Assignment 1 – Discrete Review

Answer the questions in the boxes provided on the question sheets. If you run out of room for an answer, add a page to the end of the document.

Related Readings: http://pages.cs.wisc.edu/~hasti/cs240/readings/

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Logic

1. Using a truth table, show the equivalence of the following statements.

(a) $P \vee (\neg P \wedge Q) \equiv P \vee Q$

P	Q	$\neg P$	$\neg P \wedge Q$	$P \vee (\neg P \wedge Q)$	$P \vee Q$
F	F	T	F	F	F
F	T	T	T	T	T
T	F	F	F	T	T
T	T	F	F	T	T

(b) $\neg P \vee \neg Q \equiv \neg(P \wedge Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$
F	F	T	T	T	F	T
F	T	T	F	T	F	T
T	F	F	T	T	F	T
T	T	F	F	F	T	F
				1		2

1 = 2

(c) $\neg P \vee P \equiv$ true

P	¬P	¬P ∨ P
T	F	T
F	T	T

(d) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

P	Q	R	Q∧R	P∨(Q∧R)	P∨Q	P∨R	(P∨Q)∧(P∨R)
F	F	F	F	F	F	F	F
F	F	T	F	F	F	T	F
F	T	F	F	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	T	F	F	T	T	T	T
T	T	T	T	T	T	T	T

Sets

2. Based on the definitions of the sets A and B, calculate the following: $|A|$, $|B|$, $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$

(a) $A = \{1, 2, 6, 10\}$ and $B = \{2, 4, 9, 10\}$

$|A| = 4$

$|B| = 4$

$A \cup B = \{1, 2, 4, 6, 9, 10\}$

$A \cap B = \{2, 10\}$

$A \setminus B = \{1, 6\}$

$B \setminus A = \{4, 9\}$

(b) $A = \{x \mid x \in \mathbb{N}\}$ and $B = \{x \in \mathbb{N} \mid x \text{ is even}\}$

$|A| = N$

$|B| =$ even numbers

$A \cup B = \{x \mid x \in N\} = A$

$A \cap B = \{x \in N \mid x \text{ is even}\} = B$

$A \setminus B = \{x \notin N \mid x \text{ is odd}\}$

$B \setminus A = \{ \}$

Relations and Functions

3. For each of the following relations, indicate if it is reflexive, antireflexive, symmetric, antisymmetric, or transitive.

(a) $\{(x, y) : x \leq y\}$

reflexive, antisymmetric, transitive

(b) $\{(x, y) : x > y\}$

antireflexive, antisymmetric, transitive

(c) $\{(x, y) : x < y\}$

antireflexive, antisymmetric, transitive

(d) $\{(x, y) : x = y\}$

reflexive. symmetric transitive

4. For each of the following functions (assume that they are all $f : \mathbb{Z} \to \mathbb{Z}$), indicate if it is surjective (onto), injective (one-to-one), or bijective.

(a) $f(x) = x$

bijective

(b) $f(x) = 2x - 3$

bijective

(c) $f(x) = x^2$

surjective

5. Show that $h(x) = g(f(x))$ is a bijection if $g(x)$ and $f(x)$ are bijections.

$f = \Omega \to A \quad g : A \to B$

$\forall b \in B$ $\exists$ only one $a \in A$ s.t, $g(a) = b$

since g is bijective.

For $a \in A$, $\exists$ only one $w \in \Omega$ s.t, $f(w) = a$ since f is bijection. we have $\forall b \in B$, we have unique $w \in \Omega$, s.t $h(w) = b$

