Math 207: Statistics

What Are the Chances?

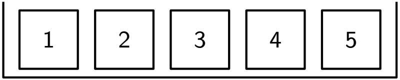
Dr. Ralph Wojtowicz

Mathematics Department

Shenandoah University

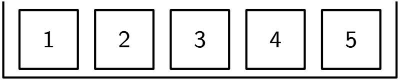
Probability

- The **probability (chance)** of something gives the percentage of time it is expected to happen, when the basic process is repeated independently and under the same conditions. This is the *frequency theory* of probability.
- Probabilities are between 0 and 1: $0\% \leq P(\text{event}) \leq 100\%$.
- If an event is impossible, its chance is 0%.
- If an event is sure to happen, its chance is 100%.
- $P(\text{event}) = 100\% \; - \; P(\text{event will not occur})$
- Examples
  - Flip a fair coin: $P(H) = 1/2 = 50\%$.
  - Roll a balanced die: $P(5) = 1/6 \approx 16.7\%$.
  - A box contains 3 red marbles and 2 blue ones. One marble is drawn at random $P(\text{red}) = 3/5 = 0.6 = 60\%$ and $P(\text{blue}) = 100\% - 60\% = 40\%$.
- Box models: In a random draw, all tickets have the same chance to be picked.



Conditional Probability

- Given two events A and B, the **conditional probability of A given B** is the probability that A **will** occur given that one knows that B **did** occur.
- Notation $P(A \mid B)$
- Examples:
  - $P(\text{die roll is a 4} \mid \text{the die roll is even}) = 1/3$
  - $P(\text{die roll is a 3} \mid \text{the die roll is even}) = 0$
  - $P(\text{blackjack on second card drawn} \mid \text{first card drawn is ace}) = 16/51$ (What else is assumed here?)
  - $P(\text{heads on 2nd coin flip} \mid \text{heads on first}) = 1/2$.
  - $P(\text{4 on second draw if you put the first ticket back} \mid \text{1 on first draw}) = 1/5$. $P(\text{4 on second draw without replacement} \mid \text{1 on first draw}) = 1/4$.



- $P(\text{third card drawn is Q}\heartsuit \mid \text{no information about first two cards}) = 1/52$.

Multiplication Rule

- **Multiplication Rule**: The probability that two things will both occur equals the probability that the first will occur times the probability that the second will occur given first did occur.
- Formula: $P(A \text{ and } B) = P(A)\,P(B \mid A)$ (assuming A is possible)
- Examples:
  - $P(\text{draw two aces in a row}) = \frac{4}{52}\,\frac{3}{51}$
  - $P(\text{head followed by tail}) = \frac{1}{2}\,\frac{1}{2}$.
  - $P(\text{4 followed by odd number}) = \frac{1}{5}\,\frac{3}{4}$ (box below without replacement)

    $P(\text{4 followed by odd number}) = \frac{1}{5}\,\frac{3}{5}$ (box below with replacement)

1	2	3	4	5

  - $P(\text{three tails in a row}) = \frac{1}{2}\,\frac{1}{2}\,\frac{1}{2} = \frac{1}{8}$
  - $P(\text{three face cards in a row}) = \frac{12}{52}\,\frac{11}{51}\,\frac{10}{50}$

Independence

- Two events A and B are **independent** if the probability of A given B is equal to the probability of A (without knowledge of B). Otherwise, the events are **dependent**.
- Formula: A and B are independent if $P(A \mid B) = P(A)$
- Examples of independent events:
  - Outcomes of two coin flips.
  - Even on first die roll, odd on second
  - Draws from a box with replacement.
- Examples of dependent events:
  - Draws from a box without replacement.
  - $A =$ a randomly selected subject is over 6' tall
$B =$ the father of a randomly selected subject is over 6' tall
  - $A =$ IBM stock price will increase next week
$B =$ Google stock price will increase next week

