PHYS 663 - Quantum Field Theory II

An Introduction to Quantum Fied Theory by *Peskin and Schroeder*

Student: **Ralph Razzouk**

Homework 8

Problem 1 - Right-Handed Neutrinos

Suppose a right-handed neutrino for each generation is added to the standard model.

(a) Show that the only new renormalizable term that can appear in the Lagrangian is

$$-\frac{1}{2}\tau_m \not{D} L_m - \frac{1}{2}\overline{N}_m \not{\partial} N_m - \frac{1}{2}M_m \overline{N}_m N_m - \left(k_{mn}\tau_m P_R N_n \tilde{\phi} + \text{ h.c. }\right).$$

(b) Do any combinations of electron-number, muon-number, and tau-number remain preserved?

(c) If there's only one generation, what's the mass matrix for the neutrino?

(d) Express the Lepton-Higgs and Lepton-gauge boson interactions in terms of these mass eigenstates.

Solution. (a) To identify allowable new terms in the Lagrangian, we must consider the representations of the fields under the Standard Model gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$

- $L_m = \begin{pmatrix} \nu_{mL} \\ e_{mL} \end{pmatrix}$ transforms as $(1, 2, -1/2)$
- N_m (right-handed neutrinos) transform as $(1, 1, 0)$ - singlets under all gauge groups
- $\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ transforms as $(1, 2, 1/2)$

Renormalizable terms must have mass dimension ≤ 4. The right-handed neutrinos N_m allow these new terms:

(a) Kinetic term: $\overline{N}_m \not{\partial} N_m$ (dimension 3)
(b) Majorana mass term: $M_m \overline{N}_m N_m$ (dimension 3)
(c) Yukawa coupling: $k_{mn}\overline{L}_m \tilde{\phi} P_R N_n + \text{h.c.}$ (dimension 4)

Here, $\tilde{\phi} = i\sigma_2 \phi^*$ is the charge conjugate of ϕ.

For the Yukawa term, we can rewrite

$$\overline{L}_m \tilde{\phi} P_R N_n = \overline{\tau}_m P_L \tilde{\phi} P_R N_n = \tau_m P_R N_n \tilde{\phi},$$

where we've used $\tau_m = \overline{L}_m P_L$ and $P_L P_R = 0$. Therefore, the complete new renormalizable contribution to the Lagrangian is:

$$-\frac{1}{2}\tau_m \not{D} L_m - \frac{1}{2}\overline{N}_m \not{\partial} N_n - \frac{1}{2}M_m \overline{N}_m N_m - \left(k_{mn}\tau_m P_R N_n \tilde{\phi} + \text{ h.c. }\right)$$

(b) The introduction of Majorana mass terms $M_m \overline{N}_m N_m$ violates lepton number conservation. To see this, note that

$$\overline{N}_m N_m = \overline{N}_m N_m = \overline{N}_{m,R} N_{m,R} + \overline{N}_{m,L} N_{m,L}$$

Since N_m is right-handed, $N_{m,L} = 0$ and $\overline{N}_{m,R} N_{m,R}$ violates lepton number by $\Delta L = 2$. The total lepton number $L = L_e + L_\mu + L_\tau$ is violated by the Majorana mass terms. However, if we consider differences of lepton numbers, such as $L_e - L_\mu$, these can be conserved if the coupling matrices have specific structures. For example, if k_{mn} and M_m are diagonal, then $L_e - L_\mu$, $L_\mu - L_\tau$, and $L_e - L_\tau$ would be conserved. But in general, these differences are also violated by off-diagonal elements in the Yukawa and mass matrices.

