MA 562 - Introduction to Differential Geometry and Topology

Introduction to Smooth Manifolds by *John M. Lee*
Student: **Ralph Razzouk**

Homework 2

Problem 2-1
Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show that for every $x \in \mathbb{R}$, there are smooth coordinate charts (U, φ) containing x and (V, ψ) containing $f(x)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth as a map from $\varphi\left(U \cap f^{-1}(V)\right)$ to $\psi(V)$, but f is not smooth in the sense we have defined in this chapter.

Solution. Since f is not continuous, it is not smooth in the sense that we have defined in this chapter. Moreover, f is smooth far from $x = 0$ since it is constant there.
Let $\epsilon > 0$, $U = (-\epsilon, \epsilon)$, $\varphi = \mathrm{id}$, $V = \left(\frac{1}{2}, \frac{3}{2}\right)$, and $\psi = \mathrm{id}$. Then U contains $x = 0$ and V contains $f(x) = 1$.
Let (U, id) and (V, id) be coordinate charts for $\mathbb{R}$, then $\mathrm{id} \circ f \circ \mathrm{id}^{-1} = f$ is the constant map on $f\left(U \cap f^{-1}(V)\right) = [0, \epsilon)$, and is therefore smooth. ■

Problem 2-3
For each of the following maps between spheres, compute sufficiently many coordinate representations to prove that it is smooth.

(a) $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ is the **nth power map for** $n \in \mathbb{Z}$, given in complex notation by $p_n(z) = z^n$.

(b) $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is the **antipodal map** $\alpha(x) = -x$.

(c) $F : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $F(w, z) = (z\bar{w} + w\bar{z}, iw\bar{z} - iz\bar{w}, z\bar{z} - w\bar{w})$, where we think of $\mathbb{S}^3$ as the subset $\left\{(w, z) : |w|^2 + |z|^2 = 1\right\}$ of $\mathbb{C}^2$.

Solution. (a) Let $z \in \mathbb{S}^1$ and let (U, θ) be an angle coordinate chart containing z, and let (V, ϕ) be an angle coordinate chart containing z^n. Then $\phi \circ p_n \circ \theta^{-1}(x) = \phi \circ p_n(\mathrm{e}^{ix}) = \phi(\mathrm{e}^{inx}) = nx + 2k\pi$, for some k, which is constant on each component of $\theta(U \cap p_n^{-1}(V))$. Note that $U \cap p_n^{-1}(V)$ is open, since p_n is continuous. Thus, p_n is smooth.

(b) Let $p \in \mathbb{S}^n$ and assume that $(\mathbb{S}^n \setminus \{N\}, \sigma)$ is the stereographic chart from the north and it contains p. Then $(\mathbb{S}^n \setminus S, \tilde{\sigma})$ contains $\alpha(p)$, where $\tilde{\sigma}$ is the stereographic projection from the south. A computation shows $\tilde{\alpha} \circ \alpha \circ \sigma^{-1}(u) = -u$, which is smooth. Thus, α is smooth.

(c) The given function F is defined over two complex variables. We can rewrite F in terms of real coordinates, which gives us

$$F(x^1, x^2, x^3, x^4) = \left(2x^1x^3 + 2x^2x^4, 2x^1x^4 - 2x^2x^3, (x^3)^2 + (x^4)^2 - (x^1)^2 - (x^2)^2\right),$$

so F is continuous since it is the restriction of a continuous function. Additionally, we have

$$\sigma_{\mathbb{S}^2} \circ F \circ \sigma_{\mathbb{S}^3}^{-1}(u^1, u^2, u^3) = \frac{\left(8u^1u^3 + 4u^2(|u|^2 - 1), 4u^1(|u|^2 - 1) - 8u^2u^3\right)}{1 + (2u^1)^2 + (2u^2)^2 - (2u^3)^2 - (|u|^2 - 1)^2},$$

which is smooth on $\sigma_{\mathbb{S}^3}\left(\mathbb{S}^3 \setminus \{N\} \cap F^{-1}\left(\mathbb{S}^2 \setminus \{N\}\right)\right)$. Here $\sigma_{\mathbb{S}^n}$ is the stereographic projection from the north of $\mathbb{S}^n$. Similar computations for different pairs of charts show that F is indeed a smooth function.

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Problem 2-5
Let $\mathbb{R}$ be the real line with its standard smooth structure, and let $\widetilde{\mathbb{R}}$ denote the same topological manifold with the smooth structure defined in Example 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is smooth in the usual sense.

(a) Show that f is also smooth as a map from $\mathbb{R}$ to $\widetilde{\mathbb{R}}$.

(b) Show that f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

Solution. In Example 1.23, $\psi(x) = x^3$, *i.e.* $\psi^{-1}(x) = x^{\frac{1}{3}}$.

(a) The coordinate representation $\psi \circ f \circ \mathrm{id}^{-1} = \psi \circ f$ is smooth since both $\psi(x) = x^3$ and f are smooth in the usual sense.

(b) The coordinate representation is $\mathrm{id} \circ f \circ \psi^{-1} = f \circ \psi^{-1}$ and $f \circ \psi^{-1}(x) = f(x^{\frac{1}{3}})$.

$\boxed{\Longrightarrow}$ Assume that f is a smooth map from $\tilde{\mathbb{R}}$ to $\mathbb{R}$. Notice that $\psi^{(j)}(0) = 0$ for all $j \neq 3$. Then, if we want $f^{(n)}(0) = 0$, then there must be an n-tuple $(m_1, \ldots, m_n)$ such that $m_j = 0$ for all $j \neq 3$. Thus, $n = 3m_3$.

