2. LCM and HCF

Prime numbers

A composite number can be uniquely expressed as a product of prime factors. For example,

- $12 = 2 \times 6 = 2 \times 2 \times 3 = 2^2 \times 3^1$
- $20 = 4 \times 5 = 2 \times 2 \times 5 = 2^2 \times 5^1$
- $124 = 2 \times 62 = 2 \times 2 \times 31 = 2^2 \times 31$ etc.

Every composite number can be expressed in a similar manner in terms of its prime factors.

Number of factors

The number of factors of a given composite number N (including 1 and the number itself) which can be resolve into its prime factors as,

$N = a^m \times b^n \times c^p \ldots$ where a, b, c are prime numbers, are **(1 + m) (1 + n) (1 + p)...**

Example: Find the total number of factors of 240.

Sol: $240 = 2 \times 2 \times 2 \times 2 \times 3 \times 5 = 2^4 \times 3^1 \times 5^1$ Comparing with the standard format for the number N, we obtain a = 2, b = 3, c = 5, m = 4, n = 1, P = 1. The total number of factors of this number including 1 and itself are,

= (1 + m) (1 + n) (l + p) ... = (1 + 4) x (1 + 1) x (1 + 1) = 5 x 2 x 2 = 20

Factors of 240 = 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 16, 20, 24,30, 40, 48, 60, 80, 120, 240 (Total 20 in numbers)

Sum of Factors

The sum of factors of the number N (as defined above) is given by the formula:

$$\frac{(a^{m+1} - 1)\,(b^{n+1} - 1)\,(c^{p+1} - 1)\ldots}{(a-1)\,(b-1)\,(c-1)\ldots}$$

Where,

a, b, c …, m, n, p... retain the same meaning.

Example: Find the sum of all the factors of 240.

Sol: The sum by the above formula = $\frac{(2^5 - 1)\,(3^2 - 1)\,(5^2 - 1)\ldots}{(2-1)\,(3-1)\,(5-1)\ldots} = \frac{31 \times 8 \times 24}{1 \times 2 \times 4} = 744$

We can see that:

1 + 2 + 3 + 4 + 5 + 6 + 8 + 10 + 12 + 15 + 16 + 20 + 24 + 30 + 40 + 48 + 60 + 80 + 120 + 240 = 744

- The number of ways in which a composite number N may be resolve two factors

  $\frac{1}{2}$ (p + 1) (q + 1) (r + 1) … if $N = a^p b^q c^r$ is not a perfect square and

$= \frac{1}{2}[(p + 1)(q + 1)(r + 1)... + 1]$ if N is a perfect square.

- The number of ways in which a composite number can be resolved into two factors which are prime to each other. If N = $a^p b^q c^r$ ... then the number of ways of resolving N into two factors prime to each other is,

  $\frac{1}{2}$ x [(1 + 1) (1 + 1) (1 + 1) ...] = 2^n -1 where n is the number of different prime factors of N.

  If P is a prime number, the coefficient of every term in the expansion of $(a + b)^p$ Except the first and the last is divisible by P.

Example: In how many ways can the number 7056 be resolved into two factors?

Sol: N = 7056 = 3^2 x 2^4 x 7^2 = 3^p x 2^q x 7^r

Note:

N is perfect square. Number of ways in which it can be resolved into two factors

= 1/2 {(p + 1) (q + 1) (r + 1) + I}

= $\frac{1}{2}$ {(2 + 1) (4 + 1) (2 + 1) + 1} = 1/2 x 46 = 23

Example: Find the number of ways in which N = 2778300 can be resolved into the factors prime to each other.

Sol: N = 2^2 x 3^4 x 5^2 x 7^3

The required number is the same as the number of ways of resolving 2 x 3 x 5 x 7 into two factors which is equal to 1/2 (1 + 1) (1 + 1) (1 + 1) (1 + 1) = 2^3 = 8.