(c) For a single generation, we have the neutrino fields ν_L (from the lepton doublet) and N_R (the right-handed neutrino). After electroweak symmetry breaking, ϕ acquires a vacuum expectation value:

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

The resulting mass terms in the Lagrangian are:

$$-\mathcal{L}_{mass} = \frac{1}{2}M\overline{N}_R N_R + k\frac{v}{\sqrt{2}}\overline{\nu}_L N_R + k\frac{v}{\sqrt{2}}\overline{N}_R \nu_L$$

We can write this in the basis of Majorana fields by defining $\nu_L^c = C\overline{\nu}_L^T$ and $N_R^c = C\overline{N}_R^T$, where C is the charge conjugation matrix. This allows us to express the mass terms as:

$$-\mathcal{L}_{mass} = \frac{1}{2}\begin{pmatrix}\overline{\nu}_L & \overline{N}_R^c\end{pmatrix}\begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}\begin{pmatrix}\nu_L^c \\ N_R\end{pmatrix} + \text{h.c.},$$

where $m_D = k\frac{v}{\sqrt{2}}$ is the Dirac mass term. The mass matrix is therefore

$$\mathcal{M} = \begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}.$$

This is the classic seesaw mechanism mass matrix, which will yield two Majorana mass eigenstates.

(d) To express the interactions in terms of mass eigenstates, we must first diagonalize the mass matrix. The mass matrix $\mathcal{M}$ can be diagonalized by a unitary transformation:

$$\mathcal{M} = U^* D U^\dagger,$$

where D is diagonal with eigenvalues m_1 and m_2.

For $M \gg m_D$, the eigenvalues are approximately:

$$m_1 \approx \frac{m_D^2}{M}$$
$$m_2 \approx M$$

The corresponding mass eigenstates are:

$$\nu_1 \approx \cos\theta\,\nu_L - \sin\theta\,N_R^c$$
$$\nu_2 \approx \sin\theta\,\nu_L + \cos\theta\,N_R^c,$$

where $\theta \approx m_D/M$ is small for $M \gg m_D$.

Now, we can express the interactions in terms of these mass eigenstates:

(a) Lepton-Higgs interactions:

$$\mathcal{L}_{Higgs} = -\frac{k}{\sqrt{2}}h\left(\overline{\nu}_L N_R + \overline{N}_R \nu_L\right)$$

In terms of mass eigenstates:

$$\mathcal{L}_{Higgs} \approx -\frac{k}{\sqrt{2}}h\left[\cos\theta\sin\theta(\overline{\nu}_1\nu_2 + \overline{\nu}_2\nu_1) + \cos^2\theta\,\overline{\nu}_1\nu_1 - \sin^2\theta\,\overline{\nu}_2\nu_2\right]$$

(b) Lepton-gauge boson interactions: The weak current coupling to the W boson is:

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} \left(\overline{\nu}_L \gamma^\mu e_L W_\mu^+ + \overline{e}_L \gamma^\mu \nu_L W_\mu^- \right)$$

In terms of mass eigenstates:

$$\mathcal{L}_W \approx \frac{g}{\sqrt{2}} \left[\cos\theta\, \overline{\nu}_1 \gamma^\mu e_L W_\mu^+ + \sin\theta\, \overline{\nu}_2 \gamma^\mu e_L W_\mu^+ + \text{h.c.} \right]$$

The Z boson coupling:

$$\mathcal{L}_Z = \frac{g}{2\cos\theta_W} \overline{\nu}_L \gamma^\mu \nu_L Z_\mu$$

In terms of mass eigenstates:

$$\mathcal{L}_Z \approx \frac{g}{2\cos\theta_W} \left[\cos^2\theta\, \overline{\nu}_1 \gamma^\mu \nu_1 + \sin^2\theta\, \overline{\nu}_2 \gamma^\mu \nu_2 + \sin\theta\cos\theta \left(\overline{\nu}_1 \gamma^\mu \nu_2 + \overline{\nu}_2 \gamma^\mu \nu_1 \right) \right] Z_\mu$$

These expressions demonstrate that the lighter mass eigenstate ν_1 couples predominantly to the weak gauge bosons, while the heavier state ν_2 has suppressed couplings proportional to $\sin\theta \approx m_D/M$.

