Sovereign Debt Auctions with Strategic Interactions *

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Abstract

In this paper, we study the impact that alternative ways of issuing sovereign debt have on borrowing decisions, the cost of debt, and welfare. We build a model of sovereign borrowing and default with auctions, disciplined with proprietary bid level data. We calibrate the model, with discriminatory price auctions, to the Portuguese economy and find that it matches standard moments in the data while also generating spreads with a volatility that significantly exceeds their mean, a documented shortcoming of previous sovereign debt models. We then perform a counterfactual, comparing the two most common types of auction: uniform and discriminatory price auctions. We find that switching to a uniform protocol constitutes a Pareto improvement, and that the difference in welfare is highest during crises (0.6% of permanent consumption). Finally, we find that accounting for dynamic effects is crucial. In a single auction setting, a risk averse government prefers the discriminatory protocol. However, with repeated auctions, the properties of the discriminatory protocol lead to over-borrowing. The effect it has on prices makes the uniform protocol a better option.

JEL Codes: D44, E43, F34, F41, G15, H63.

Keywords: Sovereign debt auctions, default risk, discretion, dilution.

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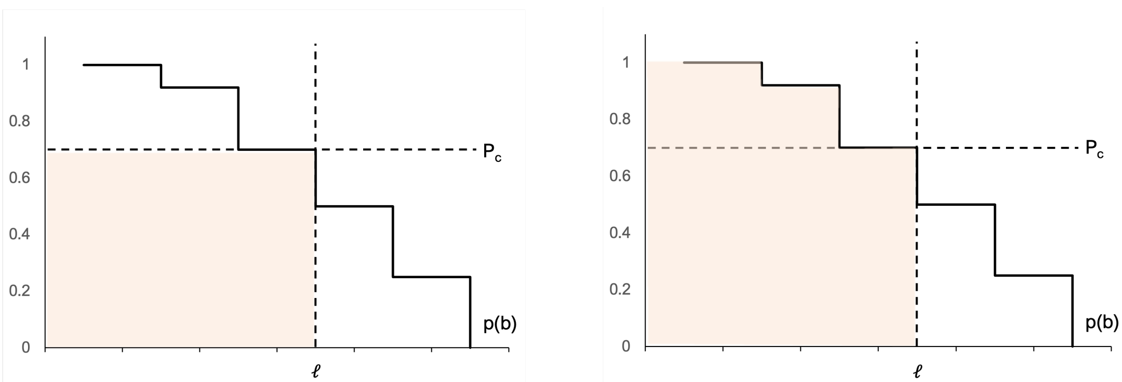
1 Introduction

Governments of both Emerging Market Economies (EMEs) and Advanced Economies (AEs) maintain enormous stocks of sovereign debt.[1] Most of this debt is issued in auctions. Because of the massive amounts involved, both policymakers and academics have been extremely interested in determining the best way to run these auctions[2]. In sovereign debt auctions, investors submit bids consisting of the highest price they are willing to pay to purchase a unit of debt, and how much they are willing to buy. Then, the government chooses which bids to accept. There is wide variation across countries in auction protocols, i.e. the set of rules determining how much each winning bid pays. OECD (2023) found 40 of 41 countries surveyed used auctions. Of those, 12 used uniform price auctions, 15 used discriminatory price auctions and 13 used both.



(a) A uniform price auction

(b) A discriminatory, or "pay-as-bid," auction

Figure 1: Comparison of uniform price and discriminatory price auctions

Figure 1 depicts how these two protocols work. Individual bids are combined into an aggregate demand function, $p(b)$. The government selects the amount issued, b', and the clearing price, P_c. In a uniform price auction, all accepted bids are executed at the marginal price. In a discriminatory price auction (pay-as-bid), all accepted bids are executed at their bidding prices. Revenue is depicted by the shaded area below the aggregate demand function. As we will show, the aggregate demand function depends on the auction protocol, and so does the amount of debt issued and the revenue raised.

[1]In 2023, AEs total government debt to GDP was over 110% (EMEs stood at nearly 70%).

[2]As stated in Chari and Weber (1992) "with such large amounts at stake, even small improvements in the Treasury's auction procedure can lead to large gains for taxpayers."

There are two reasons for this. First, the government has discretion over the quantity sold in each auction, and it chooses the quantity after observing investors' bids. The incentives to issue more or less debt differ with the auction protocol used, so different protocols lead to different issuance choices. These, in turn, lead to different lender expectations about how much the debt will be worth, which leads to different bidding behavior. Second, there is a dynamic link between debt auctions over time. The value of debt today depends not only on how much debt is issued today but also on how much debt will be issued next period, and the period after, and so forth. Therefore, the auction protocol can also affect the price of debt today through its effect on the incentives to borrow in the future.

In this paper we ask: How do outcomes (yields, borrowing and default decisions and welfare) depend on the auction protocol used? How do those differences inform what auction protocol countries facing default risk should use?

To answer these questions, we build a theory of how strategic interactions between the borrower and lenders affect outcomes, and evaluate how different auction protocols interact with default risk. We fill a gap in the literature of sovereign debt and default: the role played by how debt is issued and how primary market prices are determined. At the same time, we also contribute to the quantitative sovereign debt literature by analyzing the impact that modeling the auction has on how the model fits the data. Finally, we also contribute to the literature that studies differences in these two types of auctions, which has been centered on environments with exogenous supply and a single auction. While our application focuses on sovereign debt, our conclusions apply to other settings where a seller reserves discretion over quantity and default risk is a concern. In particular, they may also apply to auctions of corporate debt.

We begin by studying a two period environment with a single auction and default risk. The foreign investors buying the debt are symmetric and competitive. We study the two protocols. In this setting, we first show that if the debt issuance policy of the government is identical across protocols then revenue equivalence arises: both auction protocols generate the same expected revenue. However, even though ex-ante revenue is equal across auction protocols, we find that the bid functions are different. Investors bid lower prices under a discriminatory price protocol than under a uniform price protocol. Under a uniform price protocol, all winning bids are executed at the marginal price, so competition results in marginal prices exactly matching the unit value of the debt issued. On the other hand, when investors pay-as-bid, they fear their bid will not be marginal, that the government may issue more debt, and they may end up paying a high price for a low value asset. As a result, they bid lower prices. However, under a discriminatory price protocol, the average executed bid is above the marginal price of the auction. As the government is faced with different financing needs, the average executed bid has a lower variance than the marginal price of the auction. This contrasts with the uniform price auction, where the average price paid by investors is exactly the marginal price of the auction. When revenue equivalence holds, a risk neutral government is thus indifferent between protocols, but a risk averse government prefers the discriminatory price protocol because the variance of the average executed price, and that of revenue, is smaller.

When debt is endogenous, revenue equivalence need not hold. The difference in auction protocol creates different incentives for the government to borrow, which leads to different issuance policies, violating the sufficient condition, of identical debt issuance policies, that guarantees the equivalence. Even in a static environment, the choice of protocol has an impact on expected revenue, yields, borrowing, and welfare. The ranking of auction protocol depends on the government's preferences, particularly on how much it values smoothing revenue. When preferences are linear, the effects of static dilution are extreme and the government prefers the uniform price protocol. Convex preferences create a motive to smooth consumption over time, which disciplines the government's borrowing, reducing the effects of static dilution. As a result, for sufficiently concave utility, the government prefers the discriminatory price protocol. This is related to the trade-off between levels and variance of prices identified above, reminiscent of Cole et al. (2018).[3]

Having illustrated how incentives to borrow depend on the auction protocol in a two period setting with a single auction, we then move to an infinite horizon setting where fu-

[3] Their result arises from "sufficiently asymmetric information" between investors. Ours rely on debt being chosen by the government. In our setting, when utility is concave enough, the discriminatory protocol achieves lower variance in prices without the drop in the mean price that occurs under linear utility.

ture incentives to borrow are affected by each protocol. We extend a standard framework for studying government borrowing and default to allow for different auction protocols[4]. We inform our modeling decisions using proprietary bid level data for Portuguese sovereign debt auctions (as first used in Alves Monteiro (2022)). We observe that: (i) individual bid functions tend to be homogeneous in normal times; (ii) during the crisis period (2008-2014) the aggregate bid function becomes steeper and more inelastic; and, (iii) there is no evidence of persistent differences between investors. Since higher than expected government spending played a key role in the Eurozone Debt Crises (see Copelovitch et al. (2016)), we incorporate uncertainty about required government expenditures as the primary source of uncertainty regarding the government's need/desire to borrow.

In the standard sovereign debt model, long term debt creates dynamic dilution motives that make the equilibrium allocation constrained inefficient.[5] Essentially, when the government inherits legacy debt, it has an incentive to issue new claims on the resources it has already "earmarked" to pay its legacy investors. This leads the government to borrow more than it would have planned to, and lenders react by offering it lower prices in anticipation. We show that different auction result in differently shaped revenue curves for the government. Under the uniform protocol, declines in the marginal price apply to all debt issued, while under the discriminatory, they only apply to the marginal unit issued. This supercharges the dilution motives that arise under the discriminatory protocol.

In a quantitative exercise, we discipline the model using the experience of Portugal until 2011 (when it was bailed out by the European Commission and Central Bank and the International Monetary Fund). During this period, Portugal used the discriminatory price protocol for all auctions. The calibrated model is able of matching standard moments in the Portuguese economy regarding debt, spreads and business cycles statistics. Moreover, the use of a discriminatory protocol lets the model easily generate spreads whose volatility significantly exceeds their mean, a shortcoming of previous sovereign debt models, as documented in Aguiar et al. (2016).

[4]Refer to the related literature for an exposition of such standard framework.

[5]Specifically, if the government could commit ex-ante to a sequence of debt issuance choices while still lacking commitment with respect to default, it could achieve a strictly better outcome. See Aguiar and Amador (2019) for this result and a proof that the allocation with short-term debt is constrained efficient.

In a counterfactual, we compare the two protocols. We find that the uniform price protocol yields higher welfare than the discriminatory, and that these gains are highest during crises. Moreover, switching to a uniform protocol is a Pareto improvement as both the small open economy and foreign investors are better off after the switch. This is consistent with the observed switch in 2011 to a uniform price protocol for long-term debt.[6]

Accounting for dynamic effects is crucial for this welfare result. Given standard values for risk aversion of the borrowing country, we would find in the static setting with a single auction that the discriminatory price protocol is optimal. The dynamic effects of the discriminatory price protocol, however, are terrible. The uniform price protocol provides much better incentives for borrowing over time, as well as protecting investors from static dilution within an auction. These both lead to much better bond prices for the government. In fact, under the calibrated model, these gains more than justify forgoing the insurance mechanism provided by the discriminatory protocol.[7]

Related literature. This paper builds on the quantitative sovereign default literature, which is based on the classic setting of Eaton and Gersovitz (1981). Key early papers include Aguiar and Gopinath (2006), Arellano (2008), Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012). One insight of the later papers is that incorporating long term debt is crucial for being able to match the levels of debt and levels and volatility of interest rate spreads observed in EMEs. Many branches of the literature build on this workhorse model with long term debt[8]. We also build on this setting by explicitly modeling the auction protocols countries use to issue debt. We are then able to assess the role played by how debt is issued and how primary market prices are determined. We find that explicitly modeling the auction framework enriches the environment, leading to two interesting phenomena. First, the use of a discriminatory price protocol lets the model

[6]Portugal stopped issuing securities with maturity longer than one year from 2011 to 2014. When the Portuguese Treasury resumed auctioning debt at those maturities in 2014, it used the new protocol.

[7]While the fact that the gains from insurance are relatively small is consistent with the well known fact that the welfare costs of fluctuations are small, as in Lucas (1987), the existing literature is silent as to the size of the gains from reducing each kind of dilution.

[8]Conesa and Kehoe (2017) and Bocola and Dovis (2019) focus on the role of rollover risk and self-fulfilling crises. Arellano and Ramanarayanan (2012), Sánchez et al. (2018), Bocola and Dovis (2019) and Dvorkin et al. (2021) focus on the role of maturity choice.

easily generate spreads whose volatility significantly exceeds their mean (a key feature of the Eurozone countries that went through debt crises in 2008-2014, and a notable difference of those countries from the EMEs the sovereign default literature had previously focused on). Previous models (see Aguiar et al. (2016)) could not generate this pattern without producing counterfactual levels of debt or spreads[9]. Second, we find that discriminatory price protocols are prone to self-fulfilling crises even in environments where such crises would not occur under a uniform protocol. We leave the discussion of this second phenomenon to a companion paper Alves Monteiro and Fourakis (2023).

This paper also builds on the sovereign debt literature, with an emphasis on the auction framework used to issue debt. Related papers here include Cole et al. (2018), Pycia and Woodward (2023) and Cole et al. (2022). Each aims at comparing the two auction protocols. To do so, each considers a static auction model with asymmetric information across bidders and exogenous asset quality. Cole et al. (2022) in particular, identifies the insurance mechanism that we also describe for the discriminatory price protocol. Apart from Pycia and Woodward (2023), all consider exogenously random supply of debt. In Pycia and Woodward (2023), the government commits to a distribution for the supply and a reserve price before observing demand. We focus instead on incorporating different auction protocols into an infinite horizon, dynamic model of government borrowing and default. This paper is the first to consider a strategic government that has discretion over supply and can choose how much to issue after observing demand. We show that this strategic interaction between a government with discretion and optimizing investors matters. In particular, investors know that distinct protocols induce different debt issuances by the government (which may break revenue equivalence between protocols).

There is a large auction theory literature that studies multi-unit auctions. In these auctions, bidders submit both prices and quantities, generating a two dimensional strategic problem[10]. We assume that investors are infinitesimal (and therefore take aggregate bid function as given). This allows us to focus on how the auction protocols determine prices

[9]See for instance Bocola et al. (2019). Paluszynski (2023) does get closer, however, in the long run simulations, still generates counterfactual levels of debt and spreads.

[10]See Wilson (1979), Engelbrecht-Wiggans and Kahn (1998), Perry and Reny (1999) and McAdams (2006).

by aggregating bids, while avoiding, similarly to Cole et al. (2018), strategic considerations between investors. Instead, we focus on the strategic interaction that arises from having a maximizing government that chooses how much debt to issue after observing bids, reserving discretion on the quantity sold.

There is also previous work on how multi-unit auctions determine prices in equilibrium from an empirical perspective[11]. Hortaçsu (2002) presents a model of a multi-unit discriminatory auction with a finite number of symmetric bidders with independent private values. In their model, one bid affects the bid functions through changes in the distribution of the marginal price of the auction. Kastl (2011) builds on this framework by allowing for discrete-step bid functions. Our price-taking assumption allows us to abstract from this problem. Our auction framework also differs in that we assume a common valuation for the debt being auctioned. That is, the value of debt is pinned down by the future endogenous probability of default, known by investors.

2 Data: Background and Evidence

Auction data was provided by the *Portuguese Treasury and Debt Management Agency* (IGCP, Portuguese acronym). The data comprises all auctions of treasury bills (short maturities) and bonds (long maturities) held from 2003 and 2004, respectively, and up to 2020. The data comprises all individual bids (price and amount) that were placed in each auction, even if not executed. Issuance of treasury bills in the primary market is done through auctions. Treasury bonds are launched in syndicated operations[12]. New issuances of a line that has been launched are done through auctions. IGCP uses a primary dealership model to issue debt, where only primary dealers, a group of financial intermediaries, participate in the auctions. Dealers are permitted to submit multiple bids as long as the total value does not exceed the upper limit of the overall amount announced for the auction.

[11]For instance, these empirical studies compare the two protocols with different results: Barbosa et al. (2022), Kang and Puller (2008), Armantier and Lafhel (2009), Hattori and Takahashi (2022), Mariño and Marszalec (2023), Castellanos and Oviedo (2008), and Armantier and Sbaï (2009)

[12]A syndicate is a group of banks that is given the mandate to place government bonds. It follows a book building process that allows monitoring and intervention in the allocation of orders by the IGCP.

Table 1 presents summary data for the most common auctions. We observe 400 Treasury bill auctions and 161 Treasury bond auctions. Dealers (mean) refer to the average number of dealers present in the auctions of each security. Steps (mean) refer to the average number of bids submitted by a dealer. Issued (mean, M€) refer to the average amount issued by the IGCP in auctions of each security.

