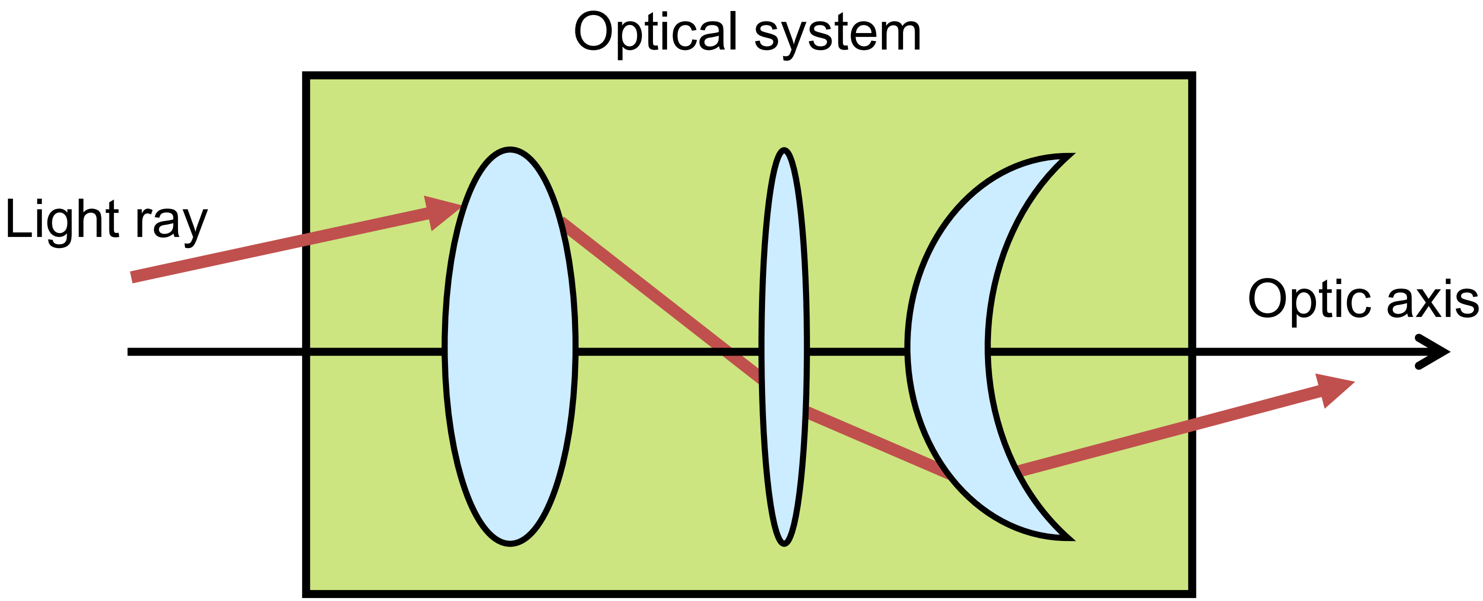
# Photonics: Geometrical Optics-2

Vandana Sharma

# Ray Transfer Matrix




We define **light rays** as directions in space, corresponding, roughly, to k-vectors of light waves.

Each **optical system** will have an **optic axis**, and all light rays will be assumed to propagate at **small angles** to it. This is called the **Paraxial Approximation**.

# Ray Matrices as Derivatives

$$\alpha_{out} = \frac{\partial \alpha_{out}}{\partial \alpha_{in}} \alpha_{in} + \frac{\partial \alpha_{out}}{\partial x_{in}} x_{in}$$

$$x_{out} = \frac{\partial x_{out}}{\partial \alpha_{in}} \alpha_{in} + \frac{\partial x_{out}}{\partial x_{in}} x_{in}$$

Since the displacements, $x_{in}$ and $x_{out}$, and angles, $\alpha_{in}$ and $\alpha_{out}$, are all assumed to be small, we can think in terms of partial derivatives.

**Angular magnification** $\frac{\partial \alpha_{out}}{\partial \alpha_{in}}$

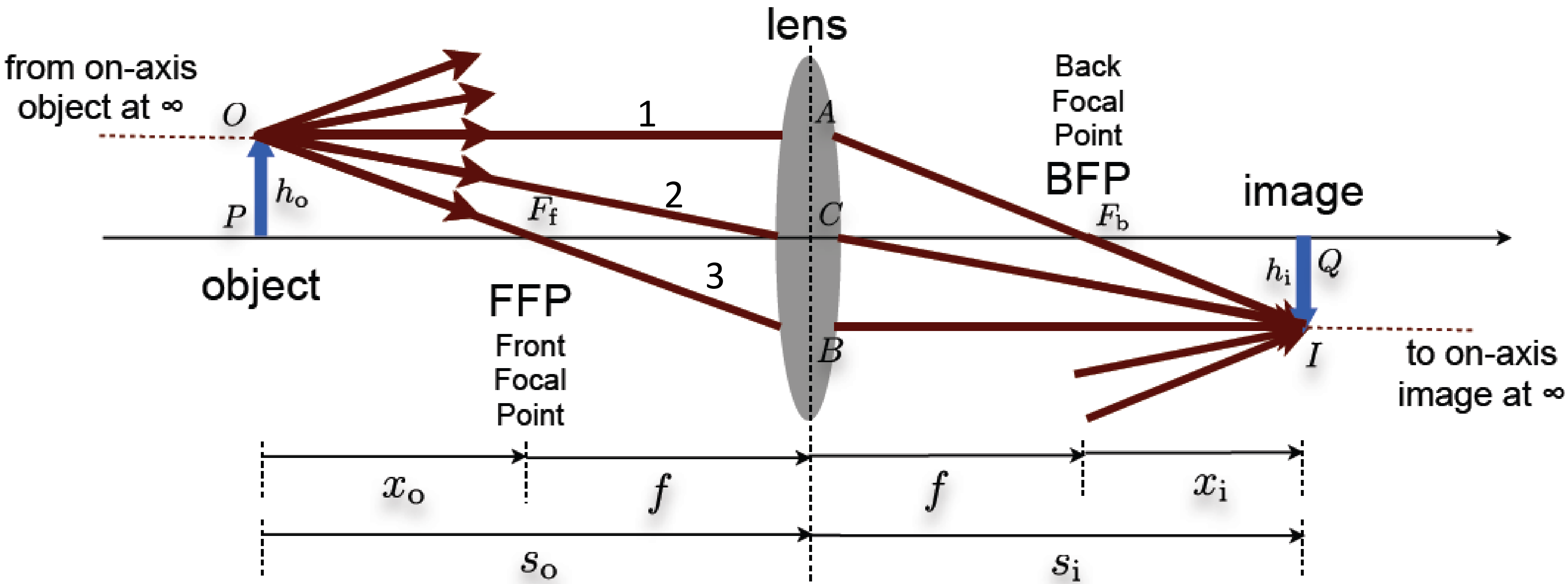
$\frac{\partial \alpha_{out}}{\partial x_{in}}$

$$\begin{bmatrix} \alpha_{out} \\ x_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \alpha_{in} \\ x_{in} \end{bmatrix}$$

$\frac{\partial x_{out}}{\partial \alpha_{in}}$

$\frac{\partial x_{out}}{\partial x_{in}}$ **Spatial magnification**

# Image formation at finite distances




$$O\hat{P}F_{\mathrm{f}} \sim B\hat{C}F_{\mathrm{f}} \Rightarrow \frac{(PO)}{(PF_{\mathrm{f}})} = \frac{(CB)}{(F_{\mathrm{f}}C)} \Rightarrow \frac{h_{\mathrm{o}}}{x_{\mathrm{o}}} = \frac{-h_{\mathrm{i}}}{f} \qquad I\hat{Q}F_{\mathrm{b}} \sim A\hat{C}F_{\mathrm{b}} \Rightarrow \frac{(QI)}{(F_{\mathrm{b}}Q)} = \frac{(CA)}{(CF_{\mathrm{f}})} \Rightarrow \frac{-h_{\mathrm{i}}}{x_{\mathrm{i}}} = \frac{h_{\mathrm{o}}}{f}$$

$$\Rightarrow \boxed{\frac{h_{\mathrm{i}}}{h_{\mathrm{o}}} = -\frac{f}{x_{\mathrm{o}}} = -\frac{x_{\mathrm{i}}}{f}} \Rightarrow \boxed{x_{\mathrm{o}}x_{\mathrm{i}} = f^2}$$

(Newton's form)

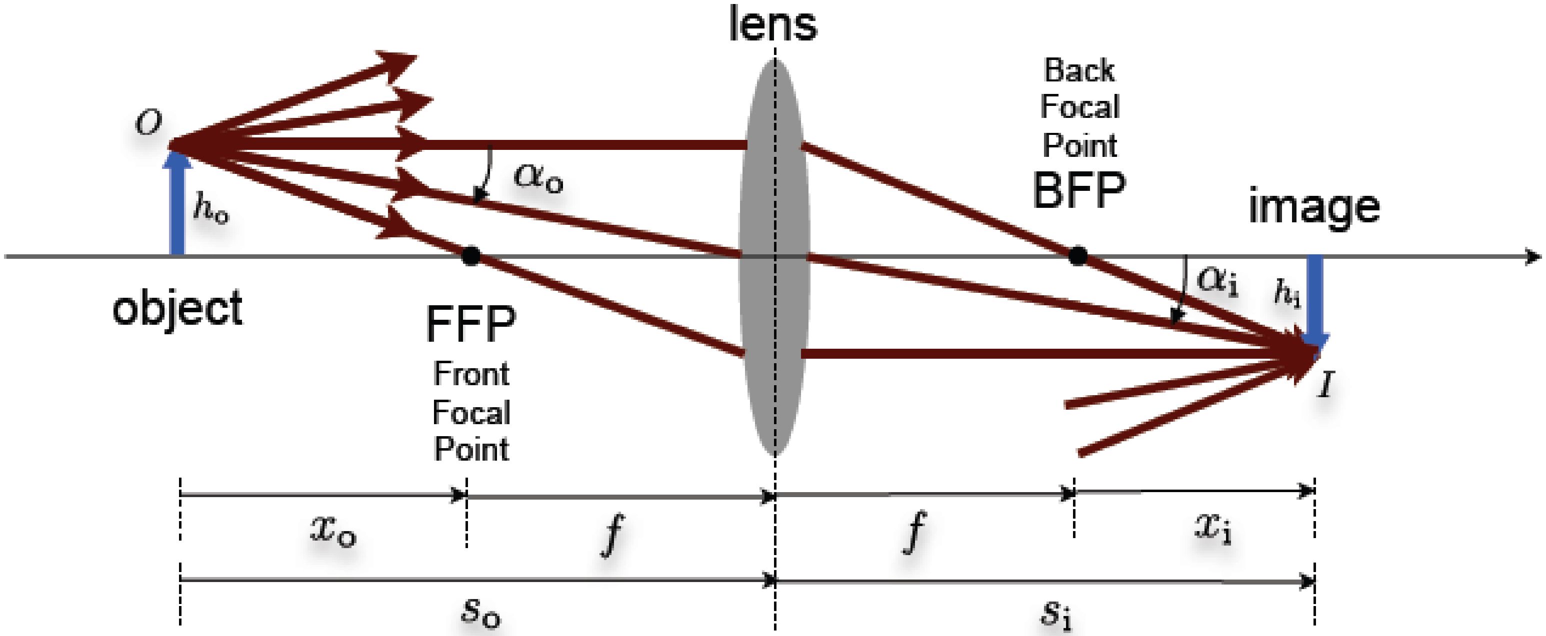
$$M_{\mathrm{T}} \equiv \frac{h_{\mathrm{i}}}{h_{\mathrm{o}}}$$

