AERO-423, Spring 2024, Homework #1
(Due date: 23:59 hours, Wednesday, February 7, 2024)

Show all the work and justify your answer!
Make sure to upload your submission to the Canvas, in time.

1. A direction cosine matrix (DCM) $C_{\mathcal{BA}}$ represents transformation from coordinate frame $\mathcal{A}$ to frame $\mathcal{B}$. The vectors $\boldsymbol{v}_{\mathcal{A}}$ and $\boldsymbol{v}_{\mathcal{B}}$ are the vector $\boldsymbol{v}$ in frames $\mathcal{A}$ and $\mathcal{B}$, respectively. Answer the following.

Part a. (2 points) Express the relationship between $\boldsymbol{v}_{\mathcal{B}}$ and $\boldsymbol{v}_{\mathcal{A}}$. Using $C_{\mathcal{BA}}$ write the expressions for both $\boldsymbol{v}_{\mathcal{B}}$ and $\boldsymbol{v}_{\mathcal{A}}$.

Solution:

$\boldsymbol{v}_{\mathcal{B}} = C_{\mathcal{BA}}\,\boldsymbol{v}_{\mathcal{A}}$

$\boldsymbol{v}_{\mathcal{A}} = C_{\mathcal{BA}}^{\mathrm{T}}\,\boldsymbol{v}_{\mathcal{B}}$

Part b. (2 points) If the direction cosine matrix C is parameterized using the 3-2-1 Euler angle sequence. The angles are ϑ_1, ϑ_2, and ϑ_3 . Show the steps in the computation of C.

Solution:

$$C = C_1(\vartheta_3)\,C_2(\vartheta_2)\,C_3(\vartheta_1) \qquad \text{where} \qquad C_1(\vartheta_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta_3 & \sin\vartheta_3 \\ 0 & -\sin\vartheta_3 & \cos\vartheta_3 \end{bmatrix}$$

$$C_2(\vartheta_2) = \begin{bmatrix} \cos\vartheta_2 & 0 & -\sin\vartheta_2 \\ 0 & 1 & 0 \\ \sin\vartheta_2 & 0 & \cos\vartheta_2 \end{bmatrix} \quad \text{and} \quad C_3(\vartheta_1) = \begin{bmatrix} \cos\vartheta_1 & \sin\vartheta_1 & 0 \\ -\sin\vartheta_1 & \cos\vartheta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \cos\vartheta_1\cos\vartheta_2 & \cos\vartheta_2\sin\vartheta_1 & -\sin\vartheta_2 \\ \sin\vartheta_3\sin\vartheta_2\cos\vartheta_1 - \cos\vartheta_3\sin\vartheta_1 & \sin\vartheta_3\sin\vartheta_2\sin\vartheta_1 + \cos\vartheta_3\cos\vartheta_1 & \sin\vartheta_3\cos\vartheta_2 \\ \cos\vartheta_3\sin\vartheta_2\cos\vartheta_1 + \sin\vartheta_3\sin\vartheta_1 & \cos\vartheta_3\sin\vartheta_2\sin\vartheta_1 - \sin\vartheta_3\cos\vartheta_1 & \cos\vartheta_3\cos\vartheta_2 \end{bmatrix}$$

Part c. (10 points) Determine the forward and inverse kinematic equations associated with the "3-2-1" Euler angle rotation sequence. Specify frame transformations and explain your approach. Is there any singularity?

Solution: Frame transformations (2pts). $\boldsymbol{\omega}$ expression and derivation (4pts). Inverse (2pts). Singularity with reasoning (2pts).

The sequence of frame transformations for "3-2-1" Euler angle sequence are:

$$B \xleftarrow{C_1(\vartheta_3)} H \xleftarrow{C_2(\vartheta_2)} G \xleftarrow{C_3(\vartheta_1)} I$$

(coordinate frames are named arbitrarily).

