Optics: Geometrical Optics

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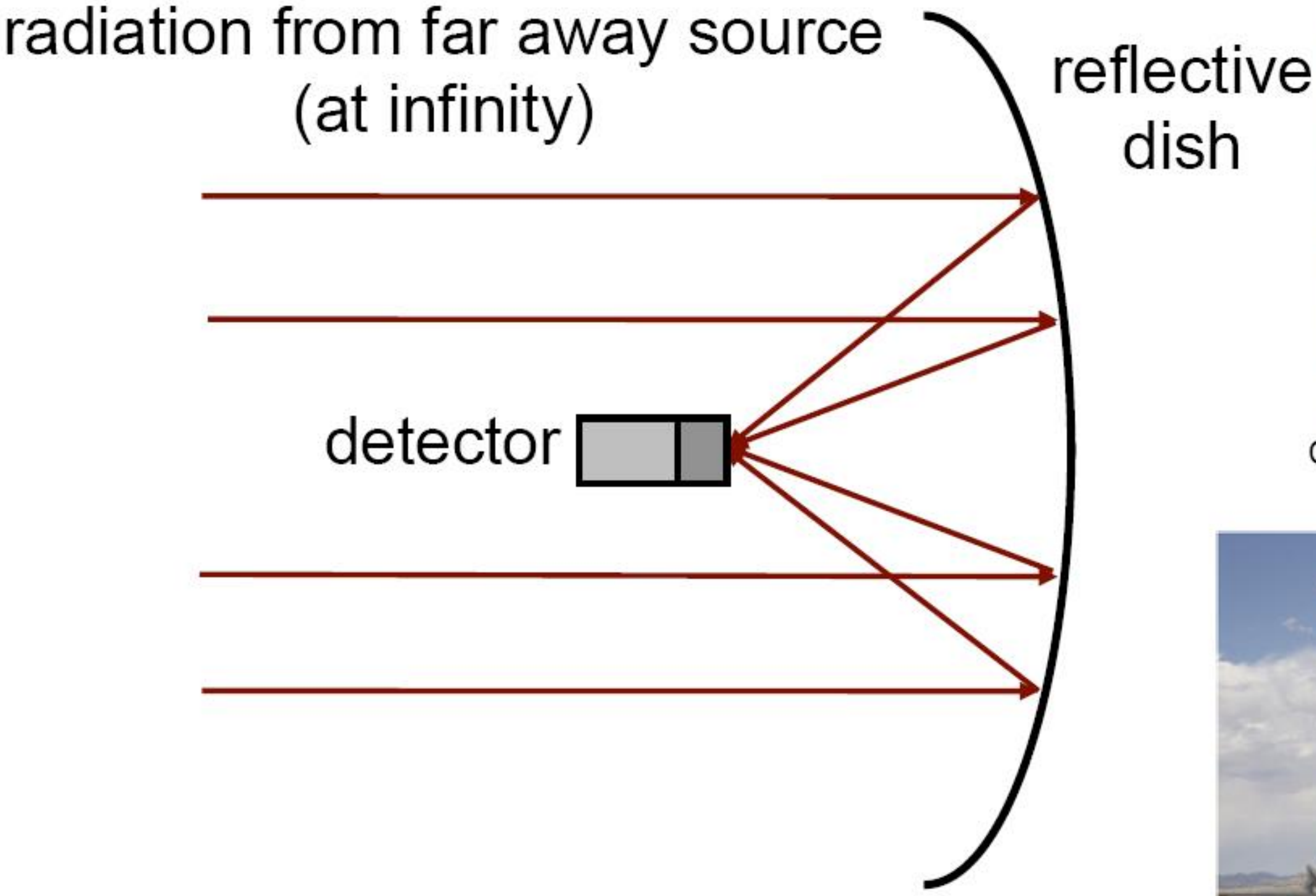
Today's Lecture

Two applications of Fermat's Principle to perfectly focus a plane wave to a point:

- Paraboloidal reflector
- Ellipsoidal refractor

- Spherical and plane waves
- Perfect focusing and collimation elements
  - paraboloid mirrors, ellipsoid and hyperboloid refractors
- Imperfect focusing: spherical elements
- The paraxial approximation
- Ray transfer matrices

Curved Reflecting Surfaces



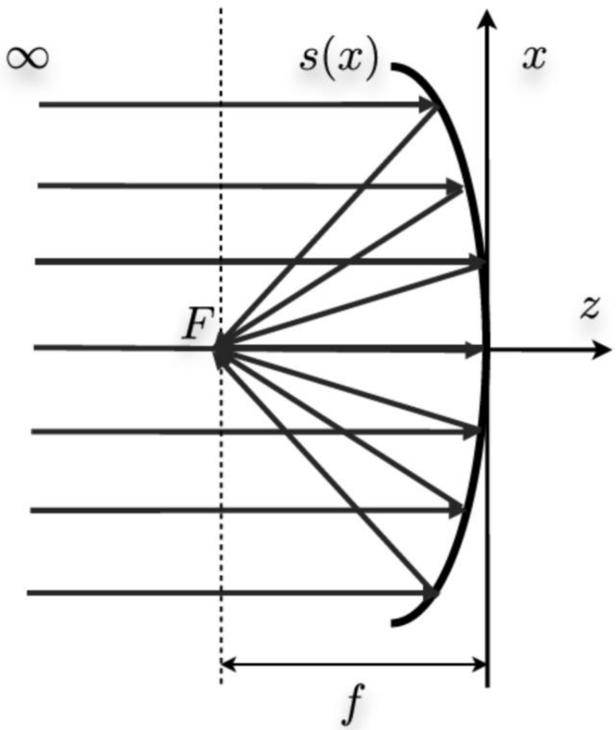
Courtesy of NASA/JPL-Caltech.

Image by hyperborea at Flickr.

Applications:

solar concentrators, satellite dishes, radio telescopes

Perfect focusing : Reflector Shape ???



focus at F

What should be the shape function *s(x)*, so that the incoming parallel ray bundle can meet at focus?

To find the answer to the above question we will invoke Fermat's principle:
The rays from infinity should follow the minimum path before they meet at F. It follows that they all must follow the same path.

$$2f = f - s + \sqrt{x^2 + (f-s)^2}$$

$$f + s = \sqrt{x^2 + (f-s)^2}$$

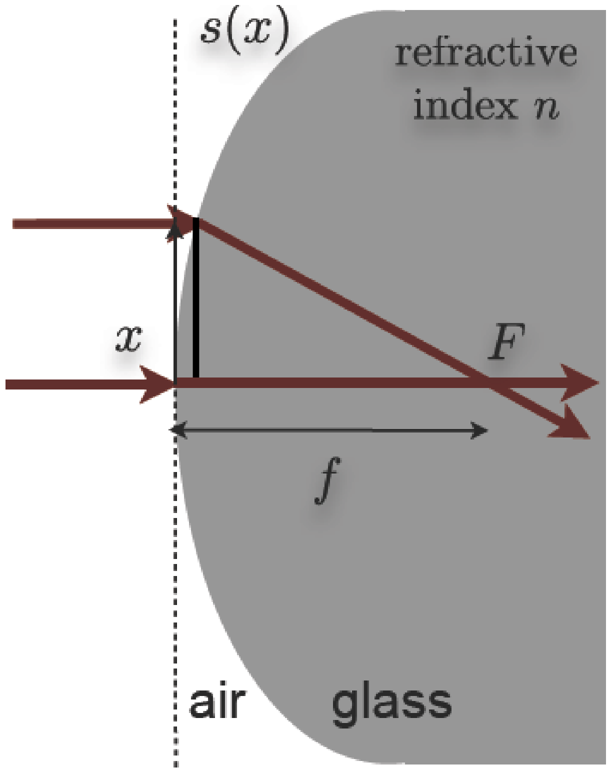
$$x^2 = (f+s)^2 - (f-s)^2$$

$$= 4sf$$

$$\boxed{s(x) = \frac{x^2}{4f}}$$

A paraboloid reflector focuses a normally incident plane wave to a point

Perfect focusing : Refractor Shape ????



What should the shape function *s(x)* be in order to focus the incoming parallel ray bundle at F?

