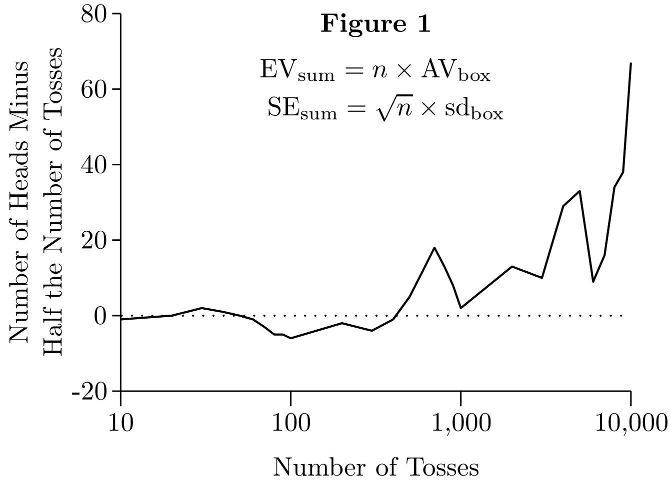
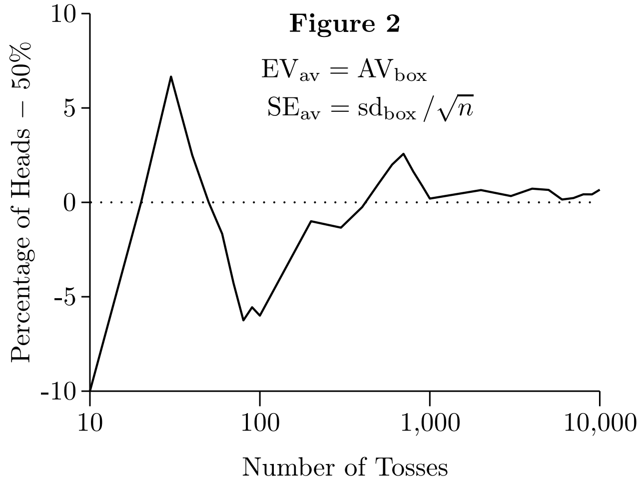
Math 207
Introduction to Statistics

Law of Large Numbers

John Kerrich was a mathematician who spent most of World War II in a German prison camp. To pass the time, he carried out experiments in probability theory. One involved tossing a coin 10,000 times. The following is a fictional discussion between Kerrich and an assistant preparing to meet the King of Denmark after the war.





Assistant. So you're going to tell the king about the law of averages.

Kerrich. Right.

Assistant. What's to tell? I mean, everyone knows about the law of averages, don't they?

Kerrich. OK. Tell me what the law of averages says.

Assistant. Well, suppose you're tossing a coin. If you get a lot of heads, then tails start coming up. Or if you get too many tails, the chance for heads goes up. In the long run, the number of heads and the number of tails even out.

Kerrich. It's not true.

Assistant. What do you mean, it's not true?

Kerrich. I mean, what you said is all wrong. First of all, with a fair coin, the chance for heads stays at 50%, no matter what happens. Whether there are two heads in a row or twenty, the chance of getting a head next time is still 50%.

Assistant. I don't believe it.

Kerrich. All right. Take a run of four heads, for example. I went through the record of my first 2,000 tosses. In 130 cases, the coin landed heads four times in a row; 69 of these runs were followed by a head, and only 61 by a tail. A run of heads just doesn't make tails more likely next time.

Assistant. You're always telling me these things I don't believe. What are you going to tell the king?

Kerrich. Well, I tossed the coin 10,000 times, and I got about 5,000 heads. The exact number was 5,067. The difference of 67 is less than 1% of the number of tosses.

Assistant. Yes, but 67 heads is a lot of heads. The king won't be impressed, if that's the best the law of averages can do.

Kerrich. What do you suggest?

Assistant. Toss the coin another 10,000 times. With 20,000 tosses, the number of heads should be quite a bit closer to the expected number. After all, eventually the number of heads and the number of tails have to even out, right?

Kerrich. You said that before, and it's wrong. Look at the data. In 1,000 tosses, the difference between the number of heads and the expected number was 2. With 2,000 tosses, the difference went up to 13.

Assistant. That was just a fluke. By toss 3,000, the difference was only 10.

Kerrich. That's just another fluke. At toss 4,000, the difference was 29. At 5,000, it was 33. Sure, it dropped back to 9 at toss 6,000, but look at Figure 1. The chance error is climbing pretty steadily from 1,000 to 10,000 tosses, and it's going straight up at the end.

Assistant. So where's the law of averages?

Kerrich. With a large number of tosses, the size of the difference between the number of heads and the expected number is likely to be quite large in absolute terms. But compared to the number of tosses, the difference is likely to be quite small. That's the law of averages. Just like I said, 67 is only a small fraction of 10,000.

Assistant. I don't understand.

Kerrich. Look. In 10,000 tosses you expect to get 5,000 heads, right?

Assistant. Right.

Kerrich. But not exactly. You only expect to get around 5,000 heads. I mean, you could just as well get 5,001 or 4,998 or 5,007. The amount off 5,000 is what we call "chance error."

Assistant. Can you be more specific?

Kerrich. Let me write an equation:

$$\text{number of heads} = \text{half the number of tosses} + \text{chance error}$$

This error is likely to be large in absolute terms, but small compared to the number of tosses. Look at Figure 2. That's the law of averages, right there.

Assistant. Hmmmm. But what would happen if you tossed the coin another 10,000 times? Then you'd have 20,000 tosses to work with.

Kerrich. The chance error would go up, but not by a factor of two. In absolute terms, the chance error gets bigger. But as a percentage of the number of tosses, it gets smaller.

Assistant. Tell me again what the law of averages says.

Kerrich. The number of heads will be around half the number of tosses, but it will be off by some amount — chance error. As the number of tosses goes up, the chance error gets bigger in absolute terms. Compared to the number of tosses, it gets smaller.

Assistant. Can you give me some idea of how big the chance error is likely to be?

Kerrich. Well, with 100 tosses, the chance error is likely to be around 5 in size. With 10,000 tosses, the chance error is likely to be around 50 in size. Multiplying the number of tosses by 100 only multiplies the likely size of the chance error by $\sqrt{100} = 10$.

Assistant. What you're saying is that as the number of tosses goes up, the difference between the number of heads and half the number of tosses gets bigger; but the difference between the percentage of heads and 50% get smaller.

Kerrich. That's it.