Math 207
Introduction to Statistics

Exam III Review Exercises

1. **Probability Rules**:

   a) **Coins**: Suppose that you flip a fair coin twice.

      i) What is the chance that both flips show tails?

      ii) What is the chance that the second flip shows heads given that the first showed heads?

   b) **Cards**: Cards are drawn without replacement from a deck.

      i) What is the chance that the first two cards are 7s?

      ii) What is the chance that the first card is a ♣ or a 7?

      iii) What is the chance that the second card is a ♣ given that the first card was?

      ii) What is the chance that the 10th card is a ♡ (given no information about the first nine cards)?

   c) **Dice**: Two fair dice are rolled.

      i) What is the chance that the sum is at least 9?

      ii) What is the chance that the sum is at least 9 or at most 3?

      iii) What is the chance that the sum is at least 9 given that one die shows a 6?

   d) **Normal Curve**: A computer program randomly generates a z-score from a normal distribution.

      i) What is the chance that the value is between -1 and 1?

      ii) What is the chance that the value is greater than 2?

2. **Mutually Exclusive Events**. Describe an event B for which A and B are mutually exclusive.

a) $A =$ the first two cards drawn from a deck are $10\clubsuit$ and $J\heartsuit$.

b) $A =$ in flipping three coins, you will get one tail and two heads

c) $A =$ in rolling two dice you will get doubles

3. **Independent Events**

a) $A =$ the first cards drawn from a deck is a $\clubsuit$

b) $A =$ the first two coin flips will both show tails

c) $A =$ in rolling a blue die and a red die, the blue die will show a 6.

d) Sets of paper tickets are shown below. Each ticket has a number on the left and a shape on the right. For each set of tickets, Determine if number and shape are independent or dependent.

i) $\boxed{1}\boxed{\triangle}$ $\boxed{1}\boxed{\square}$ $\boxed{1}\boxed{\square}$ $\boxed{1}\boxed{\star}$ $\boxed{5}\boxed{\triangle}$ $\boxed{5}\boxed{\square}$ $\boxed{5}\boxed{\square}$ $\boxed{5}\boxed{\star}$

ii) $\boxed{1}\boxed{\triangle}$ $\boxed{1}\boxed{\square}$ $\boxed{1}\boxed{\square}$ $\boxed{1}\boxed{\star}$ $\boxed{5}\boxed{\triangle}$ $\boxed{5}\boxed{\square}$ $\boxed{5}\boxed{\star}$ $\boxed{5}\boxed{\star}$

4. **Binomial Formula**. A box contains 7 blue marbles and 3 red marbles.

a) If five marbles are drawn with replacement, what is the chance that three are blue?

b) If five are drawn with replacement, what is the chance that at least three are blue?

c) If five are drawn without replacement, what is the chance that three are blue?

5. **Law of Averages**. A coin is flipped repeatedly and the percent of tails recorded.

a) You win \$1 if there are at most 45% tails. Which is better 10 tosses or 1000?

b) You win \$1 if there are between 45% and 55% tails. Which is better 10 tosses or 1000?

c) You win \$1 if there are at most 55% tails. Which is better 10 tosses or 1000?

d) You win \$1 if there are exactly 50% tails. Which is better 10 tosses or 1000?

6. **Expected Value and Standard Error for Sums**. 144 draws will be made at random with replacement from the box shown below.

$$\boxed{\boxed{1}\ \boxed{2}\ \boxed{6}}$$

a) How small can the sum be? How large?

b) How many times do you expect the ticket $\boxed{6}$ to turn up?

c) About how much do you expect the sum to be?

d) Put a plus or minus on your answer to c).

7. **Expected Value and Standard Error for Averages**. 144 draws will be made at random with replacement from the box shown below.

$$\boxed{\boxed{1}\ \boxed{2}\ \boxed{6}}$$

a) How small can the average be? How large?

b) About how much do you expect the average to be?

c) Put a plus or minus on your answer to b).

