Applied Algorithms

CSCI-B505 / INFO-I500

Lecture 6.

Amortized Analysis - 1

M. Oguzhan Kulekci

- Amortized Analysis
  - Aggregate Method
  - Accounting Method
  - Potential Method

Amortized Analysis ?

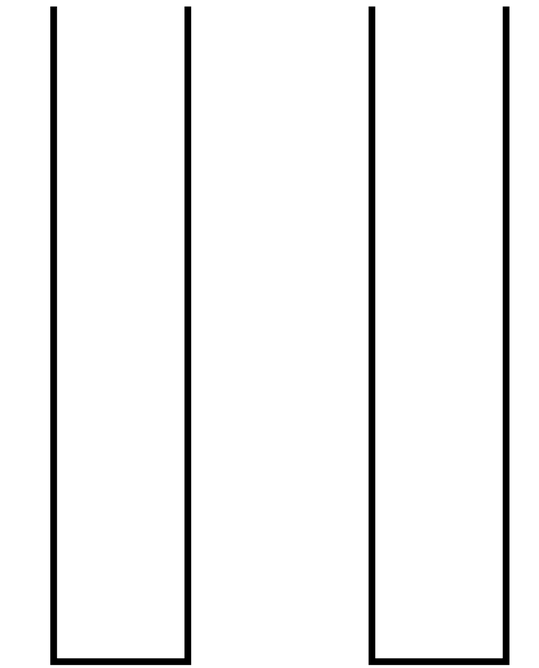
Amortize:

- *gradually write off the initial cost of (an asset) over a period*
- *reduce or pay off (a debt) with regular payments*

- In an algorithm, there may be cheap operations and expensive operations.
- Regular worst-case analysis assumes the expensive operations always dominate the execution.
- However, there can be a **deterministic** limit on the number of times *expensive* happens.

Let’s see on an example…

Queue Implementation with Two Stacks



Stack-1 Stack-2

Implement a queue by using two stacks.

Enqueue(x):
Push x into stack-1.

Dequeue():
If stack-2 is empty, then
Pop everything from stack-1 and push into stack-2;
Pop from stack-2

Enqueue(7)	Enqueue(2)	Enqueue(9)	Dequeu()	Enqueue(8)
Dequeu()	Dequeu()	Enqueue(1)	Enqueue(2)	Dequeu()

Queue Implementation with Two Stacks

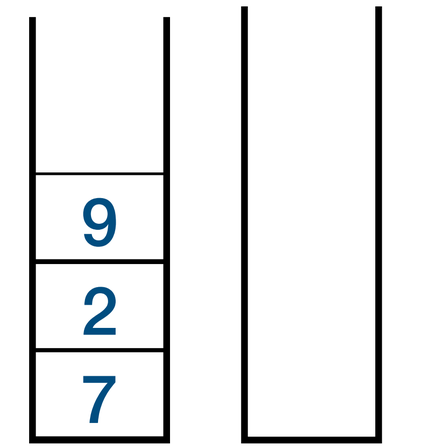
Assume n insert or fetch operations will be executed.
What will be the time-complexity on this implementation?

```
for (i=1 to n){
      operation = randomSelect(enqueue, dequeue);
      execute the operation;}
```

- Enqueue() is cheap, worst case $O(1)$-time.
- Dequeue() is expensive, worst-case $O(n)$-time.
- If I always do the expensive operation then worst-case complexity becomes $O(n^2)$!?

Is this correct ?

Queue Implementation with Two Stacks

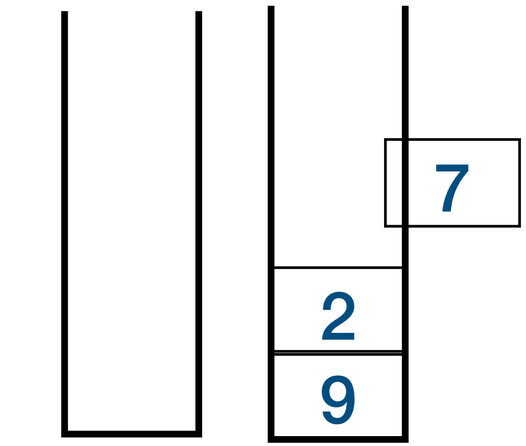


Stack-1 Stack-2

Enqueue(7)

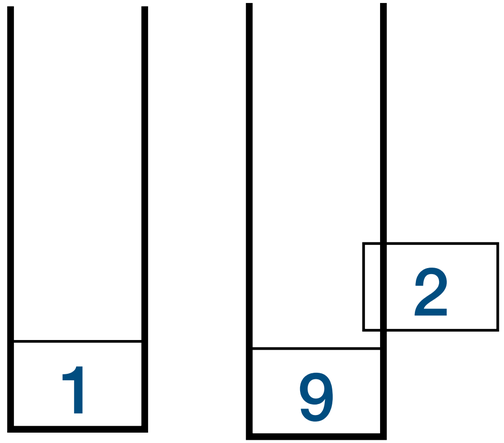
Enqueue(2)

Enqueue(9)



Stack-1 Stack-2

Dequeu()



Stack-1 Stack-2

Enqueue(1), Dequeu()

Once an expensive ‘dequeue’ happens,
some of the following ‘dequeue’s will always be cheap.

