since $x^{m+\frac{1}{3}} F_{3m+1}(x^{\frac{1}{3}}) \in C^m(\mathbb{R})$, then $F \in C^m(\mathbb{R})$ for all $m \geq 0$. Thus, $F = f \circ \psi^{-1}$ is smooth.

Therefore, f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

∎

Problem 2-6
Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. (Such a function is said to be **homogeneous of degree d**.) Show that the map $\widetilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\widetilde{P}([x]) = [P(x)]$ is well defined and smooth.

Solution. If $[x] = [y]$, then $x = \lambda y$, for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then we have

$$\tilde{P}([x]) = [P(x)] = [P(\lambda y)] = [\lambda^d P(y)] = [P(y)] = \tilde{P}([y]).$$

Thus, $\tilde{P}$ is well-defined.
Let $[x] \in U_i$ and suppose that $\tilde{P}([x]) \in U_j$. The coordinate representation $\phi_j \circ \tilde{P} \circ \phi_i^{-1}$ takes a point $(u^1, \dots, u^k) \in \phi_i(U_i)$ to

$$\frac{(P_1(\alpha), \dots, P_{j-1}(\alpha), P_{j+1}(\alpha), \dots, P_{n+1}(\alpha))}{P_j(\alpha)},$$

where $\alpha = \phi_i(u^1, \dots, u^k) = \left(u^1, \dots, u^{i-1}, 1, u^{i+1}, \dots, u^k\right)$ and P_m is the mth component of P. Since P is smooth, then each P_m is also smooth.
Thus, the coordinate representation is smooth, and therefore, so is $\tilde{P}$. ∎

Problem 2-14
Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \le f(x) \le 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Solution. Theorem 2.29 states

Theorem 2.29. Let M be a smooth manifold. If K is any closed subset of M, there is a smooth non-negative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

By Theorem 2.29, there are functions $g, h : M \to [0, \infty)$ such that $g^{-1}(0) = A$ and $h^{-1}(0) = B$. In other words, $g(A) = 0$ and $h(B) = 0$. Take $f(x) = \frac{g(x)}{g(x)+h(x)}$, and indeed $0 \le f(x) \le 1$, $f(A) = 0$, and $f(B) = 1$. ∎

Homework 3

Recommended practice/review problems (ungraded, please do not submit): Exercises 3.5, 3.7.

Problem 3-1
Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. Show that $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M.

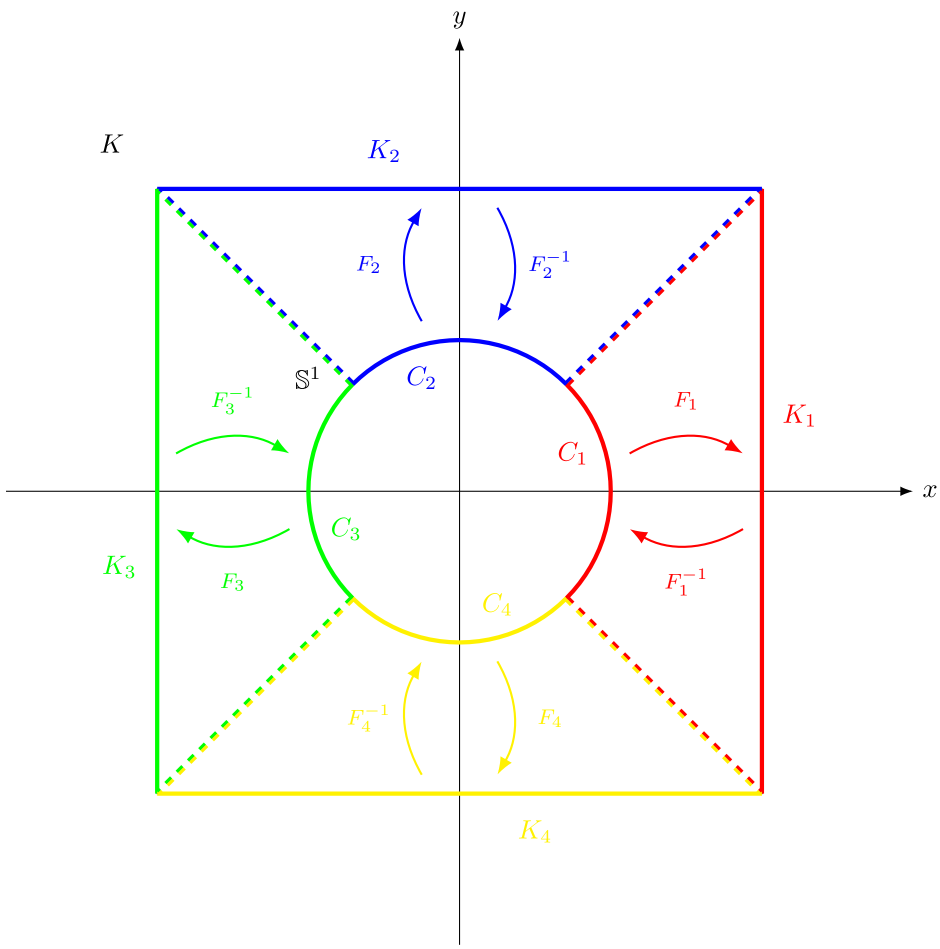
Solution. $\boxed{\implies}$ Suppose $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. We need to show that the anti-derivative is constant on each component of M. The coordinate representation of a function can be taken as the total derivative, which means all the partial derivatives are zero. If, for every $U \in M$ containing p, we have that the total derivative is zero, then its anti-derivative is going to be a constant, and that is for every point p and the small coordinate chart around p in the manifold. Since its constant around every coordinate chart around M then it will be constant around every component in M since connected components are path-connected components. Thus, F is constant on each component of M.
$\boxed{\impliedby}$ Suppose that F is constant on each component of M. The derivative of F at p, by definition, only depends on F in a neighborhood of p, which means that $\mathrm{d}F_p$ will be the derivative of a constant for each component of M, which is zero. Thus, $\mathrm{d}F_p$ is the zero map for each $p \in M$.
Therefore, $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$ if and only if F is constant on each component of M. ■

Problem 3-4
Show that $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$.

Solution. Take the tangent bundle $T\mathbb{S}^1$, which is the disjoint union of all the tangent spaces to the unit circle $\mathbb{S}^1$. Take the rotation of each tangent space from $T\mathbb{S}^1$ to, say, $(T\mathbb{S}^1)'$, by $\frac{\pi}{2}$. The rotation applied is clearly a diffeomorphism, since the rotation of every coordinate chart, for every point p, is equivalent to just changing the representation of the variables. By definition of a tangent bundle, the tangent space to every point p on the manifold $\mathbb{S}^1$ is isomorphic to $\mathbb{R}$ since it is one-dimensional. After rotating and equating every tangent space to $\mathbb{R}$, what we get is $\mathbb{S}^1 \times \mathbb{R}$. Geometrically, this is a cylinder of infinite height.
Therefore, $T\mathbb{S}^1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$. ■

Problem 3-5
Let $\mathbb{S}^1 \subseteq \mathbb{R}^2$ be the unit circle, and let $K \subseteq \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origin: $K = \{(x, y) : \max(|x|, |y|) = 1\}$. Show that there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F\left(\mathbb{S}^1\right) = K$, but there is no diffeomorphism with the same property. [Hint: let γ be a smooth curve whose image lies in $\mathbb{S}^1$, and consider the action of $dF\left(\gamma'(t)\right)$ on the coordinate functions x and y.] (Used on p. 123.)

Solution. Consider the following figure



We define four curves $C_i \in \mathbb{S}^1 \subseteq \mathbb{R}^2$, for $i = 1, 2, 3, 4$, as follows

$$C_1 = \left\{(x, y) \mid |y| \leq \frac{\sqrt{2}}{2} \text{ and } x = +\sqrt{1 - y^2}\right\},$$

$$C_2 = \left\{(x, y) \mid |x| \leq \frac{\sqrt{2}}{2} \text{ and } y = +\sqrt{1 - x^2}\right\},$$

$$C_3 = \left\{(x, y) \mid |y| \leq \frac{\sqrt{2}}{2} \text{ and } x = -\sqrt{1 - y^2}\right\},$$

$$C_4 = \left\{(x, y) \mid |x| \leq \frac{\sqrt{2}}{2} \text{ and } y = -\sqrt{1 - x^2}\right\},$$

and the four sides of the square with side length 2 centered at the origin $K_i \in K \subseteq \mathbb{R}^2$, for $i = 1, 2, 3, 4$, as

$$\begin{aligned}
K_1 &= \{(1, y) \mid -1 \leq y \leq 1\}, \\
K_2 &= \{(x, 1) \mid -1 \leq x \leq 1\}, \\
K_3 &= \{(-1, y) \mid -1 \leq y \leq 1\}, \\
K_4 &= \{(x, -1) \mid -1 \leq x \leq 1\}.
\end{aligned}$$

We aim to find four functions that map each one of those arcs on $\mathbb{S}^1$ to their respective sides on the boundary of a square of side length 2 centered at the origin. We define our function F as

$$\begin{aligned}
F &: \mathbb{R}^2 \to \mathbb{R}^2 \\
(x, y) \in C_i &\mapsto F_i(x, y) \in K_i,
\end{aligned}$$

where

$$\begin{aligned}
F_1(x, y) &= \frac{\sqrt{x^2 + y^2}}{|x|}(x, y), \\
F_2(x, y) &= \frac{\sqrt{x^2 + y^2}}{|y|}(x, y), \\
F_3(x, y) &= -\frac{\sqrt{x^2 + y^2}}{|x|}(x, y), \\
F_4(x, y) &= -\frac{\sqrt{x^2 + y^2}}{|y|}(x, y).
\end{aligned}$$

Notice that the four curves C_i trace out the entirety of $\mathbb{S}^1$, *i.e.*

$$\bigcup_{i=1}^{4} C_i = \mathbb{S}^1,$$

and the four sides K_i trace out our square

$$\bigcup_{i=1}^{4} K_i = K.$$

By construction, we have that $F(\mathbb{S}^1) = K$. Since the four functions agree on their overlaps, then by the pasting lemma, we see that F is continuous on all the C_i's. We now have to show that the inverse is also continuous. We define the inverse of the F_i's as

$$\begin{aligned}
F_1^{-1}(x, y) &= \frac{|x|}{\sqrt{x^2 + y^2}}(x, y), \\
F_2^{-1}(x, y) &= \frac{|y|}{\sqrt{x^2 + y^2}}(x, y), \\
F_3^{-1}(x, y) &= -\frac{|x|}{\sqrt{x^2 + y^2}}(x, y), \\
F_4^{-1}(x, y) &= -\frac{|y|}{\sqrt{x^2 + y^2}}(x, y).
\end{aligned}$$

Likewise, we have that the functions F_i^{-1} are also continuous. Thus, there is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$.
We will show that there is no diffeomorphism with the property $F(\mathbb{S}^1) = K$ by contradiction.
Suppose F is a diffeomorphism. Let γ be a smooth curve whose image lies in $\mathbb{S}^1$, *i.e.*

$$\gamma : \mathbb{R} \to \mathbb{S}^1.$$

Now, $\frac{\mathrm{d}\gamma}{\mathrm{d}t} \equiv \gamma'(t)$ will be a vector on $\mathbb{S}^1$. Let $\gamma(t_0)$ be the point at which $F(\gamma(t_0)) = (1,1)$. Consider the action of $\mathrm{d}F(\gamma'(t))$ to the coordinate function x. For $t < t_0$, $\mathrm{d}F(\gamma'(t))x = 0$. For $t > t_0$, $\mathrm{d}F(\gamma'(t))x = \text{constant} \neq 0$. Thus, there was a discontinuous jump at $t = t_0$. But we know that F, γ, and the coordinate function x are smooth, and the action of a the derivative on a smooth function is smooth, then we have reached a contradiction.
Therefore, there is no diffeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(\mathbb{S}^1) = K$. ∎

Problem 3-8
Let M be a smooth manifold with or without boundary and $p \in M$. Let $\mathcal{V}_pM$ denote the set of equivalence classes of smooth curves starting at p under the relation $\gamma_1 \sim \gamma_2$ if $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$ for every smooth real-valued function f defined in a neighborhood of p. Show that the map $\Psi : \mathcal{V}_pM \to T_pM$ defined by $\Psi[\gamma] = \gamma'(0)$ is well-defined and bijective. (*Used on p. 72.*)

Solution. Let the direction of the curve γ be in the counter-clockwise direction along the unit circle and let

$$\begin{aligned} \Psi : \mathcal{V}_pM &\to T_pM \\ [\gamma] &\mapsto \gamma'(0). \end{aligned}$$

Note that, by the derivative of the composition, we have $(f \circ \gamma)'(0) = \mathrm{d}f(\gamma(0))\gamma'(0) = \gamma'(0)f$.

- **Well-defined:**
  Let $[\gamma_1] = [\gamma_2]$, then

  $$\begin{aligned} (f \circ \gamma_1)'(0) &= (f \circ \gamma_2)'(0) \\ \gamma_1'(0)f &= \gamma_2'(0)f \\ \gamma_1'(0) &= \gamma_2'(0), \end{aligned}$$

  since the action of a differential operator on f is equal to the action of another differential operator on f, for all f, then the differential operators must be the same.

  Thus, $\gamma_1'(0) = \gamma_2'(0)$ which implies that $\Psi[\gamma_1] = \Psi[\gamma_2]$.

  Therefore, Ψ is well-defined.

- **Bijective:**
  - **Injective:** Let $\Psi[\gamma_1] = \Psi[\gamma_2]$. Then, $\gamma_1'(0) = \gamma_2'(0)$, and we can apply both to the same function f, and, by definition, we will get the same value. We have

    $$\begin{aligned} \gamma_1'(0)f &= \gamma_2'(0)f \\ (f \circ \gamma_1)'(0) &= (f \circ \gamma_2)'(0), \end{aligned}$$

    and by the definition of the equivalence relation on γ, we have that $\gamma_1 \sim \gamma_2$, *i.e.* $[\gamma_1] = [\gamma_2]$.

    Thus, Ψ is injective.
  - **Surjective:** We need to show that, for all $v \in T_pM$, there exists $\gamma_v \in \mathcal{V}_pM$ such that $\Psi[\gamma_v] = v$. Let $\gamma_v \in \mathcal{V}_pM$ such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. The curve γ_v has an equivalence class, and the image of the equivalence under the map Ψ is equal to $\gamma_v'(0)$, which is v.

    Thus, Ψ is surjective.

  Therefore, Ψ is bijective.

∎

Homework 4

Problem 4-4
Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the curve of Example 4.20. Show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$. (*Used on pp. 502, 542.*)

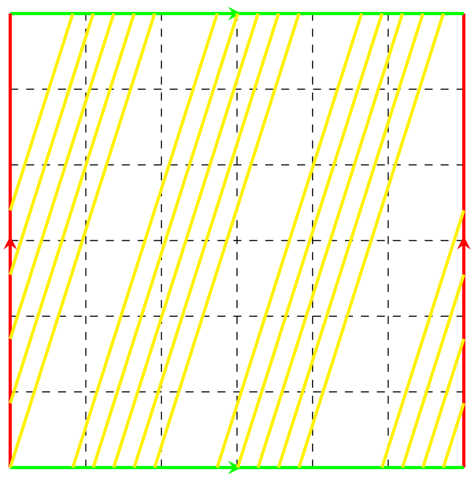
Solution. Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$ denote the torus, and let α be any irrational number. The curve γ defined in Example 4.20 is

$$\begin{aligned} \gamma : \mathbb{R} &\to \mathbb{T}^2 \\ t &\mapsto \left(\mathrm{e}^{2\pi \mathrm{i} t}, \mathrm{e}^{2\pi \mathrm{i} \alpha t}\right). \end{aligned}$$

To show that the image set $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$, we have two ways:

- Show that $\gamma(\mathbb{R})$ intersects every open set in $\mathbb{T}^2$, or
- Show that the closure of $\gamma(\mathbb{R})$ is equal to $\mathbb{T}^2$ itself, *i.e.* every point of $\mathbb{T}^2$ is a limit point of $\gamma(\mathbb{R})$.

We will proceed with the first option.



As a hand-wavy argument, it feels intuitive that, given any open set in $\mathbb{T}^2$, no matter how small, there will always be a line from $\gamma(\mathbb{R})$ that intersects it, since the irrationality of α allows $\gamma(\mathbb{R})$ to fully cover $\mathbb{T}^2$. If that was not the case, then that means there would be lines that coincide and areas left unoccupied, which contradicts the irrationality of α. What is still left unanswered is whether or not $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.
We use Dirichlet's Approximation Theorem (Lemma 4.21) to do so.

Lemma . Given $\alpha \in \mathbb{R}$ and any positive integer N, there exists integers n and m with $1 \leq n \leq N$, such that $|n\alpha - m| \leq \frac{1}{N}$.

This means we are guaranteed the existence of $n, m \in \mathbb{Z}$ such that $|n\alpha - m| < \epsilon$, for some $\epsilon > 0$. Let $\beta = n\alpha - m$. Note that

$$\mathrm{e}^{2\pi \mathrm{i} n\alpha} = \mathrm{e}^{2\pi \mathrm{i}(n\alpha - m)} = \mathrm{e}^{2\pi \mathrm{i}\beta}.$$

We have that

$$\{U \times V \mid U, V \subseteq \mathbb{S}^1 \text{ are open}\}$$

is a basis for the topology on $\mathbb{T}^2$.
Consider an open set $V \subseteq \mathbb{S}^1$, then we can find an open interval $(a, b) \subset \mathbb{R}$ such that

$$\{\mathrm{e}^{2\pi \mathrm{i} x} \mid x \in (a, b)\} \subseteq V.$$

From $|\beta| < \epsilon = b - a$, we can see that there exists an integer $k \in \mathbb{Z}$ such that $k\beta \in (a, b)$. In fact,

$$\mathrm{e}^{2\pi \mathrm{i} n\alpha k} = \mathrm{e}^{2\pi \mathrm{i} k\beta} \in V.$$

We will use the above argument now on the basis of $\mathbb{T}^2$. Given any open sets $U, V \in \mathbb{S}^1$, we can always choose $x \in \mathbb{R}$ such that $\mathrm{e}^{2\pi \mathrm{i} x} \in U$ and then use the above to find an integer k such that $\mathrm{e}^{2\pi \mathrm{i}(x+kn)\alpha} \in V$, where $\mathrm{e}^{2\pi i k n} = 1$.
We have

$$\gamma(x+kn) = \left(\mathrm{e}^{2\pi \mathrm{i}(x+kn)}, \mathrm{e}^{2\pi \mathrm{i}(x+kn)\alpha}\right) = \left(\mathrm{e}^{2\pi \mathrm{i} x}, \mathrm{e}^{2\pi \mathrm{i}(x+kn)\alpha}\right) \in U \times V.$$

Since this is true for arbitrary U and V, then $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.

∎

Problem 4-7
Suppose M and N are smooth manifolds, and $\pi : M \to N$ is a surjective smooth submersion. Show that there is no other smooth manifold structure on N that satisfies the conclusion of Theorem 4.29; in other words, assuming that $\tilde{N}$ represents the same set as N with a possibly different topology and smooth structure, and that for every smooth manifold P with or without boundary, a map $F : \tilde{N} \to P$ is smooth if and only if $F \circ \pi$ is smooth, show that Id_N is a diffeomorphism between N and $\tilde{N}$. [Remark: this shows that the property described in Theorem 4.29 is "characteristic" in the same sense as that in which Theorem A.27(a) is characteristic of the quotient topology.]

Solution. Suppose M and N are smooth manifolds and let $\tilde{N}$ be the same set as N but with a possibly different topology and smooth structure. Let $\pi : M \to N$ and $\tilde{\pi} : M \to \tilde{N}$ be surjective smooth submersions. Consider $\mathrm{Id}_N : N \to \tilde{N}$ and $\mathrm{Id}_{\tilde{N}} : \tilde{N} \to \tilde{N}$. Clearly, Id_N is a homeomorphism. We need to show smoothness of the map and its inverse.
We have that $\mathrm{Id}_{\tilde{N}}$ is smooth, then $\mathrm{Id}_{\tilde{N}} \circ \tilde{\pi} = \tilde{\pi}$ is also smooth, since $\tilde{\pi}$ is also smooth. From that, $\mathrm{Id}_N \circ \pi = \tilde{\pi}$ is smooth, and thus, so is Id_N.
Now to show that the inverse is also smooth, notice that $\mathrm{Id}_N^{-1} \circ \tilde{\pi} = \pi$ is smooth, and then so is Id_N^{-1}.
Therefore, Id_N is a diffeomorphism. ∎

Problem 4-8
This problem shows that the converse of Theorem 4.29 is false. Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) = xy$. Show that π is surjective and smooth, and for each smooth manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth; but π is not a smooth submersion.

Solution. Let

$$\begin{aligned} \pi : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto xy. \end{aligned}$$

- **Surjective**: For all $z \in \mathbb{R}$, there exists $(x, y) \in \mathbb{R}^2$ such that $\pi(x, y) = z$. In fact, let $(x, y) = (z, 1)$ or $(x, y) = (1, z)$, then $\pi(x, y) = z$.

  Thus, π is surjective.

- **Smoothness**: π is clearly smooth since it a polynomial

- $\boxed{\Longrightarrow}$ Suppose $F : \mathbb{R} \to P$ is smooth, then $F \circ \pi$ is smooth since it is a composition of smooth maps.

  $\boxed{\Longleftarrow}$ Suppose $F \circ \pi$ is smooth, then if we compose with a smooth function, say f, that acts "like" the inverse of π, then we get that $F \circ \pi \circ f = F$ is smooth. In that case, take

  $$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^2 \\ x &\mapsto (x, 1) \end{aligned}$$

  and thus, F is smooth.

  Thus, for each manifold P, a map $F : \mathbb{R} \to P$ is smooth if and only if $F \circ \pi$ is smooth.

- **Not a Smooth Submersion**: We compute

$$d\pi(x,y) = \begin{pmatrix} \partial_x(xy) \\ \partial_y(xy) \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix},$$

which has a rank 0 at $(0,0)$. Thus, π is not a smooth submersion.

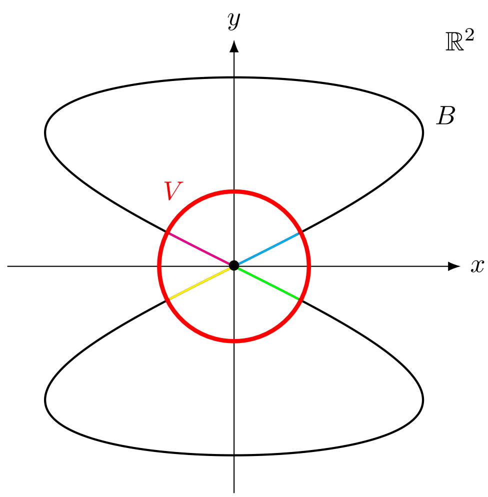
■

Problem 5-4
Show that the image of the curve $\beta : (-\pi, \pi) \to \mathbb{R}^2$ of Example 4.19 is not an embedded submanifold of $\mathbb{R}^2$. [Be careful: this is not the same as showing that β is not an embedding.]

Solution. Consider the curve

$$\begin{aligned} \beta : (-\pi, \pi) &\to \mathbb{R}^2 \\ t &\mapsto (\sin(2t), \sin(t)). \end{aligned}$$

Let $B \equiv \beta(-\pi, \pi)$ be the image of the curve. Let V be a neighborhood of 0 in $\mathbb{R}^2$, then $B \cap V$ is open in B. For small enough V, the intersection of B and V excluding the origin, *i.e.* $(B \cap V) \setminus 0$, will have four connected components.



This means that $B \cap V$ cannot be homeomorphic to any open ball in $\mathbb{R}^n$ since removing a point from $B \cap V$ created four connected components while removing a point from $\mathbb{R}^n$ creates either two connected components if $n = 1$, or one connected component otherwise. This helps us conclude that B, with the subspace topology, is not a topological manifold.
Therefore, the image of β is not an embedded submanifold of $\mathbb{R}^2$.

■

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): Exercises 4.3, 4.4, 4.7, 4.10, 4.16, 4.23, 5.10.

Problem 4.3
Verify the claims made in the preceding example.

(a) Suppose $M_1, \ldots, M_k$ are smooth manifolds. Then each of the projection maps $\pi_i : M_1 \times \cdots \times M_k \to M_i$ is a smooth submersion. In particular, the projection $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ onto the first n coordinates is a smooth submersion.

(b) If $\gamma : J \to M$ is a smooth curve in a smooth manifold M with or without boundary, then γ is a smooth immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$.

(c) If M is a smooth manifold and its tangent bundle TM is given the smooth manifold structure described in Proposition 3.18, the projection $\pi : TM \to M$ is a smooth submersion. To verify this, just note that with respect to any smooth local coordinates $\left(x^i\right)$ on an open subset $U \subseteq M$ and the corresponding natural coordinates $\left(x^i, v^i\right)$ on $\pi^{-1}(U) \subseteq TM$ (see Proposition 3.18), the coordinate representation of π is $\hat{\pi}(x, v) = x$.

(d) The smooth map $X : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$X(u, v) = \Big((2 + \cos(2\pi u)) \cos(2\pi v),\ (2 + \cos(2\pi u)) \sin(2\pi v),\ \sin(2\pi u)\Big)$$

is a smooth immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$ whose image is the doughnut-shaped surface obtained by revolving the circle $(y - 2)^2 + z^2 = 1$ in the (y, z)-plane about the z-axis (Fig. 4.1).

Solution. (a) Let $(U_1 \times \cdots \times U_k,\ \phi_1 \times \cdots \times \phi_k)$ be a smooth chart for $M_1 \times \cdots \times M_k$. The coordinate representation of π_i is then

$$\phi_i \circ \pi_i \circ (\phi_1 \times \cdots \times \phi_k)^{-1}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \mathbf{v}_i.$$

Let $\dim M_i = n_i$ and let $\mathbf{v}_i = \left(x_i^1, \ldots, x_i^{n_1}\right)$. Thus, in coordinates, $\mathrm{d}(\pi_i)_p$ is the $(n_1 + \cdots + n_k) \times n_i$ matrix $\left(\frac{\partial x_i^j}{\partial x_m^l}\right)$, where j ranges from 1 to n_i, m ranges from 1 to k, and for each m, l ranges from 1 to n_m. This matrix is of full rank since the only nonzero terms occur when $j = l$ and $i = m$ simultaneously. Therefore, π_i is a smooth submersion.

(b) Let γ be a smooth immersion. Then $\mathrm{d}\gamma_{t_0}$ is injective for every $t_0 \in J$. Then $\gamma'(t_0) = \mathrm{d}\gamma_{t_0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0}\right) \neq 0$. Conversely, suppose $\gamma'(t_0) \neq 0$ for every $t_0 \in J$. Suppose that $d\gamma t_0(\mathbf{v}) = 0$ for some $\mathbf{v} \in T_{t_0}J$. Since $T_{t_0}J$ is spanned by $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0}$, we have $\mathbf{v} = \alpha\ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t_0}$ for some $\alpha \in \mathbb{R}$. Then

$$0 = \mathrm{d}\gamma_{t_0}\left(\alpha\ \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}\right) = \alpha\, \mathrm{d}\gamma_{t_0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0}\right) = \alpha\gamma'(t_0).$$

implying $\alpha = 0$. Hence, $\mathrm{d}\gamma_{t_0}$ is injective, and therefore γ is a smooth immersion.

(c) Let (U, ϕ) be a chart on M, and let $\left(\pi^{-1}(U),\ \tilde{\phi}\right)$ be the corresponding chart on TM. Then the coordinate representation of π is

$$\phi \circ \pi \circ \tilde{\phi}^{-1}(\mathbf{x}, \mathbf{v}) = \mathbf{x}.$$

A similar computation as the one done in (a) shows that $d\pi|_p$ is surjective for every $p \in TM$, and therefore π is a submersion.

(d) The differential $\mathrm{d}X_{(u,v)}$ has the matrix representation

$$\mathrm{d}X_{(u,v)} = \begin{pmatrix} -2\pi\sin(2\pi u)\cos(2\pi v) & -2\pi(2+\cos(2\pi u))\sin(2\pi v) \\ -2\pi\sin(2\pi u)\sin(2\pi v) & 2\pi(2+\cos(2\pi u))\cos(2\pi v) \\ 2\pi\cos(2\pi u) & 0 \end{pmatrix}$$

Note that at least one of the $(1,2)$ or $(2,2)$ entries is nonzero. If $2\pi\cos(2\pi u) = 0$, then the bottom row is a row of zeros, and the determinant of the 2×2 matrix formed by the top two rows is $\pm 8\pi$. Hence, the columns are linearly independent, and thus $\mathrm{d}X_{(u,v)}$ has full rank. Therefore, X is a smooth immersion.

∎

Problem 4.4
Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.

Solution. Solved in Study Guide for Quiz 2. ∎

Problem 4.7
Prove the following proposition.

Proposition 4.6. (Elementary Properties of Local Diffeomorphisms).

(a) Every composition of local diffeomorphisms is a local diffeomorphism.

(b) Every finite product of local diffeomorphisms between smooth manifolds is a local diffeomorphism.

(c) Every local diffeomorphism is a local homeomorphism and an open map.

(d) The restriction of a local diffeomorphism to an open submanifold with or without boundary is a local diffeomorphism.

(e) Every diffeomorphism is a local diffeomorphism.

(f) Every bijective local diffeomorphism is a diffeomorphism.

(g) A map between smooth manifolds with or without boundary is a local diffeomorphism if and only if in a neighborhood of each point of its domain, it has a coordinate representation that is a local diffeomorphism.

Solution. (a) Let $F : M \to N$ and $G : N \to P$ be local diffeomorphisms and let $p \in M$. Let $U \subseteq M$ be an open set containing p such that $F|_U : U \to F(U)$ is a diffeomorphism. Let $V \subseteq N$ be an open set containing $F(p)$ such that $G|_V : V \to G(V)$ is a diffeomorphism. Then

$$G \circ F|_U \cap F^{-1}(V) : U \cap F^{-1}(V) \to (G\circ F)(U) \cap G(V)$$

is a diffeomorphism, where $U \cap F^{-1}(V) \subseteq M$ is an open set containing p. Hence, $G \circ F$ is a local diffeomorphism.

(b) Let $F_i : M_i \to N_i$ be a local diffeomorphism for $i = 1, \dots, k$. Consider the product map

$$F_1 \times \cdots \times F_k : M_1 \times \cdots \times M_k \to N_1 \times \cdots \times N_k$$

Let $p = (p_1, \dots, p_k) \in M_1 \times \cdots \times M_k$ and let U_i be an open set containing p_i such that

$$F_i|_{U_i} : U_i \to F_i(U_i)$$

is a diffeomorphism. By Exercise 2.16,

$$F_1|_{U_1} \times \cdots \times F_k|_{U_k} : U_1 \times \cdots \times U_k \to F_1(U_1) \times \cdots \times F_k(U_k)$$

is a diffeomorphism. Hence, $F_1 \times \cdots \times F_k$ is a local diffeomorphism.

(c) Let $F : M \to N$ be a local diffeomorphism. Since every point $p \in M$ has an open neighborhood U such that $F|_U : U \to F(U)$ is a diffeomorphism, we have that F is a local homeomorphism. Let $U \subseteq M$ be an arbitrary open set. Let $p' \in F(U)$ and let $p \in U$ be such that $F(p) = p'$. Since U is open, we can choose an open set V such that $p \in V \subseteq U$ and $F|_V : V \to F(V)$ is a diffeomorphism. Then $F(V) \subseteq F(U)$ is open and contains p'. Thus, $F(U)$ is open and F is an open map.

(d) Let $F : M \to N$ be a local diffeomorphism and let $U \subseteq M$ be an open submanifold. Let $p \in U$ be arbitrary, and let $V \subseteq M$ be an open set such that $F|_V : V \to F(V)$ is a diffeomorphism. By Exercise 2.16, $(F|_V)|_{U\cap V} = F|_{U\cap V} : U \cap V \to F(U \cap V)$ is a diffeomorphism. Hence, $F|_U$ is a local diffeomorphism.

(e) Let $F : M \to N$ be a diffeomorphism and let $p \in M$. Note that M itself is an open set containing p and $F|_M = F$ is a diffeomorphism. Therefore, F is a local diffeomorphism.

(f) Let $F : M \to N$ be a bijective local diffeomorphism. To see that F is smooth, note that every point of M is contained in a chart (U, ϕ) such that $F|_U$ is a diffeomorphism. Then the coordinate representation of F in the chart (U, ϕ) is smooth, so F is smooth. To see that F^{-1} is also smooth, let $F(p) \in N$ and let $U \subseteq M$ be an open set containing p such that $F|_U$ is a diffeomorphism. Then $F^{-1}|_{F(U)} : F(U) \to U$ is also a diffeomorphism. Let (V, ψ) be a smooth chart containing $F(p)$ such that $V \subseteq F(U)$. Then $F^{-1}|_V$ is a diffeomorphism onto its image, and therefore F^{-1} has a smooth coordinate representation in (V, ψ). Therefore, F^{-1} is smooth and F is a diffeomorphism.

(g) Let $F : M \to N$ be a local diffeomorphism. Let $p \in M$ and let U be an open set containing p such that $F|U : U \to F(U)$ is a diffeomorphism. We can then find a chart (V, ϕ) such that $V \subseteq U$. Then the coordinate representation of F in (V, ϕ) is a diffeomorphism, and therefore a local diffeomorphism by part (e). Conversely, suppose that every point $p \in M$ is contained in a chart (U, ϕ) such that $F(p)$ is in some chart (V, ψ) and

$$\psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \psi(F(U \cap F^{-1}(V)))$$

is a local diffeomorphism. Since F is smooth (coordinate representations are only defined for smooth maps), $\phi(U \cap F^{-1}(V))$ is open in $\mathbb{R}^n$ (or H^n). Since local diffeomorphisms are open maps, $\psi(F(U \cap F^{-1}(V)))$ is open in $\mathbb{R}^n$ (or H^n). Since the restriction of a local diffeomorphism to an open submanifold is a local diffeomorphism,

$$(\psi \circ F \circ \phi^{-1}) \circ \phi|_{U\cap F^{-1}(V)} = \psi \circ F|_{U\cap F^{-1}(V)}.$$

Similarly,

$$\psi^{-1}|_{\psi(F(U\cap F^{-1}(V)))}$$

is a local diffeomorphism, and therefore

$$F|_{U\cap F^{-1}(V)} : U \cap F^{-1}(V) \to F(U \cap F^{-1}(V))$$

is a local diffeomorphism. Then there is an open set $W \subseteq U \cap F^{-1}(V)$ containing p such that $F|_W : W \to F(W)$ is a diffeomorphism. Hence, F is a local diffeomorphism.

