11. Permutations and Combinations

Fundamental Counting Principles

The Additive Rule

Suppose that x and y are two **disjoint events, that is, they** never occur together. Further, suppose that x occurs in m ways and y occurs in n ways. **Then x or y can occur in m + n ways.** This rule can be extended to more than two mutually exclusive events also.

Example: Ramesh Kumar Joshi is taken to a toy shop "Gizmos" containing 15 distinct toy cars, 17 distinct toy dolls and 6 distinct toy guns. Find the number of ways in which Ramesh can choose a toy
Sol. If Ramesh is allowed to choose exactly one of the above mentioned toys in the shop, he will have to choose one toy out of 15 + 17 + 6 = 38 toys. There are 38 ways in which Ramesh can choose a toy.

The Product Rule

If one thing can be done in **m** different ways and a second thing can be done in **n** different ways, then the two things in succession can be done in m × n different ways. This principle is known as **product Rule**

Example: If there are 3 routes from town A to B and 5 routes from town B to C, Total how many different routes are there from A to C?

Sol. 3 x 5 = 15

- If there are **m** ways of doing a thing, n ways of doing a second thing and **p** ways of doing a third thing, then the total number of **"distinct"** ways of doing all these together is **m × n × p.**

Example: Suppose, there are five routes for going from a place A to another place B and six routes for going from the place B to a third place C. Find the number of different ways through which a person can go from A to C via B.
Sol. Since there are five different routes from A to B person can go from A to B in five different ways. After reaching B, he has six different ways of finishing the second part of his journey (i.e. going from B to C). Thus one way of going from A to B there are six different ways of completing the journey from A to C via B. Hence, the total number of differ t ways of finishing both parts of the journey (i.e. from A to B and then from B to C) = 5 times six different ways 5 × 6 = no. of ways from the first point to the second point × number of ways from the second point to the third point.

Factorial Function

Here we introduce a very useful function, called the factorial function. This function is very convenient for calculations and formulae to be used in this lecture.
1, 2, 3, ..., n

Definition: If n is a natural number, then n factorial, denoted by n! or ∟n is defined to be the product l × 2 × 3 ... (n - 1) × n

That is n! = 1 × 2 × 3 × 4 ... (n - 1) × n

As a special case, we define 0! = 1We now list the value of n! for some values of n.

0! = 1

1! = 1

2! = 1× 2 = 2

3! = 1× 2 × 3 = 6

4! = 1 × 2 × 3 × 4 = 24

5! = 1 × 2 × 3 × 4 × 5 = 120

6! = 1 × 2 × 3 × 4 × 5 × 6 = 720.

We can also define n! recursively as follows:

n! = n (n - 1)! for n ≥ 1

Thus 7! = 7 (6!) = 7 (720) = 5040;

And 8! = 8 (7!) = 8 (5040) = 40320.

Properties of Factorials:

- It is a natural number.
- If n ≥ 5, then n! ends in a zero or the unit's digit of n! always is zero.
- If n ≥1 then n! is divisible by 1, 2, 3, 4, ..., n.
- If n ≥ 1, the n! is divisible by r! for 1 ≤r ≤n In fact

  $\frac{n!}{r!} = \frac{n(n-1)\,(n-2)\,...(r+l)r\,(r-1)\,...2.1}{r(r-1)...2.1}$ = n(n − 1) (n - 2) ... (r + 1)
- Product of r(≥1) consecutive natural numbers can be written as quotient of two factorials.

  Let the r consecutive natural numbers be m, m + 1, m + 2... m + r - 1.

  Then, m × (m + 1) × (m + 2) ... x (m + r − 1) = $\frac{(m-1)!\,m(m+1)...(m+r-1)}{!} = \frac{(m+r-1)!}{(m-1)!}$

Permutations

A permutation is an arrangement of all or part of a number of things in a **definite order**. For example, the permutations of the three letters a, b, c taken all at a time **abc, acb, bca, bac, cba cab.** The permutations of the three letters a, b, c taken two at a time are **ab, ac, ba, bc, ca, cb.**

- Number of Permutations of n different things taken r at a time
- If n and r are positive integers such that 1 ≤r ≤ n, then the number of all permutations of n distinct things, taken r at a time is denoted by the symbol P (n, r) or n_P

  $n_{p_r} = \frac{n!}{(n-r)!}$ = n (n - l) (n - 2) ... (n − r + 1)

  When r = n, $n_{p_r} = n_{p_n}$ = n (n - 1) (n - 2) ... 1 = n!