Table 1: Summary Data on Treasury Bond and Bill auctions

Maturity	Auctions	Bids (mean)	Dealers (mean)	Steps (mean)	Issued (mean, M€)
3 Months	101	35.2	14.5	2.4	471.0
6 Months	88	36.4	14.7	2.4	505.6
12 Months	101	44.0	15.4	2.8	1,037.5
All Bills	400	38.7	14.8	2.5	703.1
5 Years	21	55.9	18.9	2.8	732.3
6 Years	14	56.5	18.2	3.0	754.1
10 Years	52	59.1	17.9	3.2	805.8
All Bonds	161	56.4	17.9	3.0	756.0

Other data sets are further detailed in section 5 where we calibrate the model to the Portuguese economy. In the next subsections we provide evidence on key aspects that will be used to discipline the structural model:

1. Debt agencies lack commitment to target amounts announced prior to the auction;
2. Leading up and during the crisis, investors' bids get more disperse;
3. Starting in 2008, public spending was higher than anticipated.

2.1 Lack of Commitment and Uncertainty

Lack of Commitment. The week prior to an auction, the IGCP announces the securities being issued and provides a target for the amount it expects to issue. Importantly, there is no commitment to that target. In the data we observe several instances of ex-post deviation from the target. It follows that, although there is a targeted amount, the amount issued in a given auction is uncertain from the bidder's perspective. Figure 2 highlights this lack of commitment by presenting instances of ex-post deviations from the target in

Portuguese auctions. A value of 1 represents auctions where the target is met, while the filled squares represent deviations above or below target. Even before the debt crisis, the agency would regularly deviate from the ex-ante target.

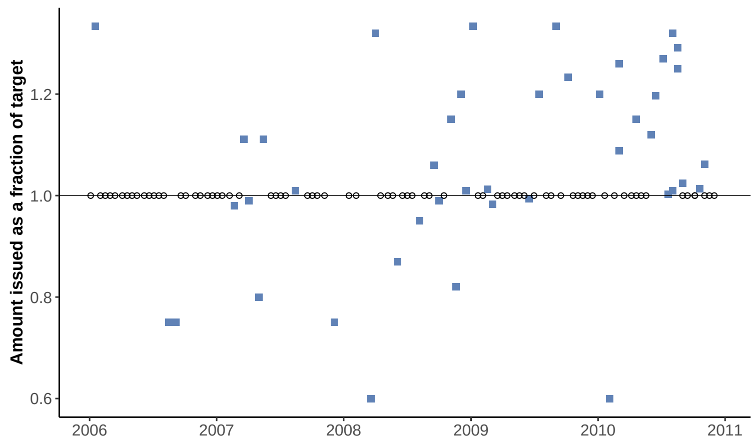


Figure 2: Amount raised as a fraction of the target in Bill Auctions

Brenner et al. (2009) surveyed treasury ministries and central banks around the world and received answers from 48 countries. One of the questions asked was "*Does the treasury (or the central bank) have the right to change the quantity of the debt that is being sold after viewing the demand?*". More than half of the countries that answered (30 out of 48) have some discretion on how much to issue, regardless of a target being announced.

Debt crisis and uncertainty. During the European debt crisis, debt management offices purposely increased the flexibility of the mechanisms used to issue debt. In April 2010, the IGCP increased this flexibility by: 1) running multiple auctions simultaneously for treasury bonds; 2) providing an interval, instead of an amount, as a target; 3) setting the target range for the sum across the auctions being ran simultaneously. April 2010 coincided with the intensifying of the crisis in Greece, with multiple downgrades of Greek debt and, ultimately, a bailout in May. In February 2011, the same type of changes, just described for treasury bonds, were also introduced in auctions of treasury bills.

The 2011 Survey of the OECD Working Party on Public Debt Management, clarifies that this was not an isolated case: "*In response to uncertainty and volatility, auction calendars have become more flexible in most jurisdictions, auctions were held more frequently and multiple series* *per auction were introduced.*" The fact that the debt issuance mechanism is more flexible implies a clear use of discretion. By providing a range as target, the government does not commit to a particular amount. This flexibility can be thought of as a way to ensure that the target (range) is met, i.e. there would be no failed auctions.

2.2 Changes in Demand

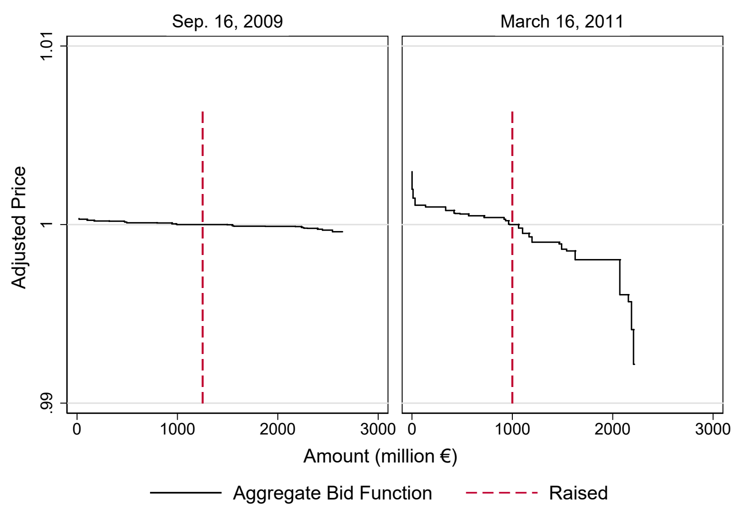


Figure 3: Demand before and during the crisis

To understand uncertainty, we look at bid level data of Portuguese debt auctions. Figure 3, highlights changes in demand during the crisis, as first documented in Alves Monteiro (2022). The figure presents the aggregate bid function for two auctions of one year treasury bills, one in each panel, together with the amount issued. Prices are normalized such that the marginal price equals 1. The left and right panels are representative of demand schedules in normal times and the crisis period. During the crisis, the demand schedule is much steeper and inelastic, with more dispersion of bids. This figure helps understand what may separate the outcomes under the two protocols. Government discretion on the amount borrowed only matters when different borrowing decisions impact the value of debt, as in the right panel. That is, discretion on the quantity sold together with default risk are the key characteristics that separate the outcomes under the two protocols.

The changes depicted in Figure 3 could be driven by differences across dealers or by dispersion within bid functions – all dealers bid a wider range of prices. In the data there is no evidence of persistent investor heterogeneity[13]. As investors do not present persistent differences, nor are there consistent differences in the levels of individual bid functions, as a simplifying assumption, we will assume that investors are symmetric.

2.3 Government Spending Uncertainty

Unanticipated large government deficits were an important driver of the sovereign debt crisis in Europe. In late 2009, the newly elected Greek government disclosed that its budget deficits were far higher than previously though, this led to the downgrade of Greek debt and to a sharp increase in spreads. Copelovitch et al. (2016) consider this to be the event that triggered the Eurozone debt crisis. For Portugal in particular, the period leading to the bailout was marked by higher public spending and lower resources to finance that spending. These led to elevated borrowing that, together with a prolonged recession, played a role in explaining the sovereign debt crisis that followed.

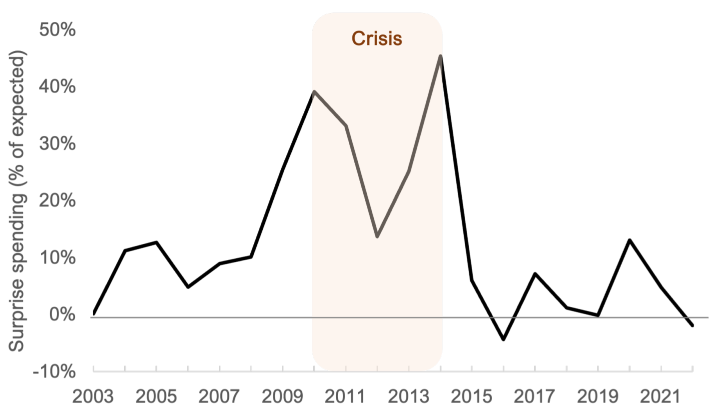


Figure 4: Deviation from expected public spending

Figure 4 highlights the increase in unanticipated public spending that became more evident after 2008. The figure plots the difference between expected and actual spending as a percentage of expected spending. In particular, expected spending is taken from the government's spending proposal submitted for the year ahead. The figure shows that the deviations between expected and actual spending are mostly positive and go up to 45%

[13]Refer to the Appendix for a detailed analysis of investors' bidding patterns.

above the 1 year ahead expectation. From 2010 and through 2014, the shaded area, these deviations were not only positive, but also higher than before and after the crisis.

The evidence provided will discipline the model introduced in the next sections. In particular, we model uncertainty regarding financing needs, through public spending surprises, as these played an important role in Portugal and Southern Europe leading up to the crisis. The fact that information regarding these financing needs is asymmetric – it is privately observed by the government – will be the driving force leading to differences in bidding across protocols. This is consistent with the fact that supply of debt is random ex ante – given the realization of the financing needs and after observing the prices demanded by investors, the government chooses how much to borrow optimally – the government reserves discretion on the quantities sold.

3 Auctions with Endogenous Issuance

We start with a simple environment to illustrate the mechanisms of each auction protocol. There are two periods, $t = \{0, 1\}$. There is a small open economy whose government borrows from a unit continuum of identical, competitive, risk neutral and deep pocketed foreign investors with discount factor R^{-1}.

The government maximizes the welfare of the small open economy, whose endowment is y in each period. Preferences over streams of consumption are $\mathbb{E}\left[u(c_0) + \beta u(c_1)\right]$, where u is strictly increasing and concave and $\beta \in (0, 1)$ denotes the discount factor.

In the first period, the economy faces a spending shock, θ, privately observed by the government. This shock takes finitely many values in $[\theta_L, \theta_H]$ and has CDF G.

The government lacks commitment. In the second period, the value of default, v^d, is drawn from a continuous distribution and CDF F (support $[\underline{v}, \overline{v}]$). We assume that $f(v^d) = F'(v^d) > 0$ on $[\underline{v}, \overline{v}]$ and $u(y) \geq \overline{v}$, so the government never defaults if it has no debt.

Timing. The government starts the first period with endowment y and debt B. Then the spending shock θ is realized. Then investors submit bid schedules. Then the government observes the bid schedules and chooses how much to borrow. In the second period, the outside option, v^d, is realized, and the government decides whether to default.

Auction Protocols. We consider the protocols most used to auction sovereign debt: the uniform price protocol (UP) and the discriminatory price protocol (DP). These determine which bids are accepted and at which prices they are executed. Investors submit bid schedules, a tuple $(p, b, K) = \left(\{p_k, b_k\}_{k\in\{1,\dots,K\}}\right)$ with $K < \infty$. A bid is a pair (p_k, b_k) representing the price p_k an investor is willing to pay for b_k units of debt. The government sorts bids in descending order of price and accepts bids until it is able to issue a chosen number of units ℓ. The lowest accepted price is the marginal price of the auction, P_c.

Under UP, all accepted bids are executed at the same price, the marginal price of the auction, P_c. We analyze the most common DP, the "*pay-as-bid*", under which accepted bids are executed at the respective bid price. The auction protocol is known by all agents before the auction. Denote the marginal price for a given ℓ as $P_c(\ell)$, where ℓ is the quantity issued. Then, the revenue in a UP auction is $\Delta(\ell)^U = P_c(\ell) \times \ell$, whereas, given aggregate bid schedule $p(b)$, revenue in a DP auction is $\Delta(\ell)^D = \int_0^\ell p(b)db$.

3.1 Optimal Bidding

In this environment, the privately observed spending shock creates uncertainty about how much the government will borrow. As a result, investors submit multiple bids. Next, we will characterize optimal bidding. First, we note that it is never optimal to bid at a price that is not in the set of marginal prices.

Proposition 1. *Bidding marginal prices is a weakly dominant strategy for investors (strongly dominant under DP).*

A bid is executed if its price is weakly greater than the marginal price $P_c(\theta; p)$. Consider consecutive marginal prices $P_{c,1} > P_{c,2}$, and a bid price p such that $P_{c,1} > p > P_{c,2}$. This bid is accepted with the same probability as one with $p = P_{c,2}$, as there is no marginal price between p and $P_{c,2}$. While the value of any debt purchased does not depend on p, the cost associated with bidding $p > P_{c,2}$ may: under a DP, the cost is $p > P_{c,2}$. Then, bidding marginal prices is a strictly dominant strategy under a DP. Under a UP, bidding marginal prices is a weakly dominant strategy, but there is no equilibrium in which investors bid a non-marginal price. Refer to the appendix for a full proof.

Proposition 1, together with the finite support of θ, implies that any equilibrium can be fully characterized by a finite number of marginal prices and total issuances. It also implies that we may reduce the problem of an individual lender to simply choosing quantities to bid for each price in the set of marginal prices. Since lenders are infinitesimal, each lender cannot influence the aggregate issuance. For each realization of θ, let $\ell(\theta; p)$ denote the quantity issued in an auction and $P_c(\theta; p)$ the associated marginal price. For any issuance ℓ, each unit has value $Q(\ell)$, the expected discounted value of a unit of debt. The payoff of submitting a bid function (p, b, K) is

$$\max_{b \in \mathbb{R}_+^K} \left\{ \sum_{k=1}^{K} \mathbb{E}_\theta \left[\underbrace{\mathbf{1}\{p_k \geq P_c(\theta)\}}_{\text{Prob. of winning bid}} \left(\underbrace{Q(\ell(\theta))}_{\text{Value per unit}} - \underbrace{\phi(p_k, P_c(\theta)|\ell(\theta))}_{\text{Cost per unit}} \right) b_k \right] \right\},$$

where $\phi(p, P_c(\theta)|\ell(\theta))$ is the price paid by a lender to purchase a unit of the bond when they bid p and the marginal price is $P_c(\theta)$. In both protocols, $\phi(\cdot)$ satisfies $\phi(p, P_c(\theta)|\ell(\theta)) \in [P_c(\theta), p]$ and is weakly decreasing in p.[14] As lenders are infinitesimal, their payoffs are separable across bids. Individual bids (p_k, b_k) then solve

$$\max_{b_k \geq 0} \mathbb{E}_\theta \left[\mathbb{1}\{p_k \geq P_c(\theta)\} \Big(Q(\ell(\theta)) - \phi(p_k, P_c(\theta)|\ell(\theta)) \Big) b_k \right].$$

In equilibrium, the expected payoff to a lender of any bids with $b_k^\star > 0$ must be 0. If this payoff were negative, the lender would prefer setting b_k to 0. If it were positive, then the lender would prefer to set b_k arbitrarily large. Therefore, equilibrium bids satisfy

$$\mathbb{E}_\theta \left[\mathbb{1}\{p_k \geq P_c(\theta)\} \Big(Q(\ell(\theta)) - \phi(p_k, P_c(\theta)|\ell(\theta)) \Big) \right] = 0.$$

[14]The functional form of $\phi(\cdot)$ represents the protocol being used in the auction. This notation allows for a more general set of functional forms and nests both the uniform and discriminatory price protocols.

This condition pins down the set of marginal prices. In particular, we must have

$$0 = \int_{P_c^{-1}(p_k)}^{\theta_H} \Big(Q(\ell(\theta)) - \phi(p_k, P_c | \ell(\theta)) \Big) dG(\theta),$$

where $P_c^{-1}(p_k) \equiv \theta(p_k)$ is the minimum θ such that the marginal price is p_k. Under the UP, this becomes

$$0 = \int_{\theta(p_k)}^{\theta_H} \Big(Q(\ell(\theta)) - P_c(\ell(\theta)) \Big) dG(\theta).$$

Therefore, for every θ which is in $supp\{G\}$, we must have $P_c(\ell(\theta)) = Q(\ell(\theta))$, and it follows that investors (in the aggregate) bid all possible realizations of the value of debt $Q(\ell(\theta))$. Prices are pinned down solely by the probability of default, as $Q(\cdot)$ is simply the probability of repayment multiplied by the lender's discount factor.

For a DP auction, $\phi(p_k, P_c | \ell(\theta)) = p_k$ and

$$0 = \int_{\theta(p_k)}^{\theta_H} \Big(Q(\ell(\theta)) - p_k \Big) dG(\theta),$$

which can be rewritten as

$$p_k = \frac{1}{1 - G(\theta(p_k))} \int_{\theta(p_k)}^{\theta_H} Q(\ell(\theta)) dG(\theta) = \mathbb{E}[Q(\ell(\theta)) | \theta \geq \theta(p_k)].$$

Here, prices are not directly pinned down by the probability of default. A bid represents a commitment to pay p_k regardless of total debt issuance. In order to break even ex-ante, each bid price, p_k, must be equal to the expected value of a bond, conditional on that bid being accepted. Investors' bids therefore depend on their beliefs about how much the government will borrow. Under a DP, investors may incur losses or profits ex-post.

