Investigation and Validation of 2D laminar flow in a pipe

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Abstract

This report investigates the fully developed incompressible 2D laminar flow in a radial pipe. An analytical velocity profile is derived from Navier-Stokes equation to validate with the simulations that were ran. This analytical equations assumes that the flow is fully developed, two dimensional and is in steady state. Further the analytical is discretised based on the number of grid points in the wall normal direction. The computational domain is increased according to the full developement of the flow and refined with three different mesh sizes. Two outlet boundary conditions were used to depict the changes in velocity profile at the outlet. These solutions were further compared with analytical solution with their respective grid points. The comparision between the two oulet conditions and analytical was performed with each mesh size. Comparision of the flow profile with changing the order of upwind scheme is assessed. The relative and absolute errors are presented for these comparisions at the outlet wall. A non isothermal simulation was performed with Dirichlet boundary condition at the top and bottom walls to study the affect of temperature on the simulation.

List of Abbreviations

CFD	Computational Fluid Dynamics
SWBLI	Shock Wave Boundary Layer Interaction
RSWBLI	Ramp-Induced Shock Wave Boundary Layer Interaction
ISWBLI	Impinged Shock Wave Boundary Layer Interaction

List of Symbols

θ	=	Compression ramp angle
ϕ	=	Angle of sweep
C_f	=	Coefficient of skin friction
L_s	=	Length of separation
x, y, z	=	Stream-wise, Wall normal and span-wise direction
Δt	=	Time step
Ω	=	Volume of the cell
ΩS	=	Surface area of a face of the cell
$\vec{F}_c$	=	Convective flux vector
$\vec{F}_v$	=	Viscous flux vector
ρ	=	Density
E	=	Energy
H	=	Enthalpy
u_i	=	Components of velocity
n_i	=	Components of unit normal
V	=	Contravariant velocity
τ_{ij}	=	Shear stress
μ	=	Dynamic viscosity
Re	=	Reynolds number
δ_{ij}	=	Kronecker delta function
Θ	=	Work done by viscous stress and heat conduction
T	=	Temperature
k	=	Thermal conductivity of the fluid (W/m·K)
γ	=	Ratio of specific heats (dimensionless)
Pr	=	Prandtl number (dimensionless, 0.71 in this work)
R	=	Non-dimensionalized gas constant
M_∞	=	Free stream Mach number (dimensionless)
S	=	Sutherland's constant

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Listings

1. Introduction

The flow between parallel plates is a well discussed problem in the literature this is an simplication of a three dimensional flow assuming the side walls are infinitely long thus there are no affects along z direction. Considering a low Reynolds Number and the flow is steady

2. Governing equations and numerical discretization

2.1 Analytical Solution for Fully-Developed Laminar Flow in a Circular Tube

2.1.1 Governing Equations

The two-dimensional incompressible Navier–Stokes equations in cylindrical coordinates (r, z) for axisymmetric flow are:

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0. \tag{2.1}$$

Momentum (radial):

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2}\right]. \tag{2.2}$$

Momentum (axial):

$$\rho\left(v_r\frac{\partial v_z}{\partial r} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2 v_z}{\partial z^2}\right]. \tag{2.3}$$

2.1.2 Assumptions

For steady, fully-developed, incompressible laminar flow in a horizontal circular tube:

1. Steady flow: $\frac{\partial}{\partial t} = 0$.
2. Fully-developed: $\frac{\partial v_z}{\partial z} = 0$ and $v_r = 0$.
3. Axisymmetric flow: $\frac{\partial}{\partial \theta} = 0$.
4. No-slip at the wall: $v_z(R) = 0$.
5. Gravity acts perpendicular to flow, thus does not appear in z-momentum.

