Linear Algebra
for data science

Eng. Fawzia faraag

What is Linear Algebra?

- It is a branch of mathematics concerned with the study of linear systems, vector spaces (or linear spaces), and linear transformations

Topic Coverage

- Systems of Linear Equations
- Matrices
- Determinants
- Vector Spaces
- Linear Transformations
- Eigenvalues and Eigenvectors

What is Linear Equation?

- Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Variables appear only to the first power.

$a_1x + a_2y = b,\qquad a_1, a_2,$ and b are constants.

linear equation in two variables x and y.

$a_1x + a_2y + a_3z = b,\qquad a_1, a_2, a_3,$ and b are constants.

linear equation in three variables x, y, and z.

What is Linear Equation?

A **linear equation in *n* variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** b is a real number.

Systems of Linear Equations

A **system of m linear equations in n variables** is a set of m equations, is linear in the same n variables:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}$$

A system of linear equations is also called a **linear system.**

The **solutions** must satisfy each equation in the system.

Systems of Linear Equations

For example, the system

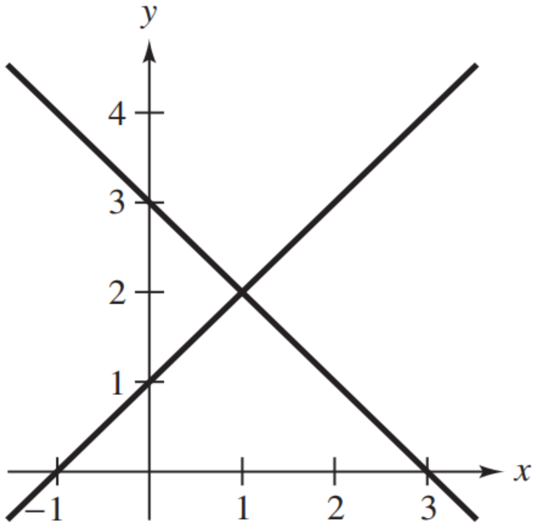
$$\begin{aligned} 3x_1 + 2x_2 &= 3 \\ -x_1 + x_2 &= 4 \end{aligned}$$

has $x_1 = -1$ and $x_2 = 3$ as a solution

Systems of Linear Equations

Systems of Two Equations in Two Variables

Unique solution

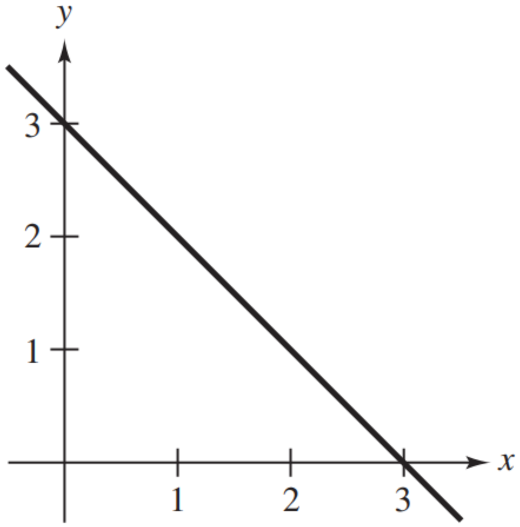


a. Two intersecting lines:

$$\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$$

$\boxed{\text{(consistent system)}}$

Infinitely many solutions

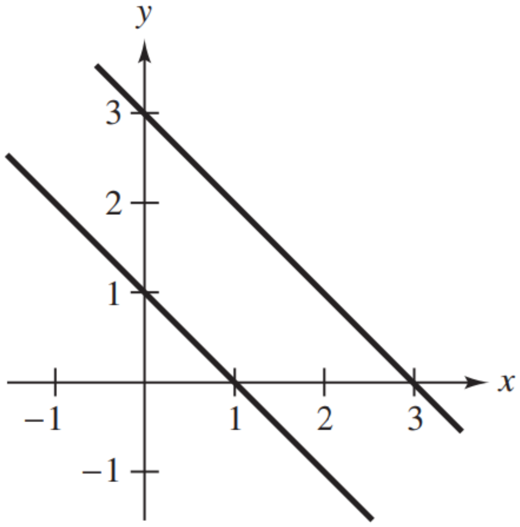


b. Two coincident lines:

$$\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$$

$\boxed{\text{(consistent system)}}$

No solution



c. Two parallel lines:

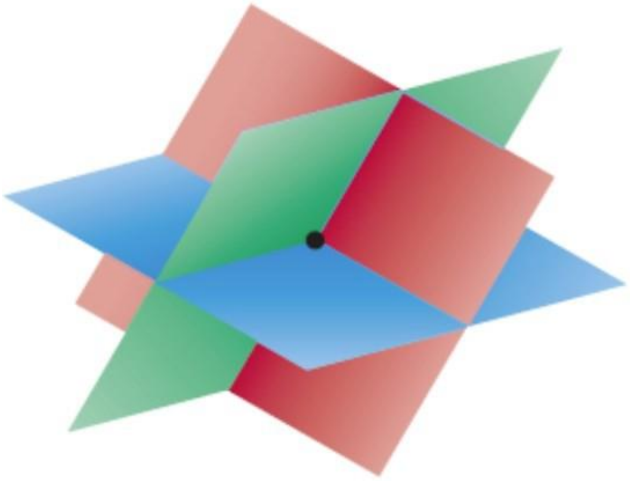
$$\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$$

$\boxed{\text{(inconsistent system)}}$

Systems of Linear Equations

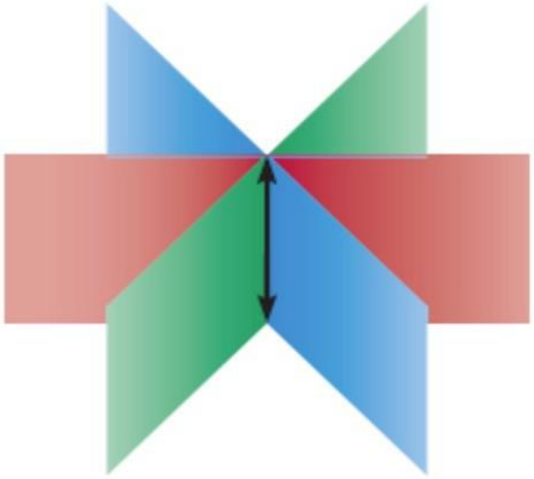
Systems of Three Equations in Three Variables

Unique solution



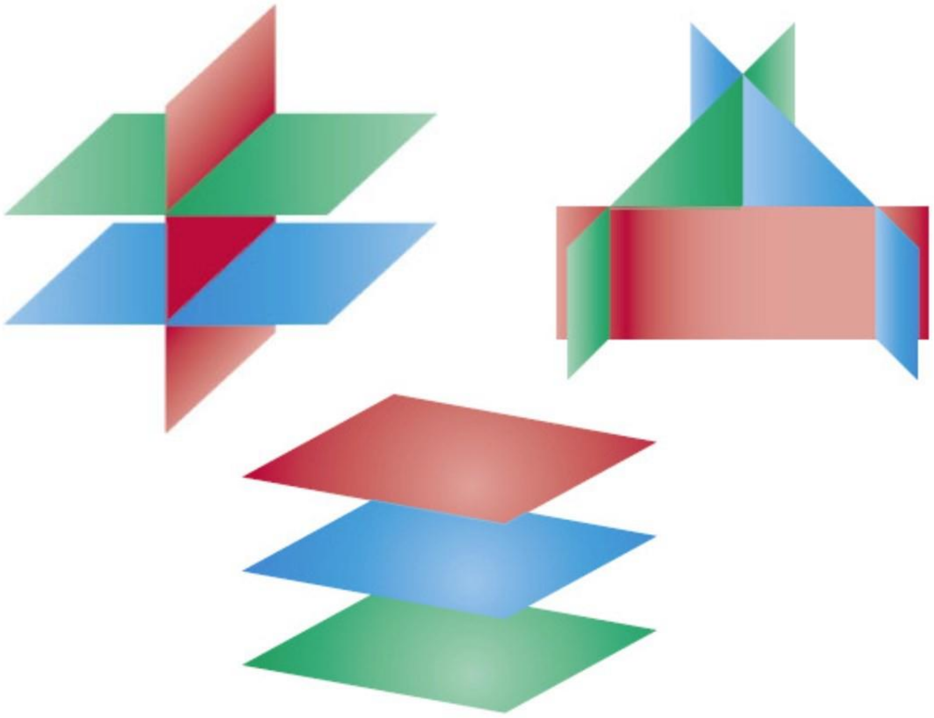
(consistent system)

Infinitely many solutions



(consistent system)

No solution



(inconsistent system)

Systems of Linear Equations

Definition of a Matrix

If m and n are positive integers, then an $m \times n$ (read "m by n") matrix is a rectangular array

$$\begin{array}{cc} & \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & \cdots & \text{Column } n \end{array} \\ \begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \vdots \\ \text{Row } m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

in which each **entry,** a_{ij}, of the matrix is a number. An $m \times n$ matrix has m rows and n columns. Matrices are usually denoted by capital letters.

Systems of Linear Equations

System

$$
\begin{aligned}
x - 4y + 3z &= 5 \\
-x + 3y - z &= -3 \\
2x - 4z &= 6
\end{aligned}
$$

Cofficient Matrix

$$
\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix}
$$

Augmented Matrix

$$
\begin{bmatrix} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{bmatrix}
$$

Systems of Linear Equations

Consider the system of 3 equations in 4 unknowns:

$$\begin{array}{ccccccccc} x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & -7 \\ x_1 & + & 2x_2 & & & - & 2x_4 & = & 9 \\ 3x_1 & + & x_2 & + & 4x_3 & & & = & 7 \end{array}$$

The matrix form is $AX = B$,

where $A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 1 & 2 & 0 & -2 \\ 3 & 1 & 4 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $B = \begin{bmatrix} -7 \\ 9 \\ 7 \end{bmatrix}$

The augmented matrix is $[A \mid B] = \left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & -7 \\ 1 & 2 & 0 & -2 & 9 \\ 3 & 1 & 4 & 0 & 7 \end{array}\right]$

Systems of Linear Equations

Elementary Row Operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

a. Interchange the first and second rows.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$	$R_1 \leftrightarrow R_2$

Systems of Linear Equations

b. Multiply the first row by $\frac{1}{2}$ to produce a new first row.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$	$\left(\frac{1}{2}\right)R_1 \rightarrow R_1$

$$\boldsymbol{R_1 = \left(\frac{1}{2}\right)R_1}$$

Systems of Linear Equations

c. Add -2 times the first row to the third row to produce a new third row.

Original Matrix	New Row-Equivalent Matrix	Notation
$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$	$R_3 + (-2)R_1 \rightarrow R_3$ $\boldsymbol{R_3 = -2R_1 + R_3}$

Notice that adding -2 times row 1 to row 3 does not change row 1.

Systems of Linear Equations

$$\text{e.g. } A=\begin{bmatrix}3 & 6 & 9\\ 1 & 1 & 2\\ 2 & 0 & 3\end{bmatrix}\xrightarrow{R_1\leftrightarrow R_2}\begin{bmatrix}1 & 1 & 2\\ 3 & 6 & 9\\ 2 & 0 & 3\end{bmatrix}\xrightarrow{\frac{1}{3}R_2}\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 2 & 0 & 3\end{bmatrix}$$

$$\xrightarrow{R_3=R_3-2R_1}\begin{bmatrix}1 & 1 & 2\\ 1 & 2 & 3\\ 0 & -2 & -1\end{bmatrix}=B.$$

The matrix *B* is row equivalent to the matrices *A*.

Systems of Linear Equations

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the properties below.

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in row-echelon form is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1.

Systems of Linear Equations

Determine whether each matrix is in **r**ow-**e**chelon **f**orm. If it is, determine whether the matrix is also in **r**educed **r**ow-**e**chelon **f**orm.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

f. $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Systems of Linear Equations

Determine whether each matrix is in **r**ow-**e**chelon **f**orm. If it is, determine whether the matrix is also in **r**educed **r**ow-**e**chelon **f**orm.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ ref ✓ rref x

b. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$ rref x ref x

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ ref ✓ rref x

d. $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ref ✓ rref ✓

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ ref x rref x

f. $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ref ✓ rref ✓

Systems of Linear Equations

Linear System

$$
\begin{aligned}
x - 2y + 3z &= 9 \\
-x + 3y \phantom{{}+3z} &= -4 \\
2x - 5y + 5z &= 17
\end{aligned}
$$

Associated Augmented Matrix

$$
\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}
$$

Carl Friedrich Gauss

$$
\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}
$$

row-**e**chelon **f**orm

Gaussian Elimination

$$
\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}
$$

reduced **r**ow-**e**chelon **f**orm

Gauss-Jordan Elimination

Systems of Linear Equations

$$AX = B$$

Unique solution

Number of leading 1's = number of unknowns (variables)

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(consistent system)

Infinitely many solutions

Number of leading 1's < number of unknowns (variables)

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(consistent system)

No solution

The system is inconsistent (0 = 1!!!)

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

(inconsistent system)

Systems of Linear Equations

Solve the system of linear equations

$$\begin{array}{rcrcrcrcr}
x & + & y & + & 2z & - & 5w & = & 3 \\
2x & + & 5y & - & z & - & 9w & = & -3 \\
2x & + & y & - & z & + & 3w & = & -11 \\
x & - & 3y & + & 2z & + & 7w & = & -5
\end{array}$$

Systems of Linear Equations

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & 1 & -1 & 3 & -11 \\ 1 & -3 & 2 & 7 & -5 \end{array}\right) \xrightarrow[R_4 = -R_1 + R_4]{R_2 = -2R_1 + R_2,\; R_3 = -2R_1 + R_3} \left(\begin{array}{cccc|c} 1 & 1 & 2 & -5 & 3 \\ 0 & 3 & -5 & 1 & -9 \\ 0 & -1 & -5 & 13 & -17 \\ 0 & -4 & 0 & 12 & -8 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 1 & 2 & -5 & 3 \\ 0 & -1 & -5 & 13 & -17 \\ 0 & 3 & -5 & 1 & -9 \\ 0 & -4 & 0 & 12 & -8 \end{array}\right) \xrightarrow{R_2 = -R_2} \left(\begin{array}{cccc|c} 1 & 1 & 2 & -5 & 3 \\ 0 & 1 & 5 & -13 & 17 \\ 0 & 3 & -5 & 1 & -9 \\ 0 & -4 & 0 & 12 & -8 \end{array}\right)$$

$$\xrightarrow[R_4 = 4R_2 + R_4]{R_1 = -R_2 + R_1,\; R_3 = -3R_2 + R_3} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 8 & -14 \\ 0 & 1 & 5 & -13 & 17 \\ 0 & 0 & -20 & 40 & -60 \\ 0 & 0 & 20 & -40 & 60 \end{array}\right) \xrightarrow{R_3 = \frac{-1}{20} R_3}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & 8 & -14 \\ 0 & 1 & 5 & -13 & 17 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 20 & -40 & 60 \end{array}\right) \xrightarrow[R_4 = -20R_3 + R_4]{R_1 = 3R_3 + R_1,\; R_2 = -5R_3 + R_2}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Let $\boxed{w = t}$, $t \in R$

$x + 2w = -5$

$x + 2t = -5$

$\boxed{x = -2t - 5}$

$y - 3w = 2$

$y - 3t = 2$

$\boxed{y = 3t + 2}$

$z - 2w = 3$

$z - 2t = 3$

$\boxed{z = 2t + 3}$

Systems of Linear Equations

$$\left(\begin{array}{cccc|c} 1 & 0 & -3 & 8 & -14 \\ 0 & 1 & 5 & -13 & 17 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 20 & -40 & 60 \end{array}\right) \xrightarrow{\begin{array}{l} R_1 = 3R_3 + R_1 \\ R_2 = -5R_3 + R_2 \\ R_4 = -20R_3 + R_4 \end{array}}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & -5 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$x + 2w = -5$

$x + 2t = -5$

$\boxed{x = -2t - 5}$

$y - 3w = 2$

$y - 3t = 2$

$\boxed{y = 3t + 2}$

$z - 2w = 3$

$z - 2t = 3$

$\boxed{z = 2t + 3}$

Let $\boxed{w = t}$, $t \in R$

Solu. $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2t-5 \\ 3t+2 \\ 2t+3 \\ t \end{pmatrix}$, $t \in R$ #

ex: let $t = 0$, let $t = 1$ . . . →

$\begin{pmatrix} -5 \\ 2 \\ 3 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -7 \\ 5 \\ 5 \\ 1 \end{pmatrix}$, Infinitely many solutions

Systems of Linear Equations

Example 2

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Systems of Linear Equations

Example 2

$$\begin{array}{cccc|c} x & y & z & w & \\ \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$x + 2w = 3$$

$$x + 2r = 3$$

$$\boxed{x = -2r + 3}$$

$$y + z = 1$$

$$y + t = 1 \longrightarrow \boxed{y = -t + 1}$$

let $\boxed{z = t}$, $\boxed{w = r}$, $t, r \in R$

Solu. $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2r + 3 \\ -t + 1 \\ t \\ r \end{pmatrix}$, $t, r \in R$

Systems of Linear Equations

Solve the system of linear equations

$$\begin{aligned} 3x + 3y + 12z &= 6 \\ x + y + 4z &= 2 \\ 2x + 5y + 20z &= 10 \\ -x + 2y + 8z &= 4 \end{aligned}$$

Systems of Linear Equations

$$\left[\begin{array}{ccc|c} 3 & 3 & 12 & 6 \\ 1 & 1 & 4 & 2 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{array}\right] \xrightarrow{R_1 = \frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 1 & 1 & 4 & 2 \\ 2 & 5 & 20 & 10 \\ -1 & 2 & 8 & 4 \end{array}\right]$$

$$\xrightarrow[R_4 = R_1 + R_4]{R_2 = -R_1 + R_2,\ R_3 = -2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 12 & 6 \\ 0 & 3 & 12 & 6 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 0 & 3 & 12 & 6 \\ 0 & 3 & 12 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 = \frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2 \\ 0 & 1 & 4 & 2 \\ 0 & 3 & 12 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_3 = -3R_2 + R_3]{R_1 = -R_2 + R_1} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Let $\boxed{z = t}$, $t \in R$

$\boxed{x = 0}$

$y + 4z = 2$

$y + 4t = 2$

$\boxed{y = -4t + 2}$

Solu. $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -4t+2 \\ t \end{pmatrix}$, $t \in R$

Systems of Linear Equations

Solve the system of linear equations

$$\begin{aligned} x + y &= 2 \\ y + z &= 2 \\ x + z &= 2 \\ x + y - 2z &= 0 \end{aligned}$$

Systems of Linear Equations

Solve the system of linear equations

$$x + y = 2$$
$$y + z = 2$$
$$x + z = 2$$
$$x + y - 2z = 0$$

$$\begin{aligned} x + y \phantom{{}+z} &= 2 \\ y + z &= 2 \\ x \phantom{{}+y} + z &= 2 \\ x + y - 2z &= 0 \end{aligned}$$

Systems of Linear Equations

$$\begin{aligned} x + y \quad &= 2 \\ y + z &= 2 \\ x \quad + z &= 2 \\ x + y - 2z &= 0 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow[R_4 = -R_1 + R_4]{R_3 = -R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$$\xrightarrow[R_3 = R_2 + R_3]{R_1 = -R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right) \xrightarrow{R_3 = \frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right)$$

$$\xrightarrow[R_4 = 2R_3 + R_4]{R_1 = R_3 + R_1,\ R_2 = -R_3 + R_2} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$\because$ # of leading 1's = # of variables

$\therefore$ this is a unique solu.