Without loss of generality, we restrict consideration to symmetric pure strategy equilibria (i.e. ones where every investor submits identical bids). This pins down individual bid quantities b_k. We thus abstract from any coordination problems between investors so that we can focus on the strategic interaction between investors and the government.

3.2 Government

In the second period, the government chooses whether to default by solving

$$W\left(B', v^d\right) = \max_{d\in\{0,1\}} \quad \left\{(1-d)u(y-B') + dv^d\right\}.$$

The default policy rule is then

$$d = \begin{cases} 1, \text{ if } \; v^d > u(y-B') \\ 0, \text{ if } \; v^d \leq u(y-B') \end{cases} .$$

Define $\underline{v}^d(B')$ by $\underline{v}^d(B') = u(y-B')$, the value of v^d that makes the government indifferent between defaulting and repaying for each B'. The value of a bond at the end of period 0 is $Q(B') = R^{-1}F(\underline{v}^d(B'))$, the discounted probability of repayment. Note that $Q(B')$ is independent of the protocol used. In a single auction environment, differences in bids across protocols do not arise due to different fundamental valuations of debt. In the first period, the government chooses borrowing B' and marginal price P_c to solve

$$U(B;p) = \max_{\{B'\geq 0,\ P_c\geq 0\}} \quad \left\{u(y+\Delta(B,B';p(.)) - B - \theta) + \beta\mathbb{E}\left[W(B',v^d)\right]\right\} \tag{1}$$

$$\text{s.t.} \quad \begin{cases} \Delta(B,B';p(.)) = p(B')B', \text{ under UP} \\ \Delta(B,B';p(.)) = \int_0^{B'} p(B,\ell)d\ell, \text{ under DP} \end{cases} ,$$

where it takes the bid function $p(.)$ as given. Let $B'(\theta;p)$ denote its optimal policy.

3.3 Equilibrium

We are now ready to define an equilibrium in this environment. All the objects, and associated problems are defined above.

Definition 1 (Equilibrium). *Given the auction protocol, an equilibrium consists of values $\{U,W\}$, a price function Q, a bid function p, and policy rules $\{d,\mathcal{B},P_c\}$, such that:*

1. *The price equation equals the discounted probability of repayment, given policy rules;*

2. *The bid function satisfies ex-ante zero profits for investors, given policy rules and prices;*

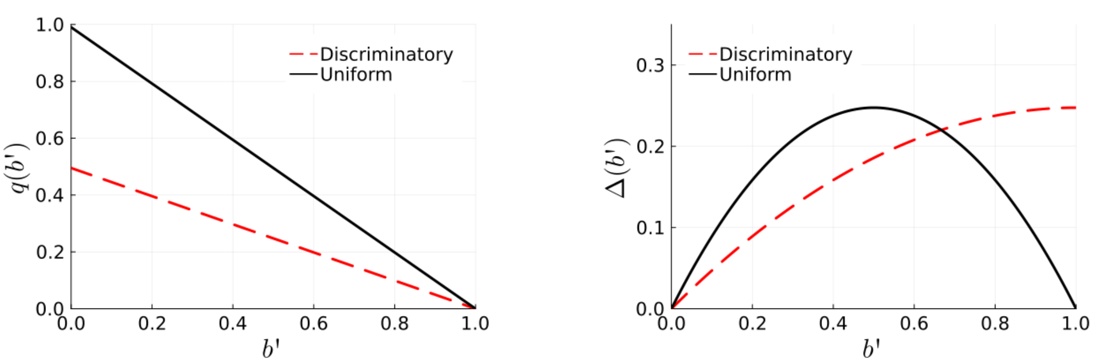
3. *The policy rules solve the government's problems given values and prices;*

4. *The auction clears, given the bid function and policy rules.*

3.4 Exogenous Borrowing

We first consider an example with exogenous borrowing, where B' is random with a distribution known by all agents. Figure 5 illustrates the bid schedules and revenue when both B' and v^d are uniformly distributed on $[0, y]$. First, under a DP, bids at each increment are smaller than under a UP. Secondly, revenue is higher (lower) under a DP for high (low) levels of B'. This occurs because in a UP the marginal price is the average price, while in a DP, the average price exceeds the marginal price.

In a DP, investors commit to paying what they bid, even if the marginal price of the auction is lower than that. This is a classic phenomenon known as the winner's curse, and is responsible for lower bids at each increment being borrowed. Each time an additional bid is accepted by the government, the value of the bonds already sold at higher bids is diluted[15]. Since this dilution takes place across states of the world (realizations of B') and not across time per se, we term it "static dilution." Because all accepted bids are executed at the marginal price under a UP, this phenomenon is completely absent in that case.



(a) Bid Schedules

(b) Revenue

Figure 5: Comparing outcomes under UP and DP

[15]The sequential execution of bids is reminiscent of a game of sequential banking as in Bizer and DeMarzo (1992). Note, however, that here the government interacts with all investors simultaneously.

Under a UP, revenue is the product of the marginal price and the quantity borrowed. Under a DP, revenue is instead the integral of the bid function (from 0 to B'). Panel (b) shows that revenue under a UP yields the familiar single-peaked Laffer curve where, for high levels of B', the decrease in price more than offsets the increase in quantity and, ultimately, revenue falls to zero. In a DP, although the marginal price is decreasing in B', all inframarginal increments are executed at their bid prices, which yields a consistently increasing Laffer Curve. In this setting, the DP provides insurance by allowing the government to transfer resources from good states (low B') to bad states (high B'). While it has lower marginal prices, it provides insurance through lower variance of average prices. This is a similar point to Cole et al. (2018), where the choice of protocol depends on how the government values different states of the world.

As B' is exogenous here, utility flows in the second period are independent of the protocol, so differences in welfare depend only on flows (and therefore auction revenue) in the first period. Let $\hat{\Delta}(B')$ and $\Delta(B')$ denote revenue under a UP and a DP, respectively.

Theorem 1 (Revenue Equivalence)**.** *If B' is a random variable independent of the auction protocol, then ex-ante expected revenue in the auction is the same under both protocols.*

This familiar result follows from the investors break-even condition: i) as B' is independent of the protocol, so will be the investor's expected payoff; ii) as investors break even in expectation and i) holds, the expected cost of winning bids must also be independent of the protocol; iii) because investors are competitive, their expected cost is the government's expected revenue. Refer to the appendix for a formal proof.

Revenue equivalence implies a risk neutral government is indifferent between protocols. However, a risk averse government prefers the DP, since it yields less volatile consumption (with the same mean). Auction revenue and therefore consumption are less volatile both because of the lower variance in prices and because revenue itself is increasing.

With endogenous borrowing, the distribution of B' depends on the bid schedule and protocol chosen, so revenue equivalence need not hold. Since $B'(\theta; p)$ may differ across protocols, $\mathbb{E}[B'Q(B')]$ may also do so. Since in any equilibrium investors break even in expectation, the cost of winning bids (and revenue for the government) may no longer be equal across protocols. In other words, introducing a government that chooses debt optimally after observing aggregate demand is enough to break revenue equivalence.

3.5 Endogenous Borrowing

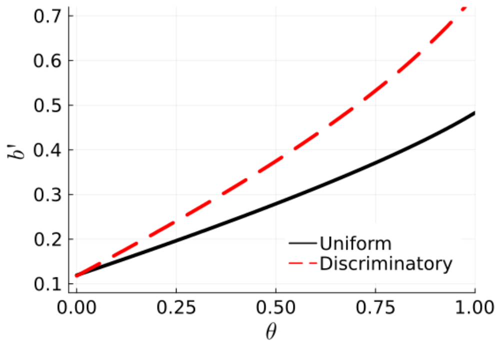
We now let the government choose B' optimally and highlight in a numerical example how each protocol affects revenue and welfare. θ is exponentially distributed with CDF $G(\theta) = 1 - \exp(-\lambda\theta)$ with $\lambda = 4$. v^d is uniformly distributed on $[0, y]$ with CDF $F(v^d) = v^d/y$. We set $y = 1$, $\beta = 0.9$, $R = 1.01$ and $B = 0$ and assume utility to be constant relative risk aversion (CRRA). We consider the cases of $\gamma = 0.5$ and *log* utility ($\gamma \to 1$).

Figure 6 depicts both cases. For the case of $\gamma = 0.5$, in panel (a) we see that under the DP the government borrows more than under the UP, and that this difference is increasing in θ. This follows from the bid schedules and revenue curves depicted in panels (c) and (e). At the first bids, DP prices are much lower due to static dilution at the margin. This static dilution at the margin then decreases along the bid schedule. The fact that all debt is issued at the marginal price under the UP substantially decreases incentives to borrow at the margin (compared to the DP). Ex-ante welfare is higher under the UP, with $\mathbb{E}[V(\theta)_{UP}] = 3.329 > \mathbb{E}[V(\theta)_{DP}] = 3.327$.

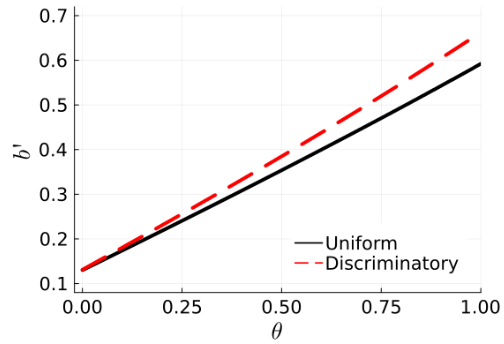
For *log* utility, in panel (b) we see that the added concavity causes the government to borrows relatively less by reinforcing the importance of consumption smoothing across both states and time. This implicit disciplining effect on borrowing limits static dilution. Panel (d) depicts the result of this discipline on the price schedule. Less dilution implies higher prices under the DP than those observed with $\gamma = 0.5$. Finally, ex-ante welfare is now higher under the DP,[16] $\mathbb{E}[V(\theta)_{UP}] = -0.500 < \mathbb{E}[V(\theta)_{DP}] = -0.498$.

This section used a simple environment to understand how different protocols affect borrowing and the cost of debt. In general, we showed that bid prices are lower under a DP than under a UP due to static dilution. With exogenous debt issuance, we recovered a

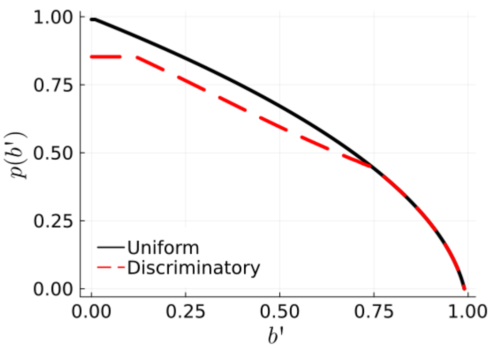
[16] To assess whether this pattern is not dependent on a specific set of functional forms and parameterizations, we perform a robustness check. For a list refer to the appendix.



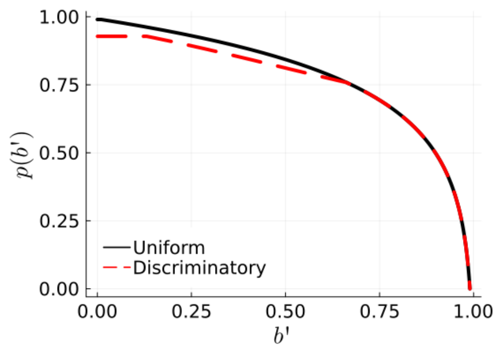
(a) Borrowing Decisions ($\gamma = 0.5$)



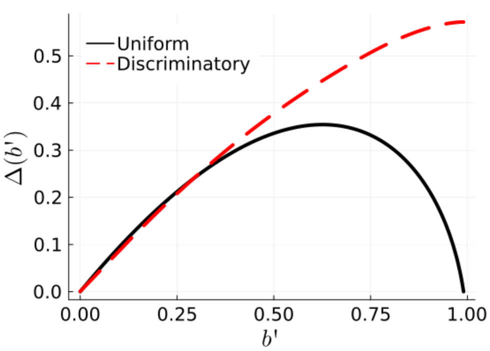
(b) Borrowing Decisions ($\gamma = 1$)



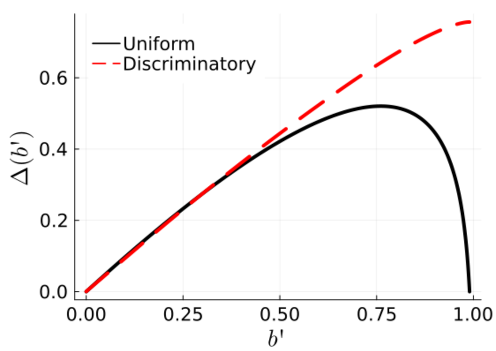
(c) Bid Schedules($\gamma = 0.5$)



(d) Bid Schedules ($\gamma = 1$)



(e) Revenue ($\gamma = 0.5$)



(f) Revenue ($\gamma = 1$)

Figure 6: Comparing optimal outcomes under UP and DP

standard revenue equivalence result and showed that a risk averse government prefers the DP due to its lower revenue variance. Endogenous debt issuance breaks revenue equivalence because it yields different protocol-specific borrowing distributions. For any θ, governments issue more debt under the DP. For low levels of risk aversion, the effects of static dilution under the DP are extreme, and the government prefers the UP instead. However, for sufficiently concave utility, consumption smoothing motives discipline borrowing, which limits static dilution under the DP, which fares better than the UP.

4 Quantitative Model

We now introduce dynamics using an infinite horizon model with long term debt. As before, the DP provides insurance across states. However, since revenue from each accepted bid is independent of how many bids are accepted, the DP supercharges the (dynamic) dilution effects inherent in models of long term defaultable debt. The government overborrows relative to the UP, and the expectation of this future dilution lowers prices significantly even when it is not close to default. The UP effectively commits the government to a framework where future dilution will be less tempting, which raises prices today. We calibrate our model to the Portuguese economy under a DP, which was used prior to the crisis. We then perform a counterfactual, solving the model under a UP and assess how the protocols interact with default risk and how they affect auction outcomes.

4.1 Environment

Time is discrete and infinite, $t = \{0, 1, 2, \ldots\}$. There is a small open economy whose government borrows from a continuum of competitive, risk neutral, deep-pocketed foreign investors with discount factor R^{-1}. The government maximizes the welfare of the small open economy. Its preferences over streams of consumption are given by

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right],$$

where u is a nice function and $\beta \in (0, 1)$ is the discount factor. There is a public exogenous state of the world $s \in \mathcal{S}$, which is a Markov process and governs the endowment $y(s)$ and expected public spending $g(s)$. The private exogenous state of the world includes $T \in \mathcal{T}$, which determines a budget surprise θ_T and is i.i.d. over time.

The government borrows using a defaultable long term bond. We follow Chatterjee and Eyigungor (2012) and Hatchondo and Martinez (2009), and model debt as a contract promising a stream of exponentially declining coupon payments. At time t, a unit of the bond promises to pay $(1-\lambda)^{t+k-1}(\lambda+\kappa)$ of the consumption good in period $t+k$. As in the two period model, the government issues debt via an auction where investors provide bid schedules. To align our notation here with that of the literature, we use B' for the government's choice variable, so the gross issuance implied is $\ell = B' - (1-\lambda)B$. At these auctions, it incurs an issuance cost $i(s, B, B') \geq 0$.[17]

If the government defaults, it transitions to bad credit standing, where it is excluded from financial markets and suffers a flow utility cost of $h(s)$. It regains good credit standing with probability η. Reentry occurs through restructuring: the face value of the pre-default debt (B) is reduced by a fraction τ. The timing of events within a period is as follows.

1. The exogenous state variables are realized at the beginning of the period.
2. If in good standing, the government chooses whether to default.

3.1. If the government entered the period in good standing and chose to repay $(d = 0)$:
   (a) The government runs an auction;
   (b) Investors submit bid functions after observing the public state s;
   (c) The government chooses B' and P_c, given the aggregate bid function.

3.2. If the government chose to default $(d = 1)$ or entered the period in bad standing, it is excluded from financial markets and cannot borrow.
   (a) Next period, with probability η the government regains access to financial markets, and with probability $(1-\eta)$ remains excluded.

4.2 Optimal Bidding

The solution to the lenders' problem is similar to what we described in the two period environment. Each lender takes as given the strategies of all other lenders, as well as the government's strategy. Lenders' actions are aggregated into a market demand curve.