2.1.3 Reduction of the Governing Equations

With $v_r = 0$, continuity (2.1) gives:

$$\frac{\partial v_z}{\partial z} = 0 \quad \Rightarrow \quad v_z = v_z(r). \tag{2.4}$$

The radial momentum equation (2.2) becomes:

$$0 = -\frac{\partial p}{\partial r}, \tag{2.5}$$

so the pressure depends only on z:

$$p = p(z). \tag{2.6}$$

The axial momentum equation (2.3) simplifies to:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz}. \tag{2.7}$$

2.1.4 Velocity Distribution

Integrating (2.7):

$$\frac{dv_z}{dr} = \frac{1}{2\mu}\frac{dp}{dz}r. \tag{2.8}$$

Integrating again:

$$v_z(r) = \frac{1}{4\mu}\frac{dp}{dz}r^2 + C_2. \tag{2.9}$$

Applying $v_z(R) = 0$:

$$C_2 = -\frac{1}{4\mu}\frac{dp}{dz}R^2. \tag{2.10}$$

Thus the velocity profile is:

$$v_z(r) = \frac{1}{4\mu}\frac{dp}{dz}(r^2 - R^2). \tag{2.11}$$

2.1.5 Volumetric Flow Rate

$$Q = \int_0^R v_z(r)(2\pi r)\,dr = -\frac{\pi R^4}{8\mu}\frac{\Delta p}{\ell}. \tag{2.12}$$

2.1.6 Mean and Maximum Velocity

$$V = \frac{Q}{\pi R^2} = \frac{R^2}{8\mu}\frac{\Delta p}{\ell}, \tag{2.13}$$

$$v_{\max} = \frac{R^2}{4\mu}\frac{\Delta p}{\ell} = 2V. \tag{2.14}$$

2.1.7 Non-Dimensional Velocity Profile

$$\frac{v_z(r)}{v_{\max}} = 1 - \left(\frac{r}{R}\right)^2. \tag{2.15}$$

2.1.8 Variable Definitions

- r: Radial distance from tube center (m).
- R: Tube radius (m).
- z: Axial coordinate (m).
- $v_z(r)$: Axial velocity as a function of radius (m/s).
- $v_{\max}$: Maximum velocity at $r = 0$ (m/s).
- V: Mean velocity over cross-section (m/s).
- Q: Volumetric flow rate (m^3/s).
- Δp: Pressure drop along tube length (Pa).
- ℓ: Length over which Δp acts (m).
- μ: Dynamic viscosity (Pa·s).
- ρ: Density (kg/m^3).

The two-dimensional incompressible Navier–Stokes equations (Cartesian coordinates x, y) and continuity are

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \rho f_x, \tag{2.16}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \rho f_y, \tag{2.17}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.18}$$

where $u(x, y, t)$ and $v(x, y, t)$ are the velocity components in the x- and y-directions, respectively, $p(x, y, t)$ is pressure, ρ is density, μ dynamic viscosity, and $\mathbf{f} = (f_x, f_y)$ body forces per unit mass.

Assumptions (single-line statement)

Assuming steady ($\partial/\partial t = 0$), unidirectional fully-developed flow in the x-direction ($v \equiv 0$, $u = u(y)$, $\partial u/\partial x = 0$), negligible body forces in x (or included in pressure gradient), and constant viscosity and density; under these assumptions all terms in (2.16) except the pressure gradient and the viscous term in y vanish.

Using those assumptions, equation (2.16) reduces to the ordinary differential equation

$$0 = -\frac{\mathrm{d}p}{\mathrm{d}x} + \mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2}, \tag{2.19}$$

while (2.17) gives hydrostatic balance (or $\partial p/\partial y = 0$ if gravity absent), and continuity is satisfied trivially by $v = 0$ and $\partial u/\partial x = 0$.

2.2 Solution for flow between parallel plates ($y = \pm h/2$)

Integrate (2.19) twice with respect to y. Write the constant streamwise pressure gradient as $G \equiv \mathrm{d}p/\mathrm{d}x$ (note G is negative for flow in positive x if pressure decreases along x):

$$\mu \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = G, \tag{2.20}$$

$$\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{G}{\mu} y + C_1, \tag{2.21}$$

$$u(y) = \frac{G}{2\mu} y^2 + C_1 y + C_2, \tag{2.22}$$

where C_1, C_2 are integration constants.

