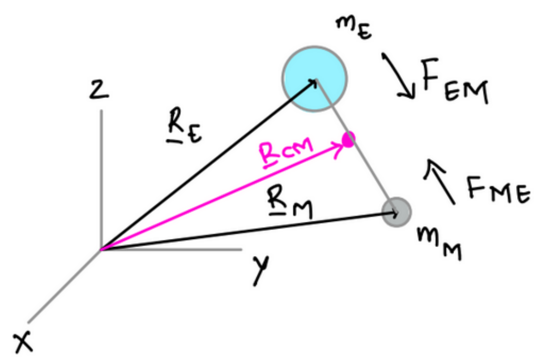
AERO-423, Spring 2024, Homework #2
(Due date: 23:59 hours, Sunday, February 25, 2024)

Show all the work and justify your answer!
Make sure to upload your submission to the Canvas, in time.

1. Masses of the Earth and the Moon are given as m_{E} and m_{M} respectively. Their respective position vectors are given as $\boldsymbol{R}_{\mathrm{E}}$ and $\boldsymbol{R}_{\mathrm{M}}$.



For the Earth-Moon system described in the figure, answer the following questions:

Part a. (1 points) Position vector of the center of mass, $\boldsymbol{R}_{\mathrm{CM}}$

Solution:

$$\boldsymbol{R}_{\mathrm{CM}} = \frac{m_{\mathrm{E}}\boldsymbol{R}_{\mathrm{E}} + m_{\mathrm{M}}\boldsymbol{R}_{\mathrm{M}}}{m_{\mathrm{E}} + m_{\mathrm{M}}}$$

Part b. (1 points) If the relative distance between the Earth and the Moon is $\boldsymbol{r} = \boldsymbol{R}_{\mathrm{E}} - \boldsymbol{R}_{\mathrm{M}}$, write the force of Moon on the Earth $\boldsymbol{F}_{\mathrm{EM}}$ and the force of Earth on the Moon $\boldsymbol{F}_{\mathrm{ME}}$. The gravitational constant is G.

Solution:

$$\boldsymbol{F}_{\mathrm{EM}} = -\frac{G m_{\mathrm{E}} m_{\mathrm{M}}}{r^3}\boldsymbol{r}$$
$$\boldsymbol{F}_{\mathrm{ME}} = \frac{G m_{\mathrm{E}} m_{\mathrm{M}}}{r^3}\boldsymbol{r}$$

Part c. (2 points) Derive relative equations of motion

Solution:

$$\boldsymbol{F}_{\mathrm{EM}} = m_{\mathrm{E}}\ddot{\boldsymbol{R}}_{\mathrm{E}} = -\frac{Gm_{\mathrm{E}}m_{\mathrm{M}}}{r^3}\boldsymbol{r}$$

$$\boldsymbol{F}_{\mathrm{ME}} = m_{\mathrm{M}}\ddot{\boldsymbol{R}}_{\mathrm{M}} = \frac{Gm_{\mathrm{E}}m_{\mathrm{M}}}{r^3}\boldsymbol{r}$$

subtracting the above two equations yield

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{R}}_{\mathrm{E}} - \ddot{\boldsymbol{R}}_{\mathrm{M}} = -\frac{G(m_{\mathrm{E}}+m_{\mathrm{M}})}{r^3}\boldsymbol{r}$$

Part d. (4 points) Prove that the equation of motion of the Earth with respect to the center of mass of the Earth-Moon system is

$$\ddot{\boldsymbol{r}} = -\frac{Gm_{\mathrm{M}}^3}{(m_{\mathrm{E}}+m_{\mathrm{M}})^2 r^3}\boldsymbol{r}$$

where $\boldsymbol{r} = \boldsymbol{R}_{\mathrm{E}} - \boldsymbol{R}_{\mathrm{CM}}$.

Also, derive the equation of motion for m_{M} with respect to $\boldsymbol{R}_{\mathrm{CM}}$ with $\boldsymbol{r}' = \boldsymbol{R}_{\mathrm{M}} - \boldsymbol{R}_{\mathrm{CM}}$.