$\therefore x = 1$

$y = 1$

$z = 1$

solu. $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Systems of Linear Equations

Solve the system of linear equations

$$\begin{array}{lllllllll} x & + & 2y & + & 3z & + & 4w & = & 5 \\ x & + & 3y & + & 5z & + & 7w & = & 11. \\ x & & & - & z & - & 2w & = & -6 \end{array}$$

Systems of Linear Equations

$$\begin{aligned} x &+ 2y + 3z + 4w = 5 \\ x &+ 3y + 5z + 7w = 11. \\ x & - z - 2w = -6 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 7 & 11 \\ 1 & 0 & -1 & -2 & -6 \end{array}\right) \xrightarrow[R_3 = -R_1 + R_3]{R_2 = -R_1 + R_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & -2 & -4 & -6 & -11 \end{array}\right)$$

$$\xrightarrow[R_3 = 2R_2 + R_3]{R_1 = -2R_2 + R_1} \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -1 \\ 0 & 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$\because 0 = 1 \quad \therefore$ No solution

Systems of Linear Equations

Find all values of a for which the system

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

(i) has no solution,
(ii) has infinitely many solutions,
(iii) has a unique solutions.

Systems of Linear Equations

$$\begin{aligned} x + y - z &= 2 \\ x + 2y + z &= 3 \\ x + y + (a^2-5)z &= a \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & (a^2-5) & a \end{array}\right] \xrightarrow[R_3=-R_1+R_3]{R_2=-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & (a^2-4) & (a-2) \end{array}\right]$$

(i) $\boxed{\text{No solution}}$

If $a^2-4=0$ & $a-2 \neq 0$

$a^2 = 4$ & $a \neq 2$

$a = \pm 2$

$\therefore \boxed{a=-2}$

Systems of Linear Equations

$$\begin{aligned} x &+ y - z = 2 \\ x &+ 2y + z = 3 \\ x &+ y + (a^2-5)z = a \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & (a^2-5) & a \end{array}\right] \xrightarrow[R_3=-R_1+R_3]{R_2=-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & (a^2-4) & (a-2) \end{array}\right]$$

(ii) $\boxed{\text{Infinitely many solution}}$

if $a^2-4=0$ & $a-2=0$

$a^2=4$ & $a=2$

$a=\pm 2$

$\therefore \boxed{a=2}$

Systems of Linear Equations

$$\begin{aligned} x &+ y &- z &= 2 \\ x &+ 2y &+ z &= 3 \\ x &+ y &+ (a^2-5)z &= a \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & (a^2-5) & a \end{array}\right] \xrightarrow[R_3 = -R_1 + R_3]{R_2 = -R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & (a^2-4) & (a-2) \end{array}\right]$$

(iii) $\boxed{\text{unique solution}}$

if $a^2 - 4 \neq 0$

$a^2 \neq 4$

$a \neq \pm 2$

$\therefore \boxed{a = \mathbb{R} \setminus \{\pm 2\}}$

Systems of Linear Equations

HOMOGENEOUS SYSTEMS OF LINEAR EQUATIONS

Systems of linear equations in which each of the constant terms is zero are called **homogeneous.**

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0. \end{aligned}}$$

The homogenous system is always consistent (has solution) which is either of following: The unique solution ($\boldsymbol{X} = \boldsymbol{O}$, Zero solution), called the ***trivial solution (or obvious)***, or an infinitely many solutions (including the trivial solution), called the ***nontrivial solution***.

Systems of Linear Equations

Consider the **homogenous** system of equations:

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & 0 \\ x_1 & + & 2x_2 & & & - & 2x_4 & = & 0 \\ 3x_1 & + & x_2 & + & 4x_3 & & & = & 0 \end{array}$$

The matrix form is $AX = O$,

$$\text{where } A = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 1 & 2 & 0 & -2 \\ 3 & 1 & 4 & 0 \end{bmatrix},\ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{The augmented matrix is } [A \mid O] = \left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 0 \\ 1 & 2 & 0 & -2 & 0 \\ 3 & 1 & 4 & 0 & 0 \end{array}\right]$$

Systems of Linear Equations

Example:

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}.$$

Solve the homogeneous linear system $AX = O$.

Find one specific nonzerosolution.

Systems of Linear Equations

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow[R_3 = -R_1 + R_3]{R_2 = -R_1 + R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_2 = -R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[R_3 = -R_2 + R_3]{R_1 = -R_2 + R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{R_3 = -R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = -R_3 + R_2}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}$$

$x + w = 0$

$x + t = 0 \longrightarrow \boxed{x = -t}$

$y - w = 0$

$y - t = 0 \longrightarrow \boxed{y = t}$

$z + w = 0$

$z + t = 0 \longrightarrow \boxed{z = -t}$

let $\boxed{w = t}$, $t \in R$

$\therefore$ solu. $X = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -t \\ t \\ -t \\ t \end{pmatrix}$, $t \in R$

* let $t = 1$

$$\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

Systems of Linear Equations

- *For the system of linear equations $AX = B$ ($B \neq O$),*

If X_1 and X_2 are two solutions, then $rX_1 + sX_2$, $r + s = 1$, is also a solution.

e.g. If X_1 and X_2 are two solutions to the system of linear equations $AX = B$ ($B \neq O$), then $3X_1 - 2X_2$ and $0.25X_1 + 0.75X_2$ are also solutions.

e.g. If X_1, X_2 and X_3 are solutions to the system of linear equations $AX = B$ ($B \neq O$), then $3X_1 + 2X_2 - 4X_3$ is also a solution (as $3 + 2 - 4 = 1$).

- *For the homogenous system of linear equations $AX = O$,*

If X_1 and X_2 are two solutions, then $rX_1 + sX_2$ is also a solution.

e.g. If X_1 and X_2 are two solutions, then $3X_1 + 2X_2$ and $10X_1 - 4X_2$ are also solutions.

Matrices

It is standard mathematical convention to represent matrices in any one of the three ways listed below.

1. An uppercase letter such as A, B, or C
2. A representative element enclosed in brackets, such as $[a_{ij}]$, $[b_{ij}]$, or $[c_{ij}]$
3. A rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Example : If

$$A = \begin{bmatrix} 2 & 5 & 3 & 7 \\ 2 & 3 & 1 & 0 \\ 6 & 1 & 9 & 4 \\ 6 & 4 & 0 & 7 \end{bmatrix} \qquad a_{13} = 3 \qquad a_{34} = 4$$

Matrices

(1) Square matrix : (#rows = #columns)

$$\begin{bmatrix} 3 & -1 & -3 \\ 2 & 4 & 0 \\ -1 & 5 & 6 \end{bmatrix}$$

Main diagonal

(2) Upper triangular matrix :

$$\begin{bmatrix} 3 & -1 & -3 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

(3) Lower triangular matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 0 \\ 2 & 7 & -6 \end{bmatrix}$$

(4) Diagonal matrix :

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

(5) Identity matrix :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(6) Zero matrix :

$$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(7) Column matrix (n – vector):

$\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$ is a 3 - **column vector**

(8) Row matrix (n – vector):

$\begin{bmatrix} 3 & 5 & 8 \end{bmatrix}$ is a 3 - **row vector**

Operations with Matrices

Definition of Matrix Addition

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times n$, then their **sum** is the $m \times n$ matrix $A + B = [a_{ij} + b_{ij}]$.

The sum of two matrices of different sizes is *undefined*.

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0+(-1) & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

Operations with Matrices

Definition of Scalar Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and c is a scalar,
then the scalar multiple of A by c is the $m \times n$ matrix $cA = [ca_{ij}]$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$$

Operations with Matrices

For the matrices A and B, find (a) $3A$, (b) $-B$, and (c) $3A - B$.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix}$$

SOLUTION

a. $3A = 3\begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(4) \\ 3(-3) & 3(0) & 3(-1) \\ 3(2) & 3(1) & 3(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix}$

b. $-B = (-1)\begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & -3 \\ 1 & -3 & -2 \end{bmatrix}$

c. $3A - B = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & -3 \\ 6 & 3 & 6 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & -4 & 3 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 12 \\ -10 & 4 & -6 \\ 7 & 0 & 4 \end{bmatrix}$

Properties of Matrix Operations

Note :

- $A + B = B + A$
- $A + (B + C) = (A + B) + C$
- $A + O = O + A = A$
- $A - A = -A + A = O$

Zero matrix

For $r, s \in R$,

- $(r + s)A = rA + sA$
- $r(A + B) = rA + rB$
- $r(sA) = (rs)A$
- $1A = A$

Operations with Matrices

Definition of Matrix Multiplication

If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the **product** AB is an $m \times p$ matrix

$$AB = [c_{ij}]$$

where

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} a_{ik}b_{kj} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdot \cdot \cdot + a_{in}b_{nj}. \end{aligned}$$

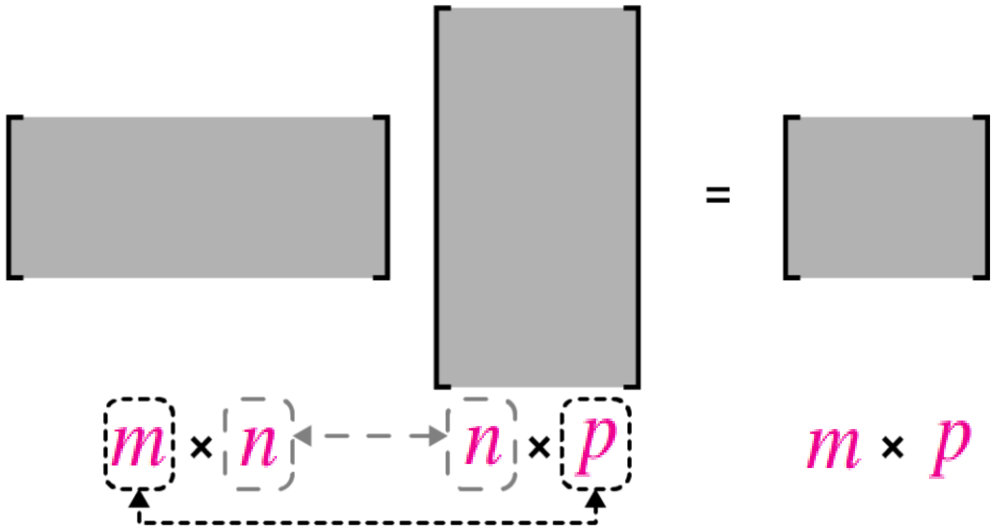
Operations with Matrices

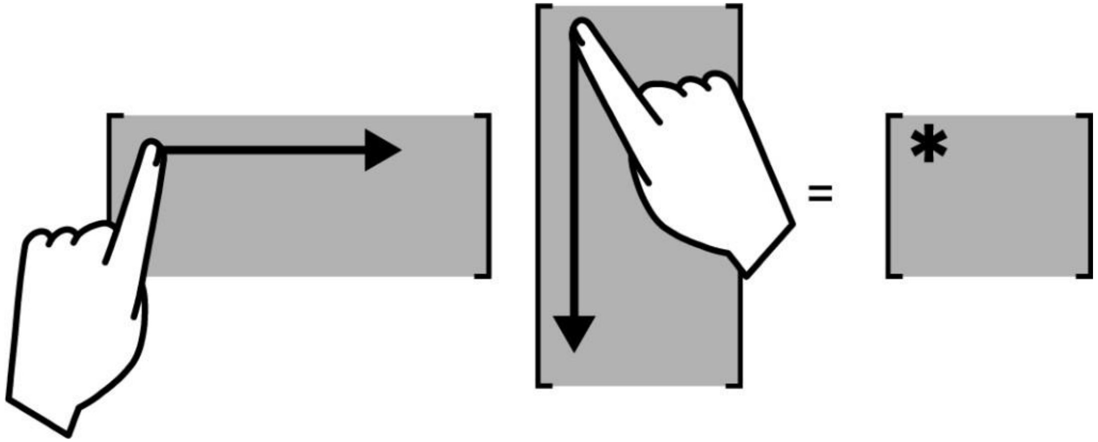
Matrix Multiplication

$$\underset{m \times n}{A} \quad \underset{n \times p}{B} \quad = \quad \underset{m \times p}{AB.}$$

Equal

Size of AB





Operations with Matrices

Find the product AB, where

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix}.$$

$$c_{11} = (-1)(-3) + (3)(-4) = -9$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

$$c_{12} = (-1)(2) + (3)(1) = 1$$

$$\begin{bmatrix} -1 & 3 \\ 4 & -2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}.$$

Operations with Matrices

Exampl :

If $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 2 & -1 \end{bmatrix}$,

then $AB = \begin{bmatrix} (2\times 4 + 1\times 2) & (2\times 6 + 1\times -1) \\ (3\times 4 + 0\times 2) & (3\times 6 + 0\times -1) \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 12 & 18 \end{bmatrix}$

Operations with Matrices

$$\underset{2 \times 3}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix}} \underset{3 \times 3}{\begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix}} = \underset{2 \times 3}{\begin{bmatrix} & & \\ & & \end{bmatrix}}$$

Operations with Matrices

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -5 & 7 & 5 \\ -7 & 6 & 6 \end{bmatrix}$$

2×3 $\quad$ 3×3 $\quad$ 2×3

Operations with Matrices

Example : If $A=\begin{bmatrix}1 & -1\\ 2 & -3\end{bmatrix}$, find A^2 and A^4.

$$A^2 = A\cdot A = \begin{bmatrix}1 & -1\\ 2 & -3\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & -3\end{bmatrix}$$

$$= \begin{bmatrix}-1 & 2\\ -4 & 7\end{bmatrix}$$

$$A^4 = A^2\cdot A^2 = ?$$

Properties of Matrix Operations

Note:

- $A(BC) = (AB)C$
- $A(B+C) = AB + AC$
- $(A+B)C = AC + BC$
- $AI_n = A$, A is an $m \times n$ matrix
- $I_m A = A$, A is an $m \times n$ matrix
- $A^n = \underbrace{AA \ldots A}_{n \text{ times}}$

Identity matrix

Zero matrix

- $AO = OA = O$
- If I is identity matrix and $(I)^n = I$.

Operations with Matrices

Show that AB and BA are not equal for the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

SOLUTION

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

$$AB \neq BA$$

Operations with Matrices

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find all matrices B such that $AB = BA$.

Let $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

$$AB = BA \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x+z & y+w \\ z & w \end{bmatrix} = \begin{bmatrix} x & x+y \\ z & z+w \end{bmatrix}$$

$$\Rightarrow \begin{cases} x + z = x \\ y + w = x + y \\ z = z \\ w = z + w \end{cases} \Rightarrow \begin{cases} z = 0 \\ x = w \end{cases} \Rightarrow \begin{cases} z = 0 \\ x = w = t \\ y = r \end{cases} \Rightarrow B = \begin{bmatrix} t & r \\ 0 & t \end{bmatrix}, \quad t, r \in \mathbf{R}.$$

Operations with Matrices

True or False

If $AC = BC$, then $A = B$.

Operations with Matrices

True or False

If $AC = BC$, then $A = B$.

False

$AC = BC$

$AC - BC = 0$

$(A-B)C = 0$

If $C = 0$
then A may be not equal to B

The Transpose of A Matrix

The **transpose** of a matrix is formed by writing its rows as columns.

A is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Size: $m \times n$

then the transpose, denoted by A^T, is the $n \times m$ matrix

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{m2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{m3} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{mn} \end{bmatrix}.$$

Size: $n \times m$

The Transpose of A Matrix

The **transpose** of a matrix is formed by writing its rows as columns.

$$\text{If } A=\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}, \text{then } A^T=\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix} \qquad \text{If } B=\begin{bmatrix} 3 & -2 & 5 \end{bmatrix}, \text{then } B^T=\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

$$C=\begin{bmatrix} 1 & -1 & 1 & -1 \\ 3 & 4 & 5 & 6 \\ 2 & -2 & 2 & -2 \\ 7 & 8 & 9 & 0 \end{bmatrix} \Rightarrow C^T=\begin{bmatrix} 1 & 3 & 2 & 7 \\ -1 & 4 & -2 & 8 \\ 1 & 5 & 2 & 9 \\ -1 & 6 & -2 & 0 \end{bmatrix}$$

Properties of Transposes

If A and B are matrices (with sizes such that the matrix operations are defined) and c is a scalar, then the properties below are true. its rows as columns.

1. $(A^T)^T = A$	Transpose of a transpose
2. $(A + B)^T = A^T + B^T$	Transpose of a sum
3. $(cA)^T = c(A^T)$	Transpose of a scalar multiple
4. $(AB)^T = B^T A^T$	Transpose of a product

If I is identity matrix and $(I)^T = I$.

Symmetric Matrices

A square matrix $A = [a_{ij}]$ is said to be symmetric if $A^T = A$

$(i.e.\ a_{ij} = a_{ji},\ 1 \le i \le n,\ 1 \le j \le n)$.

$$C = \begin{bmatrix} 1 & -1 & \boxed{3} & 4 \\ -1 & 8 & 5 & 6 \\ \boxed{3} & 5 & 9 & -2 \\ 4 & 6 & -2 & 0 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 1 & -1 & 3 & 4 \\ -1 & 8 & 5 & 6 \\ 3 & 5 & 9 & -2 \\ 4 & 6 & -2 & 0 \end{bmatrix}$$

$C^T = C \Rightarrow C$ is symmetric.

- If A is an $n \times n$ matrix, then $A + A^T$ and AA^T are both symmetric:

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T.$$

$$(AA^T)^T = (A^T)^T A^T = AA^T.$$

- If A and B are symmetric matrices, then $AB + BA$ is also symmetric:

$$(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T$$

$$= BA + AB = AB + BA.$$

Skew Symmetric Matrices

A square matrix $A = [a_{ij}]$ is said to be skew symmetric if

$A^T = -A$ (*i.e.* $a_{ij} = -a_{ji}$, $1 \le i \le n$, $1 \le j \le n$).

$$C = \begin{bmatrix} 0 & -1 & \boxed{-3} & 4 \\ 1 & 0 & 5 & -6 \\ \boxed{3} & -5 & 0 & -2 \\ -4 & 6 & 2 & 0 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 0 & 1 & 3 & -4 \\ -1 & 0 & -5 & 6 \\ -3 & 5 & 0 & 2 \\ 4 & -6 & -2 & 0 \end{bmatrix}$$

$C^T = -C \Rightarrow$

C is skew symmetric.