■

Problem 4.10
Suppose M, N, and P are smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Prove the following:

(a) If $G : P \to M$ is continuous, then G is smooth if and only if $F \circ G$ is smooth.

(b) If in addition F is surjective and $G : N \to P$ is any map, then G is smooth if and only if $G \circ F$ is smooth.

Solution. Suppose M, N, and P are smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism.

(a) $\boxed{\implies}$ Let $G : P \to M$ be continuous. Suppose G be smooth. Since local diffeomorphisms are smooth maps, their composition $F \circ G$ is also smooth.

$\boxed{\impliedby}$ Suppose $F \circ G$ is smooth. Let $p \in P$ and let $U \subseteq M$ be an open set containing $G(p)$ such that $F|_U : U \to F(U)$ is a diffeomorphism. Since G is continuous, $G^{-1}(U)$ is open, and therefore $(F \circ G)|_{G^{-1}(U)} : G^{-1}(U) \to F(U)$ is smooth. Then

$$(F|_U)^{-1} \circ (F \circ G)|_{G^{-1}(U)} = G|_{G^{-1}(U)} : G^{-1}(U) \to U$$

is smooth. Since $p \in P$ was arbitrary, we conclude that G is smooth.

(b) $\boxed{\implies}$ Suppose G is smooth. Since local diffeomorphisms are smooth maps, their composition $G \circ F$ is also smooth.

$\boxed{\impliedby}$ Suppose $G \circ F$ is smooth. Let $q \in N$ be arbitrary, and let $F(p) = q$ (using the fact that F is surjective). Let $U \subseteq M$ be an open neighborhood of p such that $F|_U : U \to F(U)$ is a diffeomorphism. Then $G \circ F|_U$ is smooth, and therefore

$$(G \circ F|_U) \circ (F|_U)^{-1} = G|_{F(U)} : F(U) \to N$$

is smooth. Since $q \in N$ was arbitrary, we conclude that G is smooth.

■

Problem 4.16
Show that every composition of smooth embeddings is a smooth embedding.

Solution. Let $F : M \to N$ and $G : N \to P$ be smooth embeddings. We want to show that their composition $G \circ F : M \to P$ is also a smooth embedding. First, we'll show that $G \circ F$ is a smooth immersion. Since both F and G are smooth immersions, their composition $G \circ F$ is also a smooth immersion by Exercise 4.4. Next, we need to show that $G \circ F$ is a topological embedding. We know that $F : M \to F(M)$ and $G : N \to G(N)$ are homeomorphisms. Then $G|_{F(M)} : F(M) \to G(F(M))$ is also a homeomorphism, which shows that $G \circ F : M \to G(F(M))$ is a homeomorphism. Therefore $G \circ F$ is a topological embedding. Since $G \circ F$ is both a smooth immersion and a topological embedding, it is a smooth embedding. ■

Problem 4.24
Give an example of a smooth embedding that is neither an open map nor a closed map.

Solution. Let $F : \mathbb{R} \to \mathbb{R}^2$ be defined by

$$F(t) = (\mathrm{e}^t, 0).$$

First, let's verify this is a smooth embedding. We have that F is clearly smooth. The derivative is

$$\mathrm{d}F_t = (\mathrm{e}^t, 0),$$

which is never zero, so F is a smooth immersion.
Moreover, F is clearly injective and continuous. Its image is $(0, \infty) \times 0$, and the inverse map

$$F^{-1} : (0, \infty) \times 0 \to \mathbb{R}, \quad (x, 0) \mapsto \ln(x)$$

is continuous. Therefore F is a topological embedding.
However, F is neither an open nor a closed map. For instance, $F(\mathbb{R}) = (0, \infty) \times 0$ is neither open nor closed in $\mathbb{R}^2$ despite $\mathbb{R}$ being both open and closed in itself. Therefore F is neither an open nor a closed map. ■

Problem 5.10
Show that spherical coordinates (Example C.38) form a slice chart for $\mathbb{S}^2$ in $\mathbb{R}^3$ on any open subset where they are defined.

Solution. Let $\psi : U \to \mathbb{R}^3$ be spherical coordinates where they are defined, namely

$$\psi(r, \theta, \phi) = \Big(r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\phi)\Big),$$

where $r > 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.
Then we have

$$\psi(U \cap \mathbb{S}^2) = \{(r, \theta, \phi) \in U : r = 1\}.$$

This shows that spherical coordinates define a slice chart for $\mathbb{S}^2$ in $\mathbb{R}^3$, since the sphere corresponds precisely to points where the first coordinate r equals 1. ∎

Homework 5

Problem 5-6
Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, and let $UM \subseteq T\mathbb{R}^n$ be the set of all unit tangent vectors to M:

$$UM = \{(x, v) \in T\mathbb{R}^n : x \in M, v \in T_x M, |v| = 1\}.$$

It is called the **unit tangent bundle of M**. Prove that UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n \approx \mathbb{R}^n \times \mathbb{R}^n$. (Used on p. 147.)

Solution. Suppose $M \subseteq \mathbb{R}^n$ is an embedded m-dimensional submanifold, then TM is an embedded $(2m)$-dimensional submanifold of $T\mathbb{R}^n$. Consider the smooth map

$$\begin{aligned} \Phi : TM &\to \mathbb{R}, \\ v &\mapsto |v|. \end{aligned}$$

Notice that the differential of Φ in the level set $\Phi^{-1}(1)$ has at least rank 1 since there is no element in the level set that maps to the origin. Additionally, the dimension of the codomain of Φ is $\dim(\mathbb{R}) = 1$, so the dimension of the level set $\Phi^{-1}(1)$ is $\dim(M) - \dim(\mathbb{R}) = 2m - 1$. Thus, the differential $\mathrm{d}\Phi$ is surjective for all points $p \in M$, and $\Phi^{-1}(1)$ is then a regular level set.
Then, by the regular level set theorem, every regular level set of a smooth map is a closed embedded submanifold whose codimension is equal to the dimension of the range, which is one in this case.
Consider now the unit tangent bundle UM. Notice that UM is the level set $\Phi^{-1}(1)$, which we showed to be an embedded $(2m-1)$-dimensional submanifold of TM. Therefore, UM is an embedded $(2m-1)$-dimensional submanifold of $T\mathbb{R}^n$. ■

Problem 5-7
Let $F : \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x,y) = x^3 + xy + y^3$. Which level sets of F are embedded submanifolds of $\mathbb{R}^2$? For each level set, prove either that it is or that it is not an embedded submanifold.

Solution. Both the domain and codomain are smooth manifolds, and the function F is a smooth map since it a polynomial. Given $F(x,y) = x^3 + xy + y^3$, we can compute the differential of F, given by

$$\begin{aligned} \mathrm{d}F(x,y) &= \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} 3x^2 + y & x + 3y^2 \end{pmatrix}. \end{aligned}$$

The constant rank level set theorem guarantees that each level set of F is a closed embedded submanifold of codimension equal to the constant rank in M. This will not necessarily be true when the domain contains a point where $\mathrm{d}F$ has rank 0, *i.e.* where both partial derivatives are zero.
Then, solving for these points, we have

$$\begin{aligned} \mathrm{d}F(x,y) = \begin{pmatrix} 0 & 0 \end{pmatrix} &\implies \begin{cases} 3x^2 + y = 0 & \implies y = -3x^2 \\ x + 3y^2 = 0 & \implies x + 3\left(-3x^2\right)^2 = x + 27x^4 = 0 \end{cases} \\ &\implies x(27x^3 + 1) = 0 \\ &\implies x = 0 \quad \text{or} \quad x = -\frac{1}{3} \\ &\implies (x,y) = (0,0) \quad \text{or} \quad (x,y) = \left(-\frac{1}{3}, -\frac{1}{3}\right). \end{aligned}$$

Plugging in these values in F, we have

$$F(0,0) = 0 \quad \text{and} \quad F\left(-\frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{27}.$$

Then, unless F is equal to one of those two preceding values, the level set will always be an embedded submanifold. Thus, for all $c \notin \{0, \frac{1}{27}\}$, the level set $F^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.
We will now check whether or not the level sets are embedded submanifolds at the critical points.

- **For** $(x,y) = (0,0)$:

  The level set $F^{-1}(0)$ is the folium of Descartes. A sketch of what that looks like is shown in the figures below.

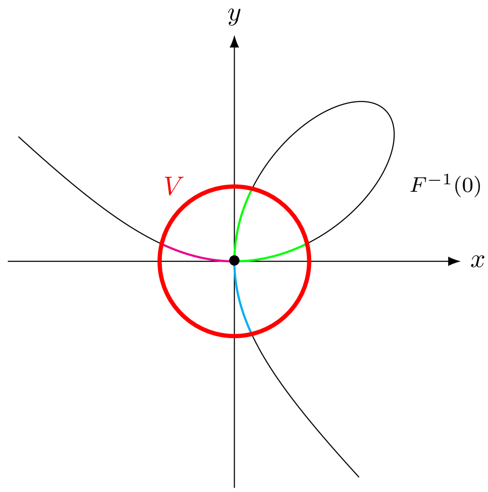


Figure 1: Connected components after removing the point $(0,0)$.

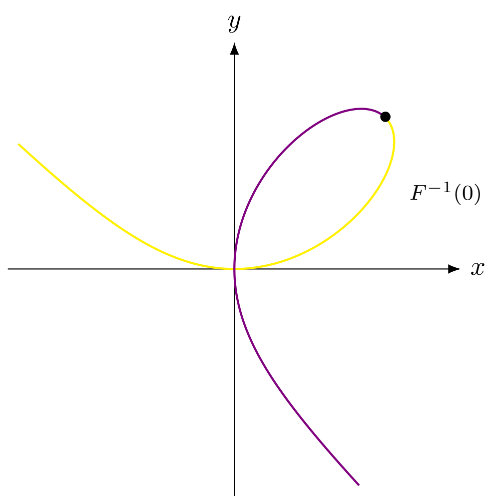


Figure 2: Connected components after removing any point $(x,y) \neq (0,0)$.

We first note that $(0,0)$ is a non-degenerate critical point of F. Notice that $F^{-1}(0)$ does not have the discrete topology, then it cannot be a zero-dimensional embedded submanifold. Additionally, notice that $F^{-1}(0) \setminus (0,0)$ has three connected components, but $F^{-1}(0) \setminus (x,y)$, for any $(x,y) \neq (0,0)$, has only two connected components, then $F^{-1}(0)$ cannot be homeomorphic to $\mathbb{R}$, so it cannot be a one-dimensional embedded submanifold. Finally, $F^{-1}(0)$ is closed and not equal to $\mathbb{R}^2$, so it cannot be a two-dimensional embedded submanifold.

Thus, $F^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$.

- **For** $(x,y) = \left(-\frac{1}{3}, -\frac{1}{3}\right)$:

The level set $F^{-1}\left(\frac{1}{27}\right)$ can be factorized and written as

$$\begin{aligned}
F(x,y) &= \frac{1}{27} \\
x^3 + y^3 + xy - \frac{1}{27} &= 0 \\
x^3 + y^3 + 3xy(x+y) - 3xy(x+y) + xy - \left(\frac{1}{3}\right)^3 &= 0 \\
(x+y)^3 - 3xy\left(x + y - \frac{1}{3}\right) - \left(\frac{1}{3}\right)^3 &= 0 \\
(x+y)^3 - \left(\frac{1}{3}\right)^3 - 3xy\left(x + y - \frac{1}{3}\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left((x+y)^2 + \frac{1}{3}(x+y) + \frac{1}{9}\right) - 3xy\left(x + y - \frac{1}{3}\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left((x+y)^2 + \frac{1}{3}(x+y) + \frac{1}{9} - 3xy\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left(y^2 + 2xy + x^2 + \frac{1}{3}x + \frac{1}{3}y + \frac{1}{9} - 3xy\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left(y^2 - xy + \frac{1}{3}y + x^2 + \frac{1}{3} + \frac{1}{9}\right) &= 0 \\
\left(x + y - \frac{1}{3}\right)\left(y^2 - \left(x - \frac{1}{3}\right)y + \left(x^2 + \frac{1}{3}x + \frac{1}{9}\right)\right) &= 0 \\
\implies x + y - \frac{1}{3} = 0 \qquad \text{or} \qquad y^2 - \left(x - \frac{1}{3}\right)y + \left(x^2 + \frac{1}{3}x + \frac{1}{9}\right) &= 0.
\end{aligned}$$

The first condition is simple that the points (x,y) belong to the straight line $y = -x + \frac{1}{3}$. To find out what the second condition refers to visually, we solve the quadratic equation for y in terms of x.

$$\begin{aligned}
y_\pm &= \frac{\left(x - \frac{1}{3}\right) \pm \sqrt{\left(x - \frac{1}{3}\right)^2 - 4(1)\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right)}}{2(1)} \\
&= \frac{\left(x - \frac{1}{3}\right) \pm \sqrt{x^2 - \frac{2}{3}x + \frac{1}{9} - 4x^2 - \frac{4}{3}x - \frac{4}{9}}}{2} \\
&= \frac{\left(x - \frac{1}{3}\right) \pm \sqrt{-3x^2 - 2x - \frac{1}{3}}}{2},
\end{aligned}$$

where we require real roots, which implies that

$$\begin{aligned} -3x^2 - 2x - \frac{1}{3} &\geq 0 \\ x^2 + \frac{2}{3}x + \frac{1}{9} &\leq 0 \\ \left(x + \frac{1}{3}\right)^2 &\leq 0, \end{aligned}$$

which has a discriminant equal to zero, and hence, x is a double root at $x = -\frac{1}{3}$. Replacing in the solution for y, we also get that the discriminant is null and y is also is a double root of value $y = -\frac{1}{3}$. Thus, $F^{-1}\left(\frac{1}{27}\right)$ is the embedding of the disjoint union between a zero-dimensional submanifold and a one-dimensional submanifold, but these manifolds must have the same dimension everywhere, and thus, $F^{-1}\left(\frac{1}{27}\right)$ is not an embedded submanifold of $\mathbb{R}^2$.

■

Problem 5-19
Suppose $S \subseteq M$ is an embedded submanifold and $\gamma : J \to M$ is a smooth curve whose image happens to lie in S. Show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$ for all $t \in J$. Give a counterexample if S is not embedded.

Solution. Let M be m-dimensional and S be k-dimensional, with $k \leq m$. Since S is an embedded submanifold of M, then, for every point $p \in S$, there exists an open neighborhood U in M centered at p such that $\varphi : U \cap S \to \mathbb{R}^k$ is a smooth embedding.
The differential of γ at t, $\gamma'(t)$, is an element of the tangent space $T_{\gamma(t)}M$. To show that $\gamma'(t)$ is in the subspace $T_{\gamma(t)}S$ of $T_{\gamma(t)}M$, for all $t \in J$, we need to show that $\gamma'(t)$ lies in the image of the differential $\mathrm{d}\varphi(p)$ of the embedding φ at the point $p = \gamma(t) \in S$.
Consider a slice chart (U, φ) as defined above. Since $\gamma(t) \in S$, then it is also in U. Let $p = \gamma(t)$ and consider the composition map $\varphi \circ \gamma : J \to \mathbb{R}^k$. By the chain rule, the differential of the composition map is

$$\mathrm{d}(\varphi \circ \gamma)(t) = \mathrm{d}\varphi(p)\gamma'(t).$$

Since φ is an embedding, its differential is an injective linear map from T_pM to $\mathbb{R}^k$. Since $\gamma(t)$ is in S, then $\varphi \circ \gamma(t)$ is a constant function of t, and its derivative with respect to t is null. Thus, $\mathrm{d}\varphi(p)\gamma'(t)$ is the kernel of the linear map, and is therefore a subspace of $T_{\gamma(t)}S$.
A counterexample to this, given that S is not embedded, is by letting the curve to be

$$\begin{aligned} \gamma : \mathbb{R} &\to \mathbb{R}^2, \\ t &\mapsto (t^2, t^3). \end{aligned}$$

and S to be the x-axis. Then γ lies entirely in S, but at $t = 0$, the differential $\gamma'(0) = (0, 0)$ is not in the subspace $T_{\gamma(0)}S = \operatorname{span}\{(1, 0)\}$, because S is not an embedded submanifold of $\mathbb{R}^2$ at the origin since it is not locally diffeomorphic to a Euclidean space in any neighborhood of the origin. ■

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): Exercises 4.24, 4.32, 5.10, 5.20, 5.36, 5.40.

Problem 4.24
Give an example of a smooth embedding that is neither an open map nor a closed map.

Solution. Solved in Homework 4 Recommended Problems. ■

Problem 4.32
Prove Theorem 4.31.

Theorem 4.31. (Uniqueness of Smooth Quotients). Suppose that M, N_1, and N_2 are smooth manifolds, and $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ are surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F : N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$:

Solution. Solved in Study Guide for Quiz 2. ■

Problem 5.10
Show that spherical coordinates (Example C.38) form a slice chart for $\mathbb{S}^2$ in $\mathbb{R}^3$ on any open subset where they are defined.

Solution. Solved in Homework 4 Recommended Problems. ■

Problem 5.20
Suppose M is a smooth manifold and $S \subseteq M$ is an immersed submanifold. Show that every subset of S that is open in the subspace topology is also open in its given submanifold topology; and the converse is true if and only if S is embedded.

Solution. Let $U \subseteq S$ be open in the subspace topology, then there exists an open set $V \subseteq M$ such that $U = S \cap V$. If $\iota : S \hookrightarrow M$ is the inclusion map, then $U = \iota^{-1}(U) = \iota^{-1}(S \cap V) = \iota^{-1}(V)$ is open in the manifold topology of S, since inclusion maps of immersed submanifolds are continuous.
Now suppose every subset open in the manifold topology of S is also open in the subspace topology from M. Then for any U open in the manifold topology of S, $\iota(U)$ is open in S with respect to the subspace topology from M. Therefore ι is a topological embedding. Since ι is also a smooth immersion, it is a smooth embedding. Hence S is embedded.
Conversely, if S is embedded, then the submanifold topology on S agrees with the subspace topology by definition. Therefore every set open in the manifold topology is open in the subspace topology. ■

Problem 5.36
Prove Proposition 5.35.

Proposition 5.35. Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J$, $\gamma(0) = p$, and $\gamma'(0) = v$.

Solution. Solved in Study Guide for Quiz 2. ■

Problem 5.40
Suppose $S \subseteq M$ is a level set of a smooth map $\Phi : M \to N$ with constant rank. Show that $T_pS = \operatorname{Ker}(\mathrm{d}\Phi_p)$ for each $p \in S$.

Solution. Solved in Study Guide for Quiz 2. ∎

Homework 6

No recommended practice problems this week. HOMEWORK INCOMPLETE.

Problem 6-2
Prove Theorem 6.18 (the Whitney immersion theorem) in the special case $\partial M = \varnothing$. [Hint: without loss of generality, assume that M is an embedded n-dimensional submanifold of $\mathbb{R}^{2n+1}$. Let $UM \subseteq T\mathbb{R}^{2n+1}$ be the unit tangent bundle of M (Problem 5-6), and let $G : UM \to \mathbb{RP}^{2n}$ be the map $G(x, v) = [v]$. Use Sard's theorem to conclude that there is some $v \in \mathbb{R}^{2n+1} \backslash \mathbb{R}^{2n}$ such that $[v]$ is not in the image of G, and show that the projection from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$ with kernel $\mathbb{R}v$ restricts to an immersion of M into $\mathbb{R}^{2n}$.]

Solution. The Whitney Immersion Theorem states that every smooth manifold M can be smoothly immersed into some Euclidean space $\mathbb{R}^n$, for sufficiently large N. We aim to prove this for the case $\partial M = \varnothing$. We use the hint given.
Without loss of generality, assume that M is an embedded n-dimensional submanifold of $\mathbb{R}^{n+1}$. Let UM be the unit tangent bundle of M and let $G : UM \to \mathbb{RP}^{2n}$ be the map $G(x, v) = [v]$. Since $\partial M = \varnothing$. then UM is a smooth manifold of dimension $2n$ and $\mathbb{RP}^{2n}$ is a smooth manifold of dimension $2n - 1$.
By Sard's theorem, almost every point of $\mathbb{RP}^{2n}$ is a regular value of G, then the preimage of any regular value is a smooth submanifold of UM of dimension 1. Thus, there exists some $v \in \mathbb{R}^{2n+1} \setminus \mathbb{R}^{2n}$ such that $[v]$ is not in the image of G.
Let $p : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}$ be the projection with kernel $\mathbb{R}v$. Then p restricts to an immersion of M into $\mathbb{R}^{2n}$.
Let i be an injective smooth map that embeds the n-dimensional submanifold M into $\mathbb{R}^{2n+1}$ and let X be a vector field on M. Then $\mathrm{d}_\pi(M)$ is a linear subspace of $\mathbb{R}^{2n}$ of dimension n and $\mathrm{d}_\pi(x)$ is a linear map from $\mathrm{d}_\pi(M)$ to $\mathbb{R}^{2n}$ that is injective at every point of M.
Therefore, $\mathrm{d}_\pi(X)$ has full rank n at every point of M, which means that $p \circ i$ is an immersion of M into $\mathbb{R}^{2n}$. ■

Problem 6-5
Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold. Show that M has a tubular neighborhood U with the following property: for each $y \in U, r(y)$ is the unique point in M closest to y, where $r : U \to M$ is the retraction defined in Proposition 6.25. [Hint: first show that if $y \in \mathbb{R}^n$ has a closest point $x \in M$, then $(y - x) \perp T_xM$. Then, using the notation of the proof of Theorem 6.24, show that for each $x \in M$, it is possible to choose $\delta > 0$ such that every $y \in E\left(V_\delta(x)\right)$ has a closest point in M, and that point is equal to $r(y)$.]

Solution. Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold and let $y \in \mathbb{R}^n$ and define the map

$$\begin{aligned} d : \mathbb{R}^n &\to \mathbb{R} \\ p &\mapsto |y - p|. \end{aligned}$$

Let $x \in M$ be a closest point to $y \in \mathbb{R}^n$. The directional derivative of d in the direction of v is

$$\nabla d \cdot v|_x = \frac{y - x}{|y - x|} \cdot v = 0.$$

Thus $y - x \perp T_xM$.
Incomplete... ■

Problem 6-10
Suppose $F : N \to M$ is a smooth map that is transverse to an embedded submanifold $X \subseteq M$, and let $W = F^{-1}(X)$. For each $p \in W$, show that $T_pW = (dF_p)^{-1}\left(T_{F(p)}X\right)$. Conclude that if two embedded

submanifolds $X, X' \subseteq M$ intersect transversely, then $T_p(X \cap X') = T_pX \cap T_pX'$ for every $p \in X \cap X'$. (Used on p. 146)

Solution. Let $X \subseteq M$ and $W = F^{-1}(X)$. It is enough to check that W is a submanifold of M in a neighborhood $X \in W$. Let (V, ψ) be a local coordinate chart of N adapted to X around $X = F_p$, then $\psi : V \to \mathbb{R}^{n+k}$ and $\psi(V \cap X) = \psi(V) \cap \mathbb{R}^n$, where $n = \dim(X)$.
Let $\pi_2 : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be the standard projection and let $g = \pi_1 \circ \psi$, then $g : V \to \mathbb{R}^k$ is a submersion and $g^{-1}(0) = V \cap X$.
Incomplete... ∎

Homework 7

Problem 7-2
Let G be a Lie group.

(a) Let $m : G\times G \to G$ denote the multiplication map. Using Proposition 3.14 to identify $T_{(e,e)}(G\times G)$ with $T_eG \oplus T_eG$, show that the differential $\mathrm{d}m_{(e,e)} : T_eG \oplus T_eG \to T_eG$ is given by

$$\mathrm{d}m_{(e,e)}(X, Y) = X + Y.$$

[Hint: compute $\mathrm{d}m_{(e,e)}(X, 0)$ and $\mathrm{d}m_{(e,e)}(0, Y)$ separately.]

(b) Let $i : G \to G$ denote the inversion map. Show that $\mathrm{d}i_e : T_eG \to T_eG$ is given by $\mathrm{d}i_e(X) = -X$. (Used on pp. 203, 522.)

Solution. (a) We have

$$\begin{aligned}\mathrm{d}m_{(e,e)}(X, Y) &= \mathrm{d}m_{(e,e)}(X, 0) + \mathrm{d}m_{(e,e)}(0, Y)\\ &= \mathrm{d}(m^1)_e(X) + \mathrm{d}(m^2)_e(Y),\end{aligned}$$

where

$$\begin{aligned} m^1 : G &\to G, \\ x &\mapsto m(x, e), \end{aligned} \qquad \text{and} \qquad \begin{aligned} m^2 : G &\to G, \\ y &\mapsto m(e, y). \end{aligned}$$

We have that $m^1 = m^2 = \mathrm{Id}_G$, thus, $\mathrm{d}m_{(e,e)}(X, Y) = X + Y$.

(b) We construct a constant map c by defining it to be the composition of three other maps m, n, and q, defined as

$$\begin{aligned} n : G &\to G \times G, \\ x &\mapsto (x, x), \end{aligned} \qquad \begin{aligned} q : G \times G &\to G \times G, \\ (x, y) &\mapsto (x, i(y)), \end{aligned}$$

and m defined as stated in the problem. Then $c = m \circ n \circ q$ is a constant map. Thus $\mathrm{d}c_e(X) = 0$. From that, we have

$$\begin{aligned} 0 &= \mathrm{d}c_e(X) \\ &= \mathrm{d}m_{(e,e)}\left(\mathrm{d}n_{(e,e)}\left(\mathrm{d}q_e(X)\right)\right) \\ &= \mathrm{d}m_{(e,e)}\left(\mathrm{d}n_{(e,e)}(X, X)\right) \\ &= \mathrm{d}m_{(e,e)}\left(X, \mathrm{d}i_e(X)\right) \\ &= X + \mathrm{d}i_e(X), \end{aligned}$$

where the last step follows from part (a). Therefore,

$$\mathrm{d}i_e(X) = -X.$$

■

Problem 7-4
Let $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ denote the determinant function. Use Corollary 3.25 to compute the differential of det, as follows.

(a) For any $A \in \mathrm{M}(n,\mathbb{R})$, show that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det\left(I_n + tA\right) = \operatorname{tr}(A)$$

where $\operatorname{tr}\left(A^i_j\right) = \sum_i A^i_i$ is the trace of A. [Hint: the defining equation (B.3) expresses $\det\left(I_n + tA\right)$ as a polynomial in t. What is the linear term?]

(b) For $X \in \mathrm{GL}(n,\mathbb{R})$ and $B \in T_X\mathrm{GL}(n,\mathbb{R}) \cong \mathrm{M}(n,\mathbb{R})$, show that

$$\mathrm{d}(\det)_X(B) = (\det(X)) \operatorname{tr}\left(X^{-1}B\right)$$

[Hint: $\det(X + tB) = \det(X)\det\left(I_n + tX^{-1}B\right)$.] (Used on p. 203)

Solution. (a) We aim to compute the derivative of $\det(\mathbb{I}_n + tA)$ with respect to t at $t = 0$. Using the hint, we can express $\det(\mathbb{I}_n + tA)$ as a polynomial in t as follows

$$\det(\mathbb{I}_n + tA) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (\mathbb{I}_n + tA)_{i,\sigma(i)}.$$

The terms in the product are

$$(\mathbb{I}_n + tA)_{i,\sigma(i)} = \delta_{i,\sigma(i)} + tA_{i,\sigma(i)},$$

where $\delta_{i,\sigma(i)}$ is the Kronecker delta. Plugging this back in the determinant expansion, we get

$$\det(\mathbb{I}_n + tA) \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \left(\delta_{i,\sigma(i)} + tA_{i,\sigma(i)}\right).$$

Expanding the product using the binomial theorem, we have

$$\prod_{i=1}^{n} \left(\delta_{i,\sigma(i)} + tA_{i,\sigma(i)}\right) = \sum_{k=0}^{n} t^k \sum_{1 \le i_1 \le \cdots \le i_k \le n} A_{i_1,\sigma(i_1)}\delta_{i_1,\sigma(i_1)} \cdots A_{i_k,\sigma(i_k)}\delta_{i_k,\sigma(i_k)}.$$

Replacing this in the determinant expression, we have

$$\det(\mathbb{I}_n + tA) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{k=0}^{n} t^k \sum_{1 \le i_1 \le \cdots \le i_k \le n} A_{i_1,\sigma(i_1)}\delta_{i_1,\sigma(i_1)} \cdots A_{i_k,\sigma(i_k)}\delta_{i_k,\sigma(i_k)}.$$

The linear term in t is given by

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{i=1}^{n} A_{i,\sigma(i)}\delta_{i,\sigma(i)},$$

which is just the trace of A, $\operatorname{tr}(A)$.
Thus,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det\left(\mathbb{I}_n + tA\right) = \operatorname{tr}(A).$$

(b) Let us view the expression $X + tB$ as the image, at time t, of a curve γ on $M(n,\mathbb{R})$, namely

$$\begin{aligned} \gamma : \mathbb{R} &\to M(n,\mathbb{R}), \\ t &\mapsto X + tB. \end{aligned}$$

The expression $\det(X+tB)$ can thus be seen as the composition of γ with the determinant function, given by

$$t \overset{\gamma}{\longmapsto} X+tB \overset{\det}{\longmapsto} \det(X+tB).$$

Computing the derivative of this composition with respect to t by an application of the chain rule yields

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(X+tB) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det\circ\gamma(t) \\
&= \mathrm{d}(\det)_{\gamma(0)}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\gamma(t)\right) \\
&= \mathrm{d}(\det)_{X+0B}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(X+tB)\right) \\
&= \mathrm{d}(\det)_X B.
\end{aligned}$$

On the other hand, using the hint and the result from part (a), we repeat the computation of this derivative as follows

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(X+tB) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(X)\,\det(\mathbb{I}_n + tX^{-1}B) && \text{(by the hint)} \\
&= \det(X)\,\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(\mathbb{I}_n + tX^{-1}B) && \text{(Leibniz product rule)} \\
&= (\det(X))\,\mathrm{tr}(X^{-1}B). && \text{(by part (a))}
\end{aligned}$$

Equating them, we obtain $\mathrm{d}(\det)_X(B) = (\det(X))\,\mathrm{tr}\left(X^{-1}B\right)$.

■

Problem 8-1
Prove Lemma 8.6 (the extension lemma for vector fields).

Lemma 8.6. (Extension Lemma for Vector Fields). Let M be a smooth manifold with or without boundary, and let $A \subseteq M$ be a closed subset. Suppose X is a smooth vector field along A. Given any open subset U containing A, there exists a smooth global vector field $\tilde{X}$ on M such that $\left.\tilde{X}\right|_A = X$ and $\mathrm{supp}(\tilde{X}) \subseteq U$.

Solution. We first construct a smooth function $\phi : U \to \mathbb{R}$ with $0 \le \phi(x) \le 1$ for all $x \in U$, such that $\phi(x) = 1$ for $x \in A$ and $\phi(x) = 0$ for x in a neighborhood of the boundary of U. This can be done using a smooth bump function. Notice that ϕ is smooth on U.
We will now extend the vector field X to a vector field $\tilde{X}$ on U.
For each $x \in \partial A$, take V_x to be a coordinate neighborhood containing x and let $e_1, \dots, e_n$ be the standard coordinate vector fields with respect to the coordinates on V_x. In that neighborhood, define a vector field $\tilde{X}_x$ given by

$$\tilde{X}_x(y) = \sum_{i=1}^{n} \phi(y) X_x^i(y) e_i,$$

where X_x^i are extensions of the coordinate components of X (restricted to V_x), satisfying the condition that $\tilde{X}$ agrees with X on $V_x \cap A$ by the extension lemma for smooth functions (Lemma 2.26). Replacing V_x by $V_x \cap U$, we may assume that $V_x \subseteq U$.
The family of sets

$$\big(A \setminus \bigcup_{x\in\partial A} V_x\big) \cup \{V_x \mid x \in \partial A\} \cup \{U \setminus A\}$$

is an open cover of U. Let $\{\theta : U \to \mathbb{R}\} \cup \{\varphi_x : U \to \mathbb{R} \mid x \in \partial A\} \cup \{\psi : U \to \mathbb{R}\}$ be a smooth partition of unity subordinate to this open cover, with $\operatorname{supp}(\theta) \subseteq A \setminus \bigcup_{x\in\partial A} V_x$, $\operatorname{supp}(\varphi_x) \subseteq V_x$, and $\operatorname{supp}(\psi) \subseteq U \setminus A$. Our construction is complete by defining the vector field $\tilde{X}$ on U by

$$\tilde{X}(y) := \sum_{x\in\partial A} \varphi_x(y)\tilde{X}_x(y) + \theta(y)X(y)$$

for all $y \in U$.
Therefore, by definition of the partition of unity, this agrees with X on A, and is smooth on all of U.

■

Problem 8-12
Let $F : \mathbb{R}^2 \to \mathbb{RP}^2$ be the smooth map $F(x, y) = [x, y, 1]$, and let $X \in \mathfrak{X}\left(\mathbb{R}^2\right)$ be defined by $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Prove that there is a vector field $Y \in \mathfrak{X}\left(\mathbb{RP}^2\right)$ that is F-related to X, and compute its coordinate representation in terms of each of the charts defined in Example 1.5.