Apply symmetry / boundary conditions. For flow between two stationary plates located at $y = \pm h/2$ we have no-slip

$$u\left(y = \tfrac{h}{2}\right) = 0, \tag{2.23}$$

$$u\left(y = -\tfrac{h}{2}\right) = 0. \tag{2.24}$$

Subtracting the two boundary conditions eliminates C_2 and yields $C_1 = 0$ (symmetry implies zero shear at the centreline). Solving for C_2 gives

$$C_1 = 0, \tag{2.25}$$

$$C_2 = -\frac{G}{8\mu} h^2. \tag{2.26}$$

Substitute back to obtain the velocity profile

$$u(y) = \frac{G}{2\mu} y^2 - \frac{G}{8\mu} h^2 = \frac{G}{2\mu}\left(y^2 - \frac{h^2}{4}\right). \tag{2.27}$$

It is conventional to express with − sign if we set $G = -\mathrm{d}p/\mathrm{d}x$ as the negative pressure gradient (so $G > 0$ when pressure decreases in x):

$$u(y) = -\frac{1}{2\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)\left(\frac{h^2}{4} - y^2\right). \tag{2.28}$$

Maximum velocity at the centerline $y = 0$:

$$u_{\max} = u(0) = -\frac{1}{8\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)h^2. \tag{2.29}$$

Volumetric flow rate per unit span (unit depth in z) is

$$Q' = \int_{-h/2}^{h/2} u(y)\,\mathrm{d}y = -\frac{1}{12\mu}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)h^3. \tag{2.30}$$

Alternatively, in terms of $u_{\max}$:

$$Q' = \frac{2}{3}u_{\max}\,h. \tag{2.31}$$

Variable definitions

x streamwise coordinate (m).

y wall-normal coordinate; plates at $y = \pm h/2$ (m).

h total channel height (distance between plates) (m).

$u(x, y, t)$ streamwise velocity component (m s^{-1}).

$v(x, y, t)$ wall-normal velocity component (m s^{-1}).

$p(x, y, t)$ pressure (Pa).

ρ fluid density (kg m^{-3}).

μ dynamic viscosity (Pa s).

G shorthand for $\mathrm{d}p/\mathrm{d}x$ (Pa m^{-1}); take care of its sign convention.

Q' volumetric flow rate per unit span (m^{2}s^{-1}).

$u_{\max}$ maximum (centerline) velocity (m s^{-1}).

Short answers to your conceptual questions

- **If I fix the wall temperature is it Dirichlet or Neumann?**
  Fixing the wall temperature is a Dirichlet boundary condition (value specified). A Neumann thermal condition would be specifying the heat flux $k\frac{\partial T}{\partial n}\big|_{\text{wall}}$.

- **Is "momentum" the convective term in Navier–Stokes?**
  The convective term is $(\mathbf{u} \cdot \nabla)\mathbf{u}$ and represents convective momentum transport (nonlinear advection of momentum). "Momentum equation" refers to the whole Navier–Stokes momentum balance, of which the convective term is one part.

- **When you call second-order upwind for momentum what does it change?**
  Second-order upwind increases the spatial accuracy of the discretisation of the convective (advection) term from first to second order. Practically this reduces numerical diffusion and gives a sharper profile for velocity/quantity gradients, at the cost of slightly larger stencil and potential small non-physical oscillations if the solution has steep discontinuities. It modifies how face values are reconstructed (quadratic/linear reconstruction instead of first-order backward).

3. Grid generation

4. Methodology

4.1 Domain specification

4.2 Initial conditions

5. Results and discussion

5.1 Grid independence

6. Conclusion and future work

References