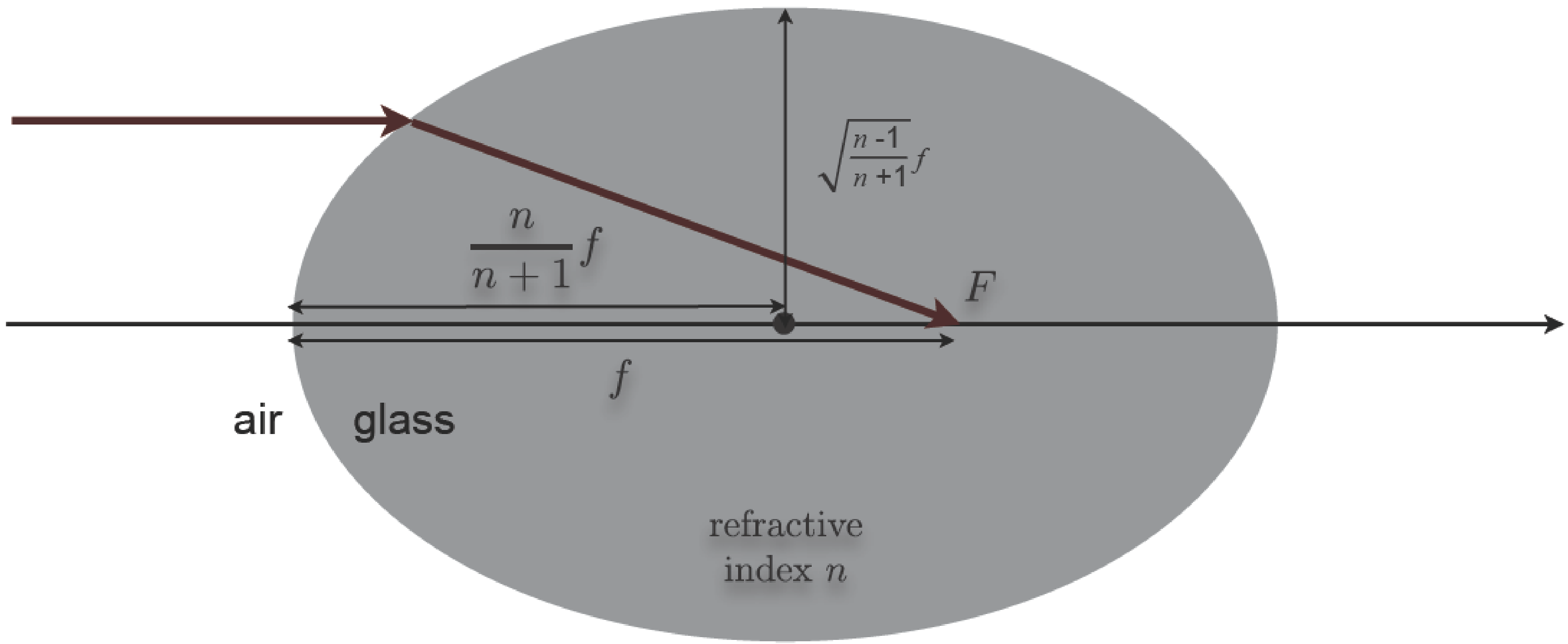
AGAIN → To find the answer w'll invoke Fermat's principle:
The rays from infinity should follow the minimum path before they meet at F. It follows that they all must follow the same path.

$$\mathrm{n}f = s + n\sqrt{x^2 + (f - s)^2}$$
$$\Rightarrow \cdots \Rightarrow \cdots \Rightarrow$$
$$(n^2-1)s^2 - 2n(n-1)fs + n^2x^2 = 0$$
$$\Rightarrow \cdots \Rightarrow \cdots \Rightarrow$$

$$\left(s - \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

A ellipsoidal refractor focuses a normally incident plane wave to a point

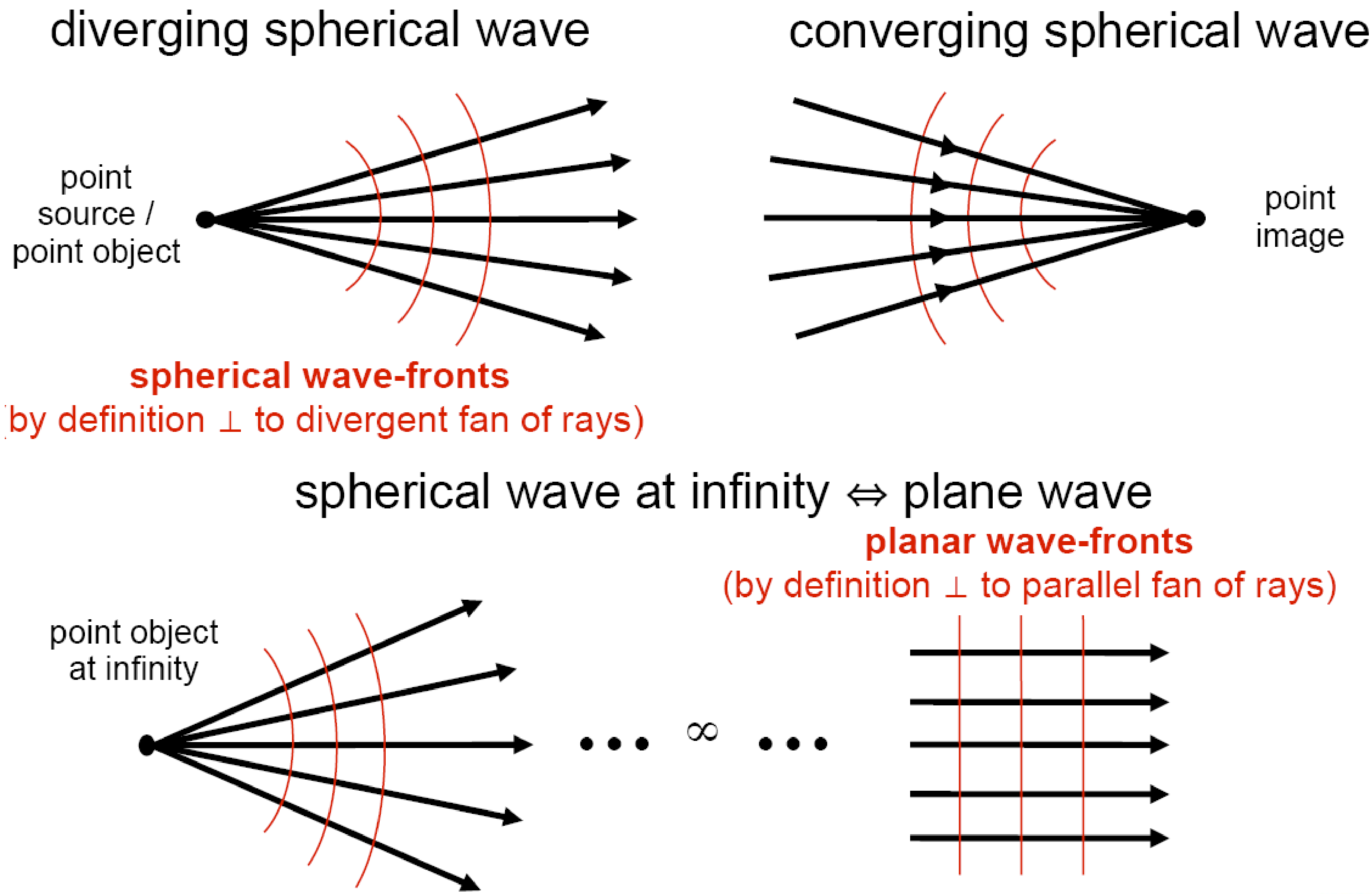
Ellipsoidal refractive concentrator



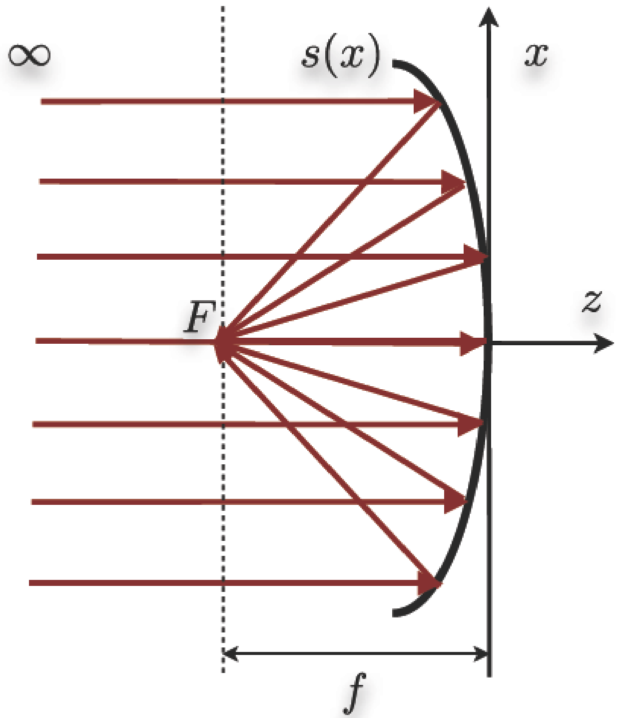
Surface Shape *s(x) is*

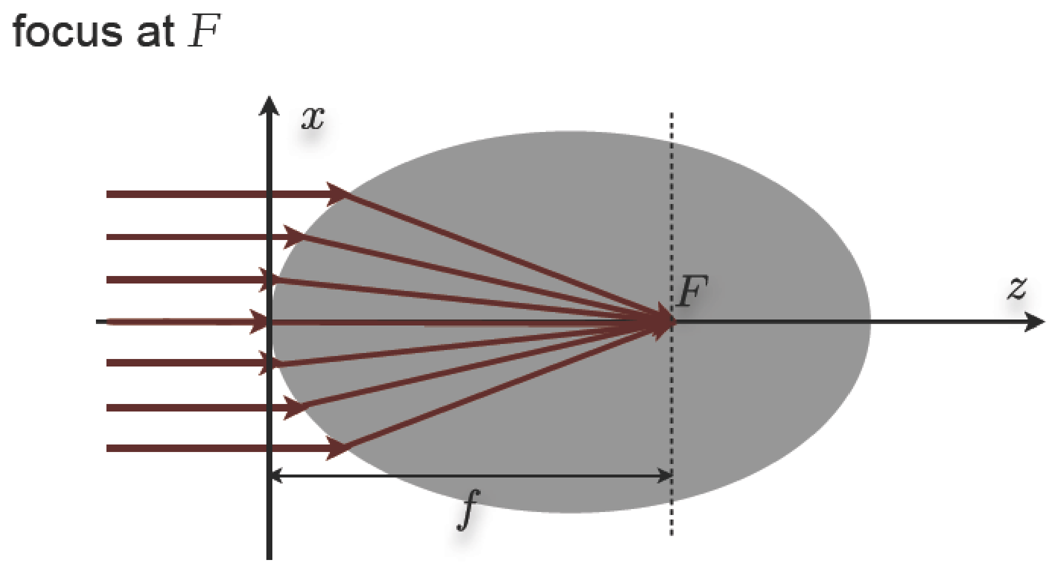
$$\left(s-\frac{n}{n+1}f\right)^2+\frac{n^2}{n^2-1}x^2=\left(\frac{n}{n+1}f\right)^2$$