HCF of Numbers

It is the highest common factor of two or more given numbers. It is also called GCF (greatest common factor). For example HCF of 10 & 15 = 5, HCF of 55 and 200 = 5, HCF of 64 and 36 = 4 etc.

Factorization method to find HCF

To find the HCF of given numbers, first resolve the numbers into their prime factors. After expressing the number in terms of the prime factors, the HCF is the product of common factors.

Example: Find the HCF of 88, 24 and 124.

Sol: 88 = 2 x 44 = 2 x 2 x 22 = 2 x 2 x 2 x 11 = 2^3 x 11^1

24 = 2 x 12 = 2 x 2 x 6 = 2 x 2 x 2 x 3 = 2^3x 3^1.

124 = 2 x 62 = 2 x 2 x 31 = 2^2 x 31^1

➔ HCF = 2^2 = 4

Division Method to find HCF

By division method, we start with the two numbers and proceed as shown below, till the remainder becomes zero.

Example: Find the HCF of 12 and 48.

Sol: 12) 48 (4

48

0

HCF = 12

Here, when the remainder is not zero, divide the previous divisor with that remainder and proceed in the same way until you get the remainder as zero. The last divisor is the required HCF.

Example: Find the HCF of 10 and 25.

Sol: 10) 25 (2

20

5) 10 (2

10

0

If there are more than two numbers, we will repeat the whole process with the HCF obtained from two numbers as the divisor and so on. The last divisor will then be the required HCF of the number.

Example: The HCF of 10, 25 and 30

Sol: We can find the HCF of 10 and 25 i.e. 5. Now we have to find, the HCF of 5 and 30 which is 5. So, the HCF of 10, 25 and 30 is 5.

Note:

If we have to find the greatest number that will exactly divide p, q and r, then required number = **HCF of p, q and r**.

Example: Find the greatest number that will exactly divide 65, 52 and 78.

Sol: Required number = HCF of 65, 52 and 78 = 13.

Note:

If we have to find the greatest number that will divide p, q and r leaving remainders a, b and c respectively, then the required number = HCF of (p - a), (q - b) and (r - c).

Example: Find the greatest number that will divide 65, 52 and 78 leaving remainders 5, 2 and 8 respectively.

Sol: Required number = HCF of (65 - 5), (52 - 2) and (78 - 8)

= HCF of 60, 50 and 70 = 10

Note:

If we have to find the greatest number that will divide p, q and r leaving the same remainder in each case, then required number is,

= HCF of the absolute values of (p - q), (q - r) and (r - p)

Example: Find the greatest number that will divide 65, 81 and 145 leaving the same remainder in each case.

Sol: Required number is,

= HCF of (81 - 65), (145 - 81) and (145 - 65)

= HCF of 16, 64 and 80 = 16

Example: How many numbers below 90 and other than unity exist, such that the HCF of that number and 90 is unity?

Sol. $90 = 3^2 \times 2 \times 5$

Number of multiples of 2 = 45

Number of multiples of J = 30

Number of multiples of 5 = 18

Number of multiples of 2 & 5 = 9

Number of multiples of 2 & 3 = 15

Number of multiples of 3 & 5 = 6

Number of multiples of 2, 3 & 5 = 3

Total = (24 + 6 + 12) + (12 + 3 + 6 + 3) = 42 + 24 = 66

24 numbers (including unity) compared with 90 have only '1' as common factor. Hence required result is 24 - 1 = 23.

Example: 3 vessels of volume 115 liters, 161 liters & 207 liters are filled with liquid completely. What is the volume of the flask that can measure liquid in all three of them exactly?

Sol: HCF of 115, 161 & 207 i.e. 23

LCM of Numbers

Lowest common multiple of two or more numbers is the smallest number which is exactly divisible by all of them e.g.,

- LCM of 5, 7, 10 = 70
- LCM of 2,4, 5 = 20
- LCM of 11, 10, 3 = 330

Factorization Method to find LCM

To find the LCM of the given numbers, first resolve all the numbers into their prime factors and then the LCM is the product of highest powers of all the prime factors.