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Problem 2 - Adjoint Representation Fermions
Suppose two Majorana fermions were added, $\tilde{W}(1,3,0)$ and $\tilde{G}(8,1,0)$ with transformation properties

$$\delta P_L\tilde{W} = -\epsilon_{abc}\omega_2^b P_L\tilde{W}^c$$
$$D_\mu P_L\tilde{W}^a = \left(\partial_\mu\delta_{ac} + g_2\epsilon_{abc}W_\mu^b\right)P_L\tilde{W}^c$$

and

$$\delta P_L\tilde{G}^\alpha = -f_{\alpha\beta\gamma}\omega_3^\beta P_L G^{-\gamma}$$
$$D_\mu P_L\tilde{G}^\alpha = \left(\partial_\mu\delta_{\alpha\gamma} + g_3 f_{\alpha\beta\gamma}G_\mu^\beta\right)P_L\tilde{G}^\gamma.$$

1. Show that the reality of ϵ_{abc} and $f_{\alpha\beta\gamma}$ cause $P_R\tilde{W}$ and $P_R\tilde{G}$ to have the same transformation as $P_L\tilde{\omega}$ and $P_L\tilde{G}$.
2. Show that these adjoint fermions do have
$$SU(3)\times SU(2)\times U(1)$$
invariant mass terms
$$-\frac{m_{\tilde{W}}}{2}\overline{\tilde{W}}\tilde{W} - \frac{m_{\tilde{G}}}{2}\overline{\tilde{G}}\tilde{G}.$$
3. Are these new Yukama interactions? What are they?

Solution. (a) We begin by examining the transformation properties of $P_R\tilde{W}$ and $P_R\tilde{G}$ to show they transform identically to their left-handed counterparts.

For the $SU(2)$ adjoint fermion $\tilde{W}$, we know that $P_L\tilde{W}$ transforms as

$$\delta P_L\tilde{W}^a = -\epsilon_{abc}\omega_2^b P_L\tilde{W}^c$$

To determine the transformation of $P_R\tilde{W}$, we recall that for a Majorana fermion, we have the condition $\tilde{W} = \tilde{W}^c = C\overline{\tilde{W}}^T$ where C is the charge conjugation matrix. This implies

$$P_R\tilde{W} = P_R\tilde{W}^c = P_R C\overline{\tilde{W}}^T = C\overline{P_L\tilde{W}}^T$$

Now, considering the transformation of $P_L\tilde{W}$, we can determine that of $P_R\tilde{W}$

$$\begin{aligned}\delta(P_R\tilde{W}^a) &= \delta(C\overline{P_L\tilde{W}^a}^T)\\ &= C\overline{\delta(P_L\tilde{W}^a)}^T\\ &= C\overline{(-\epsilon_{abc}\omega_2^b P_L\tilde{W}^c)}^T\\ &= -\epsilon_{abc}\omega_2^b C\overline{P_L\tilde{W}^c}^T\end{aligned}$$

Since ϵ_{abc} is real and $C\overline{P_L\tilde{W}^c}^T = P_R\tilde{W}^c$, we obtain

$$\delta(P_R\tilde{W}^a) = -\epsilon_{abc}\omega_2^b P_R\tilde{W}^c.$$

This demonstrates that $P_R\tilde{W}$ transforms identically to $P_L\tilde{W}$.

Similarly for $\tilde{G}$, given that $P_L\tilde{G}$ transforms as

$$\delta P_L\tilde{G}^\alpha = -f_{\alpha\beta\gamma}\omega_3^\beta P_L\tilde{G}^\gamma$$

and using the Majorana condition $\tilde{G} = \tilde{G}^c$, we derive

$$\begin{aligned}
\delta(P_R\tilde{G}^\alpha) &= \delta(C\overline{P_L\tilde{G}^\alpha}^T) \\
&= C\overline{\delta(P_L\tilde{G}^\alpha)}^T \\
&= C\overline{(-f_{\alpha\beta\gamma}\omega_3^\beta P_L\tilde{G}^\gamma)}^T \\
&= -f_{\alpha\beta\gamma}\omega_3^\beta C\overline{P_L\tilde{G}^\gamma}^T
\end{aligned}$$