Let the angular velocity in the B frame be $\boldsymbol{\omega}_B^{B/I} = \omega_1\,\hat{\boldsymbol{b}}_1 + \omega_2\,\hat{\boldsymbol{b}}_2 + \omega_3\,\hat{\boldsymbol{b}}_3$. We know that angular velocity is a linear operator and

$$\boldsymbol{\omega}^{B/I} = \boldsymbol{\omega}^{B/H} + \boldsymbol{\omega}^{H/G} + \boldsymbol{\omega}^{G/I}$$

$$\boldsymbol{\omega}^{B/I} = \begin{Bmatrix} 0 \\ 0 \\ \dot{\vartheta}_1 \end{Bmatrix}_G + \begin{Bmatrix} 0 \\ \dot{\vartheta}_2 \\ 0 \end{Bmatrix}_H + \begin{Bmatrix} \dot{\vartheta}_3 \\ 0 \\ 0 \end{Bmatrix}_B =$$

$$= C_1(\vartheta_3)C_2(\vartheta_2) \begin{Bmatrix} 0 \\ 0 \\ \dot{\vartheta}_1 \end{Bmatrix} + C_1(\vartheta_3) \begin{Bmatrix} 0 \\ \dot{\vartheta}_2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} \dot{\vartheta}_3 \\ 0 \\ 0 \end{Bmatrix}$$

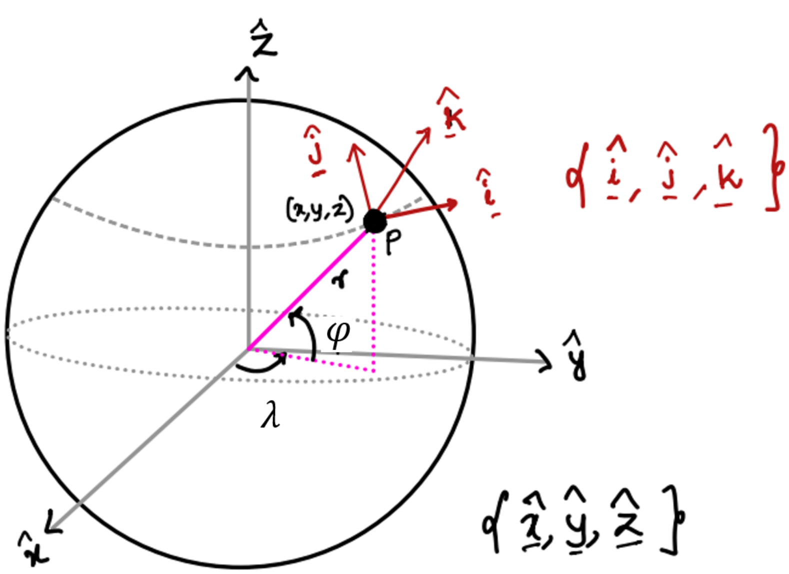
$$= \begin{bmatrix} -\sin(\vartheta_2) & 0 & 1 \\ \cos(\vartheta_2)\sin(\vartheta_3) & \cos(\vartheta_3) & 0 \\ \cos(\vartheta_2)\cos(\vartheta_3) & -\sin(\vartheta_3) & 0 \end{bmatrix} \begin{Bmatrix} \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \\ \dot{\vartheta}_3 \end{Bmatrix}$$

This is the direct kinematic equation.
Inverse kinematic equation:

$$\begin{Bmatrix} \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \\ \dot{\vartheta}_3 \end{Bmatrix} = \frac{1}{\cos\vartheta_2} \begin{bmatrix} 0 & \sin\vartheta_3 & \cos\vartheta_3 \\ 0 & \cos\vartheta_3\cos\vartheta_2 & -\sin\vartheta_3\cos\vartheta_2 \\ \cos\vartheta_2 & \sin\vartheta_2\sin\vartheta_3 & \cos\vartheta_3\sin\vartheta_2 \end{bmatrix} \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

singularity exists when $\vartheta_2 = \pi/2$ radians.

2. The position of a point P can be specified using Cartesian coordinates (x, y, z) or spherical coordinates (r, φ, λ), where r is the radial distance of P from origin of the $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ coordinate system. Angles λ and φ denote azimuth and elevation angles respectively as shown in the below figure. The unit vectors $\{\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}\}$ form the spherical coordinate system to represent any arbitrary point P on the sphere.