Solution:

$$\boldsymbol{R}_{\mathrm{CM}} = \frac{m_{\mathrm{E}}\boldsymbol{R}_{\mathrm{E}} + m_{\mathrm{M}}\boldsymbol{R}_{\mathrm{M}}}{m_{\mathrm{E}}+m_{\mathrm{M}}} \qquad \text{and} \qquad \text{from 1c} \quad \boldsymbol{r} = \boldsymbol{R}_{\mathrm{E}} - \boldsymbol{R}_{\mathrm{M}}$$

$$\Longrightarrow \boldsymbol{r} = \boldsymbol{R}_{\mathrm{E}} - \frac{\boldsymbol{R}_{\mathrm{CM}}(m_{\mathrm{E}}+m_{\mathrm{M}}) - m_{\mathrm{E}}\boldsymbol{R}_{\mathrm{E}}}{m_{\mathrm{M}}} = \frac{(m_{\mathrm{E}}+m_{\mathrm{M}})}{m_{\mathrm{M}}}(\boldsymbol{R}_{\mathrm{E}} - \boldsymbol{R}_{\mathrm{CM}})$$

we know that, from 1c)

$$\ddot{\boldsymbol{r}} = -\frac{G(m_{\mathrm{E}}+m_{\mathrm{M}})}{r^3}\boldsymbol{r}$$

By substitution, and by denoting $\boldsymbol{R}_{\mathrm{E}} - \boldsymbol{R}_{\mathrm{CM}}$ also as $\boldsymbol{r}$,

$$\frac{(m_{\mathrm{E}}+m_{\mathrm{M}})}{m_{\mathrm{M}}}\ddot{\boldsymbol{r}} = -\frac{G(m_{\mathrm{E}}+m_{\mathrm{M}})}{(\frac{m_{\mathrm{E}}+m_{\mathrm{M}}}{m_{\mathrm{M}}})^3 r^3}(\frac{m_{\mathrm{E}}+m_{\mathrm{M}}}{m_{\mathrm{M}}})\boldsymbol{r}$$

$$\Longrightarrow \ddot{\boldsymbol{r}} = -\frac{Gm_{\mathrm{M}}^3}{(m_{\mathrm{E}}+m_{\mathrm{M}})^2 r^3}\boldsymbol{r}$$

A similar equation of motion for m_{M} with respect to $\boldsymbol{R}_{\mathrm{CM}}$ is obtained by reversing the subscripts as

$$\ddot{\boldsymbol{r}}' = -\frac{Gm_{\mathrm{E}}^3}{(m_{\mathrm{E}}+m_{\mathrm{M}})^2 r'^3}\boldsymbol{r}' \qquad \text{where} \qquad \boldsymbol{r}' = \boldsymbol{R}_{\mathrm{M}} - \boldsymbol{R}_{\mathrm{CM}}$$

2. Consider a satellite orbiting a planet of radius R_p with gravitational parameter μ. At perigee, if the position (vector) $\boldsymbol{r}$ and velocity $\boldsymbol{v}$ of the satellite are given as

$$\boldsymbol{r} = 3R_p\,\hat{\boldsymbol{e}} \qquad \text{and} \qquad |\boldsymbol{v}| = \sqrt{\frac{5\mu}{12\,R_p}}$$

where $\hat{\boldsymbol{e}}$ is the unit-vector pointing to perigee. Using only the given information answer the following:

Part a. (3 points) Semi-major axis, a

Solution: Energy equation tell us that (students need to identify this to begin with - learning exercise includes this analysis from the textbook). Also, below R_p and R are same.

$$\frac{v^2}{2} = \mu\Big[\frac{1}{r} - \frac{1}{2a}\Big]$$

$$\text{where} \qquad r = |\boldsymbol{r}| = 3R \qquad \text{and} \qquad v^2 = \frac{5\mu}{12R}$$

$$\text{solving which yields} \quad a = 4R$$

Part b. (1 point) Eccentricity, e

Solution:

$$r_p = a(1-e) \implies e = 0.25$$

Part c. (1 point) Radius at apogee, r_a

Solution: $r_a = a(1+e) \implies r_a = 5R$

Part d. (1 point) Semi-latus rectum, p

Solution: $p = a(1-e^2) \implies p = \dfrac{15R}{4}$

Part e. (1 point) Magnitude of angular momentum, h

Solution: $h = \sqrt{\mu p}$ or $h = r_p v_p \implies h = \sqrt{\dfrac{15\mu R}{4}}$

Part f. (1 point) Magnitude of velocity at apogee, v_a

Solution: $r_p v_p = r_a v_a \implies v_a = \sqrt{\dfrac{3\mu}{20R}}$

Part g. (1 point) True anomaly, φ, (in degrees) at $r = 3.5\,R_p$

Solution: $r = \dfrac{p}{1 + e\cos(\varphi)} \implies \varphi = 1.28\,rad$ or $73.39°$