[17] This is included for technical reasons. See the appendix for details.

After observing that market demand curve and the private state of the world T, the government chooses B'. For the rest of this section, we suppress dependence on the public state for ease of notation. As before, lenders bid only marginal prices $P_c(.)$ in equilibrium. In general, bids $p(n)$ for incremental debt issuance at $n \geq (1-\lambda)B$ must satisfy[18]

$$0 = \mathbb{E}\left[\mathbb{1}\{p(n) \geq P_c(T)\}\Big(Q(\mathcal{B}(T)) - \phi(p(n), P_c(T)|\mathcal{B}(T))\Big)\right]. \tag{2}$$

As in the two period model, using a UP then implies

$$p(n) = Q(\mathcal{B}(n)). \tag{3}$$

Similarly, as before, the optimal strategy in the DP is to bid the expected value of the bond, conditional on the bid being accepted,

$$p(n) = \mathbb{E}[Q(\mathcal{B}(T))|p(n) \geq P_c(T)]. \tag{4}$$

As in the two period model, UP prices are pinned down by the value of debt at each state, while DP prices depend on investors' beliefs about the government's borrowing distribution. Again, without loss of generality, we restrict consideration to symmetric pure strategy equilibria. As before, this restriction pins down individual quantities, and we abstract from any coordination problem between investors.

4.3 Government's Problem

At the beginning of the period, if the country is in good standing, the government's problem is to choose whether or not to default as follows:

$$V(s,T,B) = \max_{d\in\{0,1\}} \left\{(1-d)V^R(s,T,B) + d\left(V^D(s,T,B)\right)\right\}$$

[18] In this environment, we allow the government to perform debt buybacks. This is done through reverse auctions that are conducted under the same protocol as the auctions. A set of conditions analogous to the equations below govern optimal bidding. In those conditions, certain inequalities are flipped (e.g. $\mathbb{1}\{p(n) \leq P_c(T)\}$ in the analogue of 2 and $p(n) \leq P_c(T)$ in the analogue of 4). We assume the government either borrows or buys back debt in a given auction (i.e. does not do both). This is consistent with the data.

where V^R is the repayment value function and V^D is the value under default.

The value under default is given by:

$$V^D(s,T) = (1-\beta)\ \left[u\big(y(s) - \underbrace{g(s)\times\theta_T}_{\substack{\text{Realized}\\ \text{Spending}}}\big) - h(s)\right] +$$
$$+\beta\left(\mathbb{E}\Big[\eta V\left(s', T', (1-\tau)B)\right) + (1-\eta)V^D(s',T',B)\Big| s\Big]\right)$$

where the government does not have access to financial markets and just consumes the endowment net of realized public spending. The default cost, $h(s)$, is measured in utils. The continuation value depends on whether the government regains access to financial markets, which occurs with probability η. In such cases, the government is liable for a fraction $(1-\tau)$ of the debt it had due prior to the default event. Otherwise, with probability $(1-\eta)$ the government remains in default.

Conditional on choosing to repay its debt, the government's problem is:

$$V^R\left(s, T, B\right) = \max_{\{c\geq 0,\ P_c>0,\ B'\}} \left\{(1-\beta)\ u\left(c\right) + \beta\mathbb{E}\Big[V\left(s', T', B'\right)\Big| s\Big]\right\}$$

$$\text{s.t.}\quad c + (\lambda+\kappa)B + g(s)\times\theta_T = y(s) + \Delta\Big(s, B, B'\Big)\left(1 - i(s,B,B')\right),$$

where $\Delta(\cdot)$ denotes the auction revenue (or reverse auction cost) given by

$$\Delta(s,B,B') = \int_{(1-\lambda)B}^{B'} \phi(p(s,B,n), P_c(s,B,B'))dn,$$

where $P_c(s,B,B') = p(s,B,B')$ in equilibrium since we assume bidders play symmetric pure strategies. Therefore, choosing B' implicitly chooses P_c. Auction revenue and the country's endowment are used to finance pubic spending, debt service, and consumption. When choosing borrowing and consumption, the government takes into account how its choices affect both the revenue raised today and its continuation value $\mathbb{E}[V(\cdot)|s]$.

Before we proceed, it is worth discussing how the value of a bond $Q(.)$ varies across protocols. In the two period environment, we had $Q(b) = R^{-1}F(u(y-b))$, the discounted probability of repayment, for all protocols. Here however, conditional on repayment, the government chooses a new B' every period and so the value of debt is not necessarily the same across protocols. In fact, given a protocol j, the value of debt is given by its discounted expected payments,

$$Q_j(s, B') = R^{-1}\,\mathbb{E}\left[(1-d_j')\Big((\kappa+\lambda)+(1-\lambda)Q_j(s', \mathcal{B}_j(s', B', T'))\Big) + d_j' Q_j^D(s', B')\Big| s\right].$$

Here, $d_j' = d_j(s', T', B')$ is the government's default policy, $(\kappa+\lambda)$ is debt service, $(1-\lambda)Q(s', \mathcal{B}(s', T', B'))$ is the residual value of the fraction of debt that does not mature, and $Q^D(s', B')$ is the value of a bond should the government default, which is

$$Q_j^D(s, B) = R^{-1}\left(\eta(1-\tau)Q_j\big(s, (1-\tau)B\big) + (1-\eta)\mathbb{E}\Big[Q_j^D(s', B)|s\Big]\right).$$

With probability η, the government restructures its pre-default debt stock with haircut τ and reaccesses financial markets. With complementary probability $(1-\eta)$, it remains excluded from financial markets.

These functional equations make clear that $Q(s, B')$ may differ across protocols for two reasons. First, default decisions may differ due to the differences in budget sets implied by different protocols. Second, even if default decisions are the same, the continuation value of investors' debt claims $(1-\lambda)Q_j(s', \mathcal{B}_j(s', B', T'))$ will typically differ because different protocols induce different distributions of borrowing, $\mathcal{B}(\cdot)$. Since debt is long term, the entire future path of fiscal policy affects the current value of the claim $Q_j(s, B')$, introducing a dynamic channel by which the auction protocol used in the future determines the value of a bond today (even if future default decisions are held constant).

This environment also introduces a dynamic inefficiency, dynamic dilution, through the use of long-term debt. When issuing additional debt in the future, original investors see the value of their claims fall as the probability of default increases. The dynamic nature of the environment gives us this new channel. The crucial aspect is how the protocol interacts with dilution. In particular, what are the incentives on borrowing that each protocol provides and what are the corresponding effects on prices, default and welfare.

4.4 Equilibrium

We are now ready to define an equilibrium in this environment. All the objects, and associated problems and functional equations, are defined above.

Definition 2 (Equilibrium)**.** *Given the auction protocol, a recursive equilibrium consists of value functions, $\{V,\ V^R,\ V^D\}$, price equations, $\{Q, Q^D\}$, bid function p, and policy rules, $\{d, \mathcal{B}, P_c\}$, that satisfy the following sets of conditions (for the full, detailed list, see the appendix):*

1. *Given policy rules, price equations satisfy their functional equations;*
2. *Given policy rules and prices, bid function satisfies zero profit conditions for investors;*
3. *Given values and bids, policy rules solve the government's problem;*
4. *Given bids, value functions satisfy their functional equations;*
5. *Bids and policy rules are consistent with auction clearing.*

5 Calibration

In this section, we describe the data used to discipline the model, the functional forms used in the quantitative implementation of the model, how certain parameters were estimated, and how the remaining parameters were calibrated. After that, we assess the model's fit using both targeted and untargeted moments.

5.1 Data

We use data from the Portuguese economy to perform a case study for the theory developed in this paper. As described in section 2, we have detailed data on Portuguese sovereign debt auctions. Furthermore, by using Portugal as an example we can evaluate the switch in protocol that occurred in the midst of the sovereign debt crisis. Finally, Portugal fits the core assumptions of this type of sovereign default model as it is a small open economy with a vast majority of its debt securities held by foreigners.

We use Eurostat Annual National Accounts for real and nominal GDP over the period 1995-2022. Monthly data on long-term government bond yields for Portugal and Germany are from the European Central Bank (ECB) Interest Rate Statistics, covering 1999-2022. Data on government debt securities are from BPStat general government statistics. Finally, we use annual data for realized government expenditures and revenues as well as the government's one year ahead expectation for those same measures for the period 2003-2022. Realized expenditures and revenues are obtained from the *Portuguese Public Finance Council* (CFP, Portuguese acronym). The year ahead estimates are obtained from the government's budget proposal reports, which is submitted every year in October.

Next, we describe how we selected the structural parameters of the model. Some are set based on estimates in the existing literature. Others are directly estimated based on the data. The remaining parameters are calibrated using simulated method of moments (SMM) to match key characteristics of the Portuguese economy.

5.2 Functional Forms and Parameters

In the model, a period is a year. The model is calibrated to match the experience of Portugal since joining the Euro. The annual risk-free real interest rate, r is set to 0.02, a standard value in the literature. The maturity rate λ of the bond and its coupon value κ are set to those in Paluszynski (2023) (who also studies Portugal during the same period). We set $\tau = 0.535$ to match the face value haircut applied during the 2012 Greek restructuring (as documented in Zettelmeyer et al. (2013)). The functional form of utility is constant relative risk aversion, $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$. We set $\gamma = 2$ a standard value in macroeconomics. The utility cost of default is parametrized following Bianchi and Mondragon (2022) as $h(y_t) = \max\{0, (1 - h_0) + h_1 \log y_t\}$.

Using OLS, we estimate an AR(1) process for detrended log real per capita GDP for years 1995-2019 and an AR(1) process for the detrended year ahead expectation of log real per capita public spending for years 2003-2019,

$$y_t = \mu_y + \rho_y y_{t-1} + \epsilon_t \quad \text{and} \quad g_t = \mu_g + \rho_g g_{t-1} + \nu_t.$$

We assume that the innovations of these processes are correlated and estimate their correlation using the OLS residuals, $\rho_{\epsilon,\nu} = corr(\hat{\epsilon}_t, \hat{\nu}_t)$. The spending shocks are assumed lognormal, $\theta_t \sim$ log-normal $(0, \sigma_\theta)$, and their standard deviation, σ_θ, is estimated using the log differences between the real public spending and its year ahead expectation.

Additionally, for technical reasons, we include preference shocks in the government's decision problem and an issuance cost function. The preference shocks (assumed to be Generalized Type One Extreme Value distribution with scale parameter σ_m and correlation parameter ρ_m) ensure that the model has a pure strategy equilibrium that can be robustly computed, and the issuance costs rule out a counterfactual behavior Chatterjee and Eyigungor (2015) termed "maximum dilution."[19] For more details see the appendix.

The final three parameters, β, h_0 and h_1, govern the impatience of the government and the penalty for defaulting. They are calibrated by SMM, using a standard set of moments known to be informative about them in this class of model (see e.g. Chatterjee and Eyigungor (2012)), the mean of the debt to GDP ratio and the mean and volatility of the interest rate spreads. In particular, h_0 and h_1 largely determine how much debt the government can sustain without defaulting and how those implied limits vary across exogenous states. The impatience β governs how quickly the government approaches these limits (and how quickly it is willing to back away from them during recessions).

Since h_0 controls the average penalty, it is closely tied to average indebtedness. The joint effects of h_1 and β are harder to cleanly separate. Since h_1 controls how quickly penalties change across exogenous states, it is a key determinant of the elasticity of the bond price near average debt levels (and therefore near the average implied debt limit). This makes it a key determinant of average spreads.[20] The government's impatience, β, controls the

[19]When default is imminent, the maturity structure of the debt, together with the opportunity for restructuring, gives the government an incentive to issue as much debt as possible, extracting the value of existing bondholders'. Issuance cost functions counteract these incentives.

[20]For relatively small h_1, default thresholds across nearby exogenous states are similar, so the bond price

Table 2: Parameters set independently, (a) and (b), and calibrated, (c)

(a) Literature

Parameters	Value
R	1.02
γ	2
λ	0.212
κ	0.050
η	0.154
τ	0.535

(b) Estimated

Parameters	Value
μ_y	0.005
ρ_y	0.802
σ_ϵ	0.019
μ_g	-0.388
ρ_g	0.773
σ_ν	0.054
$\rho_{\epsilon,\nu}$	0.397
σ_θ	0.115

(c) Calibrated

Parameters	Value
β	0.932
h_0	0.912
h_1	0.333

rate at which the government approaches its implied debt limits and therefore both the mean of, and variation in, the distance between realized debt levels and implied debt limits. It therefore plays a key role in determining the relative volatility of spreads as well as their average level. Table 2 summarizes the parameters set independently (panels (a) and (b)) and those that were calibrated (panel (c)).

5.3 Targeted Moments

The calibrated model closely matches the moments in the data. Table 3 compares the targeted moments for calibration in the data, for 1999Q1 – 2011Q1, and in the model, where r is the internal rate of return that makes the present discounted value of the promised future payments on a unit bond equal to the unit price:

$$r(s, B') = \frac{(\lambda + \kappa)}{Q(s, B')} - \lambda.$$

It is worth stressing that, as documented in Aguiar et al. (2016), standard sovereign debt models typically fail at matching the volatility of the spread (see e.g. Bocola and Dovis

becomes highly elastic near those thresholds, disincentivizing debt accumulation past levels that are close to risk-free. On the other hand, higher h_1 leads to thresholds that are much more spread out, yielding a less elastic bond price (all else equal), which incentivizes debt accumulation beyond risk-free levels.

Table 3: Targeted moments

Moments	Data	Model
$\mathbb{E}[b'/y]$	48.91%	49.49%
$\mathbb{E}[r - r_f]$	0.61%	0.63%
$\sigma(r - r_f)$	1.02 p.p.	1.02 p.p.

(2019) for another example of this in a model calibrated to Italy). In particular, volatility in the model tends to be lower than in the data. Even though Portugal is, in relative terms, an extreme case of this phenomenon (with spreads actually much more volatile than their mean), the model is able to match all three moments jointly. Our results suggest that the use of a DP in Portuguese sovereign debt auctions may have played a role in generating this pattern. The DP generally incentivizes higher marginal spreads, because decreases in the marginal bid accepted affects only revenue for the marginal unit. This makes the government more willing to borrow further into higher spreads. In contrast, for a UP auction (which is assumed by almost all of the quantitative sovereign default literature), decreases in the marginal price apply to revenue collected from all units sold. Simply by accounting for the actual protocol used at the time, however, we can easily match the relative volatility of spreads that those models fail to attain.[21]

5.4 Validation

Table 4 presents simulated business cycle moments under both protocols, along with their empirical counterparts.[22] In addition to the standard secondary market spread widely

[21]To our knowledge, the only alternatives for matching this high relative volatility involve relatively exotic assumptions about the nature of fundamental shocks. For example, Paluszynski (2023) calibrates a standard sovereign borrowing and default model and also finds that such a model cannot generate the observed volatility of the spreads. The author then introduces imperfectly observed rare disasters (a switching process for the long run mean of the AR(1) for GDP) and learning. This yields spread volatility that surpasses levels observed in the data while matching average levels of spreads and debt. This is the case for conditional simulations (i.e. along the series of observed shocks from 1998 to 2019). For long run simulations, however, high volatility of spreads comes at the expense of counterfactual debt and spreads.

[22]Model moments are generated from simulations that extend to 10,000 years and are repeated 1,000 times. Empirical moments involving spreads are computed using annual data and average spreads from 1999 to 2010, the year before Portugal's bailout. Empirical moments using average bid spreads were computed for treasury bond auctions. Other empirical moments were computed using annual data starting from 1995 and up to 2019.

used in the literature, we calculate two new primary market spreads, the average bid spread, r_{bid}, and the average spread on the last bid accepted, r_{marg}, computed as

$$r_{bid}(s,B,B') = \frac{(\lambda+\kappa)}{\overline{p}(s,B,B')} - \lambda \qquad \text{and} \qquad r_{marg}(s,B,B') = \frac{(\lambda+\kappa)}{p(s,B,B')} - \lambda,$$

where $\overline{p}(s,B,B')$ is the average price of bids executed in an auction and $p(s,B,B')$ is the price of the last bid accepted in an auction.