$h(x)$ is bijection

Induction

6. Prove the following by induction.

(a) $\sum_{i=1}^{n} i = n(n+1)/2$

LHS RHS LHS

$n=1, \sum_{i=1}^{1} i = 1 \quad 1(1+1)/2 = 2/2 = 1 \quad$ LHS = RHS

For n, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

For $n+1$, $\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + n+1$

$= \frac{n(n+1)}{2} + n+1$

$= (n+1)\left[\frac{n}{2}+1\right] = (n+1)\left[\frac{n+2}{2}\right]$

$\boxed{= \frac{(n+1)((n+1)+1)}{2}}$

(b) $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$

LHS RHS

$n=1, \sum_{i=1}^{1} i^2 = 1, \ 1(1+1)(2(1)+1)/6 = 1 \times 2 \times 3/6 = \underline{1}$ LHS = RHS

$n=n, \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$n+1, \sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$

$= \frac{(n+1)}{6}\left[2n^2+n+6n+6\right] = \frac{(n+1)}{6}\left[2n^2+7n+6\right] = \frac{(n+1)}{6}\left[2n^2+4n+3n+6\right]$

$= \frac{(n+1)}{6}\left[2n(n+2)+3(n+2)\right] = \frac{(n+1)}{6}\left[(n+2)(2n+3)\right]$

$\boxed{= \frac{(n+1)}{6}((n+1)+1)(2(n+1)+1)}$

(c) $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$

LHS RHS

$n=1, \sum_{i=1}^{1} i^3 = 1 \quad \frac{1(1+1)^2}{4} = \frac{4}{4} = 1$ LHS = RHS

$n=n, \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

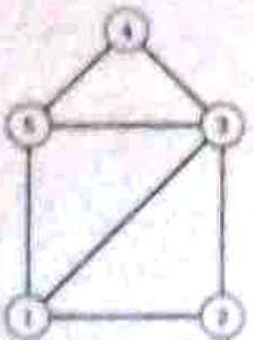
$n=n+1, \sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3$

$= \frac{(n+1)^2}{4}\left[n^2+4n+4\right] = \frac{(n+1)^2}{4}\left[(n+2)^2\right]$

$\boxed{= \frac{(n+1)^2}{4}\left[((n+1)+1)^2\right]}$

Graphs and Trees

7. Give the adjacency matrix, adjacency list, edge list, and incidence matrix for the following graph.



Adjacency matrix :-
$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency list :-

1 → 2 → 3 → 5
2 → 1 → 3
3 → 1 → 2 → 5 → 4
4 → 3 → 5
5 → 1 → 3 → 4

Edge list :- (1,2), (1,3) (1,5)
(2,3)
(3,4) (3,5)
(4,5)

Incidence Matrix :-

	(1,2)	(1,3)	(1,5)	(2,3)	(3,4)	(3,5)	(4,5)
1	1	1	1				
2	1			1			
3		1		1	1	1	
4					1		1
5			1			1	1

8. How many edges are there in a complete graph of size n? Prove by induction.

Complete graph has $\frac{n(n-1)}{2}$ number of edges

Proof by Induction :-

$n = 1$, $\frac{1(1-1)}{2} = 0$ $N = 0$

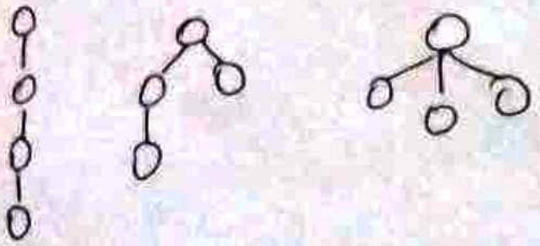
For n, $N = \frac{n(n-1)}{2}$

For $n+1$, we add another vertex

$$\frac{n(n-1)}{2} + n = \frac{n^2 - n + 2n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

$\therefore$ For complete graph we have $N = \frac{n(n-1)}{2}$

9. Draw all possible (unlabelled) trees with 4 nodes.



10. Show by induction that, for all trees, $|E| = |V| - 1$.

If $|V| = 1$, $|E| = 1 - 1 = 0$

$|V| = N$ $|E| = N - 1$

Consider tree with N+1 vertices, T has minimum 2 leaves

we get new graph by removing one of leaves v from T :

V — T

Now we have T' is a connected graph and doesn't contain simple cycle since T' is a subgraph of tree T.