Lateral magnification

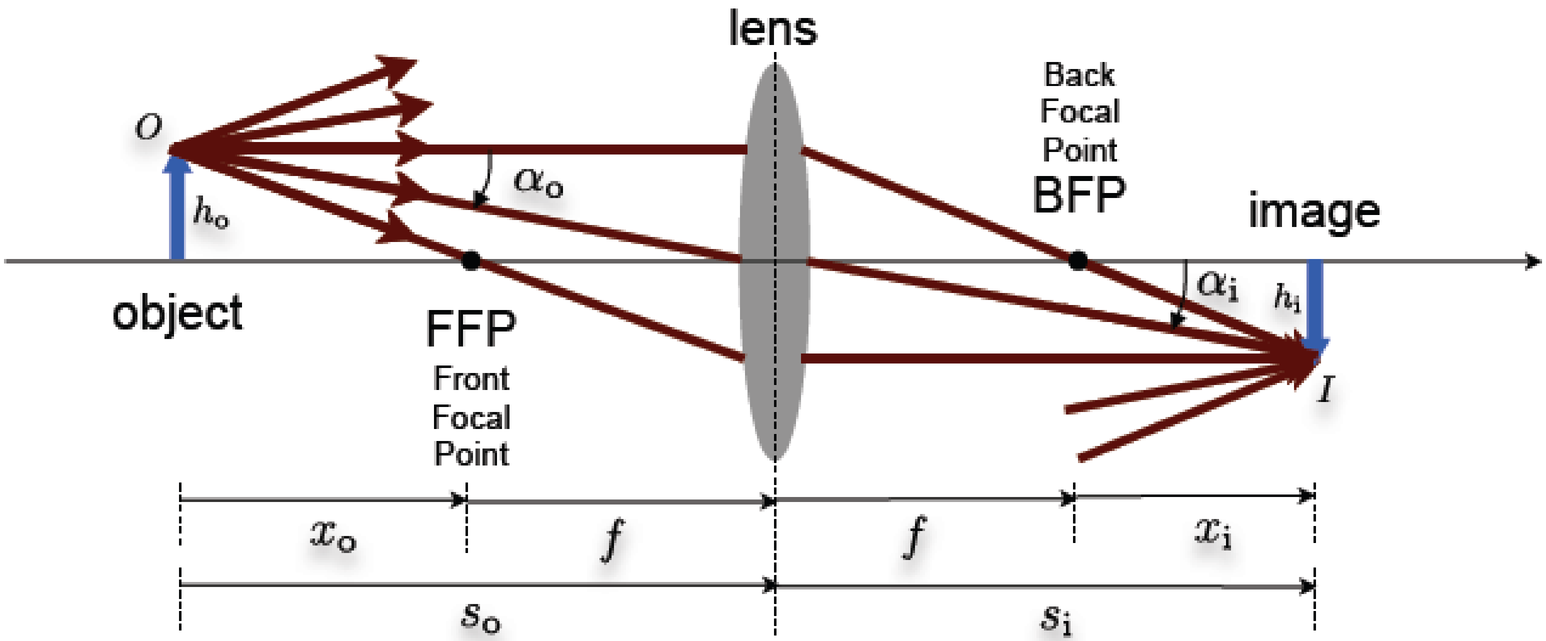
Imaging condition

$$O\hat{A}C \sim I\hat{B}C \Rightarrow \frac{(CA)}{(CB)} = \frac{(OA)}{(CQ)} \Rightarrow \frac{h_{\mathrm{o}}}{-h_{\mathrm{i}}} = \frac{s_{\mathrm{o}}}{s_{\mathrm{i}}} \Rightarrow \boxed{\frac{h_{\mathrm{i}}}{h_{\mathrm{o}}} = -\frac{s_{\mathrm{i}}}{s_{\mathrm{o}}}} \quad \text{and} \quad \frac{s_{\mathrm{i}}}{s_{\mathrm{o}}} = \frac{f}{x_{\mathrm{o}}} = \frac{f}{s_{\mathrm{o}} - f} \Rightarrow \boxed{\frac{1}{s_{\mathrm{o}}} + \frac{1}{s_{\mathrm{i}}} = \frac{1}{f}}$$

# Imaging condition using ray transfer matrices



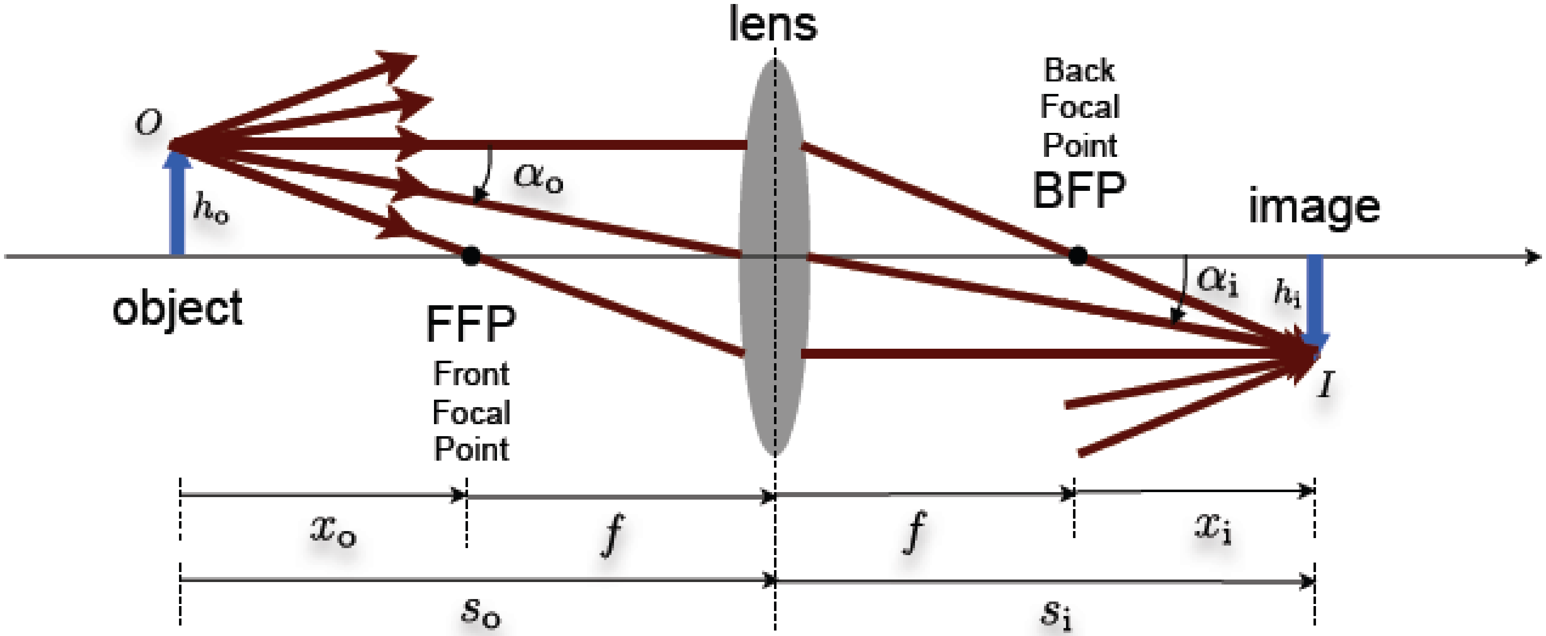
# Imaging condition using ray transfer matrices




$$\begin{pmatrix} \alpha_i \\ h_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_i & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_o & 1 \end{pmatrix} \begin{pmatrix} \alpha_o \\ h_o \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{s_o}{f} & -\frac{1}{f} \\ s_i + s_o - \frac{s_i s_o}{f} & 1 - \frac{s_i}{f} \end{pmatrix} \begin{pmatrix} \alpha_o \\ h_o \end{pmatrix}$$

# Imaging condition using ray transfer matrices

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$$\begin{pmatrix}\alpha_i\\ h_i\end{pmatrix} = \begin{pmatrix}1-\frac{s_o}{f} & -\frac{1}{f}\\ 0 & 1-\frac{s_i}{f}\end{pmatrix}\begin{pmatrix}\alpha_o\\ h_o\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{x_o}{f} & -\frac{1}{f}\\ 0 & -\frac{x_i}{f}\end{pmatrix}\begin{pmatrix}\alpha_o\\ h_o\end{pmatrix}$$

*Imaging* object point $O$ to image point $I$ requires that *all* rays departing from $O$ meet again at $I$, *independently* of ray departure angle $\alpha_o$. Equivalently, we require that a divergent spherical wave originating from $O$ must *focus*, *i.e.* converge at $I$.

Mathematically, this is expressed as

$$\frac{\partial h_i}{\partial \alpha_o} = 0 \Rightarrow s_i + s_o - \frac{s_i s_o}{f} = 0 \Rightarrow \boxed{\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}}.$$

Imaging condition

$$M_T \equiv \frac{h_i}{h_o} = 1 - \frac{s_i}{f} = -\frac{x_i}{f}$$

*Angular* magnification $\quad M_A \equiv \dfrac{\partial \alpha_i}{\partial \alpha_o} = 1 - \dfrac{s_o}{f} = -\dfrac{x_o}{f}$. $\quad$ Note $\quad M_A = \dfrac{1}{M_T}$

# Real and virtual images

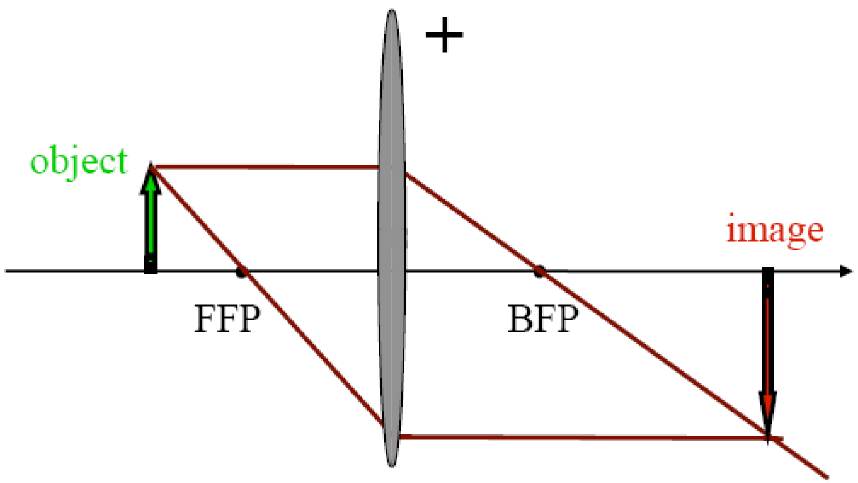



image: real & inverted; $M_T<0$

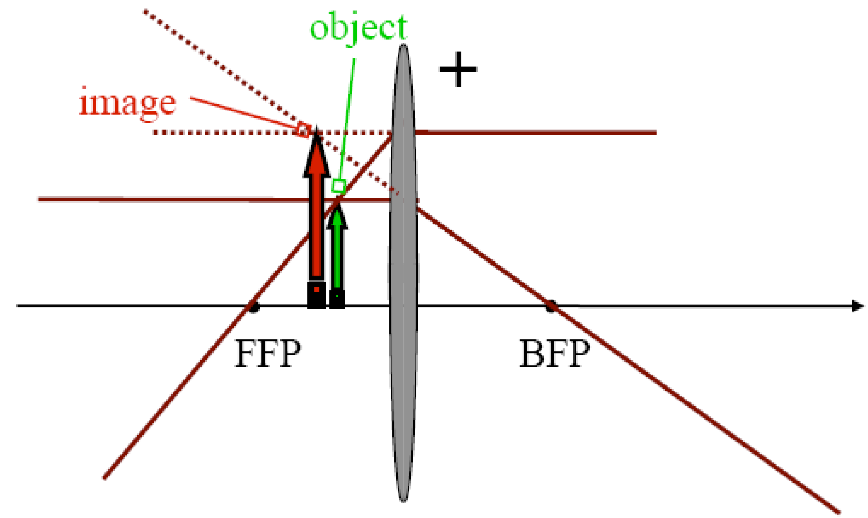



image: virtual & erect; $M_T>1$

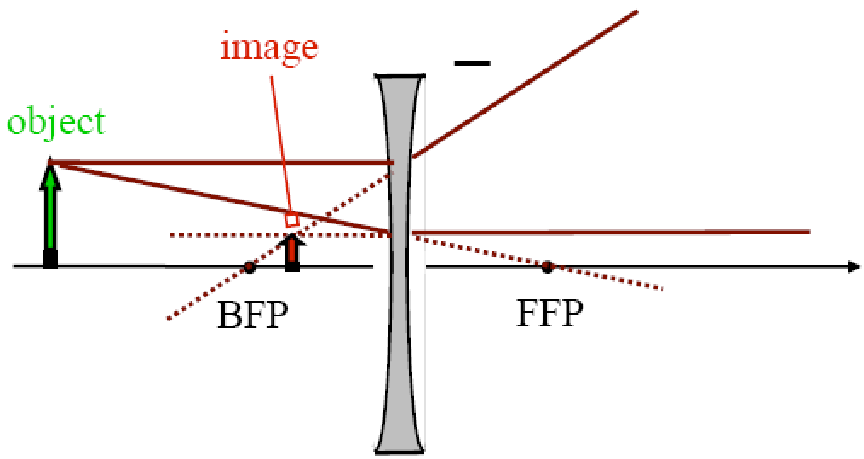



image: virtual & erect; $0<M_T<1$

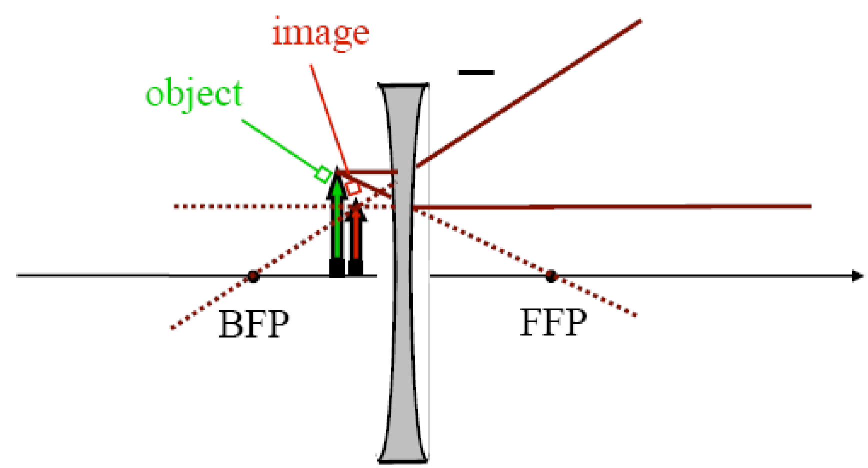
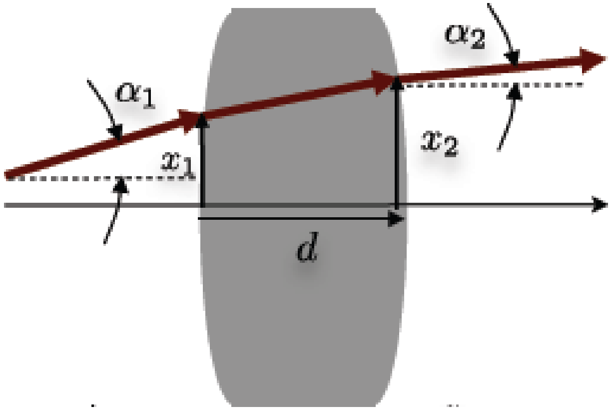



