AERO-423, Spring 2024, Homework 5
(Due date: Thursday April 25, 2024)

NAME:____________________________

Show all work and justify your answer!

For all problems use: $J_2 = 1.082645 \cdot 10^{-3}$, $\mu = 398,600.4415$ km^3/s^2, and (Earth radius) $R_\oplus = 6,378.135$ km.

1. (8 pts) A satellite is in a "Molniya" orbit with an apogee altitude of $39,906$ km over the Russian territory, with an inclination of $63.435°$, and with argument of perigee $270°$. The period of this orbit is half a sidereal day. Considering only the J_2 effect, answer the following:

a. (1 points) Is Russia toward Northern point of the orbit or is it towards the South?

b. (3 points) Find the perturbation (rate change) in the argument of perigee (degrees/day). How does the perigee position change over time?

c. (2 points) Find the perturbation (rate change) of node line (degrees/day).

c. (2 points) Change in orbital period.

Solution:

a. Argument of periapsis is $270°$. Apogee (Russia) occurs at the Northern point of this Molniya orbit.

b,c,d.

$$
\begin{aligned}
T &= 12 \text{ hours} \\
a &= \left(\mu(\frac{T}{2\pi})^2\right)^{1/3} = 26610 \text{ kms} \\
r_a &= 6,378 + 39906 = 46284 \text{ km} \\
r_p &= 2a - r_a = 6936.4 \text{ km} \\
e &= \frac{r_a - r_p}{r_a + r_p} = 0.7393 \\
p &= a(1-e^2) = 12065 \text{ km} \\
n_0 &= \sqrt{\frac{\mu}{a^3}} = 1.4544 \times 10^{-4} \text{ rad/s} \\
\dot{\omega} &= \frac{3J_2}{4}(\frac{R_E}{p})^2 n_0 (5cos^2(i) - 1) \approx 0 \\
& \implies \text{No change in the position of perigee over time.} \\
\dot{\Omega} &= -\frac{3J_2}{2}(\frac{R_E}{p})^2 n_0 cos(i) = -2.95 \times 10^{-8} \text{ rad/s} = \underline{-0.146°/day \text{ to the west}} \\
n &= n_0(1 + \frac{3J_2}{4}(\frac{R_E}{p})^2(2 - 3sin^2(i)\sqrt{1-e^2}), \text{ perturbed mean motion} \\
\Delta T &= \frac{2\pi}{n} - \frac{2\pi}{n_0} = \underline{2.64 \text{ sec.}}
\end{aligned}
$$

2. (5 pts) Compute the yearly perigee maintenance cost of a satellite that is almost in a Tundra orbit. The orbital parameters are:

$a = 42,164$ km, $e = 0.24$, $i = 70$ deg, $\omega = 270$ deg. Compute the annual perigee maintenance cost using a radial impulse in terms of percentage $\frac{\Delta m}{m_0}$ using $I_{sp} = 300$ s, $g = 9.81$ m/s^2, 1 year = 365.25 days and use $\frac{\Delta m}{m_0} = 1 - e^{-\Delta v/(g\, I_{sp})}$. Why do you think the perigee maintenance cost is low or high?

Consider only the J_2 effect.

Solution:

First, perigee maintenance using radial Δv:

$$
\begin{aligned}
p &= a(1-e^2) = 39735 \text{ km} \\
T &= 2\pi\sqrt{\frac{a^3}{\mu}} = 86164 \text{ s} = 1436 \text{ minutes} \\
\Delta v_{orbit} &= |-e\sqrt{\mu/p}\frac{3\pi J_2}{2}\frac{R_E^2}{p^2}(5cos^2 i - 1)| = 4.148 \times 10^{-5} \text{ km/s} \\
\Delta v_{year} &= \frac{3600 \times 24 \times 365.25}{T}\Delta v_{orbit} = 0.0152 \text{ km/s} \\
\frac{\Delta m}{m} &= 1 - exp(-\frac{22.79 \times 1,000}{9.81 \times 300}) = 0.51\%
\end{aligned}
$$

Close to critical inclination (frozen orbit) - low maintenance cost.