Table 4: Moments of the Ergodic Distribution

	Data	Discriminatory	Uniform
$\mathbb{E}[r-r^\star]$	0.61%	0.63%	0.26%
$\mathbb{E}[r_{bid}-r^\star]$	0.79%	0.66%	0.26%
$\mathbb{E}[r_{marg}-r^\star]$	0.82%	1.01%	0.26%
$\sigma(r-r^\star)$	1.02 p.p.	1.02 p.p.	0.14 p.p.
Default Rate	-	0.99%	0.43%
$\mathbb{E}\left[b'/y\right]$	48.91%	49.49%	53.98%
$\sigma(tb/y)$	4.35 p.p.	2.40 p.p.	2.01 p.p.
$\sigma(c)/\sigma(y)$	1.49	1.52	1.53
$corr(tb/y,y)$	-0.48	-0.12	-0.16
$corr(tb/y,r-r^\star)$	0.18	-0.14	-0.11
$corr(y,r-r^\star)$	-0.54	-0.23	-0.35
$corr(y,r_{marg}-r^\star)$	-0.76	-0.34	-0.35
$corr(y,r_{bid}-r^\star)$	-0.76	-0.58	-0.35

In this table, the simulated moments under the UP are counterfactual. They illustrate the model's prediction if Portugal switched to a UP (rather than moments from an alternative calibration where parameters were chosen to match the data in a model where the country used a UP). Average spreads under the UP are much lower and less volatile than under the DP. The average spread on the last bid accepted, $\mathbb{E}[r_{marg}-r^\star]$, highlights the willingness to borrow more on the margin under this protocol. The difference between the average spread in the secondary market, $\mathbb{E}[r-r^\star]$, and the average bid spread, $\mathbb{E}[r_{bid}-r^\star]$, highlights the extent of static dilution in the DP auction.

Under the DP, investors require spreads higher than those measured in the secondary market, a phenomenon called underpricing. At the same time, investors often overpay for their first (less often marginal) bids. Because of this, the average spread for accepted bids is lower than the spread on the marginal bid, but still exceeds the secondary market spread. The ordering of these spreads matches the data, albeit more pronounced in the model. By incorporating the observed auction protocol and data on measured forecast errors, the model generates a significant difference between secondary market spreads and marginal spreads that is similar in magnitude to what we see in the data.

High dilution incentives under the DP prevent the government from sustaining as much debt as it could under a UP, and lead it to default more often. While we opted to not estimate a default rate in the data as we believe there is no consistent way of doing so. After all, the country has not defaulted since 1891 (although it did default more frequently in the mid 1800s, in 1828, 1837, 1841, 1845, and 1852). That said, if we assume Portugal would have defaulted in 2011 had it not received a bailout, we would have 2 defaults in about the last 170 years which yields a 1.2% default rate, which is close to our 0.96%.

In the model, GDP y and government spending $G = g \times \theta$ have clear empirical counterparts, data on which was used to discipline them. The model's national accounts also yield consumption c and the trade balance $tb = y - c - G$. In order to assess the model's fit to the data, we need to account for the absence of investment in the model. Since the conceptual meaning of the trade balance in the model matches that in the data (net exports), we choose to compare the model's trade balance directly to its data counterpart and the model's consumption to the data residual $y - G - tb = c + i$, the sum of consumption and private investment. Data moments involving c in the table above use this implied residual, which is why we calculate a relative volatility of consumption well above 1, which the model matches quite well. That said, the model does not quite reproduce the observed volatility of the trade balance.

Both the data and the model display relatively weak correlations between the trade balance and either output or spreads.[23] Finally, the last three rows present the correlations of output with various spread measures. We find that the strongest association in the data is with the average spread for bids accepted in a given auction (closely followed by

[23]Of these, the strongest in the data is a somewhat moderate -0.48 between the trade balance and output, which the model admittedly does not match.

that with the spread on the marginal bid). While the model underestimates the absolute strength of these correlations, it does reproduce the ordering of their magnitudes.

Aguiar et al. (2016) mention that standard models ability to increase the volatility of the spreads relied on sufficiently high variability of output. The drawback it seemed, was that it "comes at the expense of tying the spread much too closely to output fluctuations." Here, however, that is not the case as the calibrated model generates the spread volatility observed in the data while inducing a correlation between spreads and output that is close, but smaller, than the one observed in the data.

6 Comparing Protocols

In this section we compare some of the key properties of the equilibrium under a DP, the one used in Portugal during the relevant period, and under a (counterfactual) UP. To do so, we perform a decomposition exercise: first, we compare the outcome of using either a UP or a DP auction this period, keeping all future auctions fixed under the DP; then, we compare the outcome of using the DP or the UP for all auctions in every period. This exercise allows us to disentangle the static and dynamic effects of the protocols.

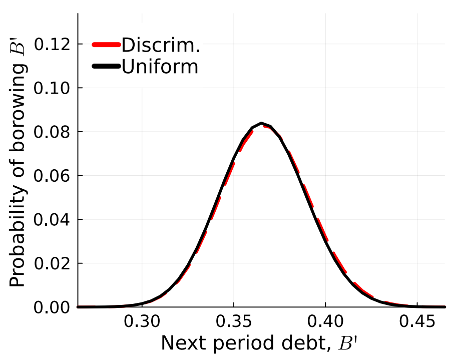
Fixed Protocol for Future Auctions

By keeping all future auctions fixed under the DP, we effectively focus on the static effects of using different protocols in the current period – the static dilution under the DP.

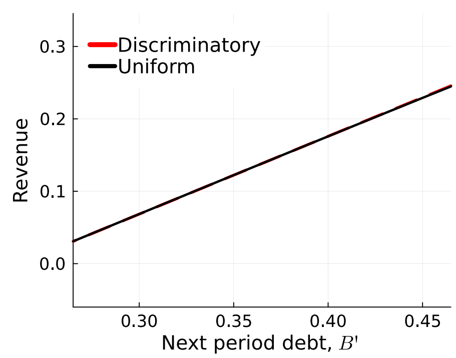
In this environment, it is useful to consider borrowing policies as a probability distribution. Specific realizations are determined by the i.i.d., privately observed spending shocks. Figure 7 compares these distributions and Laffer Curves across protocols. Panels (a) and (c) show the distribution of next period debt, B', for current debt $B \in \{0.3, 0.55\}$ at the means of y and g. Panels (b) and (d) present the corresponding Laffer Curves.

For low levels of debt (panels (a) and (b)), the borrowing distributions are very similar under both protocols. Since there is virtually no default risk for the debt levels chosen with positive probability, the bids submitted under both protocols are close to the risk-free price. In other words, marginal revenue is similar and static dilution is minimal.

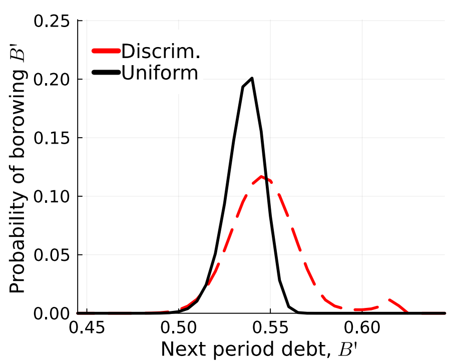
For higher levels of debt (panels (c) and (d)), we see that the government clearly borrows more under the DP. The higher initial debt means that with positive probability it is optimal for the government to borrow up to a region where the likelihood of default increases. Investors submit lower prices under a DP due to static dilution. As revenue is weakly increasing under the DP as opposed to the UP, the government has an incentive to borrow more under a DP. This is particularly significant when the government starts the period highly indebted and is faced with negative surprise spending. In such cases: 1) revenue under the UP is decreasing; and, 2) static dilution under the DP, is decreasing at the margin. These two forces rationalize the difference in borrowing at the right tail.



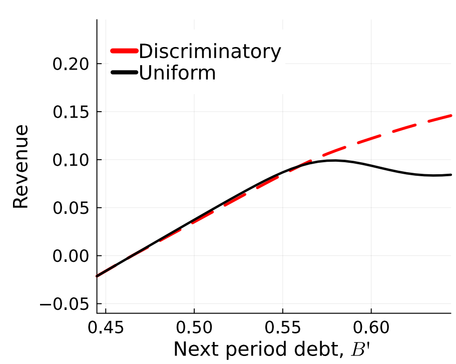
(a) Borrowing Decisions at $B = 0.3$ and $y = \bar{y}$



(b) Revenue at $B = 0.3$ and $y = \bar{y}$



(c) Borrowing Decisions at $B = 0.55$ and $y = \bar{y}$

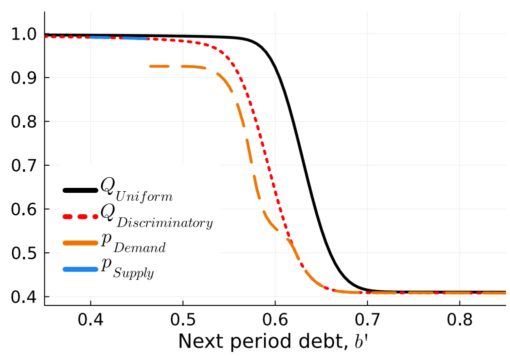


(d) Revenue at $B = 0.55$ and $y = \bar{y}$

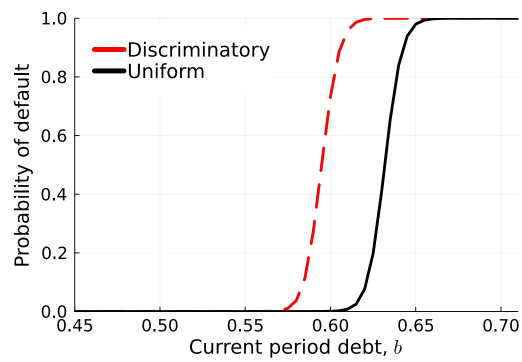
Figure 7: Comparing outcomes under UP and DP today, for fixed future DP

Full comparison of the DP and the UP

We now compare the equilibrium under the DP and the one under the UP, for all auctions, both this period and in the future. As a result, we observe both the static effects discussed above and the dynamic effects introduced by using different protocols in the future. The fact that future auctions are not fixed under one protocol means that borrowing and default decisions will also differ in future auctions. As the value of debt today, $Q(\cdot)$, is pinned by those future decision rules, it follows that it will also be different across protocols. This is in contrast with the exercise above where the function depicting the value of debt was the same and only bids were different due to static dilution under the DP. This difference in $Q(\cdot)$ captures the dynamic difference between the two protocols.



(a) Bid schedule at $y = \bar{y}$



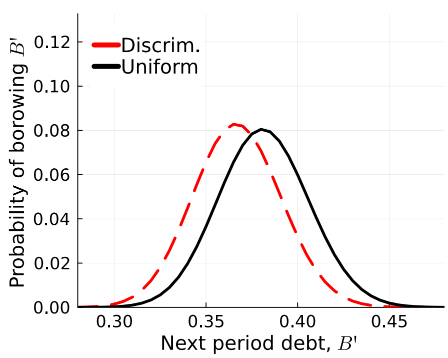
(b) Default decision at $y = \bar{y}$

Figure 8: Comparing prices and default decisions under UP and DP

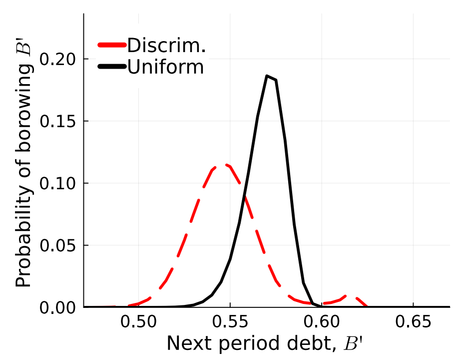
Figure 8 depicts the value of debt and bid schedules and how they relate to the default decisions. Panel (a) depicts the value of debt and bid schedules under a DP, as well as under the UP. As discussed, we see that the value of debt, $Q(\cdot)$, is different across protocols and weakly lower under a DP. This difference highlights the impact of the incentives to borrow over time provided by the different protocols. The government has an incentive to borrow more than under a DP, as revenue is always increasing. Investors internalize the fact that this incentive is present in every future auction. This supercharged dilution motive over time means that the value of debt claims is lower.

Panel (a) also depicts the bid schedules under the two protocols. Recall that the bid schedule under the UP overlaps with the value of debt. Under the DP, we recover the static dilution discussed in the first part of the exercise, as well as in the two-period environment: investors bid weakly below the value of debt for bids that might not be marginal. p_{supply} denotes the bid schedule for debt buy-backs in a reverse auction, with investors bidding weakly above the value of debt for buy-back bids that might not be marginal.

Figure 9 shows the impact of the dynamic channel on borrowing decisions. Compared to the UP, borrowing decision rules imply more dilution, the asset is worth less, investors submit even lower bids, and, as a result, the government is not able to sustain as much debt as under the UP, borrows less and defaults more (cf. Figure 8, panel (b)). Effectively, taking into account the dynamic channel of the UP shifts the borrowing distributions to the right (cf. Figure 7) as it provides the government more commitment compared to the bad incentives provided by the DP.



(a) Borrowing Decisions at $B = 0.3$ and $y = \bar{y}$



(b) Borrowing Decisions at $B = 0.55$ and $y = \bar{y}$

Figure 9: Comparing borrowing decisions under UP and DP

6.1 Government Payoffs

We now turn to comparing payoffs across protocols. Since the importance of the preference shocks for government payoffs depends on the number of choices available, we use values net of preference shocks throughout this section. Given an initial state s_0 and debt B_0, we define the government's welfare as the expected discounted utility over surprise spending states and preference shocks , as well as future public states of endowment net of expected public spending. For example, the value under a DP, for state (s_0, B_0) is:

$$\mathbb{E}_{T,\mathbf{m},s_1}\left[V^{DP}(s_0, T, \mathbf{m}, B_0)\right] = \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t (1-\beta)\big(u(c_t^{DP}) - d_t^{DP} h(s)\big) \,\Big|\, s_0\right]$$

where $c_t^{DP} \equiv c^{DP}(s, T, \mathbf{m}, B)$ denotes equilibrium consumption and $d_t^{DP} \equiv d^{DP}(s, T, \mathbf{m}, B)$ denotes the government's equilibrium default decision.

We then compute the percentage increase in the consumption path, under DP, that would make the government indifferent between this allocation and the allocation where the government follows its optimal borrowing plan under UP. That is, we compute the equivalent variation in permanent consumption (ζ). Since utility is CRRA with relative risk aversion coefficient $\gamma \neq 1$ we can define ζ as in the following equation:

$$(1+\zeta)^{1-\gamma}\mathbb{E}\left[V^{DP}(s_0, \mathbf{m}, T_0, B_0)\right] = \mathbb{E}\left[V^{UP}(s_0, \mathbf{m}, T_0, B_0)\right]$$

Solving for ζ yields:

$$\zeta = \left(\frac{\overline{V}_{UP}}{\overline{V}_{DP}}\right)^{\frac{1}{1-\gamma}} - 1$$

where $\overline{V}_{UP}$ and $\overline{V}_{DP}$ are, respectively, the average value under the UP and DP for an initial state (s_0, B_0).

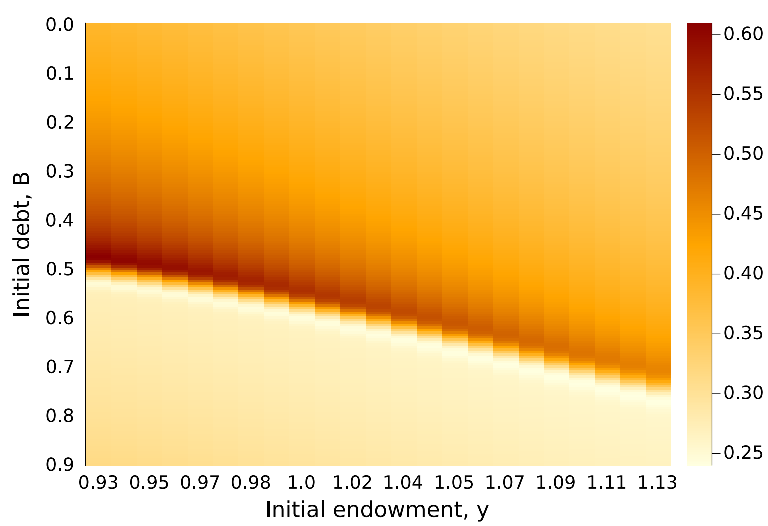


Figure 10: Equivalent variation $\zeta(y_0, B_0)$

Figure 10 shows a heat map for the equivalent variation, ζ, in percentage terms, for different levels of y_0 and B_0. The equivalent variation is strictly positive for every initial state, highlighting that, under the calibrated model, the UP is preferred to the DP. A closer inspection of the figure provides further insight. At the top right corner, with high endowment and zero debt, the difference between protocols is at the lower end of the interval. However, once we start moving down the initial endowment and increasing the level of initial debt, towards the bottom left corner, the difference starts increasing. This pattern has to do with the increase in the likelihood of default that follows from a decrease in endowment coupled with an increase in debt. As default becomes more likely the difference in the protocols increases, up to the point where default is certain. One could infer this through the differences in the bid schedules. When default is extremely unlikely, the bid schedules are closer together, however, as the likelihood of default increases, static dilution is more noticeable and, as a result, dynamic dilution also becomes more pronounced under the DP. The access to the insurance benefits of the DP are too costly.