Example: Find the LCM of 40, 120, 380.

Sol. 40 = 4 x 10 = 2 x 2 x 2 x 5 = 2^3 x 5^1;

120 = 4 x 30 = 2 x 2 x 2 x 5 x 3 = 2^3 x 5^1 x 3^1;

380 = 2 x 190 = 2 x 2 x 95 = 2 x 2 x 5 x 19 = 22 x 51 x 191;

➔ Required LCM = 23 x 51 x 31 x 191 = 2280

Division method to find LCM

Write the given numbers separately. Then divide by 2 and write the result below the numbers divisible by 2. If it is not divisible by 2 then try with 3, 5, 7… etc. Leave the others (those not divisible) untouched. Do the same for all steps till you get 1 as the remainder in each column?

Example: Find the LCM of 6, 10, 15, 24, and 39.

Sol.

2	6	10	15	24	39
2	3	5	15	12	39
2	3	5	15	6	39
3	3	5	15	3	39
5	1	5	5	1	13
13	1	1	1	1	13
	1	1	1	1	1

LCM = 2 x 2 x 2 x 3 x 5 x 13 = 1560.

Note:

If we have to find the least number which is exactly divisible by p, q and r, then the required number = LCM of p, q and r.

Example: Find the least number that is exactly divisible by 6, 5 and 7.

Sol. Required number = LCM of 6, 5 and 7 = 210. If we' e to find the least number which when divided by p, q and r leaves the remainders a, band c respectively, then if it is observed that,

(p - a) = (q – b) = (r – c) = K (say), then

The required number = (LCM of p, q and r) - (K)

Example: Find the least number which when divided by 6, 7 and 9 leaves the remainders 1, 2 and 4 respectively.

Sol. Here, (6 - 1) = (7 - 2) = (9 - 4) = 5.

Required number = (LCM of 6, 7 and 9) – 5 = 126 - 5 = 121

If we have to find the least number which when divided by p, q and r leaves the same remainder 'a' each ease, then required number = (LCM of p, q and r) + a.

Example: Find the LCM of 25 and 35 if their HCF is 5.

Sol. LCM = $\frac{\text{Product of two numbers}}{\text{HCF}} = \frac{25 \times 35}{5}$ = 175

Example: By using the rule that LCM = Product of two numbers ÷ HCF,

Sol: 442 = 2 x 17 x 13 ➔ HCF = 26 ➔ LCM = 26 x 442/26 = 442

HCF & LCM of Decimals

Example: Calculate the HCF and LCM of 0.6,0.9, 1.5, 1.2 and 3

Sol: The numbers can be written as 0.6, 0.9, 1.5, 1.2, 3.0 Consider them as 6,9, 15, 12,30

➔ HCF = 3 ➔ Required HCF = 0.3 and LCM = 18.0

Note

If the first number in the above example had been 0.61, then, the equivalent integers would have been 61, 90, 150, 120 and 300 etc.

HCF & LCM of Fractions

HCF of fractions = HCF of numerators ÷ LCM of denominators

LCM of fraction = LCM of numerators ÷ HCF of denominators

Example: Find the HCF and LCM of: $\frac{5}{16}, \frac{3}{4},$ and $\frac{7}{15}$

Sol. HCF = $\frac{\text{HCF of }(5,3,7)}{\text{LCM of }(16,4,15)} = \frac{1}{240}$

LCM = $\frac{(\text{LCM of }(5,3,7)}{\text{HCF of }(16,4,15)} = \frac{105}{1}$ = 105.

Surds:

Surds are irrational roots of a rational number. For example, $\sqrt{6}$ is a surd, since its exact value can't be found.Similarlyπ, e, $\sqrt{7}$,$\sqrt{8}$, $\sqrt[3]{9}$,$\sqrt[4]{27}$ etc. are all surds.