Part h. (1 point) Tangential and radial velocities, $v_\perp$ and v_r, at φ

Solution: $v_\perp = \dfrac{\mu}{h}(1 + e\cos(\varphi)) = \sqrt{\dfrac{4\mu}{15R}}(1 + 0.25\cos\varphi) = \dfrac{1}{7}\sqrt{\dfrac{15\mu}{R}}$

$v_r = \dfrac{\mu}{h} e\sin(\varphi) = \sqrt{\dfrac{3\mu}{196R}} = \dfrac{1}{14}\sqrt{\dfrac{3\mu}{R}}$

Part i. (1 point) Eccentric anomaly, E (in degrees)

Solution: $\tan(\frac{\varphi}{2}) = \sqrt{\frac{1+e}{1-e}} \tan(\frac{E}{2}) \implies E = 1.047198 \quad \text{or} \quad 60°$

Part j. (1 point) Mean anomaly, M (in degrees)

Solution: $M = E - e \sin E \implies M = 0.830691 \quad \text{or} \quad 47.595°$

3. (5 points) Using the mean anomaly M and eccentricity e obtained in problem 2, write a MATLAB/Python *Kepler solver function* to solve for eccentric anomaly using Newton's method. Write the equations, and comment on the choice of initial value of E, convergence, tolerance, and number of iterations. Plot the values of E (y-axis) vs iterations (x-axis). Attach your code and the plot to this submission.

Solution:

Homework #2

Problem 2

Identify the energy equation

$$\frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$\frac{5\mu}{12R_p} - \frac{\mu}{R_p} = -\frac{\mu}{2a}$$

$$a = 4R_p$$

```
syms Rp mu real % Variables

a = 4*Rp;
eccentricity = 1 - 3*Rp/a
```

```
eccentricity =
```

$$\frac{1}{4}$$

```
p = a*(1-eccentricity^2)
```

```
p =
```

$$\frac{15\ \mathrm{Rp}}{4}$$

```
Vp = sqrt(5*mu / (12*Rp) );

h = sqrt(mu * p)
```

```
h =
```

$$\sqrt{\frac{15\ \mathrm{Rp}\ \mu}{4}}$$

```
Rapogee = a*(1 + eccentricity);
Rperigee = a*(1 - eccentricity);
v_apogee = simplify( Rperigee*Vp / Rapogee )
```

```
v_apogee =
```

$$\frac{\sqrt{15}\ \sqrt{\frac{\mu}{\mathrm{Rp}}}}{10}$$

```
r = 3.5*Rp;

true_anomaly = eval(acos( (p/r - 1) / eccentricity));
fprintf("True Anomaly: %f rad (%f degrees)", true_anomaly, true_anomaly*180/pi);
```

```
True Anomaly: 1.281045 rad (73.398450 degrees)
```

```
v_perp = simplify( mu*( 1 + eccentricity*cos(true_anomaly) ) / h )
```

v_perp =

$$\frac{\sqrt{15}\ \mu}{7\ \sqrt{\mathrm{Rp}\ \mu}}$$

```
v_radial = simplify( mu*( eccentricity*sin(true_anomaly) ) / h )
```

v_radial =

$$\frac{\sqrt{3}\ \mu}{14\ \sqrt{\mathrm{Rp}\ \mu}}$$

```
eccentric_anomaly = 2*atan(tan(true_anomaly/2) * ...
    ( sqrt( (1-eccentricity)/(1+eccentricity) )));
fprintf("Eccentric Anomaly: %f rad (%f degrees)", ...
```

```
Eccentric Anomaly: 1.047198 rad (60.000000 degrees)
```

```
        eccentric_anomaly, eccentric_anomaly*180/pi);

Mean_anomaly = eccentric_anomaly - eccentricity * sin(eccentric_anomaly);
fprintf("Mean Anomaly: %f rad (%f degrees)", (Mean_anomaly), (Mean_anomaly)*180/pi);
```

```
Mean Anomaly: 0.830691 rad (47.595100 degrees)
```