Solution. Recall that two vector fields X and Y are F-related if $F_*X = Y$, where F_* is the pushforward of F. In other words, we need to find a vector Y on $\mathbb{RP}^2$ such that $F_*X_p = Y_{F(p)}$, for all $p \in \mathbb{R}^2$.
To compute the pushforward F_*X_p, for some point p, we can use the chain rule. Let $f : \mathbb{RP}^2 \to \mathbb{R}$ be a smooth function. Then

$$\begin{aligned}(F_*X_p)(f) &= X_p(f \circ F)\\ &= \frac{\partial(f\circ F)}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial(f\circ F)}{\partial y}\frac{\partial y}{\partial x} - \frac{\partial(f\circ F)}{\partial x}\frac{\partial x}{\partial y} - \frac{\partial(f\circ F)}{\partial y}\frac{\partial y}{\partial y}.\end{aligned}$$

Computing the partial derivatives of $f \circ F$ with respect to x and y, we have

$$\begin{aligned}\frac{\partial(f\circ F)}{\partial x} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x},\\ \frac{\partial(f\circ F)}{\partial y} &= \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}.\end{aligned}$$

Substituting these expressions back, we get

$$(F_*X_p)(f) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}.$$

We can now define the vector field Y on $\mathbb{RP}^2$ by $Y = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$. Then, for any smooth function $f : \mathbb{RP}^2 \to \mathbb{R}$, we have

$$Y_{F(p)}(f) = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = (F_*X_p)(f),$$

which shows that Y is F-related to X.
The charts defined in Example 1.5 are

$$\begin{aligned}\varphi_1 &: U_1 \to \mathbb{R}^2\\ [1, a, b] &\mapsto (a, b).\end{aligned}$$

$$\begin{aligned}\varphi_2 &: U_2 \to \mathbb{R}^2\\ [c, 1, d] &\mapsto (c, d).\end{aligned}$$

$$\begin{aligned}\varphi_3 &: U_3 \to \mathbb{R}^2\\ [x, y, 1] &\mapsto (x, y).\end{aligned}$$

Notice that $\varphi_3^{-1} = F$. Computing the coordinate representation of Y in terms of the charts, we have

$$X = \begin{cases} (1+a^2)\dfrac{\partial}{\partial a} + ab\dfrac{\partial}{\partial b}, & \text{for } U_1 \cap U_3 \\ -(1+c^2)\dfrac{\partial}{\partial c} - cd\dfrac{\partial}{\partial d}, & \text{for } U_2 \cap U_3 \\ x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}, & \text{for } U_3. \end{cases}$$

Therefore, there is a vector field $Y \in \mathfrak{X}\left(\mathbb{RP}^2\right)$ that is F-related to X.

■

Problem 8-13
Show that there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point. [Hint: try using stereographic projection; see Problem 1-7.]

Solution. Let (u, v) and (w, t) be the stereographic coordinates relative to the projection from the north pole and from the south pole, respectively. The maps are then

$$\varphi_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2,$$
$$\varphi_S : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2.$$

The change of coordinates map and its inverse are

$$(u, v) = \varphi_N \circ \varphi_S^{-1}(w, t) = \frac{(w, t)}{w^2 + t^2},$$
$$(w, t) = \varphi_S \circ \varphi_N^{-1}(u, v) = \frac{(u, v)}{u^2 + v^2}.$$

Consider the vector field $\frac{\partial}{\partial u} = \partial_u$ in coordinates (we could also do ∂_v), defined on the domain of φ_N. For some point p in the intersection of the two coordinate charts, $p \in \mathbb{S}^2 \setminus \{N, S\}$, we can compute ∂_u in the stereographic coordinates relative to the projection from the south pole, $(w, t) = \varphi_S \circ \varphi_N^{-1}(u, v)$, given by

$$\begin{aligned} \partial_u &= \left(t^2 - w^2\right)\frac{\partial}{\partial w} - 2wt\frac{\partial}{\partial t} \\ &= \left(t^2 - w^2\right)\partial_w - 2wt\partial_t, \qquad p \in \mathbb{S}^2 \setminus \{N, S\}. \end{aligned}$$

This vector field can be extended at the north pole. Thus,

$$X_p = \begin{cases} \left(\varphi_N^{-1}\right)_* (\partial_u), & p \in \mathbb{S}^2 \setminus \{N\} \\ \left(\varphi_S^{-1}\right)_* \left(t^2 - w^2\right)\partial_w - 2wt\partial_t, & p \in \mathbb{S}^2 \setminus \{S\} \end{cases}$$

X_p is a well-defined vector field on all of $\mathbb{S}^2$ and is smooth. By construction, we have that $X_N = 0$ and $X_p = \partial_u \neq 0$ on $\mathbb{S} \setminus \{N\}$.
Therefore, there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point. ■

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 7.2, 7.13, 7.20, 7-7, 8.9, 8-5, 8.18.

Problem 7.2
Prove Proposition 7.1.

Proposition 7.1. If G is a smooth manifold with a group structure such that the map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$ is smooth, then G is a Lie group.

Solution. To show G is a Lie group, we need to prove that both multiplication and inversion are smooth maps. Let's call the given smooth map $F(g, h) = gh^{-1}$.
First, we'll show the inversion map is smooth. Note that inversion factors as

$$G \xrightarrow{g \mapsto (e,g)} G \times G \xrightarrow{F} G$$

The first map is smooth since it's just inclusion into a product manifold with a constant component. The second map is F which is smooth by assumption. Therefore their composition, the inversion map, is smooth.
Now we can show multiplication is smooth. The multiplication map factors as

$$G \times G \xrightarrow{(g,h) \mapsto (g,h^{-1})} G \times G \xrightarrow{(g,k) \mapsto gk} G$$

The first map is smooth since we just proved inversion is smooth. The second map is smooth because it can be written as a composition

$$G \times G \xrightarrow{(g,k) \mapsto (g,k)} G \times G \xrightarrow{F} G$$

where both maps are smooth.
Therefore G is a Lie group since both multiplication and inversion are smooth maps. ■

Problem 7.13
Given a group G and a subset $S \subseteq G$, show that the subgroup generated by S is equal to the set of all elements of G that can be expressed as finite products of elements of S and their inverses.

Solution. Let X denote the set of all elements of G that can be expressed as finite products of elements of S and their inverses.
We first show that X is a subgroup. The identity is a trivial product with 0 terms, so $e \in X$ If $x, y \in X$, then by definition they are finite products of elements of S and their inverses. The concatenation of these products gives another finite product expressing xy, so $xy \in X$ If $x \in X$ is expressed as a finite product $x = s_1 s_2 \cdots s_n$ where each s_i is either an element of S or an inverse of one, then $x^{-1} = s_n^{-1} \cdots s_2^{-1} s_1^{-1}$ is also such a finite product. So X is closed under inverses. Thus, X is a subgroup of G containing S.
Now let H be any subgroup of G containing S. Then H must contain all inverses of elements of S and must be closed under products. Thus, H must contain all finite products of elements of S and their inverses. In other words, H must contain X.
Therefore, X is precisely the subgroup generated by S since X is the smallest subgroup containing S. ■

Problem 7.20
Let $S \subseteq \mathbb{T}^3$ be the image of the subgroup H of the preceding example under the obvious embedding $\mathbb{T}^2 \hookrightarrow \mathbb{T}^3$. Show that S is a Lie subgroup of $\mathbb{T}^3$ that is not closed, embedded, or dense; but its closure is a properly embedded Lie subgroup of $\mathbb{T}^3$.

Solution. Solved in Study Guide for Quiz 3. ■

Problem 7-7
Prove Proposition 7.15 (properties of the identity component of a Lie group).

Proposition 7.15. Let G be a Lie group and let G_0 be its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.

Solution. Solved in Study Guide for Quiz 3. ∎

Problem 8.9
Prove Proposition 8.8.

Proposition 8.8. Let M be a smooth manifold with or without boundary.

(a) If X and Y are smooth vector fields on M and $f, g \in C^\infty(M)$, then $fX + gY$ is a smooth vector field.

(b) $X_{(M)}$ is a module over the ring $C^\infty(M)$.

Solution. (a) Let (U, ϕ) be a chart on M. If $(x, X(x))$ and $(x, Y(x))$ are the coordinate representations of X and Y respectively in this chart, then $(x, f(x)X(x) + g(x)Y(x))$ is the coordinate representation of $fX + gY$. Since f and g are smooth functions and both X and Y are smooth vector fields, this coordinate representation is smooth. Therefore $fX + gY$ is a smooth vector field by Proposition 8.1.

(b) This follows directly from part (a). We know $\mathfrak{X}(M)$ is an abelian group under addition of vector fields. Part (a) shows it's closed under scalar multiplication by functions in $C^\infty(M)$. The remaining module axioms follow from the properties of multiplication of functions and the distributive property of vector field addition

$$(f + g)X = fX + gXf(X + Y) = fX + fY(fg)X = f(gX)1X = X$$

for all $f, g \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$.

∎

Problem 8-5
Prove Proposition 8.11 (completion of local frames).

Proposition 8.11. (Completion of Local Frames). Let M be a smooth n-manifold with or without boundary.

(a) If $(X_1, \ldots, X_k)$ is a linearly independent k-tuple of smooth vector fields on an open subset $U \subseteq M$, with $1 \le k < n$, then for each $p \in U$ there exist smooth vector fields $X_{k+1}, \ldots, X_n$ in a neighborhood V of p such that $(X_1, \ldots, X_n)$ is a smooth local frame for M on $U \cap V$.

(b) If $(v_1, \ldots, v_k)$ is a linearly independent k-tuple of vectors in T_pM for some $p \in M$, with $1 \le k \le n$, then there exists a smooth local frame (X_i) on a neighborhood of p such that $X_i|_p = v_i$ for $i = 1, \ldots, k$.

(c) If $(X_1, \ldots, X_n)$ is a linearly independent n-tuple of smooth vector fields along a closed subset $A \subseteq M$, then there exists a smooth local frame $\left(\widetilde{X}_1, \ldots, \widetilde{X}_n\right)$ on some neighborhood of A such that $\widetilde{X}_i\Big|_A = X_i$ for $i = 1, \ldots, n$.

Solution. ∎

Problem 8.18
Prove the claim in the preceding example in two ways: directly from the definition, and by using Proposition 8.16.
Let $F : \mathbb{R} \to \mathbb{R}^2$ be the smooth map $F(t) = (\cos t, \sin t)$. Then $\frac{\mathrm{d}}{\mathrm{d}t} \in \mathfrak{X}(\mathbb{R})$ is F-related to the vector field $Y \in \mathfrak{X}\left(\mathbb{R}^2\right)$ defined by

$$Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Solution. Solved in Study Guide for Quiz 3. ■

Homework 8

Problem 9-1
Suppose M is a smooth manifold, $X \in \mathfrak{X}(M)$, and γ is a maximal integral curve of X.

(a) We say γ is periodic if there is a number $T > 0$ such that $\gamma(t+T) = \gamma(t)$ for all $t \in \mathbb{R}$. Show that exactly one of the following holds:

- γ is constant.
- γ is injective.
- γ is periodic and non-constant.

(b) Show that if γ is periodic and non-constant, then there exists a unique positive number T (called the period of $\boldsymbol{\gamma}$) such that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$ for some $k \in \mathbb{Z}$.

(c) Show that the image of γ is an immersed submanifold of M, diffeomorphic to $\mathbb{R}, \mathbb{S}^1$, or $\mathbb{R}^0$.

Solution. (a) We will solve this by exhaustion.

- If the vector field X is zero everywhere, then, considering a specific point p, $X_p = 0 \implies \gamma'(t) = 0$. Thus, γ is constant.
- If the vector field X is non-zero, then γ is non-constant. We now have two cases:
  - If γ is injective, this is the second bullet point.
  - If γ is not injective, then at some time $t_1 > t_0$, we will have that $\gamma(t_1) = \gamma(t_0)$. Since a vector field is non-invariant with respect to its own flow, then γ is periodic.

(b) Let $\gamma : \mathbb{R} \to \mathbb{R}$ be periodic and non-constant. Since γ is periodic, there exists a unique positive number $T > 0$ such that $\gamma(t+T) = \gamma(t)$. Since γ is non-constant, then $\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \neq 0,\ \forall t \in \mathbb{R}$.

Let us consider all the periods T for which we will have $\gamma(t) = \gamma(t')$, from which we can form a set, say $\mathcal{T}$. This set is discrete since γ is non-constant. Additionally, this set is bounded from below since γ is periodic, otherwise, if a smallest positive number T does not exist, then there exists some infimum $\tilde{T}$ of those periods and hence a Cauchy sequence that converges to it. If T_1 and T_2 are in the set, then so is their absolute difference, and since the sequence is Cauchy, the infimum $\tilde{T}$ must be zero. Fix some time t_0 and choose a small enough period $\epsilon > 0$, then $\gamma(t_0)$ is in every neighborhood of $\gamma(t)$, $\forall t$. Pick a neighborhood of $\gamma(t)$ in M, then we can find a period p small enough such taht $t_0 + kp$, for some $k \in \mathbb{Z}$ lies in $(t-\epsilon, t+\epsilon)$. This contradicts the Hausdorff property of the manifold M. Thus, there must exists a smallest positive period T.

All the other periods will be multiples of T, otherwise we can subtract T enough times to find a yet another smaller period, which is not possible since T is the smallest. We know that $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT \in \mathcal{T}$. To show uniqueness, consider another period that is an integer multiple of T, then it would not divide $(t+T) - t$, even though $\gamma(t) = \gamma(t+T)$.

(c)
- If γ is constant, then the image of γ is an immersed submanifold M diffeomorphic to $\mathbb{R}^0$.
- If γ is non-constant and injective, then the vector field is non-zero, which means that $\gamma'(t) \neq 0$. From that, we see that γ has full rank, which is one, the dimension of the domain. Thus, γ is a smooth immersion of the open interval J into M. Thus, the image γ is an immersed submanifold M diffeomorphic to $\mathbb{R}$ due to injectivity.
- If γ is non-constant and periodic, we can image t smoothly looping on a circle with period T. To make this rigorous, we consider the smooth covering map

$$\begin{aligned} \pi : \mathbb{R} &\to \mathbb{S}^1, \\ t &\mapsto \mathrm{e}^{2\pi \mathrm{i} \frac{t}{T}}. \end{aligned}$$

Since γ is constant on the fibers of π, then there must be a map $\tilde{\gamma} : \mathbb{S}^1 \to M$ such that $\tilde{\gamma} \circ \pi = \gamma$. From part (b), $\gamma(t) = \gamma(t')$ if and only if $t - t' = kT$, which means that $\tilde{\gamma}$ is injective. Since π is a local diffeomorphism, then $\tilde{\gamma}$ is a smooth immersion. Thus, the image of γ is an immersed submanifold of M diffeomorphic to $\mathbb{S}^1$.

■

Problem 9-7
Let M be a connected smooth manifold. Show that the group of diffeomorphisms of M acts transitively on M: that is, for any $p, q \in M$, there is a diffeomorphism $F : M \to M$ such that $F(p) = q$. [Hint: first prove that if $p, q \in \mathbb{B}^n$ (the open unit ball in $\mathbb{R}^n$), there is a compactly supported smooth vector field on $\mathbb{B}^n$ whose flow θ satisfies $\theta_1(p) = q$.]

Solution. We start by proving the hint. Let $\mathbb{B}^n$ be the unit ball in $\mathbb{R}^n$ and let $p, q \in \mathbb{B}$. Define

$$L = \{p + t(q - p) \mid 0 \leq t \leq 1\}$$

to be the line connecting p and q. Since $\mathbb{B}^n$ is convex, then $L \subset \mathbb{B}^n$. We now define a constant vector field X on the convex set L by $X_\ell = q - p$. It is clear that X is smooth. From that, there is a smooth vector field $\tilde{X}$ on $\mathbb{B}^n$ such that $\tilde{X}|_L = X$ and $\text{supp}(\tilde{X}) \subset \mathbb{B}^n$. Thus, $\tilde{X}$ is a compactly supported smooth vector field on $\mathbb{B}^n$. We can define γ as

$$\gamma(t) = p + t(q - p)$$

to be the integral curve of $\tilde{X}$ since $\gamma'(t) = q - p = \tilde{X}_{\gamma(t_0)}$, $\forall t_0 \in [0, 1]$. Therefore, if θ is the flow of $\tilde{X}$, then the flow satisfies $\theta_1(p) = \gamma(1) = q$.
Now, let M be a connected smooth manifold and, for a fixed $p \in M$, define U_p to be the orbit of p under the action given, giving us

$$U_p = \{q \in M \mid \exists \text{ diffeomorphism } F : M \to M \text{ such that } F(p) = q\}.$$

- **Non-empty**: The identity on M is a diffeomorphism, so $p \in U_p$. Thus, U_p is non-empty.
- **Open**: Let $q \in U_p$. We need to show that there is a neighborhood of q contained in U_p. Since $q \in U_p$, there exists a diffeomorphism F such that $F(p) = q$. Since M is a smooth manifold, there exists a coordinate chart (V, φ) around q where V is open and $\varphi : V \to \mathbb{B}^n$ is a diffeomorphism onto its image. By the hint, we proved, for any point $r \in V$, there exists a compactly supported smooth vector field X on $\mathbb{B}^n$ whose flow θ_t takes $\varphi(q)$ to $\varphi(r)$ at time 1. Pull this vector field back to V via φ to get a compactly supported vector field $Y = (\varphi^{-1})^* X$ on V. The flow ψt of Y then takes q to r at time 1. The composition $\psi_1 \circ F$ is a diffeomorphism taking p to r. Then, $V \subseteq U_p$. Thus, U_p is open.
- **Closed**: Let q be a limit point of U_p. We need to show $q \in U_p$. Take a coordinate chart (V, φ) around q as before. Since q is a limit point, there exists some $r \in U_p \cap V$. By using the hint, we can construct a diffeomorphism taking r to q. If G is the diffeomorphism taking p to r, then composing these diffeomorphisms gives us one taking p to q. Then, $q \in U_p$. Thus, U_p is closed.

Thus, U_p is non-empty, open, and closed. Since M is connected, then $U_p = M$, and hence, this is only one orbit. Therefore, the action is transitive. ■

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 8.29, 8.34, 8.35, 8.43, 9.5, 9.37, 9.40, 9-5.

Problem 8.29
Prove part (d) of Proposition 8.28.

Proposition 8.28. (Properties of the Lie Bracket). The Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:

(d) For $f, g \in C^\infty(M)$,
$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

Solution. Solved in Study Guide for Quiz 3. ∎

Problem 8.34
Verify that the kernel and image of a Lie algebra homomorphism are Lie subalgebras.

Solution. Let $A : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism.

First, let's show $\ker(A)$ is a Lie subalgebra of $\mathfrak{g}$. Let $X, Y \in \ker A$. Then

$$A([X,Y]) = [A(X), A(Y)] = [0,0] = 0,$$

where the first equality uses that A is a Lie algebra homomorphism. Therefore $[X,Y] \in \ker(A)$, showing the kernel is closed under the Lie bracket. Since $\ker(A)$ is also a vector subspace, it is a Lie subalgebra.

Next, let's show $\operatorname{im}(A)$ is a Lie subalgebra of $\mathfrak{h}$. Let $V, W \in \operatorname{im}(A)$, then there exist $X, Y \in \mathfrak{g}$ such that $A(X) = V$ and $A(Y) = W$. Therefore

$$[V,W] = [A(X), A(Y)] = A([X,Y]),$$

showing $[V,W] \in \operatorname{im}(A)$. Since $\operatorname{im}(A)$ is also a vector subspace, it is a Lie subalgebra. ∎

Problem 8.35
Suppose $\mathfrak{g}$ and $\mathfrak{h}$ are finite-dimensional Lie algebras and $A : \mathfrak{g} \to \mathfrak{h}$ is a linear map. Show that A is a Lie algebra homomorphism if and only if $A\,[E_i, E_j] = [AE_i, AE_j]$ for some basis $(E_1, \ldots, E_n)$ of g.

Solution. $\boxed{\Longrightarrow}$ If A is a Lie algebra homomorphism, then by definition we have $A[X,Y] = [AX, AY]$ for all $X, Y \in \mathfrak{g}$. Therefore this holds in particular for any basis vectors E_i, E_j.

$\boxed{\Longleftarrow}$ Suppose $A[E_i, E_j] = [AE_i, AE_j]$ for some basis $(E_1, \ldots, E_n)$. Let $X, Y \in \mathfrak{g}$ be arbitrary vectors. We can write them in terms of the basis as

$$X = \sum_{i=1}^n \alpha_i E_i, \quad Y = \sum_{j=1}^n \beta_j E_j.$$

Then using bilinearity of the Lie bracket and linearity of A

$$\begin{aligned}
[AX, AY] &= \left[A\left(\sum_{i=1}^{n} \alpha_i E_i\right), A\left(\sum_{j=1}^{n} \beta_j E_j\right)\right] \\
&= \left[\sum_{i=1}^{n} \alpha_i AE_i, \sum_{j=1}^{n} \beta_j AE_j\right] \\
&= \sum_{i,j=1}^{n} \alpha_i\beta_j [AE_i, AE_j] \\
&= \sum_{i,j=1}^{n} \alpha_i\beta_j A[E_i, E_j] \\
&= A\left(\sum_{i,j=1}^{n} \alpha_i\beta_j [E_i, E_j]\right) \\
&= A[X, Y].
\end{aligned}$$

Therefore, A is a Lie algebra homomorphism. ■

Problem 8.43
Prove Corollary 8.41 by choosing a basis for V and applying Proposition 8.41.

Corollary 8.41. If V is any finite-dimensional real vector space, the composition of the canonical isomorphisms in (8.16) yields a Lie algebra isomorphism between $\mathrm{Lie}(\mathrm{GL}(V))$ and $\mathfrak{gl}(V)$.

Solution. Choosing a basis for V determines an isomorphism $V \to \mathbb{R}^n$, which induces a vertical Lie algebra isomorphism in the following commutative diagram

$$\begin{array}{ccccc}
\mathrm{Lie}(\mathrm{GL}(V)) & \longrightarrow & T_{\mathrm{id}}\mathrm{GL}(V) & \longrightarrow & \mathfrak{gl}(V) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Lie}(\mathrm{GL}(n,\mathbb{R})) & \longrightarrow & T_{I_n}\mathrm{GL}(n,\mathbb{R}) & \longrightarrow & \mathfrak{gl}(n,\mathbb{R})
\end{array}$$

The diagram commutes by naturality of the horizontal maps. The bottom maps are all isomorphisms by Proposition 8.41. Since the vertical maps are also isomorphisms, we conclude that the top maps are isomorphisms as well.

Therefore, the composition of the maps in (8.16) gives a Lie algebra isomorphism between Lie(GL(V)) and $\mathfrak{gl}(V)$. ■

Problem 9.5
Prove the translation lemma.

Lemma 9.4. (Translation Lemma). Let V, M, J, and γ be as in the preceding lemma. For any $b \in \mathbb{R}$, the curve $\hat{\gamma} : \widehat{J} \to M$ defined by $\widehat{\gamma}(t) = \gamma(t+b)$ is also an integral curve of V, where $\widehat{J} = \{t : t+b \in J\}$.

Solution. Let $p \in \widehat{J}$ be arbitrary and let (U, ϕ) be a chart containing $\hat{\gamma}(p)$. Then in coordinates:

$$\begin{aligned}
\mathrm{d}\hat{\gamma}p\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=p} \phi(\hat{\gamma}(t)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=p} \phi(\gamma(t+b)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=p+b} \phi(\gamma(s)) \cdot \left.\frac{\mathrm{d}}{\mathrm{d}t}(t+b)\right|_{t=p} \\
&= \mathrm{d}\gamma p + b\left(\frac{\mathrm{d}}{\mathrm{d}s}\right) \\
&= V_{\gamma(p+b)} \\
&= V_{\hat{\gamma}(p)},
\end{aligned}$$

where we used that γ is an integral curve of V in the second to last line. Since p was arbitrary, this shows that $\hat{\gamma}$ is also an integral curve of V. ∎

Problem 9.37
Suppose $v \in \mathbb{R}^n$ and W is a smooth vector field on an open subset of $\mathbb{R}^n$. Show that the directional derivative $D_v W(p)$ defined by (9.15) is equal to $(\mathscr{L}_V W)_p$, where V is the vector field $V = v^i \frac{\partial}{\partial x^i}$ with constant coefficients in standard coordinates.

Solution. Solved in Study Guide for Quiz 3. ∎

Problem 9.40
Prove Corollary 9.39.

Corollary 9.39. Suppose M is a smooth manifold with or without boundary, and $V, W, X \in \mathfrak{X}(M)$.

(a) $\mathscr{L}_V W = -\mathscr{L}_W V$.

(b) $\mathscr{L}_V [W, X] = [\mathscr{L}_V W, X] + [W, \mathscr{L}_V X]$.

(c) $\mathscr{L}_{[V,W]} X = \mathscr{L}_V \mathscr{L}_W X - \mathscr{L}_W \mathscr{L}_V X$.

(d) If $g \in C^\infty(M)$, then $\mathscr{L}_V(gW) = (Vg)W + g\mathscr{L}_V W$.

(e) If $F : M \to N$ is a diffeomorphism, then $F_*(\mathscr{L}_V X) = \mathscr{L}_{F_* V} F_* X$.

Solution. (a) From Proposition 8.28, we know that $[V, W] = -[W, V]$. From Proposition 9.38, we know that $\mathscr{L}_V W = [V, W]$. Thus, we have

$$\mathscr{L}_V W = [V, W] = -[W, V] = -\mathscr{L}_W V.$$

(b) Using the Jacobi identity and Proposition 9.38, we have

$$\begin{aligned}
\mathscr{L}_V [W, X] &= [V, [W, X]] \\
&= [[V, W], X] + [W, [V, X]] \\
&= [\mathscr{L}_V W, X] + [W, \mathscr{L}_V X].
\end{aligned}$$

(c) Using Proposition 9.38, we have

$$\begin{aligned}
\mathscr{L}_{[V,W]} X &= [[V, W], X] \\
&= [V, [W, X]] - [W, [V, X]] \\
&= \mathscr{L}_V \mathscr{L}_W X - \mathscr{L}_W \mathscr{L}_V X.
\end{aligned}$$

(d) Using Proposition 8.28, we have

$$\begin{aligned}\mathscr{L}_V(gW) &= [V, gW] \\ &= (Vg)W + g[V, W] \\ &= (Vg)W + g\mathscr{L}_V W.\end{aligned}$$

(e) Let's use Proposition 8.19, which states that if F is a diffeomorphism and X, Y are F-related vector fields, then $[X, Y]$ and $[F_X, F_Y]$ are F-related. Thus, we have

$$\begin{aligned}F_(\mathscr{L}_V X) &= F[V, X] \\ &= [F_V, F_X] \\ &= \mathscr{L}_{FV} F_X.\end{aligned}$$

■

Problem 9-5
Suppose M is a smooth, compact manifold that admits a nowhere vanishing smooth vector field. Show that there exists a smooth map $F : M \to M$ that is homotopic to the identity and has no fixed points.

Solution. Let V be a nowhere vanishing vector field on M. Since M is compact, by Theorem 9.20 (the flow box theorem), V generates a global flow $\varphi : \mathbb{R} \times M \to M$.
Let's define $F : M \to M$ by $F(p) = \varphi(1, p)$. Then F is smooth as it's the time-1 map of the flow. We can construct a homotopy $H : [0, 1] \times M \to M$ from the identity to F by

$$H(t, p) = \varphi(t, p).$$

We can see that H is a smooth map and

- When $t = 0$, $H(0, p) = \varphi(0, p) = p$ (the identity map).
- When $t = 1$, $H(1, p) = \varphi(1, p) = F(p)$.

Now we need to show F has no fixed points. Suppose, for contradiction, that p is a fixed point of F. Then

$$p = F(p) = \varphi(1, p)$$

By the properties of flows, this means the integral curve through p is periodic with period 1. But then V_p must be zero (as the curve stays at p), contradicting the fact that V is nowhere vanishing. Therefore, F has no fixed points. ■

Homework 9

Problem 10-10
Suppose M is a compact smooth manifold and $E \to M$ is a smooth vector bundle of rank k. Use transversality to prove that E admits a smooth section σ with the following property: if $k > \dim(M)$, then σ is nowhere vanishing; while if $k \leq \dim(M)$, then the set of points where σ vanishes is a smooth compact codimension-k submanifold of M. Use this to show that M admits a smooth vector field with only finitely many singular points.

Solution. Let M be a compact smooth manifold of dimension m and let $E \to M$ be a smooth vector bundle of rank k.
First, note that E always admits a zero section, which we'll call $\xi : M \to E$. Let $M_0 := \xi(M) \subseteq E$ denote the image of this zero section, which is an embedded k-codimensional submanifold of E.
By the Homotopy Transversality Theorem, there exists a homotopy $H : M \times [0,1] \to E$ such that:

(i) $H(\cdot, 0) = \xi(\cdot)$

(ii) $H(\cdot, 1)$ is transverse to M_0

Let $\sigma := H(\cdot, 1) : M \to E$ be our section that is transverse to M_0. We now consider two cases:

- **For** $k > \dim(M)$:

  In this case, dimensional considerations show that σ must be nowhere vanishing. Indeed, if σ vanishes at a point, this would create a transverse intersection between $\sigma(M)$ and M_0, but

  $$\dim(\sigma(M)) + \dim(M_0) - \dim(E) = m + m - (m + k) = 2m - (m + k) < 0.$$

  making such an intersection impossible.

- **For** $k \leq \dim(M)$:

  Let's analyze the set of points where σ vanishes. Note that

  $$\begin{aligned} \dim(E) &= k + m, \\ \operatorname{codim}(M_0) &= k, \\ \operatorname{codim}(\sigma(M)) &= k. \end{aligned}$$

  Since σ is transverse to M_0, their intersection $M_0 \cap \sigma(M)$ is a smooth submanifold with:

  $$\operatorname{codim}(M_0 \cap \sigma(M)) = \operatorname{codim}(M_0) + \operatorname{codim}(\sigma(M)) = k + k = 2k$$

Therefore, $M_0 \cap \sigma(M)$ is a smooth compact codimension-k submanifold of M.
To prove the final statement about vector fields, apply this result to the tangent bundle TM. Since $\dim(TM) = \dim(M)$, we get a vector field with zeros forming a 0-dimensional submanifold. By compactness of M, this must be a finite set of points, giving us a vector field with finitely many singular points.

■

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 10.1, 10.9, 10.11, 10.14, 10.27, 10.31.

Problem 10.1
Suppose E is a smooth vector bundle over M. Show that the projection map $\pi : E \to M$ is a surjective smooth submersion.

Solution. First, we need to show that π is surjective. This follows directly from the vector bundle axioms - each fiber $E_p = \pi^{-1}(p)$ is nonempty as it's a vector space, so every point in M has a preimage.
Now let's show π is a smooth submersion. Let $v \in E$ be arbitrary and let $p = \pi(v)$. We need to show that $\mathrm{d}\pi_v$ is surjective. Let (U, ϕ) be a local trivialization around p, so we have a diffeomorphism

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^k,$$

where k is the rank of E. Let $\mathrm{pr}_1 : U \times \mathbb{R}^k \to U$ be the projection onto the first factor. Then we have a commutative diagram:

$$\pi^{-1}(U) \xrightarrow{\phi} U \times \mathbb{R}^k \xrightarrow{\mathrm{pr}_1} U$$

and $\pi|_{\pi^{-1}(U)} = \mathrm{pr}_1 \circ \phi$. By the chain rule, we have

$$\mathrm{d}\pi_v = \mathrm{d}(\mathrm{pr1})\phi(v) \circ \mathrm{d}\phi_v.$$

Now, $\mathrm{d}\phi_v$ is an isomorphism since ϕ is a diffeomorphism. And $\mathrm{d}(\mathrm{pr1})\phi(v)$ is surjective since projection maps are submersions. Thus, $\mathrm{d}\pi_v$ is surjective. Therefore, π is a smooth submersion since v was arbitrary. ■

Problem 10.9
Show that the zero section of every vector bundle is continuous, and the zero section of every smooth vector bundle is smooth. [Hint: consider $\Phi \circ \zeta$, where Φ is a local trivialization.]

Solution. Let $\pi : E \to M$ be a vector bundle of rank k and let $\zeta : M \to E$ be its zero section, defined by $\zeta(p) = 0_p$ where 0_p is the zero vector in the fiber $\pi^{-1}(p)$.
First, let's prove continuity. Let $p \in M$ and let (U, Φ) be a local trivialization around p. Then we have

$$\Phi \circ \zeta|_U : U \to U \times \mathbb{R}^k.$$

For any $q \in U$, we can write $\Phi \circ \zeta(q) = (q, 0)$ where 0 is the zero vector in $\mathbb{R}^k$. This is clearly continuous since both component functions are continuous.
Since Φ is a homeomorphism and the composition of continuous functions is continuous, $\zeta|_U$ is continuous. As this holds for every point $p \in M$, we conclude that ζ is continuous.
For smoothness, assume now that $\pi : E \to M$ is a smooth vector bundle. Then Φ is a diffeomorphism and the map $q \mapsto (q, 0)$ is smooth. The composition of smooth functions is smooth, so $\zeta|_U$ is smooth.
Since smoothness is a local property and we've shown ζ is smooth in a neighborhood of each point, we conclude that ζ is smooth. ■

Problem 10.11
Let $E \to M$ be a smooth vector bundle.

(a) Show that if $\sigma, \tau \in \Gamma(E)$ and $f, g \in C^\infty(M)$, then $f\sigma + g\tau \in \Gamma(E)$.

(b) Show that $\Gamma(E)$ is a module over the ring $C^\infty(M)$.

Solution. (a) Let's prove that $f\sigma + g\tau$ is a smooth section. First, we verify it is a section. For any $p \in M$ we have

$$\pi(f\sigma + g\tau)(p) = \pi\big(f(p)\sigma(p) + g(p)\tau(p)\big) = p$$

since $\sigma(p)$ and $\tau(p)$ are in the fiber over p and the fiber is a vector space. For smoothness, let (U, Φ) be a local trivialization. Then we have

$$\Phi \circ (f\sigma + g\tau) = \Big(p,\ f(p)v(p) + g(p)w(p)\Big)$$

where v, w are the local representations of σ, τ respectively. Since f, g, v, w are all smooth and addition and multiplication of smooth functions are smooth, we conclude that $f\sigma + g\tau$ is smooth.