A standard deck of cards has the following properties.

- 52 cards
- 4 suits: hearts ♡, spades ♠, clubs ♣, and diamonds ♢
- Hearts and diamonds are red. Spades and clubs are black
- 13 cards of each suit: 2 – 10, J, Q, K, and ace
- 4 cards of each rank: 2 – 10, J, Q, K, and ace
- J, Q, and K are face cards

Multiplication Rule for Probabilities:

- $P(A \text{ and } B) = P(A)\, P(B \mid A)$.
- If A and B are independent, then $P(A \text{ and } B) = P(A)\, P(B)$.

Sum Rule for Probabilities:

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- If A and B are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$

Binomial Formula: The chance that an event will occur exactly k times out of n is given by the binomial formula:

$$\frac{n!}{k!\,(n-k)!}\, p^k\, (1-p)^{n-k}$$

In this formula, n is the number of trials, k is the number of times the event is to occur, and p is the probability that the event will occur on any particular trial. The assumptions:

- The value of n must be fixed in advance.
- p must be the same from trial to trial.
- The trials must be independent.

The **mean** of a list of numbers $x_1, \ldots, x_n$ is

$$\text{mean} = \frac{1}{n}\,(x_1 + x_2 + \cdots + x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

The **population standard deviation** is

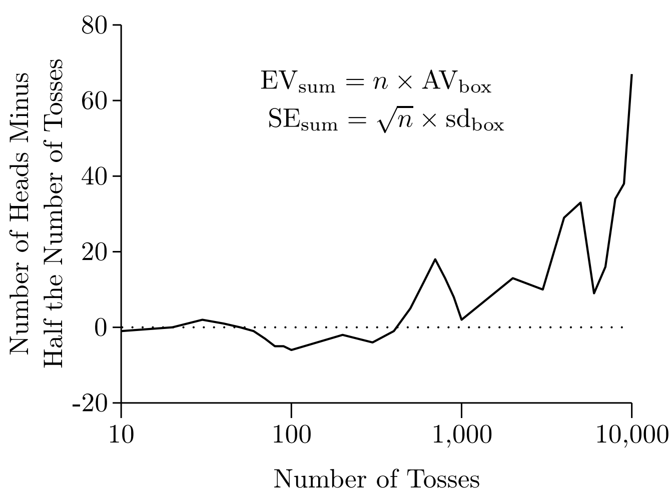
$$\text{SD} = \sqrt{\frac{1}{n}\left((x_1 - \text{mean})^2 + \cdots + (x_n - \text{mean})^2\right)}$$

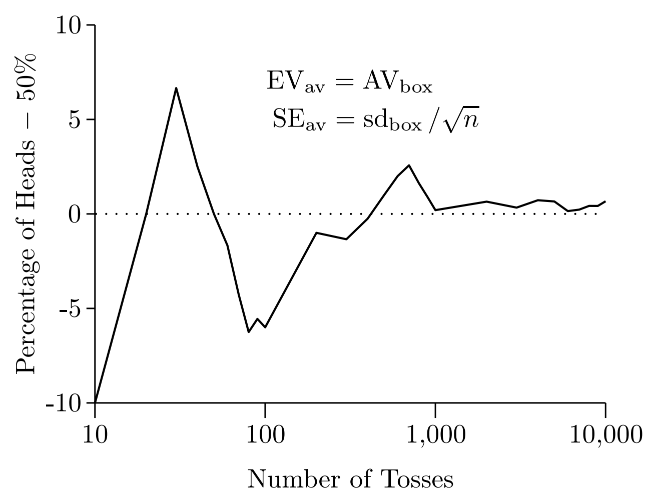
If we know the contents of a box of numbered tickets, we use SD.