Spherical and plane waves



Perfect imaging of a point source located at infinity





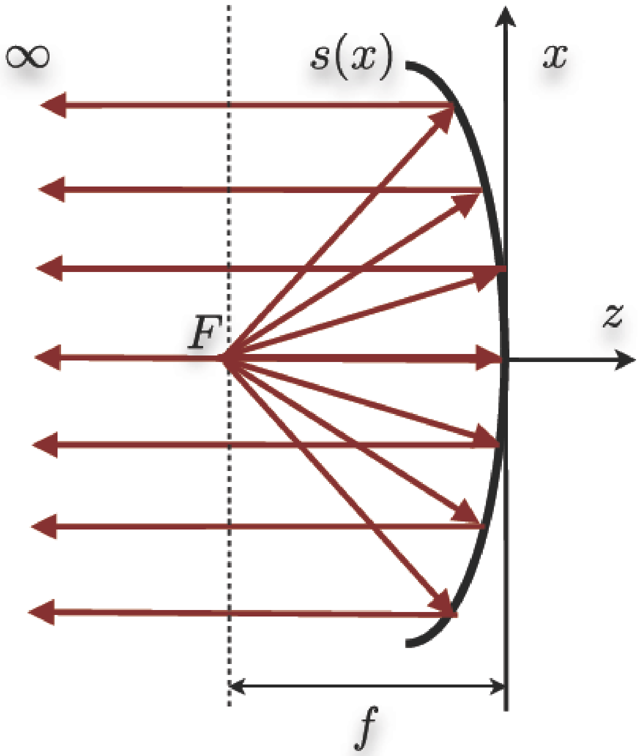
Paraboloid reflector

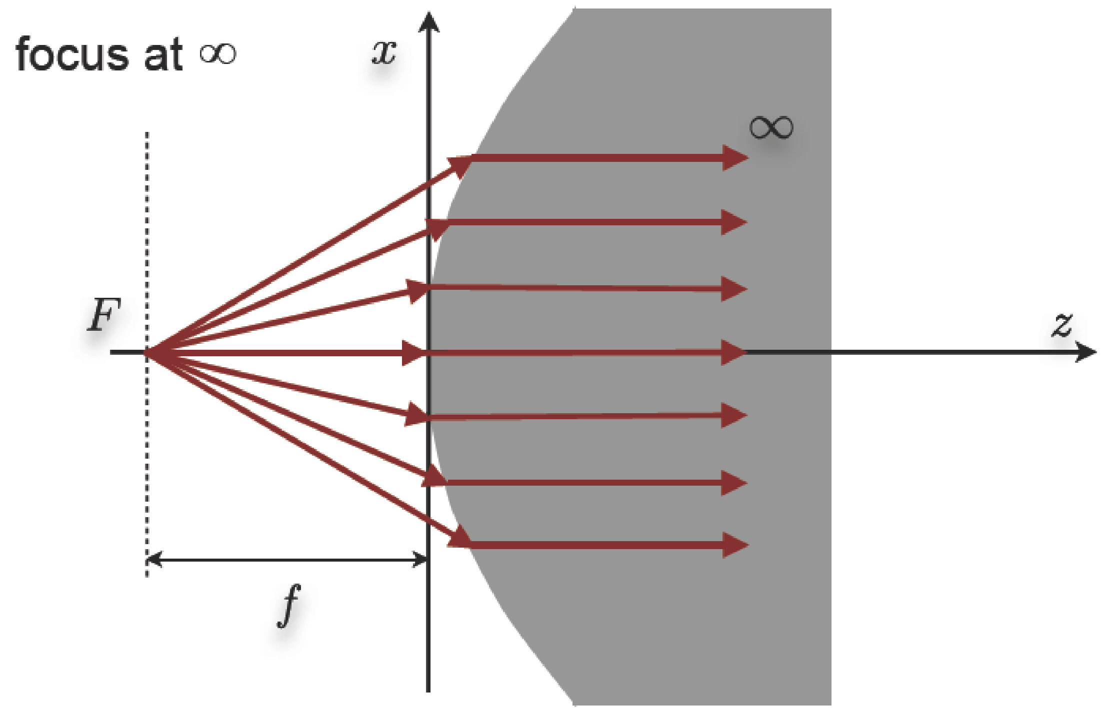
$$s(x) = \frac{x^2}{4f}$$

Ellipsoid refractor

$$\left(s - \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

Perfect imaging of a point source to infinity





Paraboloid reflector

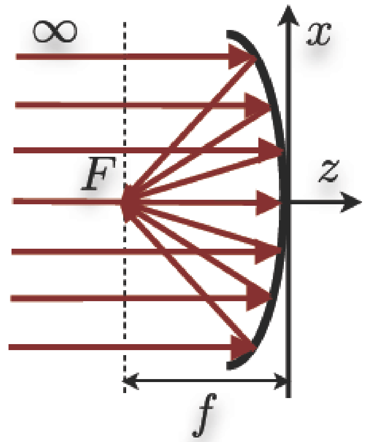
Hyperboloid refractor

$$s(x) = \frac{x^2}{4f}$$

$$\left(s + \frac{1}{n+1}f\right)^2 - \frac{1}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

Summary: Objects and images at infinity

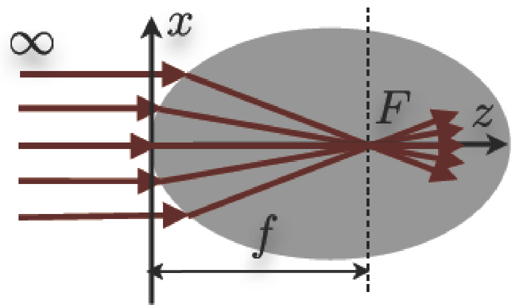
object at ∞
image at F



paraboloidal reflector

$$s = \frac{x^2}{4f}$$

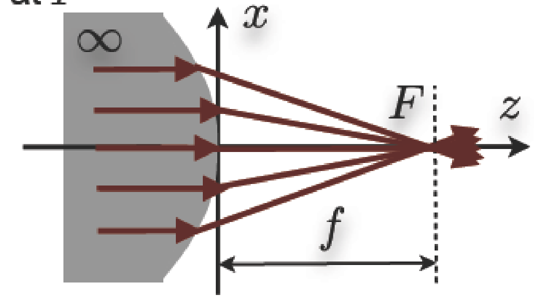
object at ∞
image at F



ellipsoidal refractor

$$\left(s - \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

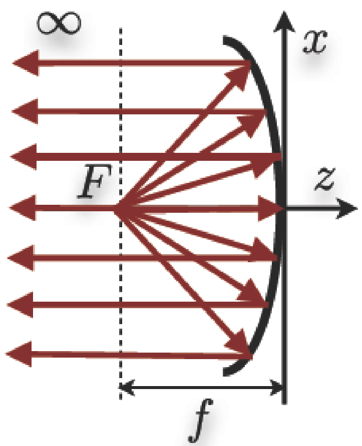
object at ∞
image at F



hyperboloidal refractor

$$\left(s - \frac{1}{n+1}f\right)^2 - \frac{1}{n^2-1}x^2 = \left(\frac{1}{n+1}f\right)^2$$

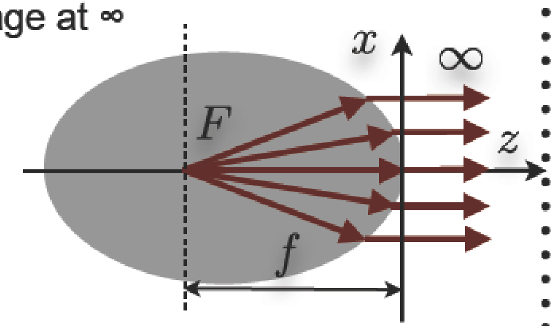
object at F
image at ∞



paraboloidal reflector

$$s = \frac{x^2}{4f}$$

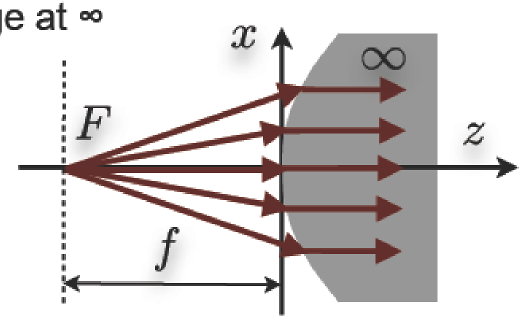
object at F
image at ∞



ellipsoidal refractor

$$\left(s + \frac{n}{n+1}f\right)^2 + \frac{n^2}{n^2-1}x^2 = \left(\frac{n}{n+1}f\right)^2$$

