SERII NUMERICE

1. Să se determine sumele seriilor:

a) $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

Rezolvare:

$$S_n = \sum_{k=1}^{\infty} \frac{1}{k(k+3)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+3}\right) =$$

$$= 1 - \frac{1}{n+3}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \Big(1 - \underbrace{\frac{1}{n+3}}_{\downarrow 0}\Big) = 1$$

Astfel, rezultă că $\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = 1$

b) $\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}}$

Rezolvare:

$$S_n = \sum_{k=1}^{\infty} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}}$$

Amplificăm cu conjugata și obținem,

$$\frac{(n+1)\sqrt{n}-n\sqrt{n+1}}{[(n+1)\sqrt{n}+n\sqrt{n+1}][(n+1)\sqrt{n}-n\sqrt{n+1}]}$$

Rezolvăm ce este la numitor:

$$[(n+1)\sqrt{n}+n\sqrt{n+1}][(n+1)\sqrt{n}-n\sqrt{n+1}] =$$

$$= [(n+1)\sqrt{n}\cdot(n+1)\sqrt{n}] + [n(n+1)\sqrt{n}\cdot\sqrt{n+1}] -$$

$$- [n(n+1)\sqrt{n}\sqrt{n+1}] - n^2\sqrt{n+1}\sqrt{n+1} =$$

$$= (n+1)(n+1)\sqrt{n^2} + \cancel{n(n+1)\sqrt{n(n+1)}} -$$

$$- \cancel{n(n+1)\sqrt{n(n+1)}} - n^2\sqrt{(n+1)^2} =$$

$$= n(n+1)(n+1) - n^2(n+1) =$$

$$= (n^2+n)(n+1) - n^3 - n^2 = \cancel{n^3} + \cancel{n^2} + n^2 +$$

$$+ n - \cancel{n^3} - \cancel{n^2} = n^2 + n = n(n+1)$$

Obținem

$$S_n = \sum_{k=1}^{n} \frac{(k+1)\sqrt{k} - k\sqrt{k+1}}{k(k+1)} =$$

$$= \sum_{k=1}^{n} \left[\frac{\cancel{(k+1)}\sqrt{k}}{k\cancel{(k+1)}} - \frac{\cancel{k}\sqrt{k+1}}{\cancel{k}(k+1)}\right] =$$

$$= \sum_{k=1}^{n} \frac{\sqrt{k}}{k} - \frac{\sqrt{k+1}}{k+1}$$

Avem

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n} - \frac{\sqrt{n+1}}{n+1} = 0$$

$$\frac{\sqrt{n}}{n} \downarrow 0 \qquad \frac{\sqrt{n+1}}{n+1} \downarrow 0$$

În concluzie

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n} + n\sqrt{n+1}} = 0$$

c) $$\sum_{n=1}^{\infty} \frac{n^2-n-1}{(n+1)!}$$

Rezolvare:

Știm că $$\frac{n}{(n+1)!} = 1$$

$$S_n = \sum_{k=1}^{n} \frac{k^2-k-1}{(k+1)!} = \sum_{k=1}^{n} \frac{k\left(k-1-\frac{1}{k}\right)}{(k+1)!}$$

$$= \sum_{k=1}^{n} \frac{k}{(k+1)!}\left(k-1-\frac{1}{k}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{k}{(k+1)!}\left(k-1-\frac{1}{k}\right)$$

$$= \lim_{n\to\infty} \underset{\downarrow 1}{\frac{k}{(k+1)!}} \lim_{n\to\infty} \left(\underset{\downarrow \infty}{k-1} - \underset{\downarrow 0}{\frac{1}{k}}\right) = \underset{\downarrow \infty}{k-1}$$

Deci $\sum_{n=1}^{\infty} \frac{n^2-n-1}{(n+1)!} = n-1 \to \infty$

d) $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$

Rezolvare:

$$S_n = \sum_{k=1}^{n} \frac{k}{2^{k-1}} = \sum_{k=1}^{n} \frac{k}{\frac{2^k}{2}} =$$

$$= \sum_{k=1}^{n} \frac{2k}{2^k}$$

$$\lim_{n\to\infty} \frac{2n}{2^n} = 2\lim_{n\to\infty} \frac{n}{2^n} \to 0$$

2. Utilizând criteriul general de convergență a lui Cauchy, să se demonstreze convergența serilor:

a) $\sum_{n=1}^{\infty} \frac{\cos 2n}{2^n}$

$a_n = \frac{\cos 2n}{2^n}$; $\lim_{n\to\infty} \frac{\cos 2n}{2^n} \Rightarrow 0 \Leftrightarrow$

a_n convergentă

Atunci:

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cos 2n}{2^n} = \lim_{n\to\infty} \frac{1-2\sin^2 n}{2^n}$

$= \lim_{n\to\infty} \left(\underbrace{\frac{1}{2^n}}_{\downarrow 0} - 2 \underbrace{\frac{\sin^2 n}{2^n}}_{\downarrow 0} \right) \Rightarrow 0$