Then we have T' is a tree by definition

since T' has n vertices we have $|V_{T'}| = n$ $|E_{T'}| = n-1$

then for T : $|V_T| = n+1$ $|E_T| =$ ~~[illegible]~~ $= n$

$|V| = |E| + 1$

Counting

11. How many 3 digit pin codes are there?

$10 \times 10 \times 10 = 1000 = \underline{10^3}$

12. What is the expression for the sum of the ith line (indexing starts at 1) of the following:

1
2 3
4 5 6
7 8 9 10

i^{th} line has numbers different from $(i-1)$ lines

sum of $(i-1)$ are $= 1+2+3+\cdots+(i-1) = \frac{(i-1)(i)}{2}$

$\therefore$ First term of ith line $= \frac{(i-1)i}{2} + 1 = \frac{i^2-i+2}{2}$

sum

$= \frac{i(i^2-i+2)}{2} + [0+1\cdots+(i-1)] = \frac{1}{2}\left[(i^3-i^2+2i) + i^2 - i\right] = \frac{1}{2}[i^3+i]$

$\boxed{\frac{1}{2}[i^3+i]}$

13. A standard deck of 52 cards has 4 suits, and each suit has card number 1 (ace) to 10, a jack, a queen, and a king. A standard poker hand has 5 cards. For the following, how many ways can the described had be drawn from a standard deck.

(a) A royal flush: all 5 cards have the same suit and are 10, jack, queen, king, ace.

$^4C_1 = \frac{4!}{(4-1)! \times 1!} = \frac{4 \times 3!}{3! \times 1} = \underline{4}$

(b) A straight flush: all 5 cards have the same suit and are in sequence, but not a royal flush.

$^4C_1 \times {}^9C_1 = \frac{4!}{3!} \times \frac{9!}{8!} = 4 \times 9 = \underline{36}$

(c) A flush: all 5 cards have the same suit, but not a royal or straight flush.

$^4C_1 \times {}^{13}C_5 - 36 - 4 = 5148 - 36 - 4 = \underline{5108}$

(d) Only one pair (2 of the 5 cards have the same number/rank, while the remaining 3 cards all have different numbers/ranks):

$^{13}C_1 \times {}^{12}C_3 \times {}^4C_2 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1 = \underline{1{,}098{,}240}$

Proofs

14. Show that $2x$ is even for all $x \in \mathbb{N}$.

(a) By direct proof.

$\frac{2x}{2} = x$, $x \in N$

As we get whole number when divided by 2

then $2x$ is even

(b) By contradiction.

~~Q~~ If $2x$ is odd, it can be written as

$2x = 2k+1$

then we have $Y = \frac{2x}{2} = \frac{2k+1}{2} = k + \frac{1}{2} \notin N$

Contradicts with $x \in N$.

Thus $2x$ is even

15. For all $x, y \in \mathbb{R}$, show that $|x + y| \leq |x| + |y|$. (Hint: use proof by cases.)

Case when $x + y \geq 0$

$x \geq 0,\ y \geq 0 \quad x + y \leq x + y$

$x \geq 0,\ y < 0 \quad x - y \leq x + y$

$x < 0,\ y \geq 0 \quad -x + y \leq x + y$

when $x + y < 0$

$x \geq 0 \quad y < 0 \quad -x - y \leq x - y$

$x < 0,\ y \geq 0 \quad -x - y \leq -x + y$

$x < 0,\ y < 0 \quad -x - y \leq -x - y$

Program Correctness (and invariants)

16. For the following algorithms, describe the loop invariant(s) and prove that they are sound and complete.

(a)

Algorithm 1: findMin

Input: a: A non-empty array of integers (indexed starting at 1)
Output: The smallest element in the array
begin
  $min \leftarrow \infty$
  for $i \leftarrow 1$ **to** $len(a)$ **do**
    if $a[i] < min$ **then**
      $min \leftarrow a[i]$
    end
  end
  return min
end

Loop Invariant :- At the begining of every iteration of the for loop, the variable min holds the smallest element in subarray to $a[1..(i-1)]$, ie first $(i-1)$ elements

Initialization :- Variable min = ∞ as before the first iteration for $i=1$, when compared to the first element it will store the value $a[1]$

After the iteration, the variable min is compared with the i^{th} element and is assigned the i^{th} element if it is less than min, i is incremented to $i+1$ and we have the smallest element among the first i elements stored in min.