Addition Rule

- **Addition Rule**: The probability that at least one of two things will both equals the probability that the first will occur plus the probability that the second will occur minus the chance that both will occur
- Formula: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- Examples:
  - $P(\text{draw a } \heartsuit \text{ or a } 10) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$
  - $P(\text{draw a } \heartsuit \text{ or a } \clubsuit) = \frac{13}{52} + \frac{13}{52} - 0$
  - Roll two dice: $P(\text{sum is less than 6 or roll is doubles}) = \frac{10}{36} + \frac{6}{36} - \frac{2}{36}$
  - Flip 10 coins: $P(\text{at least 9 tails}) = P(9 \text{ tails}) + P(10 \text{ tails}) - 0.$

Mutually Exclusive Events

- Two events A and B are **mutually exclusive** when the occurrence of one prevents the occurrence of the other: one excludes the other.
- If A and B are mutually exclusive, then
  - $P(A \text{ and } B) = 0$
  - $P(A \text{ or } B) = P(A) + P(B)$
- Examples of mutually exclusive events:
  - $A =$ first card is a 10 and $B =$ first card is a 7
  - $A =$ Apple stock price will have a net increase next week
    $B =$ Apple stock price will have a net decrease next week
  - $A =$ out of 10 coin flips, exactly 3 are H and
    $B =$ out of 10 coin flips, exactly 5 are H
- Examples of events that are not mutually exclusive:
  - $A =$ first card is a 10 and $B =$ first card is a $\heartsuit$
  - $A =$ Apple stock will increase next week and
    $B =$ IBM stock will decrease next week

Examples to Motivate Use of the Binomial Formula

- A coin is tossed four times. What is the chance of getting exactly one head?
  - List the possibilities: HTTT or THTT or TTHT or TTTH.
  - By the multiplication rule, each possibility has chance $(1/2)^4 = 1/16$.
  - By the addition rule, the final answer is $\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = 4\left(\frac{1}{16}\right) = \frac{1}{4}$
- A coin is tossed 5 times. What is the chance of getting exactly two heads?
  - List the possibilities: HHTTT or HTHTT or HTTHT or HTTTH or THHTT or THTHT or THTTH or TTHHT or TTHTH or TTTHH.
  - By the multiplication rule, each possibility has chance $(1/2)^5 = 1/32$.
  - By the addition rule, the final answer is $10\left(\frac{1}{32}\right) = 31.25\%$
- A coin is tossed 20 times. What is the chance of getting exactly 10 heads?
  - There are 184,756 possibilities but we don't want to list them!
  - The chance of each is $(1/2)^{20}$.
  - So, the final answer is $184{,}756\left(\frac{1}{2}\right)^{20} \approx 17.6\%$.

Binomial Formula

- Factorials: $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
  - Examples:
    $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
    $4! = 4 \times 3 \times 2 \times 1 = 24$
    and $0! = 1! = 1$ by definition
- The chance that an event will occur exactly k times out of n is given by the **binomial formula**:

$$\frac{n!}{k!\,(n-k)!}\; p^k \,(1-p)^{n-k}$$

  where n is the number of trials, k is the number of times the event is to occur, and p is the probability that the event will occur on any particular trial
- The factorial term counts all the possibilities (addition rule) while the rest of the formula gives the chance of each individual possibility (multiplication rule).
- Assumptions:
  - The value n must be fixed in advance.
  - p must be the same from trial to trial.
  - The trials must be independent.

Binomial Formula Examples

- A coin is tossed four times. What is the chance of getting exactly one head?
  - $n = 4$, $k = 1$ and $p = 1/2$

$$\begin{aligned}\frac{4!}{1!\,(4-1)!}\,(1/2)^1\,(1/2)^{4-1} &= \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 3!}\,(1/2)\,(1/2)^3 \\ &= \frac{4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1}\,(1/2)^4 \\ &= 4\,(1/16) = 1/4\end{aligned}$$

- A coin is tossed 5 times. What is the chance of getting exactly two heads?
  - $n = 5$, $k = 2$ and $p = 1/2$
  - $\frac{5!}{2!\,3!}\,(1/2)^2\,(1/2)^3 = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 3\cdot 2}\,(1/2)^5 = 10\,(1/2)^5$
- A box has 4 red marbles and 5 green ones. Ten marbles are drawn at random with replacement. What is the chance that exactly 2 are green?
  - $n = 10$, $k = 2$ and $p = 5/9$.
  - Answer $= 45\,(5/9)^2\,(4/9)^8$