Part a. (2 points) Express the Cartesian coordinates (x, y, z) in terms of spherical coordinates (r, φ, λ) and vice-versa. For simplification, you may consider (x, y, z) are positive values.

Solution:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} r\cos\lambda\cos\varphi \\ r\sin\lambda\cos\varphi \\ r\sin\varphi \end{Bmatrix} \qquad \text{and} \qquad \begin{Bmatrix} r \\ \varphi \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \sqrt{x^2+y^2+z^2} \\ \sin^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \sin^{-1}\left(\dfrac{y}{\sqrt{x^2+y^2}}\right) \end{Bmatrix}$$

$$\begin{Bmatrix} r \\ \varphi \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \sqrt{x^2+y^2+z^2} \\ \tan^{-1}\left(\dfrac{z}{\sqrt{x^2+y^2}}\right) \\ \tan^{-1}\left(\dfrac{y}{x}\right) \end{Bmatrix}$$

Part b. (6 points) The equations transforming the unit-vectors $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ to the unit-vectors $\{\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}\}$ are,

$$\begin{aligned} \hat{\boldsymbol{i}} &= -\sin\lambda\,\hat{\boldsymbol{x}} + \cos\lambda\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{j}} &= -\sin\varphi\cos\lambda\,\hat{\boldsymbol{x}} - \sin\varphi\sin\lambda\,\hat{\boldsymbol{y}} + \cos\varphi\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{k}} &= \cos\varphi\cos\lambda\,\hat{\boldsymbol{x}} + \cos\varphi\sin\lambda\,\hat{\boldsymbol{y}} + \sin\varphi\,\hat{\boldsymbol{z}} \end{aligned}$$

Prove that the angular velocity, $\boldsymbol{\omega}$, of the $\{\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}}\}$ frame relative to the inertial $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$ frame can be written as,

$$\boldsymbol{\omega} = a\,\hat{\boldsymbol{i}} + b\,\hat{\boldsymbol{j}} + c\,\hat{\boldsymbol{k}}$$

by finding the expressions of a, b, and c, in terms of φ, λ, and their derivative, $\dot{\varphi}$ and $\dot{\lambda}$.

HINT: Use transport theorem, e.g., $\dfrac{\mathrm{d}\hat{\boldsymbol{k}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{k}}$.

Solution:

$$1)\,\frac{\mathrm{d}\hat{\boldsymbol{k}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{k}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\cos\varphi\cos\lambda\hat{\boldsymbol{x}} + \cos\varphi\sin\lambda\hat{\boldsymbol{y}} + \sin\varphi\hat{\boldsymbol{z}}\} = (a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}) \times \hat{\boldsymbol{k}}$$

$$(-\dot{\varphi}\sin\varphi\cos\lambda - \dot{\lambda}\cos\varphi\cos\lambda)\hat{\boldsymbol{x}} + (-\dot{\varphi}\sin\varphi\sin\lambda + -\dot{\lambda}\cos\varphi\cos\lambda)\hat{\boldsymbol{y}} +$$

$$\dot{\varphi}\cos\varphi\hat{\boldsymbol{z}} = -a\hat{\boldsymbol{j}} + b\hat{\boldsymbol{i}}$$

$$\dot{\lambda}\cos\varphi\,\underbrace{(-\sin\lambda\hat{\boldsymbol{x}} + \cos\lambda\hat{\boldsymbol{y}})}_{\hat{\boldsymbol{i}}} + \dot{\varphi}\,\underbrace{(-\sin\varphi\cos\lambda\hat{\boldsymbol{x}} - \sin\varphi\cos\lambda\hat{\boldsymbol{y}} + \cos\varphi\hat{\boldsymbol{z}})}_{\hat{\boldsymbol{j}}}$$