(b) Now we show $\Gamma(E)$ is a module over $C^\infty(M)$. From part (a), $\Gamma(E)$ is closed under scalar multiplication by elements of $C^\infty(M)$ and addition. The remaining module axioms follow directly from the vector space structure of each fiber:

$$\begin{aligned}(f+g)\sigma &= f\sigma + g\sigma\\ (fg)\sigma &= f(g\sigma)\\ 1\sigma &= \sigma\\ f(\sigma+\tau) &= f\sigma + f\tau\end{aligned}$$

for all $f, g \in C^\infty(M)$ and $\sigma, \tau \in \Gamma(E)$. Therefore $\Gamma(E)$ is a module over $C^\infty(M)$.

■

Problem 10.14
Let $\pi : E \to M$ be a smooth vector bundle. Show that each element of E is in the image of a smooth global section.

Solution. Let $e \in E$ and let $p = \pi(e)$. Let (U, Φ) be a local trivialization containing p and write

$$\Phi(e) = (p, v),$$

where $v \in \mathbb{R}^k$ and k is the rank of E. Let f be a smooth bump function supported in U with $f(p) = 1$. Define a local section $\sigma_U : U \to \pi^{-1}(U)$ by

$$\sigma_U(q) = \Phi^{-1}(q,\ f(q)v).$$

This is smooth since Φ^{-1} is smooth and f is smooth. Moreover, $\sigma_U(p) = e$. Now we can extend σ_U to a global section σ by setting it to zero outside U. Specifically, define

$$\sigma(q) = \begin{cases}\sigma_U(q) & \text{if } q \in U\\ 0_q & \text{if } q \in M \setminus U\end{cases}$$

This extension is smooth since σ_U is supported in U (as f is supported in U) and the zero section is smooth. Therefore σ is a smooth global section with $\sigma(p) = e$. ■

Problem 10.27
Show that a smooth rank-k vector bundle over M is smoothly trivial if and only if it is smoothly isomorphic over M to the product bundle $M \times \mathbb{R}^k$.

Solution. $\boxed{\Longrightarrow}$ Suppose $\pi : E \to M$ is smoothly trivial. Then by definition there exists a smooth global frame $\{\sigma_1, \ldots, \sigma_k\}$ for E. Define a map

$$F : E \to M \times \mathbb{R}^k$$

by expressing each $e \in E$ uniquely in terms of the frame:

$$F(e) = (\pi(e),\ v_1,\ \ldots,\ v_k),$$

where $e = \sum_{i=1}^{k} v_i\sigma_i(\pi(e))$. This map is smooth: in any local trivialization, the coordinates v_i are smooth functions of e since the frame sections are smooth. The inverse map

$$F^{-1} : M \times \mathbb{R}^k \to E$$

given by

$$F^{-1}(p,\, v_1,\, \dots,\, v_k) = \sum_{i=1}^{k} v_i\sigma_i(p)$$

is also smooth since the σ_i are smooth. Moreover, F preserves fibers since $\pi(e) = \pi_1(F(e))$ where π_1 is the projection onto M. Therefore F is a smooth vector bundle isomorphism.
$\boxed{\Longleftarrow}$ Suppose $F : E \to M \times \mathbb{R}^k$ is a smooth vector bundle isomorphism over M. For each $i = 1, \dots, k$, define $\sigma_i : M \to E$ by

$$\sigma_i(p) = F^{-1}(p, e_i),$$

where e_i is the ith standard basis vector of $\mathbb{R}^k$. These sections are smooth since F^{-1} is smooth. They form a frame since F is a vector bundle isomorphism, meaning it restricts to a linear isomorphism on each fiber, and $\{e_1,\, \dots,\, e_k\}$ is a basis for $\mathbb{R}^k$. Therefore E is smoothly trivial. ■

Problem 10.31
Given a smooth vector bundle $E \to M$ and a smooth subbundle $D \subseteq E$, show that the inclusion map $\iota : D \hookrightarrow E$ is a smooth bundle homomorphism over M.

Solution. To show that ι is a bundle homomorphism over M, it must satisfy

- $\pi_E \circ \iota = \pi_D$ where π_E, π_D are the respective bundle projections, and
- For each $p \in M$, the restriction $\iota_p : D_p \to E_p$ is linear.

The first condition follows immediately since ι is the inclusion map

$$\pi_E(\iota(d)) = \pi_E(d) = \pi_D(d),$$

for all $d \in D$. The second condition holds since each fiber D_p is a vector subspace of E_p, so the inclusion map is automatically linear. Now we need to prove that ι is smooth. Let $d \in D$ and $p = \pi_D(d)$. Let (U, Φ) be a local trivialization for E around p in which D is a slice. This means

$$\Phi(D \cap \pi_E^{-1}(U)) = U \times (\mathbb{R}^k \times 0),$$

where k is the rank of D. Then $\Phi \circ \iota$ maps $D \cap \pi_D^{-1}(U)$ into $U \times \mathbb{R}^n$ (where n is the rank of E) by

$$\Phi \circ \iota(d) = (p,\, v,\, 0),$$

where $v \in \mathbb{R}^k$. This map is smooth since it's just the inclusion of coordinates. Since smoothness is a local property and we can cover D with such charts, we conclude ι is smooth. ■

Homework 10

Problem 9-4
For any integer $n \geq 1$, define a flow on the odd-dimensional sphere $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ by $\theta(t, z) = e^{it}z$. Show that the infinitesimal generator of θ is a smooth non-vanishing vector field on $\mathbb{S}^{2n-1}$. [Remark: in the case $n = 2$, the integral curves of X are the curves γ_z of Problem 3-6, so this provides a simpler proof that each γ_z is smooth.]

Solution. Let's compute the infinitesimal generator X of the flow θ. Recall that for a flow, the infinitesimal generator at a point p is given by

$$X_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \theta(t, p).$$

For $z = (z_1, \ldots, z_n) \in \mathbb{S}^{2n-1}$, we have

$$\begin{aligned}\theta(t, z) &= \mathrm{e}^{\mathrm{i}t} z \\ &= \mathrm{e}^{\mathrm{i}t}(z_1, \ldots, z_n) \\ &= (\mathrm{e}^{\mathrm{i}t} z_1, \ldots, \mathrm{e}^{\mathrm{i}t} z_n).\end{aligned}$$

To compute X_z, we differentiate with respect to t at $t = 0$

$$\begin{aligned}X_z &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{e}^{\mathrm{i}t} z_1, \ldots, \mathrm{e}^{\mathrm{i}t} z_n) \\ &= \left(\mathrm{i}\mathrm{e}^{\mathrm{i}t} z_1, \ldots, \mathrm{i}\mathrm{e}^{\mathrm{i}t} z_n\right)\big|_{t=0} \\ &= (\mathrm{i}z_1, \ldots, \mathrm{i}z_n) \\ &= \mathrm{i}z.\end{aligned}$$

- **Smoothness:**

  The map $z \mapsto \mathrm{i}z$ is clearly $\mathbb{R}$-linear. All components are complex-valued multiplication, which is smooth.

  Thus, X is a smooth vector field on $\mathbb{S}^{2n-1}$.

- **Non-vanishing:**

  To show X is non-vanishing, let $z \in \mathbb{S}^{2n-1}$, we have

  $$\begin{aligned}|X_z| &= |\mathrm{i}z| \\ &= |\mathrm{i}| \cdot |z| \\ &= |z| \\ &= 1.\end{aligned}$$

  The last equality follows since $z \in \mathbb{S}^{2n-1}$. Equivalently, we can show the latter by noticing that $X_z = 0$ only if $z = 0$, but $0 \notin \mathbb{S}^{2n-1}$.

  Thus, $X_z \neq 0$ for all $z \in \mathbb{S}^{2n-1}$ and has constant length 1 everywhere, so X is a non-vanishing vector field.

Therefore, the infinitesimal generator of θ is a smooth non-vanishing vector field on $\mathbb{S}^{2n-1}$.

■

Problem 10-12
Let $\pi : E \to M$ and $\widetilde{\pi} : \widetilde{E} \to M$ be two smooth rank-k vector bundles over a smooth manifold M with or without boundary. Suppose $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M such that both E and $\widetilde{E}$ admit smooth local trivializations over each U_α. Let $\{\tau_{\alpha\beta}\}$ and $\{\widetilde{\tau}_{\alpha\beta}\}$ denote the transition functions determined by the given local trivializations of E and $\widetilde{E}$, respectively. Show that E and $\widetilde{E}$ are smoothly isomorphic

over M if and only if for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ such that

$$\tilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}, \quad p \in U_\alpha \cap U_\beta.$$

Solution. $\boxed{\Longrightarrow}$ Suppose that E and $\widetilde{E}$ are smoothly isomorphic. Then there exists a smooth bundle isomorphism $F : E \to \widetilde{E}$ covering the identity on M. Let $\{\phi_\alpha\}$ and $\{\widetilde{\phi}_\alpha\}$ be the local trivializations of E and $\widetilde{E}$ respectively over $\{U_\alpha\}$.
For each $\alpha \in A$, let $\mathrm{pr}_2 : U_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$ denote the projection onto the second factor, i.e., $\mathrm{pr}_2(p, v) = v$. Define $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ by

$$\sigma_\alpha(p) = \mathrm{pr}_2 \circ \widetilde{\phi}_\alpha \circ F \circ \phi_\alpha^{-1}(p, \cdot)$$

This map is well-defined because $\widetilde{\phi}_\alpha \circ F \circ \phi_\alpha^{-1}$ maps (p, v) to (p, w) for some $w \in \mathbb{R}^k$, and pr_2 extracts this w. Since F is a smooth bundle isomorphism, each σ_α is smooth and invertible.
For $p \in U_\alpha \cap U_\beta$, we have

$$\begin{aligned}
\widetilde{\tau}_{\alpha\beta}(p) &= \widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}(p, \cdot) \\
&= \widetilde{\phi}_\alpha \circ F \circ F^{-1} \circ \widetilde{\phi}_\beta^{-1}(p, \cdot) \\
&= (\widetilde{\phi}_\alpha \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi_\beta^{-1}) \circ (\phi_\beta \circ F^{-1} \circ \widetilde{\phi}_\beta^{-1})(p, \cdot) \\
&= \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}.
\end{aligned}$$

Therefore, for each $\alpha \in A$ there exists a smooth map $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ such that

$$\tilde{\tau}_{\alpha\beta}(p) = \sigma_\alpha(p)\tau_{\alpha\beta}(p)\sigma_\beta(p)^{-1}, \quad p \in U_\alpha \cap U_\beta.$$

$\boxed{\Longleftarrow}$ Suppose we have smooth maps $\sigma_\alpha : U_\alpha \to \mathrm{GL}(k, \mathbb{R})$ satisfying the given relation. Define $F : E \to \widetilde{E}$ locally by

$$F|_{\pi^{-1}(U_\alpha)} = \widetilde{\phi}_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha} \times \sigma_\alpha) \circ \phi_\alpha$$

To show this defines a global bundle isomorphism, we need to verify that these local definitions agree on overlaps. For $p \in U_\alpha \cap U_\beta$:

$$\begin{aligned}
\widetilde{\phi}_\beta \circ F \circ \phi_\alpha^{-1}(p, v) &= (\mathrm{id}_{U_\beta} \times \sigma_\beta)(p,\, \tau_{\beta\alpha}(p)v) \\
&= (p,\, \sigma_\beta(p)\tau_{\beta\alpha}(p)v) \\
&= (p,\, \widetilde{\tau}_{\beta\alpha}(p)\sigma_\alpha(p)v) \\
&= \widetilde{\tau}_{\beta\alpha}(p)(p,\, \sigma_\alpha(p)v) \\
&= \widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}(p,\, \sigma_\alpha(p)v).
\end{aligned}$$

Thus, F is well-defined and each σ_α is smooth and invertible.
Therefore, F is a smooth bundle isomorphism covering the identity on M.

■

Problem 11-5
For any smooth manifold M; show that T^*M is a trivial vector bundle if and only if TM is trivial.

Solution. We will proceed by proving both directions at the same time.

$$\begin{aligned}
T^*M \text{ is a trivial vector bundle} &\iff T^*M \cong M \times \mathbb{R}^n \\
&\iff \text{there exists a global frame of 1-forms } \{\omega_1, \omega_2, \dots, \omega_n\} \\
&\iff \text{there exists a dual frame of vector fields } \{X_1, X_2, \dots, X_n\} \text{ where } \omega_i(X_i) = \delta_{ij} \\
&\iff TM \cong M \times \mathbb{R}^n \\
&\iff TM \text{ is a trivial vector bundle.}
\end{aligned}$$

■

Problem 11-16
Let M be a compact manifold of positive dimension. Show that every exact covector field on M vanishes at least at two points in each component of M.

Solution. Let M be a compact manifold of positive dimension and let ω be an exact covector field on M, then there exists a map $f \in C^\infty(M)$ such that $\omega = \mathrm{d}f$. The form $\mathrm{d}f$ vanishes at every global extremum of f, of which there exists at least two. ■

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 11.2, 11.3, 11.5, 11.7, 11.10, 11.12, 11.32, 11.35, 11.41.

Problem 11.2
Prove Proposition 11.1.

Proposition 11.1. Let V be a finite-dimensional vector space. Given any basis $(E_1, \ldots, E_n)$ for V, let $\varepsilon^1, \ldots, \varepsilon^n \in V^*$ be the covectors defined by

$$\varepsilon^i \left(E_j\right) = \delta^i_j,$$

where δ^i_j is the Kronecker delta symbol defined by (4.4). Then $\left(\varepsilon^1, \ldots, \varepsilon^n\right)$ is a basis for V^*, called the dual basis to $(\boldsymbol{E_j})$. Therefore, $\dim V^* = \dim V$.

Solution. First, we'll show that $(\varepsilon^1, \ldots, \varepsilon^n)$ spans V^*. Let $\omega \in V^*$ be arbitrary. Define

$$\eta = \sum_{i=1}^{n} \omega(E_i)\varepsilon^i.$$

We claim $\eta = \omega$. To verify this, let's evaluate both on an arbitrary basis vector E_j. We have

$$\begin{aligned}\eta(E_j) &= \sum_{i=1}^{n} \omega(E_i)\varepsilon^i(E_j) \\ &= \sum_{i=1}^{n} \omega(E_i)\delta^i_j \\ &= \omega(E_j).\end{aligned}$$

Since η and ω agree on a basis, they are equal as linear functionals. Thus $(\varepsilon^1, \ldots, \varepsilon^n)$ spans V^*.
Now we'll show linear independence. Suppose

$$\sum_{i=1}^{n} a_i\varepsilon^i = 0,$$

for some scalars a_i. Evaluating this on E_j gives

$$\begin{aligned}0 &= \sum_{i=1}^{n} a_i\varepsilon^i(E_j) \\ &= \sum_{i=1}^{n} a_i\delta^i_j \\ &= a_j.\end{aligned}$$

Therefore all $a_i = 0$, proving linear independence.
Since we've found a basis for V^* with the same number of elements as the dimension of V, we conclude that $\dim(V^*) = \dim(V)$. ■

Problem 11.3
Show that $A^*\omega$ is actually a linear functional on V, and that A^* is a linear map.

Solution. First, let's show that for any $\omega \in W^*$, $A^*\omega$ is a linear functional on V. Let $v_1, v_2 \in V$ and $c \in \mathbb{R}$, then

$$\begin{aligned} A^*\omega(v_1 + v_2) &= \omega(A(v_1 + v_2)) \\ &= \omega(A(v_1) + A(v_2)) \quad \text{(since } A \text{ is linear)} \\ &= \omega(A(v_1)) + \omega(A(v_2)) \quad \text{(since } \omega \text{ is linear)} \\ &= A^*\omega(v_1) + A^*\omega(v_2). \end{aligned}$$

For scalar multiplication, we have

$$\begin{aligned} A^*\omega(cv_1) &= \omega(A(cv_1)) \\ &= \omega(cA(v_1)) \quad \text{(since } A \text{ is linear)} \\ &= c\omega(A(v_1)) \quad \text{(since } \omega \text{ is linear)} \\ &= cA^*\omega(v_1). \end{aligned}$$

Therefore $A^*\omega$ is indeed a linear functional on V.
Now let's show A^* is linear. Let $\omega_1, \omega_2 \in W^*$ and $c \in \mathbb{R}$. For any $v \in V$, we have

$$\begin{aligned} A^*(\omega_1 + \omega_2)(v) &= (\omega_1 + \omega_2)(A(v)) \\ &= \omega_1(A(v)) + \omega_2(A(v)) \\ &= A^*\omega_1(v) + A^*\omega_2(v) \\ &= (A^*\omega_1 + A^*\omega_2)(v) \end{aligned}$$

For scalar multiplication, we have

$$\begin{aligned} A^*(c\omega_1)(v) &= (c\omega_1)(A(v)) \\ &= c\omega_1(A(v)) \\ &= c(A^*\omega_1)(v) \\ &= (cA^*\omega_1)(v) \end{aligned}$$

Since these equalities hold for all $v \in V$, we conclude that A^* is linear. ∎

Problem 11.5
Prove Proposition 11.4.

Proposition 11.4. The dual map satisfies the following properties

(a) $(A \circ B)^* = B^* \circ A^*$.

(b) $(\mathrm{Id}_V)^* : V^* \to V^*$ is the identity map of V^*.

Solution. (a) Let ω be in the domain of A^* and let v be in the domain of B. Then

$$\begin{aligned} ((A \circ B)^*\omega)(v) &= \omega((A \circ B)(v)) \\ &= \omega(A(B(v))) \\ &= (A^*\omega)(B(v)) \\ &= (B^*(A^*\omega))(v). \end{aligned}$$

Since these are equal for all v, we have $(A \circ B)^* = B^* \circ A^*$.

(b) Let $\omega \in V^*$ and $v \in V$. Then

$$\begin{aligned} ((\mathrm{Id}_V)^*\omega)(v) &= \omega(\mathrm{Id}_V(v)) \\ &= \omega(v) \\ &= (\mathrm{Id}_{V^*}\omega)(v). \end{aligned}$$

Since these are equal for all v, we have $(\mathrm{Id}_V)^* = \mathrm{Id}_{V^*}$.

∎

Problem 11.7
Let V be a vector space.

(a) For any $v \in V$, show that $\xi(v)(\omega)$ depends linearly on ω, so $\xi(v) \in V^{**}$.

(b) Show that the map $\xi : V \to V^{**}$ is linear.

Solution. (a) Let $\omega_1, \omega_2 \in V^*$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}\xi(v)(\omega_1 + \omega_2) &= (\omega_1 + \omega_2)(v)\\ &= \omega_1(v) + \omega_2(v)\\ &= \xi(v)(\omega_1) + \xi(v)(\omega_2).\end{aligned}$$

For scalar multiplication, we have

$$\begin{aligned}\xi(v)(c\omega_1) &= (c\omega_1)(v)\\ &= c\omega_1(v)\\ &= c\xi(v)(\omega_1).\end{aligned}$$

Therefore $\xi(v)$ is indeed a linear functional on V^*, so $\xi(v) \in V^{**}$.

(b) Let $v_1, v_2 \in V$, $c \in \mathbb{R}$, and $\omega \in V^*$. Then

$$\begin{aligned}\xi(v_1 + v_2)(\omega) &= \omega(v_1 + v_2)\\ &= \omega(v_1) + \omega(v_2)\\ &= \xi(v_1)(\omega) + \xi(v_2)(\omega)\\ &= (\xi(v_1) + \xi(v_2))(\omega).\end{aligned}$$

For scalar multiplication, we have

$$\begin{aligned}\xi(cv_1)(\omega) &= \omega(cv_1)\\ &= c\omega(v_1)\\ &= c\xi(v_1)(\omega)\\ &= (c\xi(v_1))(\omega).\end{aligned}$$

Therefore, ξ is linear since these equalities hold for all $\omega \in V^*$.

■

Problem 11.10
Suppose M is a smooth manifold and $E \to M$ is a smooth vector bundle over M. Define the dual bundle to $\boldsymbol{E}$ to be the bundle $E^* \to M$ whose total space is the disjoint union $E^* = \bigsqcup_{p \in M} E_p^*$, where E_p^* is the dual space to E_p, with the obvious projection. Show that $E^* \to M$ is a smooth vector bundle, whose transition functions are given by $\tau^*(p) = \left(\tau(p)^{-1}\right)^T$ for any transition function $\tau : U \to \mathrm{GL}(k, \mathbb{R})$ of E.

Solution. First, let's show how to give E^* a smooth structure. Let (U, Φ) be a local trivialization of E, giving a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$. We can define a local trivialization Φ^* of E^* over U by

$$\Phi^* : (\pi^*)^{-1}(U) \to U \times (\mathbb{R}^k)^*,$$

where for $\omega \in E_p^*$, $\Phi^*(\omega) = (p, \omega \circ (\Phi|_{E_p})^{-1})$. Now let's verify this gives E^* a smooth vector bundle structure:

- Each fiber E_p^* has a natural vector space structure since it's the dual space.

- The local trivializations Φ^* are fiber-preserving vector space isomorphisms because $\Phi|_{E_p}$ is a vector space isomorphism for each p.
- For smoothness, we need to check the transition functions. Let (U, Φ) and (V, Ψ) be local trivializations of E with $U \cap V \neq \emptyset$. The transition function for E^* is

$$\begin{aligned}(\Psi^* \circ (\Phi^*)^{-1})(p, \omega) &= \Psi^*(\Phi_*^{-1}(\omega)) \\ &= (p, \omega \circ (\Psi|_{E_p})^{-1} \circ (\Phi|_{E_p})) \\ &= (p, \omega \circ \tau(p)^{-1})\end{aligned}$$

In matrix form, this is multiplication by $(\tau(p)^{-1})^T$, which is smooth since τ is smooth and matrix inversion and transpose are smooth operations.

Therefore, $E^* \to M$ is a smooth vector bundle with transition functions $\tau^*(p) = (\tau(p)^{-1})^T$. ■

Problem 11.12
Prove Proposition 11.11. [Suggestion: try proving (a) $\implies$ (b) $\implies$ (c) $\implies$ (a), and (c) $\implies$ (d) $\implies$ (e) $\implies$ (b). The only tricky part is (d) $\implies$ (e); look at the proof of Proposition 8.14 for ideas.]

Proposition 11.11. (Smoothness Criteria for Covector Fields). Let M be *a* smooth manifold with or without boundary, and let $\omega : M \to T^*M$ be a rough covector field. The following are equivalent:

(a) ω is smooth.

(b) In every smooth coordinate chart, the component functions of ω are smooth.

(c) Each point of M is contained in some coordinate chart in which ω has smooth component functions.

(d) For every smooth vector field $X \in \mathfrak{X}(M)$, the function $\omega(X)$ is smooth on M.

(e) For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X) : U \to \mathbb{R}$ is smooth on U.

Solution. Let's prove this step by step following the suggested chain of implications.
$\boxed{\text{(a)} \implies \text{(b)}}$ If ω is smooth and (U, φ) is a coordinate chart, then $\varphi^* \circ \omega$ is smooth. The component functions are just compositions of this map with coordinate projections, which are smooth.
$\boxed{\text{(b)} \implies \text{(c)}}$ This is immediate since (b) is a stronger condition than (c).
$\boxed{\text{(c)} \implies \text{(a)}}$ Let $p \in M$ and let (U, φ) be a chart containing p where ω has smooth components. Let $\tilde{\omega}$ be the local representation of ω in this chart. Then $\omega = (\varphi^{-1})^* \circ \tilde{\omega} \circ \varphi$ is smooth on U. Since p was arbitrary, ω is smooth on M.
$\boxed{\text{(c)} \implies \text{(d)}}$ Let $X \in \mathfrak{X}(M)$ and let (U, φ) be a chart where ω has smooth components ω_i. Then locally

$$\omega(X) = \sum_{i=1}^{n} \omega_i X^i,$$

where X^i are the components of X. Since both ω_i and X^i are smooth, their sum and product are smooth.
$\boxed{\text{(d)} \implies \text{(e)}}$ Let $U \subseteq M$ be open and X a smooth vector field on U. By the extension lemma for vector fields, X extends to a smooth vector field $\tilde{X}$ on M. Then $\omega(\tilde{X})$ is smooth on M by (d), so $\omega(X) = \omega(\tilde{X})|_U$ is smooth on U.
$\boxed{\text{(e)} \implies \text{(b)}}$ Let (U, φ) be a coordinate chart and let $\frac{\partial}{\partial x^i}$ be the coordinate vector fields. Then the component functions of ω are $\omega_i = \omega\left(\frac{\partial}{\partial x^i}\right)$, which are smooth by (e).
Therefore, all the statements in Proposition 11.11 are equivalent. ■

Problem 11.32
Prove that if $\varphi : [c,d] \to [a,b]$ is a decreasing diffeomorphism, then

$$\int_{[c,d]} \varphi^*\omega = -\int_{[a,b]} \omega.$$

Solution. First, let's understand what $\varphi^*\omega$ means. For any 1-form ω on $[a,b]$, we can write it locally as $f(x)\,\mathrm{d}x$ where f is smooth. Then

$$\varphi^*\omega = (f \circ \varphi)\,\mathrm{d}\varphi$$

Since φ is decreasing, we have $\mathrm{d}\varphi < 0$, or more precisely, $\mathrm{d}\varphi = \varphi'(t)\,\mathrm{d}t$ where $\varphi'(t) < 0$ for all $t \in [c,d]$.
Now we can compute the integral. We have

$$\begin{aligned}
\int_{[c,d]} \varphi^*\omega &= \int_{[c,d]} (f \circ \varphi)(t)\varphi'(t)\,\mathrm{d}t \\
&= \int_c^d (f \circ \varphi)(t)\varphi'(t)\,\mathrm{d}t \\
&= \int_b^a f(x)\,\mathrm{d}x \quad \text{(using substitution } x = \varphi(t)) \\
&= -\int_a^b f(x)\,\mathrm{d}x \\
&= -\int_{[a,b]} \omega.
\end{aligned}$$

The negative sign appears because we switched the limits of integration when changing variables, since $\varphi(c) = b$ and $\varphi(d) = a$ (as φ is decreasing). ■

Problem 11.35
Prove proposition 11.34.

Proposition 11.34. (Properties of Line Integrals). Let M be a smooth manifold with or without boundary. Suppose $\gamma : [a,b] \to M$ is a piecewise smooth curve segment, and $\omega, \omega_1, \omega_2 \in \mathfrak{X}^*(M)$.

(a) For any $c_1, c_2 \in \mathbb{R}$,

$$\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2.$$

(b) If γ is a constant map, then $\int_\gamma \omega = 0$.

(c) If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ with $a < c < b$, then

$$\int_\gamma \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$

(d) If $F : M \to N$ is any smooth map and $\eta \in \mathfrak{X}^*(N)$, then

$$\int_\gamma F^*\eta = \int_{F\circ\gamma} \eta.$$

Solution. Let's prove each property one by one.

(a) For any piecewise smooth curve segment γ, we have

$$\begin{aligned}\int_\gamma (c_1\omega_1 + c_2\omega_2) &= \int_a^b (c_1\omega_1 + c_2\omega_2)(\gamma'(t))\,\mathrm{d}t \\ &= \int_a^b (c_1\omega_1(\gamma'(t)) + c_2\omega_2(\gamma'(t)))\,\mathrm{d}t \\ &= c_1\int_a^b \omega_1(\gamma'(t))\,\mathrm{d}t + c_2\int_a^b \omega_2(\gamma'(t))\,\mathrm{d}t \\ &= c_1\int_\gamma \omega_1 + c_2\int_\gamma \omega_2.\end{aligned}$$

(b) If γ is constant, then $\gamma'(t) = 0$ for all $t \in [a, b]$. We have

$$\int_\gamma \omega = \int_a^b \omega(\gamma'(t))\,\mathrm{d}t = \int_a^b 0\,\mathrm{d}t = 0.$$

(c) For a curve segment broken into two pieces, we have

$$\begin{aligned}\int_\gamma \omega &= \int_a^b \omega(\gamma'(t))\,\mathrm{d}t \\ &= \int_a^c \omega(\gamma'(t))\,\mathrm{d}t + \int_c^b \omega(\gamma'(t))\,\mathrm{d}t \\ &= \int_{\gamma_1} \omega + \int_{\gamma_2} \omega.\end{aligned}$$

(d) For a smooth map $F : M \to N$ and $\eta \in \mathfrak{X}^*(N)$

$$\begin{aligned}\int_\gamma F^*\eta &= \int_a^b (F^*\eta)(\gamma'(t))\,\mathrm{d}t \\ &= \int_a^b \eta(F_*(\gamma'(t)))\,\mathrm{d}t \\ &= \int_a^b \eta((F\circ\gamma)'(t))\,\mathrm{d}t \\ &= \int_{F\circ\gamma} \eta,\end{aligned}$$

where we used the fact that $(F \circ \gamma)'(t) = F_*(\gamma'(t))$.

■

Problem 11.41
Prove Proposition 11.40. [Remark: this would be harder to prove if we defined conservative fields in terms of smooth curves instead of piecewise smooth ones.]

Proposition 11.40. A smooth covector field ω is conservative if and only if its line integrals are path-independent, in the sense that $\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$ whenever γ and $\tilde{\gamma}$ are piecewise smooth curve segments with the same starting and ending points.

Solution. $\boxed{\Longrightarrow}$ Suppose $\omega = \mathrm{d}f$ for some $f \in C^\infty(M)$. Let γ and $\tilde{\gamma}$ be piecewise smooth curve segments from p to q. Then

$$\begin{aligned}\int_\gamma \omega &= \int_\gamma \mathrm{d}f \\ &= \int_a^b \mathrm{d}f(\gamma'(t))\,\mathrm{d}t \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(f\circ\gamma)(t)\,\mathrm{d}t \\ &= f(q) - f(p).\end{aligned}$$

Similarly, $\int_{\tilde{\gamma}} \omega = f(q) - f(p)$. Therefore $\int_\gamma \omega = \int_{\tilde{\gamma}} \omega$.
$\boxed{\Longleftarrow}$ Suppose the line integrals are path-independent. Fix a point $p_0 \in M$ and define $f : M \to \mathbb{R}$ by

$$f(p) = \int_\gamma \omega,$$

where γ is any piecewise smooth curve segment from p_0 to p. This is well-defined by our path-independence assumption.
We claim $\omega = \mathrm{d}f$. Let $p \in M$ and let $X \in T_pM$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = X$. Then

$$\begin{aligned}\mathrm{d}f_p(X) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(\gamma(t)) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_{\gamma_t} \omega,\end{aligned}$$

where γ_t is any piecewise smooth curve from p_0 to $\gamma(t)$. Choose γ_t to be the concatenation of a fixed curve γ_0 from p_0 to p with the segment of γ from p to $\gamma(t)$. Then

$$\begin{aligned}\mathrm{d}f_p(X) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_0^t \omega(\gamma'(s))\,\mathrm{d}s \\ &= \omega(\gamma'(0)) \\ &= \omega_p(X).\end{aligned}$$

Therefore $\mathrm{d}f = \omega$, proving ω is conservative. ∎

Homework 11

Problem 11-13
The length of a smooth curve segment $\gamma : [a, b] \to \mathbb{R}^n$ is defined to be the value of the (ordinary) integral

$$\mathrm{L}(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

Show that there is no smooth covector field $\omega \in \mathfrak{X}^* (\mathbb{R}^n)$ with the property that $\int_\gamma \omega = \mathrm{L}(\gamma)$ for every smooth curve γ.

Solution. For the sake of contradiction, assume such a smooth covector field ω exists.
First, let's understand what this means. For any curve γ, we have

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t)) \, \mathrm{d}t = \int_a^b |\gamma'(t)| \, \mathrm{d}t.$$

This means that for any tangent vector $\mathbf{v}$ at any point p, we must have

$$\omega_p(\mathbf{v}) = |\mathbf{v}|.$$

Now let's show this is impossible. Fix a point p and consider any two nonzero vectors $\mathbf{v}, \mathbf{w}$ at p. Since ω_p is linear, we have

$$\omega_p(\mathbf{v} + \mathbf{w}) = \omega_p(\mathbf{v}) + \omega_p(\mathbf{w}) = |\mathbf{v}| + |\mathbf{w}|$$

However, we also need

$$\omega_p(\mathbf{v} + \mathbf{w}) = |\mathbf{v} + \mathbf{w}|.$$

By the triangle inequality, $|\mathbf{v} + \mathbf{w}| \leq |\mathbf{v}| + |\mathbf{w}|$, with equality if and only if $\mathbf{v}$ and $\mathbf{w}$ are positively parallel. Since we can always find vectors that aren't positively parallel, this is a contradiction.
Therefore, no such smooth covector field can exist.
This makes geometric sense because the length functional isn't linear in the tangent vector, while covector fields must act linearly on tangent vectors. ∎

Problem 12-1
Give an example of finite-dimensional vector spaces V and W and a specific element $\alpha \in V \otimes W$ that cannot be expressed as $v \otimes w$ for $v \in V$ and $w \in W$.