The **sample standard deviation** is

$$\text{sd} = \sqrt{\frac{1}{n-1}\left((x_1 - \text{mean})^2 + \cdots + (x_n - \text{mean})^2\right)}$$

The figures below show how the standard error changes as the sample size increases. The figure on the right illustrates the Law of Averages.





Math 207
Introduction to Statistics

Exam III Review Exercises (Solutions)

1. **Probability Rules**:

a) **Coins**: Suppose that you flip a fair coin twice.

i) What is the chance that both flips show tails?

Use the *and* rule: $\frac{1}{2}\frac{1}{2} = \frac{1}{4}$.

ii) What is the chance that the second flip shows heads given that the first showed heads?

This is a *conditional probability*: $\frac{1}{2}$.

b) **Cards**: Cards are drawn without replacement from a deck.

i) What is the chance that the first two cards are 7s?

Use the *and* rule: $\frac{4}{52}\frac{3}{51}$.

ii) What is the chance that the first card is a ♣ or a 7?

Use the *or* rule: $\frac{13}{52} + \frac{4}{52} - \frac{1}{52}$.

iii) What is the chance that the second card is a ♣ given that the first card was?

This is a *conditional probability*: $\frac{12}{51}$.

ii) What is the chance that the 10th card is a ♡ (given no information about the first nine cards)?

$\frac{13}{52}$.

c) **Dice**: Two fair dice are rolled.

i) What is the chance that the sum is at least 9?

Write out the table of all 36 possibilities for two dice. There are 10 with a sum of at least 9. So, the answer is $\frac{10}{36}$.

ii) What is the chance that the sum is at least 9 or at most 3?

Write out the table of all 36 possibilities for two dice. There are 10 with a sum of at least 9 and there are three more with a sum of at most 3. So, the answer is $\frac{12}{36}$.

iii) What is the chance that the sum is at least 9 given that at least one die shows a 6?

There are 11 possibilities in which at least one die shows a 6. Of these, 7 have a sum of at least 9. So, the answer is $\frac{7}{11}$.

d) **Normal Curve**: A computer program randomly generates a z-score from a normal distribution.

i) What is the chance that the value is between -1 and 1?

The normal curve is a histogram so area = percent. The area between ± 1 is 68%.

ii) What is the chance that the value is greater than 2?

The area to the right of 2 is $1 - \text{pnorm}(2) = 2.28\%$.

2. **Mutually Exclusive Events**. Describe an event B for which A and B are mutually exclusive.

a) $A =$ the first two cards drawn from a deck are $10\clubsuit$ and $J\heartsuit$.

$B =$ the first card is $\clubsuit$.

b) $A =$ in flipping three coins, you will get one tail and two heads

$B =$ the first two coin flips are tails.

c) $A =$ in rolling two dice you will get doubles

$B =$ the sum of the dice is 7.

3. **Independent Events**

a) $A =$ the first card drawn from a deck is a $\clubsuit$

$B =$ the first card drawn is a 10.

b) $A =$ the first two coin flips will both show tails

$B =$ the 100th flip is tails.

c) $A =$ in rolling a blue die and a red die, the blue die will show a 6.

$B =$ the red die will show a 3.

d) Sets of paper tickets are shown below. Each ticket has a number on the left and a shape on the right. For each set of tickets, Determine if number and shape are independent or dependent.

i) $\boxed{1}\boxed{\triangle}$ $\boxed{1}\boxed{\square}$ $\boxed{1}\boxed{\square}$ $\boxed{1}\boxed{\star}$ $\boxed{5}\boxed{\triangle}$ $\boxed{5}\boxed{\square}$ $\boxed{5}\boxed{\square}$ $\boxed{5}\boxed{\star}$

independent

ii) $\boxed{1}\boxed{\triangle}$ $\boxed{1}\boxed{\square}$ $\boxed{1}\boxed{\square}$ $\boxed{1}\boxed{\star}$ $\boxed{5}\boxed{\triangle}$ $\boxed{5}\boxed{\square}$ $\boxed{5}\boxed{\star}$ $\boxed{5}\boxed{\star}$

dependent

4. **Binomial Formula**. A box contains 7 blue marbles and 3 red marbles.

a) If five marbles are drawn with replacement, what is the chance that three are blue?