Queue Implementation with Two Stacks

Once an expensive dequeue happens, the following ones are always cheap.

- How many times an item is inserted into the stack-1 and stack-2 ?
- How many times it is popped from stack-1 and stack-2 ?
- For n enquee/dequee, **at most** $4n$ push/pop are achieved
- $\frac{4n}{n} = 4 \in O(1)$ **per each enqueue/dequeu operation.**

This is **NOT average-case** analysis,
but a **worst-case** analysis.

Binary Counter

Assume we have a k-bit binary counter, and the cost of incrementing this counter is defined as being equal to the number of bits flipped. What is the cost of incrementing this counter n times?

A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	COST
0	0	0	0	0	0	
0	0	0	0	0	**1**	**1**
0	0	0	0	**1**	**0**	**2**
0	0	0	0	1	**1**	**1**
0	0	0	**1**	**0**	**0**	**3**
0	0	0	1	0	**1**	**1**
.....						

INCREMENT(A)

```
1  i = 0
2  while i < A.length and A[i] == 1
3        A[i] = 0
4        i = i + 1
5  if i < A.length
6        A[i] = 1
```

Binary Counter

A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	COST
0	0	0	0	0	0	
0	0	0	0	0	**1**	**1**
0	0	0	0	**1**	**0**	**2**
0	0	0	0	1	**1**	**1**
0	0	0	**1**	**0**	**0**	**3**
0	0	0	1	0	**1**	**1**
.....						

Regular worst-case analysis:

- At most how many bits can be flipped ?
  - All of the k bits, e.g., 011111 $\rightarrow$100000
- Thus, if we consider n increments then it makes $O(n \cdot k)$

```
INCREMENT(A)
1  i = 0
2  while i < A.length and A[i] == 1
3      A[i] = 0
4      i = i + 1
5  if i < A.length
6      A[i] = 1
```

However, it is not possible to have consecutive increments with k-bits flip! So…

Binary Counter

A[5]	A[4]	A[3]	A[2]	A[1]	A[0]	COST
0	0	0	0	0	0	
0	0	0	0	0	**1**	**1**
0	0	0	0	**1**	**0**	**2**
0	0	0	0	1	**1**	**1**
0	0	0	**1**	**0**	**0**	**3**
0	0	0	1	0	**1**	**1**
0	0	0	1	**1**	**0**	**2**

A[0] flips at each increment

A[1] flips once at each 2 increments

A[2] flips once at each 4 increments

.....

A[k] flips once at each 2^i increments

So total cost of n increment operations is

$$n + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \ldots + \left\lfloor \frac{n}{2^{k-1}} \right\rfloor < n \sum_{i=0}^{\infty} \frac{1}{2^i} = 2n$$

n increment operations cost less than 2n flips.

Thus, each increment costs 2, which makes $O(1)$ time per increment.

Aggregate Method

- Compute the total cost of n operations.
- Divide this cost by n to compute the cost of one operation.
- This is the **aggregate** method of amortized analysis.
- We have two alternative approaches, ***accounting*** and ***potential***

Accounting Method

- Again we assume n operations will be achieved.
- We compute an **amortized cost** per operation
- Before each operation, we deposit in an account the **amortized cost** of that operation.
- Each operation drops exactly the regular **required** amount from the account, where the excess amount from cheap operations are expected to **amortize** the expensive ones.
- If there appears a case that there is not enough money in the account (**bankruptcy**), then the operation can not be performed. Thus, it should be strictly avoided.
- **What should we assume the amortized cost to avoid the bankruptcy ?**

Accounting Method

$$\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} \hat{c}_i$$

- The regular cost of the i^{th} operation is c_i
- The amortized cost of i^{th} operation is $\hat{c}_i$
- **For any $k = 1 \ldots n$, the total regular cost should never exceed the total amortized cost.**

Accounting Method

Queue with Two Stacks

	Stack operations	Stack-1	Stack-2	Actual Cost	Deposit	Remaining Balance
Enqueue(7)	1-push	7		1	$2	$1
Enqueue(2)	1-push	7,2		1	$2	$2
Enqueue(9)	1-push	7,2,9		1	$2	$3
Dequeue()	3-pop, 3-push, 1-pop		9,2	7	$0	**BANKRUPTCY** !
Dequeue()	1-pop		2	1		
Enqueue(8)	1-push	8	2	1		
Dequeue()	1-pop	8		1		
Enqueue(7)	1-push	8,7		1		
Dequeue()	2-pop, 2-push, 1-pop		2,7	5		

Assume the amortized cost for enqueue is **$2** , and dequeue is **free** !