Termination :- Terminates when ~~d~~ $i = len(a) + 1$, returns min which has smallest value in the array.

(b)

Algorithm 2: InsertionSort

Input: a: A non-empty array of integers (indexed starting at 1)
Output: a sorted from largest to smallest

```
begin
    for i ← 2 to len(a) do
        val ← a[i]
        for j ← 1 to i − 1 do
            if val > a[j] then
                shift a[j..i − 1] to a[j + 1..i]
                a[j] ← val
                break
            end
        end
    end
    return a
end
```

Initialization state, when the loop statement is true before the first iteration

Maintaenance true will be maintained in next loops

Termination, after last iteration when we obtain output

The outer for loop invariant is the A[1], A[2]..A[i]
and initialization starts at A[m] where 1<m<n

For the nested for loop, invariant is A[A 1], A[2].
.. A[i-1]

If variable val value a[i] is larger than nested for loop value then shift happens.

By end of the loop we get a sorted list of numbers.

Recurrences

17. Solve the following recurrences.

(a) $c_0 = 1; c_n = c_{n-1} + 4$

$c_n = c_{n-1} + 4$
$= (c_{n-2} + 4) + 4$
$= c_{n-2} + 2 * 4$
$= c_{n-3} + 3 \times 4$
$= c_{n-n} + n \times 4$
$= c_0 + 4n$
$c_n = 1 + 4n$

$c_1 = 1 + 4(1) = 5$
$c_2 = 1 + 4(2) = 9$
$c_3 = 1 + 4(3) = 13$

(b) $d_0 = 4; d_n = 3 \cdot d_{n-1}$

$d_n = 3 \times d_{n-1}$
$= 3 \times (3 \times d_{n-2})$
$= 3^2(d_{n-2})$
$= 3^2(3 \times d_{n-3})$
$= 3^3(d_{n-3})$
$= 3^n(d_{n-n})$
$= 3^n(d_0)$
$d_n = 4 \times 3^n$

$d_1 = 4 \times 3^1 = 12$
$d_2 = 4 \times 3^2 = 36$
$d_3 = 4 \times 3^3 = 108$

(c) $T(1) = 1; T(n) = 2T(n/2) + n$ (An upper bound is sufficient.)

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$= 2\left(2T\left(\frac{n}{2^2}\right) + \frac{n}{2}\right) + n$$

$$= 2^2 T\left(\frac{n}{2^2}\right) + 2n$$

$$= 2^2\left(2T\left(\frac{n}{2^3}\right) + \frac{n}{2^2}\right) + 2n$$

$$= 2^3 T\left(\frac{n}{2^3}\right) + 3n$$

$$= 2^{\log_2 n} T\left(\frac{n}{n}\right) + \log_2 n \cdot n = n(\log_2 n + 1)$$

$$= O(n \log n)$$

(d) $f(1) = 1; f(n) = \sum_1^{n-1}(i \cdot f(i))$
(Hint: compute $f(n+1) - f(n)$ for $n > 1$)

$$f(n) - f(n-1)$$

$$= \sum_{i=1}^{n-1} i f(i) - \sum_{i=1}^{n-2} i f(i) = (n-1) f(n-1)$$

$$f(n) = n f(n-1)$$

$$= n \times (n-1) \times f(n-2)$$

$$= n \times (n-1) \times (n-2) f(n-3)$$

$$= n! \times f(1)$$

$$= n!$$