- If A is an $n \times n$ matrix, then $A - A^T$ is skew symmetric:

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$

- If A and B are skew symmetric, then $AB + BA$ is symmetric:

$$(AB + BA)^T = (AB)^T + (BA)^T = B^T A^T + A^T B^T$$
$$= (-B)(-A) + (-A)(-B) = BA + AB = AB + BA.$$

The Inverse of a Matrix

Definition of the Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that $AB = BA = I_n$, where I_n is the identity matrix of order n. The matrix B is the (multiplicative) **inverse** of A.

A matrix that does not have an inverse is **noninvertible** (or **singular**).

The inverse of A is denoted by A^{-1} .

The Inverse of a Matrix

Show that B is the inverse of A, where

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}.$$

SOLUTION

Using the definition of an inverse matrix, show that B is the inverse of A by showing that $AB = I = BA$.

$$AB = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1+2 & 2-2 \\ -1+1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The Inverse of a Matrix

Finding the Inverse of a Matrix

Let A be a square matrix of order n.

1. Write the $n \times 2n$ matrix that consists of A on the left and the $n \times n$ identity matrix I on the right to obtain $[A \quad I]$. This process is called **adjoining** matrix I to matrix A.

2. If possible, row reduce A to I using elementary row operations on the entire matrix $[A \quad I]$. The result will be the matrix $[I \quad A^{-1}]$. If this is not possible, then A is noninvertible (or singular).

3. Check your work by multiplying to see that $AA^{-1} = I = A^{-1}A$.

The Inverse of a Matrix

Find the inverse of the matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

The Inverse of a Matrix

$$\begin{array}{cc} \quad A & \qquad\qquad I \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = 6R_1 + R_3]{R_2 = -R_1 + R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = 4R_2 + R_3]{R_1 = R_2 + R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \xrightarrow{R_3 = -R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow[R_2 = R_3 + R_2]{R_1 = R_3 + R_1}$$

$$\left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \;\middle|\; \underbrace{\begin{matrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{matrix}}_{A^{-1}}\right]$$

$$A^{-1} = \begin{pmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{pmatrix}$$

check: $AA^{-1} = A^{-1}A = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The Inverse of a Matrix

Show that the matrix has no inverse.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix}$$

The Inverse of a Matrix

$$\begin{array}{cc} A & I \\ \left[\begin{matrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{matrix}\right. & \left|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \end{array} \xrightarrow{\substack{R_2 = -3R_1 + R_2 \\ R_3 = 2R_1 + R_3}}$$

$$\left[\begin{matrix} 1 & 2 & 0 \\ 0 & -7 & 2 \\ 0 & 7 & -2 \end{matrix}\right| \left.\begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{matrix}\right] \xrightarrow{R_2 = \frac{-1}{7} R_2}$$

$$\left[\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & \frac{-2}{7} \\ 0 & 7 & -2 \end{matrix}\right| \left.\begin{matrix} 1 & 0 & 0 \\ \frac{3}{7} & \frac{-1}{7} & 0 \\ 2 & 0 & 1 \end{matrix}\right] \xrightarrow{R_3 = -7R_2 + R_3}$$

$$\Big[\underbrace{\begin{matrix} 1 & 2 & 0 \\ 0 & 1 & \frac{-2}{7} \\ 0 & 0 & 0 \end{matrix}}_{\neq I}\Big| \underbrace{\begin{matrix} 1 & 0 & 0 \\ \frac{3}{7} & \frac{-1}{7} & 0 \\ -1 & 1 & 1 \end{matrix}}_{\neq A^{-1}}\Big]$$

$\boxed{A}$ has no inverse or is noninvertible

is singular

The Inverse of a Matrix

If A is a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

adjoint of A

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A is invertible if and only if $ad - bc \neq 0$.

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

$$ad - bc = (3)(2) - (-1)(-2) = 4.$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Properties of Inverse

<u>Note:</u>

- The inverse of a matrix is *unique*.
- If A and B are nonsingular matrices, then
  - A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.
  - A^T is nonsingular and $(A^T)^{-1} = (A^{-1})^T$.
  - AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.
  - A is nonsingular and $(A^n)^{-1} = (A^{-1})^n$.
  - A is nonsingular and $(nA)^{-1} = \frac{1}{n}A^{-1}$.
  - I is identity matrix and $(I)^{-1} = I$.

Properties of Inverse

<u>Note:</u>

- If A and B are nonsingular matrices, then
  - $(A+B)^{-1} \neq A^{-1} + B^{-1}$
  - $(A+B)^{-1} = A^{-1} - A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1} - A^{-1}BA^{-1}BA^{-1}BA^{-1} + \ldots$

Properties of Inverse

True or False

If A and B are invertible square matrices then $(A+B)^{-1} = A^{-1} + B^{-1}$

Properties of Inverse

True or False

If A and B are invertible square matrices then $(A+B)^{-1} = A^{-1} + B^{-1}$

False

let $A = B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore A^{-1} = B^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\boxed{A^{-1}+B^{-1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}$

$\boxed{(A+B)^{-1}} = \frac{1}{4}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}}$

$\neq$

Properties of Inverse

Example : Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$.

(i) Find $(A^2)^{-1}$.

(ii) Find the matrix X such that $A^T X = B$.

(iii) Find the matrix Y such that $YA = B^T$.

Properties of Inverse

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = -R_1 + R_3]{R_2 = -R_1 + R_2}$$

(In the first matrix, A is written above the left block and I above the right block.)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_3 = -2R_2 + R_3} \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} & \underbrace{\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 1 & 1 \end{matrix}}_{A^{-1}} \end{array}\right]$$

Properties of Inverse

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_3 = -2R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$

$$\underbrace{}_{I} \qquad \underbrace{}_{A^{-1}}$$

(i) $(A^2)^{-1} = (A^{-1})^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ -1 & -2 & 1 \end{pmatrix}$$

Properties of Inverse

(ii) $A^T X = B \rightarrow (A^T)^{-1}(A^T)X = (A^T)^{-1}B$

$$(A^T)^{-1} = (A^{-1})^T$$

$X = (A^T)^{-1}B$

$X = (A^{-1})^T B$

$$X = \underbrace{\begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{(A^{-1})^T} \underbrace{\begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}}_{B}$$

$$X = \begin{pmatrix} 1 & -4 \\ 1 & 5 \\ 0 & 2 \end{pmatrix}$$

(iii) $YA = B^T \rightarrow YAA^{-1} = B^T A^{-1}$

$Y = B^T A^{-1}$

$$Y = \overbrace{\begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 2 \end{pmatrix}}^{B^T} \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2 & 1 & 1 \end{pmatrix}}^{A^{-1}}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ -4 & 5 & 2 \end{pmatrix}$$

Periodic Matrix

In linear algebra, a *periodic matrix* is a square matrix $\boldsymbol{A}$ that satisfies the condition:

$$\boldsymbol{A}^{\boldsymbol{k}} = \boldsymbol{I}, \text{ for } k \in Z^{+} (\text{positive integers})$$

The smallest such k is the *order*.

Example:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \rightarrow \quad A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\therefore A \text{ is periodical with order} = 4$$

Periodic Matrix

Example: Prove that the following matrix is periodical and find the order.

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Periodic Matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

row 1
col 1

$$A^2 = AA = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_{3\times 3}$$

$\therefore$ A is periodical matrix with order = 3

Adjoint Matrix

The *adjoint* (or *adjugate*) matrix of an $n \times n$ matrix is the **transpose** of the *cofactor* matrix.

Steps: Given matrix $A_{n \times n}$.

1 Find the minor M_{ij} for each element A_{ij}.

2 Compute the cofactor matrix C.

3 The adjoint matrix $\mathrm{adj}(A) = C^T$.

Adjoint Matrix

Steps: Given matrix $A_{3\times3} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$.

1- Find the minor M_{ij} for each element A_{ij}.

$$M_{11} = \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}, \dots, M_{23} = \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$

2- Compute the cofactor matrix $C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$, where $C_{ij} = (-1)^{i+j}(M_{ij})$

$$C_{11} = (-1)^2 M_{11}, \dots, C_{12} = (-1)^3 M_{12}$$

3- The adjoint matrix $\operatorname{adj}(A) = C^T$.

Adjoint Matrix

Example: Find the adjoint of matrix A.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

Adjoint Matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$\boxed{1}$ Find the minor M_{ij} :

$M_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2-0=2$, $M_{12} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1=3$

$M_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 0-1=-1$, $M_{21} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2-0=2$

$M_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2-0=2$, $M_{23} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0-1=-1$

$M_{31} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1-0=1$, $M_{32} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1-0=1$

$M_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1-2=-1$

Adjoint Matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$\boxed{1}$ Find the minor M_{ij} :

$M_{11} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$, $M_{12} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3$

$M_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$, $M_{21} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2$

$M_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2 - 0 = 2$, $M_{23} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

$M_{31} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1$, $M_{32} = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 - 0 = 1$

$M_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$

$\boxed{2}$ Compute cofactor matrix C : $\rightsquigarrow C_{ij} = (-1)^{i+j}(M_{ij})$

$$C = \begin{pmatrix} \overset{+}{2} & \overset{-}{3} & \overset{+}{-1} \\ \overset{-}{2} & \overset{+}{2} & \overset{-}{-1} \\ \overset{+}{1} & \overset{-}{1} & \overset{+}{-1} \end{pmatrix} = \begin{pmatrix} 2 & -3 & -1 \\ -2 & 2 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$\boxed{3}$ The adjoint matrix of A $\mathrm{adj}(A) = C^T$:

$$\mathrm{adj}(A) = \begin{pmatrix} 2 & -2 & 1 \\ -3 & 2 & -1 \\ -1 & 1 & -1 \end{pmatrix}$$

Determinants

Definition of the Determinant of a 2 × 2 Matrix

Every square matrix can be associated with a real number called its *determinant.*

The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Example : If

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix},$$

$$\det(A) = |A| = 2(2) - 1(-3)$$

$$= 4 + 3 = 7.$$

Determinants

The Determinant of a 3 × 3 Matrix

The determinant of the matrix $A = \begin{array}{ccc} + & - & + \\ \end{array}$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is given by

$$|A| = \det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Note that *odd* positions (where $i + j$ is odd) have negative signs, and *even* positions (where $i + j$ is even) have positive signs.

Determinants

Minors and Cofactors of a Square Matrix

Minor of a_{21}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Delete row 2 and column 1.

Cofactor of a_{21}

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

Minor of a_{22}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Delete row 2 and column 2.

Cofactor of a_{22}

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

Note that *odd* positions (where $i + j$ is odd) have negative signs, and *even* positions (where $i + j$ is even) have positive signs.

Determinants

Minors and Cofactors of a Square Matrix

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3×3 matrix

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

4×4 matrix

Sign Pattern for Cofactors

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$n \times n$ matrix

Note that *odd* positions (where $i + j$ is odd) have negative signs, and *even* positions (where $i + j$ is even) have positive signs.

Example

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Determinants

$$A = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix} \end{array}.$$

$$\boxed{|A|} = -2\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix}$$

$$= -2\left[(3)(1) - (2)(4)\right] + \left[(3)(0) - (4)(-1)\right]$$

$$= -2\left[3 - 8\right] + \left[0 + 4\right]$$

$$= -2\left[-5\right] + \left[4\right] = 10 + 4 = \boxed{14}$$

$|A|$
det(A)

Example

Find the determinant of

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix}.$$

Determinants

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{bmatrix} 1 & -2 & \overset{+}{3} & 0 \\ -1 & 1 & \overset{-}{0} & 2 \\ 0 & 2 & \overset{+}{0} & 3 \\ 3 & 4 & \overset{-}{0} & -2 \end{bmatrix}. \quad |A| = 3 \begin{vmatrix} -1 & 1 & 2 \\ \overset{-}{0} & \overset{+}{2} & \overset{-}{3} \\ 3 & 4 & -2 \end{vmatrix}$$

$$|A| = 3\left[2 \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} - 3 \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right]$$

$$= 3\left[2\big((-1)(-2) - (3)(2)\big) - 3\big((-1)(4) - (3)(1)\big) \right]$$

$$= 3\left[2(2 - 6) - 3(-4 - 3) \right]$$

$$= 3\left[2(-4) - 3(-7) \right]$$

$$= 3\left[-8 + 21 \right] = 3[13] = \boxed{39}$$

|A|

det(A)

Determinants

The determinant of a 3 × 3 matrix (alternative method)

$$- a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Subtract these three products.

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

Add these three products.

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$+ \longrightarrow |A|$$

Determinants

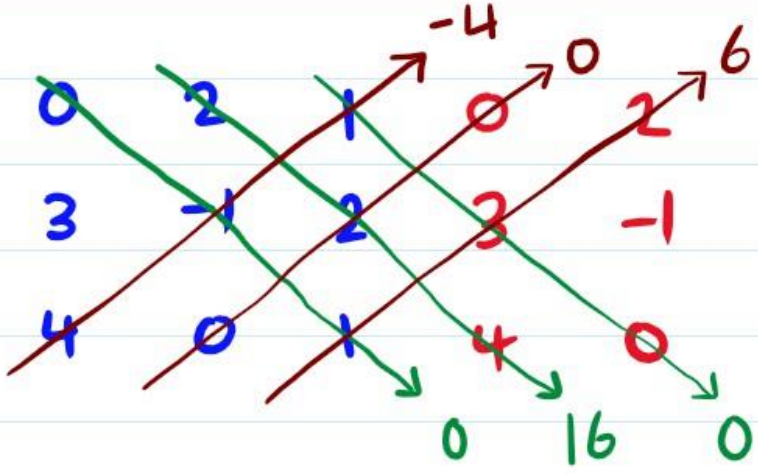
Example

Find the determinant of

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$

Determinants

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}.$$



$\boxed{|A|} = 0 + 16 + 0 - (-4) - 0 - 6$

$= 16 + 4 - 6 = \boxed{14}$

Properties of Determinants

Upper Triangular Matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular Matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Determinant of a Triangular Matrix

If A is a triangular matrix of order n, then its determinant is the **product of the entries on the main diagonal**. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdots a_{nn}.$$

The determinant of the lower triangular matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

is $|A| = (2)(-2)(1)(3) = -12.$

Properties of Determinants

Conditions That Yield a Zero Determinant

If A is a square matrix and any one of the conditions below is true, then $\det(A) = 0$.

1. An entire row (or an entire column) consists of zeros.
2. Two rows (or columns) are equal.
3. One row (or column) is a multiple of another row (or column).

$$\begin{vmatrix} 0 & 0 & 0 \\ 2 & 4 & -5 \\ 3 & -5 & 2 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & -2 & 4 \\ 0 & 1 & 2 \\ 1 & -2 & 4 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & -6 \\ -2 & 0 & 6 \end{vmatrix} = 0.$$

The first row has all zeros.

The first and third rows are the same.

The third column is a multiple of the first column.

Properties of Determinants

Determinant of a Transpose

If $\boldsymbol{A}$ is a square matrix, then $\mathbf{det}(A) = \boldsymbol{det}(\boldsymbol{A}^{\boldsymbol{T}})$.

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 0 & 0 \\ -4 & -1 & 5 \end{bmatrix} \qquad \begin{aligned} |A| &= 2(-1)^3 \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix} \\ &= (2)(-1)(3) \\ &= -6. \end{aligned}$$

$$A^T = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 0 & -1 \\ -2 & 0 & 5 \end{bmatrix} \qquad \begin{aligned} |A^T| &= 2(-1)^3 \begin{vmatrix} 1 & -1 \\ -2 & 5 \end{vmatrix} \\ &= (2)(-1)(3) \\ &= -6. \end{aligned}$$

Properties of Determinants

Notes:

If A is an $n \times n$ invertible matrix, then

$$\mathbf{det}\left(\boldsymbol{A}^{-1}\right) = \frac{\mathbf{1}}{\boldsymbol{det}(\boldsymbol{A})} \qquad |A^{-1}| = \frac{1}{|A|}.$$

If a matrix B results from a matrix A by interchanging two rows (columns) of A, then $|B| = -|A|$.

e.g. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}.$ $\qquad \begin{vmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{vmatrix}$

Interchange the first two rows. Interchange the first two columns.

Properties of Determinants

Notes:

If a matrix B is obtained from a matrix A by multiplying **one** row (column) of A by a number r , then $|B| = r\,|A|$.

e.g. $$\begin{vmatrix} \boxed{10 \quad 20} \\ 3 \quad\;\; 4 \end{vmatrix} = 10\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (10)(-2) = -20.$$

If a matrix B is obtained from a matrix A by adding a multiple of one row (column) of A to another row (column) of A, then $|B| = |A|$.

e.g. $$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \underset{R_1 + 10R_2}{\begin{vmatrix} 31 & 42 \\ 3 & 4 \end{vmatrix}} = -2.$$

Properties of Determinants

Notes:

- $|AB| = |A|\,|B|$
- $|ABC\ldots| = |A|\,|B|\,|C|\,\ldots|..|$
- $|A^n| = |A|^n$
- If A is an $n\times n$ matrix and r is a scalar, then
  - $|r\,A| = r^{\,n}\,|A|$, $|r\,A^{-1}| = r^{\,n} / |A|$, $|(rA)^{-1}| = 1 / (r^{\,n}\,|A|)$
- A square matrix A is *invertible* (*nonsingular*) if and only if $|A| \neq 0$
- $|A+B| \neq |A| + |B|$

Examples

Example 1:

If A and B are 5×5 matrices with $|A| = 8$ and $|B| = 2$. Find

(i) $|A^2|$ (ii) $|-A|$ (iii) $|A^T B^{-1}|$ (iv) $|2A^{-1}B^4A|$.

$$(i)\ \left|A^2\right| = |A|^2 = 8^2 = 64.$$

$$(ii)\ |-A| = (-1)^5|A| = -8.$$

$$(iii)\ \left|A^T B^{-1}\right| = \left|A^T\right|\left|B^{-1}\right| = |A|\frac{1}{|B|} = 4.$$

$$(iv)\ \left|2A^{-1}B^4A\right| = 2^5\left|A^{-1}\right|\left|B^4\right||A| = 32\frac{1}{|A|}|B|^4|A| = 32|B|^4 = 512.$$

Examples

Example 2: Answer each of the following as **True or False**. Justify your answer.

(i) If A and B are $n\times n$ matrices, then $|AB| = |BA|$.

(ii) If A and B are $n\times n$ matrices, then $|A + B| = |A| + |B|$.

(iii) If A and B are $n\times n$ matrices such that $AB^2 = I_n$, then A is nonsingular.

(i) $|AB| = |A||B| = |B||A| = |BA|$, $\boxed{\text{TRUE.}}$

(ii) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$.

$|A+B| = 24, |A| = 2, |B| = 12$, $\boxed{\text{FALSE}}$, using counter example.

