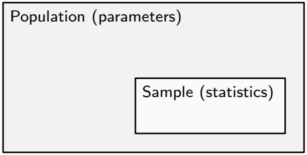
Math 207: Statistics

Chapter 23: The Accuracy of Averages



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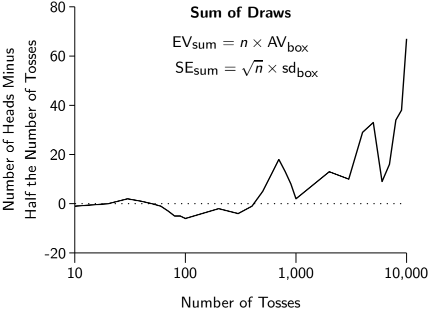
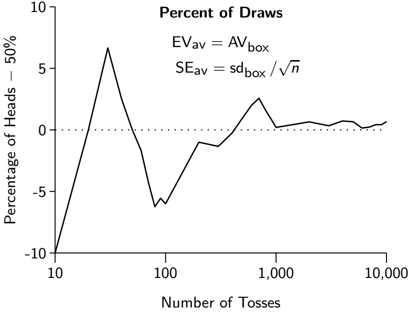
Expected Value, Standard Error, Central Limit Theorem

- Many statistics problems are modeled as samples from a box of numbered tickets.
- Solution procedure:
  - Formulate a box model.
  - Compute the average and SD of the contents of the box.
  - Determine if you are computing a sum or average (% in a 0/1 box is an average).
  - Use the appropriate formulas to compute the EV and SE for the sample.
  - Use the normal curve to compute chances of the sample being in a specified range.





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  $z_2 = (4.5 - 4)/0.4 = 1.25$
- Area under the normal curve: $\texttt{pnorm}(1.25) - \texttt{pnorm}(-1.25) = 0.789 = 78.9\%$

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- Estimate $\mathrm{SD}_{\mathrm{box}} = 0.8$.
- $\mathrm{SE}_{\mathrm{av}} = \frac{\mathrm{SD}_{\mathrm{box}}}{\sqrt{n}} \approx \frac{0.8}{\sqrt{100}} = 0.08.$

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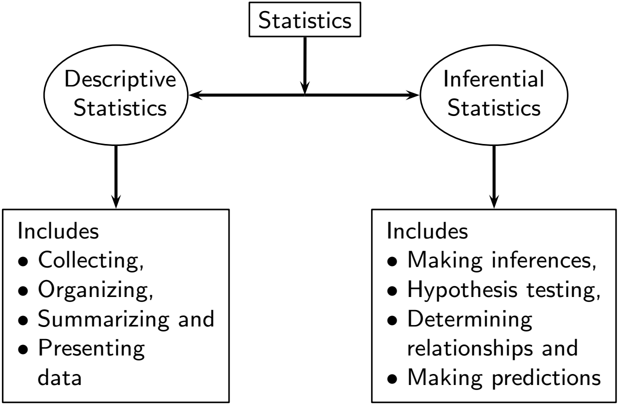
- Estimate the true average of the tickets to be 10.2 but also give a plus or minus on the estimate.
- Estimate $\text{SD}_{\text{box}} = 0.8$.
- $\text{SE}_{\text{av}} = \frac{\text{SD}_{\text{box}}}{\sqrt{n}} \approx \frac{0.8}{\sqrt{100}} = 0.08$.
- If we assume that the samples follow a normal curve, then

$$10.2 \pm 2 \cdot (0.08) = 10.2 \pm 0.16$$

  is 95% confidence interval on the average of the box.

Inferential Statistics

- In real situations, we typically don't know the contents of the box!
- That is where inferential statistics is used.



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- Prior to taking the sample, we are 95% confident that this procedure will give an interval that contains the true average.

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- Now compute a $\pm$ on this estimate:
  - $\text{SD}_{\text{box}} \approx \$53{,}000$.
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- Now compute a $\pm$ on this estimate:
  - $\text{SD}_{\text{box}} \approx \$53{,}000$.
  - $\text{SE}_{\text{av}} = \$53{,}000/\sqrt{1000} \approx \$1{,}700$.
- 95% confidence interval:

$$\$53{,}000 \ \pm \ 2 \cdot (1{,}700) = \$53{,}000 \ \pm \$3{,}400$$