object at F
image at ∞

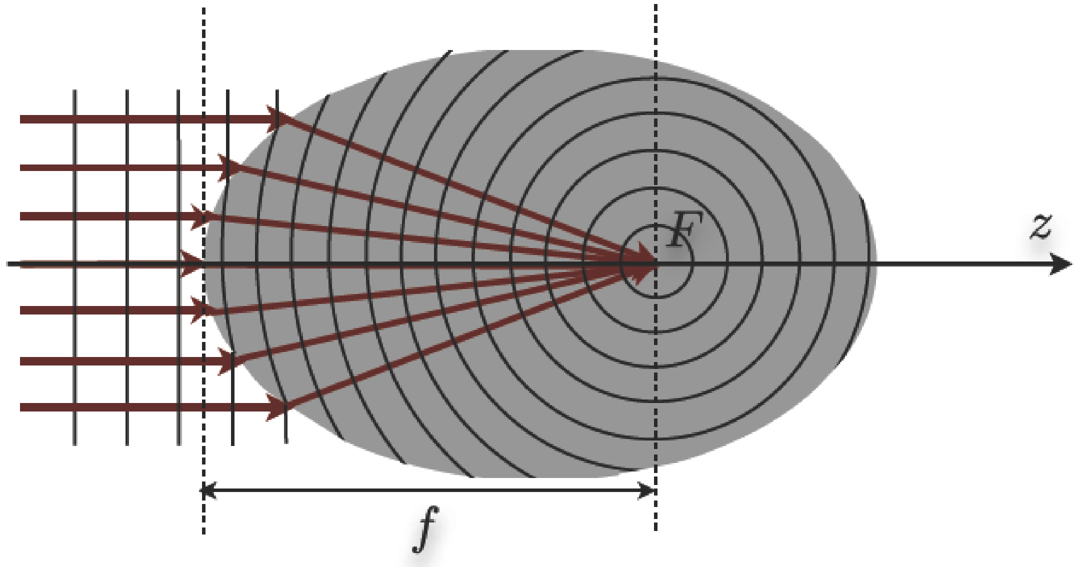


hyperboloidal refractor

$$\left(s + \frac{1}{n+1}f\right)^2 - \frac{1}{n^2-1}x^2 = \left(\frac{1}{n+1}f\right)^2$$

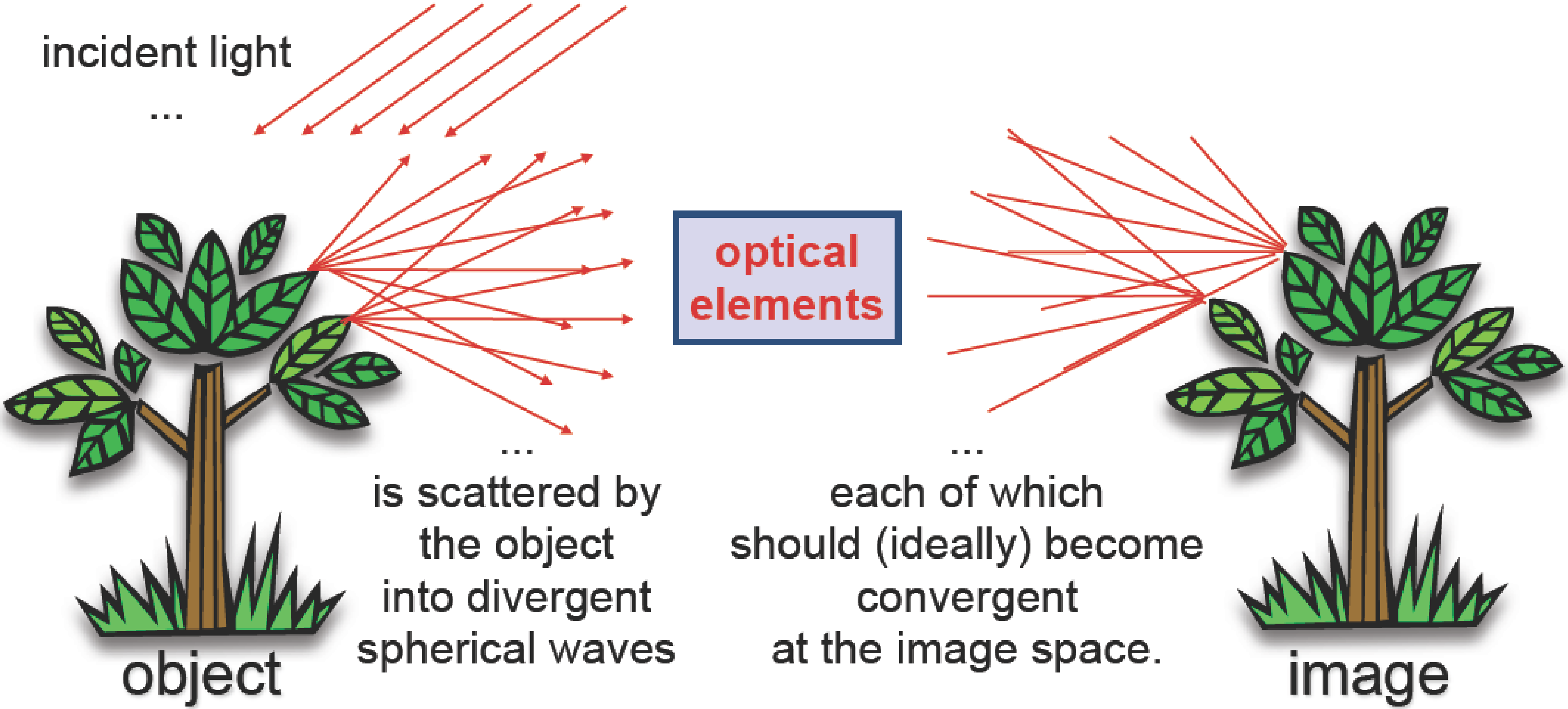
Assignment 2 : Derive all → Submit by 1st November 12:00 Hrs along with Assignment 1

Focusing: from planar to spherical wavefronts



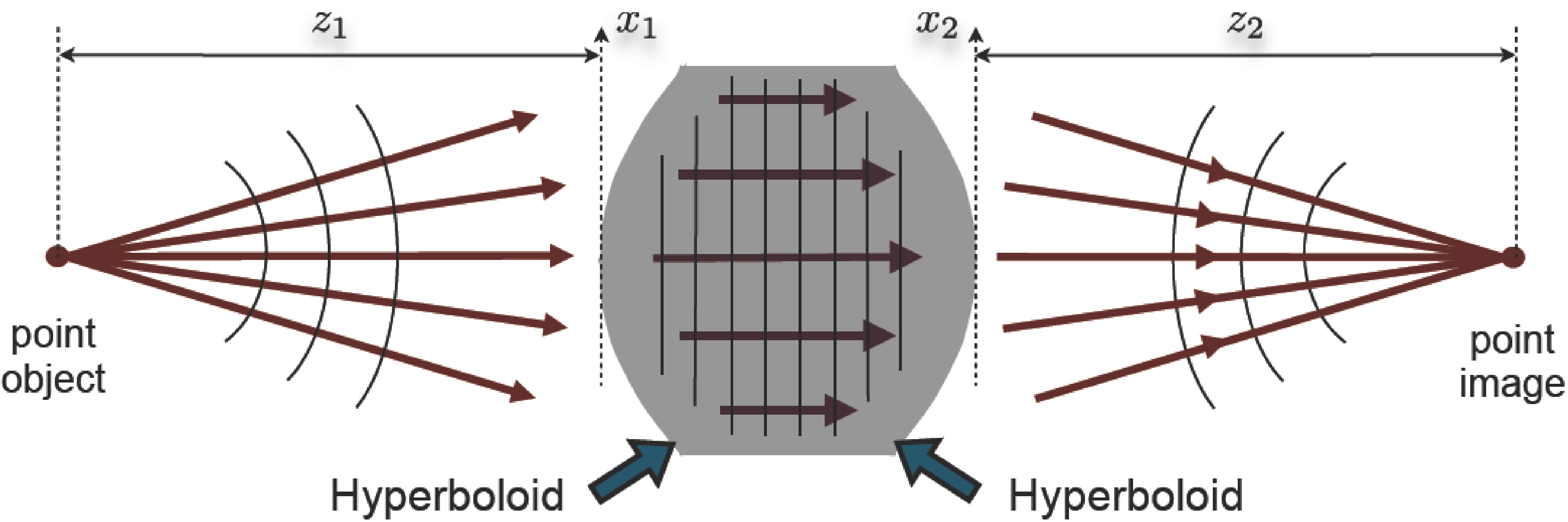
- The wavefronts are spaced by λ in air, by λ/n in the dielectric medium
- The wavefronts remain continuous at the interface
- Refraction at the curved interface causes the wavefronts to bend
- The elliptical shape of the refractive interface at on-axis incidence works out exactly so the planar wavefronts become spherical inside the dielectric medium therefore perfect focusing results
- Any shape other than elliptical or off-axis incidence would have resulted in a non-spherical wavefront, therefore imperfect focusing => such imperfectly focused wavefronts are called *aberrated.*

The need of "perfect imagers"



Perfect imaging on-axis

The purpose of the simplest imaging system is to convert a diverging spherical wave to a converging spherical wave, i.e to image a point object to a point image.