Thus 8_{p_3} denotes the numbers of permutations of 8 different things taken 3 at a time, and s_{p_s} denotes the number of permutations of 5 different things taken 5 at a time.

- In permutations, the order of arrangement is taken into account; when the order is changed, a different permutation is obtained.

Example: Find the values of 5_{P_1},$5_{P_2}5_{P_3}$, 5_{P_4}, 5_{P_5} and 10_{P_7}

Sol.

5_{P_1} = 5, 5_{P_2} = 5 × 4 = 20, 5_{P_3} = 5 × 4 ×3 = 60,

5_{P_4} = 5 × 4 × 3 ×2 = 120,

5_{P_5} = 5! = 5 × 4 × 3 ×2 1 = 120

10_{P_7} = 10 ×9 × 8 ×7 × 6 × 5 × 4 = 604800.

Example: Find the total number of ways in which 4 persons can take their places in a cab having 6 seats

Sol. The number of ways in which 4 persons can take their places in a cab having 6 seats

6_{p_4} = 6 × 5 × 4 × 3 = 360 ways

Important:

- The total number of permutations of n different things taken all at a time is n!
- The total number of arrangements of n different things taken r at a time, in which a particular thing always, occurs = r × $n-1_{p_{r-1}}$
- The total number of permutations of n different things taken r at a time in which a particular thing never occurs = $n-1_{P_r}$
- The total number of permutations of n dissimilar things taken r at a time with repetitions = n^r, Circular Permutation
- The number of permutations of n things taken all at a time when p of them are alike and of one kind, q of them are alike and of second kind, all other being different, is $\frac{n!}{p!q!}$

  Remark: The above theorem can be extended if in addition to the above r things are alike and of third kind and so on, then total permutation = $\frac{n!}{p!q!r!}$

Example: In how many ways 6 people can stand in a queue?

Sol. The number of ways in which 6 people can stand in a queue = 6! = 720.

Example: How many 4 digits number (repetition is not allowed) can be made by using digits 1 - 7 if

4 will always be there in the number?

Sol. Total digits (n) = 7

Total ways of making the number if 4 is always there = r $\times n - 1_{p_{r-1}}$ = 4 × 6_{p_3} = 480.

Example: How many different 3 letter words can be made by 5 vowels, if vowel 'A' will never be included?

Sol. Total letters (n) = 5 So total number of ways = $n - 1_{P_r}$ = $5 - 1_{P_3}$ = 4_{P_3} = 24.

Example: How many 3 digits number can be made by using digits 1 to 7 if repetition is allowed?

Sol. Total digits (n) = 7

So total ways = 7^3 = 343

Example: Find the number of permutations of the letters of the world ASSAS5 ATION.

Sol. We have 13 letters in all of which 3 are A's, 4 are S's, 2 are I's and 2 are N's

The number of arrangements = $\frac{13!}{3!4!2!2!}$

Example: Find the number of permutations of the letters of the words 'DADDY DID A DEADLY DEED'.

Sol. We have 19 letters in all of which 9 are Ds; 3 are As; 2 are ys; are Es and the rest are all distinct. Therefore the number of arrangements = $\frac{19!}{9!3!2!3!}$

Example: How many different word 'ORDINATE'?

- So that THE VOWELS OCCUPY ODD PLACES.
- Beginning with 'O'
- Beginning with 'O' and ending with 'E'.

Sol.

- ORDINATE contains 8 letters: 4 odd places, 4 vowels.

  ➔ Number of arrangements of the vowels 4! Also number of arranging consonants is 4!

  Number of words = 4! × 4! = $(4 \times 3 \times 2 \times 1)^2$ = 576.
- When O is fixed we have only seven letters at our disposal

  ➔ Number of words = 7! = 5040
- When we have only six letters at our disposal, leaving 'O' and 'E' which are fixed Number of permutations = 6! = 720.

Circular Permutation

So far we have discussed permutation of objects (or things) in a row. This type of permutations is generally known as **linear permutations.** If we arrange the objects along a closed curve viz. a circle, the permutations are known as **circular permutations.**

As we have seen in the earlier sections of this chapter that every linear arrangement has a beginning and an end, but there is nothing like a beginning or an end in a circular permutation. Thus, in a circular permutation, we consider one object as fixed and the remaining objects are arranged as in case of linear arrangements.
The number of ways of arranging n distinct objects around a circle is (n - 1)!& the total number when taken r at a time will be $\frac{n_{p_r}}{r}$

Note: In the above, anti-clockwise and clockwise order of arrangements are considered as distinct permutations.