6.2 Lender's Payoffs

We next discuss how we compare lender's welfare. Consistently with how we have proceeded with government's welfare, we define lender's welfare as the beginning of period value to the lender of holding a bond. That is,

$$Q_{ante}(s_0, B) = \mathbb{E}_{T,m,s'} \left[(1-d)\big((\kappa+\lambda) + (1-\lambda)Q(s', B')\big) + dQ^D(s_0, B) | s_0 \right]$$

where d, B' are policy rules and $Q(\cdot)$ and $Q^D(\cdot)$ are as described before.

As lenders are risk neutral, the payoffs are already in units of consumption. We compute the equivalent variation in consumption as the difference in $Q_{ante}(\cdot)$:

$$\zeta_L(s_0, B_0) = Q^{UP}_{ante}(s_0, B) - Q^{DP}_{ante}(s_0, B)$$

where Q^{UP}_{ante} and Q^{DP}_{ante} denote the ex-ante value under a UP and a DP, respectively.

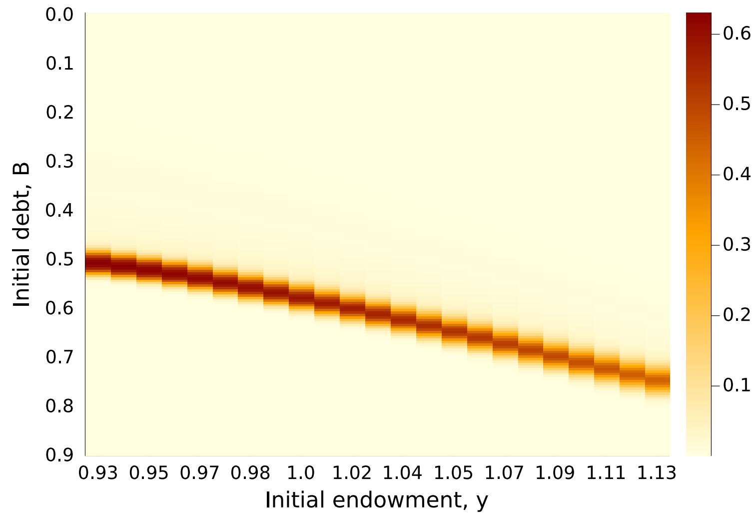


Figure 11: Equivalent variation $\zeta_L(s_0, B_0)$

Figure 11 shows a heat map for the equivalent variation, ζ_L, for different levels of y_0 and B_0. As expected, differences in prices are very close to zero absent default risk. These differences became meaningful as the country nears default. As the likelihood of default increases, static dilution within an auction, and the corresponding effect on dynamic dilution, get more pronounced, leading to lower prices under the DP. Noticeably, ζ_L is non-negative. As such, the value to the lenders is also larger under the UP.

6.3 Discussion

Under the calibrated model, the insurance component of the DP is more than offset by the dilution effects. In fact, the UP protects investors from being diluted within an auction, while at the same time provides better incentives on government's borrowing over time. The result that the insurance component is more than offset is consistent with the known fact that the welfare costs of fluctuations are small, as in Lucas (1987), and as such the benefits of insurance are limited for aggregate shocks.

The world under a UP is better than the world under a DP. The increase in government's welfare does not come at the expense of lenders, instead, using the UP when default risk is a concern is a Pareto improvement. This is consistent with the switch observed in the data: Portugal switched from a DP to a UP. In particular, Portugal stopped issuing securities with maturity longer than one year from 2011 to 2014 and the switch occurred upon the return of the Portuguese Treasury to financial markets, for those same maturities.

It is possible to reconcile the timing of the change observed in the data with the results depicted in the heat maps above. If we assume that changing the protocol of the auctions involves switching costs, then the government would wait for a state such that the gain from switching is greater than the costs. The gains are larger when the government is faced with a combination of high debt and low endowment. The same is true for lenders[24]. This is consistent with switching protocols during the crisis.

We have not explained why would Portugal use a DP in the first place. We do not model an environment where default risk is not a driving force governing the value of debt. Prior to the 2010s, it was not expected that Europe would have a sovereign debt crisis as this type of event was typically associated with emerging markets. This explanation is consistent with the flat bid schedules that we observe in the data for the early 2000s (recall Figure 3). In such an environment, the two protocols are close to equivalent and using the DP does not seem as outlandish as when default risk is a concern.

7 Conclusion

In this paper, we built a theoretical model of sovereign borrowing and default with auctions and asymmetric information on government's public spending. We disciplined the model with proprietary bid level data for Portuguese sovereign debt auctions, data on differences between realized and expected public spending, as well as institutional details relevant for modeling sovereign debt issuances. We then validated the model calibrated to the Portuguese economy. The calibrated model under the DP, was capable of matching standard moments in the Portuguese economy regarding debt, spreads and business cycles statistics. The use of a DP enabled the model to easily generate spreads whose volatility significantly exceeds their mean, a shortcoming of this class of models.

We compared the two most widely used auction protocols to issue sovereign debt, when

[24]If we were to consider that lenders could influence the government's choice of which protocol to use.

default risk is a concern. Letting the government choose how much to borrow after observing the demand for debt, we took into account how different protocols affect not only investors' but also government's decisions over time. We found that the benefits of switching from a DP to a UP are increasing in the likelihood of default and go up to 0.6% of permanent consumption. Moreover, switching to a UP is a Pareto improvement as both the small open economy and foreign lenders are better off after the switch. This result is consistent with the change in protocol observed in the data: Portugal switched from a DP to a UP in the aftermath of the sovereign debt crisis.

Finally, we found that dynamics are key in separating outcomes under the two protocols. Even though for reasonable parameter values the DP performs better than the UP under a single auction setting, in the model calibrated to Portugal, the UP is preferred. In fact, when default risk is a concern, the UP protects investors from static dilution within an auction and at the same time provides better incentives on borrowing over time.

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Appendix

Appendix A: Full Definition of Equilibrium

Given the auction protocol, a stationary symmetric recursive equilibrium consists of:

1. Value functions V, V^R, V^D;
2. Price functions Q, Q^D;
3. Bid price function p;
4. Policy functions B', ℓ, P_c, d.

such that the following conditions are satisfied:

1. Default decision optimality: given V^R and V^D, d solves the government's default or repayment decision and V is the resulting value function;
2. Borrowing decision optimality: given V and p, $\{B', P_c, \ell\}$ solve the government's repayment problem and V^R is the resulting value function;
3. Asset pricing in good standing: given d, B' and Q^D, Q satisfies the functional equation defining the value of debt while in good standing;
4. Value of default: given V, V^D is the value function for the government upon default;
5. Asset pricing in default: given Q, Q^D satisfies the functional equation defining the value of a defaulted bond;
6. Bid optimality: given Q, B' and P_c, p satisfies the bid optimality condition of ex-ante zero profits for investors;
7. Auction clearing: given p, P_c and B', the sum of accepted bid quantities equals the debt issuance, $\ell \equiv B' - (1 - \lambda)B$.

Appendix B: Omitted Proofs

Theorem 1 (Revenue Equivalence). *If B' is a random variable independent of the auction protocol, then ex-ante expected revenue in the auction is the same under both protocols.*

Proof:

Let B' be a continuous random variable with cdf G. Let, as before, Δ^D and Δ^U denote, respectively, revenue under the discriminatory and uniform price protocols.

$$
\begin{aligned}
\mathbb{E}[\Delta^D(b')] &= \int_0^{b_H} \left[\int_0^{b'} p(b)db \right] dG(b') \\
&= \int_0^{b_H} p(b) \left[\int_b^{b_H} dG(b') \right] db \\
&= \int_0^{b_H} p(b)[1 - G(b)]db \\
&= R^{-1} \int_0^{b_H} \left[\int_b^{b_H} F(\underline{v}^d(\tilde{b}))dG(\tilde{b}) \right] db \\
&= R^{-1} \int_0^{b_H} F(\underline{v}^d(\tilde{b}))g(\tilde{b}) \left[\int_0^{\tilde{b}} db \right] d\tilde{b} \\
&= R^{-1} \int_0^{b_H} \tilde{b}F(\underline{v}^d(\tilde{b}))dG(\tilde{b}) \\
&= \mathbb{E}[\Delta^U(b')] \qquad \square
\end{aligned}
$$

The first equality defines expected revenue under a discriminatory price protocol. For the second equality we proceed by changing the order of integration. After simplifying the expression, and substituting $p(b)$ by its equilibrium expression, for the fifth equality we perform another change in the order of integration. Simplifying yields the definition of expected revenue under the uniform price protocol.

Proposition 1. *Bidding marginal prices is a dominant strategy for investors.*

Proof: Let $\mathcal{P}$ denote the set of marginal prices chosen by the government. A marginal price $P_c(\theta; p)$, depends on the realization of θ, for a given aggregate bid schedule. As θ is drawn from a discrete distribution with finite support, the set $\mathcal{P}$ is itself finite.

Let the shock θ take two possible values, further, let the associated marginal prices be $P_{c,1} > P_{c,2}$. Consider a bid with price p. Suppose for the sake of contradiction that a bid function (p_1, p_2) that has $p_1 = P_{c,1} > p_2 > P_{c,2}$ dominates (p_1, p_2') that has $p_1 = P_{c,1} > p_2' = P_{c,2}$.

The first bid is the same across bid functions so we can focus on the second bid. First note that the bid with price p_2 is accepted with the same probability of a bid with price $p_2' = P_{c,2}$, as there is no marginal price chosen in between $P_{c,1}$ and $P_{c,2}$. It follows that

$$Pr[P_c(\theta) \leq p_2] = Pr[P_c(\theta) \leq p_2'] = Pr[P_c(\theta) \leq P_{c,2}]$$

Second, the value of debt only depends on the marginal price, that is not affected by an individual infinitesimal dealer. In particular we have $Q(\ell(P_{c,1})) \geq Q(\ell(P_{c,2}))$. That is, bidding p_2 instead of p_2' does not affect the value of debt, the investors payoff if the bid is executed. The cost associated with bidding $p_2 > P_{c,2}$, however, is greater than bidding $p_2' = P_{c,2}$. The dealer's unitary profit for this bid is then:

$$Pr[P_c(\theta) \leq P_{c,2}] \Big(Q(\ell(P_{c,2})) - \phi(p_2, P_c(\theta) | \ell(P_{c,2})) \Big)$$

For a discriminatory price protocol $\phi(p_2, \cdot) = p_2$ and

$$Pr[P_c(\theta) \leq P_{c,2}] \Big(Q(\ell(P_{c,2})) - p_2 \Big) < Pr[P_c(\theta) \leq P_{c,2}] \Big(Q(\ell(P_{c,2})) - p_2' \Big) \tag{5}$$

Let the uniform price protocol be the limiting case when $\epsilon \to 0$ as follows: $\epsilon p_2 + (1 - \epsilon) P_{c,2}$, then in the limit,

$$\epsilon p_2 + (1 - \epsilon) P_{c,2} - (\epsilon p_2' + (1 - \epsilon) P_{c,2}) \to 0 \tag{6}$$

and the government is indifferent between bidding p_2 and p_2'.

Note that (5) and (6) contradict $p = (p_1, p_2)$ dominating $p' = (p_1, p_2')$. It follows that bidding marginal prices is a strictly dominant strategy under a discriminatory price protocol and a weakly dominant strategy under a uniform price protocol. □

Appendix (For Online Publication)

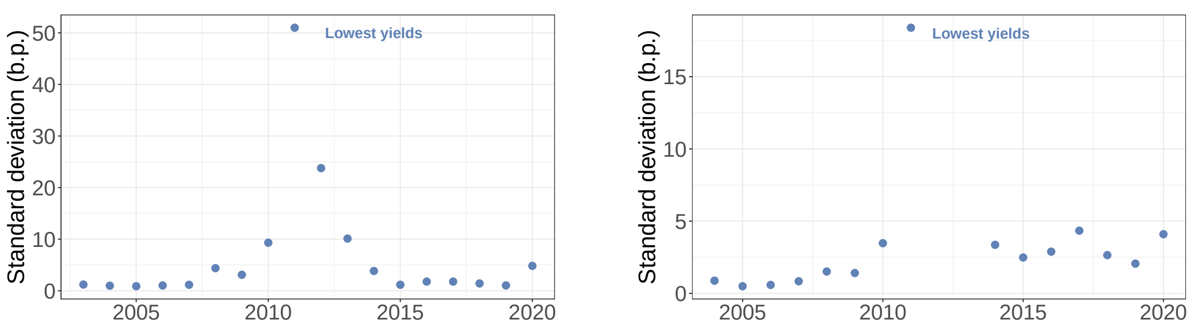
Appendix A: Investor Heterogeneity

To assess differences across dealers we first look at the variation in the price of the first bid, with the lowest yield (highest price). The first bid in a dealer's bid function is the most likely to be executed as it has the lowest yield. As such, we argue that this bid is also the most informative regarding dealer's characteristics.

Figure 12 presents the standard deviation of the lowest bid yield across dealers, $\{1, \ldots, N\}$, at a given auction, for treasury bills and treasury bonds. More precisely, each point represents the average of such standard deviations across auctions, $\{1, \ldots, M_t\}$, for a given year, t as follows:

$$\overline{SD}_t = \frac{1}{M_t} \sum_{j=1}^{M_t} \sqrt{\sum_{i=1}^{N} \frac{(p_{i1j} - \overline{p}_{1j})^2}{N}}$$

We separate this analysis for bills and bonds as the set of dealers participating in bills and bond auctions are potentially different. Moreover, for bonds, we see a change in protocol during the crisis as well as a hiatus on issuances.



(a) Treasury Bills

(b) Treasury Bonds

Figure 12: Variation between dealers' lowest bid yields

For both securities we see that prior to the crisis the standard deviation is very small. We then see a temporary increase during the crisis period followed by a return to zero afterwards. This pattern is more clear for treasury bills given the continued issuance of these securities during the crisis. Apart from that, the main difference with respect to treasury bonds is that after the crisis the variation does not quite go back to zero, instead it remains at slightly higher levels than before the crisis. This change in pattern occurs at the same time as the protocol for treasury bond auctions switched from discriminatory to uniform price.

Figure 13 presents the standard deviation of bid yields, within a bid function, for each dealer. More precisely, each point represents the average of such standard deviations across auctions, $\{1, \ldots, M_t\}$, for a given dealer, i, and a given year, t, as follows:

$$\overline{SD}_{i,t} = \frac{1}{M_t} \sum_{j=1}^{M_t} \sqrt{\sum_{k=1}^{K_j} \frac{(p_{ikj} - \overline{p}_{ij})^2}{N}}$$

The time series of the average standard deviation as a similar trend across all dealers (across panels): 1) increasing towards 2008; 2) a drop in 2009 before the crisis; 3) higher from 2010 to 2012; 4) a decrease starting in 2013 particularly accentuated in 2014; and, 5) almost flat bid functions from 2014 onward. Note, however, that some dealers have more disperse bid functions than others during the crisis period.