Since $f_{\alpha\beta\gamma}$ is real and $C\overline{P_L\tilde{G}^\gamma}^T = P_R\tilde{G}^\gamma$, we obtain

$$\delta(P_R\tilde{G}^\alpha) = -f_{\alpha\beta\gamma}\omega_3^\beta P_R\tilde{G}^\gamma$$

Therefore, $P_R\tilde{G}$ transforms identically to $P_L\tilde{G}$. The reality of the structure constants ϵ_{abc} and $f_{\alpha\beta\gamma}$ is crucial for this result.

(b) We need to show that the mass terms

$$-\frac{m_{\tilde{W}}}{2}\overline{\tilde{W}}\tilde{W} - \frac{m_{\tilde{G}}}{2}\overline{\tilde{G}}\tilde{G}$$

are invariant under $SU(3)\times SU(2)\times U(1)$.

For $\tilde{W}$, we consider an infinitesimal $SU(2)$ transformation

$$\begin{aligned}
\delta(\overline{\tilde{W}}^a\tilde{W}^a) &= \delta(\overline{\tilde{W}}^a)\tilde{W}^a + \overline{\tilde{W}}^a\delta(\tilde{W}^a) \\
&= \overline{\delta(\tilde{W}^a)}\tilde{W}^a + \overline{\tilde{W}}^a\delta(\tilde{W}^a)
\end{aligned}$$

Using the transformation property $\delta\tilde{W}^a = -\epsilon_{abc}\omega_2^b\tilde{W}^c$, we have

$$\begin{aligned}
\delta(\overline{\tilde{W}}^a\tilde{W}^a) &= -\epsilon_{abc}\omega_2^b\overline{\tilde{W}}^c\tilde{W}^a - \epsilon_{acd}\omega_2^c\overline{\tilde{W}}^a\tilde{W}^d \\
&= -\epsilon_{abc}\omega_2^b\overline{\tilde{W}}^c\tilde{W}^a - \epsilon_{cad}\omega_2^c\overline{\tilde{W}}^a\tilde{W}^d
\end{aligned}$$

Renaming indices in the second term ($c\to b$, $a\to c$, $d\to a$), we get

$$\begin{aligned}
\delta(\overline{\tilde{W}}^a\tilde{W}^a) &= -\epsilon_{abc}\omega_2^b\overline{\tilde{W}}^c\tilde{W}^a - \epsilon_{bca}\omega_2^b\overline{\tilde{W}}^c\tilde{W}^a \\
&= -\omega_2^b(\epsilon_{abc}+\epsilon_{bca})\overline{\tilde{W}}^c\tilde{W}^a
\end{aligned}$$

Since $\epsilon_{abc} = -\epsilon_{bac} = -\epsilon_{acb} = -\epsilon_{bca}$, we have $\epsilon_{abc}+\epsilon_{bca} = 0$, thus

$$\delta(\overline{\tilde{W}}^a\tilde{W}^a) = 0.$$

This confirms that $\overline{\tilde{W}}\tilde{W}$ is $SU(2)$ invariant.

For $\tilde{G}$, applying the same procedure with $\delta\tilde{G}^\alpha = -f_{\alpha\beta\gamma}\omega_3^\beta\tilde{G}^\gamma$, we obtain

$$\begin{aligned}
\delta(\overline{\tilde{G}}^\alpha\tilde{G}^\alpha) &= -f_{\alpha\beta\gamma}\omega_3^\beta\overline{\tilde{G}}^\gamma\tilde{G}^\alpha - f_{\alpha\delta\epsilon}\omega_3^\delta\overline{\tilde{G}}^\alpha\tilde{G}^\epsilon \\
&= -f_{\alpha\beta\gamma}\omega_3^\beta\overline{\tilde{G}}^\gamma\tilde{G}^\alpha - f_{\delta\alpha\epsilon}\omega_3^\delta\overline{\tilde{G}}^\alpha\tilde{G}^\epsilon
\end{aligned}$$