image: virtual & erect; $0<M_T<1$

# Thick lens

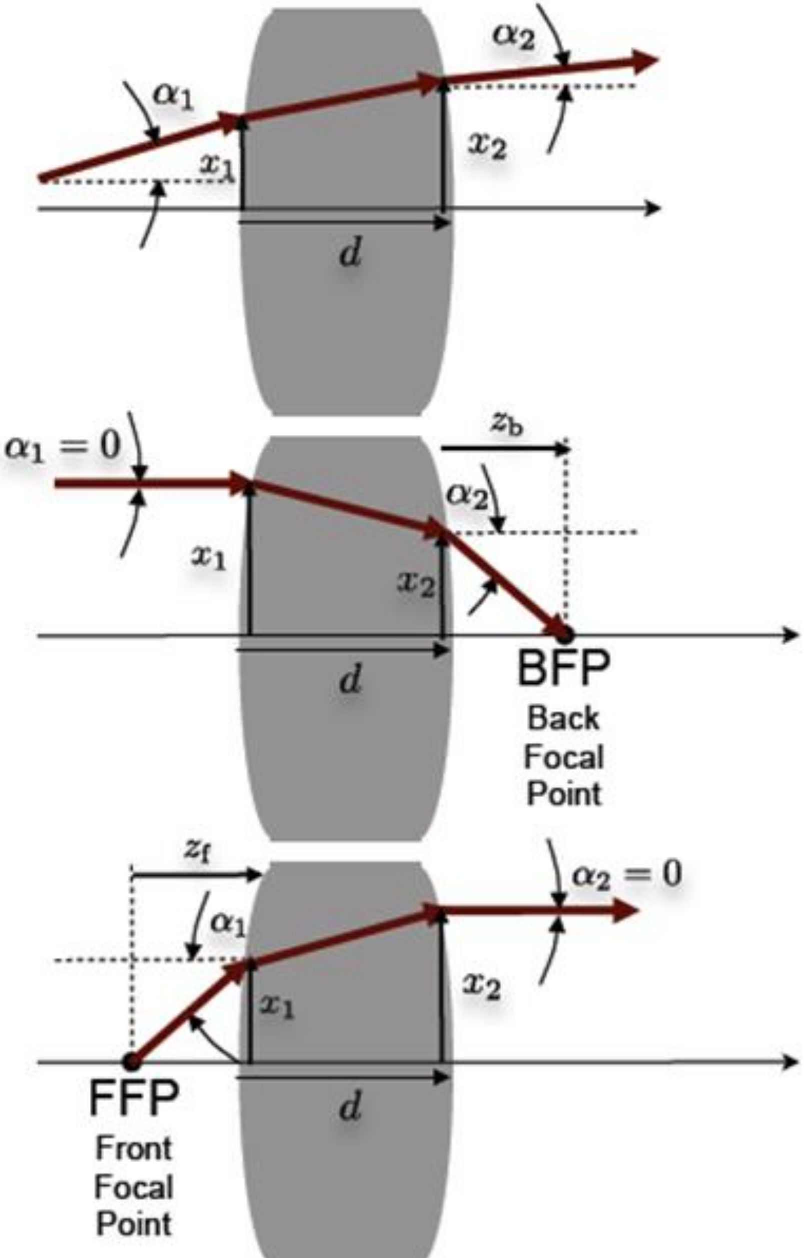



$$\begin{pmatrix} \alpha_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1-n}{R_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{n-1}{R_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{n-1}{n}\frac{d}{R_2} & -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}\right] \\ \frac{d}{n} & 1 - \frac{n-1}{n}\frac{d}{R_1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix}$$

We define $\frac{1}{\mathrm{EFL}} \equiv (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}$

# Thick lens




$$\begin{pmatrix} \alpha_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1-n}{R_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{d}{n} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{n-1}{R_1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{n-1}{n}\frac{d}{R_2} & -\left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}\right] \\ \frac{d}{n} & 1 - \frac{n-1}{n}\frac{d}{R_1} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ x_1 \end{pmatrix}$$

We define $\frac{1}{\mathrm{EFL}} \equiv (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{(n-1)^2 d}{nR_1R_2}$

On–axis object at infinity $\Rightarrow \alpha_1 = 0$.
Where do the rays focus? *i.e.*,$z_b = ?$ : $x_2 = 0$

$$\begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_b & 1 \end{pmatrix} \begin{pmatrix} 1 + \frac{n-1}{n}\frac{d}{R_2} & -\frac{1}{\mathrm{EFL}} \\ \frac{d}{n} & 1 - \frac{n-1}{n}\frac{d}{R_1} \end{pmatrix} \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \Rightarrow$$

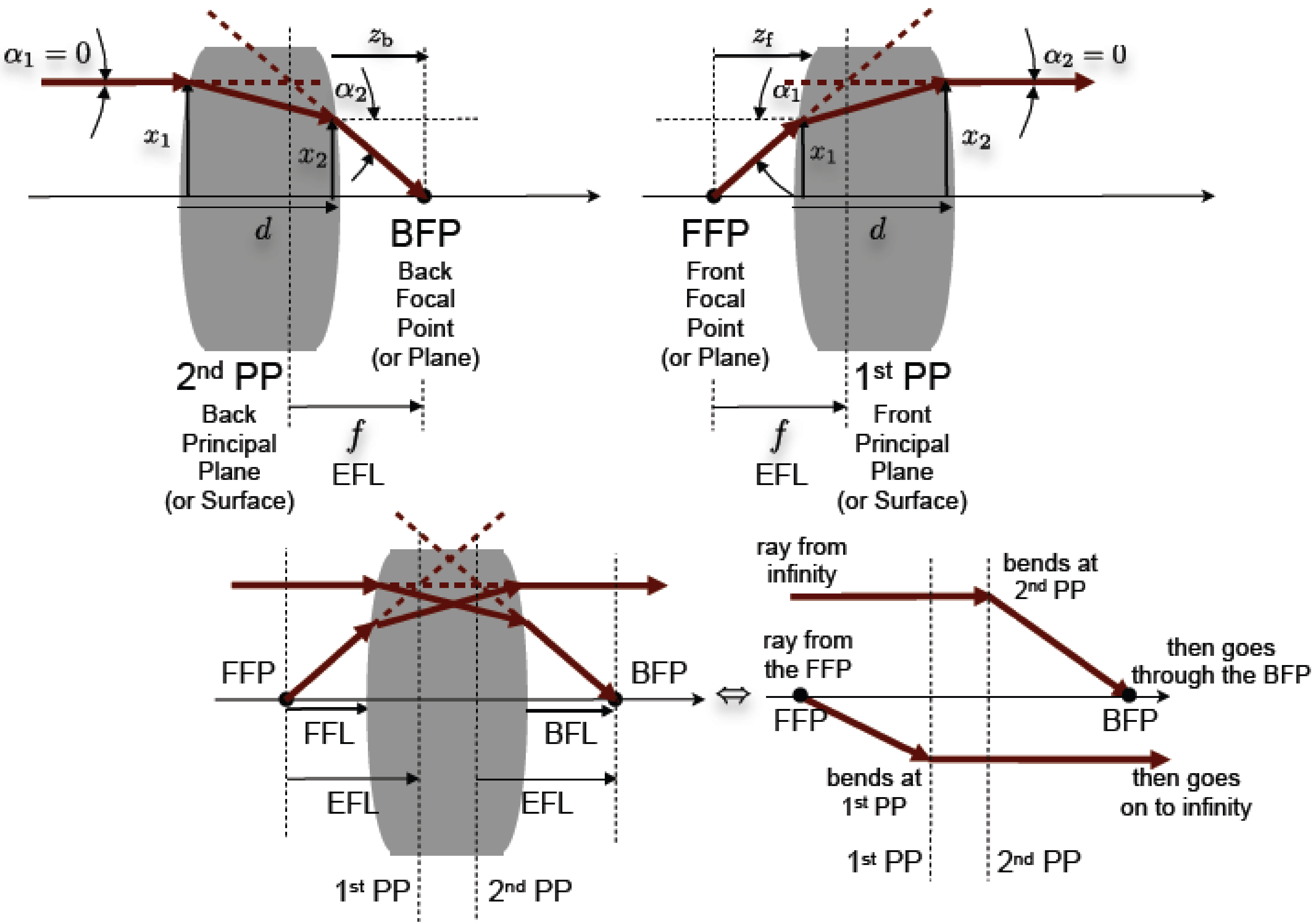
$$\begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{x_1}{\mathrm{EFL}} \\ x_1\left(-\frac{z_b}{\mathrm{EFL}} + 1 - \frac{n-1}{n}\frac{d}{R_1}\right) \end{pmatrix} \Rightarrow \begin{cases} \alpha_2 = -\frac{x_1}{\mathrm{EFL}} \\ z_b = (\mathrm{EFL})\left(1 - \frac{n-1}{n}\frac{d}{R_1}\right) \end{cases}$$

$$\Rightarrow \begin{cases} P \equiv \frac{1}{\mathrm{EFL}} & \text{is the } \textbf{power} \text{ of the thick lens;} \\ f \equiv (\mathrm{EFL}) & \text{is the } \textbf{effective focal length}; \\ z_b \equiv (\mathrm{BFL}) & \text{is the } \textbf{back focal length}. \end{cases}$$

Similarly, by requiring an on–axis point object at finite distance $z_f$ to produce an on–axis image at infinity, *i.e.* $\alpha_2 = 0$, we find

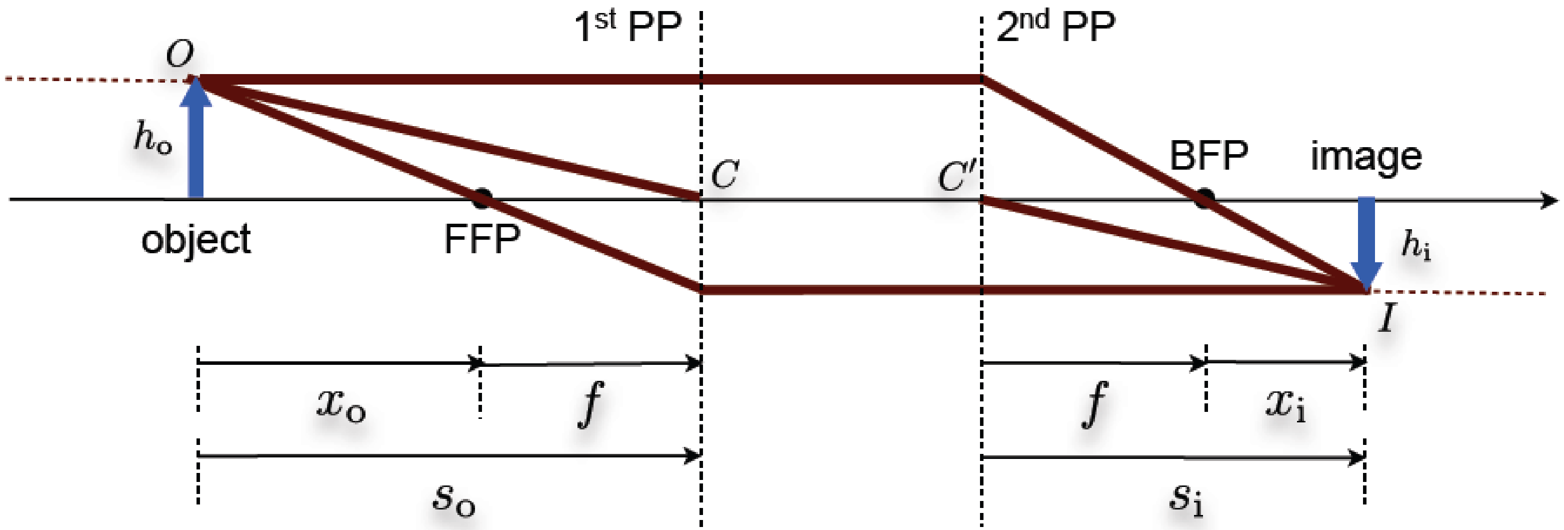
$$z_f \equiv (\mathrm{FFL}) = (\mathrm{EFL})\left(1 + \frac{n-1}{n}\frac{d}{R_2}\right) \qquad x_2 = \alpha_1\,(\mathrm{EFL})$$

# Focal Lengths and Principal Planes



# Image formation with composite elements

composite element (e.g., thick lens)




To find the imaging condition for the composite element we can use the principal planes as follows:

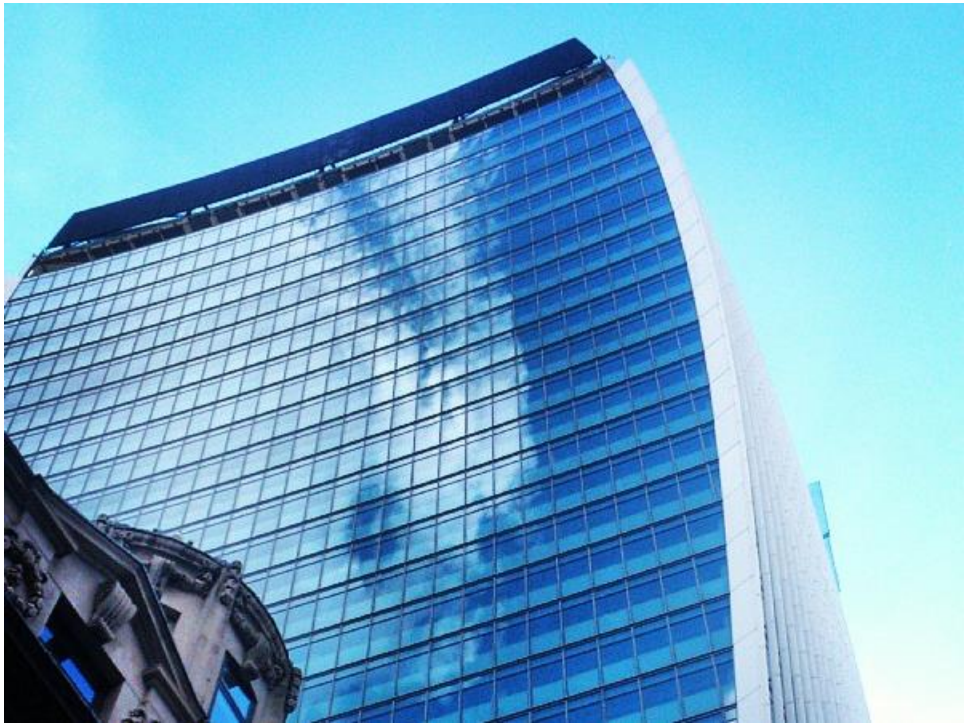
- trace an on-axis ray from infinity through $O$ to the 2nd PP then bend so that it goes through the BFP;
- trace a ray from $O$ through the FFP then bend at the 1st PP so that it goes to infinity on-axis;
- the intersection of the traced rays is the image point $I$;
- the ray from $O$ through the intersection of the 1st PP with the optical axis should emerge at the intersection $C$ of the 2nd PP with the optical axis and also go through the image point $I$; moreover, if the indices of refraction to the left and right of the composite are the same, then $OC \| C'I$.
- It is easy to see that the similar triangle arguments that we used in the case of the single thin lens apply here as well; therefore, the imaging condition and magnification relations remain the same with the notation as shown above.

$$x_{\rm o}x_{\rm i} = f^2; \qquad \frac{1}{s_{\rm o}} + \frac{1}{s_{\rm i}} = \frac{1}{f}; \qquad M_{\rm T} = -\frac{x_{\rm i}}{f} = -\frac{f}{x_{\rm o}} = -\frac{s_{\rm i}}{s_{\rm o}}; \qquad M_{\rm A} = \frac{1}{M_{\rm T}}.$$

# Optics –Aberration

# Disasters

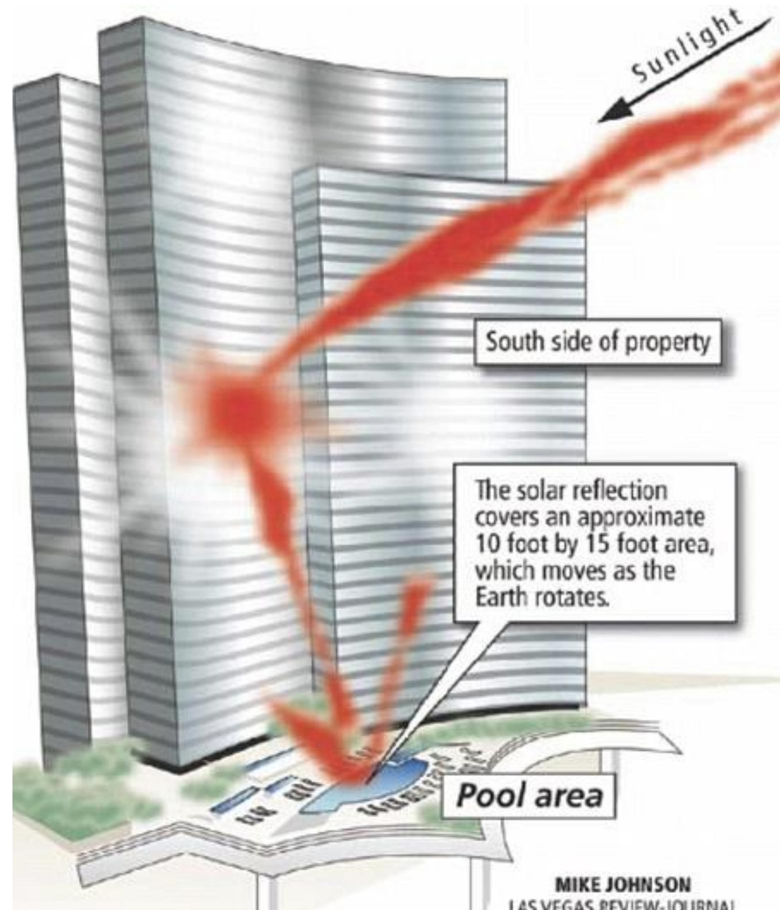
# Accidental Cylindrical Lenses



London skyscraper focuses sunlight and has melted a Jaguar (car) parked nearby even before it’s completed.



“Las Vegas Death Ray”: Vdara Hotel in Las Vegas focuses sun onto sunbathers at its pool.




# Aberrations


Aberrations are distortions that occur in images, usually due to imperfections in lenses, some unavoidable, some avoidable.

Most aberrations can't be modeled with ray matrices. Designers beat them with lenses of multiple elements, that is, several lenses in a row. Some zoom lenses can have as many as a dozen or more elements.

# Aberrations

- Chromatic

– is due to the fact that the refractive index of lenses, etc. varies with wavelength; therefore, focal lengths, imaging conditions, etc. are wavelength dependent

- Geometrical

– are due to the deviation of non-paraxial rays from the approximations we have used so far to derive focal lengths, imaging conditions, etc.; therefore, rays going through imaging systems typically do not focus perfectly but instead scatter around the “paraxial” (or “Gaussian”) focus

Aberrations are failures to focus to a "point“. Some are failures of paraxial assumption

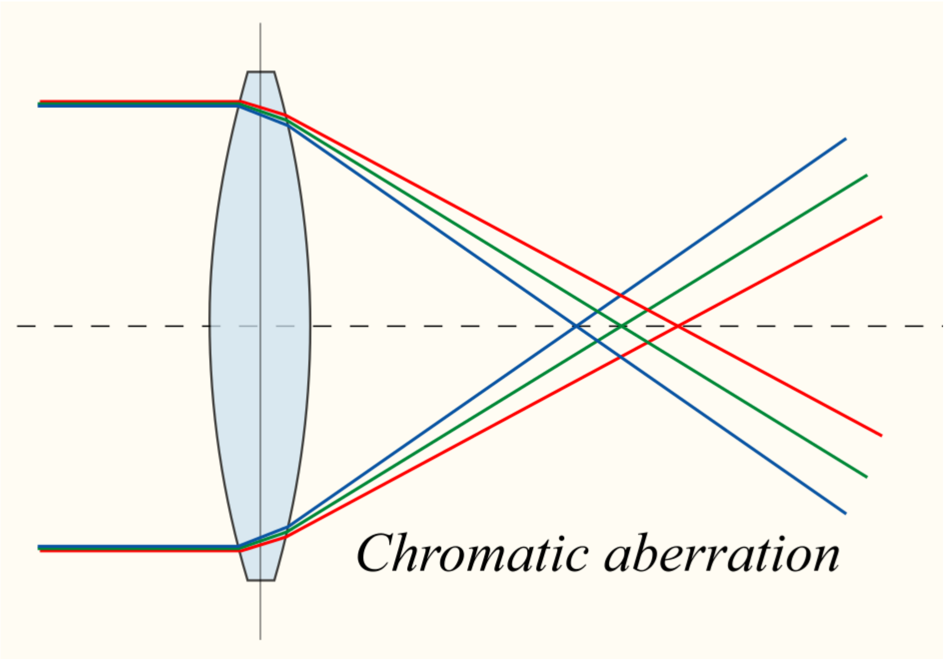
$$sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$ (Formalism developed by Seidel)

# Chromatic Aberration

Because the lens material has a different refractive index for each wavelength, the lens will have a different focal length for each wavelength. Recall the lens-maker's formula:

$$\boxed{1/f(\lambda) = (n(\lambda)-1)(1/R_1 - 1/R_2)}$$

Image with chromatic aberration

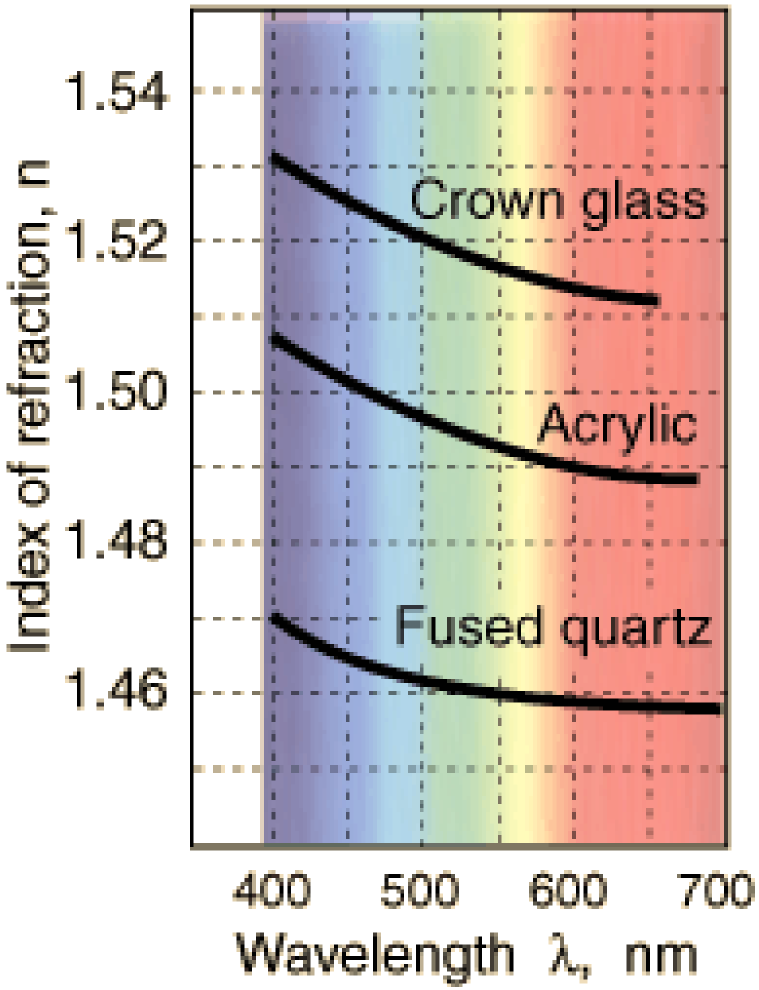


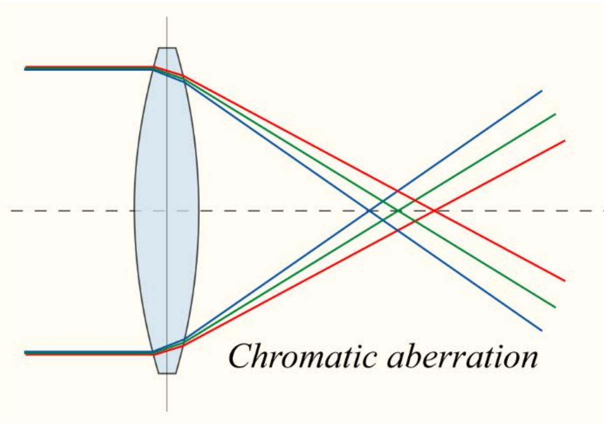

The refractive index is larger for blue than red, so the focal length is less for blue than red.

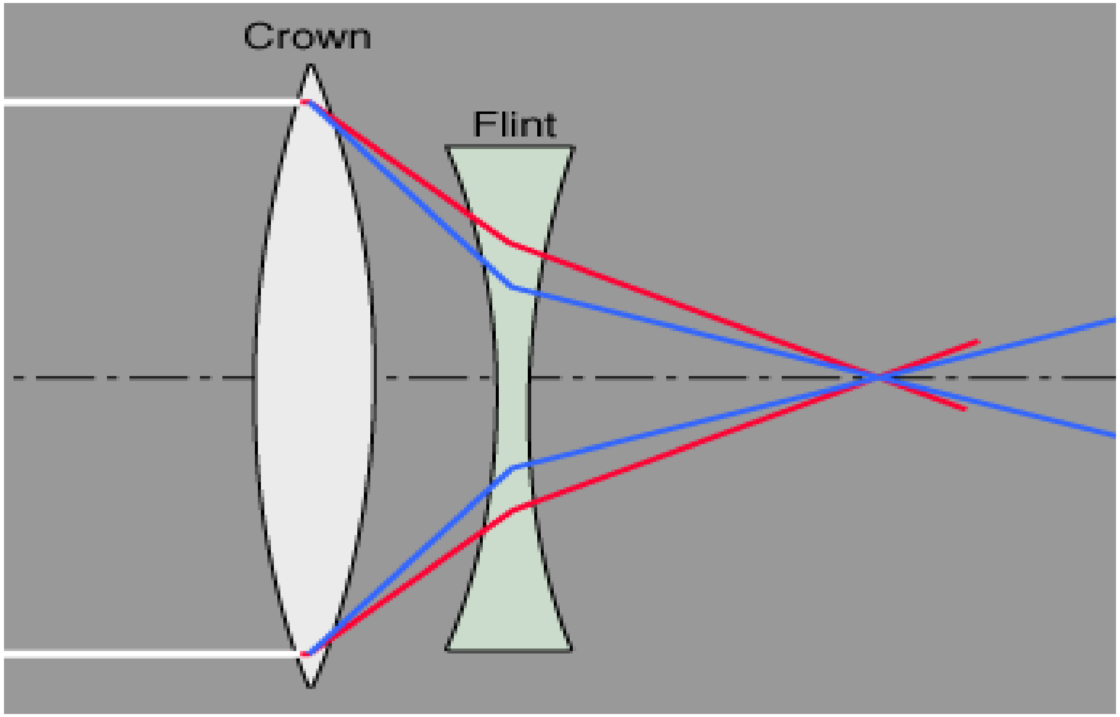
You can model spherical aberration using ray matrices, but only one color at a time.