3. (6 points) The following are the estimated attitudes of a spacecraft using three different coordinates (at different instances).

i. quaternion

$$\boldsymbol{q} = \begin{Bmatrix}0.0921, & 0.2306, & 0.4191, & 0.8733\end{Bmatrix}^T$$

ii. Principal axis-angle ($\boldsymbol{e}$ and ϕ)

$$\boldsymbol{e} = \begin{Bmatrix}0.1382, & 0.4075, & 0.9027\end{Bmatrix}^T \qquad \phi = 61.1403^\circ$$

iii. Gibbs vector

$$\boldsymbol{\rho} = \begin{Bmatrix}0.0890, & 0.2976, & 0.4994\end{Bmatrix}^T$$

If the true attitude quaternion is

$$\boldsymbol{q}_{\mathrm{t}} = \begin{Bmatrix}0.0742, & 0.2363, & 0.4418, & 0.8623\end{Bmatrix}^T$$

a. (3 points) Compute the attitude errors in the estimated attitudes.

b. (3 points) If $\boldsymbol{r} = \begin{Bmatrix}0.1250, & -0.5735, & -0.8096\end{Bmatrix}^{\mathrm{T}}$ is a measured inertial direction, compute the direction errors of $\boldsymbol{r}$ that is measured using the estimated attitudes (i, ii, iii).

Report all angles in degrees.

Solution:

- True DCM ($\boldsymbol{T}$) from $\boldsymbol{q}_t$

$$\boldsymbol{T} = (q_4^2 - \boldsymbol{q}_v^{\mathrm{T}}\boldsymbol{q}_v)\, I + 2\boldsymbol{q}_v\boldsymbol{q}_v^{\mathrm{T}} - 2q_4\,[\boldsymbol{q}_v\times] = \begin{bmatrix} 0.4980 & 0.7969 & -0.3420 \\ -0.7268 & 0.5987 & 0.3368 \\ 0.4731 & 0.0809 & 0.8773 \end{bmatrix}$$

- DCM from $\boldsymbol{q}$

$$\boldsymbol{C_q} = \begin{bmatrix} 0.5423 & 0.7745 & -0.3256 \\ -0.6896 & 0.6317 & 0.3542 \\ 0.4800 & 0.0324 & 0.8767 \end{bmatrix}$$

- DCM from $\{\boldsymbol{e}, \phi\}$

$$\boldsymbol{C_e} = C_1 = I\ \cos\phi + (1-\cos\phi)\,\boldsymbol{e}\,\boldsymbol{e}^{\mathrm{T}} - [\boldsymbol{e}\times]\ \sin\phi = \begin{bmatrix} 0.4925 & 0.8197 & -0.2924 \\ -0.7614 & 0.5686 & 0.3113 \\ 0.4214 & 0.0693 & 0.9042 \end{bmatrix}$$

- DCM from $\boldsymbol{\rho}$

$$\boldsymbol{q}_\rho = \frac{1}{\sqrt{1+\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\rho}}} \begin{Bmatrix}\boldsymbol{\rho} \\ 1\end{Bmatrix} = \begin{Bmatrix}0.0767 \\ 0.2565 \\ 0.4304 \\ 0.8620\end{Bmatrix}$$

$$\boldsymbol{C_\rho} = \begin{bmatrix} 0.4978 & 0.7814 & -0.3762 \\ -0.7027 & 0.6177 & 0.3531 \\ 0.5083 & 0.0886 & 0.8566 \end{bmatrix}$$

a. Attitude errors

- $$\Delta_{\boldsymbol{q}} = T\,C_{\boldsymbol{q}}^T = \begin{bmatrix} 0.9986 & 0.0389 & -0.0350 \\ -0.0401 & 0.9986 & -0.0342 \\ 0.0336 & 0.0356 & 0.9988 \end{bmatrix}$$
$$\mathrm{tr}(\Delta_{\boldsymbol{q}}) = 1 + 2\cos\varphi_{\boldsymbol{q}} \implies \boxed{\varphi_{\boldsymbol{q}} = 3.6^\circ}$$

- $$\Delta_{\boldsymbol{e}} = T\,C_{\boldsymbol{e}}^T = \begin{bmatrix} 0.9985 & -0.0326 & -0.0442 \\ 0.0343 & 0.9986 & 0.0397 \\ 0.0428 & -0.0411 & 0.9982 \end{bmatrix}$$
$$\mathrm{tr}(\Delta_{\boldsymbol{e}}) = 1 + 2\cos\varphi_{\boldsymbol{e}} \implies \boxed{\varphi_{\boldsymbol{e}} = 3.907^\circ}$$