(iii) $|AB^2| = |I_n| \Rightarrow |A||B|^2 = |I_n| = 1 \Rightarrow |A| \neq 0$, $\boxed{\text{TRUE.}}$

Examples

Example 3: Evaluate $\begin{vmatrix} 2 & 3 & 5 & 7 \\ 0 & 0 & 3 & 4 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}$.

$$\begin{vmatrix} 2 & 3 & 5 & 7 \\ 0 & 0 & 3 & 4 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \underset{R_2 \leftrightarrow R_3}{=} -\begin{vmatrix} 2 & 3 & 5 & 7 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -1 \end{vmatrix} = -(2)(4)(3)(-1) = 24.$$

Examples

Example 4: By reduction to triangular form, evaluate

$$\begin{vmatrix} 1 & -3 & -4 & -2 \\ -1 & 4 & 3 & 4 \\ 2 & -5 & 1 & 8 \\ 0 & 1 & 0 & -1 \end{vmatrix}.$$

Examples

$$\begin{vmatrix} 1 & -3 & -4 & -2 \\ -1 & 4 & 3 & 4 \\ 2 & -5 & 1 & 8 \\ 0 & 1 & 0 & -1 \end{vmatrix}.$$

$$\xrightarrow[R_3 = -2R_1 + R_3]{R_2 = R_1 + R_2} = \begin{vmatrix} 1 & -3 & -4 & -2 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 1 & 9 & 12 \\ 0 & 1 & 0 & -1 \end{vmatrix} \xrightarrow[R_4 = -R_2 + R_4]{R_3 = -R_2 + R_3}$$

$$= \begin{vmatrix} 1 & -3 & -4 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & \boxed{10} & 10 \\ 0 & 0 & 1 & -3 \end{vmatrix} \xrightarrow{\text{10 Common factor from Row 3}}$$

$$= 10 \begin{vmatrix} 1 & -3 & -4 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 1 & -3 \end{vmatrix} \xrightarrow{R_4 = -R_3 + R_4}$$

$$= 10 \begin{vmatrix} 1 & -3 & -4 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{vmatrix} = 10(1)(1)(1)(-4)$$

$$= \boxed{-40}$$

Examples

Example 5: If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, Find $\begin{vmatrix} 2g & a & 3d \\ 2h & b & 3e \\ 2i & c & 3f \end{vmatrix}$.

Examples

Example 5: If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2$, Find $\begin{vmatrix} 2g & a & 3d \\ 2h & b & 3e \\ 2i & c & 3f \end{vmatrix}$.

$$\begin{vmatrix} 2g & a & 3d \\ 2h & b & 3e \\ 2i & c & 3f \end{vmatrix} = 3\begin{vmatrix} 2g & a. & d \\ 2h & b. & e \\ 2i & c. & f \end{vmatrix} = 6\begin{vmatrix} g & a & d \\ h & b & e \\ i & c & f \end{vmatrix}$$

$$= -6\begin{vmatrix} d. & a & g \\ e & b & h \\ f & c & i \end{vmatrix} = 6\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = 6\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6 \times 2 = 12.$$

Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix A with a nonzero determinant $|A|$, then the solution of the system is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the ith column of A_i is the column of constants in the system of equations.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

If the determinant $|A| \neq 0$, then you know the system has a unique solution

$$x_1 = \frac{|A_1|}{|A|} \text{ and } x_2 = \frac{|A_2|}{|A|}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad |A_1| = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \text{ and } |A_2| = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}$$

Examples

Use Cramer's Rule to solve the system of linear equations.

$$\begin{aligned} 4x_1 - 2x_2 &= 10 \\ 3x_1 - 5x_2 &= 11 \end{aligned}$$

Examples

$$|A| = \begin{vmatrix} 4 & -2 \\ 3 & -5 \end{vmatrix} = (4)(-5) - (3)(-2) = -20 + 6$$
$$= -14$$

$$|A_1| = \begin{vmatrix} 10 & -2 \\ 11 & -5 \end{vmatrix} = (10)(-5) - (11)(-2) = -50 + 22$$
$$= -28$$

$$|A_2| = \begin{vmatrix} 4 & 10 \\ 3 & 11 \end{vmatrix} = (4)(11) - (3)(10) = 44 - 30$$
$$= 14$$

$$\therefore \boxed{x_1} = \frac{|A_1|}{|A|} = \frac{-28}{-14} = \boxed{2}$$

$$\boxed{x_2} = \frac{|A_2|}{|A|} = \frac{14}{-14} = \boxed{-1}$$

Examples

Use Cramer's Rule to solve the system of linear equations for x.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \phantom{{}+ 5z} &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Examples

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{vmatrix} = 1\begin{vmatrix} -2 & 3 \\ -5 & 5 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= [(-2)(5) - (-5)(3)] + 3[(1)(5) - (2)(3)]$$

$$= [-10 + 15] + 3[5 - 6] = 5 - 3 = 2$$

$$\boxed{|A_1|} = \begin{vmatrix} 9 & -2 & 3 \\ -4 & 3 & 0 \\ 17 & -5 & 5 \end{vmatrix} = 4\begin{vmatrix} -2 & 3 \\ -5 & 5 \end{vmatrix} + 3\begin{vmatrix} 9 & 3 \\ 17 & 5 \end{vmatrix}$$

$$= 4[(-2)(5) - (-5)(3)] + 3[(9)(5) - (17)(3)]$$

$$= 4[-10 + 15] + 3[45 - 51]$$

$$= 4(5) + 3(-6) = 20 - 18 = \boxed{2}$$

$$\boxed{|A_2|} = \begin{vmatrix} 1 & 9 & 3 \\ -1 & -4 & 0 \\ 2 & 17 & 5 \end{vmatrix} = 1\begin{vmatrix} 9 & 3 \\ 17 & 5 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= [(9)(5) - (17)(3)] - 4[(1)(5) - (2)(3)]$$

$$= [45 - 51] - 4[5 - 6]$$

$$= -6 - 4(-1) = -6 + 4 = \boxed{-2}$$

$$\boxed{|A_3|} = \begin{vmatrix} 1 & -2 & 9 \\ -1 & 3 & -4 \\ 2 & -5 & 17 \end{vmatrix} = 1\begin{vmatrix} -2 & 9 \\ -5 & 17 \end{vmatrix} + 3\begin{vmatrix} 1 & 9 \\ 2 & 17 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & -5 \end{vmatrix}$$

$$= [(-2)(17) - (-5)(9)] + 3[(1)(17) - (2)(9)] + 4[(1)(-5) - (2)(-2)]$$

$$= [-34 + 45] + 3[17 - 18] + 4[-5 + 4]$$

$$= 11 + 3(-1) + 4(-1) = 11 - 3 - 4 = \boxed{4}$$

Examples

$$|A| = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{vmatrix} = 1\begin{vmatrix} -2 & 3 \\ -5 & 5 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= \left[(-2)(5) - (-5)(3)\right] + 3\left[(1)(5) - (2)(3)\right]$$

$$= \left[-10 + 15\right] + 3\left[5 - 6\right]$$

$$\boxed{|A_1|} = \begin{vmatrix} 9 & -2 & 3 \\ -4 & 3 & 0 \\ 17 & -5 & 5 \end{vmatrix} = 4\begin{vmatrix} -2 & 3 \\ -5 & 5 \end{vmatrix} + \dots$$

$$= 4\left[(-2)(5) - (-5)(3)\right] + 3\left[(9)(5) - (17)(3)\right]$$

$$= 4\left[-10 + 15\right] + 3\left[45 - 51\right]$$

$$= 4(5) + 3(-6) = 20 - 18 = \boxed{2}$$

$$\boxed{|A_2|} = \begin{vmatrix} 1 & 9 & 3 \\ -1 & -4 & 0 \\ 2 & 17 & 5 \end{vmatrix} = 1\begin{vmatrix} 9 & 3 \\ 17 & 5 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$$

$$= \left[(9)(5) - (17)(3)\right] - 4\left[(1)(5) - (2)(3)\right]$$

$$\dots 1] - 4\left[5 - 6\right]$$

$$\dots 4(-1) = -6 + 4 = \boxed{-2}$$

$$\dots \begin{vmatrix} \dots 2 & 9 \\ \dots 3 & -4 \\ \dots 5 & 17 \end{vmatrix} = 1\begin{vmatrix} -2 & 9 \\ -5 & 17 \end{vmatrix} + 3\begin{vmatrix} 1 & 9 \\ 2 & 17 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & -5 \end{vmatrix}$$

$$= \left[\dots - (-5)(9)\right] + 3\left[(1)(17) - (2)(9)\right] + 4\left[(1)(-5) - (2)(-2)\right]$$

$$= \left[-34 + 45\right] + 3\left[17 - 18\right] + 4\left[-5 + 4\right]$$

$$= 11 + 3(-1) + 4(-1) = 11 - 3 - 4 = \boxed{4}$$

$$\therefore x = \frac{|A_1|}{|A|} = \frac{2}{2} = 1$$

$$y = \frac{|A_2|}{|A|} = \frac{-2}{2} = -1$$

$$z = \frac{|A_3|}{|A|} = \frac{4}{2} = 2$$

Solution

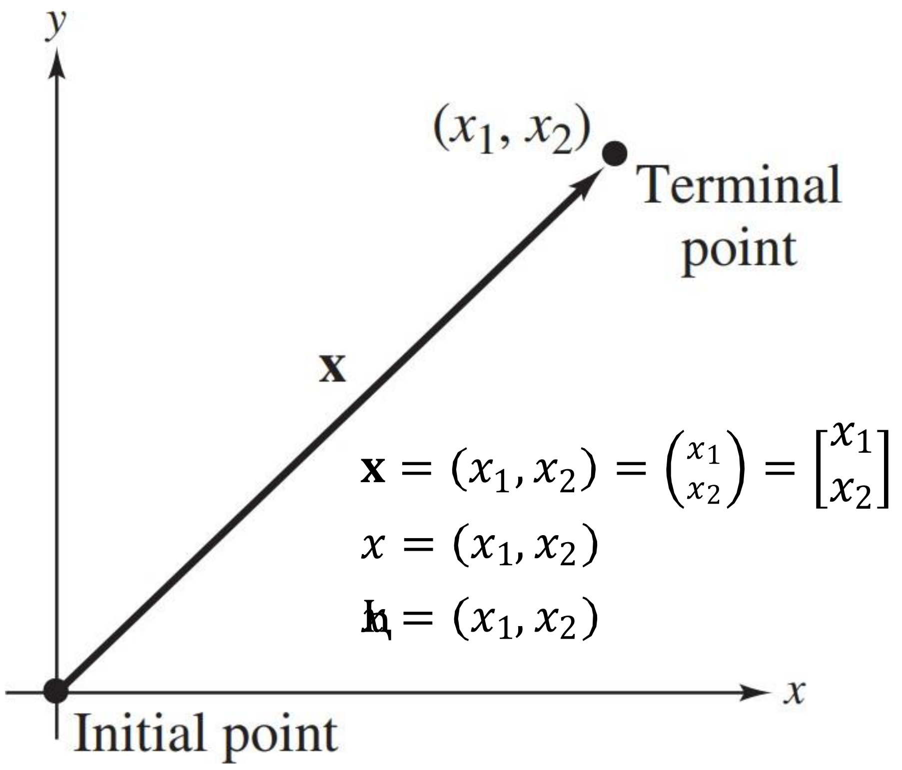
Vectors in the plane

Geometric Representation

Geometrically, a **vector in the plane** is represented by a **directed line segment** with its **initial point** at the origin and its **terminal point** at (x_1, x_2).

The term vector derives from the *Latin* word *vectus*, meaning "to carry." The idea is that if you were to carry something from the origin to the point (x_1, x_2).

Vectors are represented by lowercase letters set in boldface type (such as **u**, **v**, **w**, and **x**). Non-bold italic, as in v, or non-bold italic accented by a right arrow, as in $\vec{h}$, or $\vec{v}$.



Vectors in R^n

Vectors in R^2

R^2 = 2-space = set of all ordered pairs of real numbers.

a 2-vector $\mathbf{u} = (-3, 5) = \begin{pmatrix} -3 \\ 5 \end{pmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ is vector in R^2 or vector in the plane.

Vectors in R^3

R^3 = 3-space = set of all ordered triples of real numbers.

a 3-vector $\mathbf{v} = (1,0,-2) = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ is vector in R^3 or vector in the space.

Vectors in R^n

Vectors in R^n

R^n = n-space = set of all ordered n-tuples of real numbers.

An n-vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is called a vector in R^n,

where $v_1, v_2, \ldots, v_n \in R$ are called the components of the vector.

Vectors in R^n

Vector Equality

Two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbf{R}^n$are said to be equal if $u_i = v_i, 1 \leq i \leq n$.

Example

- Two vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$ in $\mathbf{R}^3$ are equal.
- Two vectors $\mathbf{s} = (-3, 2, 0, 7)$ and $\mathbf{w} = (-3, 2, 0, 7)$ in $\mathbf{R}^4$ are equal.

Vectors in R^n

Vector Addition

The sum of two vectors $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbf{R}^n$ are a vector in $\mathbf{R}^n$ defined by:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

Example

Two vectors $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$ in $\mathbf{R}^3$

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 - 2 \\ 0 + 2 \\ 5 - 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

Vectors in R^n

Scalar Multiplication

The product of a vector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ in $\mathbf{R}^n$ by a real scalar r is defined by $r\mathbf{u} = \begin{bmatrix} ru_1 \\ ru_2 \\ \vdots \\ ru_n \end{bmatrix}$.

Example

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$ be vectors in $\mathbf{R}^3$. Find $3\mathbf{u} - 2\mathbf{v}$.

$$3\mathbf{u} - 2\mathbf{v} = 3\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} - 2\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 15 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 21 \end{bmatrix}$$

Vectors in R^n

Properties (1/2):

Let **u**, **v**, and **w** be vectors in R^n, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v}$ is a vector in R^n.	Closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	Associative property of addition
4. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	Additive identity property
5. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ is the zero vector in R^n	Additive inverse property
6. $c\mathbf{u}$ is a vector in R^n.	Closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	Distributive property
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$	Distributive property
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$	Associative property of multiplication
10. $1(\mathbf{u}) = \mathbf{u}$	Multiplicative identity property

Vectors in R^n

Properties (2/2):

Let $\mathbf{v}$ be a vector in R^n, and let c be a scalar. Then the properties below are true.

1. The additive identity is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
2. The additive inverse of $\mathbf{v}$ is unique. That is, if $\mathbf{v} + \mathbf{u} = \mathbf{0}$, then $\mathbf{u} = -\mathbf{v}$.
3. $0\mathbf{v} = \mathbf{0}$
4. $c\mathbf{0} = \mathbf{0}$
5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.
6. $-(-\mathbf{v}) = \mathbf{v}$

is the zero vector in R^n is the zero value (scalar)

Linear Combination

Linear Combination

A vector $\mathbf{x}$ in R^n is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ in R^n if there exist scalars $c_1, c_2, \cdots, c_k \in R$ such that

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k.$$

The vector $\mathbf{x}$ is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$.

Example:

The vector $\mathbf{x} = (-1, -2, -2)$ is a linear combination of

$\mathbf{u} = (0, 1, 4)$, $\mathbf{v} = (-1, 1, 2)$, and $\mathbf{w} = (3, 1, 2)$

since $\mathbf{x} = \mathbf{u} - 2\mathbf{v} - \mathbf{w}$.

Linear Combination

Example 1:

Determine whether the vector $\mathbf{x} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is a linear combination of the vectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

Linear Combination

Example 1:

$X = aX_1 + bX_2 \Rightarrow \begin{bmatrix} 5 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 3 \\ -4 \end{bmatrix}$

$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} a + 3b \\ 2a - 4b \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -4 & 0 \end{array}\right] \xRightarrow{R_2 = -2R_1 + R_2} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -10 & -10 \end{array}\right] \xRightarrow{R_2 = \frac{1}{-10} R_2}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right] \xRightarrow{R_1 = -3R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \begin{array}{l} \rightarrow a \\ \rightarrow b \end{array}$$

$\therefore a = 2 , b = 1$

$\therefore X = 2X_1 + X_2$

$\therefore$ X linear combination of X_1 and X_2

Linear Combination

Example 2:

Determine whether the vector $\mathbf{v} = \begin{bmatrix} 10 \\ 1 \\ 4 \end{bmatrix}$ is a linear combination of the vectors $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$.

Linear Combination

Example 2:

$$V = a u_1 + b u_2 + c u_3 \Rightarrow \begin{bmatrix} 10 \\ 1 \\ 4 \end{bmatrix} = a\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c\begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2a + b - 2c \\ 3a + 2b + 2c \\ 5a + 4b + 3c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{array}\right] \xrightarrow{R_1 = \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{array}\right] \xrightarrow[R_3 = -5R_1 + R_3]{R_2 = -3R_1 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & \frac{1}{2} & 5 & -14 \\ 0 & \frac{3}{2} & 8 & -21 \end{array}\right] \xrightarrow{R_2 = 2R_2} \left[\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & 1 & 10 & -28 \\ 0 & \frac{3}{2} & 8 & -21 \end{array}\right] \xrightarrow[R_3 = \frac{-3}{2}R_2 + R_3]{R_1 = \frac{-1}{2}R_2 + R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & 19 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & -7 & 21 \end{array}\right] \xrightarrow{R_3 = \frac{1}{-7}R_3} \left[\begin{array}{ccc|c} 1 & 0 & -6 & 19 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow[R_2 = -10R_3 + R_2]{R_1 = 6R_3 + R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{l} \to a \\ \to b \\ \to c \end{array} \qquad \therefore a = 1 , \ b = 2 , \ c = -3$$

$$V = a u_1 + b u_2 + c u_3$$

$$\therefore V = u_1 + 2u_2 - 3u_3$$

$\therefore$ v is linear Combination of u_1, u_2, and u_3

Linear Combination

Example 3:

Determine whether the vector $\mathbf{v} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is a linear combination of the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$.

Linear Combination

Example 3:

$$v = a u_1 + b u_2 + c u_3 \Rightarrow \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix} = \begin{bmatrix} a + 2c \\ -2a + 5b \\ 2a + 5b + 8c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right] \xrightarrow[R_3 = -2R_1 + R_3]{R_2 = 2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 5 & 4 & 1 \\ 0 & 5 & 4 & 3 \end{array}\right] \xrightarrow{R_2 = \frac{1}{5} R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & \boxed{1} & \frac{4}{5} & \frac{1}{5} \\ 0 & 5 & 4 & 3 \end{array}\right] \xrightarrow{R_3 = -5R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & \frac{4}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$0 = 2$

$\therefore$ No solution

$\therefore$ v is NOT a linear combination of u_1, u_2, and u_3.

Linear Combination

Example 4:

Find the value of m such that the vector $(m, 7, -4)$ is linear combination of vectors $(-2, 2, 1)$ and $(2, 1, -2)$.