Concluzie

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cos 2n}{2^n} = 0 \Rightarrow$ seria este convergentă.

b) $\sum_{n=1}^{\infty} \frac{\left(1-\frac{1}{n}\right)^n}{n^2}$

$a_n = \frac{\left(1-\frac{1}{n}\right)^n}{n^2}$; $\lim_{n\to\infty} \frac{\left(1-\frac{1}{n}\right)^n}{n^2} = 0$

a_n convergente

Atunci,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\left(1-\frac{1}{n}\right)^n}{n^2}$

$\lim_{n\to\infty} \left(1 - \underbrace{\frac{1}{n}}_{\downarrow 0}\right)^n = 1 \neq 0$

c) $\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\ln n \cdot \ln(n+1)}$

3. Să se studieze convergența absolută a seriilor:

a) $\sum_{n=1}^{\infty} \sin \frac{x}{n^2} \quad (x \in \mathbb{R})$

$a_n = \sin \frac{x}{n^2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sin \frac{x}{n^2} =$

$= \lim_{n\to\infty} \left(\frac{(n+1)\frac{1}{n^2}}{2}\right)\left(\sin \frac{n\frac{1}{n^2}}{2}\right) / \sin\left(\frac{\frac{1}{n^2}}{2}\right)$

$= \frac{1}{2}$ → serie neconvergentă

b) $\sum_{n=1}^{\infty} \frac{\sin n^2}{2^n}$

$a_n = \frac{\sin n^2}{2^n}$; $|\sin t| \le 1$

$|a_{n+1} + a_{n+2} + \dots + a_{n+p}| \le |a_{n+1}| + |a_{n+2}| + \dots$

$\dots + |a_{n+p}| \le \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^{n+p}} =$

$= \frac{1}{2^{n+1}} \left(1 + \frac{1}{2} + \dots + \frac{1}{2^{p-1}}\right) = \frac{1}{2^{n+1}} \cdot \frac{1 - 1/2^p}{1 - 1/2}$

$< \frac{1}{2^{n+1}} \cdot \frac{1}{1 - 1/2} = \frac{1}{2^n}$

Fie $\varepsilon > 0$. Există $n_\varepsilon \in N^*$ astfel încât $\frac{1}{2^n} < \varepsilon$, $(\forall)$ $n \ge n_\varepsilon$. Rezultă:

$$|a_{n+1} + a_{n+2} + \dots + a_{n+p}| < \varepsilon \quad (\forall)\ n, p \in N^*,\ n \ge n_\varepsilon$$

Pe baza criteriului general a lui Cauchy, seria este convergentă.

c) $\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$

$a_n = (-1)^n \frac{1}{\ln n}$

Observăm că $\frac{1}{\ln n}$ este descrescător

Aplicăm criteriul logaritmic

$$\lim_{n\to\infty} \frac{\ln \frac{1}{(-1)^n \frac{1}{\ln n}}}{\ln n}$$

4. Să se studieze natura seriilor cu termeni pozitivi:

a) $$\sum_{n=1}^{\infty} \frac{n^2}{3^n+4^n}$$

$$a_n = \frac{n^2}{3^n+4^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{3^{n+1}+4^{n+1}} \cdot \frac{3^n+4^n}{n^2} =$$

$$= \frac{(n+1)^2}{7} \cdot \frac{1}{n^2} = \frac{1+\frac{1}{n}+\frac{1}{n^2}}{7}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{7} =$$

$$= \frac{1}{7} \Rightarrow \text{serie divergentă}$$

b) $$\sum_{n=1}^{\infty} \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

$$a_n = \frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{\left(2+\frac{1}{n+1}\right)^{n+1}} \cdot \frac{\left(2+\frac{1}{n}\right)^n}{n^2} =$$

$$= \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{\quad}$$

c) $$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$a_n = \frac{2^n}{n!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} =$$

$$= \lim_{n\to\infty} \frac{2 \cdot n!}{(n+1)!} = 2 \lim_{n\to\infty} \frac{1}{n+1} = 2$$

$\Rightarrow$ serie divergente

d) $$\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$$

$$a_n = \left(\frac{n}{n+1}\right)^{n^2} = \left(\frac{\cancel{n}}{\cancel{n}\left(1+\frac{1}{n}\right)}\right)^{n^2} =$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)^{n^2}$$

e) $\sum_{n=1}^{\infty} \frac{n}{n^2+n+1}$

$$a_n = \frac{n}{n(n+1+\frac{1}{n})} = \frac{1}{n+1+\frac{1}{n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1+\underbrace{\frac{1}{n}}_{0}} = 1$$

$\to$ serie divergente

f) $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$

g) $\sum_{n=1}^{\infty} \left(\frac{1 \cdot 4 \cdot 7 \ldots (3n-2)}{2 \cdot 5 \cdot 8 \ldots (3n-1)} \right)^2$

