MA 2110 - Probability

Random Variables - 23, August 2016

1. You roll n dice; all those that show a six are rolled again. Let X be the number of resulting sixes. What is the distribution of X? Find its mean and variance.

2. You roll a fair die repeatedly until a number larger than 4 is observed. If N is the total number of times that you roll the die, find $P(N = k)$, for $k = 1, 2, 3, \ldots$.

3. For what values of c_1 and c_2 are the following two functions probability mass functions?

   (a) $p_X(x) = c_1 x,\ 1 \leqslant x \leqslant n.$

   (b) $p_X(x) = \dfrac{c_2}{x(x+1)},\ x \geqslant 1.$

4. You roll 5 dice. Let X be the smallest number shown and Y the largest.

   (a) Find the distribution of X and $E(X)$.

   (b) Find the distribution of Y and $E(Y)$.

5. You take an exam that contains 20 multiple-choice questions. Each question has 4 possible options. You know the answer to 10 questions, but you have no idea about the other 10 questions so you choose answers randomly. Your score X on the exam is the total number of correct answers. Find the PMF of X ? What is $P(X > 15)$?

6. Let X have density $f_X(x) = cx^{-d}$, $x > 1$. Find (a) c, (b) $E(X)$. In each case state for what values of d your answer holds.

7. (a) Find the constant c such that the function

$$f(x) = \begin{cases} cx^2 & 0 < x < 3 \\ 0 & otherwise \end{cases}$$

   is a density function.

   (b) Compute $P(1 < X < 2)$.

   (c) Find the distribution function F_X of X.

   (d) Use F_X to find $P(1 < X \leqslant 2)$.

8. Which of the following can be density functions? For those that are, find the value of c, and the distribution function $F(x)$.

   (a) $f(x) = \begin{cases} cx(1-x), & 0 < x < 1\ , \\ 0, & \text{otherwise}\ . \end{cases}$

   (b) $f(x) = \begin{cases} cx^{-1}, & x \geqslant 1\ , \\ 0, & \text{otherwise}\ . \end{cases}$

   (c) $f(x) = c\exp(-x^2 + 4x)$, $-\infty < x < \infty$.

   (d) $f(x) = ce^x(1 + e^x)^{-2}$, $-\infty < x < \infty$.

9. Which of the following can be distribution functions? For those that are, find the density $f(x)$.

   (a) $F(x) = \begin{cases} 1 - e^{-x^2}, & x \geqslant 0\ , \\ 0, & \text{otherwise}\ . \end{cases}$

   (b) $F(x) = \begin{cases} \exp(-x^{-1}) & x \geqslant 0\ , \\ 0, & \text{otherwise}\ . \end{cases}$

   (c) $F(x) = e^x(e^x + e^{-x})^{-1}$, $-\infty < x < \infty$.

   (d) $F(x) = e^{-x^2} + e^x(e^x + e^{-x})^{-1}$, $-\infty < x < \infty$.

10. Let X have the density $f_X(x) = \dfrac{e^x}{2\sinh a}$, $-a < x < a$. Show that $E(X) = a\coth a - 1$.

11. A fair coin and a fair 6-sided die are thrown repeatedly until the the first time 6 appears on the die. Let X be the number of heads obtained (we are including the heads that may have occurred together with the first 6) in the count. The generating function of X is $P_X(s)$. What is $P_X(s)$?

12. I am new to basketball and so I am practising to shoot the ball through the basket. If the probability of success at each throw is 0.2, how many times would I fail on average before a success occurs

13. The number of misprints per page of text is commonly modeled by a Poisson distribution. It is given that the parameter of this distribution is $\lambda = 0.6$ for a particular book. Find the probability that there are exactly 2 misprints on a given page in the book. How about the probability that there are 2 or more misprints?

14. Four points are chosen on the unit sphere. What is the probability that the origin lies inside the tetrahedron determined by the four points?

15. Your dart is equally likely to hit any point of a circular dart board. Its height above the bull is Y (negative if below the bull), and its distance from the bull is R. Find the density and distribution of Y and of R. What is $E(R)$?

16. An urn contains one carmine ball and one magenta ball. A ball is drawn at random; if it is carmine the game is over. If it is magenta then the ball is returned to the urn together with one extra magenta ball. This procedure is repeated until 10 draws have been made or a carmine ball is drawn, whichever is sooner. Let X be the number of draws. Find $p_X(x)$ and $E(X)$.

    Now suppose the game can only be terminated by the appearance of a carmine ball. Let Y be the number of draws. Find the distribution $p_Y(y)$ and $E(Y)$.

17. My book has f pages, with n characters on each page. Each character is wrong with probability p, independently of the others. If I proofread the book once, I detect any error with probability δ independently of any other detections and of other proofreadings.

    (a) Show that the number of errors remaining has a binomial distribution.

    (b) I wish to proofread enough times that the chance of no errors remaining exceeds $\frac{1}{2}$. If $f = 2^8, n = 2^9, p = 2^{-8}, \delta = \frac{1}{2}$, show that 10 readings will do.

18. (**Negative Binomial** Distribution) A close relative of the binomial distribution arises when we ask the opposite question. The question above is 'Given n independent Bernoulli trials, what is the chance of k successes?' Suppose we ask instead 'Given we must have k successes, what chance that we need n trials?' Find the p.m.f. of such an observation / random variable.

19. A **de Moivre** trial is one where the outcome is one of 3 possibilities, viz., S, F, T with associated probabilities $P(S) = p, P(F) = q, P(T) = r$, where $p + q + r = 1$. You perform a sequence of independent de Moivre trials.

    (a) Let X be the number of trials up to and including the first trial at which you have recorded at least one success and at least one failure. Find the distribution of X, and its mean.

    (b) Let Y be the number of trials until you have recorded at least j successes and at least k failures. Find the distribution of Y.