Solution. Let's take $V = W = \mathbb{R}^2$ for simplicity. Consider the element $\alpha \in \mathbb{R}^2 \otimes \mathbb{R}^2$ given by

$$\alpha = e_1 \otimes e_1 + e_2 \otimes e_2,$$

where $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$.
Let's prove by contradiction that α cannot be written as $v \otimes w$ for any $v, w \in \mathbb{R}^2$. Suppose there exist vectors $v = (a, b)$ and $w = (c, d)$ such that $\alpha = v \otimes w$. Then

$$e_1 \otimes e_1 + e_2 \otimes e_2 = (ae_1 + be_2) \otimes (ce_1 + de_2)$$

Expanding the right side, we have

$$e_1 \otimes e_1 + e_2 \otimes e_2 = ac(e_1 \otimes e_1) + ad(e_1 \otimes e_2) + bc(e_2 \otimes e_1) + bd(e_2 \otimes e_2).$$

Since $\{e_i \otimes e_j\}$ forms a basis of $V \otimes W$, comparing coefficients, we have

$$\begin{aligned} ac &= 1, \\ ad &= 0, \\ bc &= 0, \\ bd &= 1. \end{aligned}$$

From the second and third equations, either $a = c = 0$ or $b = d = 0$. But then either $ac = 0$ or $bd = 0$, contradicting the first and last equations.
Therefore, α cannot be written as a simple tensor $v \otimes w$.
This example is sometimes called the identity element when identifying $V \otimes W$ with $\mathrm{End}(V)$ (when $V = W$), since it acts as the identity transformation. ■

Problem 12-4
Let $V_1, \ldots, V_k$ and W be finite-dimensional real vector spaces. Prove that there is a canonical (basis-independent) isomorphism

$$V_1^* \otimes \cdots \otimes V_k^* \otimes W \cong \mathrm{L}\left(V_1, \ldots, V_k; W\right).$$

(In particular, this means that $V^* \otimes W$ is canonically isomorphic to the space $\mathrm{L}(V; W)$ of linear maps from V to W.)

Solution. We construct a natural map

$$\Phi : V_1^* \otimes \cdots \otimes V_k^* \otimes W \to \mathrm{L}(V_1, \ldots, V_k; W).$$

For simple tensors, we define

$$\Phi(\omega_1 \otimes \cdots \otimes \omega_k \otimes w)(v_1, \ldots, v_k) = \omega_1(v_1) \cdots \omega_k(v_k) w.$$

We extend this linearly to all tensors. This map is clearly linear since all operations involved are linear. To prove Φ is an isomorphism, let's show it's well-defined, prove injectivity, compare dimensions to conclude bijectivity.

- **Well-defined:** We verify Φ maps to multilinear maps. For each i, fixing all inputs except v_i, we get

  $$v_i \mapsto \omega_1(v_1) \cdots \omega_i(v_i) \cdots \omega_k(v_k) w,$$

  which is linear since ω_i is linear.

- **Injectivity:** Suppose $\Phi(\alpha) = 0$ for some $\alpha \in V_1^* \otimes \cdots \otimes V_k^* \otimes W$. We can write

  $$\alpha = \sum_j \omega_{1j} \otimes \cdots \otimes \omega_{kj} \otimes w_j.$$

  Let $\{e_{i\ell}\}$ be bases for V_i and $\{\omega_{im}\}$ be the dual bases. Then evaluating $\Phi(\alpha)$ on basis elements must give zero:

  $$\sum_j \omega_{1j}(e_{1\ell_1}) \cdots \omega_{kj}(e_{k\ell_k}) w_j = 0,$$

  for all choices of indices ℓ_i. This implies $\alpha = 0$.

- **Equal dimensionality:** Both spaces have the same dimension

  $$\begin{aligned} \dim(V_1^* \otimes \cdots \otimes V_k^* \otimes W) &= (\dim V_1) \cdots (\dim V_k)(\dim W), \\ \dim(\mathrm{L}(V_1, \ldots, V_k; W)) &= (\dim V_1) \cdots (\dim V_k)(\dim W). \end{aligned}$$

  Since Φ is injective between spaces of the same finite dimension, it must be an isomorphism.

The isomorphism is canonical because we never made any basis choices in constructing Φ. ■

Problem 1
Do the following (simpler) variant of Exercise 12.18 from page 317: Show that T^kT^*M is a smooth vector bundle over M, and determine its rank.

Solution. First, recall that T^kT^*M means we take the tangent bundle of the cotangent bundle k times. Let's analyze this inductively.

- For $k = 0$, we just have T^*M which is clearly a smooth vector bundle of rank n (where n is the dimension of M).
- For $k = 1$, TT^*M is the tangent bundle of the manifold T^*M. Since T^*M is a smooth manifold (it's a vector bundle over M), its tangent bundle is a smooth vector bundle over T^*M. Moreover, since T^*M has dimension $2n$, TT^*M has rank $2n$.
- For general k, each time we take the tangent bundle, we double the dimension of the base manifold and make the rank equal to this new dimension.

Therefore, T^kT^*M is a smooth vector bundle over a manifold of dimension 2^kn, and has rank 2^kn.
To verify smoothness more carefully: If (U, ϕ) is a chart on M, we get induced charts on each iterated tangent bundle. The transition functions between these charts are compositions of derivatives of the original transition functions, which are smooth since the original transition functions are smooth.
The rank calculation follows from the fact that if $\pi : E \to M$ is a vector bundle of rank r over a manifold M of dimension n, then TE is a vector bundle of rank $2r$ over a manifold of dimension $n + r$. ∎

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 12.3, 12.17, 12.20, 12.23, 12.26.

Problem 12.3
Show that the tensor product operation is bilinear and associative: $F \otimes G$ depends bilinearly on F and G, and $(F \otimes G) \otimes H = F \otimes (G \otimes H)$.

Solution.

- **Bilinearity:** Let F, F_1, F_2, G, G_1, G_2 be tensor fields. For any test tensor field A, we have

$$\begin{aligned}((F_1+F_2)\otimes G)(A) &= (F_1+F_2)(G(A))\\ &= F_1(G(A)) + F_2(G(A))\\ &= (F_1\otimes G)(A) + (F_2\otimes G)(A)\\ &= (F_1\otimes G + F_2\otimes G)(A).\end{aligned}$$

  Similarly,

$$\begin{aligned}(F\otimes(G_1+G_2))(A) &= F\big((G_1+G_2)(A)\big)\\ &= F\big(G_1(A)+G_2(A)\big)\\ &= F\big(G_1(A)\big) + F\big(G_2(A)\big)\\ &= (F\otimes G_1)(A) + (F\otimes G_2)(A)\\ &= (F\otimes G_1 + F\otimes G_2)(A).\end{aligned}$$

  Also, for any scalar c,

$$\begin{aligned}((cF)\otimes G)(A) &= (cF)(G(A))\\ &= c\big(F(G(A))\big)\\ &= c(F\otimes G)(A).\end{aligned}$$

  These establish bilinearity.

- For associativity, let's check the action on a test tensor field A

$$\begin{aligned}\big((F\otimes G)\otimes H\big)(A) &= (F\otimes G)(H(A))\\ &= F\big(G(H(A))\big)\\ &= F\big((G\otimes H)(A)\big)\\ &= \big(F\otimes(G\otimes H)\big)(A).\end{aligned}$$

  Since this holds for all test tensor fields A, we conclude that $(F \otimes G) \otimes H = F \otimes (G \otimes H)$.

The key insight is that tensor product of tensor fields corresponds to composition of their actions on test tensor fields, and composition of functions is associative. ■

Problem 12.17
Show that the following are equivalent for a covariant k-tensor α:

(a) α is alternating.

(b) For any vectors $v_1, \ldots, v_k$ and any permutation $\sigma \in S_k$,

$$\alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = (\operatorname{sgn}\sigma)\alpha\left(v_1, \ldots, v_k\right)$$

(c) With respect to any basis, the components $\alpha_{i_1 \ldots i_k}$ of α change sign whenever two indices are interchanged.

Solution. We will show (a) $\implies$ (b) $\implies$ (c) $\implies$ (a).

$\boxed{\text{(a)} \implies \text{(b)}}$ Let α be alternating. First note that S_k is generated by transpositions. Therefore, it suffices to prove the formula for adjacent transpositions $(i, i+1)$. By definition of alternating, we have

$$\alpha(v_1, \ldots, v_i, v_i, \ldots, v_k) = 0$$

for all vectors. Replacing v_i with $v_i + tv_{i+1}$ for some scalar t, we get

$$\begin{aligned} 0 &= \alpha(v_1, \ldots, v_i + tv_{i+1}, v_i + tv_{i+1}, \ldots, v_k) \\ &= \alpha(v_1, \ldots, v_i, v_i, \ldots, v_k) \\ &\quad + t[\alpha(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) \\ &\quad + \alpha(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k)] \\ &\quad + t^2(\text{terms}). \end{aligned}$$

Since this holds for all t, the coefficient of t must be zero, then

$$\alpha(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) = -\alpha(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k).$$

$\boxed{\text{(b)} \implies \text{(c)}}$ Let $\{e_i\}$ be any basis. The components are given by

$$\alpha_{i_1 \ldots i_k} = \alpha(e_{i_1}, \ldots, e_{i_k}).$$

Interchanging two indices corresponds to interchanging the corresponding basis vectors, which by (b) changes the sign.

$\boxed{\text{(c)} \implies \text{(a)}}$ Let vectors $v_1, \ldots, v_k$ be given with $v_i = v_{i+1}$ for some i. We can write these in terms of basis vectors

$$v_j = \sum_{\ell} v_j^{\ell} e_{\ell},$$

then

$$\alpha(v_1, \ldots, v_k) = \sum \alpha_{i_1 \ldots i_k} v_1^{i_1} \cdots v_k^{i_k}.$$

Since $v_i = v_{i+1}$, their components are equal. By hypothesis (c), terms with $i_i = i_{i+1}$ appear twice with opposite signs and thus cancel out. Therefore, α is alternating. ∎

Problem 12.20

Prove Proposition 12.19.

Proposition 12.19. (Smoothness Criteria for Tensor Fields). Let M be a smooth manifold with or without boundary, and let $A : M \to T^k T^* M$ be a rough section. The following are equivalent.

(a) A is smooth.

(b) In every smooth coordinate chart, the component functions of A are smooth.

(c) Each point of M is contained in some coordinate chart in which A has smooth component functions.

(d) If $X_1, \ldots, X_k \in \mathfrak{X}(M)$, then the function $A(X_1, \ldots, X_k) : M \to \mathbb{R}$, defined by

$$A(X_1, \ldots, X_k)(p) = A_p\left(X_1|_p, \ldots, X_k|_p \right)$$

is smooth.

(e) Whenever $X_1, \ldots, X_k$ are smooth vector fields defined on some open subset $U \subseteq M$, the function $A(X_1, \ldots, X_k)$ is smooth on U.

Solution. We will show (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (e) $\implies$ (a).

$\boxed{\text{(a)} \implies \text{(b)}}$ If A is smooth, then in any coordinate chart (U, ϕ), its components are

$$A_{i_1 \ldots i_k} = A\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right)$$

which are compositions of smooth maps and thus smooth.

$\boxed{\text{(b)} \implies \text{(c)}}$ This is immediate since every point is in some coordinate chart.

$\boxed{\text{(c)} \implies \text{(d)}}$ Let $X_1, \ldots, X_k \in \mathfrak{X}(M)$ and let $p \in M$. Choose a coordinate chart (U, ϕ) containing p where A has smooth components. Then locally

$$A(X_1, \ldots, X_k) = \sum A_{i_1 \ldots i_k} X_1^{i_1} \cdots X_k^{i_k}$$

where X_j^i are the components of X_j. Since both A's components and the vector fields' components are smooth, their product is smooth.

$\boxed{\text{(d)} \implies \text{(e)}}$ This follows immediately since restriction of smooth functions are smooth.

$\boxed{\text{(e)} \implies \text{(a)}}$ We need to show A is a smooth section. Let (U, ϕ) be a coordinate chart and consider the coordinate vector fields $\frac{\partial}{\partial x^i}$. By hypothesis (e), the functions

$$A_{i_1 \ldots i_k} = A\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right)$$

are smooth on U. These are precisely the component functions of A in this chart. Therefore A has smooth components in every chart, which implies A is smooth.

The key insight is that smoothness of a tensor field can be characterized either through its local coordinate expressions or through its action on smooth vector fields. ■

Problem 12.23

Prove Proposition 12.22.

Proposition 12.22. Suppose M is a smooth manifold with or without boundary, $A \in \mathcal{T}^k(M), B \in \mathcal{T}^l(M)$, and $f \in C^\infty(M)$. Then fA and $A \otimes B$ are also smooth tensor fields, whose components in any smooth local coordinate chart are

$$\begin{aligned}
(fA)_{i_1 \ldots i_k} &= fA_{i_1 \ldots i_k}, \\
(A \otimes B)_{i_1 \ldots i_{k+l}} &= A_{i_1 \ldots i_k} B_{i_{k+1} \ldots i_{k+l}}.
\end{aligned}$$

Solution. First, we'll prove that fA is a smooth tensor field. Let (U, φ) be a smooth coordinate chart for M. Let's consider how fA acts on vector fields and covector fields. For any smooth vector fields $X_1, \ldots, X_k$ and smooth functions $g_1, \ldots, g_k$, we have

$$\begin{aligned}
(fA)(g_1 X_1, \ldots, g_k X_k) &= f \cdot A(g_1 X_1, \ldots, g_k X_k) \\
&= f \cdot g_1 \cdots g_k \cdot A(X_1, \ldots, X_k).
\end{aligned}$$

This shows that fA is indeed a tensor field. Moreover, in local coordinates we have

$$\begin{aligned}
(fA)_{i_1 \ldots i_k} &= (fA)\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) \\
&= f \cdot A\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) \\
&= fA_{i_1 \ldots i_k}.
\end{aligned}$$

Since both f and the components of A are smooth functions, the components of fA are also smooth.
For the tensor product, let's verify that $A\otimes B$ is multilinear. For any vector fields $X_1, \ldots, X_{k+l}$ and functions $g_1, \ldots, g_{k+l}$, we have

$$\begin{aligned}(A \otimes B)(g_1X_1, \ldots, g_{k+l}X_{k+l}) &= A(g_1X_1, \ldots, g_kX_k) \cdot B\Big(g_{k+1}X_{k+1}, \ldots, g_{k+l}X_{k+l}\Big) \\ &= \Big(g_1 \cdots g_k \cdot A(X_1, \ldots, X_k)\Big) \cdot \Big(g_{k+1} \cdots g_{k+l} \cdot B(X_{k+1}, \ldots, X_{k+l})\Big) \\ &= g_1 \cdots g_{k+l} \cdot (A \otimes B)(X_1, \ldots, X_{k+l}).\end{aligned}$$

This verifies that $A \otimes B$ is indeed a tensor field. In local coordinates we have

$$\begin{aligned}(A \otimes B)_{i_1 \ldots i_{k+l}} &= (A \otimes B)\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_{k+l}}}\right) \\ &= A\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right) \cdot B\left(\frac{\partial}{\partial x^{i_{k+1}}}, \ldots, \frac{\partial}{\partial x^{i_{k+l}}}\right) \\ &= A_{i_1 \ldots i_k} B_{i_{k+1} \ldots i_{k+l}}.\end{aligned}$$

Since the components of both A and B are smooth functions, their product is also smooth, making $A \otimes B$ a smooth tensor field. ■

Problem 12.26
Prove Proposition 12.25.

Proposition 12.25. (Properties of Tensor Pullbacks). Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, A and B are covariant tensor fields on N, and f is a real-valued function on N.

(a) $F^*(fB) = (f \circ F)F^*B$.

(b) $F^*(A \otimes B) = F^*A \otimes F^*B$.

(c) $F^*(A + B) = F^*A + F^*B$.

(d) F^*B is a (continuous) tensor field, and is smooth if B is smooth.

(e) $(G \circ F)^*B = F^*(G^*B)$.

(f) $(\mathrm{Id}_N)^* B = B$.

Solution. Solved in Study Guide for Quiz 4. ■

Homework 12

Problem 14-1
Show that covectors $\omega^1, \dots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$.

Solution. $\boxed{\Longrightarrow}$ First, suppose the covectors are linearly dependent. Then there exist scalars $c_1, \dots, c_k$, not all zero, such that

$$c_1\omega^1 + \cdots + c_k\omega^k = 0.$$

Without loss of generality, assume $c_1 \neq 0$. Then

$$\omega^1 = -\frac{c_2}{c_1}\omega^2 - \cdots - \frac{c_k}{c_1}\omega^k.$$

Substituting this into the wedge product

$$\begin{aligned}\omega^1 \wedge \cdots \wedge \omega^k &= \left(-\frac{c_2}{c_1}\omega^2 - \cdots - \frac{c_k}{c_1}\omega^k\right) \wedge \omega^2 \wedge \cdots \wedge \omega^k \\ &= -\frac{c_2}{c_1}\omega^2 \wedge \omega^2 \wedge \cdots \wedge \omega^k - \cdots - \frac{c_k}{c_1}\omega^k \wedge \omega^2 \wedge \cdots \wedge \omega^k \\ &= 0,\end{aligned}$$

since each term contains a repeated covector, and $\omega^i \wedge \omega^i = 0$ for any covector ω^i.
$\boxed{\Longleftarrow}$ Conversely, suppose $\omega^1 \wedge \cdots \wedge \omega^k = 0$. Let's extend $\omega^1, \dots, \omega^k$ to a basis $\omega^1, \dots, \omega^k, \omega^{k+1}, \dots, \omega^n$ of the dual space. Each ω^i can be written in terms of the dual basis $e^1, \dots, e^n$

$$\omega^i = \sum_{j=1}^n a_{ij}e^j$$

The wedge product $\omega^1 \wedge \cdots \wedge \omega^k$ can be written as a sum of basic k-forms $e^{i_1} \wedge \cdots \wedge e^{i_k}$ with coefficients given by $k \times k$ minors of the matrix (a_{ij}). Since this wedge product is zero, all these minors must be zero, which implies that the rows of (a_{ij}) are linearly dependent. Therefore, the covectors $\omega^1, \dots, \omega^k$ are linearly dependent.
Therefore, covectors $\omega^1, \dots, \omega^k$ on a finite-dimensional vector space are linearly dependent if and only if $\omega^1 \wedge \cdots \wedge \omega^k = 0$. ■

Problem 14-9
Let M, N be smooth manifolds, and suppose $\pi : M \to N$ is a surjective smooth submersion with connected fibers. We say that a tangent vector $v \in T_pM$ is **vertical** if $d\pi_p(v) = 0$. Suppose $\omega \in \Omega^k(M)$. Show that there exists $\eta \in \Omega^k(N)$ such that $\omega = \pi^*\eta$ if and only if $v \lrcorner \omega_p = 0$ and $v \lrcorner d\omega_p = 0$ for every $p \in M$ and every vertical vector $v \in T_pM$. [Hint: first, do the case in which $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is projection onto the first n coordinates.]

Solution. In this case, let's write coordinates $(x^1, \dots, x^n, y^1, \dots, y^m)$ on $\mathbb{R}^{n+m}$ and $(x^1, \dots, x^n)$ on $\mathbb{R}^n$. The vertical vectors are those of the form $\sum_{j=1}^m b_j \frac{\partial}{\partial y^j}$.
A general k-form ω on $\mathbb{R}^{n+m}$ can be written as

$$\omega = \sum_{I,J} f_{I,J}(x,y)\,\mathrm{d}x^I \wedge \mathrm{d}y^J,$$

where I and J are multi-indices with $|I| + |J| = k$. For ω to be a pullback, it must not contain any dy^j terms, so $f_{I,J} = 0$ for all $J \neq \emptyset$. Moreover, the coefficients cannot depend on y. Therefore

$$\omega = \sum_{|I|=k} f_I(x)\,\mathrm{d}x^I = \pi^\eta,$$

where $\eta = \sum_{|I|=k} f_I(x)\,\mathrm{d}x^I$ is a k-form on $\mathbb{R}^n$.

- **For necessity:** If $\omega = \pi^*\eta$, then ω has no $\mathrm{d}y^j$ terms, so contracting with vertical vectors gives zero. Also, $\mathrm{d}\omega = \pi^* d\eta$ has at most one $\mathrm{d}y^j$ in each term, so contracting with vertical vectors again gives zero.

- **For sufficiency:** If $v \lrcorner \omega = 0$ for all vertical vectors, then ω cannot have any terms with more than one $\mathrm{d}y^j$. If also $v \lrcorner \mathrm{d}\omega = 0$, then the coefficients cannot depend on y.

- **For the general case:** Let $p \in M$. Near p, we can choose coordinates $(x^1, \dots, x^n)$ on N and extend them to coordinates $(x^1, \dots, x^n, y^1, \dots, y^m)$ on M such that π looks like the projection $\mathbb{R}^{n+m} \to \mathbb{R}^n$ locally.

The conditions $v \lrcorner \omega = 0$ and $v \lrcorner \mathrm{d}\omega = 0$ for vertical vectors are tensorial, so they hold in one coordinate system if and only if they hold in all coordinate systems.
By our local result, in each coordinate neighborhood U_α, there exists $\eta_\alpha \in \Omega^k(\pi(U_\alpha))$ such that $\omega|U\alpha = \pi^\eta_\alpha$.
These local forms must agree on overlaps because π is surjective and has connected fibers. Therefore, they piece together to give a global form $\eta \in \Omega^k(N)$ such that $\omega = \pi^\eta$.
The key insight is that the condition about contractions with vertical vectors forces the form to be "horizontal" (no $\mathrm{d}y$ terms) and "basic" (coefficients independent of y), which are exactly the conditions needed for a form to be a pullback. ■

Problem 15-3
Suppose $n \geq 1$, and let $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd. [Hint: consider the map $F : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ given by $F(x) = -x$, and use Corollary 15.34.] (Used on pp. 393, 435.)
(Since we havent introduced Riemannian manifolds yet, you should probably ignore the hint in the book. You might try to use the results of Ex. 15.13(b) instead, along with, say, stereographic coordinates.)

Solution. First, recall that a diffeomorphism is orientation-preserving if and only if its differential has positive determinant at every point. The antipodal map α is smooth, and at each point $x \in \mathbb{S}^n$, we need to compute $\det(\mathrm{d}\alpha_x)$. Using a stereographic projection $\phi : \mathbb{S}^n \setminus N \to \mathbb{R}^n$ from the north pole $N = (0, \dots, 0, 1)$, given by

$$\phi(x^1, \dots, x^{n+1}) = \left(\frac{x^1}{1 - x^{n+1}}, \dots, \frac{x^n}{1 - x^{n+1}} \right).$$

Consider the composition $\phi \circ \alpha \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. This is easier to analyze than α directly. For $y \in \mathbb{R}^n$, we write out this composition

$$(\phi \circ \alpha \circ \phi^{-1})(y) = -\frac{y}{|y|^2}$$

This is because the antipodal map of a point, when viewed through stereographic projection, is the inversion through the unit sphere scaled by -1. Computing the differential of this map at a point $y \in \mathbb{R}^n$, we have

$$\mathrm{d}(\phi \circ \alpha \circ \phi^{-1})_y(v) = -\frac{|y|^2 v - 2(y \cdot v)y}{|y|^4}$$

The determinant of this linear transformation is

$$\det(\mathrm{d}(\phi \circ \alpha \circ \phi^{-1})_y) = \frac{(-1)^n}{|y|^{2n}}$$

Since ϕ is a diffeomorphism (away from the north pole), the original map α is orientation-preserving if and only if this determinant is positive.

- When n is odd, $(-1)^n = -1$, and $|y|^{2n}$ is positive, so the determinant is negative.

- When n is even, $(-1)^n = 1$, and $|y|^{2n}$ is positive, so the determinant is positive.

However, we need to be careful: ϕ itself changes orientation. In fact, stereographic projection is orientation-preserving when n is even and orientation-reversing when n is odd (this can be proven by computing its Jacobian). Therefore, accounting for this, we have

- When n is odd, α is orientation-preserving.
- When n is even, α is orientation-reversing.

Therefore, α is orientation-preserving if and only if n is odd. ∎

Problem 15-12
Show that every orientation-reversing diffeomorphism of $\mathbb{R}$ has a fixed point.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be an orientation-reversing diffeomorphism. Being orientation-reversing means that, for every point $x \in \mathbb{R}$, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) < 0.$$

This means f is strictly decreasing. Being a diffeomorphism means f is smooth and bijective.
Consider the function $g(x) = f(x) - x$. We will show this has a zero, which means f has a fixed point. Since f is strictly decreasing, $g'(x) = f'(x) - 1 < -1$ for all x. This means g is also strictly decreasing. Since f is bijective from $\mathbb{R}$ to $\mathbb{R}$ and strictly decreasing, then

$$\lim_{x\to\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \infty.$$

Thus,

$$\lim_{x\to\infty} g(x) = -\infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = \infty.$$

By the Intermediate Value Theorem, since g is continuous and takes both positive and negative values, there must exist some $c \in \mathbb{R}$ such that $g(c) = 0$. Thus, $f(c) = c$, so c is a fixed point of f.
Therefore, every orientation-reversing diffeomorphism of $\mathbb{R}$ has a fixed point. ∎

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 14.4, 14.6, 14.14, 14.17, 14.22, 14.25, 15.4, 15.8, 15.10, 15.12, 15.13, 15.14, 15.16.

Problem 14.4
Prove Proposition 14.3.

Proposition 14.3. (Properties of Alternation). Let α be a covariant tensor on a finite-dimensional vector space.

(a) $\operatorname{Alt}(\alpha)$ is alternating.

(b) $\operatorname{Alt}(\alpha) = \alpha$ if and only if α is alternating.

Solution. (a) To show $\operatorname{Alt}(\alpha)$ is alternating, we need to prove that for any permutation σ:

$$(\operatorname{Alt}(\alpha))(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)(\operatorname{Alt}(\alpha))(v_1, \ldots, v_k)$$

We can write this out using the definition of alternation:

$$\begin{aligned}(\operatorname{Alt}(\alpha))(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) &= \frac{1}{k!}\sum_{\tau\in S_k} \operatorname{sgn}(\tau)\,\alpha\Big(v_{\sigma(\tau(1))}, \ldots, v_{\sigma(\tau(k))}\Big)\\ &= \frac{1}{k!}\sum_{\tau\in S_k} \operatorname{sgn}(\tau)\,\alpha\Big(v_{(\sigma\circ\tau)(1)}, \ldots, v_{(\sigma\circ\tau)(k)}\Big)\\ &= \frac{1}{k!}\sum_{\eta\in S_k} \operatorname{sgn}(\sigma^{-1}\circ\eta)\,\alpha\Big(v_{\eta(1)}, \ldots, v_{\eta(k)}\Big),\end{aligned}$$

where we made the substitution $\eta = \sigma\circ\tau$. Since

$$\operatorname{sgn}(\sigma^{-1}\circ\eta) = \operatorname{sgn}(\sigma^{-1})\operatorname{sgn}(\eta) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\eta),$$

we have

$$\begin{aligned}(\operatorname{Alt}(\alpha))(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) &= \operatorname{sgn}(\sigma)\frac{1}{k!}\sum_{\eta\in S_k} \operatorname{sgn}(\eta)\,\alpha\Big(v_{\eta(1)}, \ldots, v_{\eta(k)}\Big)\\ &= \operatorname{sgn}(\sigma)(\operatorname{Alt}(\alpha))(v_1, \ldots, v_k).\end{aligned}$$

(b) $\boxed{\Longrightarrow}$ If $\operatorname{Alt}(\alpha) = \alpha$, then by part (a), α is alternating.

$\boxed{\Longleftarrow}$ Assume α is alternating. Then for any permutation σ, we have

$$\begin{aligned}(\operatorname{Alt}(\alpha))(v_1, \ldots, v_k) &= \frac{1}{k!}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\,\alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\\ &= \frac{1}{k!}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma)\,\alpha(v_1, \ldots, v_k)\\ &= \frac{1}{k!}\sum_{\sigma\in S_k} \alpha(v_1, \ldots, v_k)\\ &= \alpha(v_1, \ldots, v_k).\end{aligned}$$

Therefore, $\operatorname{Alt}(\alpha) = \alpha$.

■

Problem 14.6
Show that

$$\delta^I_J = \begin{cases} \operatorname{sgn}\sigma & \text{if neither } I \text{ nor } J \text{ has a repeated index and } J = I_\sigma \text{ for some } \sigma \in S_k, \\ 0 & \text{if } I \text{ or } J \text{ has a repeated index or } J \text{ is not a permutation of } I. \end{cases}$$

Solution. First, let's recall the definition of δ^I_J. For multiple indices $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_k)$, we have

$$\delta^I_J = \det(\delta^{i_q}_{j_p})_{1 \le p,q \le k}.$$

- **Case 1:** If I has a repeated index, then two rows of the matrix $(\delta^{i_q}_{j_p})$ would be identical, making its determinant zero. Similarly, if J has a repeated index, then two columns would be identical.
- **Case 2:** If J is not a permutation of I, then there exists some index in either I or J that doesn't appear in the other. This means there will be either a row or column of all zeros in $(\delta^{i_q}_{j_p})$, making its determinant zero.
- **Case 3:** If neither I nor J has repeated indices and $J = I_\sigma$ for some $\sigma \in S_k$, then we need to show $\delta^I_J = \operatorname{sgn}\sigma$.

Let's work this out explicitly. For any entry in the matrix $(\delta^{i_q}_{j_p})$, we have

$$\delta^{i_q}_{j_p} = \begin{cases} 1 & \text{if } i_q = j_p \\ 0 & \text{otherwise} \end{cases}$$

Since $J = I_\sigma$, we have $j_p = i_{\sigma(p)}$. Therefore, $\delta^{i_q}_{j_p} = 1$ if and only if $q = \sigma(p)$. This means $(\delta^{i_q}_{j_p})$ is exactly the permutation matrix corresponding to σ.
We know that the determinant of a permutation matrix equals the sign of the permutation it represents. Therefore

$$\det(\delta^{i_q}_{j_p}) = \operatorname{sgn}\sigma.$$

■

Problem 14.14
Show that $\Lambda^k T^*M$ is a smooth subbundle of $T^k T^*M$, and therefore is a smooth vector bundle of rank $\binom{n}{k}$ over M.

Solution. Solved in Study Guide for Quiz 4. ■

Problem 14.17
Prove Lemma 14.17.

Lemma 14.17. Suppose $F : M \to N$ is smooth.

(a) $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = (F^*\omega) \wedge (F^*\eta)$.

(c) In any smooth chart,

$$F^*\left(\sum_I{}' \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I{}' (\omega_I \circ F)\, d\left(y^{i_1} \circ F\right) \wedge \cdots \wedge d\left(y^{i_k} \circ F\right).$$

Solution. (a) Let $\omega, \eta \in \Omega^k(N)$ and $a, b \in \mathbb{R}$. For any $p \in M$ and $\mathbf{v}_1, \dots, \mathbf{v}_k \in T_pM$, we have

$$\begin{aligned} F^*(a\omega + b\eta)_p(\mathbf{v}_1, \dots, \mathbf{v}_k) &= (a\omega + b\eta)_{F(p)}\Big(\mathrm{d}F_p\mathbf{v}_1, \dots, \mathrm{d}F_p\mathbf{v}_k\Big) \\ &= a\omega_{F(p)}\Big(\mathrm{d}F_p\mathbf{v}_1, \dots, \mathrm{d}F_p\mathbf{v}_k\Big) + b\eta_{F(p)}\Big(\mathrm{d}F_p\mathbf{v}_1, \dots, \mathrm{d}F_p\mathbf{v}_k\Big) \\ &= a(F^*\omega)_p(\mathbf{v}_1, \dots, \mathbf{v}_k) + b(F^*\eta)_p(\mathbf{v}_1, \dots, \mathbf{v}_k). \end{aligned}$$

Therefore, $F^*(a\omega + b\eta) = aF^*\omega + bF^*\eta$.

(b) Let $\omega \in \Omega^r(N)$ and $\eta \in \Omega^s(N)$. For any $p \in M$ and $\mathbf{v}_1, \dots, \mathbf{v}_{r+s} \in T_pM$, we have

$$\begin{aligned} [F^*(\omega \wedge \eta)]_p(\mathbf{v}_1, \dots, \mathbf{v}_{r+s}) &= (\omega \wedge \eta)_{F(p)}\Big(\mathrm{d}F_p\mathbf{v}_1, \dots, \mathrm{d}F_p\mathbf{v}_{r+s}\Big) \\ &= \frac{1}{r!s!} \sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)\omega_{F(p)}\Big(\mathrm{d}F_p\mathbf{v}_{\sigma(1)}, \dots, \mathrm{d}F_p\mathbf{v}_{\sigma(r)}\Big) \cdot \eta_{F(p)}\Big(\mathrm{d}F_p\mathbf{v}_{\sigma(r+1)}, \dots, \mathrm{d}F_p\mathbf{v}_{\sigma(r+s)}\Big) \\ &= \frac{1}{r!s!} \sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)(F^*\omega)_p\Big(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(r)}\Big) \cdot (F^*\eta)_p\Big(\mathbf{v}_{\sigma(r+1)}, \dots, \mathbf{v}_{\sigma(r+s)}\Big) \\ &= [(F^*\omega) \wedge (F^*\eta)]_p(\mathbf{v}_1, \dots, \mathbf{v}_{r+s}). \end{aligned}$$

(c) This follows from (a) and (b) since

$$\begin{aligned} F^*(\omega_I \, \mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}) &= (F^*\omega_I)F^*(\mathrm{d}y^{i_1}) \wedge \cdots \wedge F^*(\mathrm{d}y^{i_k}) \\ &= (\omega_I \circ F)\, \mathrm{d}(y^{i_1} \circ F) \wedge \cdots \wedge \mathrm{d}(y^{i_k} \circ F) \end{aligned}$$

Then by linearity from (a), the result follows for sums.

■

Problem 14.22
Let X be a smooth vector field on M.

(a) Show that if ω is a smooth differential form, then $i_X\omega$ is smooth.

(b) Verify that $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$ is linear over $C^\infty(M)$ and therefore corresponds to a smooth bundle homomorphism $i_X : \Lambda^k T^*M \to \Lambda^{k-1}T^*M$.

Solution. (a) Let $\omega \in \Omega^k(M)$ and $p \in M$. In local coordinates around p, we can write

$$\omega = {\sum_I}' \omega_I \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

and

$$X = \sum_{j=1}^n X^j \frac{\partial}{\partial x^j}$$

Then

$$\begin{aligned} i_X\omega &= {\sum_I}' \omega_I i_X \left(\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}\right) \\ &= {\sum_I}' \omega_I \sum_{j=1}^k (-1)^{j-1} X^{i_j} \, \mathrm{d}x^{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x^{i_j}} \wedge \cdots \wedge \mathrm{d}x^{i_k} \end{aligned}$$

Since ω_I and X^j are smooth functions, and the wedge product of smooth forms is smooth, $i_X\omega$ is smooth.