Example: How many ways can 7 Australians sit down at a round table?
Sol. 7 Australians can take their seat at the round table in (7 - 1)! 6! ways,

Example: In how many ways can 12 persons among whom two are brothers be arranged along a circle so that there is exactly one person between the two brothers?
Sol. One person between the two brothers can be chosen in 10_{p_1} = 10 ways. The remaining 9 persons can be arranged in 9_{p_9} = 9! The two brothers can be arranged in 2! Ways. Therefore, the total number of ways = (9!) (10) (2!) = (10!) (2!)
Important

- If there be no difference between clockwise andanticlockwise arrangements, the total no. of circular permutations of n things taken all at a time is $\frac{(n-1)!}{2}$ and the total number when taken r at a time will be $\frac{n_{p_r}}{2r}$

Example: In how many ways can a garland of 10 different flowers be made?
Sol. Since, there is no difference between clockwise and anticlockwise arrangements, the total no. of ways = (10 - 1)! /2 = 9! /2.

Combinations

A combination is a grouping or selection of all or part of a number of things without reference to the arrangement of the things selected. Thus the combinations of the three letters a, b, c taken 2 at a time are ab, ac, bc. Note that ab and ba are 1 combination but 2 permutations of the letters a, b.

The symbol n_{C_r} represents the number of combinations (selections, groups) of n different things taken r at a time. Thus 9_{C_4}denotes the number of combinations of 9 different things taken 4 at time.

Note: The sym C(n, r) having the same meanings as $n_{C_r}$is sometimes used.

Difference between Permutation & Combination

In a combination only a group is made and the order in which the objects are arrangement in a definite order is considered.

Examples

- ab and ba are two different permutations, but each represents the same combination.
- abc, acb, bac, cab, cab, cba are six different permutations but each one f them represents the same combination, namely a group of three objects a, b and c.

Remark: We use the word **arrangements for permutations** and **'selections'** for combinations.

Number of combinations of n different things taken r at a time

$$n_{C_r} = \frac{n_{p_r}}{r!} = \frac{n!}{r!(n-r)!} = \frac{n\,(n-1)(n-2)\ldots(n-r+l)}{n!}$$

Examples: The number of handshakes that may be exchanged among a party of 12 students if each student shakes hands once with each other student is

$$12_{C_2} = \frac{12!}{2!(12-2)!} = \frac{12!}{2!10!} = \frac{12\times 11}{1\times 2} = 66$$

The following formulas very useful in simplifying calculations:

$n_{C_r} = n_{C_{n-r}}$(complementary combination).

This formula indicates at the number of selections of r out of n things is the same as the number of selections of n - r out of n things. Like given in the following cases:

$5_{C_1} = \frac{5}{1} = 5$ $5_{C_2} = \frac{5\times 4}{1\times 2} = 10$ $5_{C_3} = \frac{5!}{5!} = 1$ $9_{C_7} = \frac{9\times 8}{1\times 2} = 36$, $22_{C_{22}} = \frac{25\times 24\times 23}{1\times 2\times 3} = 2300$

Note that in each case the numerator and denominator have the same number of factors.

Example: In how many ways a hockey team of eleven can be elected from 16 players?

Sol. Total number of ways = $16_{C_{11}} = \frac{16!}{11!\times 5!} = \frac{16\times 15\times 14\times 13\times 12}{5\times 4\times 3\times 2\times 1} = 4368$.

Practice Exercise

The following exercise is to be taken by the students as a homework assignment. Only after going through the theory provided. A sincere student shall stand to gain 01 of speed and conceptual clarity after having solved this exercise.

DIRECTIONS: For the following questions, four options are given. Choose the best option.

1. The letters is of the word ENTRANCE are arranged in all possible ways. The number of arrangements having the E's and N's together is

   1) 7! 2) 6! 3) 8! 4) 9!

2. A picnic party of four persons is to be selected from 8 Males and 3 females so as to include at least one Female. The possible numbers of ways are?

   1) 168 2) 84 3) 8 4) 260

3. How many numbers, each consisting of four different digits, can be formed with the digits 0, 1, 2, 3?

   1) 15 2) 19 3) 18 4) 24

4. How many numbers lying between 3000 and 4000 and which are divisible by 5 can be made with the digits 3, 4, 5, 6, 7 and 8? (Digits are not to be repeated in any number)

   1) 11 2) 12 3) 13 4) 14

5. The total number of ways of answering 5 objective type questions. Each question having 4 choice is

   1) 256 2) 512 3) 1024 4) 4096

6. A "necklace" is a string with several beads on it. It is allowed to rotate a necklace but not to turn it over. How many different necklaces can be made using 13 different beads?