Types of Surds

Pure Surds: The surds which are made up of only an irrational number are known as pure surds. For example.$\sqrt{6}$,$\sqrt{7}$,$\sqrt{8}$ etc

Mixed Surds: Surds which are made up of partly rational and partly irrational numbers are called mixed surds. For example $3\sqrt{3}$, $6\sqrt[4]{27}$etc. are mixed surds.

Example: Convert $2\sqrt{8}$ to a pure surd.

Sol: $2\sqrt{8} = \sqrt{2^2 \times 8} = \sqrt{8 \times 4}$ = 52

Rationalization of Surds

In order to rationalize a given surd, we multiply and divide it by the conjugate of denominator and then simplify [conjugate of (a + $\sqrt{b}$) is (a - $\sqrt{b}$) and vice versa].

Example: Rationalize $\frac{6+\sqrt{2}}{1-\sqrt{3}}$

Sol: $\frac{6+\sqrt{2}}{1-\sqrt{3}} = \frac{6+\sqrt{2\ 1}\ 1+\sqrt{3}}{(1-\sqrt{3})(\ 1+\sqrt{3})} = \frac{6+6\sqrt{3}+\sqrt{2}+\sqrt{6}}{1-3} = \frac{6+6\sqrt{3}+\sqrt{2}+\sqrt{6}}{-2}$

Factorization of Algebraic Expressions

The factors of a given algebraic expression consist of two or more algebraic expressions which when multiplied together produce the given expression.

Different Types with Examples

The factorization of algebraic expressions can be done in many ways. It depends on the terms contained in the expression. Based on this, following are the standard types of factorization. You are expected to understand each of them thoroughly.

Common Monomial Factor → ac + ad = a(c + d)

Example: factorize $3x^2 + 6x^3 + 12x^4$

Sol: $3x^2 + 6x^3 + 12x^4 = 3x^2(1 + 2x + 4x^2)$

Example: factorize $9s^3t + 15s^2t^3 + 3s^2t^2$

Sol: $9s^3t + 15s^2t^3 + 3s^2t^2 = 3s^2t\ (3S + 5t^2 - t)$

Difference of two squares

$a^2 - b^2 = (a + b)(a - b)$

Example: Factories $1 - x^8$

Sol: $(1 + x^4)(1 - x^4) = (1 + x^4)(1 + x^2)(1 - x^2)$

So $1 - x^8 = (1 + x^4)(1 + x^2)(1 + x)(1 - x)$

Perfect Square Trinomials
$a^2 + 2ab + b^2 = (a + b)^2$
$a^2 - 2ab + b^2 = (a - b)^2$

Example: factorize $9x^4 - 24x^2y + 16y^2$

Sol: $9x^4 - 24x^2y + 16y^2 = (3x^2 - 4y)^2$

Other Trinomials

$x^2 + (a + b)x + ab = (x + a)(x + b)$

$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$

Example: factorize $x^2 - 7xy + 12y^2$

Sol: $x^2 - 7xy + 12y^2 = x^2 - 3xy - 4xy + 12y^2 = (x - 3y)(x - 4y)$

Sum or the difference of two cube

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example: factorize: $x^6 - 7x^3 - 8$

Sol: $x^6 - 7x^3 - 8 = x^6 - 8x^3 + x^3 - 8$

$= (x^3 - 8)(x^3 + 1) = (x^3 - 2^3)(x^3 + 1)$

$= (x - 2)(x^2 + 2x + 4)(x + 1)(x^2 - x + 1)$

Grouping of terms

$ac + bc + ad + bd = c(a + b) + d(a + b) = (a + b)(c + d)$

Example: factorize $x^3 - x^2y + xy^2 + y^3$

Sol: $x^3 - x^2y + xy^2 + y^3 = x^2(x + y) + y^2(x + y) = (x + y)(x^2 + y^2$

Factors of $\mathbf{a^n \pm b^n}$

$a^n + b^n$ has (a + b) as a factor if and only if n is a positive odd integer.