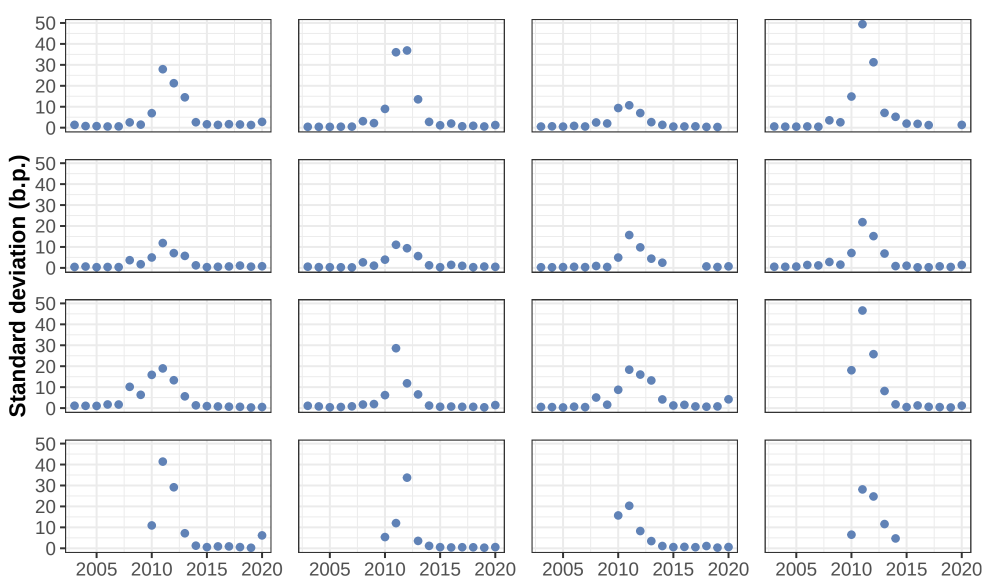


Figure 13: Average standard deviation of bids between dealers

Having said this, the standard deviation across and within investors is at roughly the same magnitude. This leads us to conclude that the pattern we see in Figure 3 is due to all investors bidding at a wider range of prices, as well as some of them bidding at higher prices and others at lower prices. That is, steeper aggregate bid functions are due, in part, to steeper individual bid functions.

No persistent heterogeneity. Finally, we evaluate whether there are persistent differences across investors. To do so, we rank the first bids, with the lowest yields, across dealers in a given auction. As before, we use the first bid as it is the most informative about differences across dealers. We postulate that if there were persistent differences across investors, we would see a persistent pattern in this ranking. For instance, well informed dealers would likely bid closer to the marginal price of the auction and consistently be ranked lower. Figure 14 depicts the relative ranking over time for each dealer across treasury bill auctions. We focus on treasury bills to highlight this fact due to the continued issuance of this type of securities during the crisis[25]. One can see that a persistent pattern does not seem to exist, in fact, ranking over time seems to be independent of dealer.

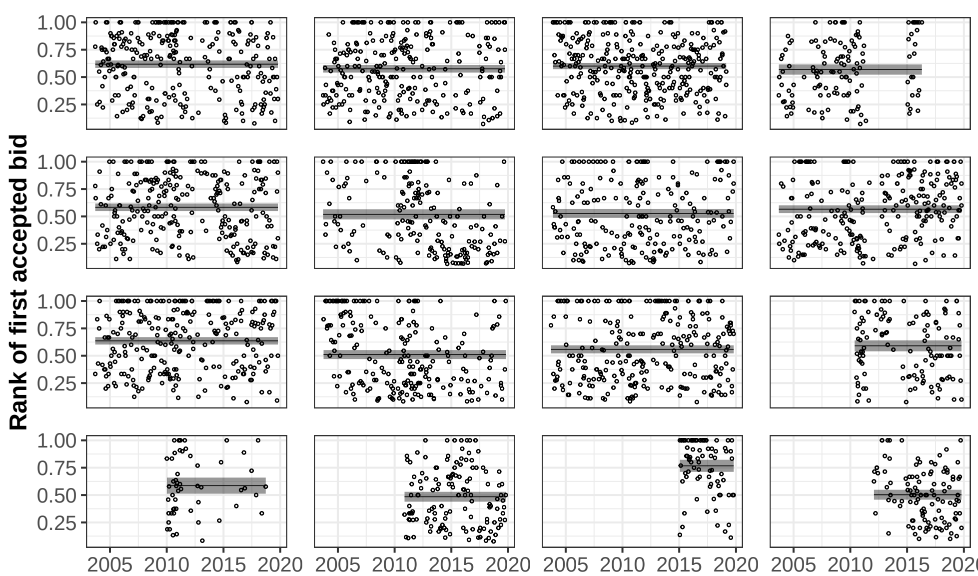


Figure 14: Rank of first bid (if accepted) over time

The horizontal bars in Figure 14 represent the dealer fixed effect, α_i, in the following regression:

$$R_{it} = \alpha_i + \epsilon_{it}$$

[25]The same pattern emerges for treasury bonds.

where the dependent variable R_{it} depicts dealers' i ranking in the auction ran at time t. Year fixed effects were not included due to lack of significance. The dealer fixed effects, as seen on the figure are fairly close to each other, with few exceptions, mostly on the lower panels. Particularly, those exceptions tend to be more significant for dealers that participate in auctions during shorter periods of time[26]. Overall, individual and time fixed effects account for less than 5% of the variation of rankings across investors and over time[27].

Appendix B: Two Period Environment, Alternative Specification

Consider the environment described in section 3. Instead of unexpected spending as a random variable, consider a preference shock in the first period. In particular, preferences over streams of consumption are as follows:

$$\mathbb{E}\left[\theta u(c_0) + \beta u(c_1)\right]$$

The taste shock, θ, is privately observed by the government. It is drawn from a continuous distribution with support on $[\theta_L, \theta_H]$ with $\theta_L < \theta_H$ and cdf G. We further assume that $g(\theta) = G'(\theta) > 0$ on $[\theta_L, \theta_H]$.

We use the same parameterization as before with one difference. Suppose that $v^d = y(1 - \exp(-z))$ with z distributed exponentially with cdf $F(z) = 1 - exp(-\mu z)$ and $z = -ln(1 - v^d/y)$. Then $F(v^d) = 1 - exp(\mu ln(1 - v^d/y)) = 1 - (1 - v^d/y)^{\mu}$ and $F'(v^d) = (\mu/y)(1 - v^d/y)^{\mu-1}$. For $\mu = 1$ this collapses into an uniform distribution on the interval $[0, y]$.

Commitment to a Borrowing Rule

We first tie the hands of the government. Suppose the government could commit to a borrowing rule, θ is observed ex-post and the government commits to it. In particular, the

[26]From all participating investors, 6 were not included in the plot as they participated for even shorter periods of time, making it harder to highlight trends in their ranking.

[27]We further test for linear trends within investors across time and verify that they are either not significant or explain less than 10% of the variation of the ranking over time.

government commits to the optimal borrowing rule under UP, regardless of the protocol being used. That is, $b(\theta) = b(\theta)_{UP}$. By fixing the distribution of b' across protocols, utility in the second period is independent of the protocol. Furthermore, we recover revenue equivalence. With linear utility, welfare is pinned down by

$$\mathbb{E}[\theta\Delta(\theta)] = \mathbb{E}[\theta]\mathbb{E}[\Delta(\theta)] + \mathbb{C}(\theta, \Delta(\theta))$$

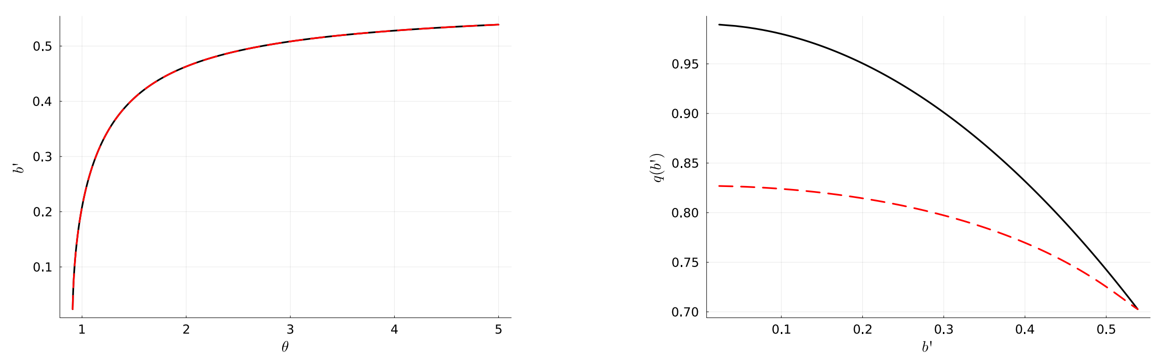
In particular, the difference in welfare across protocols is determined by the covariance term. This term is the insurance component that stems from the curvature introduced by the multiplicative taste shock, θ.

In Figure 15 below we can see how the protocols compare. Panel (a) illustrates the commitment to a borrowing rule as a function of θ. Panel (b) shows us that static dilution is still present, with the bid schedule under a DP lower than the one under a UP. Panel (c) highlights the potential benefits of insurance, higher revenue in bad states at the expense of relative lower revenue in good states. Panel (d) shows us that welfare tends to be higher under DP, particularly when there are large financing needs in the first period.

Ex-ante welfare is higher under DP:

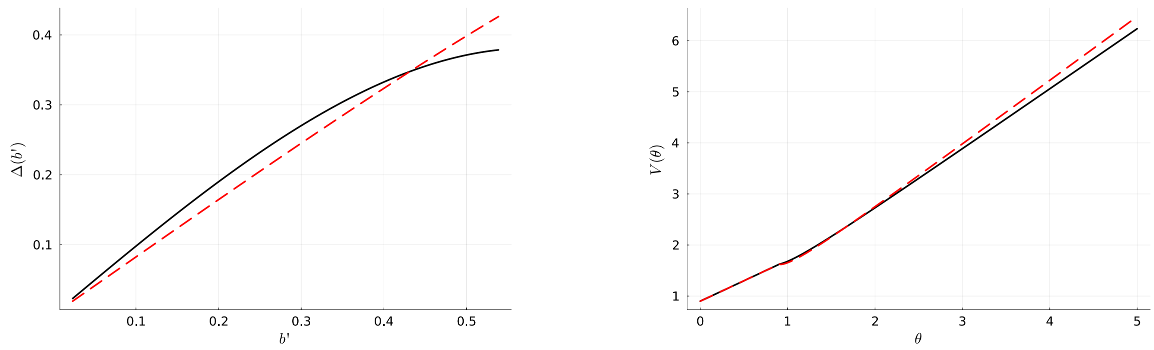
$$\mathbb{E}[V(\theta)_{UP}] = 1.754 \quad \mathbb{E}[V(\theta)_{DP}] = 1.761$$

It follows that the covariance term is larger under DP.



(a) Borrowing Decisions

(b) Bid Schedules



(c) Revenue

(d) Value functions

Figure 15: Comparing outcomes under commitment to $b(\theta)_{UP}$

To evaluate the cost of not committing to the borrowing rule, we let the government choose optimally for each realization of θ, given the price schedule. Note that welfare under a UP will be unchanged as the government was already choosing optimally. As such, the difference in welfare under the DP measures the static dilution that arises from the lack of commitment. This specification of the environment allows us to get a closed form solution detailed below.

Closed Form Solution

The equilibrium under a uniform price protocol is fairly standard to solve. Under the specified functional forms, we can also find a closed-form solution for the fixed point problem described above, between investors' and government's strategies, under a discriminatory price protocol.

Consider linear preferences, such that $u(x) = x$. Let us first characterize the equilibrium under a discriminatory price protocol. An equilibrium requires an actuarially fair price for investors and attaining the maximum in problem (1), respectively:

$$p(b) = \frac{1}{1 - G(\theta(b))} \int_{\theta(b)}^{\theta_H} Q(b(\theta)) dG(\theta)$$
$$\theta p(b(\theta)) = \beta F(y - b(\theta))$$

These two conditions together give us a single optimality condition:

$$\frac{\theta}{1-G(\theta)}\int_{\theta}^{\theta_H} F(y-b(x))dG(x) = \beta R F(y-b(\theta))$$

Before we solve the equation above, let us just point out that the solution relies on the fact that for a small enough θ the government does not borrow, $b(\theta) = 0$, and so the probability of repayment equals one. As $\theta \to 0$ the benefit of borrowing goes to zero as U only depends on consumption in the second period. Below we show that borrowing is non-decreasing in θ.

Proposition 2 (Monotonicity of b). *If u is strictly increasing and concave, $\beta \in (0,1)$, and $f(v^d) = F'(v^d) > 0$ on $[\underline{v}, \overline{v}]$, then $b(\theta; p)$ is non-decreasing in θ.*

Proof: Suppose, for the sake of contradiction that $b(\theta)$ is strictly decreasing in θ.

A government chooses b such that:

$$\theta u'(y+\Delta(b)-b_0)\Delta'(b) = \beta F(u(y-b))u'(y-b) \tag{7}$$

Let $\theta_2 > \theta_1 > 0$ and $b(\theta_1)$ and $b(\theta_2)$ be the optimal choices associated with θ_1 and θ_2, respectively. Note that the marginal benefit of borrowing (left-hand side) is non-increasing in b. First, $u'(y+\Delta(b)-b_0)$ is non-increasing in $\Delta(b)$ as u is concave by assumption; $\Delta(b)$ is a concave function of b as the price schedule is a non-increasing function of b and so $\Delta'(b)$ is also non-increasing. Further, optimality requires that $\Delta'(b) \geq 0$ along the equilibrium path. As $\theta_2 > \theta_1$ and $b(\theta_1) > b(\theta_2)$, it follows that:

$$\theta_2 u'(y+\Delta(b(\theta_2))-b_0)\Delta'(b(\theta_2)) > \theta_1 u'(y+\Delta(b(\theta_1))-b_0)\Delta'(b(\theta_1)) \tag{8}$$

Pick a value of y (or $\overline{v}$) such that $F(u(y-b(\theta_2))) = 1$, that is, $u(y-b(\theta_2)) > u(y-b(\theta_1)) \geq \overline{v}$. Then, for $\theta \in \{\theta_1, \theta_2\}$ the government never defaults and $Q(b(\cdot)) = R^{-1}$.

The marginal cost of borrowing, when default is a zero probability event, is non-decreasing in b, as u is concave and $F(\cdot)$ is constant and equal to 1. As $b(\theta_1) > b(\theta_2)$, it follows that:

$$\beta F(u(y-b(\theta_2)))u'(y-b(\theta_2)) \leq \beta F(u(y-b(\theta_1)))u'(y-b(\theta_1))$$

Note that equation (7) then requires:

$$\theta_2 u'(y+\Delta(b(\theta_2))-b_0)\Delta'(b(\theta_2)) \leq \theta_1 u'(y+\Delta(b(\theta_1))-b_0)\Delta'(b(\theta_1))$$

which contradicts equation (8). $\square$

Under a discriminatory price protocol, we have established that, with linear preferences, for any θ that has the government borrowing in equilibrium, it must be that:

$$\theta \frac{R^{-1}}{1-G(\theta)} \int_{\theta}^{\theta_H} F(\underline{v}^d(b(x))dG(x) = \beta F(y-b(\theta))$$

Let $n(\theta) \equiv F(y-b(\theta))$, denote the probability of repayment at θ. Then, the equation above is

$$\theta \int_{\theta}^{\theta_H} n(x)\frac{dG(x)}{1-G(\theta)} = \beta R\, n(\theta)$$

Set $N(\theta) \equiv \int_{\theta}^{\theta_H} n(x)dG(x)$ and $N'(\theta) = -n(\theta)dG(\theta)$. Then,

$$\theta N(\theta) = -\beta R \frac{1-G(\theta)}{dG(\theta)} N'(\theta) \iff \frac{N'(\theta)}{N(\theta)} = -\frac{\theta}{\beta R}\frac{dG(\theta)}{1-G(\theta)}$$

For an exponentially distributed θ, we have $G(x) = 1-exp(-\lambda x)$ and $dG(x) = \lambda exp(-\lambda x)$.

$$\frac{N'(\theta)}{N(\theta)} = -\frac{\theta}{\beta R}\frac{1-G(\theta)}{dG(\theta)} \iff \log(N(\theta)) = -\frac{\theta^2\lambda}{2\beta R} + C \implies N(\theta) = K.exp\left(-\frac{\theta^2\lambda}{2\beta R}\right)$$

where $K = exp(C)$. Taking derivatives we get:

$$N'(\theta) = -K\frac{\theta}{\beta R} exp\left(-\frac{\theta^2\lambda}{2\beta R}\right)$$

Recalling that $N'(\theta) = -n(\theta)dG(\theta)$, by definition this is equivalent to:

$$n(\theta) = K\frac{\theta}{\beta R} exp\left(\lambda\theta - \frac{\theta^2\lambda}{2\beta R}\right)$$

This must be true for some $K > 0$. To determine the value of K, we make some assumptions about the nature of the equilibrium. In general, an equilibrium of the kind we posit always exists. We look for equilibria in which 1) there is a $\hat{\theta}$, such that the government's first order condition holds at $b' = 0$ (and therefore $n(\hat{\theta}) = 1$), and 2) at this $\hat{\theta}$, it is the case that $n'(\hat{\theta}) = 0$. The second condition selects a specific θ. In particular, it selects the lowest possible one. We begin solving for $\hat{\theta}$ and K by examining the implications of $n'(\hat{\theta}) = 0$. The derivative of $n(\theta)$ is:

$$n'(\theta) = K\frac{1}{\beta R}exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right) + K\frac{\theta}{\beta R}\left(\lambda - \frac{\lambda\theta}{\beta R}\right)exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right)$$