(b) Let $\omega \in \Omega^k(M)$, $f \in C^\infty(M)$, and $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1} \in T_pM$. Then

$$\begin{aligned} i_X(f\omega)_p(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) &= (f\omega)_p(X_p, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) \\ &= f(p)\omega_p(X_p, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) \\ &= f(p)(i_X\omega)_p(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) \\ &= (f i_X\omega)_p(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) \end{aligned}$$

Therefore, i_X is $C^\infty(M)$-linear. This implies it corresponds to a smooth bundle homomorphism $i_X : \Lambda^k T^*M \to \Lambda^{k-1}T^*M$.

■

Problem 14.25
Suppose M is a smooth manifold and $X \in \mathfrak{X}(M)$. Show that interior multiplication $i_X : \Omega^*(M) \to \Omega^*(M)$ is an antiderivation of degree -1 whose square is zero.

Solution. We need to show the following:

- The interior multiplication i_X has degree -1 (maps $\Omega^k(M)$ to $\Omega^{k-1}(M)$). This is clear from the definition as $i_X\omega(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) = \omega(X, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1})$.
- For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$

$$i_X(\omega \wedge \eta) = (i_X\omega) \wedge \eta + (-1)^k \omega \wedge (i_X\eta).$$

Let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$. Take $\mathbf{v}_1, \ldots, \mathbf{v}_{k+l-1} \in T_pM$. Then

$$\begin{aligned} i_X(\omega \wedge \eta)(\mathbf{v}_1, \ldots, \mathbf{v}_{k+l-1}) &= (\omega \wedge \eta)(X, \mathbf{v}_1, \ldots, \mathbf{v}_{k+l-1}) \\ &= \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\, \eta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}), \end{aligned}$$

where $X_1 = X$ and $X_{j+1} = \mathbf{v}_j$ for $j \geq 1$.
This equals

$$\begin{aligned} &\frac{1}{(k-1)!l!} \sum_{\sigma \in S_{k+l-1}} \operatorname{sgn}(\sigma)\, (i_X\omega)\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k-1)}\right)\, \eta\left(\mathbf{v}_{\sigma(k)}, \ldots, \mathbf{v}_{\sigma(k+l-1)}\right) \\ &+ \frac{(-1)^k}{k!(l-1)!} \sum_{\sigma \in S_{k+l-1}} \operatorname{sgn}(\sigma)\, \omega\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right)\, (i_X\eta)\left(\mathbf{v}_{\sigma(k+1)}, \ldots, \mathbf{v}_{\sigma(k+l-1)}\right) \\ &= [(i_X\omega) \wedge \eta + (-1)^k \omega \wedge (i_X\eta)](\mathbf{v}_1, \ldots, \mathbf{v}_{k+l-1}). \end{aligned}$$

- The composition $i_X \circ i_X = 0$. Take $\omega \in \Omega^k(M)$. Then for any $\mathbf{v}_1, \ldots, \mathbf{v}_{k-2} \in T_pM$:

$$\begin{aligned} (i_X \circ i_X\omega)(\mathbf{v}_1, \ldots, \mathbf{v}_{k-2}) &= (i_X\omega)(X, \mathbf{v}_1, \ldots, \mathbf{v}_{k-2}) \\ &= \omega(X, X, \mathbf{v}_1, \ldots, \mathbf{v}_{k-2}) = 0, \end{aligned}$$

where the last equality follows because ω is alternating and we have a repeated vector X.

Therefore, the interior multiplication i_X is an antiderivation of degree -1 whose square is zero. ■

Problem 15.4
Suppose M is an oriented smooth n-manifold with or without boundary, and $n \geq 1$. Show that every local frame with connected domain is either positively oriented or negatively oriented. Show that the connectedness assumption is necessary.

Solution. Let $(E_1, \ldots, E_n)$ be a local frame defined on a connected domain U. Let $p \in U$ and let $(F_1, \ldots, F_n)$ be a positively oriented frame defined near p. There exists a smooth matrix $A = (a^i_j)$ such that

$$E_i = \sum_{j=1}^{n} a^j_i F_j.$$

Then at each point $q \in U$, we have

$$\det(A(q)) = [E_1, \ldots, E_n]_q / [F_1, \ldots, F_n]_q.$$

Since both frames are smooth and the denominator is never zero (as F_i form a frame), $\det(A)$ is a smooth function on U. Moreover, $\det(A)$ never vanishes as $(E_1, \ldots, E_n)$ is also a frame.
By the connectedness of U and since $\det(A)$ is continuous and never zero, the intermediate value theorem implies that $\det(A)$ must be either always positive or always negative on U. Therefore, $(E_1, \ldots, E_n)$ must be either positively or negatively oriented throughout U.
To show the connectedness assumption is necessary, consider $M = \mathbb{R}^n$ and let $U = (-1, 0) \cup (0, 1)$. Define a frame $(E_1, \ldots, E_n)$ on U by:

$$E_i = \begin{cases} \frac{\partial}{\partial x^i}, & \text{if } x > 0, \\ -\frac{\partial}{\partial x^i}, & \text{if } x < 0. \end{cases}$$

This frame is positively oriented on $(0, 1)$ but negatively oriented on $(-1, 0)$. ∎

Problem 15.8
Prove Proposition 15.7.

Proposition 15.7. (Product Orientations). Suppose $M_1, \ldots, M_k$ are orientable smooth manifolds. There is a unique orientation on $M_1 \times \cdots \times M_k$, called the prod***uct*** orientation, with the following property: if for each $i = 1, \ldots, k, \omega_i$ is an orientation form for the given orientation on M_i, then $\pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k$ is an orientation form for the product orientation.

Solution.

- **Existence:** Let ω_i be orientation forms on M_i for $i = 1, \ldots, k$. Define

$$\Omega = \pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k.$$

We must show Ω is nowhere vanishing and thus defines an orientation on the product manifold. Let $p = (p_1, \ldots, p_k) \in M_1 \times \cdots \times M_k$. For each i, let $\left(E^i_1, \ldots, E^i_{n_i}\right)$ be a local frame for TM_i near p_i such that $\omega_i \left(E^i_1, \ldots, E^i_{n_i}\right) > 0$.

Let $\tilde{E}^i_j = (d\pi_i)^{-1}(E^i_j)$ be the corresponding lifted vector fields on the product manifold. Then

$$\left(\tilde{E}^1_1, \ldots, \tilde{E}^1_{n_1}, \ldots, \tilde{E}^k_1, \ldots, \tilde{E}^k_{n_k}\right)$$

forms a local frame near p. We have

$$\begin{aligned} \Omega_p \left(\tilde{E}^1_1, \ldots, \tilde{E}^k_{n_k}\right) &= (\pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k)_p \left(\tilde{E}^1_1, \ldots, \tilde{E}^k_{n_k}\right) \\ &= \prod_{i=1}^{k} (\pi_i^* \omega_i)_p \left(\tilde{E}^i_1, \ldots, \tilde{E}^i_{n_i}\right) \\ &= \prod_{i=1}^{k} \omega_i \left(E^i_1, \ldots, E^i_{n_i}\right) > 0. \end{aligned}$$

Therefore, Ω is nowhere vanishing and defines an orientation.

- **Uniqueness:** Suppose there were two orientations satisfying the property. Let Ω_1 and Ω_2 be the resulting forms from this construction. Their ratio would be a smooth positive function on the product manifold. But by the construction above, this ratio must be 1, so they define the same orientation.

  This orientation is the product orientation by construction, as it's determined by the orientations of the factor manifolds via their orientation forms.

■

Problem 15.10
Prove Proposition 15.9.

Proposition 15.9. Let M be a connected, orientable, smooth manifold with or without boundary. Then M has exactly two orientations. If two orientations of M agree at one point, they are equal.

Solution. First, let's show M has exactly two orientations. Let $\mathcal{O}$ be some orientation on M and let ω be an orientation form for $\mathcal{O}$. Then $\{-\omega, \omega\}$ are two different orientation forms since they differ by a negative scalar at each point. These define two distinct orientations.
To show there are no others, let η be any orientation form on M. Since both η and ω are nowhere-vanishing n-forms, there exists a smooth function f such that $\eta = f\omega$. Since η and ω are both nowhere-vanishing, f must be nowhere-vanishing. By the connectedness of M and continuity of f, either $f > 0$ everywhere or $f < 0$ everywhere. Therefore, η defines the same orientation as either ω or $-\omega$.
Now, let's prove that if two orientations agree at one point, they are equal. Let ω_1 and ω_2 be orientation forms representing the two orientations. If they agree at some point p, then at p, we have

$$\omega_2 = f\omega_1,$$

where $f(p) > 0$. As before, by connectedness of M, we must have $f > 0$ everywhere, which means ω_1 and ω_2 define the same orientation on all of M. ■

Problem 15.12
Prove Proposition 15.11.

Proposition 15.11. (Orientations of Codimension-0 Submanifolds). Suppose M is an oriented smooth manifold with or without boundary, and $D \subseteq M$ is a smooth codimension-0 submanifold with or without boundary. Then the orientation of M restricts to an orientation of D. If ω is an orientation form for M, then $\iota_D^*\omega$ is an orientation form for D.

Solution. First, note that since D is a codimension-0 submanifold of M, we have $\dim(D) = \dim(M) = n$ and $T_pD = T_pM$ for all $p \in D$.
Let ω be an orientation form for M and let $\iota_D : D \hookrightarrow M$ be the inclusion map. We need to show that $\iota_D^*\omega$ is an orientation form for D.
At each point $p \in D$, we have

$$\begin{aligned}(\iota_D^*\omega)_p(\mathbf{v}_1, \dots, \mathbf{v}_n) &= \omega_{\iota_D(p)}\left(\mathrm{d}(\iota_D)_p\mathbf{v}_1, \dots, \mathrm{d}(\iota_D)_p\mathbf{v}_n\right) \\ &= \omega_p(\mathbf{v}_1, \dots, \mathbf{v}_n),\end{aligned}$$

for all $\mathbf{v}_1, \dots, \mathbf{v}_n \in T_pD = T_pM$, since $\mathrm{d}(\iota_D)_p$ is just the identity map.
Since ω is nowhere-vanishing on M, $\iota_D^*\omega$ is nowhere-vanishing on D. Moreover, $\iota_D^*\omega$ is smooth since both ι_D and ω are smooth.
Therefore, $\iota_D^*\omega$ is an orientation form for D, and the orientation it defines is the restriction of the orientation of M to D. ■

Problem 15.13
Suppose M and N are oriented positive-dimensional smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Show that the following are equivalent.

(a) F is orientation-preserving.

(b) With respect to any oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.

(c) For any positively oriented orientation form ω for N, the form $F^*\omega$ is positively oriented for M.

Solution. Solved in Study Guide for Quiz 4. ■

Problem 15.14
Show that a composition of orientation-preserving maps is orientation-preserving.

Solution. ■

Problem 15.16
Suppose $F : M \to N$ and $G : N \to P$ are local diffeomorphisms and $\mathcal{O}$ is an orientation on P. Show that $(G \circ F)^*\mathcal{O} = F^*\left(G^*\mathcal{O}\right)$.

Solution. ■

Homework 13

Problem 16-4
Suppose M is an oriented compact smooth manifold with boundary. Show that there does not exist a retraction of M onto its boundary. [Hint: if the retraction is smooth, consider an orientation form on ∂M.]

Solution. Suppose, for contradiction, that there exists a smooth retraction $r : M \to \partial M$. Let ω be a volume form on M such that $\operatorname{supp}(\omega)$ is compact. By the properties of an orientation form, ω is a top-dimensional form on M. Define $\eta = r^*\omega$, the pullback of ω by the retraction r. Note that η is a top-dimensional form on ∂M. Since $r(x) = x$ for all $x \in \partial M$, we have $\eta = \omega|_{\partial M}$. Now, consider the integral $\int_{\partial M} \eta$. By the pullback property and the definition of r, we have

$$\begin{aligned}\int_{\partial M} \eta &= \int_{\partial M} r^*\omega \\ &= \int_M (r_*\eta) \\ &= \int_M 0.\end{aligned}$$

However, this contradicts the fact that ω is a volume form and thus has a non-zero integral over M.
Therefore, no such smooth retraction can exist. ■

Problem 16-5
Suppose M and N are oriented, compact, connected, smooth manifolds, and $F, G : M \to N$ are homotopic diffeomorphisms. Show that F and G are either both orientation-preserving or both orientation-reversing. [Hint: use Theorem 6.29 and Stokes's theorem on $M \times I$.]

Solution. Let $H : M \times I \to N$ be a homotopy between F and G, where $I = [0, 1]$. Consider the map $\det(\mathrm{d}H) : M \times I \to \mathbb{R}$. By Theorem 6.29, $\det(\mathrm{d}H)$ is continuous. Define

$$\alpha(t) = \int_M \det(\mathrm{d}H_t)\ \mathrm{d}V_M,$$

where $H_t(x) = H(x, t)$ and $\mathrm{d}V_M$ is the volume form on M. By Stokes's theorem and the fact that M is compact, $\alpha(t)$ is constant for all $t \in I$.

- At $t = 0$, $\alpha(0) = \int_M \det(\mathrm{d}F)\ \mathrm{d}V_M$, the sign of which determines whether F preserves or reverses orientation.
- At $t = 1$, $\alpha(1) = \int_M \det(\mathrm{d}G)\ \mathrm{d}V_M$, the sign of which determines the orientation of G.

Since $\alpha(t)$ is constant, $\det(\mathrm{d}F)$ and $\det(\mathrm{d}G)$ must have the same sign.
Therefore, F and G are either both orientation-preserving or both orientation-reversing. ■

Problem 16-6
The Hairy Ball Theorem: *There exists a nowhere-vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.* ("You cannot comb the hair on a ball.") Prove this by showing that the following are equivalent:

(a) There exists a nowhere-vanishing vector field on $\mathbb{S}^n$.

(b) There exists a continuous map $V : \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in \mathbb{S}^n$.

(c) The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is homotopic to $\mathrm{Id}_{\mathbb{S}^n}$.

(d) The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving.

(e) n is odd.

[Hint: use Problems 9-4, 15-3, and 16-5.]

Solution. The equivalence follows from several key observations:

- (a) $\implies$ (b): Suppose there exists a nowhere-vanishing vector field V on $\mathbb{S}^n$. Then for each $x \in \mathbb{S}^n$, we have $V(x) \neq 0$. Define a map

$$\begin{aligned} V' &: \mathbb{S}^n \to \mathbb{S}^n, \\ x &\mapsto V'(x) = \frac{V(x)}{|V(x)|}. \end{aligned}$$

  Then $V'(x)$ is a unit vector for all $x \in \mathbb{S}^n$, and since $V(x)$ is tangent to $\mathbb{S}^n$ at x, we have $V'(x) \perp x$ for all $x \in \mathbb{S}^n$.

- (b) $\implies$ (c): Suppose there exists a continuous map $V : \mathbb{S}^n \to \mathbb{S}^n$ satisfying $V(x) \perp x$ for all $x \in \mathbb{S}^n$. Define a homotopy

$$\begin{aligned} H &: \mathbb{S}^n \times [0,1] \to \mathbb{S}^n, \\ (x,t) &\mapsto H(x,t) = \cos(\pi t)x + \sin(\pi t)V(x). \end{aligned}$$

  Then $H(x,0) = x$ and $H(x,1) = -x = \alpha(x)$ for all $x \in \mathbb{S}^n$. Moreover, since $V(x) \perp x$, we have $|H(x,t)| = 1$ for all $(x,t) \in \mathbb{S}^n \times [0,1]$, so H is a homotopy between $\mathrm{Id}_{\mathbb{S}}^n$ and α.

- (c) $\implies$ (d): Suppose α is homotopic to $\mathrm{Id}_{\mathbb{S}}^n$. Then the induced maps on homology groups are equal, *i.e.* $\alpha_* = \mathrm{Id}_{H_n(\mathbb{S}^n)}$. Since $H_n(\mathbb{S}^n) \cong \mathbb{Z}$, this means that α_* acts as multiplication by some integer k on $H_n(\mathbb{S}^n)$. Since α is a homeomorphism, it must be that $k = \pm 1$. If $k = 1$, then α is orientation-preserving.

- (d) $\implies$ (e): Suppose α is orientation-preserving. Then the induced map on the top homology group is multiplication by 1. In particular, this means that the degree of α is 1 . Since the degree of α is also equal to $(-1)^{n+1}$, we must have n odd.

- (e) $\implies$ (a): Assume n is odd. We will construct a nowhere-vanishing vector field on $\mathbb{S}^n$. Define $V : \mathbb{S}^n \to \mathbb{R}^{n+1}$ by

$$V(x_1, \ldots, x_{n+1}) = (-x_2, x_1, -x_3, x_2, \ldots, (-1)^{n+1}x_n, x_{n-1}, (-1)^{n+2}x_{n+1}).$$

  We need to verify two properties:

  - $V(x)$ is tangent to $\mathbb{S}^n$ at x.
  - $V(x) \neq 0$ for all $x \in \mathbb{S}^n$.

  The first property follows from the construction. For the second, note that the coordinates of $V(x)$ have different signs, ensuring a non-zero vector.

  Thus, when n is odd, a nowhere-vanishing vector field exists on $\mathbb{S}^n$.

Therefore, we have showed that the statements are equivalent. ■

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 16.7.

Problem 16.7
Prove parts (a) and (b) of Proposition 16.6.

Proposition 16.6. (Properties of Integrals of Forms) Suppose M and N are non-empty oriented smooth n-manifolds with or without boundary, and ω, η are compactly supported n-forms on M.

(a) **Linearity:** If $a, b \in \mathbb{R}$, then

$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta.$$

(b) **Orientation Reversal:** If $-M$ denotes M with the opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega.$$

Solution. Solved in Study Guide for Quiz 4. ■

Homework 14

Problem 17-1
Let M be a smooth manifold with or without boundary, and let $\omega \in \Omega^p(M), \eta \in \Omega^q(M)$ be closed forms. Show that the de Rham cohomology class of $\omega \wedge \eta$ depends only on the cohomology classes of ω and η, and thus there is a well-defined bilinear map $\cup : H^p_{\mathrm{dR}}(M) \times H^q_{\mathrm{dR}}(M) \to H^{p+q}_{\mathrm{dR}}(M)$, called the **cup product**, given by $[\omega] \cup [\eta] = [\omega \wedge \eta]$.

Solution. Let $\omega' \in [\omega]$ and $\eta' \in [\eta]$ be other representatives of the same cohomology classes. Then

$$\omega' - \omega = \mathrm{d}\alpha, \text{ for some } \alpha \in \Omega^{p-1}(M)$$

and

$$\eta' - \eta = \mathrm{d}\beta, \text{ for some } \beta \in \Omega^{q-1}(M).$$

Consider $\omega' \wedge \eta'$, then

$$\begin{aligned} \omega' \wedge \eta' &= (\omega + \mathrm{d}\alpha) \wedge (\eta + \mathrm{d}\beta) \\ &= \omega \wedge \eta + \omega \wedge \mathrm{d}\beta + \mathrm{d}\alpha \wedge \eta + \mathrm{d}\alpha \wedge \mathrm{d}\beta. \end{aligned}$$

Recall that $\mathrm{d}^2 = 0$ and $d\alpha \wedge \mathrm{d}\beta = 0$ due to the graded anticommutativity of the exterior derivative. Also, $\mathrm{d}(\omega \wedge \mathrm{d}\beta) = \mathrm{d}\omega \wedge \mathrm{d}\beta = 0$ since ω is closed.
Therefore,

$$\omega' \wedge \eta' - \omega \wedge \eta = \mathrm{d}(\alpha \wedge \eta + \omega \wedge \beta).$$

This implies $[\omega' \wedge \eta'] = [\omega \wedge \eta]$ in $H^{p+q}_{\mathrm{dR}}(M)$, proving the cup product is well-defined. ■

Problem 17-8
Suppose M is a compact, connected, orientable, smooth n-manifold.

(a) Show that there is a one-to-one correspondence between orientations of M and orientations of the vector space $H^n_{\mathrm{dR}}(M)$, under which the cohomology class of a smooth orientation form is an oriented basis for $H^n_{\mathrm{dR}}(M)$.

(b) Now suppose M and N are smooth n-manifolds with given orientations. Show that a diffeomorphism $F : M \to N$ is orientation preserving if and only if $F^* : H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$ is orientation preserving.

Solution. (a) Let $\omega \in \Omega^n(M)$ be an orientation form. Since M is compact and connected, $[\omega] \in H^n_{\mathrm{dR}}(M)$ is a non-zero basis element.

- **Injective:** Suppose two orientations ω_1 and ω_2 are mapped to the same cohomology class. Then $\omega_1 = f\omega_2$ for some non-zero smooth function f. Since f is non-zero everywhere (due to the connectedness of M), the sign of f determines the same orientation. This proves injectivity.
- **Surjective:** Suppose $v \in H^n_{\mathrm{dR}}(M)$ is a non-zero cohomology class. Since $\dim H^n_{\mathrm{dR}}(M) = 1$, there exists a unique smooth volume form ω' such that $[\omega'] = v$. The form ω' defines an orientation on M. This proves surjectivity.

(b) We will prove both directions of the if and only if statement.

$\boxed{\Longrightarrow}$ Assume F is orientation preserving. Let ω_N be a volume form representing the orientation of N. Since F is orientation preserving, $F^*\omega_N$ is a volume form on M representing the orientation of M. By the previous theorem on the correspondence between manifold and cohomology orientations, this means $F^*\omega_N$ corresponds to the same orientation in $H^n_{\mathrm{dR}}(M)$ as ω_N does in $H^n_{\mathrm{dR}}(N)$. In other words, $[F^\omega_N] = F^*$ in $H^n_{\mathrm{dR}}(M)$, which means $F^{:} H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$ preserves orientation.

$\boxed{\Longleftarrow}$ Conversely, suppose $F^* : H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$ preserves orientation. Let ω_N be a volume form representing the orientation of N. Then $F^*\omega_N$ must correspond to the orientation of M under the correspondence we established in the previous theorem. By the properties of pullback and orientation, this means the sign of $\det(\mathrm{d}F)$ at each point must be positive. The condition that $\det(\mathrm{d}F)$ has a consistent positive sign is precisely the definition of an orientation-preserving diffeomorphism.

Therefore, a diffeomorphism F is orientation preserving if and only if F^* is orientation preserving.

■

Problem 17-13
Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the 2-torus. Consider the two maps $f, g : \mathbb{T}^2 \to \mathbb{T}^2$ given by $f(w, z) = (w, z)$ and $g(w, z) = (z, \bar{w})$. Show that f and g have the same degree, but are not homotopic. [Suggestion: consider the induced homomorphisms on the first cohomology group or the fundamental group.]

Solution. Consider the induced maps on $H^1(\mathbb{T}^2; \mathbb{R})$. Let α, β be the dual basis of the first cohomology group.

- For f^*, we have

$$f^*(\alpha) = \alpha,$$
$$f^*(\beta) = \beta.$$

- For g^*, we have

$$g^*(\alpha) = \beta,$$
$$g^*(\beta) = -\alpha.$$

Both maps have degree ± 1 on $H^1(\mathbb{T}^2; \mathbb{R})$. However, f and g are not homotopic. We will prove this by contradiction. Let f and g be homotopic, then their induced maps on $\pi_1(\mathbb{T}^2)$ would be conjugate. f_* is the identity on $\pi_1(\mathbb{T}^2)$ and g_* interchanges the generators of $\pi_1(\mathbb{T}^2)$.
Therefore, f and g are not homotopic. ■

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 17.22, 17.24, 17.29, 17.37.

Problem 17.22
Show that $\eta \in \Omega^n(\mathbb{S}^n)$ is exact if and only if $\int_{\mathbb{S}^n} \eta = 0$.

Solution. $\boxed{\Longrightarrow}$ First, suppose $\eta = \mathrm{d}\omega$ for some $\omega \in \Omega^{n-1}(\mathbb{S}^n)$. Then by Stokes' theorem, we have

$$\int_{\mathbb{S}^n} \eta = \int_{\mathbb{S}^n} \mathrm{d}\omega = \int_{\partial \mathbb{S}^n} \omega = 0,$$

since $\mathbb{S}^n$ has no boundary.
$\boxed{\Longleftarrow}$ Suppose $\int_{\mathbb{S}^n} \eta = 0$. Let p be any point in $\mathbb{S}^n$ and let $U = \mathbb{S}^n \setminus \{p\}$. Then U is diffeomorphic to $\mathbb{R}^n$ and thus all forms on U are exact since $H^n(\mathbb{R}^n) = 0$ for $n > 0$.
Therefore, there exists $\alpha \in \Omega^{n-1}(U)$ such that $\eta|_U = \mathrm{d}\alpha$. By the extension lemma for differential forms, we can extend α to a form $\tilde{\alpha} \in \Omega^{n-1}(\mathbb{S}^n)$.
Now, $\eta - \mathrm{d}\tilde{\alpha}$ vanishes on U since

$$(\eta - \mathrm{d}\tilde{\alpha})|_U = \eta|_U - \mathrm{d}(\tilde{\alpha}|_U) = \eta|_U - \mathrm{d}\alpha = 0.$$

Therefore, $\eta - \mathrm{d}\tilde{\alpha} = f\omega$ where ω is the volume form on $\mathbb{S}^n$ and f is a smooth function supported near p. Since $\int_{\mathbb{S}^n} \eta = 0$ and $\int_{\mathbb{S}^n} \mathrm{d}\tilde{\alpha} = 0$ by Stokes' theorem, we have

$$0 = \int_{\mathbb{S}^n} f\omega.$$

This means we can write $f\omega = \mathrm{d}\beta$ for some $(n-1)$-form β (this is a local computation near p where f is supported).
Thus

$$\eta = \mathrm{d}\tilde{\alpha} + f\omega = \mathrm{d}\tilde{\alpha} + \mathrm{d}\beta = \mathrm{d}(\tilde{\alpha} + \beta).$$

Therefore, η is exact. ■

Problem 17.24
Check that the statement and proof of Corollary 17.23 remain true if $\mathbb{R}^n \backslash \{x\}$ is replaced by $\mathbb{R}^n \backslash \bar{B}$ for some closed ball $\bar{B} \subseteq \mathbb{R}^n$.

Solution. Let B be a closed ball in $\mathbb{R}^n$ and let $M = \mathbb{R}^n \setminus \bar{B}$. We need to show

- $H^0_{\mathrm{dR}}(M)$ and $H^{n-1}_{\mathrm{dR}}(M)$ are 1-dimensional.
- All other de Rham groups are trivial.
- A closed $(n-1)$-form η is exact if and only if $\int_S \eta = 0$ for some (hence every) $(n-1)$-sphere S centered at the center of B.

Let S be any $(n-1)$-sphere in M centered at the center of B. The retraction of M onto S works exactly as in the punctured case: we can radially deform M onto S. Therefore, the inclusion $\iota : S \hookrightarrow M$ is still a homotopy equivalence.
This means $\iota^* : H^p_{\mathrm{dR}}(M) \to H^p_{\mathrm{dR}}(S)$ is an isomorphism for each p. By Theorem 17.21, $H^p_{\mathrm{dR}}(S)$ is 1-dimensional for $p = 0$ and $p = n-1$, and zero otherwise. Therefore, the same is true for $H^p_{\mathrm{dR}}(M)$.
For the last part, let η be a closed $(n-1)$-form on M. Then η is exact if and only if $\iota^*\eta$ is exact on S (since ι^* is an isomorphism). By Exercise 17.22, $\iota^*\eta$ is exact if and only if $\int_S \iota^*\eta = \int_S \eta = 0$.
Therefore, all conclusions remain valid when replacing a point with a closed ball. ■

Problem 17.29
Prove Theorem 17.28.

Theorem 17.28. (Compactly Supported Cohomology of $\mathbb{R}^n$). For $n \geq 1$, the compactly supported de Rham cohomology groups of $\mathbb{R}^n$ are

$$H_c^p(\mathbb{R}^n) \cong \begin{cases} 0 & \text{if } 0 \leq p < n, \\ \mathbb{R} & \text{if } p = n \end{cases}$$

Solution. First, let's show $H_c^p(\mathbb{R}^n) = 0$ for $0 \leq p < n$.
Let $\omega \in \Omega_c^p(\mathbb{R}^n)$ be a closed form with $p < n$. Let $K = \operatorname{supp}(\omega)$. Choose $R > 0$ large enough so that $K \subseteq B_R(0)$. Consider the map $h : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$h(t,x) = (1-t)x$$

Let $\alpha = \int_0^1 i_{X_t}(\omega \circ h_t)\,\mathrm{d}t$ where $X_t = x^i \frac{\partial}{\partial x^i}$ is the radial vector field and $h_t(x) = h(t,x)$. Then, by Cartan's magic formula, we have

$$\begin{aligned} \mathrm{d}\alpha + \alpha\,\mathrm{d} &= h^*\omega - \omega \\ \omega &= \mathrm{d}(-\alpha) + h^*\omega \end{aligned}$$

Note $h^*\omega$ has support in a smaller ball than ω. Iterating this process, we can show ω is exact.
For $p = n$, we need to show $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$. First, any n-form on $\mathbb{R}^n$ can be written as $f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ for some smooth function f. If it's closed, it's automatically closed since $\mathrm{d}\omega = 0$ for any n-form on an n-dimensional manifold. Define a map

$$\Phi : H_c^n(\mathbb{R}^n) \to \mathbb{R}, [\omega] \mapsto \int_{\mathbb{R}^n} \omega$$

This is well-defined because if $\omega = \mathrm{d}\eta$ for some compactly supported $(n-1)$-form η, then by Stokes' theorem, we have

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \mathrm{d}\eta = \int_{\partial\mathbb{R}^n} \eta = 0$$

Φ is clearly linear. It's surjective because for any $c \in \mathbb{R}$, we can find a compactly supported n-form ω with $\int_{\mathbb{R}^n} \omega = c$. For injectivity, if $\int_{\mathbb{R}^n} \omega = 0$, then by a similar argument to the first part using Cartan's formula, we can show ω is exact.
Therefore, Φ is an isomorphism and $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$. ∎

Problem 17.37
Prove Proposition 17.36.

Proposition 17.36. (Properties of the Degree). Suppose M, N, and P are compact, connected, oriented, smooth n-manifolds.

(a) If $F : M \to N$ and $G : N \to P$ are both smooth maps, then $\deg(G \circ F) = (\deg G)(\deg F)$.

(b) If $F : M \to N$ is a diffeomorphism, then $\deg F = +1$ if F is orientation-preserving and -1 if it is orientation-reversing.

(c) If two smooth maps $F_0, F_1 : M \to N$ are homotopic, then they have the same degree.

Solution. Solved in Study Guide for Quiz 4. ∎

Homework 15

Problem 16-18
Let (M, g) be an oriented Riemannian n-manifold. This problem outlines an important generalization of the operator $* : C^\infty(M) \to \Omega^n(M)$ defined in this chapter.

(a) For each $k = 1, \ldots, n$, show that g determines a unique inner product on $\Lambda^k\left(T_p^* M\right)$ (denoted by $\langle \cdot, \cdot \rangle_g$, just like the inner product on $T_p M$) satisfying

$$\left\langle \omega^1 \wedge \cdots \wedge \omega^k, \eta^1 \wedge \cdots \wedge \eta^k \right\rangle_g = \det\left(\left\langle \left(\omega^i\right)^\#, \left(\eta^j\right)^\# \right\rangle_g\right)$$

whenever $\omega^1, \ldots, \omega^k, \eta^1, \ldots, \eta^k$ are covectors at p. [Hint: define the inner product locally by declaring $\left\{ \varepsilon^I\big|_p : I \text{ is increasing} \right\}$ to be an orthonormal basis for $\Lambda^k\left(T_p^* M\right)$ whenever $\left(\varepsilon^i\right)$ is the coframe dual to a local orthonormal frame, and then prove that the resulting inner product is independent of the choice of frame.]

(b) Show that the Riemannian volume form $\mathrm{d}V_g$ is the unique positively oriented n-form that has unit norm with respect to this inner product.

(c) For each $k = 0, \ldots, n$, show that there is a unique smooth bundle homomorphism $* : \Lambda^k T^* M \to \Lambda^{n-k} T^* M$ satisfying

$$\omega \wedge *\eta = \langle \omega, \eta \rangle_g dV_g$$

for all smooth k-forms ω, η. (For $k = 0$, interpret the inner product as ordinary multiplication.) This map is called the Hodge star operator. [Hint: first prove uniqueness, and then define $*$ locally by setting

$$*\left(\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}\right) = \pm \varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_{n-k}}$$

in terms of an orthonormal coframe $\left(\varepsilon^i\right)$, where the indices $j_1, \ldots, j_{n-k}$ are chosen so that $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ is some permutation of $(1, \ldots, n)$.]

(d) Show that $* : \Lambda^0 T^* M \to \Lambda^n T^* M$ is given by $*f = f\, \mathrm{d}V_g$.

(e) Show that $* * \omega = (-1)^{k(n-k)} \omega$ if ω is a k-form.

Solution. (a) Let (ε^i) be the coframe dual to a local orthonormal frame and let $\{\varepsilon^I|_p : I \text{ is increasing}\}$ to be an orthonormal basis for $\Lambda^k(T_p^* M)$.

To prove independence of the choice of frame, let $(\tilde{\varepsilon}^i)$ be another orthonormal coframe. Then $\tilde{\varepsilon}^i = \sum_j A_{ij} \varepsilon^j$ where (A_{ij}) is an orthogonal matrix. For multi-indices I and J,

$$\begin{aligned}
\left\langle \tilde{\varepsilon}^I, \tilde{\varepsilon}^J \right\rangle_g &= \left\langle \sum_{i_1} A_{i_1, j_1} \varepsilon^{i_1}, \ldots, \sum_{i_k} A_{i_k, j_k} \varepsilon^{i_k} \right\rangle_g \\
&= \det\left(\left(A_{i_\alpha j_\beta}\right)\right)_{\alpha, \beta = 1}^k \langle \varepsilon^I, \varepsilon^J \rangle_g \\
&= \det(A) \langle \varepsilon^I, \varepsilon^J \rangle_g .
\end{aligned}$$

This shows the inner product is well-defined and independent of the choice of frame.