Linear Combination

Example 4:

$$\begin{bmatrix} m \\ 7 \\ -4 \end{bmatrix} = a\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} m \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} -2a + 2b \\ 2a + b \\ a - 2b \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 2 & m \\ 2 & 1 & 7 \\ 1 & -2 & -4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 2 & 1 & 7 \\ -2 & 2 & m \end{array}\right] \xrightarrow[R_3 = 2R_1 + R_3]{R_2 = -2R_1 + R_2}$$

$$\left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 5 & 15 \\ 0 & -2 & m-8 \end{array}\right] \xrightarrow{R_2 = \frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 1 & 3 \\ 0 & -2 & m-8 \end{array}\right] \xrightarrow[R_3 = 2R_2 + R_3]{R_1 = 2R_2 + R_1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & m-2 \end{array}\right]$$

$\therefore m - 2 = 0$

$\boxed{m = 2}$

Vector Spaces

Definition of a Vector Space

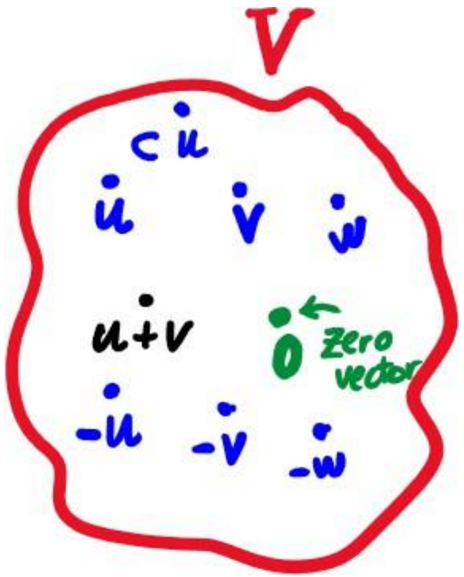
Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then V is a vector space.

Addition:

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. V has a **zero vector** $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

Scalar Multiplication:

6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1(\mathbf{u}) = \mathbf{u}$



Vector Spaces

The set $V = R^2$ with the Standard Operations of Addition and Scalar Multiplication is a Vector Space.

$$u = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + v = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

$$w = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \qquad 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad c \in R$$

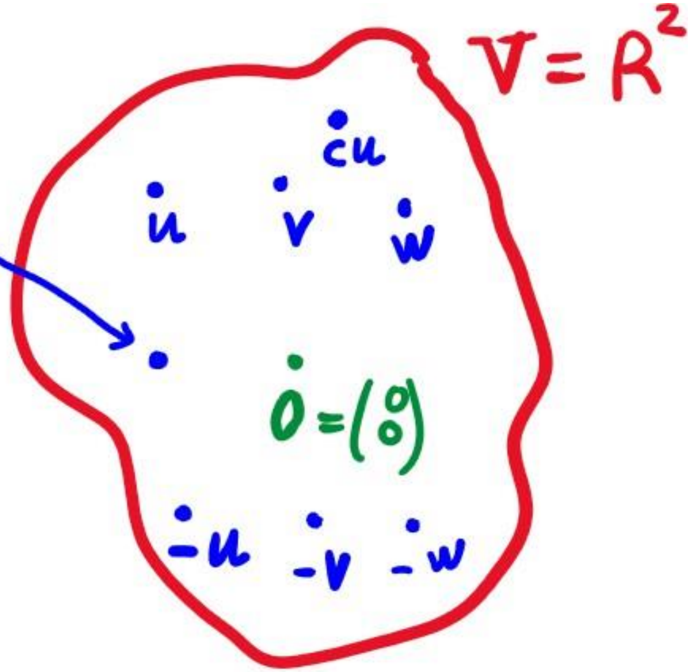
$$cu + cv = c(u+v)$$

$$u + 0 = u$$

$$-u = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

$$u + (-u) = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_1 \\ x_2 - x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$



Vector Spaces

The set $V = R^3$ with the Standard Operations of Addition and Scalar Multiplication is a Vector Space.

The set $V = R^n$ with the Standard Operations of Addition and Scalar Multiplication is a Vector Space.

The set $V = M_{m \times n}$ of all $m \times n$ matrices with the matrix addition and scalar multiplication is a Vector Space.

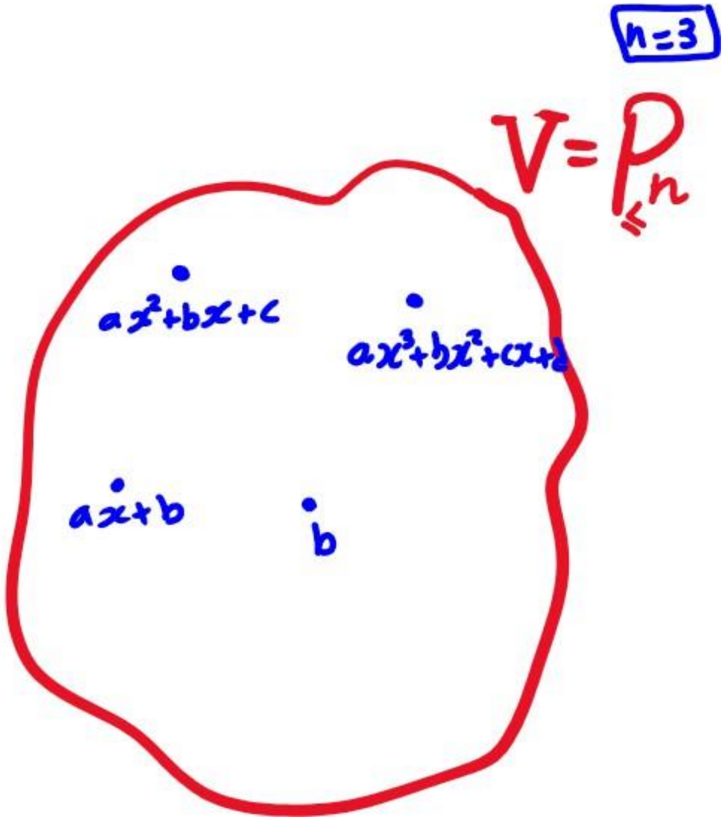
Vector Spaces

The set $V = P_n$ of all polynomials of degree at most n together with the polynomial addition and scalar multiplication is a Vector Space.

$u = x^2 + 3x$

$v = 3x^2 + 4x$

$u + v = 7x$

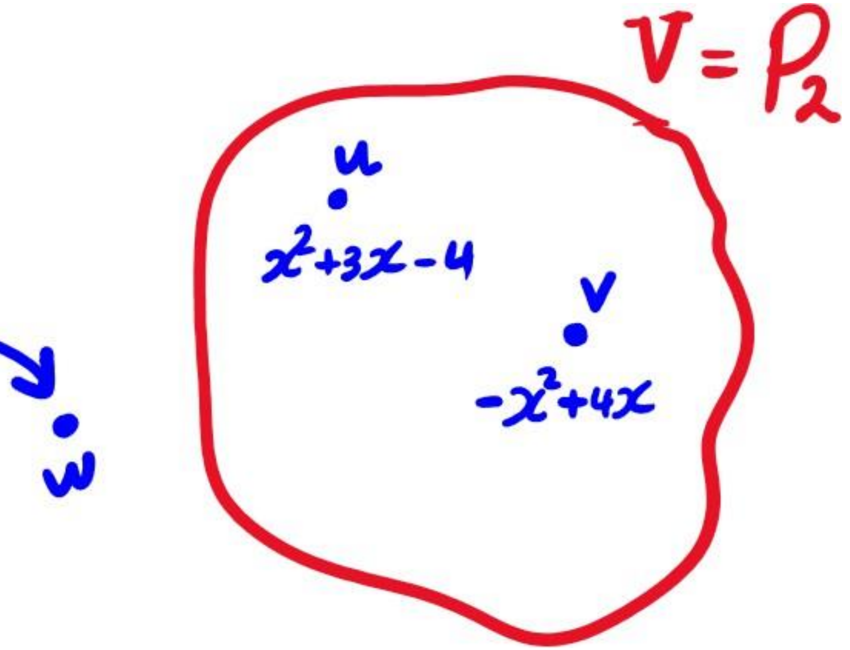


Vector Spaces

The set $V = P_2$ of Second-Degree Polynomials is Not a Vector Space.

$$u + v = \underline{\cancel{x^2} + 3x - 4} + (\cancel{-x^2} + 4x)$$

$$= \underline{\underline{7x - 4}}$$



Vector Spaces

Example 1:

Show that the set $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x > y \right\}$, together with the standard operations of vector addition and scalar multiplication is **not** a vector space.

Vector Spaces

Example 1:

Show that the set $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x > y \right\}$, together with the standard operations of vector addition and scalar multiplication is **not** a vector space.

$\boxed{1}$ $\vec{0} \notin V$ $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $0 \not> 0$

$\boxed{2}$ $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \in V \rightsquigarrow -1(u) = -1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \notin V$ $-2 < -1$

Example 2:

Determine whether $V = \{(x, x): x \text{ is a real number}\}$ together with the standard operations, is a vector space. If it is not, identify at least one of the ten vector space axioms that fails.

Vector Spaces

Example 2:

$\forall \vec{u}, \vec{v}, \vec{w} \in V \quad , \forall c, d \in \mathbb{R}$

$\vec{u} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix}$

$\vec{w} = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix}$

$\boxed{1}\ \vec{u} + \vec{v} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ u_1+v_1 \end{pmatrix} \in V$

$\boxed{2}\ \vec{u} + \vec{v} = \vec{v} + \vec{u}$

$\begin{pmatrix} v_1 \\ v_1 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} v_1+u_1 \\ v_1+u_1 \end{pmatrix}$

$\boxed{3}\ \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \longrightarrow \begin{pmatrix} u_1+v_1+w_1 \\ u_1+v_1+w_1 \end{pmatrix} \in V$

$\boxed{4}\ \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in V \quad \vec{0} + \vec{u} = \vec{u} = \vec{u} + \vec{0} \rightsquigarrow \vec{0}$ Additive Identity

$\boxed{5}\ \vec{u} + (-\vec{u}) = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} + \begin{pmatrix} -u_1 \\ -u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0} \rightsquigarrow -\vec{u}$ Additive Inverse of $\vec{u}$

$\boxed{6}\ c\vec{u} = c\begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} cu_1 \\ cu_1 \end{pmatrix} \in V$

$\boxed{7}\ c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

$c\begin{pmatrix} u_1+v_1 \\ u_1+v_1 \end{pmatrix} \quad c\begin{pmatrix} u_1 \\ u_1 \end{pmatrix} + c\begin{pmatrix} v_1 \\ v_1 \end{pmatrix}$

$\begin{pmatrix} c(u_1+v_1) \\ c(u_1+v_1) \end{pmatrix} \quad \begin{pmatrix} cu_1 \\ cu_1 \end{pmatrix} + \begin{pmatrix} cv_1 \\ cv_1 \end{pmatrix} = \begin{pmatrix} cu_1+cv_1 \\ cu_1+cv_1 \end{pmatrix}$

$\boxed{8} \quad \boxed{9}$

$\boxed{10}\ 1\vec{u} = 1\begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_1 \end{pmatrix} = \vec{u}$

Vector Spaces

Example 3:

Determine whether $V = R^3$ together with the two operations defined as:

$\forall\, x, y, z \in V,$ and $\forall\, r \in R$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \oplus \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x + y' \\ y + z' \\ z + x' \end{bmatrix} \text{ and } r \otimes \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$$ is a vector space.

$\oplus$ vector addition	$\otimes$ or $\odot$ scalar multiplication

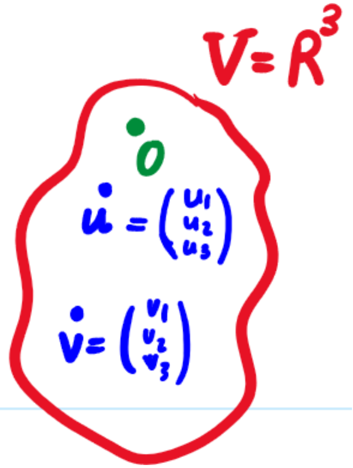
Is $(V, \oplus, \otimes)$ a vector space?

Vector Spaces

Example 3:

$\vec{0} \in V \qquad \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\vec{u} + \vec{0} = \vec{u}$ ✓ $\qquad \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \vec{u}$



$\vec{u} + (-\vec{u}) = \vec{0}$ X

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} -u_1 \\ -u_2 \\ -u_3 \end{pmatrix} = \begin{pmatrix} u_1 - u_2 \\ u_2 - u_3 \\ u_3 - u_1 \end{pmatrix} \neq \vec{0}$$

$\therefore$ V is Not a vector space.

$\vec{u} + \vec{v} \neq \vec{v} + \vec{u}$

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_2 \\ u_2 + v_3 \\ u_3 + v_1 \end{pmatrix}$$

$\neq$

$$\vec{v} + \vec{u} = \begin{pmatrix} v_1 + u_2 \\ v_2 + u_3 \\ v_3 + u_1 \end{pmatrix}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -4 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{u} + \vec{v} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \neq \vec{v} + \vec{u} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

Vector Spaces

Example 4:

Consider the set of numbers $V = \{x \colon x \in R, \ x > 0\}$ together with the two operations defined as: $\forall\, x, y \in V$, and $\forall\, r \in R$: $x \oplus y = x + y$ and $r \otimes x = rx$

Is $(V, \oplus, \otimes)$ a vector space?

Vector Spaces

Example 4:

Consider the set of numbers $V = \{x\!: x \in R,\ x > 0\}$ together with the two operations defined as: $\forall\, x, y \in V$, and $\forall\, r \in R$: $x \oplus y = x + y$ and $r \otimes x = rx$

Is $(V, \oplus, \otimes)$ a vector space?

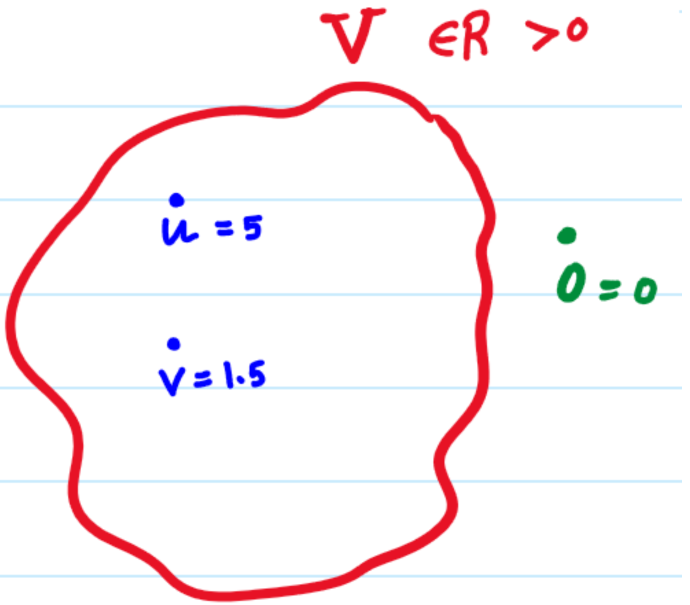
$\because$ $x = 0 \notin V$

$u \in V$ $\quad u + 0 = u$, but $0 \notin V$

let $r = -2$ $\quad ru = -2 \otimes u = -10 \notin V$

$u = 5$

$\therefore$ V is NOT a vector space.



Example 5:

Show that the set $V = R^3$, together with the operations

$r \otimes \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ 0 \end{bmatrix}$ and the standard addition operation on R^3 is **not** a vector space.

Vector Spaces

Example 5:

Show that the set $V = R^3$, together with the operations

$r \otimes \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} rx \\ ry \\ 0 \end{bmatrix}$ and the standard addition operation on R^3 is **not** a vector space.

Let $\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V$, let $r = 1$ (scalar)

$1 \otimes \vec{u} = 1 \otimes \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \neq \vec{u}$

$\therefore$ V is NOT a vector space.

Example 6:

Let V be the set of all positive real numbers.

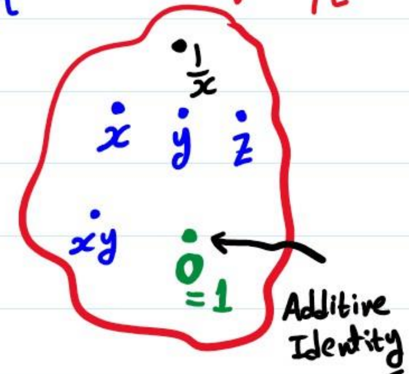
Determine whether V is a vector space with the operations shown below.

$\forall\ x, y \in V$, and $\ \forall\ c \in R :$ $\quad x \oplus y = xy$ and $c \otimes x = x^c$

Example 6: (1/2)

Let $x, y, z \in V \rightsquigarrow \mathbb{R}^{>0}$, Let c, d (scalars) $\in \mathbb{R}$

$V = \mathbb{R}^{>0}$



$\boxed{1}$ $x + y = xy \in V$ ✓

$\boxed{2}$ $x + y = y + x$

$xy = yx$ ✓

$\boxed{3}$ $x + (y + z) = (x + y) + z$

$x + (yz) = (xy) + z$

$xyz = xyz$ ✓

$\boxed{4}$ $x + 0 = x$

$x0 = x$

$0 = 1$ ✓

$x + \underset{=}{1} = 1(x) = x^1 = x$

Vector Spaces

Example 6: (2/2)

$\boxed{5}$ $x + (-x) = 0$

$(x)(-x) = 1 \rightsquigarrow (-x) = \frac{1}{x}$

$x + \frac{1}{x} = (x)\left(\frac{1}{x}\right) = 1$

Additive inverse of x

$\boxed{6}$ $cx = x^c \in V$

$c \in \mathbb{R}$: 1, 0, -3, ...

$\boxed{7}$ $c(x+y) = cx + cy$ ✓

$c(xy) = x^c + y^c$

$(xy)^c = (x^c)(y^c)$

$= (xy)^c$

$\boxed{8}$ $(c+d)x = cx + dx$

$x^{(c+d)} = x^c + x^d$ ✓

$= (x^c)(x^d)$

$= x^{(c+d)}$

$\boxed{9}$ $c(dx) = (cd)x$

$c(x^d) = x^{cd}$ ✓

$(x^d)^c$

x^{cd}

$\boxed{10}$ $1(x) = x^1 = x$ ✓

$\therefore$ V is a vector space.

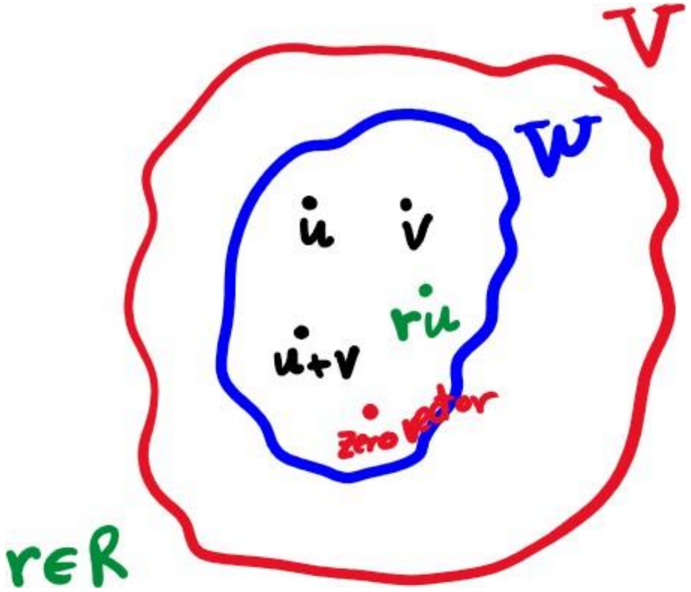
Subspaces

Definition of a Subspace of a Vector Space

A nonempty subset W of a vector space V is a subspace of V when W is a vector space under the operations of addition and scalar multiplication defined in V.