$\boxed{\Longleftarrow}$ Let $F = f \circ \psi^{-1}$. We aim to prove that F is smooth, but we first have to prove a little proposition.

Proposition 1. If $F \in C^k(\mathbb{R})$, then so is $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}})$.

Proof. We will prove this by induction on k.

- For $k = 0$, we have $x^{\frac{1}{3}} F(x^{\frac{1}{3}})$, and the result is clear since $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}})$ is continuous.
- Inductive step: Let $F \in C^k(\mathbb{R})$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{k+\frac{1}{3}} F(x^{\frac{1}{3}})\right] = x^{(k-1)+\frac{1}{3}} F(x^{\frac{1}{3}}) + \frac{1}{3} x^{(k-1)+\frac{1}{3}} x^{\frac{1}{3}} F'(x^{\frac{1}{3}}).$$

  By induction, we have

$$x^{(k-1)+\frac{1}{3}} F(x^{\frac{1}{3}}) \in C^{k-1}(\mathbb{R}).$$

  Since $xF'(x) \in C^{k-1}(\mathbb{R})$, then $\frac{1}{3} x^{(k-1)+\frac{1}{3}} F'(x^{\frac{1}{3}}) \in C^{k-1}(\mathbb{R})$, by induction. Since its derivative is $C^{k-1}(\mathbb{R})$, then it must be that $x^{k+\frac{1}{3}} F(x^{\frac{1}{3}}) \in C^k(\mathbb{R})$.

■

Now, suppose that $f^{(n)}(0) = 0$ for all n not integral multiples of 3. We can use the Taylor remainder theorem to write f up to the $3m$th term as

$$f(x) = f(0) + \frac{f^{(3)}(0)}{3!} x^3 + \frac{f^{(6)}(0)}{6!} x^6 + \ldots + + \frac{f^{(3m)}(0)}{(3m)!} x^{3m} + x^{3m+1} F_{3m+1}(x),$$

for some smooth function F_{3m+1}, then

$$f(x^{\frac{1}{3}}) = f(0) + \frac{f^{(3)}(0)}{3!} x + \frac{f^{(6)}(0)}{6!} x^2 + \ldots + + \frac{f^{(3m)}(0)}{(3m)!} x^m + x^{m+\frac{1}{3}} F_{3m+1}(x^{\frac{1}{3}}).$$

Since $x^{m+\frac{1}{3}} F_{3m+1}(x^{\frac{1}{3}}) \in C^m(\mathbb{R})$, then $F \in C^m(\mathbb{R})$ for all $m \geq 0$. Thus, $F = f \circ \psi^{-1}$ is smooth.

Therefore, f is smooth as a map from $\widetilde{\mathbb{R}}$ to $\mathbb{R}$ if and only if $f^{(n)}(0) = 0$ whenever n is not an integral multiple of 3.

■

Problem 2-6
Let $P : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth function, and suppose that for some $d \in \mathbb{Z}$, $P(\lambda x) = \lambda^d P(x)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^{n+1} \setminus \{0\}$. (Such a function is said to be **homogeneous of degree d**.) Show that the map $\widetilde{P} : \mathbb{RP}^n \to \mathbb{RP}^k$ defined by $\widetilde{P}([x]) = [P(x)]$ is well defined and smooth.

Solution. If $[x] = [y]$, then $x = \lambda y$, for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then we have

$$\tilde{P}([x]) = [P(x)] = [P(\lambda y)] = [\lambda^d P(y)] = [P(y)] = \tilde{P}([y]).$$

Thus, $\tilde{P}$ is well-defined.
Let $[x] \in U_i$ and suppose that $\tilde{P}([x]) \in U_j$. The coordinate representation $\phi_j \circ \tilde{P} \circ \phi_i^{-1}$ takes a point $(u^1, \ldots, u^k) \in \phi_i(U_i)$ to

$$\frac{(P_1(\alpha), \ldots, P_{j-1}(\alpha), P_{j+1}(\alpha), \ldots, P_{n+1}(\alpha))}{P_j(\alpha)},$$

where $\alpha = \phi_i(u^1, \ldots, u^k) = \left(u^1, \ldots, u^{i-1}, 1, u^{i+1}, \ldots, u^k\right)$ and P_m is the mth component of P. Since P is smooth, then each P_m is also smooth.
Thus, the coordinate representation is smooth, and therefore, so is $\tilde{P}$. ■

Problem 2-14
Suppose A and B are disjoint closed subsets of a smooth manifold M. Show that there exists $f \in C^\infty(M)$ such that $0 \le f(x) \le 1$ for all $x \in M$, $f^{-1}(0) = A$, and $f^{-1}(1) = B$.

Solution. Theorem 2.29 states

Theorem 1. Let M be a smooth manifold. If K is any closed subset of M, there is a smooth non-negative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.

By Theorem 2.29, there are functions $g, h : M \to [0, \infty)$ such that $g^{-1}(0) = A$ and $h^{-1}(0) = B$. In other words, $g(A) = 0$ and $h(B) = 0$. Take $f(x) = \frac{g(x)}{g(x)+h(x)}$, and indeed $0 \le f(x) \le 1$, $f(A) = 0$, and $f(B) = 1$. ■