- $$\Delta_{\boldsymbol{\rho}} = T\,C_{\boldsymbol{\rho}}^T = \begin{bmatrix} 0.9993 & 0.0215 & 0.0307 \\ -0.0207 & 0.9994 & -0.0279 \\ -0.0313 & 0.0272 & 0.9991 \end{bmatrix}$$
$$\mathrm{tr}(\Delta_{\boldsymbol{\rho}}) = 1 + 2\cos\varphi_{\boldsymbol{\rho}} \implies \boxed{\varphi_{\boldsymbol{\rho}} = 2.6688^\circ}$$

b. direction errors

$$\cos\varepsilon_{\boldsymbol{q}} = \boldsymbol{r}^{\mathrm{T}}\,\Delta_{\boldsymbol{q}}\,\boldsymbol{r} \to \boxed{\varepsilon_{\boldsymbol{q}} = 3.5255^\circ}$$
$$\cos\varepsilon_{\boldsymbol{e}} = \boldsymbol{r}^{\mathrm{T}}\,\Delta_{\boldsymbol{e}}\,\boldsymbol{r} \to \boxed{\varepsilon_{\boldsymbol{e}} = 3.6915^\circ}$$
$$\cos\varepsilon_{\boldsymbol{\rho}} = \boldsymbol{r}^{\mathrm{T}}\,\Delta_{\boldsymbol{\rho}}\,\boldsymbol{r} \to \boxed{\varepsilon_{\boldsymbol{\rho}} = 2.4978^\circ}$$

4. (3+3 points) Determine the attitude (quaternion and DCM) using the Davenport's q-method and the TRIAD method. The inertial ($\boldsymbol{r}$) and the observed ($\boldsymbol{b}$) unit-vectors are as follows:

$$\begin{aligned}
\boldsymbol{r}_1^{\mathrm{T}} &= \left\{0.1732, \quad 0.3293, \quad -0.9282\right\} \\
\boldsymbol{r}_2^{\mathrm{T}} &= \left\{-0.4056, \quad -0.5613, \quad 0.7214\right\} \\
\boldsymbol{b}_1^{\mathrm{T}} &= \left\{-0.3306, \quad -0.3173, \quad 0.8888\right\} \\
\boldsymbol{b}_2^{\mathrm{T}} &= \left\{0.5563, \quad 0.5208, \quad -0.6475\right\}
\end{aligned}$$

Use the weights: $[\alpha_1, \alpha_2] = [1/3, 1/4]$.

Solution:

TRIAD:
Attitude matrix is simply

$$\begin{aligned}
C_{I\to B} &= \left[\,\boldsymbol{b}_1 \quad \boldsymbol{b}_1\times\boldsymbol{b}_2 \quad \boldsymbol{b}_1\times(\boldsymbol{b}_1\times\boldsymbol{b}_2)\,\right]\left[\,\boldsymbol{r}_1 \quad \boldsymbol{r}_1\times\boldsymbol{r}_2 \quad \boldsymbol{r}_1\times(\boldsymbol{r}_1\times\boldsymbol{r}_2)\,\right]^{-1} \\
&= \begin{bmatrix} 0.0219 & -0.9997 & 0.0055 \\ -0.9885 & -0.0208 & 0.1500 \\ -0.1499 & -0.0088 & -0.9887 \end{bmatrix}
\end{aligned}$$

$$\boldsymbol{q}_{\text{opt}} = \begin{Bmatrix} 0.7126 \\ -0.6975 \\ -0.0506 \\ 0.0557 \end{Bmatrix}$$

q-Method:
$\boldsymbol{q}_{\text{opt}}$ is computed from the relationship

$$K\,\boldsymbol{q}_{\text{opt}} = \lambda_{\max}\,\boldsymbol{q}_{\text{opt}}$$

where

$$K = \begin{bmatrix} B + B^{\mathrm{T}} - \mathrm{tr}[B]\,I_{3\times3} & \sum_{i=1}^{n} \alpha_i [\boldsymbol{b}_i \times]^T \boldsymbol{r}_i \\ \left(\sum_{i=1}^{n} \alpha_i [\boldsymbol{b}_i \times]^T \boldsymbol{r}_i \right)^{\mathrm{T}} & \mathrm{tr}[B] \end{bmatrix}$$

where $B = \sum_{i=1}^{n} \alpha_i \boldsymbol{b}_i \boldsymbol{r}_i^{\mathrm{T}}$. Therefore, using MATLAB, the eigenvalues and eigenvectors can be computed using `[W,L] = eig(K)`. And the attitude are

$$\boldsymbol{q}_{\text{opt}} = \begin{Bmatrix} 0.7126 \\ -0.6975 \\ -0.0506 \\ 0.0557 \end{Bmatrix}$$

$$C_{I\to B} = \begin{bmatrix} 0.0219 & -0.9997 & 0.0055 \\ -0.9885 & -0.0208 & 0.1500 \\ -0.1499 & -0.0088 & -0.9887 \end{bmatrix}$$