Test for a Subspace

1. If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
2. If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.



Subspaces

Example 1:

Show that the set $W = \left\{ \begin{bmatrix} x \\ x \end{bmatrix} : x \in R \right\}$ is a subapace of the vector space R^2 with the standard operations of vector addition and scalar multiplication.

Subspaces

Example 1:

$\boxed{1}$ let $x=0 \Rightarrow \begin{bmatrix}0\\0\end{bmatrix} \in W$, W is nonempty

$\forall$

$\boxed{2}$ for every $\begin{bmatrix}a\\a\end{bmatrix}, \begin{bmatrix}a'\\a'\end{bmatrix} \in W$

$$\begin{bmatrix}a\\a\end{bmatrix} + \begin{bmatrix}a'\\a'\end{bmatrix} = \begin{bmatrix}a+a'\\a+a'\end{bmatrix} \in W$$

scalar ثابت

$\boxed{3}$ for every $\begin{bmatrix}a\\a\end{bmatrix} \in W$ and $r \in R$

$$r\begin{bmatrix}a\\a\end{bmatrix} = \begin{bmatrix}ra\\ra\end{bmatrix} \in W$$

$\therefore$ W is a subspace of R^2

Subspaces

Example 2:

Show that the set $W = \{(x_1, 0, x_3): x_1 \text{and } x_3 \in R\}$ is a subapace of the vector space R^3 with the standard operations.

Example 2:

$\boxed{1}$ let $x_1 = x_3 = 0$, $\therefore (0,0,0) \in W$, W is nonempty

$\boxed{2}$ for every $(a,0,b)$, $(\grave{a},0,\grave{b}) \in W$

$(a,0,b) + (\grave{a},0,\grave{b}) = (a+\grave{a}, 0, b+\grave{b}) \in W$

$\boxed{3}$ for every $(a,0,b) \in W$, $r \in R$

$r(a,0,b) = (ra, 0, rb) \in W$

$\therefore$ W is a subspace of a vector space R^3

Subspaces

Example 3:

Is the set $W = \{(x_1, 1, x_3): x_1 \text{and } x_3 \in R\}$ a subapace of R^3 with the standard operations ?

Example 3:

Is the set $W = \{(x_1, 1, x_3) : x_1 \text{and } x_3 \in R\}$ a subapace of R^3 with the standard operations ?

$\boxed{1}$ let $r = 0$ $\rightsquigarrow$ $r(x_1, 1, x_3) = (rx_1, r1, rx_3)$

$= (0, 0, 0) \notin W$

or

$\boxed{2}$ for every $(a, 1, b), (\grave{a}, 1, \grave{b}) \in W$

$(a, 1, b) + (\grave{a}, 1, \grave{b}) = (a + \grave{a}, 2, b + \grave{b}) \notin W$

$\therefore$ W is **not** a subspace of R^3

Subspaces

Example 4:

Is the set $W = \{(x_1, x_2): x_1 \geq 0 \text{ and } x_3 \geq 0\}$ a subapace of R^2 with the standard operations ?

Subspaces

Example 4:

Is the set $W = \{(x_1, x_2): x_1 \geq 0 \text{ and } x_3 \geq 0\}$ a subapace of R^2 with the standard operations ?

For every $(a,b) \in W$, $r \in R$

$r(a,b) = (ra, rb)$ if $r = -1$, $a > 0$, $b > 0$

$= (-a, -b) \notin W$

$\therefore W$ is not a subspace of R^2

Subspaces

Example 5:

Show that $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : c = a + b \right\}$ is a subapace of the vector space $\mathbf{R}^3$.

Subspaces

Example 5:

$$W = \left\{ \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}, a,b \in R \right\}$$

$\boxed{1}$ let $a,b=0 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, W is nonempty

$\boxed{2}$ for every $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}, \begin{bmatrix} a' \\ b' \\ a'+b' \end{bmatrix} \in W$

$$\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} + \begin{bmatrix} a' \\ b' \\ a'+b' \end{bmatrix} = \begin{bmatrix} a+a' \\ b+b' \\ a+a'+b+b' \end{bmatrix} \in W$$

$\boxed{3}$ for every $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix} \in W$, $r \in R$

$$r \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} = \begin{bmatrix} ra \\ rb \\ r(a+b) \end{bmatrix} = \begin{bmatrix} ra \\ rb \\ ra+rb \end{bmatrix} \in W$$

$\therefore$ W is a subspace of R^3

Subspaces

Example 6:

Show that $W = \left\{ \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} : a, b \in \mathbf{R} \right\}$ is not a subapace of the vector space $\mathbf{R}^3$

Subspaces

Example 6:

Show that $W = \left\{ \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} a, b \in \mathbf{R} \right\}$ is not a subapace of the vector space $\mathbf{R}^3$

$\boxed{1}$ let $a = b = 0 \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in W$ but $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$

or

$\boxed{2}$ for every $\begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix}, \begin{bmatrix} a' \\ b' \\ a'+b'+1 \end{bmatrix} \in W$

$$\begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} + \begin{bmatrix} a' \\ b' \\ a'+b'+1 \end{bmatrix} = \begin{bmatrix} a+a' \\ b+b' \\ a+a'+b+b'+2 \end{bmatrix} \notin W$$

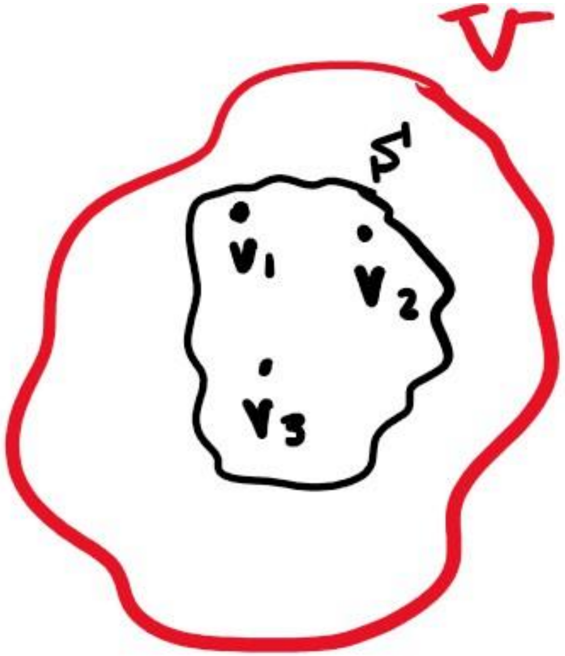
$\therefore$ W is not a subspace of R^3

Spanning Sets

Definition of a Spanning Set of a Vector Space

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of a vector space V. The set S is a **spanning set** of V when every vector in V can be written as a linear combination of vectors in S. In such cases it is said that S **spans** V.

i.e., $\text{Span}(S) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} \ : \ c_1, c_2, \cdots, c_k \in R\}$



Note that:

Span(S) or Span$\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is a subspace of V.

$W = \text{Span}(S)$

Spanning Sets

Example 1:

Show that $W = \left\{ \begin{bmatrix} a+b \\ a-2b \\ 2a+3b \end{bmatrix} + a, b \in R \right\}$ is a subapace of the vector space R^3.

Example 1:

Show that $W = \left\{ \begin{bmatrix} a+b \\ a-2b \\ 2a+3b \end{bmatrix} \colon a, b \in R \right\}$ is a subapace of the vector space R^3.

$$\begin{bmatrix} a+b \\ a-2b \\ 2a+3b \end{bmatrix} = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix} + \begin{bmatrix} b \\ -2b \\ 3b \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$W = \left\{ a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} : a, b \in R \right\}$$

$$W = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\} \quad \& \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \in R^3$$

$\therefore$ W is a subspace of R^3

Spanning Sets

Example 2:

Let $W = \left\{ \begin{bmatrix} x-y \\ 2x \\ 3x+4y \end{bmatrix} \mid x, y \in R \right\}$ be a subset of the vector space R^3.

a) Show that W is a subspace of R^3.

b) Find a subset S of R^3 that spans W.

c) Is $\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \in W$? Explain.

Spanning Sets

Example 2:

$$\begin{bmatrix} x-y \\ 2x \\ 3x+4y \end{bmatrix} = \begin{bmatrix} x \\ 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ 0 \\ 4y \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

$$W = \left\{ x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} : x, y \in R \right\}$$

$\in R^3$

$$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \right\}$$

$\therefore$ W is asubspace of R^3 $\Longrightarrow$ (1)

$$S' = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \right\}, \quad W = \text{span}(S')$$

$\Rightarrow$ (2)

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 0 & 2 \\ 3 & 4 & -5 \end{array}\right] \xrightarrow[R_3 = -3R_1 + R_3]{R_2 = -2R_1 + R_2} \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 2 & -4 \\ 0 & 7 & -14 \end{array}\right] \xrightarrow[R_3 = \frac{1}{7}R_3]{R_2 = \frac{1}{2}R_2}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{array}\right] \xrightarrow[R_3 = -R_2 + R_3]{R_1 = R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore x = 1, \ y = -2$

$$\therefore \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \in W \Rightarrow (3)$$

Spanning Sets

Example 3:

Let $W = \left\{ \begin{bmatrix} a+b \\ b \\ a-3b \end{bmatrix} \mid a, b \in R \right\}$ be a subset of the vector space R^3.

a) Show that W is a subspace of R^3.

b) Find a set S such that $W = Span(S)$.

c) Is the vector $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 9 \end{bmatrix}$ in W? Explain.

Spanning Sets

Example 4:

Is the polynomial $x^3 + x$ belongs to span$\{x^3 - x, 3x^3 + 2x, x^2\}$.

Spanning Sets

Example 4:

Is the polynomial $x^3 + x$ belongs to span$\{x^3 - x, 3x^3 + 2x, x^2\}$.

Is $x^3+x = a(x^3-x) + b(3x^3+2x) + c(x^2)$

$$\begin{array}{cc} & \begin{array}{cccc} a & b & c & \end{array} \\ \begin{array}{c} x^3 \\ x^2 \\ x \\ \text{const.} \end{array} & \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \xrightarrow{R_3 = R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 = \frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = -3R_2 + R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{-1}{5} \\ 0 & 1 & 0 & \frac{2}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \therefore\ a = \frac{-1}{5},\ b = \frac{2}{5},\ c = 0$$

$\therefore\ x^3 + x \in \text{span}\{x^3 - x, 3x^3 + 2x, x^2\}$

Spanning Sets

Example 5:

Find a spanning set for the subspace $W = \left\{ \begin{bmatrix} a \\ 2a \end{bmatrix} : a \in R \right\}$.

Spanning Sets

Example 5:

Find a spanning set for the subspace $W = \left\{ \begin{bmatrix} a \\ 2a \end{bmatrix} : a \in R \right\}$.

$$\begin{bmatrix} a \\ 2a \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$W = \left\{ a \begin{bmatrix} 1 \\ 2 \end{bmatrix} : a \in R \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

$$S' = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Spanning Sets

Example 6:

Find a spanning set for the subspace $W = \left\{ \begin{bmatrix} 2a \\ 4a \end{bmatrix} : a \in R \right\}$.

Spanning Sets

Example 6:

Find a spanning set for the subspace $W = \left\{ \begin{bmatrix} 2a \\ 4a \end{bmatrix} : a \in R \right\}$.

$$\begin{bmatrix} 2a \\ 4a \end{bmatrix} = \underbrace{2a}_{b} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$W = \left\{ b \begin{bmatrix} 1 \\ 2 \end{bmatrix} : b \in R \right\} = \mathrm{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

$$= a \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \mathrm{span} \left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$$

is also True

Spanning Sets

Example 7:

Find a spanning set for the subspace $W = \left\{ \begin{bmatrix} a \\ 2b \end{bmatrix} : a, b \in R \right\}$.

Spanning Sets

Example 7:

Find a spanning set for the subspace $W = \left\{ \begin{bmatrix} a \\ 2b \end{bmatrix} : a, b \in R \right\}$.

$$\begin{bmatrix} a \\ 2b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$W = \left\{ a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \end{bmatrix} : a, b \in R \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

$$S' = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$$

$$a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

is also True

Spanning Sets

Example 8:

Find a spanning set for the subspace $W = \left\{ \begin{bmatrix} a - b \\ b - 2a \end{bmatrix} : a, b \in R \right\}$.

Spanning Sets

Example 8:

Find a spanning set for the subspace $W = \left\{\begin{bmatrix} a-b \\ b-2a \end{bmatrix} : a, b \in R\right\}$.

$$\begin{bmatrix} a-b \\ b-2a \end{bmatrix} = a\begin{bmatrix} 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$W = \left\{a\begin{bmatrix} 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix} : a, b \in R\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

$$S = \left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

Spanning Sets

Example 9:

$$W = \left\{ \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} : a, b \in R \right\} \text{ is subspace of } R^3.$$ **(True/False)**

Spanning Sets

Example 9:

$$W = \left\{ \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} : a, b \in R \right\} \text{ is subspace of } R^3.$$ **(True/False)**

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$$

Spanning Sets

Example 10:

$W = \left\{ \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix} : a \in R \right\}$ is subspace of R^3. **(True/False)**

Spanning Sets

Example 10:

$W = \left\{ \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix} : a \in R \right\}$ is subspace of R^3. **(True/False)**

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin W$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Linear Independence

Definition of Linear Dependence and Linear Independence

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is **linearly independent** when the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

has only the trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

If there are also nontrivial solutions, then S is **linearly dependent.**

Linear Independence

Testing for Linear Independence and Linear Dependence

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a set of vectors in a vector space V. To determine whether S is linearly independent or linearly dependent, use the steps below.

1. From the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$, write a system of linear equations in the variables $c_1, c_2, \ldots,$ and c_k.
2. Determine whether the system has a unique solution.
3. If the system has only the trivial solution, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$, then the set S is linearly independent. If the system also has nontrivial solutions, then S is linearly dependent.

Linear Independence

Example 1:

Determine whether the set of vectors in R^3

is linearly independent or linearly dependent.

$$S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

Linear Independence

Example 1:

$c_1 V_1 + c_2 V_2 + c_3 V_3 = 0$ ← Zero vector

$c_1(1,2,3) + c_2(0,1,2) + c_3(-2,0,1) = (0,0,0)$ ← Zero vector from R^3

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_3 = -3R_1 + R_3]{R_2 = -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 7 & 0 \end{array}\right] \xrightarrow{R_3 = -2R_2 + R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \xrightarrow{R_3 = -R_3} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 = 4R_3 + R_2]{R_1 = 2R_3 + R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\because c_1 = c_2 = c_3 = 0$ (Trivial solution)

$\therefore$ S is linearly Independent

Linear Independence

Example 2:

Determine whether the set of vectors in P^2 is linearly independent or linearly dependent. $S = \{1 + x - 2x^2,\ \ 2 + 5x - x^2,\ \ x + x^2\}$

Linear Independence

Example 2:

Determine whether the set of vectors in P^2 is linearly independent or linearly dependent. $S = \{1 + x - 2x^2,\ 2 + 5x - x^2,\ x + x^2\}$

$$c_1(1+x-2x^2) + c_2(2+5x-x^2) + c(x+x^2) = \overset{0x^2+0x+0}{0}$$

Zero vector in P^2

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 1 & 5 & 1 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] \end{array} \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_3 = 2R_1 + R_3]{R_2 = -R_1 + R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right] \xrightarrow{R_2 = \frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & -3 & 1 & 0 \end{array}\right] \xrightarrow[R_3 = 3R_2 + R_3]{R_1 = -2R_2 + R_1}$$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & \frac{-2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Let $c_3 = t$, $t \in R$

$c_2 + \frac{1}{3}c_3 = 0$

$c_2 + \frac{1}{3}t = 0 \quad \therefore c_2 = -\frac{1}{3}t$

$c_1 - \frac{2}{3}c_3 = 0$

$c_1 - \frac{2}{3}t = 0 \quad \therefore c_1 = \frac{2}{3}t$

$\therefore$ the solution is $\begin{bmatrix} \frac{2}{3}t \\ -\frac{1}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$, $t \in R$ $\rightarrow$ infinitly many solution

$\therefore S$ is linearly dependent

Linear Independence

Example 3:

Let $S = \{X_1, X_2, X_3\}$ be linearly independent set. Show that

$T = \{X_1 + X_2, X_1 - X_3, X_2 + 2X_3\}$ is linearly independent set.

Linear Independence

Example 3:

$$c_1(X_1+X_2) + c_2(X_1-X_3) + c_3(X_2+2X_3) = 0 \rightsquigarrow 0X_1 + 0X_2 + 0X_3$$

Zero vector

$$\begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array}\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \end{array} \xrightarrow{R_2 = -R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 = -R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow[R_3 = R_2 + R_3]{R_1 = -R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_2 = R_3 + R_2]{R_1 = -R_3 + R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \therefore c_1 = c_2 = c_3 = 0 \quad \text{(Trivial solution)}$$

$\therefore$ T is linearly Independent

Linear Independence

Example 4:

Find the conditions that must be achieved by the scalar numbers c, k be the set of vectors $\{(1,2,0), (c, k, 2), (1,0,1)\}$ linearly dependent.

Linear Independence

Example 4:

$$c_1(1,2,0)+c_2(c,k,2)+c_3(1,0,1)=0 \rightarrow R^3$$

$(0,0,0)$ zero vector in R^3

$$\left[\begin{array}{ccc|c} 1 & c & 1 & 0\\ 2 & k & 0 & 0\\ 0 & 2 & 1 & 0\end{array}\right] \xrightarrow{R_2=-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & c & 1 & 0\\ 0 & k-2c & -2 & 0\\ 0 & 2 & 1 & 0\end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|c} 1 & c & 1 & 0\\ 0 & 2 & 1 & 0\\ 0 & k-2c & -2 & 0\end{array}\right] \xrightarrow{R_2=\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & c & 1 & 0\\ 0 & 1 & \frac{1}{2} & 0\\ 0 & k-2c & -2 & 0\end{array}\right] \xrightarrow[R_3=(2c-k)R_2+R_3]{R_1=-cR_2+R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1-\frac{1}{2}c & 0\\ 0 & 1 & \frac{1}{2} & 0\\ 0 & 0 & \boxed{c-\frac{1}{2}k-2} & 0\end{array}\right]$$

$=0$ linearly dependent

$\neq 0$ linearly independent

$$\therefore\ c-\frac{1}{2}k-2=0$$

$$c=\frac{1}{2}k-2$$

Linear Independence

Note:

The set of n vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v}_{\boldsymbol{n}}\}$ in $\mathbf{R}^n$ is linearly dependent

if the determinant $|\mathbf{v_1 v_2} \cdots \mathbf{v_n}| = 0$,

othrerwise S is linearly independent.