# Chromatic aberration can be minimized using additional lenses.

In an **Achromat**, the second lens cancels the dispersion of the first.



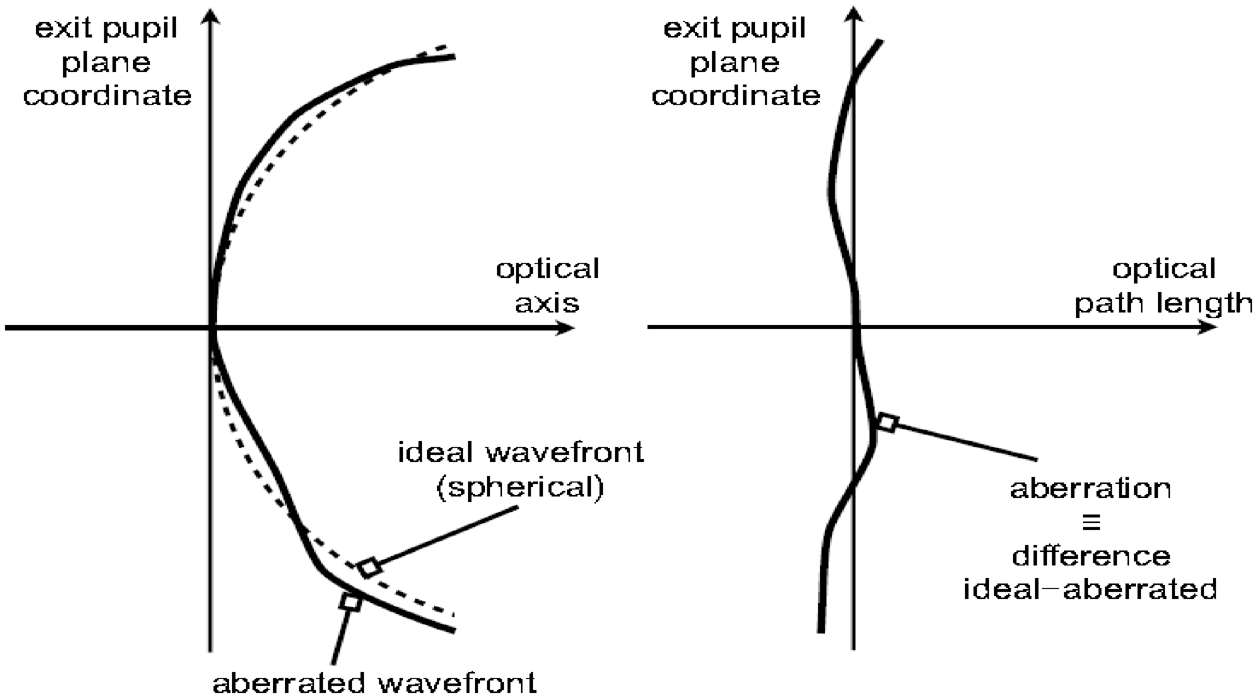






Simple achromats use two lenses of different materials.

Both materials have positive dispersion, but one has a negative focal length.

# Geometrical Aberrations




Geometrical aberration is the deviation of the wavefront produced by an optical system at the exit pupil, with respect to the ideal spherical wavefront that would have produced a point image.
Generally, computing aberrations is a complicated geometrical/algebraic exercise.
Traditionally, and to gain intuition, aberrations have been studied as successive terms in a perturbation (Taylor) expansion of the aberrated  wavefronts in the rotationally symmetric case.
Here, we will consider only 3rd order aberrations (also k/a Seidel Abbration).

$sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$ (Formalism developed by Seidel)

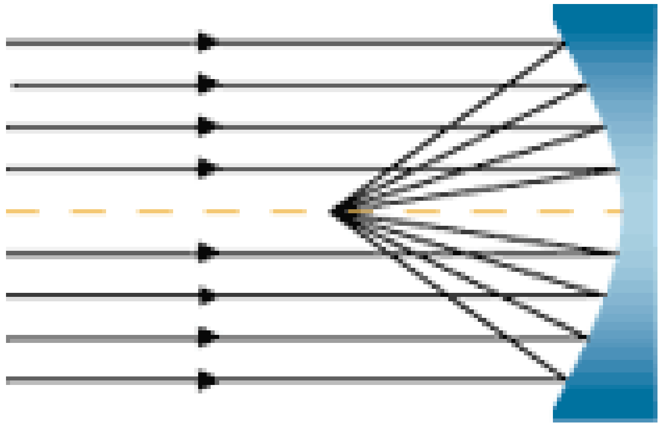
# Geometrical (Seidal) Aberrations

1. Spherical
2. Coma
3. Astigmatism
4. Field Curvature
5. Distortion

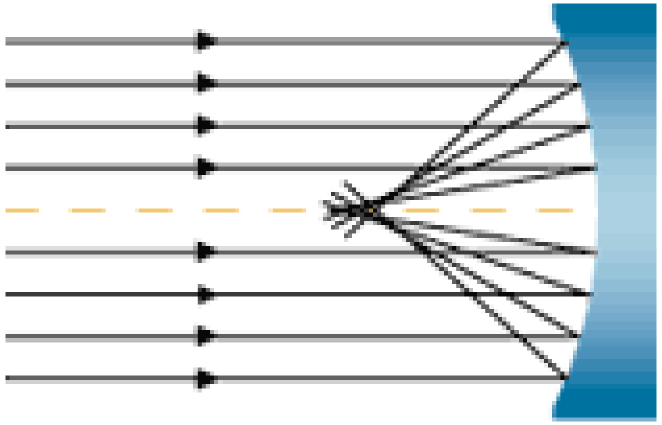
# Spherical Aberration from Curved Mirrors

For all rays to converge to a point a distance $f$ away from a curved mirror requires a paraboloidal surface. But this only works for $\theta_{in} = 0$.

Paraboloidal surface

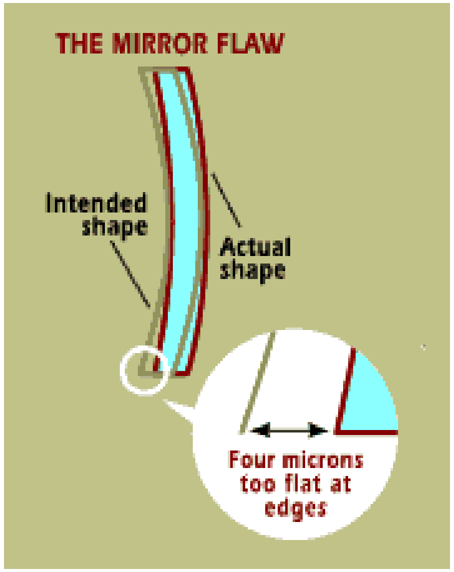


Spherical surface

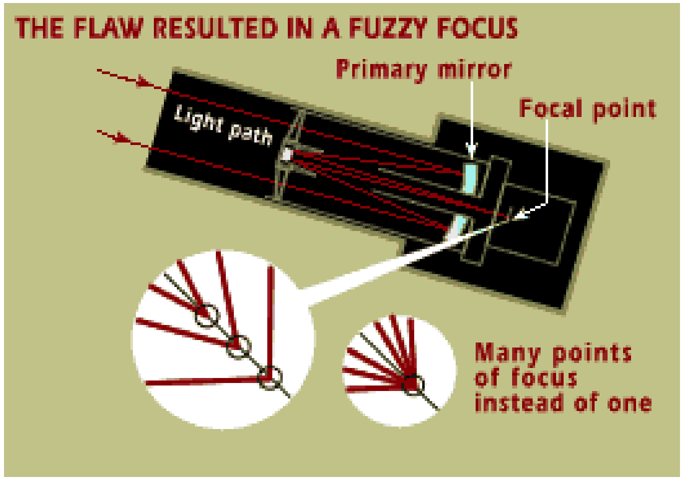


Spherical surfaces work better with off-axis rays and are easier to fabricate, so are usually used instead.

# Spherical Aberration in the Hubble Space Telescope




The Hubble Space Telescope required two hyperboloidal mirrors.




But the primary was off by 4μm at its edges.

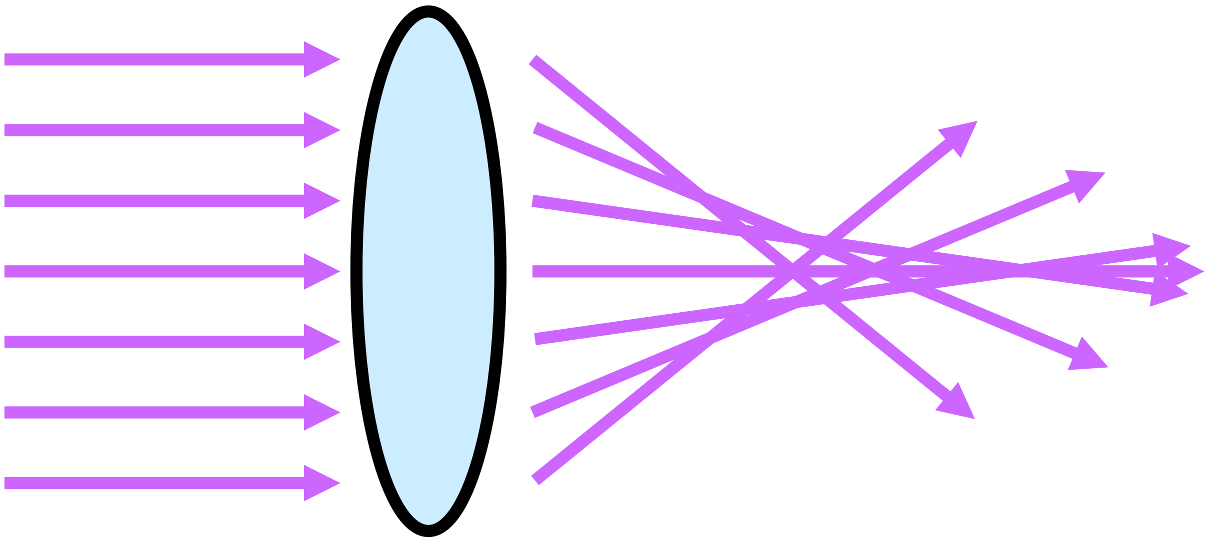
Hubble images before and after correction of its spherical aberration

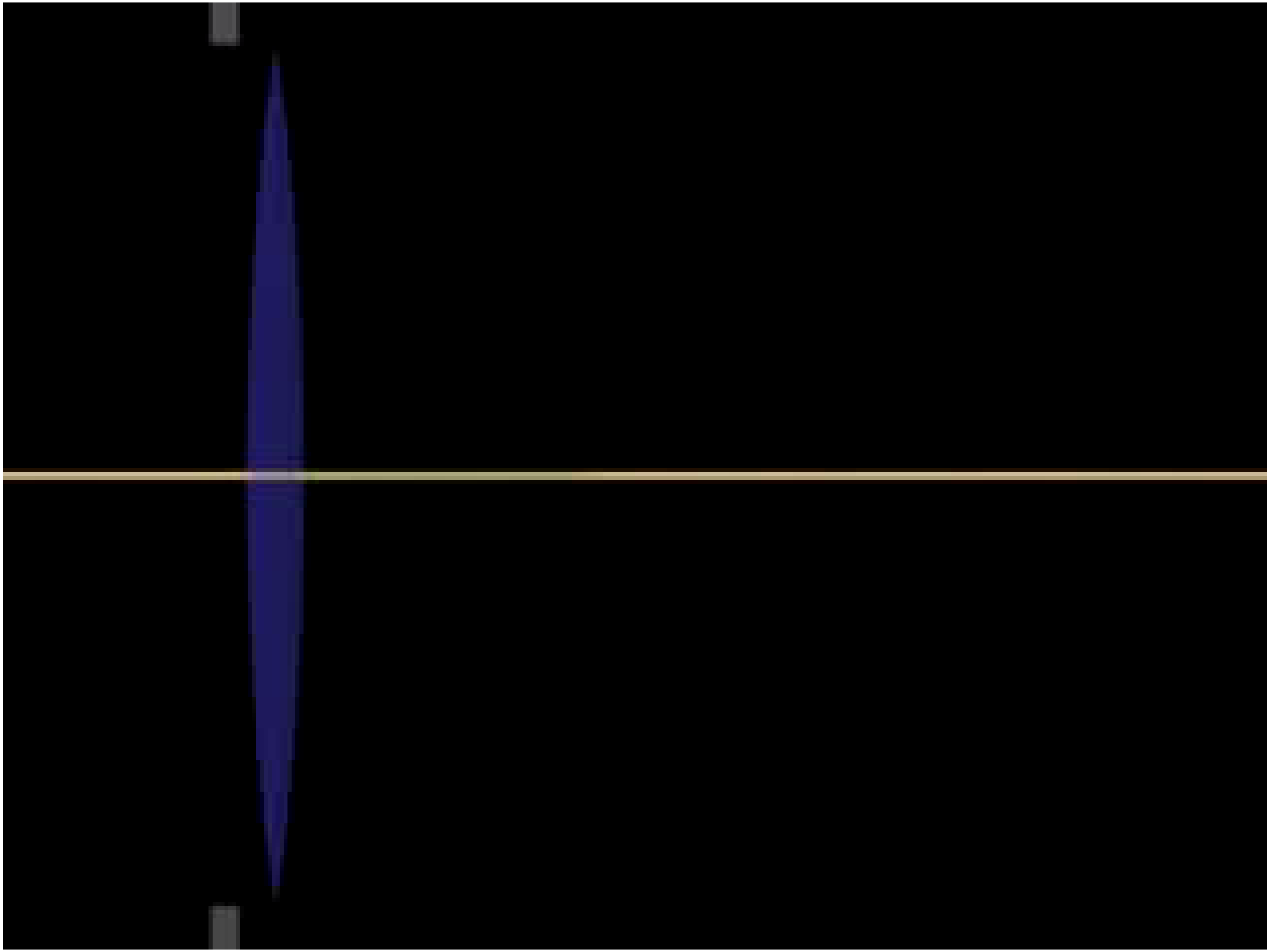
Reference: http://hubblesite.org/the_telescope/nuts_.and._bolts/optics/costar/