$$= b\hat{\boldsymbol{i}} - a\hat{\boldsymbol{j}}$$

$$\implies a = -\dot{\varphi} \quad \& \quad b = \dot{\lambda}\cos\varphi$$

$$2)\,\frac{\mathrm{d}\hat{\boldsymbol{j}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{j}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{-\sin\varphi\cos\lambda\hat{\boldsymbol{x}} - \sin\varphi\sin\lambda\hat{\boldsymbol{y}} + \cos\varphi\hat{\boldsymbol{z}}\} = (a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}) \times \hat{\boldsymbol{j}}$$

$$(-\dot{\varphi}\cos\varphi\cos\lambda + \dot{\lambda}\sin\varphi\sin\lambda)\hat{\boldsymbol{x}} + (-\dot{\varphi}\cos\varphi\sin\lambda - \dot{\lambda}\sin\varphi\cos\lambda)\hat{\boldsymbol{y}} + \dot{\varphi}\sin\varphi\hat{\boldsymbol{z}} = -c\hat{\boldsymbol{j}} + a\hat{\boldsymbol{k}}$$

$$\dot{\lambda}\sin\varphi\underbrace{(\sin\lambda\hat{\boldsymbol{x}} - \cos\lambda\hat{\boldsymbol{y}})}_{-\hat{\boldsymbol{i}}} + \dot{\varphi}\underbrace{(-\cos\varphi\cos\lambda\hat{\boldsymbol{x}} - \cos\varphi\sin\lambda\hat{\boldsymbol{y}} - \sin\varphi\hat{\boldsymbol{z}})}_{-\hat{\boldsymbol{k}}} = -c\hat{\boldsymbol{j}} + a\hat{\boldsymbol{k}}$$

$$\implies a = -\dot{\varphi} \quad \& \quad c = \dot{\lambda}\sin\varphi$$

Therefore,

$$\boldsymbol{\omega} = -\dot{\varphi}\hat{\boldsymbol{i}} + \dot{\lambda}\cos\varphi\hat{\boldsymbol{j}} + \dot{\lambda}\sin\varphi\hat{\boldsymbol{k}}$$

This can be done using any two coordinate derivates. Allow partial marks.

Part c. (3 points) Validate that

$$\frac{\mathrm{d}\hat{\boldsymbol{i}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{i}} \qquad \frac{\mathrm{d}\hat{\boldsymbol{j}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{j}} \qquad \frac{\mathrm{d}\hat{\boldsymbol{k}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{k}}$$

Solution: Using the $\boldsymbol{w}$ and derivatives shown in the above question, the students are to show that their math is correct.

$$\frac{\mathrm{d}}{\mathrm{d}t}\{-\sin\lambda\hat{\boldsymbol{x}} + \cos\lambda\hat{\boldsymbol{y}}\} = (a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}) \times \hat{\boldsymbol{i}}$$

$$-\dot{\lambda}(\cos\lambda\hat{\boldsymbol{x}} - \sin\lambda\hat{\boldsymbol{y}}) = (-\dot{\varphi}\hat{\boldsymbol{i}} + \dot{\lambda}\cos\varphi\hat{\boldsymbol{j}} + \dot{\lambda}\sin\varphi\hat{\boldsymbol{k}}) \times \hat{\boldsymbol{i}}$$

$$-\dot{\lambda}(\cos\lambda\hat{\boldsymbol{x}} - \sin\lambda\hat{\boldsymbol{y}}) = \dot{\lambda}(\sin\varphi\hat{j} - \cos\varphi\hat{k})$$

substitute in RHS $\quad \hat{\boldsymbol{j}} = -\sin\varphi\cos\lambda\hat{\boldsymbol{x}} + -\sin\varphi\sin\lambda\hat{\boldsymbol{y}} + \cos\varphi\hat{\boldsymbol{z}};$

and $\qquad \hat{\boldsymbol{k}} = \cos\varphi\cos\lambda\hat{\boldsymbol{x}} + \cos\varphi\sin\lambda\hat{\boldsymbol{y}} + \sin\varphi\hat{\boldsymbol{z}}$

$$\begin{aligned}
\text{RHS} &= \dot{\lambda}(\sin\varphi\hat{j} - \cos\varphi\hat{k}) \\
&= \dot{\lambda}(-\hat{\boldsymbol{x}}\cos\lambda(\sin^2\varphi + \cos^2\varphi) + \hat{\boldsymbol{y}}\sin\lambda(\sin^2\varphi + \cos^2\varphi) + \hat{\boldsymbol{z}}(\sin\varphi\cos\varphi - \cos\varphi\sin\varphi) \\
&\implies \text{RHS} = \dot{\lambda}(-\cos\lambda\hat{\boldsymbol{x}} + \sin\lambda\hat{\boldsymbol{y}}) \\
&= \text{LHS}
\end{aligned}$$