Use the binomial formula with $n = 5$, $k = 3$ and $p = 7/10$.

$$\frac{5!}{3!\,2!}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2 = \frac{5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2 = 10\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2.$$

b) If five are drawn with replacement, what is the chance that at least three are blue?

Add the results from the binomial formula for three cases. Each has $n = 5$. Use $k = 3$, $k = 4$ and $k = 5$.

$$\frac{5!}{3!\,2!}\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2 = 10\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2$$
$$\frac{5!}{4!\,1!}\left(\frac{7}{10}\right)^4\left(\frac{3}{10}\right)^1 = 5\left(\frac{7}{10}\right)^4\left(\frac{3}{10}\right)$$
$$\frac{5!}{5!\,0!}\left(\frac{7}{10}\right)^5\left(\frac{3}{10}\right)^0 = \left(\frac{7}{10}\right)^5$$

Add these three numbers: $10\left(\frac{7}{10}\right)^3\left(\frac{3}{10}\right)^2 + 5\left(\frac{7}{10}\right)^4\left(\frac{3}{10}\right) + \left(\frac{7}{10}\right)^5$.

c) If five are drawn without replacement, what is the chance that three are blue?

The chance of BBBRR without replacement is $\frac{7}{10}\frac{6}{9}\frac{5}{8}\frac{3}{7}\frac{2}{6}$. From a) we know that there are 10 combinations like this. So the answer is $10\,\frac{7}{10}\frac{6}{9}\frac{5}{8}\frac{3}{7}\frac{2}{6}$.

5. **Law of Averages**. A coin is flipped repeatedly and the percent of tails recorded.

a) You win \$1 if there are at most 45% tails. Which is better 10 tosses or 1000?

Stop at 10. Otherwise, the average will probably be closer to 50%.

b) You win \$1 if there are between 45% and 55% tails. Which is better 10 tosses or 1000?

Stop at 1000 so that the average will probably be close to 50%.

c) You win \$1 if there are at most 55% tails. Which is better 10 tosses or 1000?

Stop at 1000 so that the average will probably be closer to 50%.

d) You win \$1 if there are exactly 50% tails. Which is better 10 tosses or 1000?

Stop at 10 so that there are fewer possibilities.

6. **Expected Value and Standard Error for Sums**. 144 draws will be made at random with replacement from the box shown below.

$$\boxed{\boxed{1}\quad\boxed{2}\quad\boxed{6}}$$

a) How small can the sum be? How large?

The smallest the sum can be is $144 \cdot 1 = 144$. The largest is $144 \cdot 6 = 864$.

b) How many times do you expect the ticket $\boxed{6}$ to turn up?

About 1/3 of the draws will be a 6. $144/6 = 24$.

c) About how much do you expect the sum to be?

The average of the box is 3. The SD (population standard deviation) is $\sqrt{14/3} \approx 2.16$. The expected value for the sum is $144 \cdot 3 = 432$.

d) Put a plus or minus on your answer to c).

The standard error for the sum is $\sqrt{144} \cdot 2.16 = 12 \cdot 2.16 \approx 25.9$.

7. **Expected Value and Standard Error for Averages**. 144 draws will be made at random with replacement from the box shown below.

$$\boxed{\boxed{1}\quad\boxed{2}\quad\boxed{6}}$$

a) How small can the average be? How large?

If you draw 1 every time, the average is 1. If you draw 6 every time, the average is 6.

b) About how much do you expect the average to be?

As shown in #6, the average of the box is 3 and the SD is 2.16. So, the expected value of the average is 3.

c) Put a plus or minus on your answer to b).

The standard error for the average is $2.16/\sqrt{144} = 2.16/12 = 0.18$.