# Spherical aberration in lenses

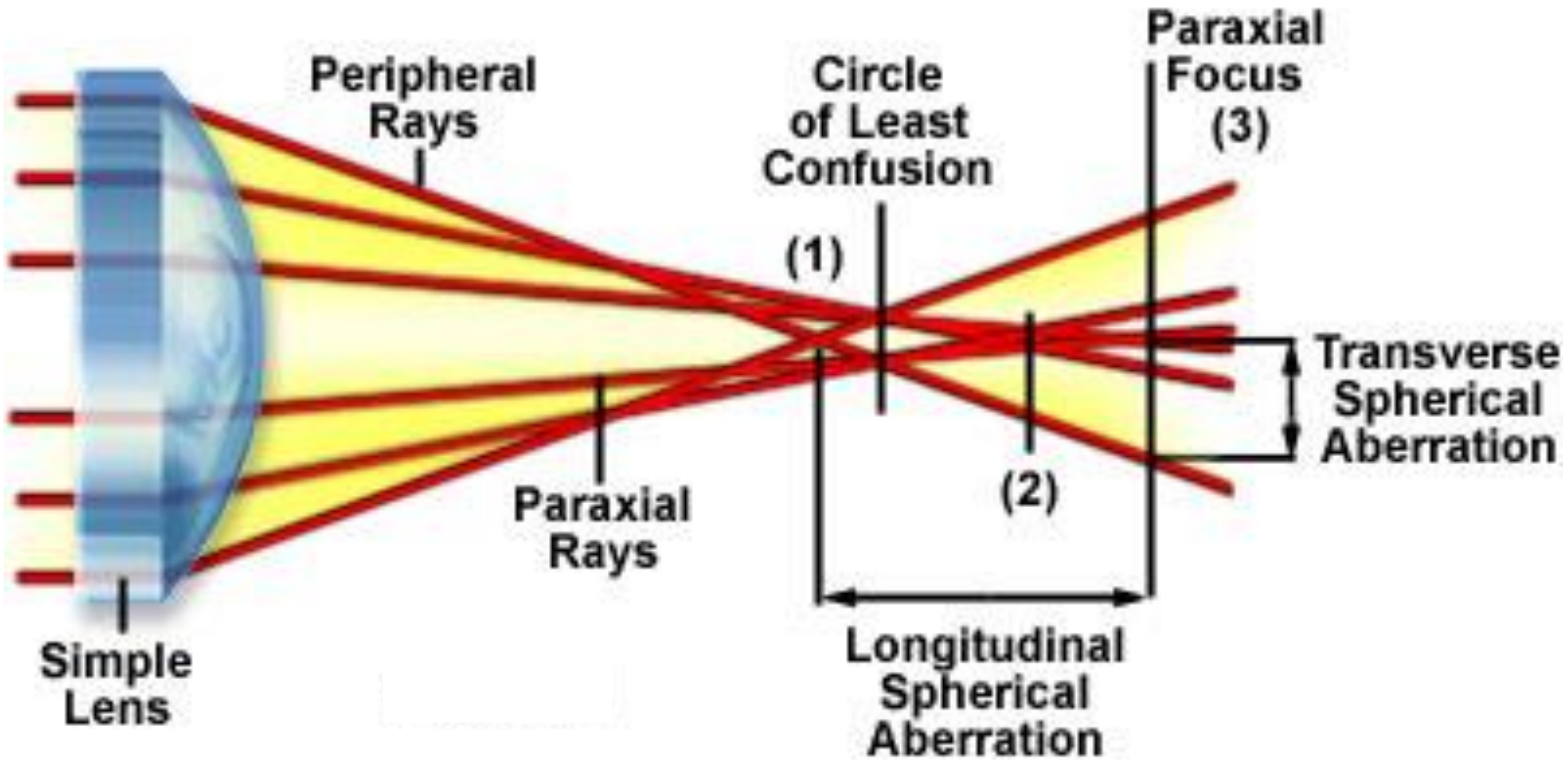
So we use spherical surfaces, which work better for a wider range of input angles.

Nevertheless, off-axis rays see a different focal length, so lenses have spherical aberration, too.



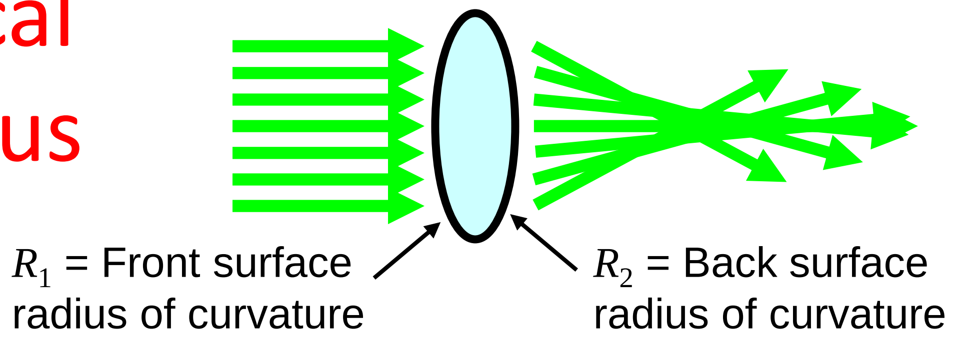


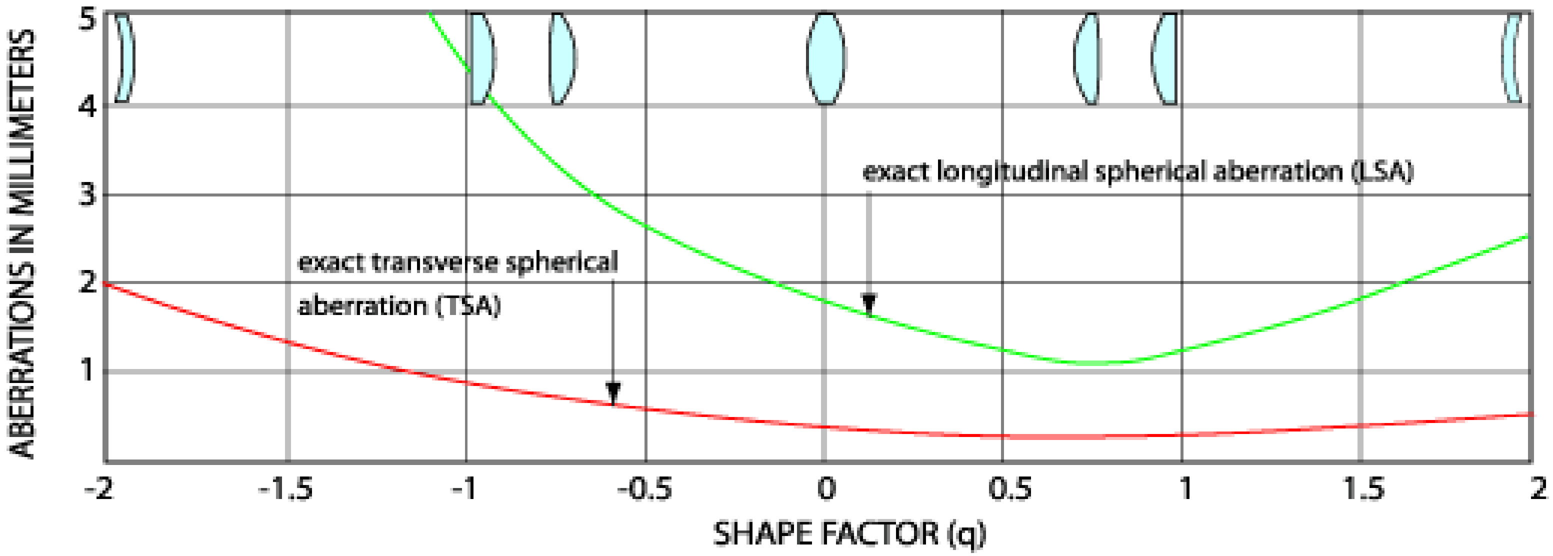
## Longitudinal and Transverse Spherical Aberration



# Minimizing Spherical Aberration in a Focus

$$q \equiv \frac{R_2 + R_1}{R_2 - R_1}$$

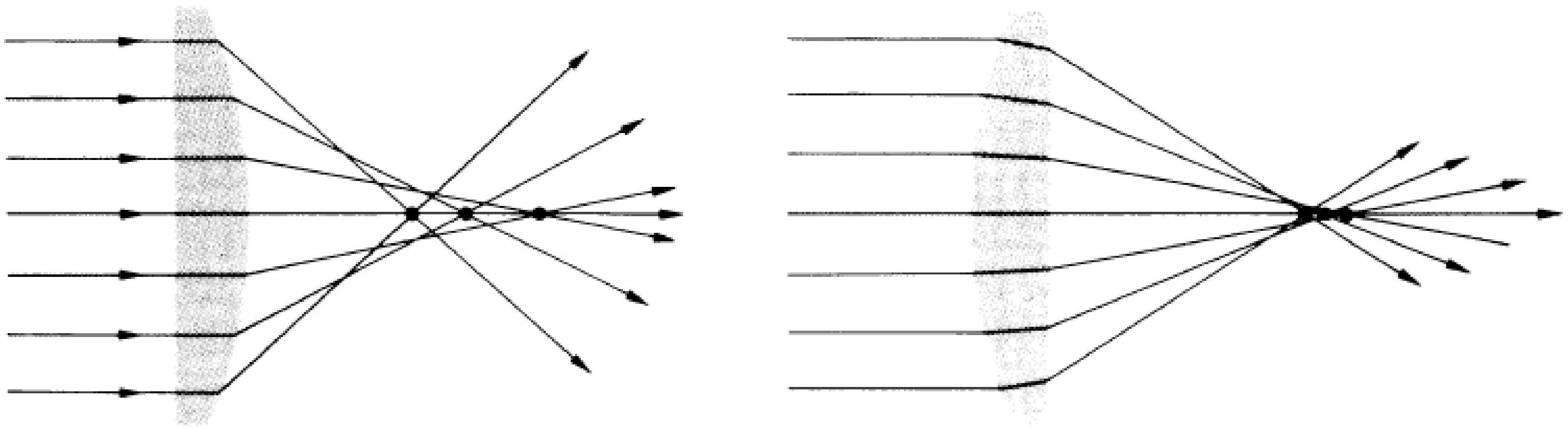



Plano-convex lenses (with their flat surface facing the focus) are best for minimizing spherical aberration when focusing.