$$\left(s+\frac{z_1}{n+1}f\right)^2-\frac{x_1^2}{n^2-1}=\left(\frac{z_1}{n+1}\right)^2$$

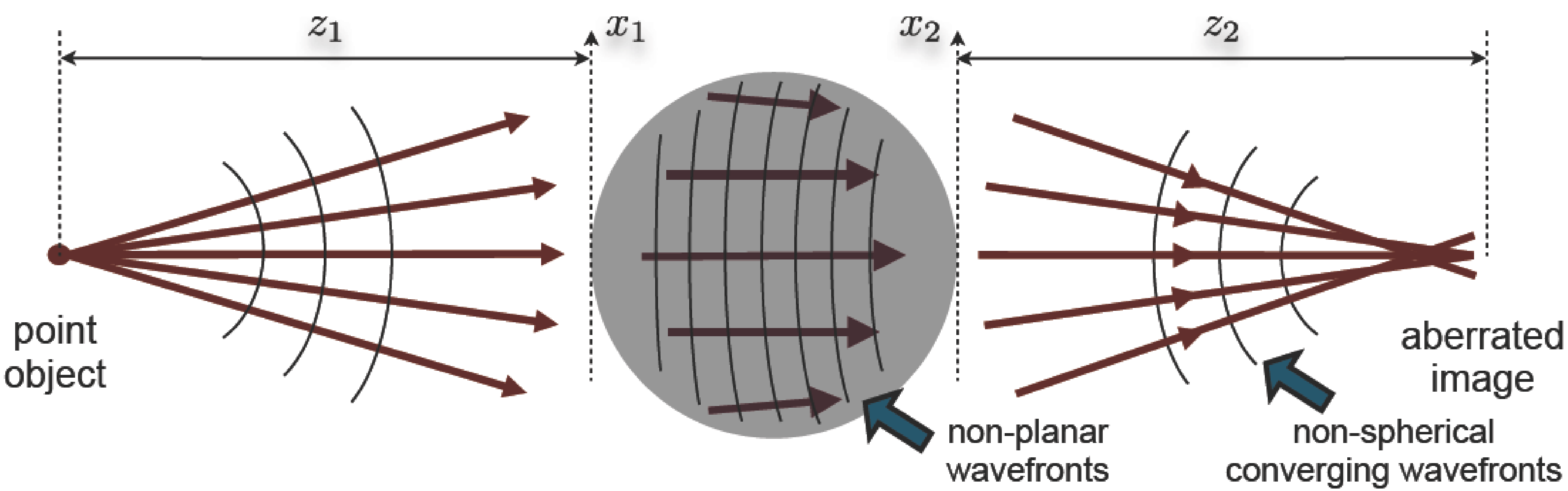
$$\left(s+\frac{z_2}{n+1}f\right)^2-\frac{x_2^2}{n^2-1}=\left(\frac{z_2}{n+1}\right)^2$$

The ideal imaging element is referred to as asphere (not a sphere) or as aspheric lens.

It works perfectly on axis and reasonably well in a limited range of angles.
Manufacturing constraints usually limit refractive elements to spherical surfaces.

Aberrated imaging with spherical elements

If asphere is replaced by a sphere, the refracted wavefront inside the sphere is not planar; Neither the refracted wavefront after the sphere is spherical. → NO perfect IMAGE

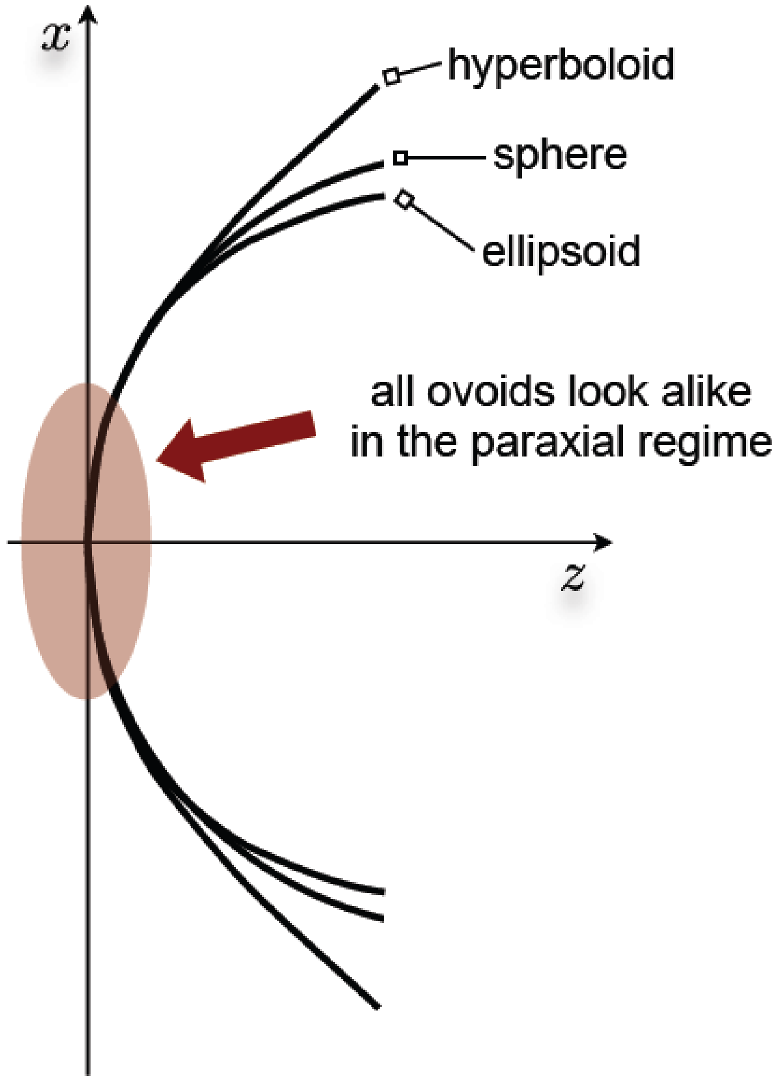


The image will be aberrated because the converging rays fail to focus at a single point and we call it **spherical aberration.**

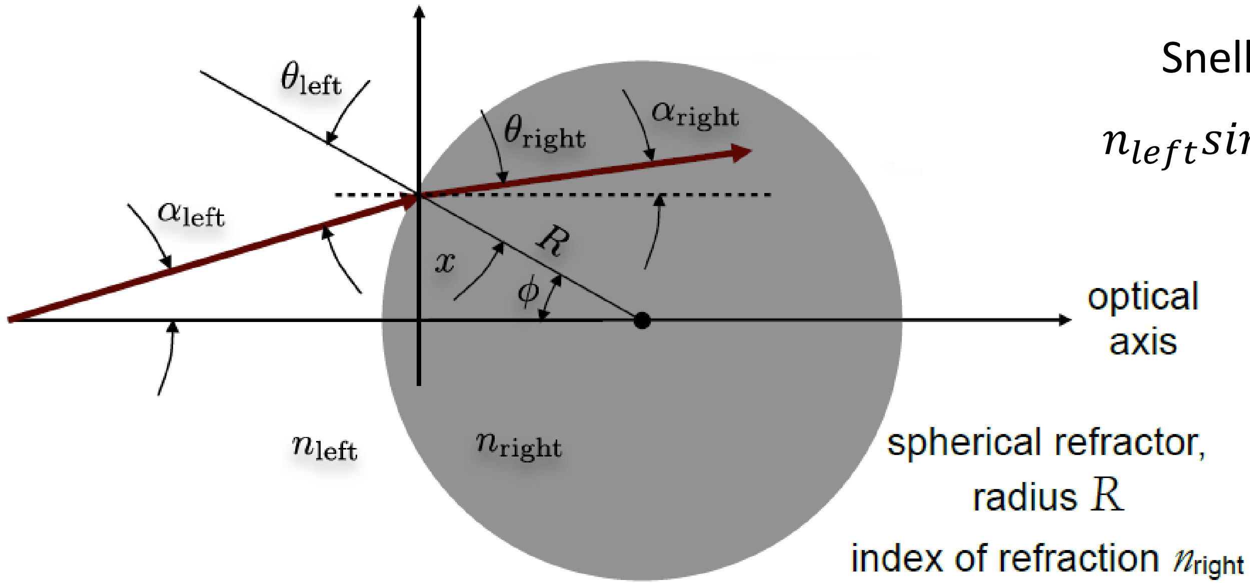
Paraxial Approximation

Each **optical system** will have an **optic axis**, and all light rays will be assumed to propagate at **small angles** to it. This is called the **Paraxial Approximation**.