   1) 13! 2) 13!/2 3) 12! 4) 12!/2

7. The number of ways of distributing 10 different books among 4 students (S_1, S_2, S_3, S_4) such that S_1 and S_2 get 2 books each and S_3 and S_4 get 3 books each is

   1) 12600 2) 15200 3) $10C_4$ 4) $\frac{10!}{2!2!3!3!}$

8. Each of two friends has 20 stamps and 10 postcards. We call an exchange fair if they exchange a stamp for a stamp or a postcard for a postcard. How many ways are there to carry out one fair exchange between these two friends?

   1) 350 2) 400 3) 600 4) 500

9. 4 men and 3 women are to be seated in a row so that the women occupy the even places. How many .such arrangements are possible?

   1) 7! 2) 44 4) 30 4) 6!

10. In how many ways can 4 men and 4 women be seated at a round table if each woman is to be

between two men?

1) 288 2) 144 3) 1440 4) 2880

11. By stringing together 9 different coloured beads, how many different bracelets can be made?

1) 20160 2) 40320 3) 80640 4) 10080

12. How many parallelograms are formed by a set of 4 parallel lines intersecting another set of 7 parallel lines?

1) 121 2) 139 3) 115 4) 126

13. In how many ways can 3 women be selected out of 15 women; if one particular woman is always included and two particular women are always excluded?

1) 66 2) 77 3) 88 4) 99

14. In how many ways can a person choose 1 or more out of 4 electrical appliances?

1) 10 2) 12 3) 14 4) 15

15. A father has 2 apples and 3 pears. Each weekday (Monday through Friday) he gives one of the fruits to his daughter. In how many ways can this be done?

1) 120 2) 10 3) 24 4) 12

16. How many ways are there to place a set of chess pieces on the first row of chess board. The set consists of a king, a queen identical rooks, knights & bishops?

1) 8! 2) 8^8 3)5040 4) 4280

17. Seven nouns, five verbs, and two adjectives are written on a black board. We can form a sentence by choosing one word of each type, and we do not care about how much sense the Sentence makes. How many ways are there to do this?

1) $7^2 \times 5^2 \times 2^2$ 2) $7^1 \times 5^1 \times 2^1 \times 3!$ 3) $7! \times 5! \times 2!$ 4) $2^7 \times 2^5 \times 2^2$

18. The numb of arrangements of the letters of the word SALOON, if the two O's do not come together, is

1) 360 2) 720 3) 240 4) 120

19. A class is composed of two brothers and six other boys. In how many ways can all the boys be seated at a round table so that the two brothers are not seated besides each other?

1) 720 2) 1440 3) 3600 4) 4320

20. Three dice are rolled. Find the number of possible outcomes in which at one dice shows 4?

1) 91 2) 100 3) 64 4) 1

21. Six boys and six girls sit along a line alternately in x ways and along a circle (again alternately) in y ways, then find the relation between x and y.

1) $x = 12y$ 2) $x = 2y$ 3) $x = 6y$ 4) None of these

22. Three men have 4 coats, 5 waist coats and 6 caps. In how many ways car. They wear them?
    1) 17280 2) 172800 3) 1728000 4) 1728

23. How many different words can be formed of the letters of the word "MATHEMATICS", so that no two vowels are together?
    1) 1058400 2) 105840 3) 105.84000 4) None of these

24. In the above question if the relative position of the vowels and consonant remains unaltered then how many different words can be formed?
    1) 15120 2) 151200 3) 15200 4) 15210

25. How many 6 digit number can be formed from the digits 1, 2, 3, 4, 5, 6 which are divisible by 4 and digits are not repeated?
    1) 192 2) 122 3) 140 4) 242

Answers:

1. 2	**2.** 4	**3.** 3	**4.** 2	**5.** 3
6. 3	**7.** 4	**8.** 4	**9.** 2	**10.** 2
11. 1	**12.** 4	**13.** **1**	**14.** 4	**15.** 2
16. 3	**17.** 2	**18.** 3	**19.** 3	**20.** 2
21. 2	**22.** 2	**23.** 1	**24.** 1	**25.** 1