Linear Independence

Example 5: (Recall Ex. 1)

Determine whether the set of vectors in R^3

is linearly independent or linearly dependent.

$S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$

Linear Independence

Example 5: (Recall Ex. 1)

Determine whether the set of vectors in R^3

is linearly independent or linearly dependent.

$S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$

$$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = -1 \neq 0$$

$\therefore$ S is linearly Independent

Linear Independence

Example 6:

Determine whether the set of vectors in R^4 is linearly independent or linearly dependent.

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 2 \end{pmatrix} \right\}$$

Linear Independence

Example 7:

The set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set of R^3 . **(True/False)**

Linear Independence

Example 7:

The set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a linearly independent set of R^3 . (True/False)

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Linear Independence

Span R^n :

To span R^n, a set of vectors must be able to generate any vector in R^n through linear combinations. For this to happen, the vectors must be:

- In R^n (i.e., each vector has n components), and Linearly independent. There must be n such independent vectors.

For Example:

- If you have 3 linearly independent vectors, they span R^3.
- If you have fewer than 3, or 3 dependent vectors, they do not span R^3.

Linear Independence

Example 8:

Which of the following does not span R^3?

a) $x = (2,2,2), y = (0,0,3), z = (0,1,1)$

b) $x = (2,-1,3), y = (4,1,2), z = (8,-1,8)$

c) Neither a nor b span R^3.

d) Both a and b span R^3.

Linear Independence

Example 8:

Which of the following does not span R^3?

a) $x = (2,2,2), y = (0,0,3), z = (0,1,1)$

b) $x = (2,-1,3), y = (4,1,2), z = (8,-1,8)$

c) Neither a nor b span R^3.

d) Both a and b span R^3.

$$\begin{vmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 3 & 1 \end{vmatrix} = -6 \neq 0 \qquad \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 0$$

Basis and Dimension

Definition of Basis

A set of vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of a vector space R^n is said to form a basis for a subspace W of R^n if:

1) $W = \text{Span}(S)$.
2) S is linearly independent set.

Note that:

A set of $\boldsymbol{n}$ vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v}_{\boldsymbol{n}}\}$ in a vector space R^n is a **basis** for R^n if S is linearly independent.

Basis and Dimension

Example 1:

Let $W = \left\{ \begin{bmatrix} a - b \\ 2a \\ 3a + 4b \end{bmatrix} : a, b \in R \right\}$ be a subset of the vector space R^3.

Find a basis for W.

Basis and Dimension

Example 1:

$$\begin{bmatrix} a-b \\ 2a \\ 3a+4b \end{bmatrix} = \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} + \begin{bmatrix} -b \\ 0 \\ 4b \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

$$W = \left\{ a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} : a, b \in R \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \right\}$$

W is a subspace of R^3

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 3 & 4 & 0 \end{array}\right] \xrightarrow[R_3 = -3R_1 + R_3]{R_2 = -2R_1 + R_2} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 7 & 0 \end{array}\right] \xrightarrow{R_2 = \frac{1}{2}R_2}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 0 \end{array}\right] \xrightarrow[R_3 = -7R_2 + R_3]{R_1 = R_2 + R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{(Trivial Solution)}$$

$\therefore S$ is linearly Independent

$$\therefore S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \right\} \text{ is a basis for } W.$$

Basis and Dimension

Example 2:

Show that the set $S = \{(1, 1), (1, -1)\}$ is a basis for R^2.

Example 2:

Show that the set $S = \{(1, 1), (1, -1)\}$ is a basis for R^2.

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = (1)(-1) - (1)(1) = -2 \neq 0$$

$\therefore$ S is linearly Independent

$\therefore$ S is a basis for R^2

Basis and Dimension

Example 3:

Show that $X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ are bases for $\mathbf{R}^3$.

Basis and Dimension

Example 3:

Show that $X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ are bases for $\mathbf{R}^3$.

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 6 \neq 0$$

$\therefore$ X_1, X_2, and X_3 are linearly Independent

$\therefore$ $S' = \{X_1, X_2, X_3\}$ is a basis for R^3

Basis and Dimension

The basis

$$S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

is the **standard basis** for R^3. This can be generalized to n-space. That is, the vectors

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, \ldots, 0) \\
\mathbf{e}_2 &= (0, 1, \ldots, 0) \\
&\vdots \\
\mathbf{e}_n &= (0, 0, \ldots, 1)
\end{aligned}$$

form the **standard basis** for R^n.

$R^3 \rightarrow e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$R^2 \rightarrow e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Basis and Dimension

Definition of the Dimension of a Vector Space

If a vector space V has a basis consisting of n vectors, then the number n is the **dimension** of V, denoted by $\dim(V) = n$. When V consists of the zero vector alone, the dimension of V is defined as zero.

Note that:

The dimension of R^n with the standard operations is n.

Basis and Dimension

Example 4:

Find the dimension of the subspace W of R^4 spanned by

$$S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = \{(-1, 2, 5, 0), (3, 0, 1, -2), (-5, 4, 9, 2)\}.$$

Basis and Dimension

Example 4:

① $W = \text{span}\{(-1,2,5,0), (3,0,1,-2), (-5,4,9,2)\}$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$

② $\left[\begin{array}{ccc|c} -1 & 3 & -5 & 0 \\ 2 & 0 & 4 & 0 \\ 5 & 1 & 9 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow{R_1 = -R_1} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 2 & 0 & 4 & 0 \\ 5 & 1 & 9 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow[R_3 = -5R_1 + R_3]{R_2 = -2R_1 + R_2}$

$\left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & 16 & -16 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow[R_3 = \frac{1}{16}R_3,\ R_4 = -\frac{1}{2}R_4]{R_2 = \frac{1}{6}R_2} \left[\begin{array}{ccc|c} 1 & -3 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 = -R_2 + R_3,\ R_4 = -R_2 + R_4]{R_1 = 3R_2 + R_1}$

$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore v_1 = (-1, 2, 5, 0)$ and $v_2 = (3, 0, 1, -2)$ form a basis for W

$\therefore \dim(W) = 2$

Basis and Dimension

Example 5:

Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}$.

Find a subset of S that forms a basis for the subspace $W = \text{span}(S)$.

Basis and Dimension

Example 5:

$$W = \text{span}(S') = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & 0 \\ 2 & 2 & 10 & 6 & 0 \\ 2 & 1 & 7 & 4 & 0 \end{array}\right] \xrightarrow[R_3 = -2R_1 + R_3]{R_2 = -2R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & 0 \\ 0 & -4 & -12 & -8 & 0 \\ 0 & -5 & -15 & -10 & 0 \end{array}\right] \xrightarrow[R_3 = \frac{-1}{5}R_3]{R_2 = \frac{-1}{4}R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 11 & 7 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 1 & 3 & 2 & 0 \end{array}\right] \xrightarrow[R_3 = -R_2 + R_3]{R_1 = -3R_2 + R_1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ are form a basis for W

$$\therefore \dim(W) = 2$$

Basis and Dimension

Example 6:

Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : c = 2a + 3b,\ \ a, b \in R \right\}$ be a subset of R^3.

1) Show that W is a subspace of R^3.
2) Find a set S such that $W = \operatorname{span}(S)$.
3) Find a basis for W.
4) What is $\dim(W)$?

Basis and Dimension

Example 6:

$$\begin{bmatrix} a \\ b \\ 2a+3b \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 2a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 3b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$W = \left\{ a\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} : a, b \in R \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$\therefore$ W is a subspac of R^3 $\longrightarrow$ (1) ✓

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}, \quad W = \text{span}(S) \longrightarrow (2)$$

$$c_1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow{R_3 = -2R_1 + R_3} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{array}\right] \xrightarrow{R_3 = -3R_2 + R_3}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore$ $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ are form a basis for W $\longrightarrow$ (3)

$\therefore \dim(W) = 2 \longrightarrow (4)$

Basis and Dimension

Example 7:

The set $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$ is a basis for R^2. **(True/False)**

Basis and Dimension

Example 7:

The set $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$ is a basis for R^2. **(True/False)**

$$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = (1)(6) - (2)(3) = 6 - 6 = 0$$

Basis and Dimension

Example 8:

Which of the following does not a basis for R^3?

a) $\{(1,1,1), (1,1,0), (1,0,0)\}$

b) $\{(1,0,0), (0,2,0), (0,0,3)\}$

c) $\{(1,1,2), (1,2,1), (0,-1,1)\}$

d) All are bases.

Example 8:

Which of the following does not a basis for R^3?

a) $\{(1,1,1), (1,1,0), (1,0,0)\}$

b) $\{(1,0,0), (0,2,0), (0,0,3)\}$

c) $\{(1,1,2), (1,2,1), (0,-1,1)\}$

d) All are bases.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1 \neq 0$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$$

$$\begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 0 \end{array}$$

$$3-(3)=0$$

Basis and Dimension

Example 9:

The dimension of the subspace $W = \left\{ \begin{bmatrix} a \\ 2a \\ 3b \end{bmatrix} : a, b \ \in R \right\}$ is

a) 0

b) 1

c) 2

d) 3

Basis and Dimension

Example 9:

The dimension of the subspace $W = \left\{ \begin{bmatrix} a \\ 2a \\ 3b \end{bmatrix} : a, b \in R \right\}$ is

a) 0

b) 1

c) 2

d) 3

$$\begin{bmatrix} a \\ 2a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3b \end{bmatrix}$$

$$a\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Basis and Dimension

Example 10:

The dimension of the subspace $W = \left\{ \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} : a \in R \right\}$ is

a) 0

b) 1

c) 2

d) 3

Example 10:

The dimension of the subspace $W = \left\{ \begin{bmatrix} a \\ 2a \\ 3a \end{bmatrix} : a \in R \right\}$ is

a) 0

b) 1

c) 2

d) 3

$$a \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Basis and Dimension

Example 11:

The dimension of the subspace $W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix} \right\}$ is

a) 0

b) 1

c) 2

d) 3

Basis and Dimension

Example 11:

The dimension of the subspace $W = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix} \right\}$ is

a) 0

b) 1

c) 2

d) 3

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Basis and Dimension

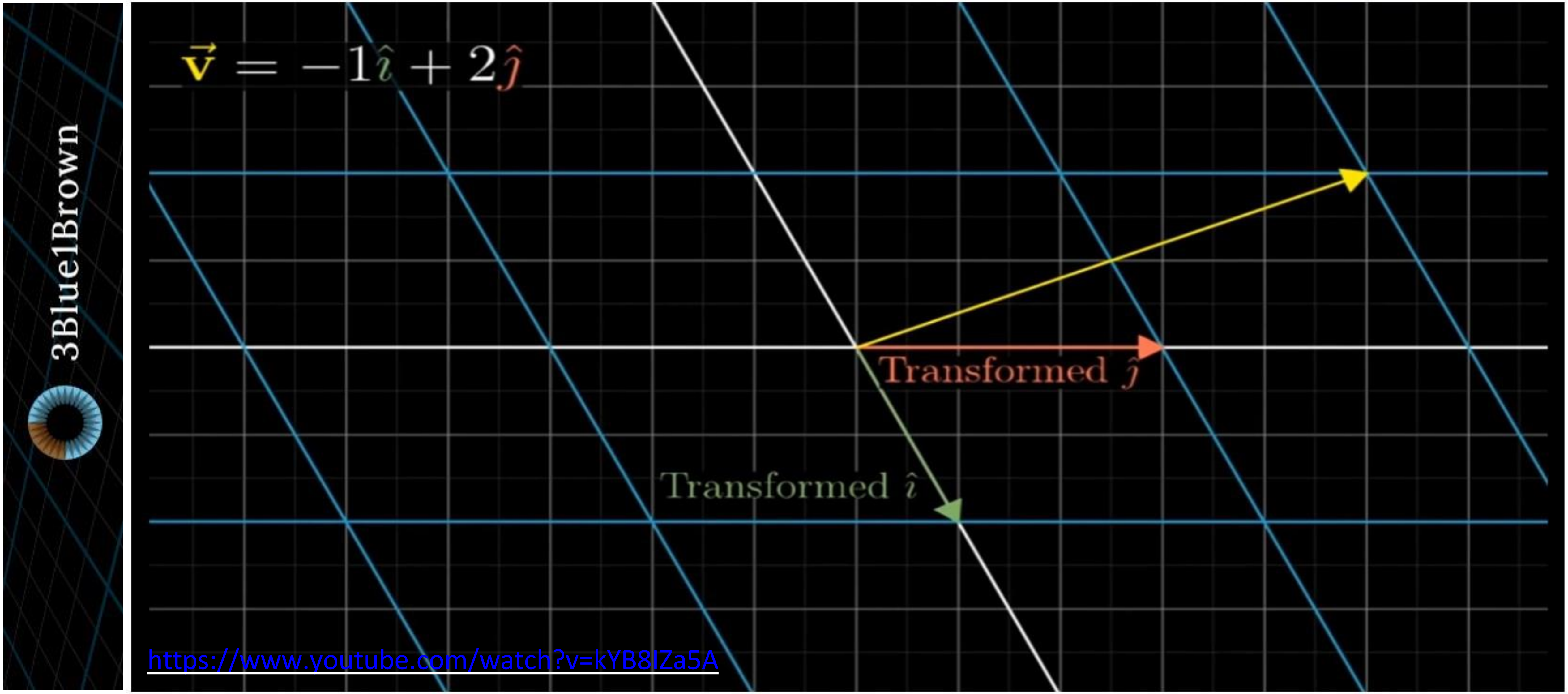
Rank of a matrix

The matrix A has a rank n if the reduced row-echelon form of A has n nonzero rows.

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \approx \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \quad \mathrm{rank}(A) = 3$$

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \approx \approx \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \quad \mathrm{rank}(B) = 1$$

Linear Transformations



https://www.youtube.com/watch?v=kYB8IZa5A

Linear Transformations

Definition of a Linear Transformation

Let V and W be vector spaces. The function $T: V \to W$ is a **linear transformation** of V into W when the two properties below are true for all $\mathbf{u}$ and $\mathbf{v}$ in V and for any scalar c in R.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

2. $T(c\mathbf{u}) = cT(\mathbf{u})$

Linear Transformations

Properties of Linear Transformations

Let T be a **linear transformation** of V into W, where $\mathbf{u}$ and $\mathbf{v}$ in V. Then the properties listed below are true.

1. $T(\mathbf{0}_V) = \mathbf{0}_W$
2. $T(-\mathbf{u}) = -T(\mathbf{u})$
3. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$
4. $T(r\mathbf{u} + s\mathbf{v}) = rT(\mathbf{u}) + sT(\mathbf{v})$, for r and $s \in R$

Linear Transformations

Example 1:

Show that the mapping $T: R^2 \to R^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + 2y \end{bmatrix}$ is a linear transformation.

Linear Transformations

Example 1:

Show that the mapping $T: R^2 \to R^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + 2y \end{bmatrix}$ is a linear transformation.

$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

ثابت
scalar

$\boxed{1}$ $\forall \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in R^2$, $c \in R$

$$T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}\right) = \begin{bmatrix} (x_1 + x_2) - (y_1 + y_2) \\ (x_1 + x_2) + 2(y_1 + y_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - y_1 + x_2 - y_2 \\ x_1 + 2y_1 + x_2 + 2y_2 \end{bmatrix} = \begin{bmatrix} x_1 - y_1 \\ x_1 + 2y_1 \end{bmatrix} + \begin{bmatrix} x_2 - y_2 \\ x_2 + 2y_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) \to (1)$$

$\boxed{2}$ $T\left(c \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)$

$$= T\left(\begin{bmatrix} c x_1 \\ c y_1 \end{bmatrix}\right) = \begin{bmatrix} c x_1 - c y_1 \\ c x_1 + 2c y_1 \end{bmatrix} = \begin{bmatrix} c(x_1 - y_1) \\ c(x_1 + 2y_1) \end{bmatrix}$$

$$= c \begin{bmatrix} x_1 - y_1 \\ x_1 + 2y_1 \end{bmatrix} = c\,T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) \to (2)$$

$\therefore$ T is a linear Transformation from R^2 to R^2

Linear Transformations

Example 2:

Show that the mapping $T: R^3 \to R^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y + z \end{bmatrix}$

is a linear transformation.

Example 2:

Show that the mapping $T: R^3 \to R^2$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y + z \end{bmatrix}$ is a linear transformation.

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \checkmark$$

$$\forall \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in R^3 \text{ , } c \in R$$

$$\boxed{1}\; T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2 \\ (y_1+y_2)+(z_1+z_2) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_2 \\ y_1+z_1 + y_2+z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1+z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2+z_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \longrightarrow (1)$$

$$\boxed{2}\; T\left(c\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cy_1 \\ cz_1 \end{bmatrix}\right) = \begin{bmatrix} cx_1 \\ cy_1+cz_1 \end{bmatrix}$$

$$= \begin{bmatrix} c\,x_1 \\ c\,(y_1+z_1) \end{bmatrix} = c\begin{bmatrix} x_1 \\ y_1+z_1 \end{bmatrix}$$

$$= c\,T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) \longrightarrow (2)$$

$\therefore$ T is a linear Transformation from R^3 to R^2

Linear Transformations

Example 3:

Show that the mapping $T: R^3 \to R^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y+z \\ z+1 \end{bmatrix}$

is not a linear transformation.

Linear Transformations

Example 3:

Show that the mapping $T: R^3 \rightarrow R^3$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y+z \\ z+1 \end{bmatrix}$ is not a linear transformation.

$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq$ zero vector in R^3 $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ T is NOT a linear Transformation.

Linear Transformations

Example 4:

Let $T: R^3 \rightarrow R^3$ defined by

$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$

Show that T is a linear transformation.

Linear Transformations

Example 4: Let $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$

Show that T is a linear transformation.

$T(0,0,0) = (0,0,0)$ ✓

$\forall (x_1, x_2, x_3), (y_1, y_2, y_3) \in R^3$, $c \in R$

$\boxed{1}$ $T((x_1,x_2,x_3)+(y_1,y_2,y_3)) = T((x_1+y_1, x_2+y_2, x_3+y_3))$

$= (x_1+y_1 - x_2 - y_2 + x_3 + y_3, 2x_1 + 2y_1 + x_2 + y_2 - x_3 - y_3, -x_1 - y_1 - 2x_2 - 2y_2 + 2x_3 + 2y_3)$

$= (x_1 - x_2 + x_3 + y_1 - y_2 + y_3, 2x_1 + x_2 - x_3 + 2y_1 + y_2 - y_3, -x_1 - 2x_2 + 2x_3 - y_1 - 2y_2 + 2y_3)$

$= (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3) +$

$(y_1 - y_2 + y_3, 2y_1 + y_2 - y_3, -y_1 - 2y_2 + 2y_3)$

$= T(x_1,x_2,x_3) + T(y_1,y_2,y_3) \longrightarrow (1)$

$\boxed{2}$ $T(c(x_1,x_2,x_3)) = T(cx_1, cx_2, cx_3)$

$= (cx_1 - cx_2 + cx_3, 2cx_1 + cx_2 - cx_3, -cx_1 - 2cx_2 + 2cx_3)$

$= (c(x_1 - x_2 + x_3), c(2x_1 + x_2 - x_3), c(-x_1 - 2x_2 + 2x_3))$

$= c(x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$

$= c\,T(x_1,x_2,x_3) \longrightarrow (2)$

$\therefore T$ is a linear Transformation from R^3 to R^3

Linear Transformations

Example 5:

Determine whether the mapping $T: R^2 \to R$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + y^2$ is a linear transformation.