Spheres, ellipsoids, hyperboloids and paraboloids in the paraxial approximation



Paraxial approximation: Refraction from a sphere



Snell's Law at the interface:

$$n_{left} sin\theta_{left} = n_{right} sin\theta_{right}$$

From the geometry: $\theta_{left} = \alpha_{left} + \emptyset, \theta_{right} = \alpha_{right} + \emptyset$

=> $n_{left}\left(sin\alpha_{left} cos\emptyset + cos\alpha_{left} sin\emptyset\right) = n_{right}(sin\alpha_{right} cos\emptyset + cos\alpha_{right} sin\emptyset)$

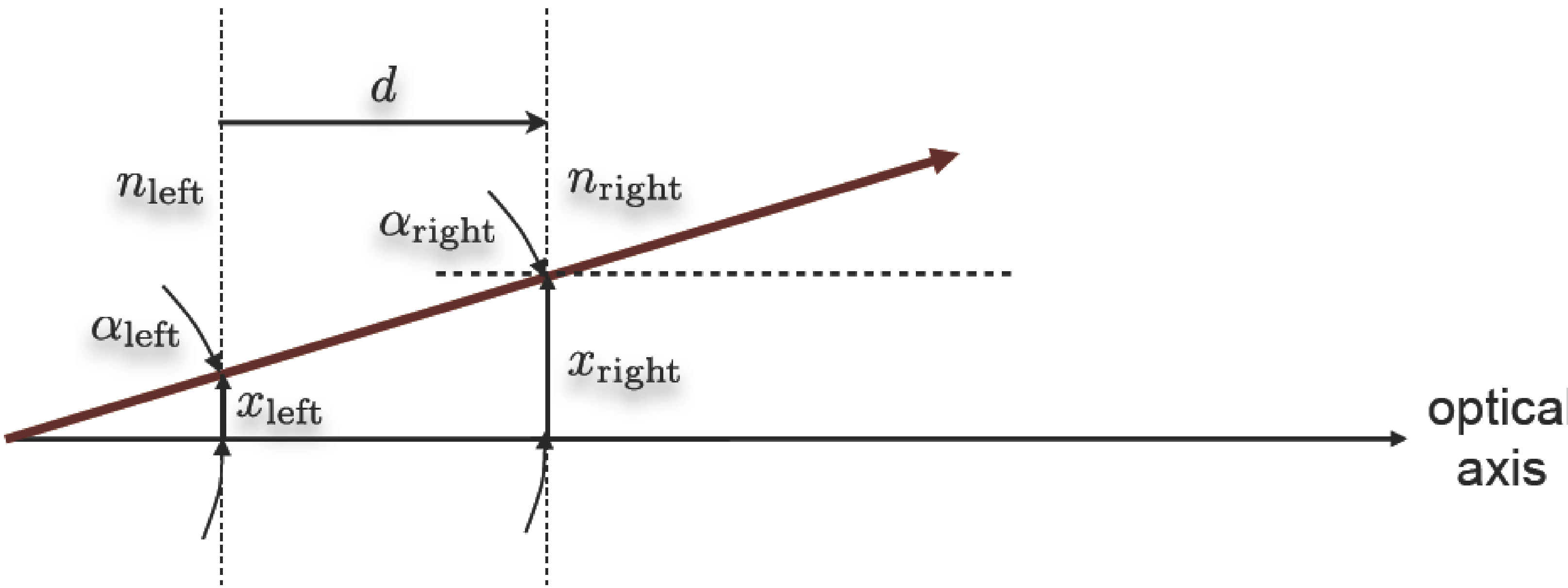
Assume: x<<R, $\alpha_{left} \ll 1, \alpha_{right} \ll 1$ <= This assumption is ***the Paraxial Approximation.***

$sin\alpha_{left} \approx \alpha_{left}$; $\cos\alpha_{left} \approx 1$ $\quad$ $sin\alpha_{right} \approx \alpha_{right}$; $\cos\alpha_{right} \approx 1$

$sin\emptyset = \frac{x}{R}$; $\cos\emptyset \approx 1$

$$n_{left}\left(\alpha_{left} + \frac{x}{R}\right) = n_{right}\left(\alpha_{right} + \frac{x}{R}\right) \Rightarrow n_{right}\alpha_{right} = n_{left}\alpha_{left} + \frac{n_{left} - n_{right}}{R} x$$

Free space propagation: paraxial approximation



Consider two points, separated by distance d, along a ray propagating in free space of uniform index of refraction $n_{left} = n_{right} = n$

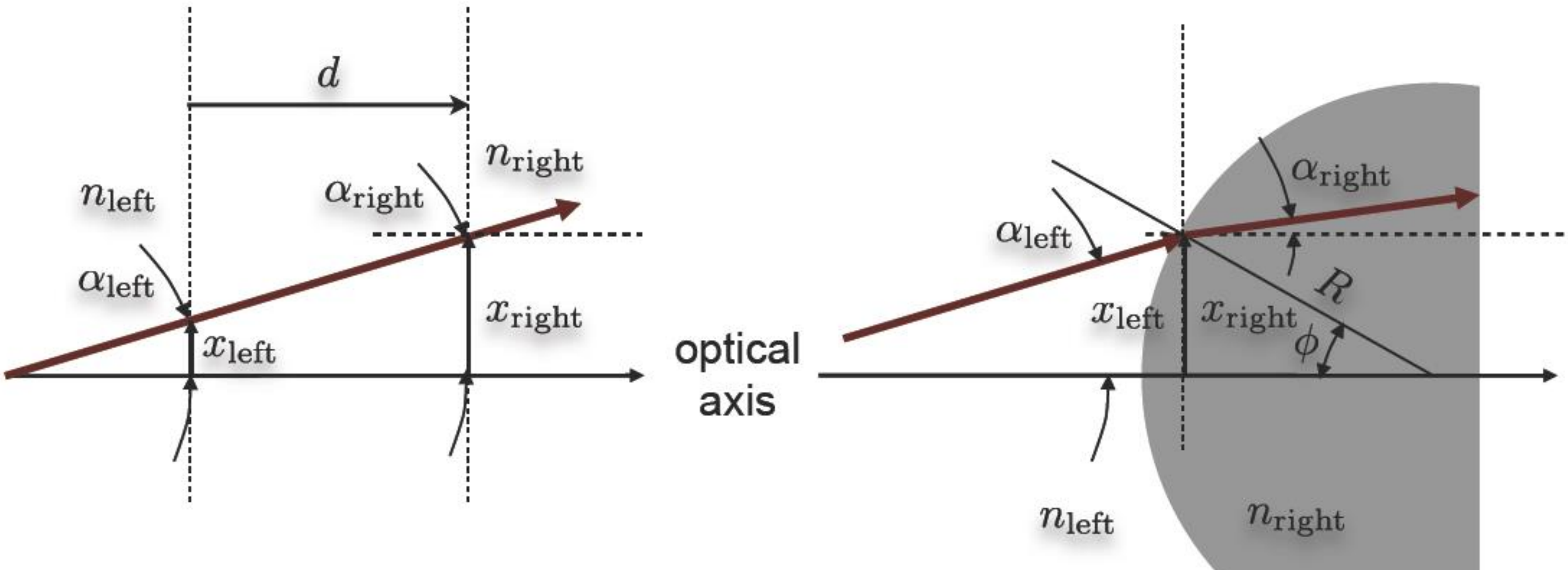
$$n_{left}\alpha_{left} = n_{right}\alpha_{right}$$

From the geometry we find

$$x_{right} = x_{left} + d tan\alpha_{left} \approx x_{left} + d\alpha_{left}$$

Since $tan\alpha_{left} \approx \alpha_{left}$ in the paraxial approximation $\alpha_{left} \ll 1$

Ray Transfer Matrices



$$n_{\text{right}}\alpha_{\text{right}} = n_{\text{left}}\alpha_{\text{left}}$$

$$x_{\text{right}} = x_{\text{left}} + d\alpha_{\text{left}}$$

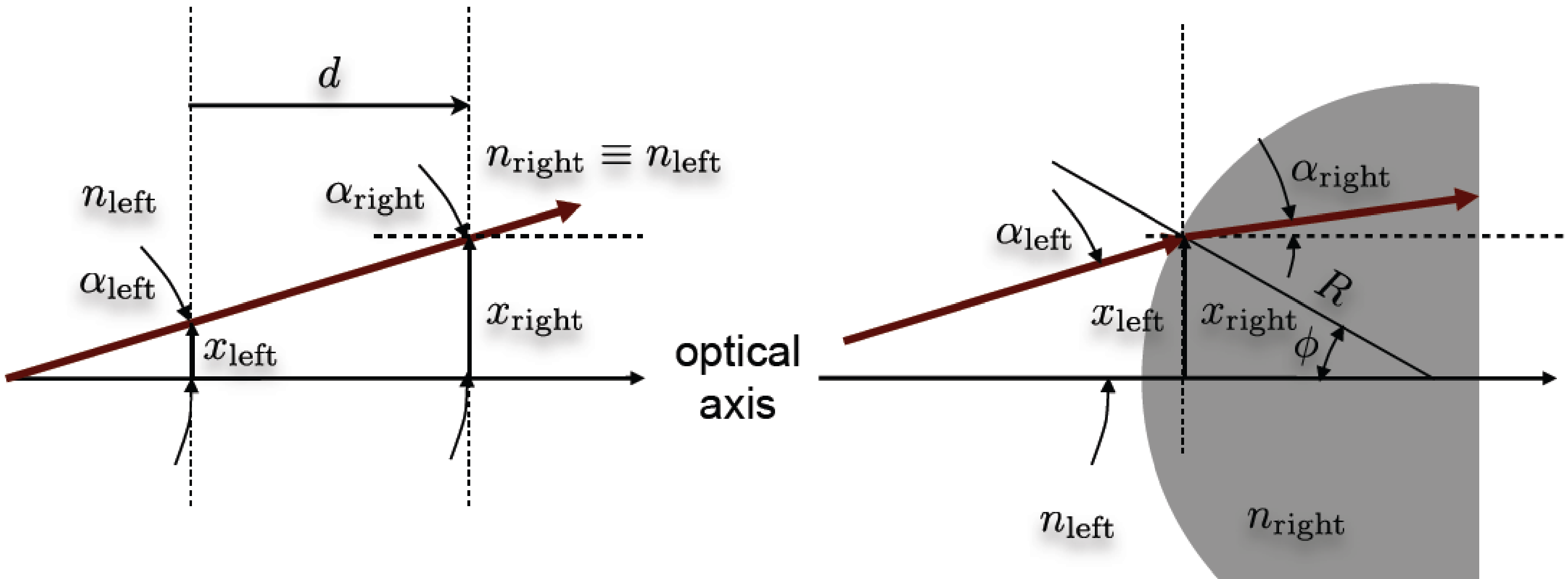
$$n_{\text{right}}\alpha_{\text{right}} = n_{\text{left}}\alpha_{\text{left}} + \frac{n_{\text{left}} - n_{\text{right}}}{R} x_{\text{left}}$$