(b) The Riemannian volume form $\mathrm{d}V_g$ is locally given by $\varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ where (ε^i) is an orthonormal coframe. By definition, this has unit norm. It's also positively oriented. Uniqueness follows from the fact that any other such form would differ by a positive scalar multiple, which would change its norm.

(c) For uniqueness, suppose $*_1$ and $*_2$ are two such operators. Then for any k-forms ω and η,

$$\begin{aligned}
\omega \wedge *_1 \eta &= \langle \omega, \eta \rangle_g dV_g = \omega \wedge *_2 \eta \\
\omega \wedge (*_1 \eta - *_2 \eta) &= 0
\end{aligned}$$

This implies $*_1\eta = *_2\eta$ for all η, so $*_1 = *_2$.

For existence, define $*$ locally as suggested in the hint. Then verify that

$$\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} \wedge *(\varepsilon^{j_1} \wedge \cdots \wedge \varepsilon^{j_k}) = \delta_{i_1 j_1} \cdots \delta_{i_k j_k} \, \mathrm{d}V_g,$$

which is equivalent to the required property.

(d) For $f \in C^\infty(M)$, we have

$$f \wedge *g = \langle f, g \rangle_g \, \mathrm{d}V_g = fg \, \mathrm{d}V_g$$

for all $g \in C^\infty(M)$. This implies $*f = f \, \mathrm{d}V_g$.

(e) In an orthonormal coframe, applying $*$ twice to a basis k-form gives

$$* * (\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}) = (-1)^{k(n-k)} (\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k})$$

The sign comes from the number of transpositions needed to bring $(i_1, \ldots, i_k, j_1, \ldots, j_{n-k})$ back to $(1, \ldots, n)$ after reversing it. This property then extends linearly to all k-forms.

∎

Recommended Problems

Recommended practice/review problems (ungraded, please do not submit): 13.7, 13.10, 13.13, 13.15, 13.22, 13.23, 13.27, 14.28, 15.30, 16.31.

Problem 13.7
Prove Proposition 13.6.

Proposition 13.6. Suppose (M, g) is a Riemannian manifold with or without boundary, and (X_j) is a smooth local frame for M over an open subset $U \subseteq M$. Then there is a smooth orthonormal frame (E_j) over U such that $\operatorname{span}\left(E_1|_p, \dots, E_j|_p\right) = \operatorname{span}\left(X_1|_p, \dots, X_j|_p\right)$ for each $j = 1, \dots, n$ and each $p \in U$.

Solution. We will construct E_j inductively using the Gram-Schmidt orthonormalization process.
Let $E_1 = \frac{X_1}{|X_1|}$. This is well-defined and smooth since X_1 is nowhere zero (being part of a frame). Now assume we have constructed orthonormal vector fields $E_1, \dots, E_k$ for some $k < n$ such that

$$\operatorname{span}(E_1|_p, \dots, E_j|_p) = \operatorname{span}(X_1|_p, \dots, X_j|_p),$$

for all $j \le k$ and $p \in U$. Define

$$Y_{k+1} = X_{k+1} - \sum_{i=1}^{k} g(X_{k+1}, E_i) E_i,$$

then Y_{k+1} is smooth since it's a sum of smooth vector fields with smooth coefficients. Moreover, Y_{k+1} is nowhere zero because $X_{k+1}|_p$ is linearly independent of $\{X_1|_p, \dots, X_k|_p\}$ at each p, and the projection removes only components in $\operatorname{span}(E_1|_p, \dots, E_k|_p) = \operatorname{span}(X_1|_p, \dots, X_k|_p)$. Therefore, we can define

$$E_{k+1} = \frac{Y_{k+1}}{|Y_{k+1}|}.$$

E_{k+1} is smooth because Y_{k+1} is smooth and nowhere zero. By construction, E_{k+1} is orthogonal to $E_1, \dots, E_k$ and $\operatorname{span}(E_1|_p, \dots, E_{k+1}|_p) = \operatorname{span}(X_1|_p, \dots, X_{k+1}|_p)$.
By induction, we obtain a smooth orthonormal frame $(E_1, \dots, E_n)$ with the desired spanning property. ■

Problem 13.10
Prove Proposition 13.9.

Proposition 13.9. (Pullback Metric Criterion). Suppose $F : M \to N$ is a smooth map and g is a Riemannian metric on N. Then F^*g is a Riemannian metric on M if and only if F is a smooth immersion.

Solution. Recall that, for any $p \in M$ and $\mathbf{v}, \mathbf{w} \in T_pM$, we have

$$(F^*g)_p(\mathbf{v}, \mathbf{w}) = g_{F(p)}(\mathrm{d}F_p\mathbf{v}, \mathrm{d}F_p\mathbf{w}).$$

$\boxed{\Longrightarrow}$ Suppose F^*g is a Riemannian metric. Then, for any $p \in M$ and nonzero $\mathbf{v} \in T_pM$, we have

$$0 < (F^*g)_p(\mathbf{v}, \mathbf{v}) = g_{F(p)}(\mathrm{d}F_p\mathbf{v}, \mathrm{d}F_p\mathbf{v}).$$

Since g is positive definite, this implies $\mathrm{d}F_p\mathbf{v} \neq 0$. Therefore, $\mathrm{d}F_p$ is injective for all $p \in M$, which means F is a smooth immersion.
$\boxed{\Longleftarrow}$ Suppose F is a smooth immersion. We need to show F^*g is symmetric, bilinear, positive definite, and smooth. The first, second, and fourth properties follow immediately from the corresponding properties of g since pullback preserves these properties.
For positive definiteness, let $\mathbf{v}$ be a nonzero vector in T_pM. Then $\mathrm{d}F_p\mathbf{v} \neq 0$ since F is an immersion. Thus,

$$(F^*g)_p(\mathbf{v}, \mathbf{v}) = g_{F(p)}(\mathrm{d}F_p\mathbf{v}, \mathrm{d}F_p\mathbf{v}) > 0,$$

since g is positive definite.
Therefore, F^*g is a Riemannian metric on M. ■

Problem 13.13
Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are isometric Riemannian manifolds. Show that g is flat if and only if $\widetilde{g}$ is flat.

Solution. Let $F : M \to \widetilde{M}$ be an isometry. Then F is a diffeomorphism and $F^*\widetilde{g} = g$. Recall that a metric is flat if and only if its Riemann curvature tensor vanishes everywhere. Let $\widetilde{R}$ be the Riemann curvature tensor of $\widetilde{M}$ and R be the Riemann curvature tensor of M. By the naturality of the Riemann curvature tensor under isometries, we have

$$F^*\widetilde{R} = R.$$

Thus, for any $p \in M$ and $\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y} \in T_pM$, we have

$$\begin{aligned} R_p(\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}) &= (F^*\widetilde{R})_p(\mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}) \\ &= \widetilde{R}_{F(p)}(\mathrm{d}F_p\mathbf{v}, \mathrm{d}F_p\mathbf{w}, \mathrm{d}F_p\mathbf{x}, \mathrm{d}F_p\mathbf{y}). \end{aligned}$$

Since F is a diffeomorphism, $\mathrm{d}F_p$ is an isomorphism. This means $R_p = 0$ if and only if $\widetilde{R}_{F(p)} = 0$. Therefore, g is flat if and only if $\widetilde{g}$ is flat. ∎

Problem 13.15
Complete the preceding proof by showing (a) ⇒ (b) ⇒ (c) ⇒ (d).

Theorem 13.14. For a Riemannian manifold (M, g), the following are equivalent:

(a) g is flat.

(b) Each point of M is contained in the domain of a smooth coordinate chart in which g has the coordinate representation $g = \delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$.

(c) Each point of M is contained in the domain of a smooth coordinate chart in which the coordinate frame is orthonormal.

(d) Each point of M is contained in the domain of a commuting orthonormal frame.

Solution. $\boxed{\text{(a)} \implies \text{(b)}}$ Let $p \in M$. Since g is flat, the Riemann curvature tensor vanishes in a neighborhood U of p. By Frobenius's theorem, we can find coordinates $(x^1, \dots, x^n)$ on a smaller neighborhood such that $\{\frac{\partial}{\partial x^i}\}$ form an orthonormal frame. Then, in these coordinates, we have

$$g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}.$$

Therefore, $g = \delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$ in these coordinates.
$\boxed{\text{(b)} \implies \text{(c)}}$ If $g = \delta_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j$ in some coordinates $(x^1, \dots, x^n)$, then

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}.$$

This means $\{\frac{\partial}{\partial x^i}\}$ forms an orthonormal coordinate frame.
$\boxed{\text{(c)} \implies \text{(d)}}$ If $\{\frac{\partial}{\partial x^i}\}$ is an orthonormal coordinate frame in some chart, then it's automatically a commuting frame since coordinate vector fields always commute; hence,

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

Therefore, $\{\frac{\partial}{\partial x^i}\}$ is a commuting orthonormal frame around each point. ∎

Problem 13.22
Prove Proposition 13.21.

Proposition 13.21. (The Normal Bundle to a Riemannian Submanifold). Let (M, g) be a Riemannian n-manifold with or without boundary. For any immersed k-dimensional submanifold $S \subseteq M$ with or without boundary, the normal bundle NS is a smooth rank- $(n-k)$ subbundle of $TM \mid s$. For each $p \in S$, there is a smooth frame for NS on a neighborhood of p that is orthonormal with respect to g.

Solution. First recall that for each $p \in S$

$$N_pS = \{\mathbf{v} \in T_pM : g_p(\mathbf{v}, \mathbf{w}) = 0, \forall \mathbf{w} \in T_pS\}.$$

Let's show NS is a smooth subbundle. Let $p \in S$ and let $(E_1, \ldots, E_k)$ be a local orthonormal frame for TS near p. We can extend this to an orthonormal frame $(E_1, \ldots, E_n)$ for TM using Proposition 13.6. Then $(E_{k+1}, \ldots, E_n)$ forms a local frame for NS near p. Indeed, these vectors are orthogonal to TS by construction, are linearly independent, and any vector orthogonal to TS must be in their span (by dimension). This shows NS has constant rank $n-k$. The frame varies smoothly since the E_i are smooth.
For the second part of the proposition, we already constructed an orthonormal frame $(E_{k+1}, \ldots, E_n)$ for NS near any point $p \in S$. To show NS is a smooth subbundle, let's verify it's the kernel of a smooth bundle map of constant rank. Consider the map

$$\Phi : TM|_S \to T^*S \otimes TM|_S$$

defined by $\Phi(\mathbf{v})(\mathbf{w}) = g(\mathbf{v}, \mathbf{w})$ for $\mathbf{v} \in T_pM$ and $\mathbf{w} \in T_pS$. Then Φ is smooth since g is smooth and has constant rank since g is non-degenerate. Additionally, $\ker \Phi = NS$ by definition.
Therefore, NS is a smooth subbundle of $TM|_S$. ■

Problem 13.23
Suppose $\gamma : [a, b] \to M$ is a piecewise smooth curve segment and $a < c < b$. Show that

$$\mathrm{L}_g(\gamma) = \mathrm{L}_g\left(\gamma|_{[a,c]}\right) + \mathrm{L}_g\left(\gamma|_{[c,b]}\right).$$

Solution. Recall that the length of a piecewise smooth curve γ is defined as

$$\mathrm{L}_g(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t.$$

Using the additivity property of definite integrals, we have

$$\begin{aligned}
\mathrm{L}_g(\gamma) &= \int_a^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t \\
&= \int_a^c \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t + \int_c^b \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t \\
&= \mathrm{L}_g\left(\gamma|_{[a,c]}\right) + \mathrm{L}_g\left(\gamma|_{[c,b]}\right).
\end{aligned}$$

Note that this holds at the point c even if γ is not smooth there since the length is well-defined for piecewise smooth curves and the integral is unaffected by the behavior at a single point. ■

Problem 13.27
Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are connected Riemannian manifolds and $F : M \to \widetilde{M}$ is a Riemannian isometry. Show that $d_{\tilde{g}}(F(p), F(q)) = d_g(p, q)$ for all $p, q \in M$.

Solution. Let $p, q \in M$. By the definition of Riemannian distance, we have

$$d_g(p,q) = \inf\{\mathrm{L}_g(\gamma) \mid \gamma \text{ is a piecewise smooth curve from } p \text{ to } q\}.$$

Since F is a Riemannian isometry, then F is a diffeomorphism and $F^*\widetilde{g} = g$. Let $\gamma : [a,b] \to M$ be any piecewise smooth curve from p to q. Consider $F \circ \gamma$, which is a piecewise smooth curve from $F(p)$ to $F(q)$. For its length, we have

$$\begin{aligned}
\mathrm{L}_{\widetilde{g}}(F \circ \gamma) &= \int_a^b \sqrt{\widetilde{g}_{F(\gamma(t))}\Big((F\circ\gamma)'(t),\, (F\circ\gamma)'(t)\Big)}\,\mathrm{d}t \\
&= \int_a^b \sqrt{\widetilde{g}_{F(\gamma(t))}\Big(\mathrm{d}F_{\gamma(t)}\gamma'(t),\, \mathrm{d}F_{\gamma(t)}\gamma'(t)\Big)}\,\mathrm{d}t \\
&= \int_a^b \sqrt{g_{\gamma(t)}\Big(\gamma'(t),\, \gamma'(t)\Big)}\,\mathrm{d}t \\
&= \mathrm{L}_g(\gamma).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
d_{\widetilde{g}}(F(p), F(q)) &= \inf\{\mathrm{L}_{\widetilde{g}}(\widetilde{\gamma}) \mid \widetilde{\gamma} \text{ is a piecewise smooth curve from } F(p) \text{ to } F(q)\} \\
&= \inf\{\mathrm{L}_g(\gamma) \mid \gamma \text{ is a piecewise smooth curve from } p \text{ to } q\} \\
&= \mathrm{d}_g(p,q),
\end{aligned}$$

where the second equality follows because F is a diffeomorphism, so every curve in $\widetilde{M}$ from $F(p)$ to $F(q)$ is of the form $F \circ \gamma$ for some curve γ in M from p to q. ■

Problem 14.28
Prove that diagram (14.27) commutes, and use it to give a quick proof that curl $\circ$ grad $\equiv 0$ and div $\circ$ curl $\equiv 0$ on $\mathbb{R}^3$. Prove also that the analogues of the left-hand and right-hand squares commute when $\mathbb{R}^3$ is replaced by $\mathbb{R}^n$ for any n.

Solution. Let us verify the diagram commutes by checking each square.

- For the leftmost square, take $f \in C^\infty(\mathbb{R}^3)$. Then $\flat \circ \mathrm{grad}(f)$ maps to a 1-form

  $$(\flat \circ \mathrm{grad}(f))_i = \sum_j g_{ij} \frac{\partial f}{\partial x^j} = \frac{\partial f}{\partial x^i},$$

  which matches $\mathrm{d}f$ component by component. Thus, the left square commutes.

- For the middle square, let $X \in \mathfrak{X}(\mathbb{R}^3)$. Then

  $$\begin{aligned}
  \beta(\mathrm{curl}(X))_{ij} &= (\mathrm{curl}(X))_k \epsilon^k_{ij} \\
  &= \epsilon^k_{ij}(\partial_i X_j - \partial_j X_i) \\
  &= (\flat \circ \mathrm{d}X^)_{ij},
  \end{aligned}$$

  where ϵ^k_{ij} is the Levi-Civita symbol. Thus, the middle square commutes.

- For the rightmost square, take $X \in \mathfrak{X}(\mathbb{R}^3)$. Then

  $$* \circ \mathrm{div}(X) = \partial_i X^i = \mathrm{d} \circ \beta(X).$$

  Thus, the right square commutes.

Now we can prove curl $\circ$ grad $\equiv 0$ using the diagram. For any $f \in C^\infty(\mathbb{R}^3)$:

$$\operatorname{curl} \circ \operatorname{grad}(f) \cong \mathrm{d} \circ \mathrm{d}f = 0,$$

since $\mathrm{d} \circ \mathrm{d} = 0$. Similarly,

$$\operatorname{div} \circ \operatorname{curl}(X) \cong \mathrm{d} \circ \mathrm{d}(X^\flat) = 0.$$

For general $\mathbb{R}^n$, the left square commutes by the same calculation as in step 1. The right square commutes because divergence and exterior derivative correspond under the metric isomorphism.
The power of this diagram is that it connects vector calculus operators to exterior calculus operators, allowing us to use algebraic properties like $\mathrm{d} \circ \mathrm{d} = 0$ to prove vector calculus identities that would otherwise require tedious component calculations. ■

Problem 15.30
Suppose (M, g) and $(\widetilde{M}, \widetilde{g})$ are positive-dimensional Riemannian manifolds with or without boundary, and $F : M \to \widetilde{M}$ is a local isometry. Show that $F^*\omega_{\tilde{g}} = \omega_g$.

Solution. First, note that since F is a local isometry, for any $p \in M$ and any orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM, the set $\{\mathrm{d}F_p(e_1), \dots, \mathrm{d}F_p(e_n)\}$ is an orthonormal basis of $T_{F(p)}\widetilde{M}$.
Let $p \in M$ and let $\{e_1, \dots, e_n\}$ be an oriented orthonormal basis of T_pM. Then we have

$$\begin{aligned}(F^*\omega_{\tilde{g}})_p(e_1, \dots, e_n) &= (\omega_{\tilde{g}})_{F(p)}(\mathrm{d}F_p(e_1), \dots, \mathrm{d}F_p(e_n)) \\ &= 1,\end{aligned}$$

where the last equality follows because $\{\mathrm{d}F_p(e_1), \dots, \mathrm{d}F_p(e_n)\}$ is oriented and orthonormal. Similarly, by definition of the volume form, we have

$$(\omega_g)_p(e_1, \dots, e_n) = 1.$$

Since both forms agree on an oriented orthonormal basis at each point and both are n-forms, we conclude that $F^*\omega_{\tilde{g}} = \omega_g$. ■

Note: The proof uses the key fact that volume forms are characterized by their values on oriented orthonormal bases, and local isometries preserve both the orthonormal property and orientation.

Problem 16.31
Show that divergence operator on an oriented Riemannian manifold does not depend on the choice of orientation, and conclude that it is invariantly defined on all Riemannian manifolds.

Solution. Recall that for a vector field X on an oriented Riemannian manifold (M, g), the divergence is defined by the equation

$$\mathscr{L}_X\omega_g = (\operatorname{div}_g X)\omega_g,$$

where ω_g is the Riemannian volume form.
Now suppose we change the orientation of M. This changes the volume form to $-\omega_g$. Then for any vector field X, we have

$$\begin{aligned}\mathscr{L}_X(-\omega_g) &= -\mathscr{L}_X\,\omega_g \quad \text{(linearity of Lie derivative)} \\ &= -(\operatorname{div}_g X)\,\omega_g \quad \text{(by definition)} \\ &= (\operatorname{div}_g X)(-\omega_g).\end{aligned}$$

This shows that the divergence defined using $-\omega_g$ gives the same result as the divergence defined using ω_g. Therefore, the divergence operator does not depend on the choice of orientation.
To see that divergence is invariantly defined on any Riemannian manifold (not necessarily orientable), we can use the following argument:
Every Riemannian manifold has an oriented double cover. The divergence operator on the double cover descends to a well-defined operator on the original manifold since:

1. The divergence is local (it only depends on the behavior of X in arbitrarily small neighborhoods)
2. We just proved it is independent of orientation
3. The divergence commutes with local isometries (as shown in a previous exercise)

Therefore, the divergence operator is well-defined on any Riemannian manifold, regardless of orientability. ■

Note: This shows that divergence is a fundamental geometric operator that depends only on the Riemannian metric, not on any additional structure like orientation.

Study Guide for Quiz 1

The following is a list of problems you should practice and understand how to solve before the quiz.

Problem 1
Let M be a topological space that is locally Euclidean of dimension n. Recall that an atlas is any set $\mathcal{A} = \{(U_\lambda, \phi_\lambda)\}_{\lambda\in\Lambda}$ (where Λ is some index set) of coordinate charts $\phi_\lambda : U_\lambda \to \phi_\lambda(U_\lambda) \subseteq \mathbb{R}^n$ on M such that $\bigcup_{\lambda\in\Lambda} U_\lambda = M$. Show that there exists a unique maximal atlas on M. (Note: this problem has nothing to do with smoothness.)

Solution. An atlas $\mathcal{A}$ is called maximal if, for all atlases $\mathcal{B}$ on M with $[\mathcal{A}] = [\mathcal{B}]$, we have that $\mathcal{B} \subset \mathcal{A}$. Every atlas is contained in a maximal atlas. We can show that there always exists a maximal atlas on M by Zorn's lemma.

Lemma (Zorn's Lemma). Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

In fact, atlases form a partially ordered set with respect to containment "$\subset$".
To prove that this maximal atlas is unique, suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are two maximal atlases on M such that there exists an atlas $\mathcal{B}$ on M with $[\mathcal{A}_1] = [\mathcal{B}]$ and $[\mathcal{A}_2] = [\mathcal{B}]$. Notice that

$$\varphi \circ \psi^{-1} = \left(\varphi \circ \phi^{-1}\right) \circ \left(\phi \circ \psi^{-1}\right)$$

for the transition functions of charts with overlaps $\varphi \in \mathcal{A}_1$, $\psi \in \mathcal{A}_2$, and $\phi \in \mathcal{B}$. Since $\mathcal{B}$ is an atlas, its charts cover M and then $\mathcal{A}_1$ and $\mathcal{A}_2$ coincide.
Therefore, there exists a unique maximal atlas on M. ■

Problem 1-8
By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An **angle function** on a subset $U \subseteq \mathbb{S}^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function θ on an open subset $U \subseteq \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure. (Used on pp. $37, 152, 176$.)

Solution. $\boxed{\Longrightarrow}$ Let θ be an angle function on a subset $U \subseteq \mathbb{S}^1$. We will prove this by contradiction. Suppose that $U = \mathbb{S}^1$. Since the image of a connected and compact space under a continuous function is connected and compact, we have that $\theta\left(\mathbb{S}^1\right) = [a, b] \subseteq \mathbb{R}$ for some $a \leq b$. We will now show that

$$\begin{aligned} \tilde{\theta} : \mathbb{S}^1 &\to [a, b], \\ z &\mapsto \theta(p) \end{aligned}$$

is a homeomorphism, which will give us a contradiction, since $[a, b]$ is simply connected, but $\mathbb{S}^1$ is not. It suffices to show that $\tilde{\theta}$ is bijective, since a continuous bijection with compact domain has a continuous inverse. By definition, $\tilde{\theta}$ is surjective. If $\tilde{\theta}(z) = \tilde{\theta}(z')$, then $z = e^{i\theta(z)} = e^{i\theta\left(z'\right)} = z'$, so $\tilde{\theta}$ is injective, and therefore bijective. Thus, $\tilde{\theta}$ is a homeomorphism which means that $\mathbb{S}^1$ is simply connected, but we know it is not, which is a contradiction. Therefore, $U \neq \mathbb{S}^1$.

$\boxed{\Longleftarrow}$ Let $U \subset \mathbb{S}^1$ be a proper open subset. Suppose that $p \in \mathbb{S}^1 \backslash U$. We will construct an angle function θ on $\mathbb{S}^1 \setminus \{p\}$. The restriction of θ to U is therefore also an angle function. Note that $\theta(z) = -i \ln(z)$ is an angle function, where we must take a branch of the complex logarithm along the ray through p and the origin. Therefore, there exists an angle function θ on an open subset $U \subseteq \mathbb{S}^1$.
Let $\theta : U \to \mathbb{R}$ be an angle function. We proved that θ is injective in the $\boxed{\Longrightarrow}$ direction above. Hence, by the theorem on invariance of domain, θ is an open map and is therefore a homeomorphism onto its image.

This shows that (U,θ) is a chart for $\mathbb{S}^1$. Suppose that $U = \mathbb{S}^1 \setminus \{N\}$. Since $\theta^{-1}(x) = e^{i\theta\left(\theta^{-1}(x)\right)} = e^{ix}$ corresponds to $(\cos(x),\sin(x)) \subseteq \mathbb{R}^n$. Let σ be the stereographic projection from the north. Then

$$\sigma \circ \theta^{-1}(x) = \frac{\cos(x)}{1-\sin(x)} = \tan\left(\frac{x}{2}+\frac{\pi}{4}\right),$$

which is a diffeomorphism from any interval of the form $\left(\frac{\pi}{2}+2\pi n, \frac{5\pi}{2}+2\pi n\right)$ to $\mathbb{R}$, where $n \in \mathbb{Z}$. Now, let $U \subset \mathbb{S}^1$ be an arbitrary proper open subset. By rotating $\mathbb{S}^1$, we may assume that $N \notin U$. Then $\sigma \circ \theta^{-1}$ is the restriction of the above formula on each connected component of U. Thus, (U,σ) is smoothly compatible with the stereographic coordinate charts, and is thus a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure. ■

Problem 3
Let $F : M \to N$ be a diffeomorphism and let $O \subseteq M$ be an open submanifold. Show that $F|_O : O \to F(O)$ is a diffeomorphism.

Solution. O is an open submanifold of M, then $F(O)$ is also an open submanifold of N, since F being a diffeomorphism (a homeomorphism suffices) means it is an open map.
Since $F(O)$ is an open submanifold of N, it automatically inherits a unique smooth structure from the smooth structure of N.
By the uniqueness of the smooth structures inherited on O from M and on $F(O)$ from N, in addition to the fact that the map F preserves the smooth structure of M to the smooth structure of N, then $F|_O$ will map the smooth structure of O to the smooth structure of $F(O)$.
Therefore, $F|_O$ is a diffeomorphism.

■

Problem 3-2
Prove Proposition 3.14 (the tangent space to a product manifold).

Proposition 3.14. (The Tangent Space to a Product Manifold). Let $M_1,\ldots,M_k$ be smooth manifolds, and for each j, let $\pi_j : M_1 \times \cdots \times M_k \to M_j$ be the projection onto the M_j factor. For any point $p = (p_1,\ldots,p_k) \in M_1 \times \cdots \times M_k$, the map

$$\alpha : T_p\left(M_1 \times \cdots \times M_k\right) \to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

defined by

$$\alpha(v) = \left(d\left(\pi_1\right)_p(v),\ldots,d\left(\pi_k\right)_p(v)\right)$$

is an isomorphism. The same is true if one of the spaces M_i is a smooth manifold with boundary.

Solution. For each $i = 1,\ldots,k$, let

$$\begin{aligned}\beta_i : M_i &\to M_1 \times \cdots \times M_k,\\ x &\mapsto \left(p_1,\ldots,p_{i-1},x,p_{i+1},\ldots,p_k\right),\end{aligned}$$

and let

$$\begin{aligned}\beta : T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k &\to T_p\left(M_1 \times \cdots \times M_k\right),\\ v_1 \oplus \cdots \oplus v_k &\mapsto d\left(\beta_1\right)_{p_1}\left(v_1\right) + \cdots + d\left(\beta_k\right)_{p_k}\left(v_k\right).\end{aligned}$$

It is straightforward to check that β is a linear map. Let α be the map defined in Proposition 3.14, that is

$$\begin{aligned}\alpha : T_p(M_1 \times \cdots \times M_k) &\to T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k\\ v &\mapsto \alpha(v) = (d(\pi_1)_p(v),\ldots,d(\pi_k)_p(v)).\end{aligned}$$

Note that $\pi_i \circ \beta_i$ is the identity on M_i if $i = j$, and is the constant map at p_j otherwise. It follows from this observation and part (b) of Proposition 3.6 that $\alpha \circ \beta$ is the identity on $T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$. Then α is surjective. Since $T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$ and $T_p(M_1 \times \cdots \times M_k)$ have the same dimension (namely $\sum_i \dim(M_i)$), and α is a surjection, then α is an isomorphism.
If one of the M_i is a smooth manifold with boundary, then $M_1 \times \cdots \times M_k$ is a smooth manifold with boundary. Hence, it has a well defined tangent space at each point, and differentials of maps are defined. The same proof then applies in this case. ■

Problem 3.17
Let (x, y) denote the standard coordinates on $\mathbb{R}^2$. Verify that $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$, where
$$\tilde{x} = x, \quad \tilde{y} = y + x^3$$
Let p be the point $(1, 0) \in \mathbb{R}^2$ (in standard coordinates), and show that
$$\left.\frac{\partial}{\partial x}\right|_p \neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p$$
even though the coordinate functions x and $\tilde{x}$ are identically equal.

Solution. The map
$$\begin{aligned} \phi : \mathbb{R}^2 &\to \mathbb{R}^2, \\ (x, y) &\mapsto \left(x, y + x^3\right) \end{aligned}$$
is smooth with a smooth inverse
$$\begin{aligned} \phi^{-1} : \mathbb{R}^2 &\to \mathbb{R}^2, \\ (x, y) &\mapsto \left(x, y - x^3\right). \end{aligned}$$
Therefore, $(\tilde{x}, \tilde{y})$ are global smooth coordinates on $\mathbb{R}^2$. Now,
$$\begin{aligned} \left.\frac{\partial}{\partial x}\right|_p &= \left.\frac{\partial \tilde{x}}{\partial x}\right|_p \frac{\partial}{\partial \tilde{x}} + \left.\frac{\partial \tilde{y}}{\partial x}\right|_p \frac{\partial}{\partial \tilde{y}} \\ &= \frac{\partial}{\partial \tilde{x}} + 3\frac{\partial}{\partial \tilde{y}} \\ &\neq \left.\frac{\partial}{\partial \tilde{x}}\right|_p. \end{aligned}$$
■

Study Guide for Quiz 2

The following is a list of problems you should practice and understand how to solve before the quiz.

Problem 4.4
Show that a composition of smooth submersions is a smooth submersion, and a composition of smooth immersions is a smooth immersion. Give a counterexample to show that a composition of maps of constant rank need not have constant rank.

Solution.

- Let $F : M \to N$ and $G : N \to O$ be smooth submersions and fix a point $p \in M$. The composition map $G \circ F : M \to O$ is smooth ? - ? and its differential at p is a linear map. Then

  $$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p : T_pM \to T_{(G\circ F)(p)}O$$

  is surjective since both linear maps

  $$\mathrm{d}F_p : T_pM \to T_{F(p)}N \quad \text{and} \quad \mathrm{d}G_{F(p)} : T_{F(p)}N \to T_{(G\circ F)(p)}O$$

  are surjective by assumption. Since p was an arbitrary point, then $G \circ F$ is a smooth submersion.

- The same argument as above but the words "submersion" and "surjective" are replaced by "immersion" and "injective", respectively.

- The maps

  $$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}^2 \\ t &\mapsto (t, t^2) \end{aligned}$$

  and

  $$\begin{aligned} g : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto y \end{aligned}$$

  have rank one everywhere (constant rank). The composition map

  $$\begin{aligned} g \circ f : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto t^2 \end{aligned}$$

  has rank one for all $t \neq 0$ but rank zero and $t = 0$.

  Therefore, the composition of maps of constant rank need not have constant rank.

■

Problem 4.32
Prove Theorem 4.31.

Theorem 4.31. (Uniqueness of Smooth Quotients). Suppose that M, N_1, and N_2 are smooth manifolds, and $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ are surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F : N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$:

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ N_1 \arrow[rr, dashed, "F"'] & & N_2 \end{tikzcd}$$

Solution. Theorem 4.30 states

Theorem 4.30. (Passing Smoothly to the Quotient). Suppose M and N are smooth manifolds and $\pi : M \to N$ is a surjective smooth submersion. If P is a smooth manifold with or without boundary and $F : M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\tilde{F} : N \to P$ such that $\tilde{F} \circ \pi = F$.

Thus, by Theorem 4.30, there are smooth maps $F : N_1 \to N_2$ and $G : N_2 \to N_1$ such that $F \circ \pi_1 = \pi_2$ and $G \circ \pi_2 = \pi_1$. Then $F \circ G \circ \pi_2 = \pi_2$. Since π_2 is surjective, $F \circ G = \mathrm{Id}$.
A similar argument shows that $G \circ F = \mathrm{Id}$.
Hence, $G = F^{-1}$ and F is a diffeomorphism. ■

Problem 5.36
Prove Proposition 5.35.

Proposition 5.35. Suppose M is a smooth manifold with or without boundary, $S \subseteq M$ is an immersed or embedded submanifold, and $p \in S$. A vector $v \in T_pM$ is in T_pS if and only if there is a smooth curve $\gamma : J \to M$ whose image is contained in S, and which is also smooth as a map into S, such that $0 \in J, \gamma(0) = p$, and $\gamma'(0) = v$.