$a^n + b^n = (a + b)(a^{n-1} \pm a^{n-2}b + a^{n-3}b^2 - \ldots ab^{n-2} + b^{n-1})$

Example: Factorize: $x^3 + 8y^6$

Sol: $x^3 + 8y^6 = x^3 + (2y^2)^3 = (x + 2y^2)[x^2 - x(2y^2) + (2y^2)^2]$

$= (x + 2y^2)(x^2 - 2xy^2 + 4y^4)$

HCF of polynomials

The HCF of two or more given polynomials is the polynomial of highest degree and largest numerical coefficients which is factor polynomials.

Method to find HCF of polynomial: The following method is suggested for finding the HCF of several polynomials:

a) Write each polynomial as a product of prime factors.
b) The HCF is the product obtained by taking each factor to the lowest power to which it occurs in both the polynomials.

Note

Two or more polynomials are relatively prime if their HCF is 1.

LCM of Polynomials

The LCM of two or more given polynomials is the polynomial of lowest degree and smallest numerical coefficients for which each of the given polynomials will be its factor.

Method to find LCM of polynomial: The following procedure is suggested for determining the LCM of several polynomials:

a) Write each polynomial as a product of prime factors.
b) The LCM is the product obtained by taking each factor to the highest power to which it occurs.

Example: Find the HCF and LCM of $9x^4y^2$ and $12\ x^3y^3$

Sol: (i) = $3^2 \times x^4\ \times y^2$ and

= $2^2 \times 3 \times x^3 \times y^3$

$\therefore$ HCF of $3x^3y^2$ and

LCM = $3^2 \times 2^2 \times x^4 \times y^3\ =\ 36x^4y^3$

So, HCF = $3x^3y^2$, LCM = $36x^4y^3$

Special Products

The following are some of the products which occur frequently in mathematics and the student should become familiar with them as soon as possible.

Product of a monomial and a binomial

$a\ (c + d) = ac + ad$

Product of the sum and the difference of two terms

$(a + b)\ (a - b) =\ a^2 -\ b^2$

Square of a binomial

$(a\ +\ b)^2 = a^2 + 2ab + b^2$

$(a\ -\ b)^2 = a^2 + 2ab + b^2$

Product of two binomials

$(x + a)\ (x + b) = x^2\ + (a + b)x + ab$

$(ax + b)\ (cx + d) = acx^2\ + (ad + bc)x + bd$

$(a + b)\ (c + d) = ac + bc + ad + bd$

Cube of a binomial

$(a + b)^3 = a3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Square of a trinomial

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc.$

Products that give answers of the form $a^n \pm b^n$

It may be verified by multiplication that

$(a - b)\ (a^2 + ab + b^2) = a^3 - b^3$

$(a - b)\ (a^3 + a^2b + ab^2 + b^3) = (a^4 - b^4)$

$(a - b)\ (a^4 + a^3b + a^2b^2 + ab^3 + b^4) = a^5 - b^5$

$(a - b)\ (a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5) = a^6 - b^6$ and so on

These may be summarized as

$(a - b)\ (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}) = a^n - b^n$

Where n is any positive integer (1, 2, 3, 4, ...)
Similarly, it may be verified that

$(a + b)\ (a^2 - ab + b^2) = a^3 + b^3$

$(a + b)\ (\ a^4\ a^3\ b + a^2b^2 - ab^3 + b^4\ ) = (a^5 + b^5)$

$(a + b)\ (\ a^6\ a^5\ b + a^4\ b^2 - a^3\ b^3 + a^2\ b^4 - ab^5 + b^6) = (a^7 + b^7)$ and so on. These maybe summarized as $(a + b)\ (a^{n-1} - a^{n-2}b - a^{n-3}b^2 - \ldots - ab^{n-2} + b^{n-1}) = a^n + b^n$ where n is any positive odd integer $(1, 3, 5, 7, \ldots)$

Example: Simplify $(x + y + z + 1)^2$

Sol: $[(x + y) + (z + 1)]^2 = (x + y)^2 + 2(x + y)(z + 1) + (z + 1\ )^2$

$= x^2 + 2xy + y^2 + 2xz + 2x + 2yz + 2y + z^2 + 2z + 1$

Practice Exercise

DIRECTIONS: For the following questions, four options are given. Choose the best option.