$$x_{\text{right}} = x_{\text{left}}$$

or, in matrix form:

$$\begin{pmatrix} n_{\text{right}}\alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{d}{n_{\text{left}}} & 1 \end{pmatrix} \begin{pmatrix} n_{\text{left}}\alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix} \qquad \begin{pmatrix} n_{\text{right}}\alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n_{\text{right}} - n_{\text{left}}}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_{\text{left}}\alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix}$$

Ray Transfer Matrices

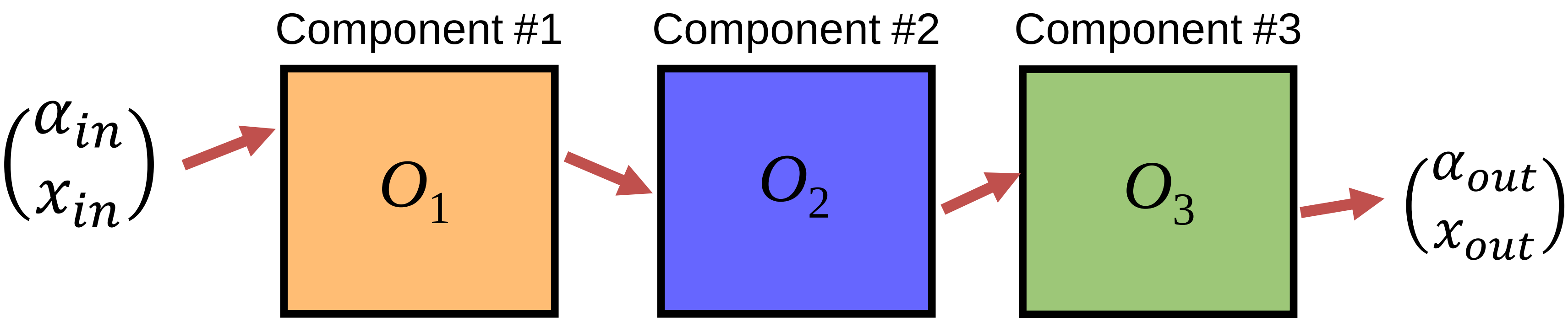


$$\begin{pmatrix} n_{\text{right}}\alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{d}{n_{\text{left}}} & 1 \end{pmatrix} \begin{pmatrix} n_{\text{left}}\alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix} \qquad \begin{pmatrix} n_{\text{right}}\alpha_{\text{right}} \\ x_{\text{right}} \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n_{\text{right}} - n_{\text{left}}}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_{\text{left}}\alpha_{\text{left}} \\ x_{\text{left}} \end{pmatrix}$$

Propagation through uniform space: distance d, index of refraction n_{left}

Refraction at spherical interface: radius R, indices n_{left}, n_{right}

By using the elemental matrices, we may ray trace through an arbitrarily long cascade of optical elements provided the paraxial approximation remains valid throughout.



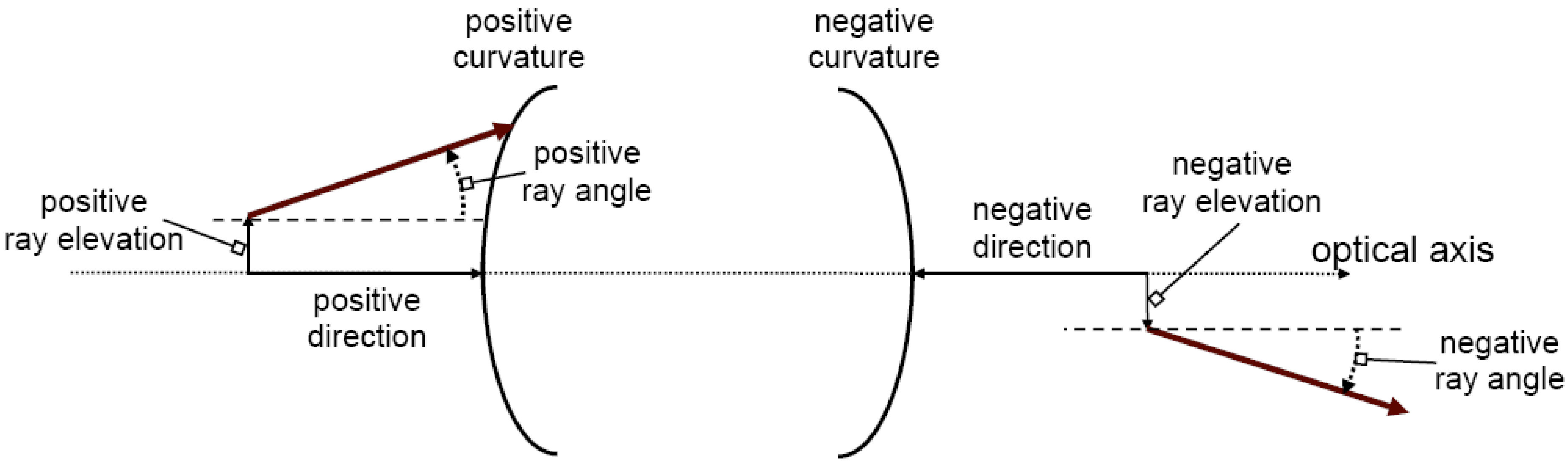
$$\begin{pmatrix}\alpha_{out}\\ x_{out}\end{pmatrix} = O_3\left\{O_2\left(O_1\begin{pmatrix}\alpha_{in}\\ x_{in}\end{pmatrix}\right)\right\} = O_3 O_2 O_1 \begin{pmatrix}\alpha_{in}\\ x_{in}\end{pmatrix}$$

By using the elemental matrices, we may ray trace through an arbitrarily long cascade of optical elements provided the paraxial approximation remains valid throughout.

Assignment 3: Write code in MATLAB and get the output matrix. Input angle, elevation and refractive indices are known and also the radius of curvature.

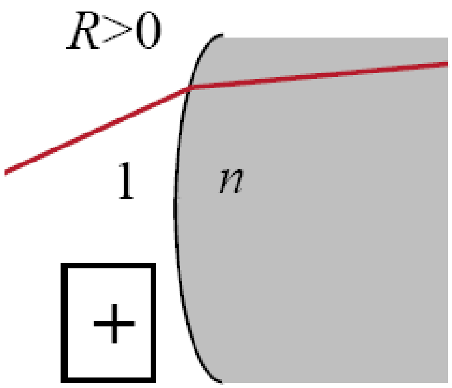
Sign conventions

- Light travels from left to right.
- A radius of curvature is positive if the surface is convex towards the left.
- Longitudinal distances are positive if pointing to the right.
- Lateral distances are positive if pointing up.
- Ray angles are positive if the ray direction is obtained by rotating the +z axis counterclockwise through an acute angle.



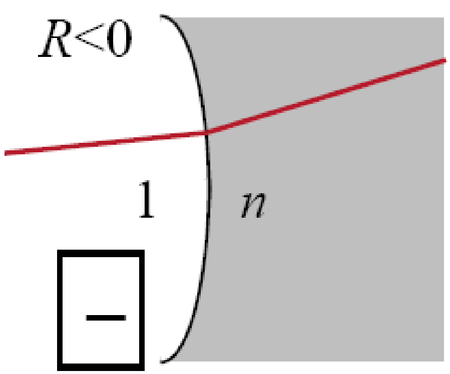
Types of refraction from spherical surfaces

Positive power bends rays "inwards"



Positive power

Negative power bends rays "outwards"