Solution. Let $\dim(M) = n$ and $\dim(S) = k$ and let $v \in T_pM$.
$\boxed{\Longrightarrow}$ Suppose that $v \in T_pS$. By Proposition 5.22, there is an open neighbourhood V of p such that $\iota|_V : V \hookrightarrow M$ is an embedding, and by Theorem 5.8, there is a smooth chart (U, φ) for M centered at p such that $V \cap U$ is a k-slice in U. Letting $\left(x^1, \ldots, x^n\right)$ be the coordinates, we have

$$v = v^1 \left.\frac{\partial}{\partial x^1}\right|_p + \cdots + v^n \left.\frac{\partial}{\partial x^n}\right|_p,$$

for some $v^i \in \mathbb{R}$. On one hand, we have $v(x^i) = v\left(x^i|_S\right) = 0$ if $i = k+1, \ldots, n$ (we are viewing the x^i as the coordinate functions of φ here). On the other hand,

$$\left(v^1 \left.\frac{\partial}{\partial x^1}\right|_p + \cdots + v^n \left.\frac{\partial}{\partial x^n}\right|_p\right)\left(x^i\right) = v^i,$$

so we have

$$v = v^1 \left.\frac{\partial}{\partial x^1}\right|_p + \cdots + v^k \left.\frac{\partial}{\partial x^k}\right|_p.$$

Let $\tilde{\gamma} : J \to \varphi(U)$ be the curve defined by $\tilde{\gamma}(t) = \left(v^1 t, \ldots, v^k t, 0, \ldots, 0\right)$ for some interval $J \subseteq \mathbb{R}$ containing 0 . Letting $\gamma = \varphi^{-1} \circ \tilde{\gamma}$, we have that the image of γ is in $S, \gamma(0) = p$, and $\gamma'(0) = v$. In fact, the image of γ is contained in V, which is embedded in M. Corollary 5.30 implies that $\gamma : J \to V$ is also smooth. Since V is an open submanifold of S, we have that $\gamma : J \to S$ is smooth.
$\boxed{\Longleftarrow}$ Suppose there is a smooth curve $\gamma : J \to M$ such that $\gamma(J) \subseteq S, \gamma$ is smooth as a map into $S, 0 \in J, \gamma(0) = p$, and $\gamma'(0) = v$. Let $\tilde{\gamma} : J \to S$ be the curve as γ, but viewed as mapping into S. Then, for a smooth function f, we have

$$\begin{aligned}
d\iota_p\left(\tilde{\gamma}'(0)\right) f &= \tilde{\gamma}(0)(f \circ \iota) \\
&= d\tilde{\gamma}\left(\left.\frac{d}{dt}\right|_0\right)(f \circ \iota) \\
&= \left.\frac{d}{dt}\right|_0 (f \circ \iota \circ \tilde{\gamma}) \\
&= \left.\frac{d}{dt}\right|_0 (f \circ \gamma) \\
&= \gamma'(0)(f) \\
&= vf.
\end{aligned}$$

Hence, $v = d\iota_p\left(\tilde{\gamma}'(0)\right) \in T_pS$. ■

Problem 5.40
Suppose $S \subseteq M$ is a level set of a smooth map $\Phi : M \to N$ with constant rank. Show that $T_pS = \operatorname{Ker}(\mathrm{d}\Phi_p)$ for each $p \in S$.

Solution. By the constant rank level set theorem, S (and every other level set) is a properly embedded submanifold of M of codimension r, where r is the rank of Φ.
Recall that we identify T_pS with the subspace $\mathrm{d}\iota_p(T_pS) \subseteq T_pM$, where $\iota : S \hookrightarrow M$ is the inclusion map. Since $\Phi \circ \iota$ is constant on $S \cap U$, it follows that $\mathrm{d}\Phi_p \circ \mathrm{d}\iota_p$ is the zero map from T_pS to $T_{\Phi(p)}N$, and therefore, $\operatorname{Im}(\mathrm{d}\iota_p) = T_pS \subseteq \operatorname{Ker}(\mathrm{d}\Phi_p)$.
By the rank-nullity,

$$\dim\left(\operatorname{Ker}\left(\mathrm{d}\Phi_p\right)\right) = \dim\left(T_pM\right) - \dim\left(\operatorname{Im}\left(\mathrm{d}\Phi_p\right)\right) = \dim\left(T_pM\right) - r = \dim\left(T_pS\right)$$

Hence, $T_pS = \operatorname{Ker}(\mathrm{d}\Phi_p)$. ■

Problem 6-11
Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, and G is transverse to an embedded submanifold $X \subseteq P$. Show that F is transverse to the submanifold $G^{-1}(X)$ if and only if $G \circ F$ is transverse to X.

Solution. Let $F : M \to N$ and $G : N \to P$ be smooth maps and G is transverse to an embedded submanifold $X \subseteq P$.
$\boxed{\Longrightarrow}$ Assume F is transverse to the submanifold G^{-1}. Apply $\mathrm{d}G_{F(p)}$ on everything to get

$$\mathrm{d}G_{F(p)}[T_{G(p)}N] = \mathrm{d}G_{G(p)}[\mathrm{d}F_p[T_pM] + T_{F(p)}G^{-1}[X]] = \mathrm{d}(G \circ F)_p[T_pM] + \mathrm{d}G_{F(p)}[T_{F(p)}G^{-1}[X]].$$

We also know that $T_{g(F(p))}P = \mathrm{d}G_{F(p)}[T_{F(p)}N] + T_{(G\circ F)(p))}X$. Adding $T_{(G\circ F)(p)}X$ on both sides of the above equation, we get

$$T_{(G\circ F)(p)}P = \mathrm{d}(G \circ F)_p[T_pM] + \mathrm{d}G_{F(p)}[T_{F(p)}G^{-1}[X]] + T_{(G\circ F)(p)}X.$$

However, it is easy to check that $\mathrm{d}G_{F(p)}[T_{F(p)}G^{-1}[X]] \subseteq T_{(G\circ F)(p)}X$. Thus, we get that

$$T_{(G\circ F)(p)}P = \mathrm{d}(G \circ F)_p[T_pM] + T_{(G\circ F)(p)}X.$$

Therefore, $G \circ F$ is transverse to X.
$\boxed{\Longleftarrow}$ Assume that $(G \circ F)$ is transverse to X. We need to show that G being transverse to X implies that

$$T_{F(p)}G^{-1}[X] = (\mathrm{d}G_{F(p)})^{-1}\left[T_{(G\circ F)(p)}X\right].$$

This is essentially Problem 6.10 in Lee. With this, take $w \in T_{F(p)}N$, then $\mathrm{d}G_{F(p)}(w) \in T_{(G\circ F)(p)}P$ and $(g \circ f)$ being transverse to X gives us that $v \in T_pM$ and $z \in T_{(G\circ F)(p)}S$ such that

$$\mathrm{d}G_{F(p)}(w) = \mathrm{d}G_{F(p)}\,\mathrm{d}F_p(v) + z,$$

so that $w - \mathrm{d}f_p(v) \in (\mathrm{d}G_{F(p)})^{-1}\left[T_{(G\circ F)(p)}X\right] = T_{F(p)}G^{-1}[X]$. Then $w = \mathrm{d}F_p(v) + z'$, where $z' \in T_{F(p)}G^{-1}[S]$ which shows that

$$T_{F(p)}N = \mathrm{d}F_p[T_pM] + T_{F(p)}G^{-1}[S].$$

Thus, F is transverse to the submanifold G^{-1}. ■

Study Guide for Quiz 3

The following is a list of problems you should practice and understand how to solve before the quiz.

Problem 7.20
Let $S \subseteq \mathbb{T}^3$ be the image of the subgroup H of the preceding example under the obvious embedding $\mathbb{T}^2 \hookrightarrow \mathbb{T}^3$. Show that S is a Lie subgroup of $\mathbb{T}^3$ that is not closed, embedded, or dense; but its closure is a properly embedded Lie subgroup of $\mathbb{T}^3$.

Solution. Let us explicitly identify S under the natural embedding $\mathbb{T}^2 \hookrightarrow \mathbb{T}^3$, which gives

$$S = \{(e^{2\pi it}, e^{2\pi i\alpha t}, 1) : t \in \mathbb{R}\} \subseteq \mathbb{T}^3.$$

- S **is a Lie subgroup:** Consider the smooth homomorphism $\phi : \mathbb{R} \to \mathbb{T}^3$ given by

$$t \mapsto (e^{2\pi it}, e^{2\pi i\alpha t}, 1).$$

  S is the image of this one-parameter subgroup. Thus, S is a Lie subgroup of $\mathbb{T}^3$

- S **is not closed:** Since α is irrational, by Kronecker's theorem, we can find a sequence $\{t_n\}$ in $\mathbb{R}$ such that

$$e^{2\pi it_n} \to 1$$
$$e^{2\pi i\alpha t_n} \to e^{i\theta} \text{ for some } \theta \neq 0$$

  Then $(e^{2\pi it_n}, e^{2\pi i\alpha t_n}, 1)$ converges to $(1, e^{i\theta}, 1)$, but $(1, e^{i\theta}, 1) \notin S$ as it cannot be of the form $(e^{2\pi it}, e^{2\pi i\alpha t}, 1)$ for any t. Thus, S is not closed.

- S **is not embedded:** Consider the map $\phi : \mathbb{R} \to S$ given by $t \mapsto (e^{2\pi it}, e^{2\pi i\alpha t}, 1)$. This map is not a homeomorphism onto its image. For any $\epsilon > 0$, we can find $t \neq t'$ with $|t - t'|$ arbitrarily large such that

$$d((e^{2\pi it}, e^{2\pi i\alpha t}, 1), (e^{2\pi it'}, e^{2\pi i\alpha t'}, 1)) < \epsilon,$$

  where d is the metric on $\mathbb{T}^3$. This shows the topology induced on S from $\mathbb{T}^3$ is not the same as its manifold topology. Thus, S is not embedded.

- S **is not dense:** Note that $S \subseteq \mathbb{T}^2 \times \{1\}$. $\mathbb{T}^2 \times \{1\}$ is a proper closed subset of $\mathbb{T}^3$. Thus, S cannot be dense in $\mathbb{T}^3$.

- **Closure of** S **is embedded:** We have

$$\overline{S} = \{(e^{2\pi is}, e^{2\pi it}, 1) : s, t \in \mathbb{R}\} = \mathbb{T}^2 \times \{1\}.$$

  This is a properly embedded 2-dimensional torus in $\mathbb{T}^3$ and is a Lie subgroup as it is the product of Lie subgroups. The embedding $\mathbb{T}^2 \times \{1\} \hookrightarrow \mathbb{T}^3$ is proper because $\mathbb{T}^2$ is compact. Thus, $\overline{S}$ is a properly embedded Lie subgroup of $\mathbb{T}^3$

■

Problem 7-7
Prove Proposition 7.15 (properties of the identity component of a Lie group).

Proposition 7.15. Let G be a Lie group and let G_0 be its identity component. Then G_0 is a normal subgroup of G, and is the only connected open subgroup. Every connected component of G is diffeomorphic to G_0.

Solution. We will prove each statement in turn.

- **G_0 is a normal subgroup of G:** Take any $g \in G$ and $h \in G_0$ and consider the map $\phi_g : G \to G$ given by $\phi_g(x) = gxg^{-1}$. The latter is a continuous automorphism. Since ϕ_g is continuous, it maps connected sets to connected sets. Additionally, G_0 is connected and contains the identity e. Thus, $\phi_g(G_0)$ is connected and contains $\phi_g(e) = e$. By definition, G_0 is the largest connected subset containing e. Hence, $\phi_g(G_0) \subseteq G_0$ and this shows that $gG_0g^{-1} \subseteq G_0$ for all $g \in G$. Therefore, G_0 is normal in G.
- **G_0 is the only connected open subgroup:** Suppose H is another connected open subgroup. Since H is open and G_0 is connected, if H intersects G_0, then H must contain G_0. We have that H contains the identity e, so H does intersect G_0. Thus, $G_0 \subseteq H$, but H is connected and contains e, so by definition of G_0, we must have $H \subseteq G_0$. Therefore, $H = G_0$.
- **Every connected component is diffeomorphic to G_0:** Let C be any connected component of G. Take any $g \in C$. Consider the left translation map $L_g : G \to G$ given by $L_g(x) = gx$. We have that L_g is a diffeomorphism of G and L_g maps G_0 to gG_0. Since G_0 is connected, gG_0 is connected. Additionally, gG_0 contains g and is contained in C (since C is a maximal connected set). Thus, $gG_0 = C$. Therefore, L_g restricts to a diffeomorphism from G_0 to C.

Therefore, Each connected component is of the form gG_0 for some $g \in G$, and the left translation by g provides the diffeomorphism between G_0 and that component.

■

Problem 8.18
Prove the claim in the preceding example in two ways: directly from the definition, and by using Proposition 8.16.
Let $F : \mathbb{R} \to \mathbb{R}^2$ be the smooth map $F(t) = (\cos t, \sin t)$. Then $\frac{\mathrm{d}}{\mathrm{d}t} \in \mathfrak{X}(\mathbb{R})$ is F-related to the vector field $Y \in \mathfrak{X}\left(\mathbb{R}^2\right)$ defined by

$$Y = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

Solution.

- **Directly from the definition:** For vector fields to be F-related, we need to show that $dF_p(X_p) = Y_{F(p)}$ for all $p \in \mathbb{R}$. We proceed by computing both sides.

  Computing $dF_t(\frac{d}{dt})$, we have

$$\mathrm{d}F_t\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(-\sin(t)\frac{\partial}{\partial x} + \cos(t)\frac{\partial}{\partial y}\right).$$

  Computing $Y_{F(t)}$, we have

$$\begin{aligned} Y_{F(t)} &= \cos(t)\frac{\partial}{\partial y} - \sin(t)\frac{\partial}{\partial x} \\ &= \left(-\sin(t)\frac{\partial}{\partial x} + \cos(t)\frac{\partial}{\partial y}\right). \end{aligned}$$

  Since these are equal for all $t \in \mathbb{R}$, we have shown that $\frac{d}{dt}$ and Y are F-related.

- **Using Proposition 8.16:** By Proposition 8.16, we need to show that for any smooth function f defined on an open subset of $\mathbb{R}^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ F) = (Yf) \circ F.$$

  The left hand side, for any $t \in \mathbb{R}$, is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(f \circ F)(t) &= \frac{\mathrm{d}}{\mathrm{d}t} f(\cos(t), \sin(t)) \\ &= \frac{\partial f}{\partial x}(-\sin(t)) + \frac{\partial f}{\partial y}(\cos(t)). \end{aligned}$$

The right hand side is

$$\begin{aligned}(Yf)\circ F(t) &= \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)\circ F(t)\\ &= \cos(t)\frac{\partial f}{\partial y}(\cos(t),\sin(t)) - \sin(t)\frac{\partial f}{\partial x}(\cos(t),\sin(t))\\ &= \frac{\partial f}{\partial x}(-\sin(t)) + \frac{\partial f}{\partial y}(\cos(t)).\end{aligned}$$

Since these expressions are equal for all $t \in \mathbb{R}$ and all smooth functions f, we have shown that $\frac{\mathrm{d}}{\mathrm{d}t}$ and Y are F-related.

■

Problem 8.29
Prove part (d) of Proposition 8.28.

Proposition 8.28. (Properties of the Lie Bracket). The Lie bracket satisfies the following identities for all $X, Y, Z \in \mathfrak{X}(M)$:

(d) For $f, g \in C^\infty(M)$,

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

Solution. For vector fields V and W, the Lie bracket $[V,W]$ is defined by

$$[V,W](h) = V(W(h)) - W(V(h)).$$

for any $h \in C^\infty(M)$. Computing $[fX, gY](h)$ for an arbitrary $h \in C^\infty(M)$ gives

$$[fX, gY](h) = (fX)(gY(h)) - (gY)(fX(h)).$$

Using the product rule for vector fields acting as derivations, meaning, for any smooth functions $a, b \in C^\infty(M)$, they satisfy $X(ab) = X(a)b + aX(b)$, then we have

- **First Term:**

$$\begin{aligned}(fX)(gY(h)) &= f\cdot X(gY(h))\\ &= f\cdot\Big(X(g)Y(h) + g\cdot X\big(Y(h)\big)\Big)\\ &= fX(g)Y(h) + fgX\big(Y(h)\big).\end{aligned}$$

- **Second Term:**

$$\begin{aligned}(gY)(fX(h)) &= g\cdot Y(fX(h))\\ &= g\cdot\Big(Y(f)X(h) + f\cdot Y\big(X(h)\big)\Big)\\ &= gY(f)X(h) + fgY\big(X(h)\big).\end{aligned}$$

Substituting these back, we get

$$\begin{aligned}[fX, gY](h) &= fX(g)Y(h) + fgX\big(Y(h)\big) - gY(f)X(h) - fgY\big(X(h)\big)\\ &= fg\Big(X\big(Y(h)\big) - Y\big(X(h)\big)\Big) + fX(g)Y(h) - gY(f)X(h)\\ &= fg[X,Y](h) + (fXg)Y(h) - (gYf)X(h).\end{aligned}$$

Since this equality holds for all $h \in C^\infty(M)$, then

$$[fX, gY] = fg[X,Y] + (fXg)Y - (gYf)X.$$

■

Problem 9.37
Suppose $v \in \mathbb{R}^n$ and W is a smooth vector field on an open subset of $\mathbb{R}^n$. Show that the directional derivative $D_v W(p)$ defined by (9.15) is equal to $(\mathscr{L}_V W)_p$, where V is the vector field $V = v^i \partial/\partial x^i$ with constant coefficients in standard coordinates.

Solution. The directional derivative of W at p in the direction v is defined by

$$D_v W(p) = \lim_{t \to 0} \frac{W(p + tv) - W(p)}{t}.$$

In standard coordinates, let $W = W^j \frac{\partial}{\partial x^j}$, then

$$W(p + tv) = W^j(p + tv) \frac{\partial}{\partial x^j}.$$

and the directional derivative becomes

$$D_v W(p) = \lim_{t \to 0} \frac{W^j(p + tv) - W^j(p)}{t} \frac{\partial}{\partial x^j} = \sum_i v^i \frac{\partial W^j}{\partial x^i}(p) \frac{\partial}{\partial x^j}.$$

For vector fields V and W, the Lie derivative is given by

$$\mathcal{L}_V W = [V, W].$$

In our case, $V = v^i \frac{\partial}{\partial x^i}$ with constant coefficients. Let's compute

$$\begin{aligned}
[V, W] &= V\left(W^j\right) \frac{\partial}{\partial x^j} - W\left(V^i\right) \frac{\partial}{\partial x^i} \\
&= \left(v^i \frac{\partial W^j}{\partial x^i}\right) \frac{\partial}{\partial x^j} - \left(W^i \frac{\partial}{\partial x^i}(v^k)\right) \frac{\partial}{\partial x^k}.
\end{aligned}$$

Since v^i are constant, then

$$\frac{\partial}{\partial x^i}(v^k) = 0.$$

Thus,

$$(\mathscr{L}_V W)_p = \left(v^i \frac{\partial W^j}{\partial x^i}\right)(p) \frac{\partial}{\partial x^j}$$

By comparing, we have

$$D_v W(p) = \sum_i v^i \frac{\partial W^j}{\partial x^i}(p) \frac{\partial}{\partial x^j} = \left(v^i \frac{\partial W^j}{\partial x^i}\right)(p) \frac{\partial}{\partial x^j} = (\mathscr{L}_V W)_p$$

Therefore, $D_v W(p) = (\mathscr{L}_V W)_p$.
Note: The key insight is that, since V has constant coefficients, the term involving derivatives of V in the Lie bracket vanishes, leaving us with exactly the directional derivative formula.

■

Study Guide for Quiz 4

The following is a list of problems you should practice and understand how to solve before the quiz.

Problem 11.10
Suppose M is a smooth manifold and $E \to M$ is a smooth vector bundle over M. Define the dual bundle to $\boldsymbol{E}$ to be the bundle $E^* \to M$ whose total space is the disjoint union $E^* = \bigsqcup_{p \in M} E_p^*$, where E_p^* is the dual space to E_p, with the obvious projection. Show that $E^* \to M$ is a smooth vector bundle, whose transition functions are given by $\tau^*(p) = \left(\tau(p)^{-1}\right)^T$ for any transition function $\tau : U \to \mathrm{GL}(k, \mathbb{R})$ of E.

Solution. To show that $E^* \to M$ is a smooth vector bundle, we need to verify that it satisfies the definition of a smooth vector bundle.

- **Local Trivialization:** Let $(U_\alpha, \varphi_\alpha)$ be a local trivialization of E. For each U_α, we can define a map $\varphi_\alpha^* : \pi^{-1}(U_\alpha) \to U_\alpha \times (\mathbb{R}^k)^*$ by
$$\varphi_\alpha^*(f) = \left(p, f \circ (\varphi_\alpha|_{E_p})^{-1}\right),$$
where $f \in E_p^*$ and $p = \pi(f)$. This provides a local trivialization for E^*.

- **Smooth Transition Functions:** Let $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ be a transition function for E. The corresponding transition function for E^* is
$$\varphi_{\alpha\beta}^* = \varphi_\beta^* \circ (\varphi_\alpha^*)^{-1}.$$
For any $p \in U_\alpha \cap U_\beta$ and $f \in (\mathbb{R}^k)^*$, we have
$$\varphi_{\alpha\beta}^*(p, f) = \left(p, f \circ \tau_{\alpha\beta}(p)^{-1}\right) = \left(p, \tau_{\alpha\beta}^*(p) \circ f\right).$$
This is equivalent to saying that $\tau_{\alpha\beta}^*(p) = (\tau_{\alpha\beta}(p)^{-1})^T$, which is smooth because $\tau_{\alpha\beta}$ is smooth and matrix inversion and transposition are smooth operations.

- **Fiber Structure:** We need to verify that the fibers E_p^* are vector spaces and that the local trivializations are linear isomorphisms on each fiber. This follows directly from the definition of E^* and the construction of φ_α^*.

Therefore, $E^* \to M$ is indeed a smooth vector bundle with transition functions given by $\tau^*(p) = \left(\tau(p)^{-1}\right)^T$ for any transition function $\tau : U \to \mathrm{GL}(k, \mathbb{R})$ of E. ■

Problem 12.36
Prove Proposition 12.25.

Proposition 12.25. (Properties of Tensor Pullbacks). Suppose $F : M \to N$ and $G : N \to P$ are smooth maps, A and B are covariant tensor fields on N, and f is a real-valued function on N.

(a) $F^*(fB) = (f \circ F)F^*B$.

(b) $F^*(A \otimes B) = F^*A \otimes F^*B$.

(c) $F^*(A + B) = F^*A + F^*B$.

(d) F^*B is a (continuous) tensor field, and is smooth if B is smooth.

(e) $(G \circ F)^*B = F^*(G^*B)$.

(f) $(\mathrm{Id}_N)^* B = B$.

Solution. (a) For any $p \in M$ and $X_1, \ldots, X_k \in T_pM$,

$$\begin{aligned}(F^*(fB))_p(X_1, \ldots, X_k) &= (fB)_{F(p)}\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k)\Big)\\ &= f(F(p)) \cdot B_{F(p)}\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k)\Big)\\ &= (f \circ F)(p) \cdot (F^*B)_p(X_1, \ldots, X_k)\\ &= ((f \circ F)F^*B)_p(X_1, \ldots, X_k).\end{aligned}$$

Thus, $F^*(fB) = (f \circ F)F^*B$.

(b) For any $p \in M$ and $X_1, \ldots, X_k, Y_1, \ldots, Y_l \in T_pM$,

$$\begin{aligned}(F^*(A \otimes B))_p\Big(X_1, \ldots, X_k, Y_1, \ldots, Y_l\Big) &= (A \otimes B)F(p)\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k), \mathrm{d}F_p(Y_1), \ldots, \mathrm{d}F_p(Y_l)\Big)\\ &= A_{F(p)}\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k)\Big) \cdot B_{F(p)}\Big(\mathrm{d}F_p(Y_1), \ldots, \mathrm{d}F_p(Y_l)\Big)\\ &= (F^*A)_p(X_1, \ldots, X_k) \cdot (F^*B)_p(Y_1, \ldots, Y_l)\\ &= (F^*A \otimes F^*B)_p(X_1, \ldots, X_k, Y_1, \ldots, Y_l).\end{aligned}$$

Thus, $F^*(A \otimes B) = F^*A \otimes F^*B$.

(c) For any $p \in M$ and $X_1, \ldots, X_k \in T_pM$,

$$\begin{aligned}(F^*(A+B))_p(X_1, \ldots, X_k) &= (A+B)_{F(p)}\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k)\Big)\\ &= A_{F(p)}\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k)\Big) + B_{F(p)}\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k)\Big)\\ &= (F^*A)_p(X_1, \ldots, X_k) + (F^*B)_p(X_1, \ldots, X_k)\\ &= (F^*A + F^*B)_p(X_1, \ldots, X_k).\end{aligned}$$

Thus, $F^*(A+B) = F^*A + F^*B$.

(d) Let $F : M \to N$ be a smooth map. We want to show that the pullback F^*B of a covariant tensor field B is itself a tensor field.

For a point $p \in M$, the pullback F^*B at p is defined as

$$(F^*B)_p(X_1, \ldots, X_k) = B_{F(p)}\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k)\Big),$$

where $X_1, \ldots, X_k$ are vector fields on M at p.

We need to show that $(F^*B)_p$ satisfies the tensor field requirements: Multilinearity over $\mathcal{C}^\infty(M)$ and coordinate independence.

- **Multilinearity:** Let $X_1, \ldots, X_k$ be vector fields on M and $f \in \mathcal{C}^\infty(M)$. Then

$$\begin{aligned}(F^*B)_p(fX_1, X_2, \ldots, X_k) &= B_{F(p)}\Big(\mathrm{d}F_p(fX_1), \mathrm{d}F_p(X_2), \ldots, \mathrm{d}F_p(X_k)\Big)\\ &= B_{F(p)}\Big(f(p)\,\mathrm{d}F_p(X_1), \mathrm{d}F_p(X_2), \ldots, \mathrm{d}F_p(X_k)\Big)\\ &= f(p)(F^*B)_p(X_1, X_2, \ldots, X_k)\end{aligned}$$

  Similar properties hold for other arguments due to the multilinearity of the original tensor B.

- **Smoothness:** If B is a smooth tensor field, then F^*B is smooth because the composition $B \circ F$ is smooth (composition of smooth maps), the differential dF_p is smooth, and the evaluation of a smooth tensor field is a smooth operation.

- **Continuity:** Continuity follows directly from the smoothness of F and B.

Therefore, F^*B is a tensor field on M, and is smooth if B is smooth.

(e) For any $p \in M$ and $X_1, \ldots, X_k \in T_pM$,

$$\begin{aligned}((G \circ F)^*B)_p (X_1, \ldots, X_k) &= B_{G(F(p))}\Big(\mathrm{d}(G \circ F)_p(X_1), \ldots, \mathrm{d}(G \circ F)_p(X_k)\Big) \\ &= B_{G(F(p))}\Big(\mathrm{d}G_{F(p)}(\mathrm{d}F_p(X_1)), \ldots, \mathrm{d}G_{F(p)}(\mathrm{d}F_p(X_k))\Big) \\ &= (G^*B)_{F(p)}\Big(\mathrm{d}F_p(X_1), \ldots, \mathrm{d}F_p(X_k)\Big) \\ &= (F^*(G^*B))_p(X_1, \ldots, X_k).\end{aligned}$$

Thus, $(G \circ F)^*B = F^*(G^*B)$.

(f) For any $q \in N$ and $Y_1, \ldots, Y_k \in T_qN$,

$$\begin{aligned}((\mathrm{Id}_N)^*B)_q (Y_1, \ldots, Y_k) &= B_{\mathrm{Id}_N(q)}\Big(\mathrm{d}(\mathrm{Id}_N)_q(Y_1), \ldots, \mathrm{d}(\mathrm{Id}_N)_q(Y_k)\Big) \\ &= B_q(Y_1, \ldots, Y_k).\end{aligned}$$

Thus, $(\mathrm{Id}_N)^* B = B$.

■

Problem 14.14
Show that Λ^kT^*M is a smooth subbundle of T^kT^*M, and therefore is a smooth vector bundle of rank $\binom{n}{k}$ over M.

Solution. To show Λ^kT^*M is a smooth subbundle of T^kT^*M, we need to demonstrate it's a smooth vector subbundle of constant rank.
Let (U, φ) be a coordinate chart on M with coordinates $(x^1, \ldots, x^n)$. Then a local frame for T^kT^*M over U is given by

$$\{\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k} : 1 \leq i_1, \ldots, i_k \leq n\}$$

A local frame for Λ^kT^*M over U is then given by

$$\{\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

We can express each basis element of Λ^kT^*M in terms of the basis elements of T^kT^*M using the alternation map:

$$\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k} = \mathrm{Alt}(\mathrm{d}x^{i_1} \otimes \cdots \otimes \mathrm{d}x^{i_k})$$

The alternation map varies smoothly with respect to the coordinates because it's a linear combination of permutations with constant coefficients. Therefore, the transition functions between different coordinate charts are smooth.
The rank of Λ^kT^*M at each point is $\binom{n}{k}$ because this is the number of possible strictly increasing k-tuples $(i_1, \ldots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq n$.
Therefore, Λ^kT^*M is a smooth subbundle of T^kT^*M of rank $\binom{n}{k}$.

■

Problem 15.13
Suppose M and N are oriented positive-dimensional smooth manifolds with or without boundary, and $F : M \to N$ is a local diffeomorphism. Show that the following are equivalent.

(a) F is orientation-preserving.

(b) With respect to any oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.

(c) For any positively oriented orientation form ω for N, the form $F^*\omega$ is positively oriented for M.

Solution. We will show that (a) $\implies$ (b) $\implies$ (c) $\implies$ (a).

$\boxed{\text{(a)} \implies \text{(b)}}$ Suppose F is orientation-preserving. Let (U, ϕ) and (V, ψ) be positively oriented charts for M and N respectively, with $F(U) \subset V$. The local representation of F is given by $\tilde{F} = \psi \circ F \circ \phi^{-1}$. Since F is orientation-preserving, $\tilde{F}$ must preserve orientation in $\mathbb{R}^n$. This means that the Jacobian determinant of $\tilde{F}$ must be positive. But this Jacobian is precisely the Jacobian matrix of F with respect to these charts.

$\boxed{\text{(b)} \implies \text{(c)}}$ Let ω be a positively oriented orientation form for N. In local coordinates, $\omega = f\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ where $f > 0$. The pullback $F^*\omega$ in local coordinates is given by

$$F^*\omega = (f \circ F)\det(JF)\,\mathrm{d}y^1 \wedge \cdots \wedge \mathrm{d}y^n$$

where JF is the Jacobian matrix of F. Since $f > 0$ and $\det(JF) > 0$ by assumption, $F^*\omega$ is positively oriented for M.

$\boxed{\text{(c)} \implies \text{(a)}}$ Let ω be a positively oriented orientation form for N. By assumption, $F^*\omega$ is positively oriented for M. This means that $F^*\omega$ defines the same orientation as the given orientation on M. By definition, this makes F orientation-preserving. ■

Problem 16.7
Prove parts (a) and (b) of the following proposition.

Proposition . (Properties of Integrals of Forms) Suppose M and N are non-empty oriented smooth n-manifolds with or without boundary, and ω, η are compactly supported n-forms on M.

(a) **Linearity:** If $a, b \in \mathbb{R}$, then
$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta.$$

(b) **Orientation Reversal:** If $-M$ denotes M with the opposite orientation, then
$$\int_{-M} \omega = -\int_M \omega.$$

Solution. (a) **Linearity:** Let (U_α, ϕ_α) be an oriented atlas for M. By the definition of integration on manifolds, we have

$$\begin{aligned}
\int_M a\omega + b\eta &= \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (a\omega + b\eta)_\alpha \\
&= \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (a\omega_\alpha + b\eta_\alpha) \\
&= \sum_\alpha \left(a\int_{\phi_\alpha(U_\alpha)} \omega_\alpha + b\int_{\phi_\alpha(U_\alpha)} \eta_\alpha \right) \\
&= a\sum_\alpha \int_{\phi_\alpha(U_\alpha)} \omega_\alpha + b\sum_\alpha \int_{\phi_\alpha(U_\alpha)} \eta_\alpha \\
&= a\int_M \omega + b\int_M \eta.
\end{aligned}$$

Here, we used the linearity of integration in $\mathbb{R}^n$ and the fact that finite sums commute with scalar multiplication.

(b) **Orientation Reversal:** Consider an oriented atlas (U_α, ϕ_α) for M, where each coordinate chart $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{R}^n$ is oriented. Let ω be a compactly supported n-form on M. For a local coordinate chart (U, ϕ), consider $\omega|_U = f, dx^1 \wedge \cdots \wedge dx^n$. Choose a smooth partition of unity ρ_α subordinate to the atlas, where each ρ_α has compact support. We have

$$\begin{aligned}\int_{-M} \omega &= \sum_\alpha \int_M \rho_\alpha(-\omega) \\ &= -\sum_\alpha \int_M \rho_\alpha \omega \\ &= -\int_M \omega.\end{aligned}$$

Therefore, $\int_{-M} \omega = -\int_M \omega$ for any compactly supported n-form ω on M.

■

Problem 17.37
Prove Proposition 17.36.

Proposition 17.36. (Properties of the Degree) Suppose M, N, and P are compact, connected, oriented, smooth n-manifolds.

(a) If $F : M \to N$ and $G : N \to P$ are both smooth maps, then $\deg(G \circ F) = (\deg G)(\deg F)$.

(b) If $F : M \to N$ is a diffeomorphism, then $\deg F = +1$ if F is orientation-preserving and -1 if it is orientation-reversing.

(c) If two smooth maps $F_0, F_1 : M \to N$ are homotopic, then they have the same degree.

Solution. (a) Let ω be an n-form on P such that $\int_P \omega = 1$. Then

$$\begin{aligned}\deg(G \circ F) &= \int_M (G \circ F)^* \omega \\ &= \int_M F^*(G^*\omega) \\ &= (\deg(F)) \cdot \int_N G^*\omega \\ &= (\deg(F)) \cdot (\deg(G)) \cdot \int_P \omega \\ &= (\deg(F))(\deg(G)).\end{aligned}$$

(b) Let ω_M and ω_N be volume forms on M and N respectively, normalized so that $\int_M \omega_M = \int_N \omega_N = 1$. Then

$$\begin{gathered}F^*\omega_N = \pm\omega_M \\ \int_M F^*\omega_N = \pm \int_M \omega_M \\ \deg(F) = \pm 1.\end{gathered}$$

The sign is positive if F is orientation-preserving and negative if it is orientation-reversing. Therefore, $\deg(F) = +1$ or -1 accordingly.

(c) Let $H : M \times [0,1] \to N$ be a smooth homotopy between F_0 and F_1. Let ω be a closed n-form on N with $\int_N \omega = 1$. Then

$$\begin{aligned}\deg(F_1) - \deg(F_0) &= \int_M F_1^*\omega - \int_M F_0^*\omega \\ &= \int_M (F_1^*\omega - F_0^*\omega) \\ &= \int_M \mathrm{d}(H^*\Omega),\end{aligned}$$

where Ω is an $(n-1)$-form on $N \times [0,1]$ such that $\mathrm{d}\Omega = \pi^*\omega$ (with $\pi : N \times [0,1] \to N$ the projection). By Stokes' theorem, this integral is zero, so $\deg(F_1) = \deg(F_0)$.

■

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