Renaming indices ($\delta\to\beta$, $\alpha\to\gamma$, $\epsilon\to\alpha$), we get

$$\begin{aligned}
\delta(\overline{\tilde{G}}^\alpha\tilde{G}^\alpha) &= -f_{\alpha\beta\gamma}\omega_3^\beta\overline{\tilde{G}}^\gamma\tilde{G}^\alpha - f_{\beta\gamma\alpha}\omega_3^\beta\overline{\tilde{G}}^\gamma\tilde{G}^\alpha \\
&= -\omega_3^\beta(f_{\alpha\beta\gamma}+f_{\beta\gamma\alpha})\overline{\tilde{G}}^\gamma\tilde{G}^\alpha
\end{aligned}$$

Using the property $f_{\alpha\beta\gamma} = -f_{\beta\alpha\gamma} = -f_{\alpha\gamma\beta} = -f_{\beta\gamma\alpha}$, we have $f_{\alpha\beta\gamma} + f_{\beta\gamma\alpha} = 0$, resulting in

$$\delta(\overline{\tilde{G}}^{\alpha}\tilde{G}^{\alpha}) = 0$$

Therefore, $\overline{\tilde{G}}\tilde{G}$ is $SU(3)$ invariant.

Since both $\tilde{W}$ and $\tilde{G}$ have zero hypercharge, they are automatically $U(1)$ invariant. Additionally, $\tilde{W}$ is an $SU(3)$ singlet and $\tilde{G}$ is an $SU(2)$ singlet, so the mass terms are fully invariant under the entire $SU(3) \times SU(2) \times U(1)$ gauge group.

(c) Yes, there are new Yukawa interactions possible with these adjoint fermions. These interactions must be gauge invariant and renormalizable. The possible Yukawa interactions involve the standard model fermions, the Higgs field, and these new adjoint fermions.

For $\tilde{W}$, which transforms as $(1, 3, 0)$, we can construct a Yukawa interaction with the lepton doublet $L(1, 2, -1/2)$ and the Higgs doublet $\phi(1, 2, 1/2)$

$$\mathcal{L}_{\tilde{W}} = y_{\tilde{W}}\overline{L}\sigma^a\tilde{W}^a\phi + \text{h.c.},$$

where σ^a are the Pauli matrices, which connect the $SU(2)$ indices of L and ϕ with the adjoint index of $\tilde{W}$.

For $\tilde{G}$, which transforms as $(8, 1, 0)$, we can construct Yukawa interactions with quark fields. For example, with the quark doublet $Q(3, 2, 1/6)$ and anti-quark singlet $\bar{u}(\bar{3}, 1, -2/3)$

$$\mathcal{L}_{\tilde{G},1} = y_{\tilde{G},1}\overline{Q}T^{\alpha}\tilde{G}^{\alpha}\bar{u} + \text{h.c.},$$

where T^{α} are the $SU(3)$ generators (Gell-Mann matrices), which connect the color indices of Q and $\bar{u}$ with the adjoint index of $\tilde{G}$.

Similarly, we can construct a Yukawa interaction with Q and $\bar{d}(\bar{3}, 1, 1/3)$

$$\mathcal{L}_{\tilde{G},2} = y_{\tilde{G},2}\overline{Q}T^{\alpha}\tilde{G}^{\alpha}\bar{d} + \text{h.c.}.$$

These interactions respect all gauge symmetries of the Standard Model and are renormalizable. They could potentially lead to interesting phenomenology, including lepton and baryon number violation, depending on the assignment of lepton and baryon numbers to $\tilde{W}$ and $\tilde{G}$.

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