Collect terms to rewrite this as:

$$n'(\theta) = K\frac{1}{\beta R}\left(1 + \lambda\theta\left(1 - \frac{\theta}{\beta R}\right)\right)exp\left(\lambda\theta - \frac{\lambda\theta^2}{2\beta R}\right)$$

Since the collection of terms outside the big parentheses are all positive, we see that this is a parabola that opens down. Setting it equal to 0 yields:

$$1 + \lambda\theta - \frac{\lambda\theta^2}{\beta R} = 0$$

We will want the right root of this (so that $n'(\theta)$ is appropriately negative for $\theta \geq \hat{\theta}$). The above can be rewritten as:

$$\theta^2 - \beta R\theta - \frac{\beta R}{\lambda} = 0$$

Then $\hat{\theta}$ is given by:

$$\hat{\theta} = \frac{\beta R + \sqrt{(\beta R)^2 + 4\frac{\beta R}{\lambda}}}{2}$$

Finally, having solved for $\hat{\theta}$ in terms of parameters, we can quickly solve for K as the solution to:

$$1 = n(\hat{\theta}) = K\frac{\hat{\theta}}{\beta R}exp\left(\lambda\hat{\theta} - \frac{\lambda\hat{\theta}^2}{2\beta R}\right)$$

So:

$$K = \frac{\beta R}{\hat{\theta}} exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right)$$

Then, $n(\theta)$ becomes:

$$n(\theta) = \frac{\beta R}{\hat{\theta}} exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right)\frac{\theta}{\beta R} exp\left(\lambda\theta - \frac{\theta^2\lambda}{2\beta R}\right)$$

which can be simplified to:

$$\begin{aligned} n(\theta) &= \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right) \\ &= \frac{\theta}{\hat{\theta}}\, exp\left(-\frac{\lambda}{\beta R}(\theta - \hat{\theta})\left(\frac{\theta + \hat{\theta}}{2} - \beta R\right)\right) \end{aligned}$$

Given a functional form of $F(\cdot)$, this can then be mapped back to choices of b' using the definition:

$$n(\theta) = F(y - b'(\theta))$$

Suppose that $v^d = y(1 - \exp(-z))$ where z is distributed exponentially with cdf $F(z) = 1 - exp(-\mu z)$ and $z = -ln(1 - v^d/y)$. Then $F(v^d) = 1 - exp(\mu ln(1 - v^d/y)) = 1 - (1 - v^d/y)^\mu$ and $F'(v^d) = (\mu/y)(1 - v^d/y)^{\mu-1}$. When $v^d = y - b(\theta)$, the term $1 - v^d/y$ becomes:

$$1 - \frac{y - b(\theta)}{y} = \frac{y - y + b(\theta)}{y} = \frac{b(\theta)}{y}$$

Using the optimality condition (7) we get:

$$\begin{aligned} &\beta F(y - b(\theta)) = \frac{\theta R^{-1}}{exp(-\lambda\theta)} N(\theta) \iff \\ &\iff \beta\left(1 - \left(\frac{b(\theta)}{y}\right)^\mu\right) = \frac{\theta R^{-1}}{exp(-\lambda\theta)}\frac{\beta R}{\hat{\theta}} exp\left(\frac{\lambda\hat{\theta}^2}{2\beta R} - \lambda\hat{\theta}\right) exp\left(-\frac{\theta^2\lambda}{2\beta R}\right) \\ &\iff 1 - \left(\frac{b(\theta)}{y}\right)^\mu = \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right) \\ &\iff b(\theta) = y\left(1 - \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right)^{\frac{1}{\mu}} \end{aligned}$$

Recall that $p(b(\theta)) = \frac{\beta}{\theta} F(y - b(\theta))$, it then follows that:

$$\begin{aligned}
p(b(\theta)) &= \frac{\beta}{\theta}\left(1 - \left(\frac{b(\theta)}{y}\right)^{\mu}\right) \\
&= \frac{\beta}{\theta}\left(1 - \left[\frac{y\left(1 - \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right)^{\frac{1}{\mu}}}{y}\right]^{\mu}\right) \\
&= \frac{\beta}{\theta}\left(\frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right) \\
&= \frac{\beta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right) \\
&= p(\theta)
\end{aligned}$$

Under a uniform price protocol, an equilibrium with positive borrowing requires:

$$\begin{aligned}
&\theta\Delta'(b(\theta)) = \beta F(y - b(\theta)) \iff \\
\iff &\theta\left[Q(b(\theta)) + \frac{\partial Q(b(\theta))}{db(\theta)} b(\theta)\right] = \beta F(y - b(\theta)) \\
\iff &\theta\left[R^{-1}F(y - b(\theta)) + R^{-1}F'(y - b(\theta))b(\theta)\right] = \beta F(y - b(\theta)) \\
\iff &\theta\left[1 - \frac{F'(y - b(\theta))}{F(y - b(\theta))} b(\theta)\right] = \beta R \\
\iff &\theta \frac{F'(y - b(\theta))}{F(y - b(\theta))} b(\theta) = \theta - \beta R \\
\iff &\theta \frac{\frac{\mu}{y}\left(\frac{b(\theta)}{y}\right)^{\mu - 1}}{1 - \left(\frac{b(\theta)}{y}\right)^{\mu}} b(\theta) = \theta - \beta R \\
\iff &\theta \frac{\mu}{y^{\mu}} b(\theta)^{\mu} = (\theta - \beta R) - (\theta - \beta R) b(\theta)^{\mu} \frac{1}{y^{\mu}} \\
\iff &b(\theta)^{\mu}\left(\theta \frac{\mu}{y^{\mu}} + \theta \frac{1}{y^{\mu}} - \beta R \frac{1}{y^{\mu}}\right) = \theta - \beta R \\
\iff &b(\theta) = \left(\frac{\theta - \beta R}{\theta(1 + \mu) - \beta R}\right)^{\frac{1}{\mu}} y
\end{aligned}$$

We have established that under a uniform price protocol investors only bid marginal prices, hence:

$$\begin{aligned}
p(b(\theta)) &= R^{-1}F(y - b(\theta)) \\
&= R^{-1}\left(1 - \left(\frac{\theta - \beta R}{\theta(1+\mu) - \beta R}\right)\right) \\
&= R^{-1}\left(\frac{\mu\theta}{\theta(1+\mu) - \beta R}\right) \\
&= p(\theta)
\end{aligned}$$

Summing up, for a **uniform price auction**:

$$b(\theta) = \left(\frac{\theta - \beta R}{\theta(1+\mu) - \beta R}\right)^{\frac{1}{\mu}} y$$

$$p(\theta) = R^{-1}\left(\frac{\mu\theta}{\theta(1+\mu) - \beta R}\right)$$

And for a **discriminatory price auction**:

$$b(\theta) = y\left(1 - \frac{\theta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)\right)^{\frac{1}{\mu}}$$

$$p(\theta) = \frac{\beta}{\hat{\theta}}\, exp\left(\lambda(\theta - \hat{\theta}) - \frac{\lambda}{2\beta R}(\theta^2 - \hat{\theta}^2)\right)$$

Appendix C: Robustness

Utility

Let us first see what happens under different utility functions. All other parameters are the same as before.

$$\begin{aligned}
\text{Linear Utility}: \ \mathbb{E}[V(\theta)_{UP}] &= 1.754 > \mathbb{E}[V(\theta)_{DP}] = 1.698 \\
\text{Log Utility}: \ \mathbb{E}[V(\theta)_{UP}] &= -0.0986 < \mathbb{E}[V(\theta)_{DP}] = -0.0973 \\
\text{CRRA}, \gamma = 2: \ \mathbb{E}[V(\theta)_{UP}] &= -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259 \\
\text{CRRA}, \gamma = 4: \ \mathbb{E}[V(\theta)_{UP}] &= -0.8065 < \mathbb{E}[V(\theta)_{DP}] = -0.8059 \\
\text{CRRA}, \gamma = 8: \ \mathbb{E}[V(\theta)_{UP}] &= -0.5146 < \mathbb{E}[V(\theta)_{DP}] = -0.5137
\end{aligned}$$

Distribution of θ

Let us keep CRRA with $\gamma = 2$ and v^d uniformly distributed on $[\underline{v}, \overline{v}]$.

$$\begin{aligned}
\theta \sim Exp(1): \ \mathbb{E}[V(\theta)_{UP}] &= -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259 \\
\theta \sim U(0,5): \ \mathbb{E}[V(\theta)_{UP}] &= -3.5773 < \mathbb{E}[V(\theta)_{DP}] = -3.5762 \\
\theta \sim U(0,10): \ \mathbb{E}[V(\theta)_{UP}] &= -5.6877 < \mathbb{E}[V(\theta)_{DP}] = -5.6851 \\
\theta \sim N(3,2): \ \mathbb{E}[V(\theta)_{UP}] &= -4.1615 < \mathbb{E}[V(\theta)_{DP}] = -4.1599
\end{aligned}$$

Distribution of v^d

Let us keep CRRA with $\gamma = 2$ and θ exponentially distributed with $\lambda = 1$.

$$\begin{aligned}
v^d \sim U(\underline{v}, \overline{v}): \ \mathbb{E}[V(\theta)_{UP}] &= -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259 \\
v^d \sim N(u(0.2) = -5, 1.5): \ \mathbb{E}[V(\theta)_{UP}] &= -2.0254 < \mathbb{E}[V(\theta)_{DP}] = -2.0248
\end{aligned}$$

Output Growth

Let us keep CRRA with $\gamma = 2$, θ exponentially distributed with $\lambda = 1$ and v^d uniformly distributed.

$$y_1 = y_0: \ \mathbb{E}[V(\theta)_{UP}] = -2.0265 < \mathbb{E}[V(\theta)_{DP}] = -2.0259$$
$$y_1 = 1.05 \times y_0: \ \mathbb{E}[V(\theta)_{UP}] = -1.9677 < \mathbb{E}[V(\theta)_{DP}] = -1.9671$$
$$y_1 = 0.95 \times y_0: \ \mathbb{E}[V(\theta)_{UP}] = -2.0896 < \mathbb{E}[V(\theta)_{DP}] = -2.0891$$

Budget Deficits

Instead of considering a multiplicative taste shock we now look at what would happen if instead uncertainty is regarding a budget deficit, θ as follows:

$$c = y + \Delta(b(\theta)) - b_0 - \theta$$

Let us keep CRRA with $\gamma = 2$ and v^d uniformly distributed. θ is exponentially distributed with $\lambda = 1$ and truncated to the interval $[0, 1]$.

$$\mathbb{E}[V(\theta)_{UP}] = -2.8976 < \mathbb{E}[V(\theta)_{DP}] = -2.8952$$

Appendix D: Computational Details

Environment: Additional Elements

The private exogenous state includes a vector $\mathbf{m}$ of preference shocks for the government that is i.i.d. over time. These preference shocks enter additively in the government's decision problems. They are unbounded and therefore ensure that every feasible action is played with positive probability in equilibrium. Introducing these shocks is like introducing randomization, ensuring convergence – that an equilibrium exists [28]. These shocks are otherwise small. The preference shocks $\mathbf{m}$ are distributed according to a Generalized Type One Extreme Value distribution with scale parameter σ_m and correlation

[28] These preference shocks have the same role as the m shock introduced in Chatterjee and Eyigungor (2012).

parameter ρ_m. These distributions are chosen for their computational tractability[29].

When the government issues debt, it incurs an issuance cost $i(s,B,B') \geq 0$. This is a standard feature in models with long term debt and positive recovery rates (see Dvorkin et al. (2021) or Chatterjee and Eyigungor (2015)). Without these adjustment costs, the government has an incentive to issue very large amounts of debt when default is imminent in order to extract the value of existing bondholders' securities. This type of "maximum" dilution behavior is counterfactual. As such, issuance costs are added to the model to prevent it from occurring in equilibrium. Quantitatively, the amount spent financing the issuance costs ends up being small. The issuance cost function is as in Fourakis (2023)[30]. This function imposes a strict limit on the one period ahead default probability from which issuing costs are positive and is continuous in the scale of the issuance. The purpose of these issuance costs is to prevent a behavior Chatterjee and Eyigungor (2015) termed "maximum dilution."

Solving the Model

The set of objects used to solve the model numerically and assess convergence are as follows:

1. The continuation value functions $W(s,T,B,B')$ and $W^D(s,T,B)$, given by:

$$W(s,T,B,B') = \mathbb{E}[V(s',T',\mathbf{m},B,B')|s]$$
$$W^D(s,T,B) = \mathbb{E}[V^D(s',T',\mathbf{m},B)|s]$$

2. The price functions $Q(s,B')$ and $Q^D(s,B)$ and the expected probability of default

[29]Specifically, both choice probabilities and ex ante expected values can be written analytically in terms of the values associated with the choices. We set ρ_m following Dvorkin et al. (2021). We then set the scale parameter at a small number that still ensures convergence, half of that of Dvorkin et al. (2021).

[30]A sine wave shifted and scaled to rise from 0 to 1 as it travels from the threshold, p_d, to 1:

$$i\left(s,B,B'\right) = \begin{cases} 0 & B' \leq \hat{B} \text{ or } \Pr\left(d'^{\star}=1\right) \leq p_d \\ \frac{1}{2}\left(1+\sin\left(\pi\left(\frac{\Pr(d'^{\star}=1)-p_d}{1-p_d}-\frac{1}{2}\right)\right)\right) & B' > \hat{B} \text{ and } \Pr\left(d'^{\star}=1\right) > p_d \end{cases}$$

where $\hat{B} = \max\{(1-\lambda)B, 0\}$.

$\delta(s, B')$.

In short, these are the continuation value functions, the price functions, and the expected probability of default. Note that there are other price and value functions (including the bid function in the discriminatory price protocol), but they can be derived based on the above set of objects and within-period optimization. We use the above set as the list to assess convergence.

These objects are defined on grids of their arguments. In particular, we have the following sets that we will need to define grids for:

1. $s \in \mathcal{S}$, that defines GDP and expected public spending.
   (a) For the grid of GDP values, $y(s)$, we use 23 points evenly spaced in logs spread across a space spanning six of the logged variable's long run standard deviations and centered at its mean:

$$[\mathbb{E}[\log(y(s))] - 3\sigma[\log(y(s))], \mathbb{E}[\log(y(s))] + 3\sigma[\log(y(s))]]$$

   (b) For the grid of expected public spending values, $g(s)$, we use 17 points evenly spaced in logs spread across a space spanning six of the logged variable's long run standard deviations and centered at its mean:

$$[\mathbb{E}[\log(g(s))] - 3\sigma[\log(g(s))], \mathbb{E}[\log(g(s))] + 3\sigma[\log(g(s))]]$$

2. $B \in \mathcal{B}$: for the grid of b we use 241 evenly spaced points on [0, 1.2].
3. $T \in \mathcal{T}$, that defines surprise budget spending: for the grid of $\theta(T)$ we use 31 points evenly spaced spanning six of the logged variable's long run standard deviations and centered at one (the average log is zero).

Given a guess for the set of objects listed above, in order to generate a new guess, the iteration proceeds as follows:

1. Using the baseline set of objects, and given the restructuring structure upon regain-

ing access to financial markets, generate new guesses for $W^D(s, T, B)$ and $Q^D(s, B)$.

2. Using the baseline set of objects, and those defined in the previous step, solve the government's problem when it enters a period in good standing. Use the solution to generate new guesses of $W(s, T, B, B')$, $Q(s, B')$ and $\delta(s, B')$.

3. Check the sup-norm distance between all objects. If it is less than 10^{-5}, stop. Otherwise, update guesses using rules of the form

$$f_{next}(\cdot) = \xi_j f_{old}(\cdot) + (1 - \xi_j) f_{new}(\cdot)$$

where $j \in \{V, Q\}$, and return to step 1.

This type of rule updates the old guess by moving fraction $(1 - \xi_j)$ of the distance towards the new guess. In general, to ensure convergence, updates of the the price functions tend to require more smoothing than those of the value functions. Moreover, solving the government's problem in good standing under a discriminatory price protocol also requires smoothing for the update of bid schedules and auction revenue.