1. A person has a number of pegs to peg in a row. At first he tried to peg 5, then 6, then 8, then 12 in each, but had always 1 left. On trying 13 he had none left. What is the smallest number of pegs that he could have had?

   1) 381 2) 481 3) 255 4) 581 5) None of these

2. A heap of stones can be made up exactly into groups of 25. But when made into groups of

18, 27 and 32, there are always 11 left. The least number of stones that may be contained in such A heap is…

1) 575 2) 875 3) 255 4) 325 5) None of these

3. A number x when divided by 289 leaves 18 as the remainder. The same number when divided by 17 leaves y as the remainder. The value of y is…

1) 2 2) 3 3) 1 4) 5 5) None of these

4. If $(x + (1 / x)) = 4$, then the value of $x^4 + 1/x^4$ is…

1) 124 2) 64 3) 194 4) Cannot determined (5) None of these

5. The greatest number that will divide 2629 and 2483 leaving remainders 4 and 8 respectively is…

1) 75 2) 65 3) 80 4) 60 5) None of these

6. The least number which when divided by 35 leaves a remainder 25, when divided by 45 leaves a remainder of 35 and when divided by 55 leaves remainder of 45 is…

1) 3465 2) 3645 3) 3655 4) 3455 5) None of these

7. The value of $\frac{1}{4+\frac{1}{4+\frac{1}{4+\ldots}}}$ is…

1) 0.351 2) 0.452 3) 1.258 4) 0.235 5) None of these

8. An equivalent fraction with a rational denominator for$(8 - 5\sqrt{2}) \div (3 - 2\sqrt{2})$ is…

1) $(\frac{2+\sqrt{2}}{14})$ 2) $4 + \sqrt{2}$ 3) $\sqrt{2}$ 4) $3 + \sqrt{2}$ 5) None of these

9. When a number is divided by 8 or 7 it leaves a remainder 1 but when the same number is divided by 9 the remainder is 5. What will be the remainder when it is divided by 72?

1) 41 2) 25 3) 67 4) 48 5) None of these

10. In an election for the President, 261 valid votes are east for the 5 contestants. The least number of votes a candidate requires to win the election is…

1) 53 2) 54 3) 257 4) 72 5) None of these

11. If ab = ac = ad = bd = be = cd = 1, then what can be the values of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$?

(a) a + b + c + d (b) 3a + b (c) 2(a + d)

1) Only (a) 2) (a) & (b) 3) (a) and (c) 4)(a), (b) & (c) 5) None of these

12. If x, y and z are real numbers and $(x + y + z)^3 = 343$ and $xy + yz + xz = 10$, then what

will be greatest value of x?

1) 2 2) 7 3) $\sqrt{29}$ 4) 5 5) None of these

13. I have a certain number of beads which lie between 600 and 900. If 2 beads are taken away the remainder can be equally divided among 3, 4, 5, 6, 7 or 12 boys. The number of beads I have is…

1) 729 2) 842 3) 576 4) 961 5) None of these

14. The lowest number which is exactly divisible by 7, 8, 9 but leaves a remainder of 5 when divided by 11 is…

1) 504 2) 1008 3) 7056 4) None of these (5) 512

15. A gardener planted saplings in such a way that every row had as many saplings as every column. If in all there were 729 trees, then how many saplings were there in each row?

1) 26 2) 27 3) 28 4) 29 5) None of these

Answers:

1. 2	**2.** 2	**3.** 3	**4.** 3	**5.** 1
6. 4	**7.** 4	**8.** 2	**9.** 1	**10.** 1
11. 4	**12.** 4	**13.** 2	**14.** 4	**15.** 2