Other two are derived similarly (LHS and RHS are available in part b). Pasting here:

$$1)\,\frac{\mathrm{d}\hat{\boldsymbol{k}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{k}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{\cos\varphi\cos\lambda\hat{\boldsymbol{x}} + \cos\varphi\sin\lambda\hat{\boldsymbol{y}} + \sin\varphi\hat{\boldsymbol{z}}\} = (a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}) \times \hat{\boldsymbol{k}}$$

$$(-\dot{\varphi}\sin\varphi\cos\lambda - \dot{\lambda}\cos\varphi\cos\lambda)\hat{\boldsymbol{x}} + (-\dot{\varphi}\sin\varphi\sin\lambda + -\dot{\lambda}\cos\varphi\cos\lambda)\hat{\boldsymbol{y}} +$$

$$\dot{\varphi}\cos\varphi\hat{\boldsymbol{z}} = -a\hat{\boldsymbol{j}} + b\hat{\boldsymbol{i}}$$

$$\dot{\lambda}\cos\varphi\underbrace{(-\sin\lambda\hat{\boldsymbol{x}} + \cos\lambda\hat{\boldsymbol{y}})}_{\hat{\boldsymbol{i}}} + \dot{\varphi}\underbrace{(-\sin\varphi\cos\lambda\hat{\boldsymbol{x}} - \sin\varphi\cos\lambda\hat{\boldsymbol{y}} + \cos\varphi\hat{\boldsymbol{z}})}_{\hat{\boldsymbol{j}}}$$

$$= b\hat{\boldsymbol{i}} - a\hat{\boldsymbol{j}}$$

$$\text{where} \quad a = -\dot{\varphi} \quad \& \quad b = \dot{\lambda}\cos\varphi$$

$$2)\,\frac{\mathrm{d}\hat{\boldsymbol{j}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \hat{\boldsymbol{j}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\{-\sin\varphi\cos\lambda\hat{\boldsymbol{x}} - \sin\varphi\sin\lambda\hat{\boldsymbol{y}} + \cos\varphi\hat{\boldsymbol{z}}\} = (a\hat{\boldsymbol{i}} + b\hat{\boldsymbol{j}} + c\hat{\boldsymbol{k}}) \times \hat{\boldsymbol{j}}$$

$$(-\dot{\varphi}\cos\varphi\cos\lambda + \dot{\lambda}\sin\varphi\sin\lambda)\hat{\boldsymbol{x}} + (-\dot{\varphi}\cos\varphi\sin\lambda - \dot{\lambda}\sin\varphi\cos\lambda)\hat{\boldsymbol{y}} +$$

$$\dot{\varphi}\sin\varphi\hat{\boldsymbol{z}} = -c\hat{\boldsymbol{j}} + a\hat{\boldsymbol{k}}$$

$$\dot{\lambda}\sin\varphi\underbrace{(\sin\lambda\hat{\boldsymbol{x}} - \cos\lambda\hat{\boldsymbol{y}})}_{-\hat{\boldsymbol{i}}} + \dot{\varphi}\underbrace{(-\cos\varphi\cos\lambda\hat{\boldsymbol{x}} - \cos\varphi\sin\lambda\hat{\boldsymbol{y}} - \sin\varphi\hat{\boldsymbol{z}})}_{-\hat{\boldsymbol{k}}}$$

$$= -c\hat{\boldsymbol{j}} + a\hat{\boldsymbol{k}}$$

$$\text{where} \quad a = -\dot{\varphi} \quad \& \quad c = \dot{\lambda}\sin\varphi$$