5. Să se discute natura seriilor după valorile parametrului $a > 0$

a) $$\sum_{n=1}^{\infty} \frac{a^n}{n}$$

$$\frac{a_{n+1}}{a_n} = \frac{a^{n+1}}{n+1} \cdot \frac{n}{a^n} = \frac{a \cdot \cancel{n}}{\cancel{n}(1+\frac{1}{n})} =$$

$$= \frac{a}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty} \frac{a}{1+\frac{1}{n}} = a \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = a$$

$\Rightarrow$ divergentă

b) $$\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\dots(a+n)}$$

$$a_n = \frac{n!}{(a+1)(a+2)(a+3)\dots(a+n)} = \frac{n!}{(a+n)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(a+n+1)!} \quad \dots$$

b) $f_n : (-1, 1] \to R$, $f_n(x) = \dfrac{x^n}{1+x^{2n}}$, $n \in N^*$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{1+x^{2n}} =$$

$$= \frac{\lim x^n}{\lim (1+x^{2n})} = \frac{f(x)}{\lim 1 + \lim x^{2n}} =$$

$$= \frac{f(x)}{1+[f(x)]^2}$$

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

c) $f_n : [0, 1] \to R$, $f_n(x) = x^n(1-x^n)$,

$n \in N^*$

SERII DE PUTERI

Să se studieze raza de convergență și domeniul de convergență pentru următoarele serii de puteri:

a) $$\sum_{n\geq 1} \frac{n+1}{n!} x^n$$

Seria derivată: $\sum_{n\geq 1} (n+1) \cdot \overbrace{\frac{1}{n!}}^{a_n} x^n$

$\lim_{n\to\infty} \frac{1}{n!} \Rightarrow 0$, raza de convergență este

$$R = \lim_{n\to\infty}\left(\frac{a_n}{a_{n+1}}\right) = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \Rightarrow 0$$

b) $$\sum \frac{n^2}{2^n+3^n} (x-1)^n$$

c) $\sum \frac{2^{n+1}}{e^n} x^n$

$a_n = \frac{2^{n+1}}{e^n} = \frac{2^n \cdot 2}{e^n}$

$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{\frac{2^{n+1}}{e^n}}{\frac{2^{n+2}}{e^{n+1}}} =$

$= \lim \frac{2^{n+1}}{e^n} \cdot \frac{e^{n+1}}{2^{n+2}} = \lim \frac{\cancel{2^n} \cdot 2}{\cancel{e^n}} \cdot \frac{\cancel{e^n} \cdot e}{\cancel{2^n} \cdot 2^2}$

$= \lim \frac{2e}{2} = \lim e = e > 0$

$\Rightarrow$ serie converge $\forall x$

INTEGRALE IMPROPRII

Să se demonstreze convergența integralelor improprii și să se calculeze valoarea acestora:

a) $\int_1^{\infty} \frac{1}{x \ln x} dx$

$f(x) = \frac{1}{x \ln x}$

$I_1 = \int_1^{0} f(x)$ $\qquad$ $I_2 = \int_1^{\infty} f(x)$

$\lim\limits_{\substack{x \to 0 \\ x > 0}} I_1 = \lim \frac{1}{x \ln x} \Rightarrow$ nu există

$\lim\limits_{x \to \infty} I_2 = \lim \frac{1}{x \ln x} = 0$

b) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

$f(x) = \frac{1}{\sqrt{1-x^2}} > 0 \quad (\forall)\ x \in (0, \infty)$

$J_1 = \int_0^1 f(x)\,dx\,;\ J_2 = \int_1^\infty f(x)\,dx$

$\lim\limits_{\substack{x \to 0 \\ x > 0}} \frac{1^{\sqrt{(1-x^2)}}}{\sqrt{1-x^2}} = \lim \sqrt{1-x^2} \cdot \frac{1}{\sqrt{(1-x^2)^2}}$

$= \lim \frac{1-x^2}{1-x^2} = 1$

$\lim\limits_{x \to \infty} f(x) = 1$

$\Rightarrow \int_0^1$ convergentă

$\int_0^1 \frac{1^{\sqrt{1-x^2}}}{\sqrt{1-x^2}} = \int_0^1 \frac{1-x^2}{1-x^2} = 1$

c) $\int_0^\infty \frac{1}{x^2+4}\,dx$

$f(x) = \frac{1}{x^2+4}$

$J_1 = \int_0^1 f(x)\,dx\ ;\ J_2 = \int_0^\infty f(x)$

$\lim\limits_{\substack{x \to 0 \\ x > 0}} f(x) = \lim\limits_{\substack{x \to 0 \\ x > 0}} \frac{1}{x^2+4} = \lim \frac{x^2-4}{(x^2+4)(x^2-4)} =$

$= \lim \frac{x^2-4}{x^4 + \cancel{4x^2} - 4x^2 - 16} = \lim \frac{x^2-4}{x^4-16} =$

$= \lim \frac{x^2-4}{(x^2-4)}$