Negative power

AGAIN

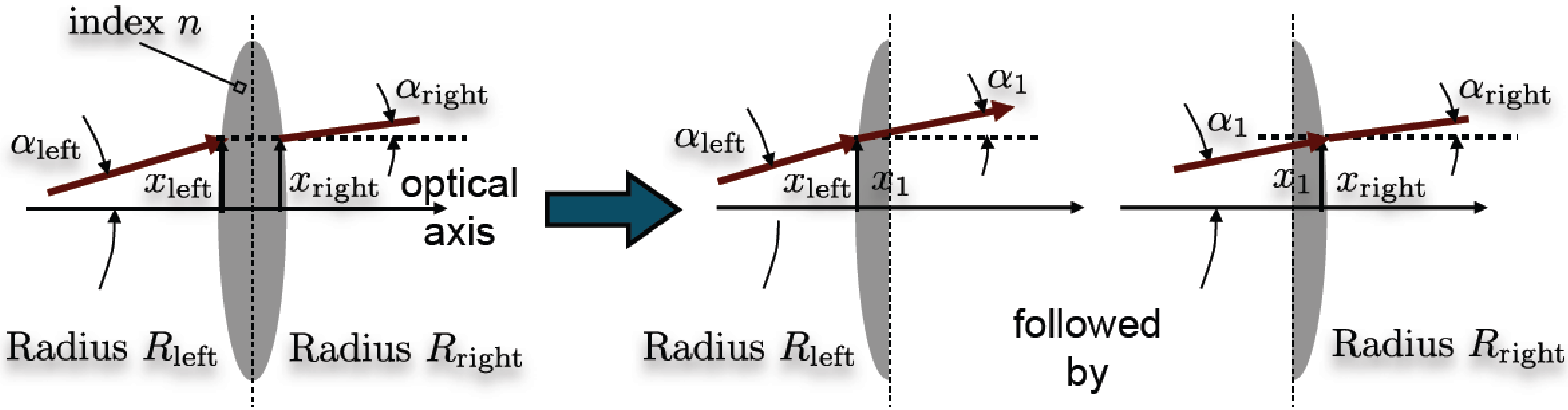
Free space propagation

$$\begin{pmatrix} n_{right}\alpha_{right} \\ x_{right} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dfrac{d}{n_{left}} & 1 \end{pmatrix} \begin{pmatrix} n_{left}\alpha_{left} \\ x_{left} \end{pmatrix}$$

Refraction in curved medium

$$\begin{pmatrix} n_{right}\alpha_{right} \\ x_{right} \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n_{right} - n_{left}}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_{left}\alpha_{left} \\ x_{left} \end{pmatrix}$$

Example: Thin lens in Air



$$\begin{pmatrix} n_{right}\alpha_{right} \\ x_{right} \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n_{right} - n_{left}}{R} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n_{left}\alpha_{left} \\ x_{left} \end{pmatrix}$$

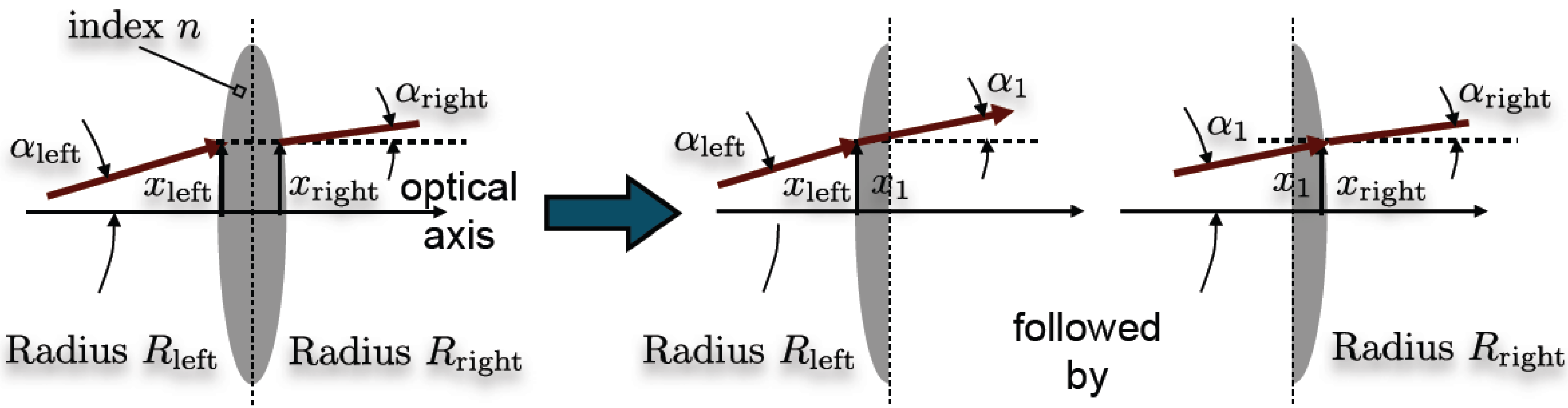
$$\begin{pmatrix} \alpha_{right} \\ x_{right} \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{1-n}{R_{right}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix}$$

$$\begin{pmatrix} n\alpha_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{n-1}{R_{left}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n\alpha_{left} \\ x_{left} \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{right} \\ x_{right} \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{1-n}{R_{right}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\dfrac{n-1}{R_{left}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} n\alpha_{left} \\ x_{left} \end{pmatrix}$$

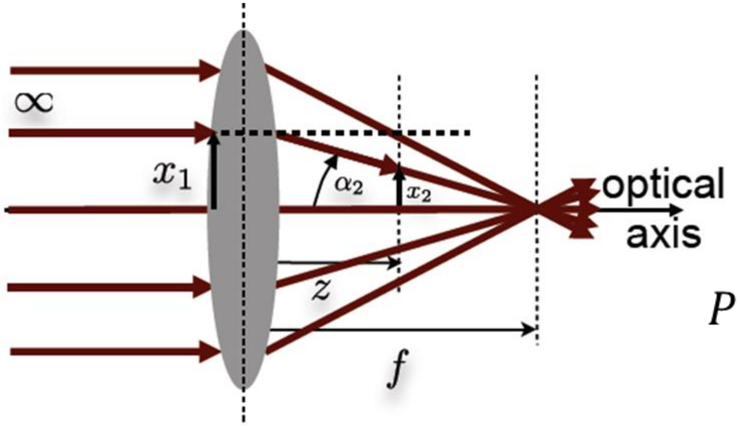
$$= \begin{pmatrix} 1 & -(n-1)\left(\dfrac{1}{R_{left}} - \dfrac{1}{R_{right}}\right) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{left} \\ x_{left} \end{pmatrix}$$

Example: Thin lens in Air



$$\begin{pmatrix} \alpha_{right} \\ x_{right} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{left} \\ x_{left} \end{pmatrix}$$

$P = \frac{1}{f} = (n-1)(\frac{1}{R_{left}} - \frac{1}{R_{right}})$ Lens Maker's Equation



$$\begin{pmatrix}\alpha_{right}\\ x_{right}\end{pmatrix} = \begin{pmatrix}1 & -\frac{1}{f}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{left}\\ x_{left}\end{pmatrix}$$

$P = \frac{1}{f} = (n-1)(\frac{1}{R_{left}} - \frac{1}{R_{right}})$ Lens Maker's Equation

If a ray arrives from infinity at an angle $\alpha_1 = 0$ and at elevation x_1. The ray is refracted and propagates further a distance z to the right of the lens.
The aim is to find its elevation x_2 and angle of propagation α_2 as a function of z.

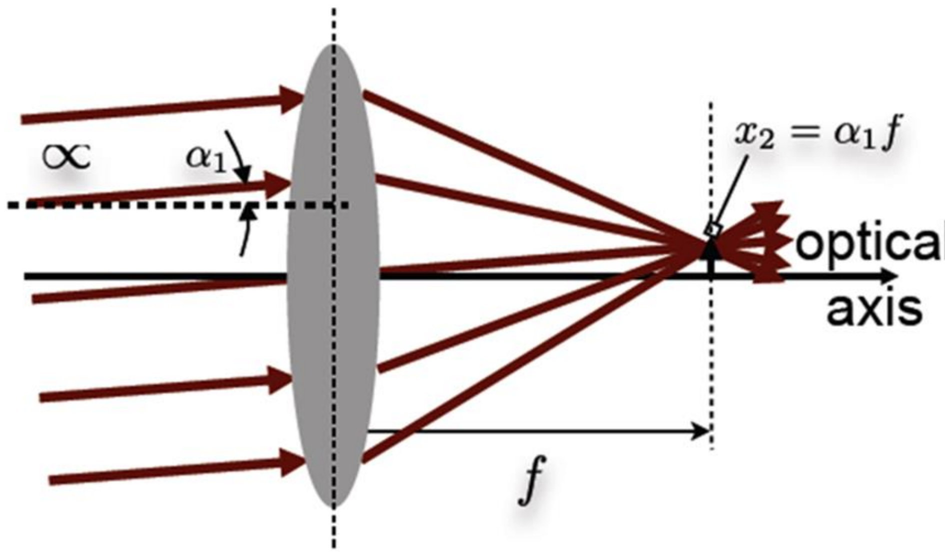
$$\begin{pmatrix}\alpha_2\\ x_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\begin{pmatrix}1 & -\frac{1}{f}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1\\ x_1\end{pmatrix} = \begin{pmatrix}1 & -\frac{1}{f}\\ z & 1-\frac{z}{f}\end{pmatrix}\begin{pmatrix}\alpha_1\\ x_1\end{pmatrix}$$

$$x_2 = \alpha_1 z + x_1\left(1-\frac{z}{f}\right) = x_1\left(1-\frac{z}{f}\right)$$, (since α_1=0)

At z=f => x_2 = 0 for all x_1. <= All the rays from infinity converge to the optical axis independent of the elevation x_1 at arrival.

$P = \frac{1}{f}$ is the lens power measured in Diopters (m^{-1})

Image of the off-axis source at infinity



$$x_2 = \alpha_1 z + x_1 \left(1 - \frac{z}{f}\right)$$

At $\alpha_1 \neq 0$, the rays still meet at focus (z = f) at the right of the lens at an elevation $x_2 = \alpha_1 f$.