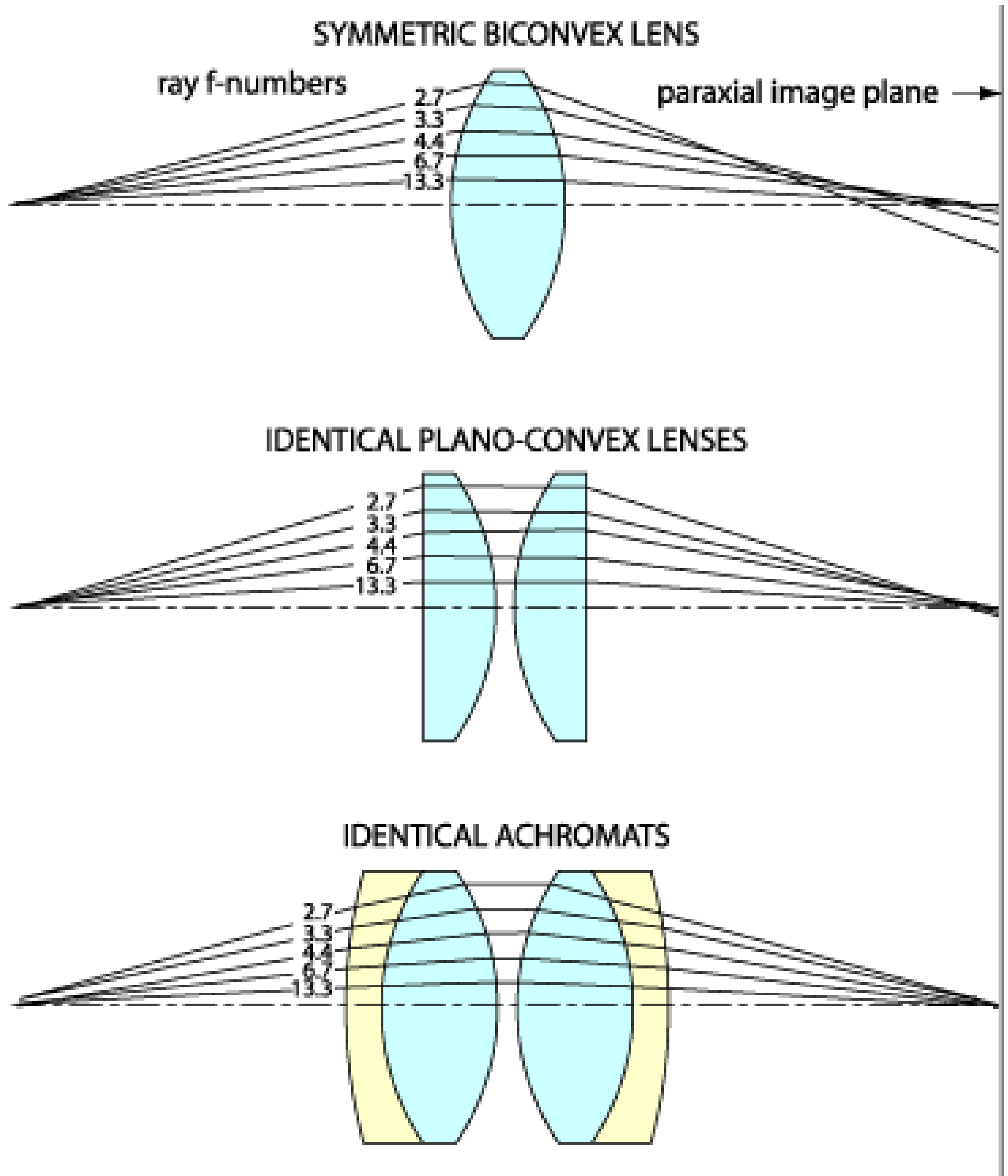
One-to-one imaging works best with a symmetrical lens ($q = \infty$).

# How to orient an asymmetric lens for spherical aberration-free focusing of plane wave?

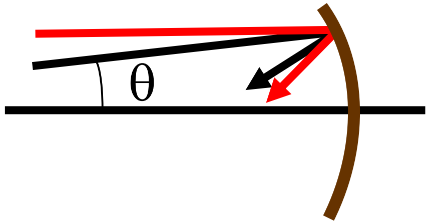


From Hecht → Chapter 6

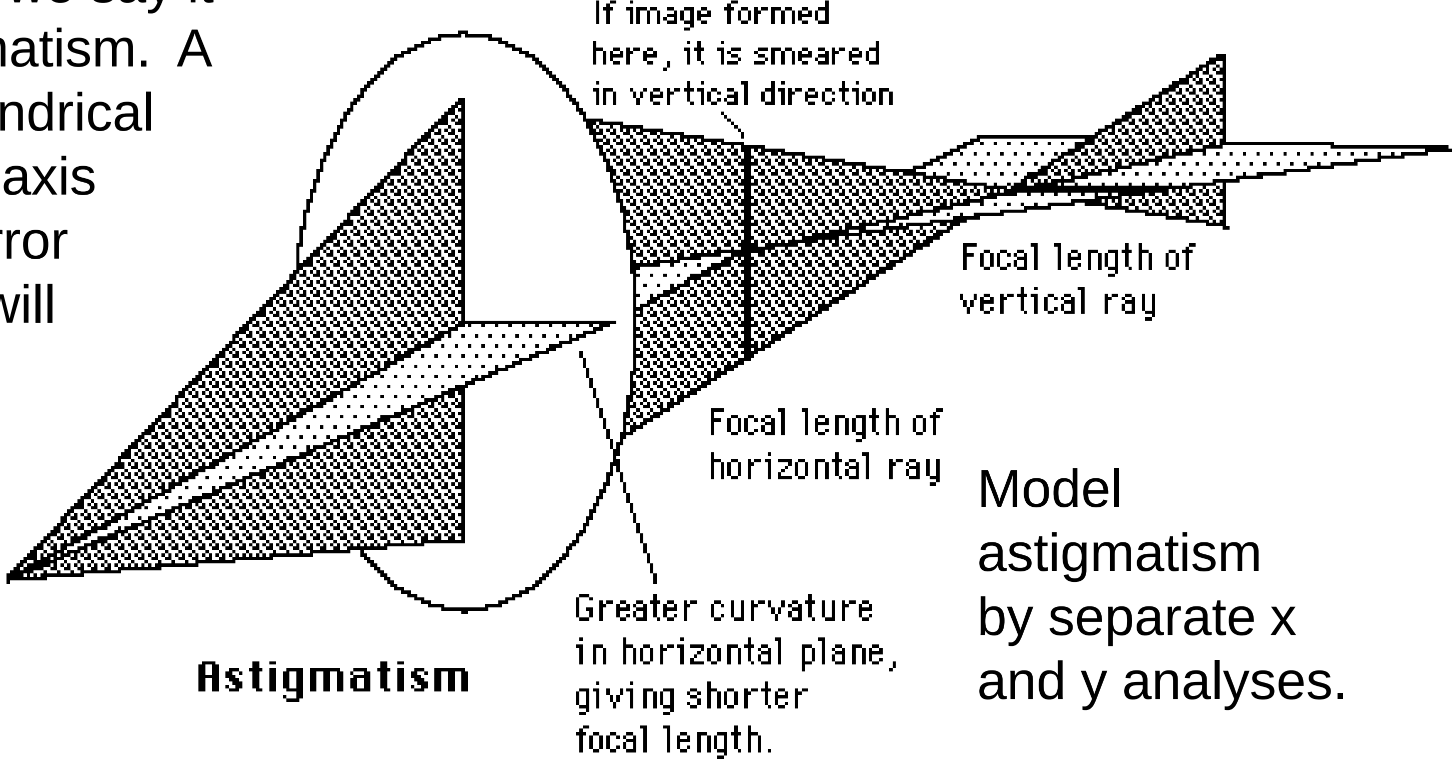
# Spherical aberration can be minimized using additional lenses also.



# Astigmatism



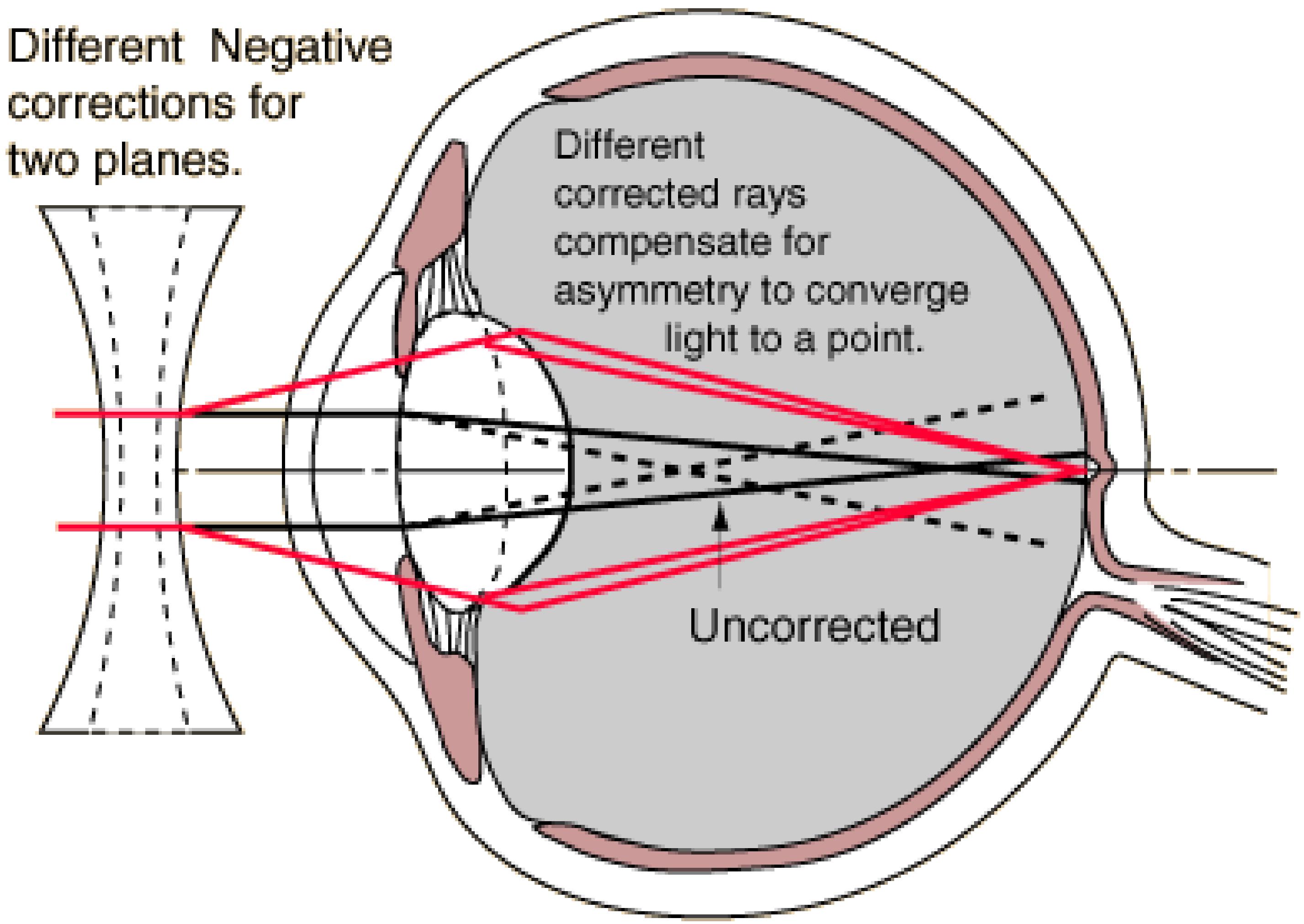

When the optical system lacks perfect cylindrical symmetry, we say it has astigmatism. A simple cylindrical lens or off-axis curved-mirror reflection will cause this problem.




Model astigmatism by separate x and y analyses.

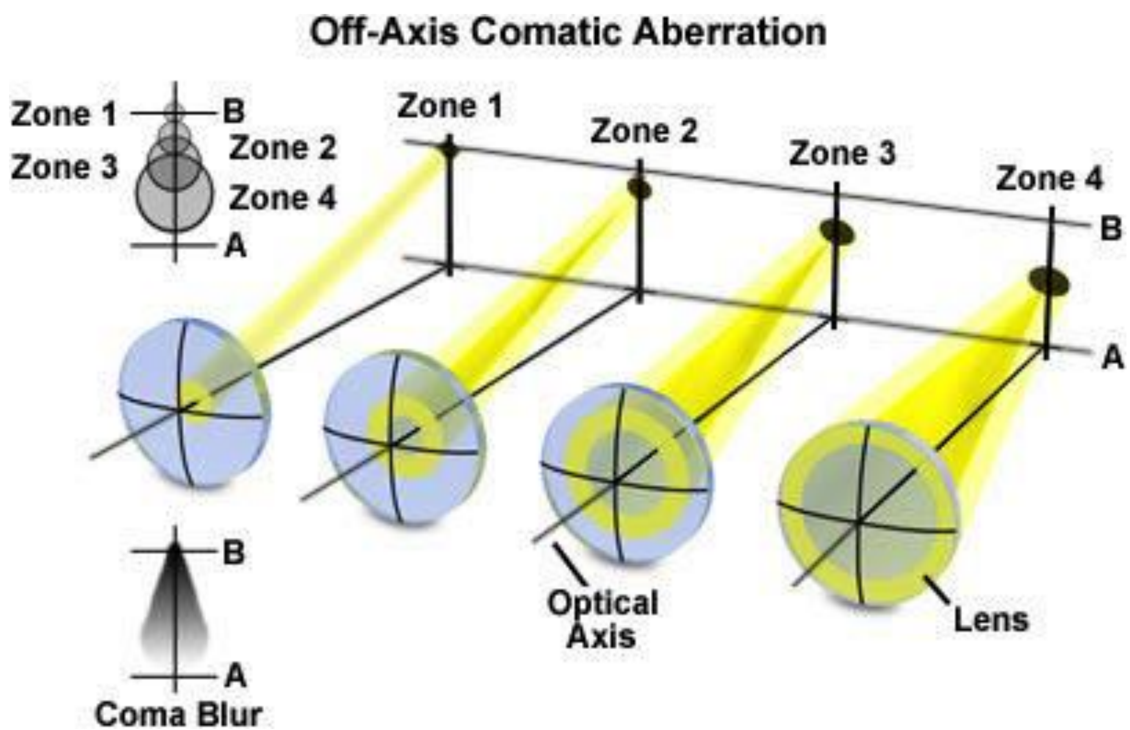
Cure astigmatism with another cylindrical lens or off-axis curved mirror.

# Astigmatism is a common problem in the eye.

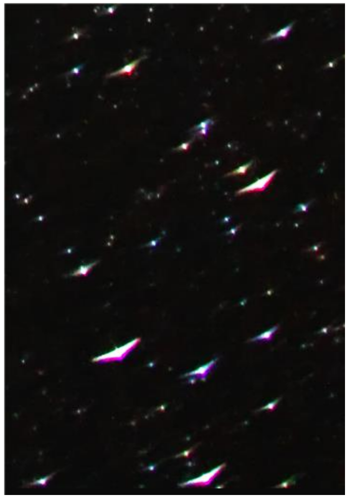


# Coma

Coma causes off-axis rays from a point of light to focus in different planes and positions, according to the lens radius they pass through.