Not a good choice ! We may face a bankruptcy

Accounting Method

Queue with Two Stacks

	Stack operations	Stack-1	Stack-2	Actual Cost	Deposit	Remaining Balance
Enqueue(7)	1-push	7		1	$4	$3
Enqueue(2)	1-push	7,2		1	$4	$6
Enqueue(9)	1-push	7,2,9		1	$4	$9
Dequeue()	3-pop, 3-push, 1-pop		9,2	7	**$0**	$2
Dequeue()	1-pop		2	1	**$0**	$1
Enqueue(8)	1-push	8	2	1	$4	$4
Dequeue()	1-pop	8		1	**$0**	$3
Enqueue(7)	1-push	8,7		1	$4	$6
Dequeue()	2-pop, 2-push, 1-pop		7	5	**$0**	$1

If we the amortized cost for enqueue is **$4** , and dequeue is free, then it seems no bankruptcy !
We know it takes no more than 4 stack operations per each item in the queue. So, we pay $4 dollars at the enqueue phase, and use the remaining $3 during the later dequeue operations.

Accounting Method

Binary Counter

```
INCREMENT(A)
1  i = 0
2  while i < A.length and A[i] == 1
3      A[i] = 0
4      i = i + 1
5  if i < A.length
6      A[i] = 1
```

$1 \rightarrow 0$ flipping, FREE, no cost

$0 \rightarrow 1$ flipping, **$? amortized cost**

- There are two different flip operations as $1 \rightarrow 0$ and $0 \rightarrow 1$.
- At each increment **some number of** ones flip into 0, and **one** zero at the end flips to 1.
- Assume $1 \rightarrow 0$ is free, so no worries on '*some number of*'
- What should be the amortized cost of $0 \rightarrow 1$ to accommodate this?

Potential Method

$$\sum_{i=1}^{i=n} \hat{c}_i \quad = \quad \sum_{i=1}^{i=n} c_i + \phi(D_i) - \phi(D_{i-1}) \quad = \quad \phi(D_n) - \phi(D_0) + \sum_{i=0}^{i=n} c_i$$

- Again we consider n operations, but the focus is on the used data structure D.
- Function $\phi(D)$ that defines the potential energy of the data structure D.
- The amortized cost $\hat{c}_i = c_i + \phi(D_i) - \phi(D_{i-1})$, where c_i is the actual real cost.
- If $\phi(D_n) \geq \phi(D_0)$ can be maintained, then the amortized cost is fine since potential never goes negative.
- **The issue is to propose such a ϕ function.**

Potential Method

Queue with two stacks

Assume $\phi = 2 \cdot x$, where x is the number of items in stack-1.

After enqueue operation the potential increases by $2 = \phi(k+1) - \phi(k)$.

Therefore, the amortized cost is $\hat{c}_E = c_E + \phi(k+1) - \phi(k) = 1 + 2 = 3$.

How about amortized cost of $\hat{c}_D$. Two cases:

1) Stack-2 is not empty. Then, no change in the potential and the amortized cost is equal to actual cost of 1.
2) Stack-2 is empty, and stack-1 has k elements. Then the potential difference is $-2k$. The actual cost is $2k+1$ (why?). So, amortized cost is $-2k + 2k + 1 = 1$.

Potential Method

Binary counter

ϕ is the number of set bits (equal to 1) in the counter.

After an increment, assume t_i bits are flipped from 1 to 0. Then the actual cost is $t_i + 1$.

$$\phi(D_i) - \phi(D_{i-1}) = [\phi(D_{i-1}) - t_i + 1] - \phi(D_{i-1}) = 1 - t_i \, .$$

Then the amortized cost is $t_i + 1 + 1 - t_i = 2$.

Reading assignment

- Read chapter 14 Amortized Analysis from Cormen.
- Read the paper ‘Amortized Computational Complexity’ by Tarjan, which dates back to 1985, on a very nice review of what amortized analysis is. Here is the link https://www.cs.princeton.edu/courses/archive/spr06/cos423/Handouts/Amortized.pdf
- Yet another paper I suggest you to look at is Amortized Efficiency of List Update and Paging Rules available at https://scholar.google.com/scholar?output=instlink&q=info:gEIaOowSipkJ:scholar.google.com/&hl=en&as_sdt=0,15&scillfp=3816853723830843295&oi=lle
- We will study some further examples in the next lecture.