Kepler equation solver using Newton Raphson Method

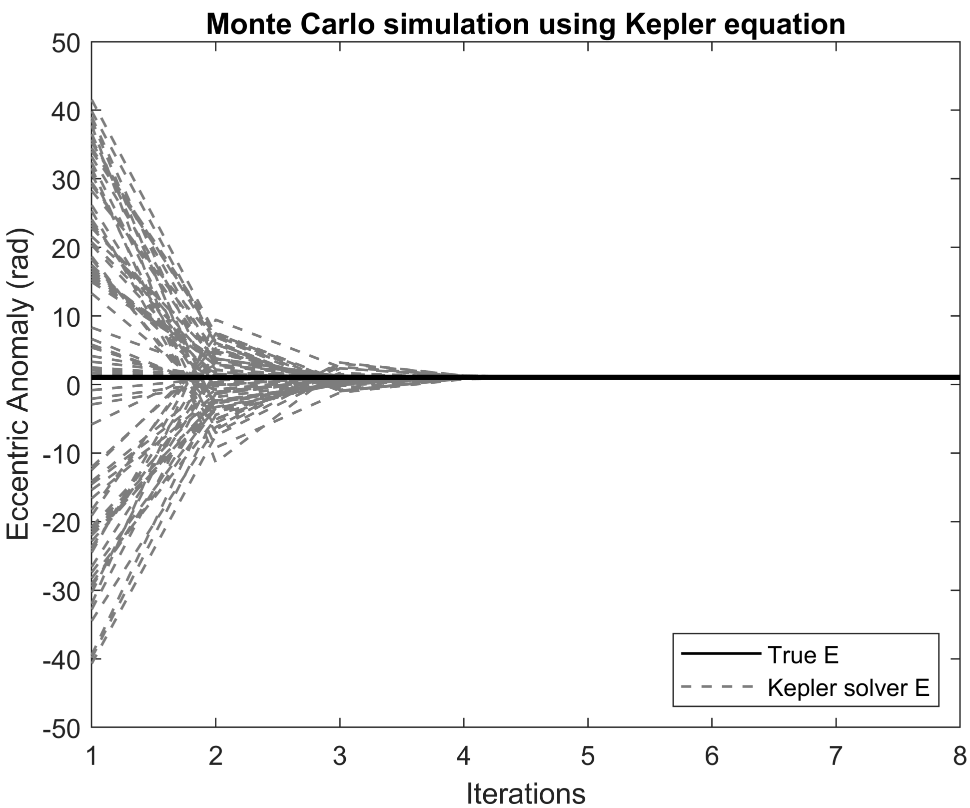
In root finding methods, a good initial guess is important. Generally, a guess close to the solution converges quicker for a convex optimization approach such as Newton's method. This can be seen in the plot as number of iterations until convergence is seemingly related to the choice of your guess.

Keep in mind that: Less computations are important in resource constrained space applications. Divergence should never occur. These are some topics for you to explore.

```
E0 = Mean_anomaly/2;
maxIter = 100;
tolerance = 1e-6;

% Monte Carlo simulation
nSims = 100;
prtrbs = randi([-100 100], nSims, 1); % 100 random values of E0
E0 = prtrbs * E0;

Ecc_anomaly = {};
iters = {};

for ii = 1:nSims
    [iters{ii}, Ecc_anomaly{ii}] = keplerSolver(Mean_anomaly, E0(ii), ...
                                       eccentricity, maxIter, tolerance);
```

```
end
```

```
% Plots
figure
plot(1:8, eccentric_anomaly*ones(8, 1), 'k', 'LineWidth', 1);
    hold on
for ii = 1:nSims
    plot(1:iters{ii}, Ecc_anomaly{ii}(1:iters{ii}), '--','Color',[.5,.5,.5], ...
                    'LineWidth', 1);
end
plot(1:8, eccentric_anomaly*ones(8, 1), 'k', 'LineWidth', 2);
hold off;
lgd = legend('True E', 'Kepler solver E');
lgd.Location = 'southeast';
title('Monte Carlo simulation using Kepler equation');
ylabel('Eccentric Anomaly (rad)'); xlabel('Iterations');
```




```
function [i, E] = keplerSolver(M, E0, ecc, maxIter, tolerance)
    i = 1;
    E = zeros(maxIter, 1);
    E(1) = E0;
    tol = 1;

    while (tol > tolerance && i < maxIter)
        f = E(i) - ecc*sin(E(i)) - M;
        df = 1-ecc*cos(E(i));
        E(i+1) = E(i) - f/df;
```

```
        tol = abs(E(i+1)-E(i));
        i = i+1;
        if (i > maxIter)
            fprintf('Maximum Iterations reached, no convergence.');
            break;
        end
    end

%     fprintf('tolerance reached: %d,\t final value E: %d\n', tol, E(i));
end
```