This creates a "comet-like" blur.

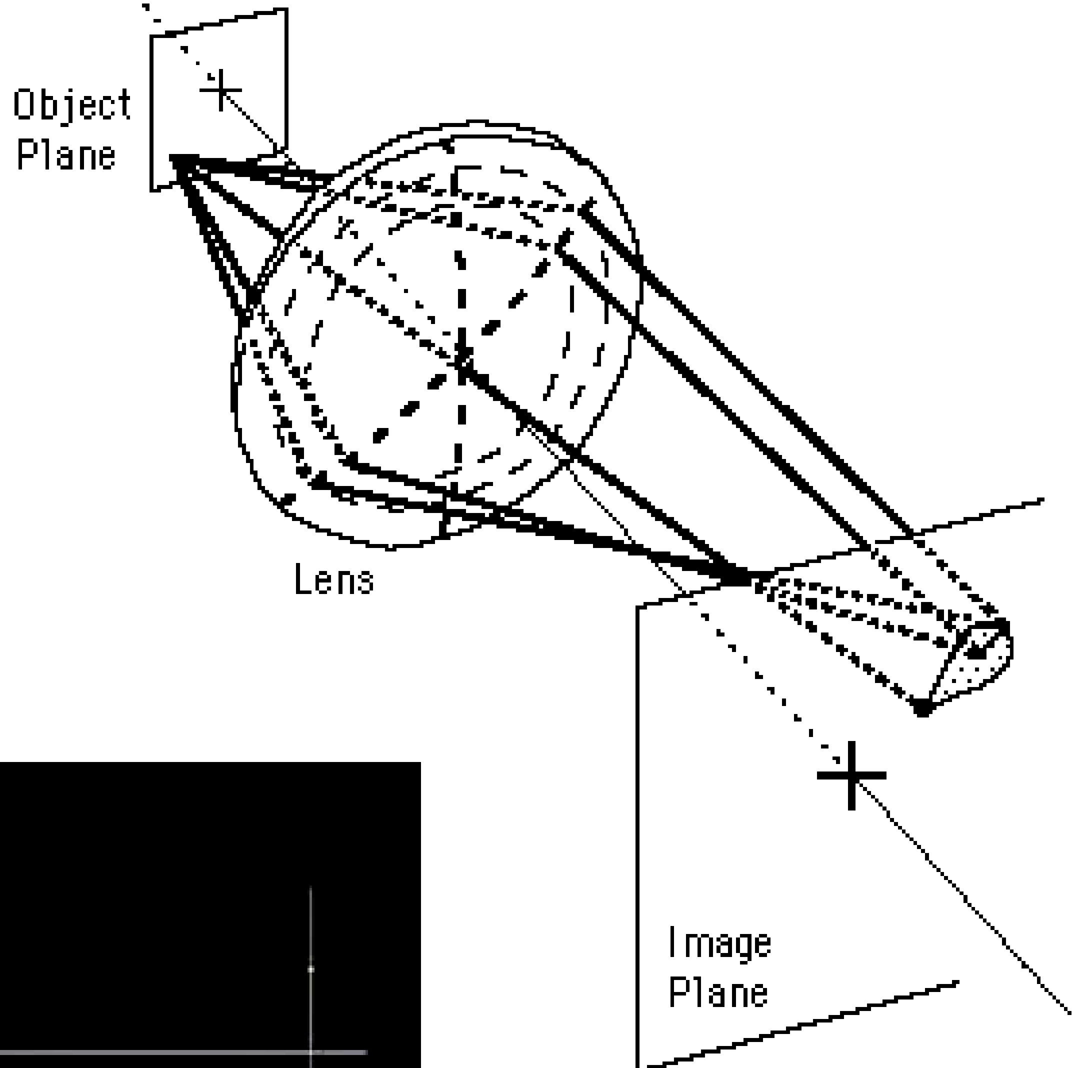


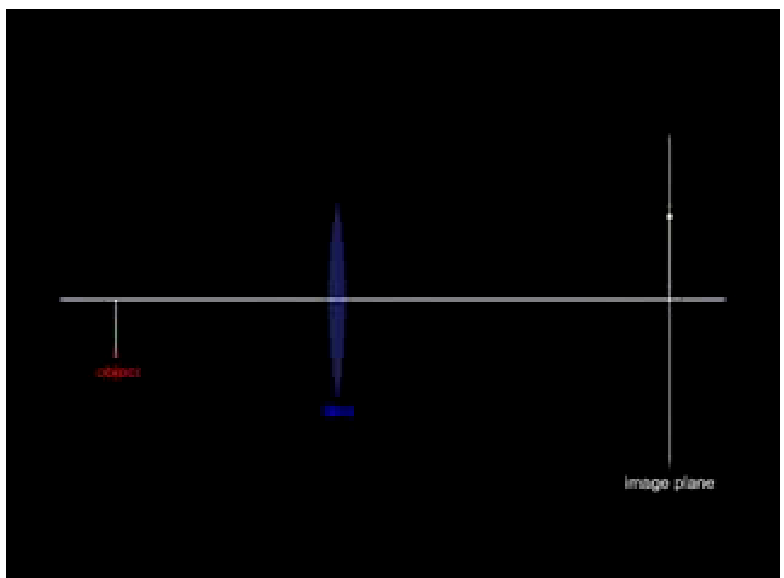
Coma in an astronomical image

A lens with coma may produce a sharp image in the center of the field, but it becomes increasingly blurred toward the edges. For a single lens, coma can be caused or partially corrected by tilting the lens.

# Coma

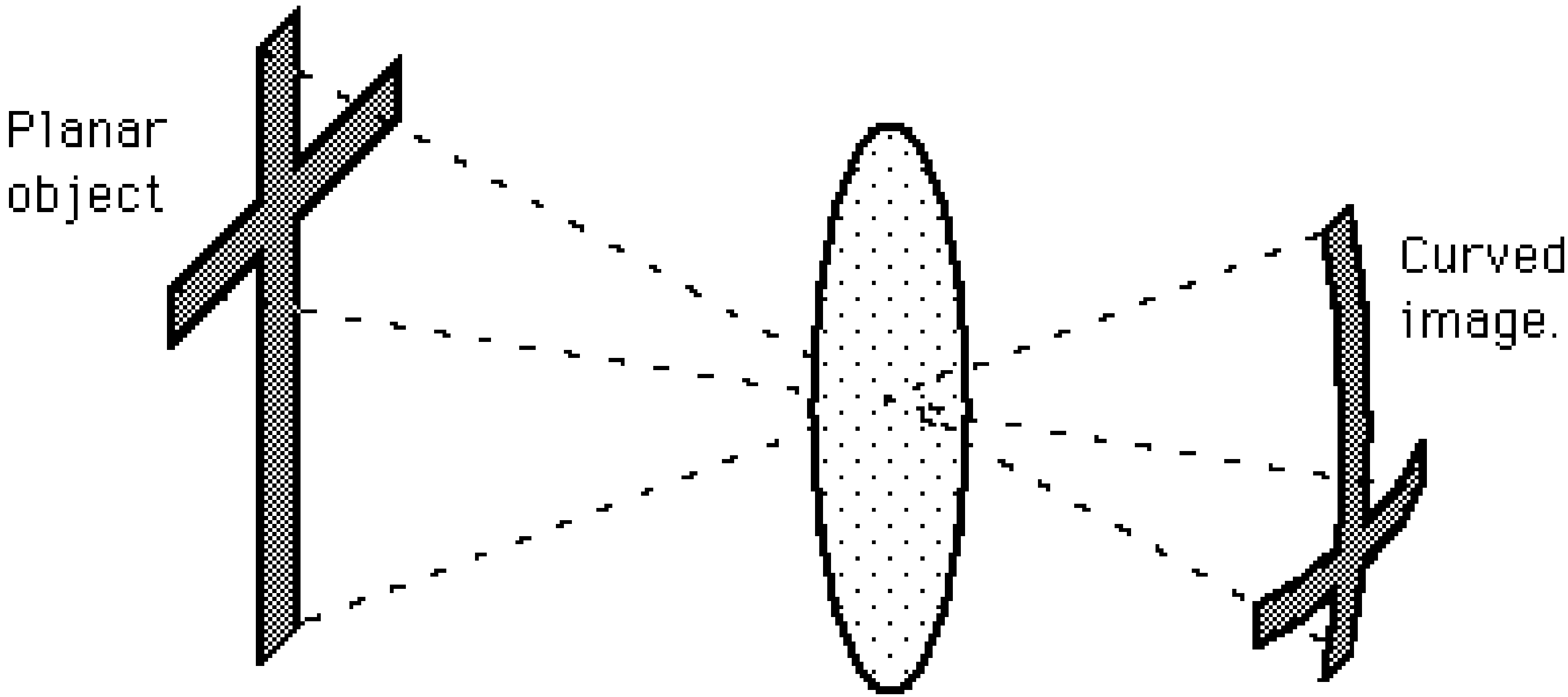
Coma causes rays from an off-axis point of light in the object plane to create a trailing "comet-like" blur directed away from the optic axis. A lens with considerable coma may produce a sharp image in the center of the field, but it becomes increasingly blurred toward the edges. For a single lens, coma can be caused or partially corrected by tilting the lens.



# Curvature of Field

Curvature of field causes a planar object to project a curved (non-planar) image. Rays at a large angle see the lens as having an effectively smaller diameter and an effectively smaller focal length, forming the image of the off-axis points closer to the lens.



Petzval condition: eliminates field curvature. For two lenses, indices of refraction $n_1$, $n_2$, focal lengths $f_1$, $f_2$, respectively:

$$\frac{1}{n_1 f_1} + \frac{1}{n_2 f_2} = 0$$

Or $n_1 f_1 + n_2 f_2 = 0$

As an example -> suppose we combine two thin lenses, one positive and one negative such that $f_1 = -f_2$ and $n_1 = n_2$.

Since,

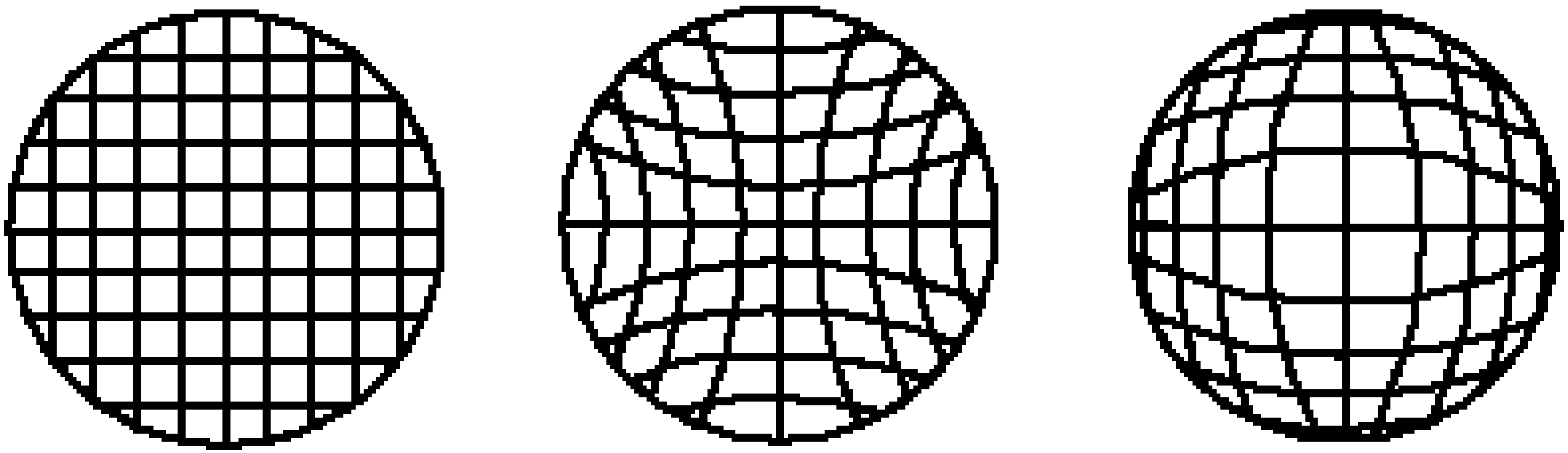
$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$$

$$f = \frac{f_1^2}{d}$$

d is thickness of the lenses along optic axis

the system can satisfy the Petzval condition, have a flat field.

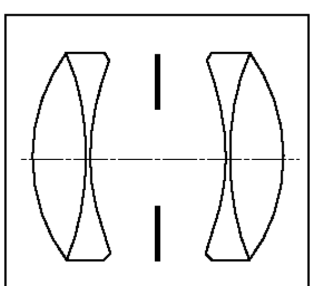
# Pincushion and Barrel Distortion



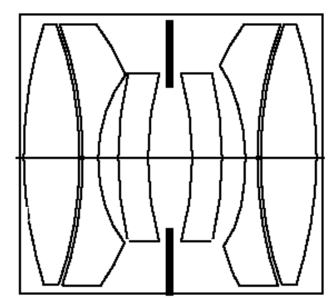
Undistorted Image

Pincushion Distortion

Barrel Distortion



These distortions are fixed by an “orthoscopic doublet” or a “Zeiss orthometer.”



# Barrel and Pincushion Distortion

Barrel

Pincushion