Linear Transformations

Example 5:

Determine whether the mapping $T: R^2 \to R$ by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + y^2$ is a linear transformation.

$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 0$ ✓

$\forall \begin{bmatrix} x \\ y \end{bmatrix} \in R^2$, $c \in R$

$\boxed{1}$ $T\left(c\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} cx \\ cy \end{bmatrix}\right) = (cx)^2 + (cy)^2$

$= c^2x^2 + c^2y^2$

$= c^2(x^2+y^2)$

$= c^2\, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

$\neq c\, T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

$\therefore$ T is NOT linear Transformation

Linear Transformations

Example 6:

Let $T: R^2 \rightarrow R^2$ be a linear transformation such that

$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$. Find $T\left(\begin{bmatrix}3\\-2\end{bmatrix}\right)$.

Linear Transformations

Example 6:

Let $T: R^2 \rightarrow R^2$ be a linear transformation such that $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\2\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$.

Find $T\left(\begin{bmatrix}3\\-2\end{bmatrix}\right)$.

$$T\left(\begin{bmatrix}3\\-2\end{bmatrix}\right) = T\left(3\begin{bmatrix}1\\0\end{bmatrix} - 2\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 3T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) - 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= 3\begin{bmatrix}1\\2\end{bmatrix} - 2\begin{bmatrix}2\\3\end{bmatrix}$$

$$= \begin{bmatrix}3\\6\end{bmatrix} - \begin{bmatrix}4\\6\end{bmatrix} = \boxed{\begin{bmatrix}-1\\0\end{bmatrix}}$$

Linear Transformations

Example 7:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and define the mapping $T: R^2 \to R^2$ by $T(\mathbf{u}) = A\mathbf{u}$, for every $\mathbf{u} \in R^2$.

1. Show that T is a linear transformation.

2. Find $T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$.

Example 7:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and define the mapping $T: R^2 \to R^2$ by $T(\mathbf{u}) = A\mathbf{u}$, for every $\mathbf{u} \in R^2$.

1. Show that T is a linear transformation.

2. Find $T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$.

$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

$\forall u, v \in R^2$, $c \in R$

$\boxed{1}$ $T(u+v) = A(u+v) = Au + Av$

$= T(u) + T(v) \longrightarrow (1)$

$\boxed{2}$ $T(cu) = A(cu) = c(Au)$

$= c\,T(u) \longrightarrow (2)$

$\therefore$ T is a linear Transformation

Linear Transformations

Example 7:

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and define the mapping $T: R^2 \to R^2$ by $T(\mathbf{u}) = A\mathbf{u}$, for every $\mathbf{u} \in R^2$.

1. Show that T is a linear transformation.

2. Find $T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$.

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = A\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \boxed{\begin{pmatrix} 9 \\ 3 \end{pmatrix}}$$

Linear Transformations

Example 8:

Let $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix}$ and define the mapping $T: R^2 \rightarrow R^3$ by $T(\mathbf{u}) = A\mathbf{u}$, for every $\mathbf{u} \in R^2$.

1. Show that T is a linear transformation.

2. Find $T(\mathbf{u})$ when $\mathbf{u} = (2, -1)$.

Linear Transformations

Linear Transformation Given by a Matrix

Let A be an $m \times n$ matrix. The function T defined by

$$T(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation from R^n into R^m.

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 1 & 0 & 0 \end{bmatrix} \quad \underline{2} \times \underline{4}$$

$T(v) = Av$

$\therefore T$ is a linear Transformation $R^4 \longrightarrow R^2$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} \quad 3 \times 3$$

$T(u) = Au$

$\therefore T$ is a linear Transformation $R^3 \longrightarrow R^3$

Linear Transformations

Example 9: True / False

The transformation $T: R^2 \to R^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y - 1 \end{bmatrix}$ is a linear transformation.

Linear Transformations

Example 9: True / False

The transformation $T: R^2 \to R^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y-1 \end{bmatrix}$ is a linear transformation.

$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \neq$ zero vector

Linear Transformations

Example 10: True / False

If $S, T: R^2 \to R^2$ are linear transformation defined by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ and $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix}$, then $(S+T)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Linear Transformations

Example 10: True / False

If $S, T: R^2 \to R^2$ are linear transformation defined by $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$ and $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x \end{bmatrix}$, then $(S+T)\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

$$S\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Linear Transformations

Example 11: Choose The Correct Answer

If $T: R^2 \to R^2$ is defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a - b \\ a + 3b \end{bmatrix}$, then $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) =$

$a)\ \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$b)\ \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$c)\ \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$d)\ \begin{bmatrix} 2 \\ 8 \end{bmatrix}$

Linear Transformations

Example 11: Choose The Correct Answer

If $T: R^2 \to R^2$ is defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a - b \\ a + 3b \end{bmatrix}$, then $T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) =$

$a)\ \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$b)\ \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$c)\ \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

$\boxed{d)\ \begin{bmatrix} 2 \\ 8 \end{bmatrix}}$

$$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2(2) - 2 \\ 2 + 3(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

Linear Transformations

Example 12: Choose The Correct Answer

If $T: R^3 \to R^2$ is defined by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a - 3c \\ 3b + 2c \end{bmatrix}$, then $T\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right) =$

$a)\ \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

$b)\ \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$c)\ \begin{bmatrix} 8 \\ 3 \end{bmatrix}$

$d)\ \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Linear Transformations

Example 12: Choose The Correct Answer

If $T: R^3 \to R^2$ is defined by $T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a - 3c \\ 3b + 2c \end{bmatrix}$, then $T\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right) =$

$a)\ \begin{bmatrix} 3 \\ 8 \end{bmatrix}$

$b)\ \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$c)\ \begin{bmatrix} 8 \\ 3 \end{bmatrix}$

$d)\ \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$a=3$
$b=2$
$c=1$

$$T\left(\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2(3) - 3(1) \\ 3(2) + 2(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Linear Transformations

Example 13: Choose The Correct Answer

If $T: R^2 \to R^2$ is defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$ is a linear transformation. The matrix A such that $T\,(X) = AX, X \in R^2$ is

$a)\ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$b)\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$c)\ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

$d)\ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

Linear Transformations

Example 13: Choose The Correct Answer

If $T: R^2 \to R^2$ is defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ x+y \end{bmatrix}$ is a linear transformation. The matrix A such that $T\,(X) = AX, X \in R^2$ is

$a)\ \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$b)\ \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$c)\ \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

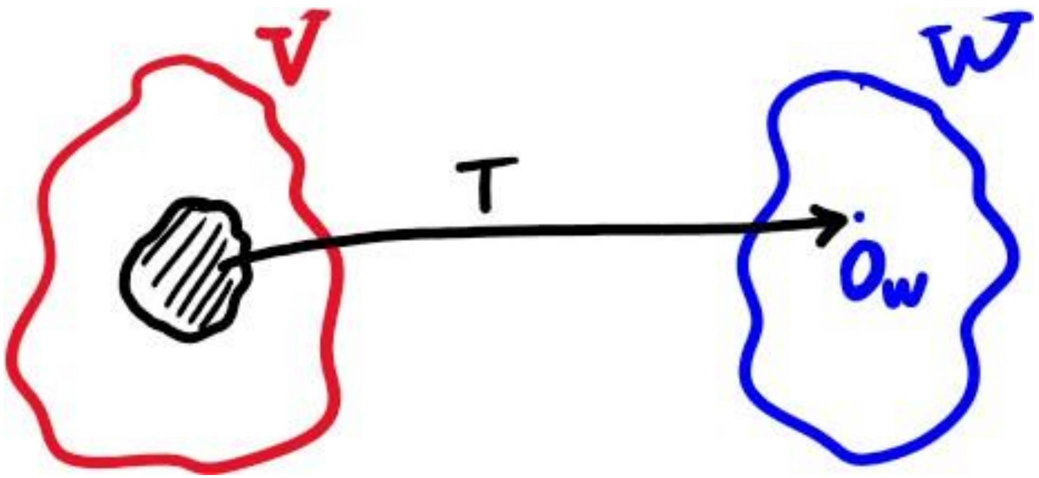
$d)\ \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$

Kernel (Null Space) and Range

Definition of Kernel of a Linear Transformation

The kernel of a linear transformation $T: V \to W$, denoted $\ker(T)$, is the set of all vectors $\mathbf{u} \in V$ such that:

$$T(\mathbf{u}) = \mathbf{0}_W$$



Kernel (Null Space) and Range

Definition of Null Space of a Linear Transformation

If you view a matrix A as representing a linear transformation $T(\mathbf{u}) = A\mathbf{u}$, then:

$$\mathrm{Null}(T) = \ker(T)$$

Can write $\mathrm{N}(T)$

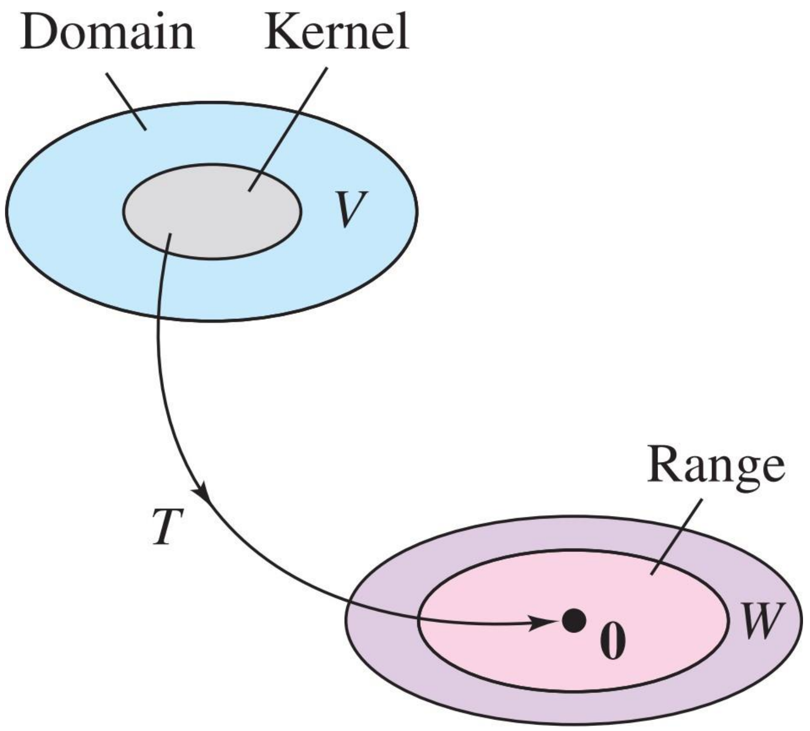
Kernel (Null Space) and Range

Definition of Range (or Image) of a Linear Transformation

The range (or image) of a linear transformation $T: V \to W$, denoted Range(T), is the set of all vectors in W that can be reached by applying T to some vector in V.

$$\text{Range}(T) = \{T(\mathbf{u}) | \mathbf{u} \in \boldsymbol{V}\}$$

Can write $\text{R}(T)$



Let T be linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.

The $\ker(T)$ is a subspace of $\mathbf{R}^n$.

The range(T) is a subspace of $\mathbf{R}^m$.

$\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(\text{domain})$.

$\text{nullity}(T) \quad + \text{rank}(T) \qquad = n.$

Kernel (Null Space) and Range

Example 1: $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y + z \end{bmatrix}$

Ker(T)

$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x - z \\ y + z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ← solve

$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$

$x \quad y \quad z \quad =$

$\therefore \boxed{z = t}$, $t \in \mathbb{R}$

$y + z = 0$

$y + t = 0 \Rightarrow \boxed{y = -t}$

$x - z = 0$

$x - t = 0 \Rightarrow \boxed{x = t}$

Null(T) = N(T)

$\therefore \operatorname{Ker}(T) = \left\{ \begin{bmatrix} t \\ -t \\ t \end{bmatrix} , t \in \mathbb{R} \right\}$

$= \left\{ t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} , t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

R(T), image(T)

$\operatorname{Range}(T) = \operatorname{span}\left\{ T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right\}$

$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$

Basis

Kernel (Null Space) and Range

Example 2:

Let $T: R^3 \to R^3$ defined by

$T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$

1) Show that T is a linear transformation.

2) Find the $\ker(T)$.

Kernel (Null Space) and Range

Example 2:

Let $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + x_2 - x_3, -x_1 - 2x_2 + 2x_3)$

1) Show that T is a linear transformation.

2) Find the $\ker(T)$.

Ker(T)

$T(x_1, x_2, x_3) = (0, 0, 0)$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \end{array}\right) \xrightarrow[R_3 = R_1 + R_3]{R_2 = -2R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right) \xrightarrow[R_3 = \frac{1}{3} R_3]{R_2 = \frac{1}{3} R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow[R_3 = R_2 + R_3]{R_1 = R_2 + R_1} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array} \quad \therefore x_3 = t, \ t \in R$$

$x_1 = 0$ $\qquad$ $x_2 - x_3 = 0$

$x_1 = 0$ $\qquad$ $x_2 - t = 0 \to x_2 = t$

Null(T) = N(T)

$$\text{Ker}(T) = \left\{ \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}, t \in R \right\}$$

$$= \left\{ t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, t \in R \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Kernel (Null Space) and Range

Example 3:

Let $T: R^3 \to R^3$ be linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ x+y+z \end{bmatrix}.$$ Find $\mathrm{N}(T)$ and $\mathrm{R}(T)$.

Kernel (Null Space) and Range

Example 3:

Let $T: R^3 \to R^3$ be linear transformation defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ x+y+z \end{bmatrix}$. Find N($T$) and R($T$).

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ x+y+z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x=0 \checkmark \\ x+y=0 \\ 0+y=0 \Rightarrow y=0 \checkmark \end{array}$$

$$\begin{array}{ccc} x & y & z \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \quad \begin{array}{l} x+y+z=0 \\ 0+0+z=0 \Rightarrow z=0 \checkmark \end{array}$$

$$N(T) = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} = \{\vec{0}\} \qquad \overbrace{\dim(N(T))}^{\text{nullity}(T)} = 0$$

$$R(T) = \text{span}\left\{T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\right\}$$

$$= \text{span}\{\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{Basis}}\} \qquad \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$$

$$= R^3 \qquad \overbrace{\dim(R(T))}^{\text{rank}(T)} = 3$$

Kernel (Null Space) and Range

Example 4:

Define $T: R^3 \to R^2$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ y + z \end{bmatrix}$

1) Show that T is a linear transformation.

2) Find the null space, N(T).

2) Find the ragne of T, R(T).

Kernel (Null Space) and Range

Example 4: $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ y + z \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ y + z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \end{array} \xrightarrow{R_1 = \frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_1 = \frac{1}{2}R_2 + R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad \therefore z = t \quad , \quad t \in \mathbb{R}$$

$$y + z = 0$$

$$y + t = 0 \longrightarrow y = -t$$

$$x + \frac{1}{2}z = 0$$

$$x + \frac{1}{2}t = 0 \longrightarrow x = -\frac{1}{2}t$$

$$\therefore N(T) = \left\{ \begin{bmatrix} -\frac{1}{2}t \\ -t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} , t \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} \right\}$$

$$R(T) = \text{span}\left\{ T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{Basis}} \right\} = \mathbb{R}^2$$

Kernel (Null Space) and Range

Example 5:

Define $T: R^4 \to R^3$ by $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}$, where $a, b, c,$ and $d \in R$.

1) Find the Null(T), its basis, and its dimension.

2) Find the Range(T), its basis, and its dimension.

Kernel (Null Space) and Range

Example 5: $T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix}$

<u>Null(T)</u> $\quad T\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b-c \\ a+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right) \end{array} \xrightarrow{R_3 = -R_1 + R_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_3 = R_2 + R_3]{R_1 = -R_2 + R_1}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_3 = -R_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow[R_2 = R_3 + R_2]{R_1 = -R_3 + R_1}$$

$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \end{array}$$

$\therefore d = t, \quad t \in R$

$c - d = 0$

$c - t = 0 \Rightarrow c = t$

$b - d = 0$

$b - t = 0 \Rightarrow b = t$

$a + d = 0$

$a + t = 0 \Rightarrow a = -t$

$$Null(T) = \left\{ \begin{bmatrix} -t \\ t \\ t \\ t \end{bmatrix}, t \in R \right\}$$

$$= \left\{ t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, t \in R \right\} = span\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$\therefore$ Basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, $\overset{nullity(T)}{\dim(Null(T))} = 1$

$$Range(T) = span\left\{ T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) \right\}$$

$$= span\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (1)(-1)(1) = -1 \neq 0$$

$$= span\underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{Basis} = R^3$$

$\dim(Range(T)) = 3$

rank(T)

Kernel (Null Space) and Range

Example 6:

Define $T: R^3 \to R^3$ by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$.

1) Show that T is a linear transformation.

2) Find the null space, $\mathrm{N}(T)$.

3) Find the ragne of T, $\mathrm{R}(T)$.

Kernel (Null Space) and Range

Example 6: $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$

$\boxed{1}$ $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$

$$= \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad 3\times 3$$

$\because T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\therefore$ T is a linear Transformation from $R^3 \longrightarrow R^3$.

$\boxed{2}$ $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$\therefore \boxed{y = t}, \; t \in R$

$\boxed{z = 0}$

$\boxed{x = 0}$

$$\therefore N(T) = \left\{ \begin{bmatrix} 0 \\ t \\ 0 \end{bmatrix}, t \in R \right\}$$

$$= \left\{ t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, t \in R \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$R(T) = \text{span}\left\{ T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Kernel (Null Space) and Range

Example 7: True / False

The null space for the linear transformation $T: R^2 \to R^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ y - x \end{bmatrix} \text{ is } N(\mathrm{T}) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}.$$

Kernel (Null Space) and Range

Example 7: True / False

The null space for the linear transformation $T: R^2 \to R^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ y - x \end{bmatrix} \text{ is } N(\mathrm{T}) = \left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}.$$

$$\begin{array}{cc} x & y \\ \end{array}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} \boxed{1} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$y = t \quad , \ t \in R$

$x - t = 0$

$x = t$

$$N(T) = \left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, t \in R \right\}$$

Kernel (Null Space) and Range

Example 8: True / False

The range for the linear transformation $T: R^2 \to R^2$ defined by

$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ y - x \end{bmatrix}$ is R^2.

Kernel (Null Space) and Range

Example 8: True / False

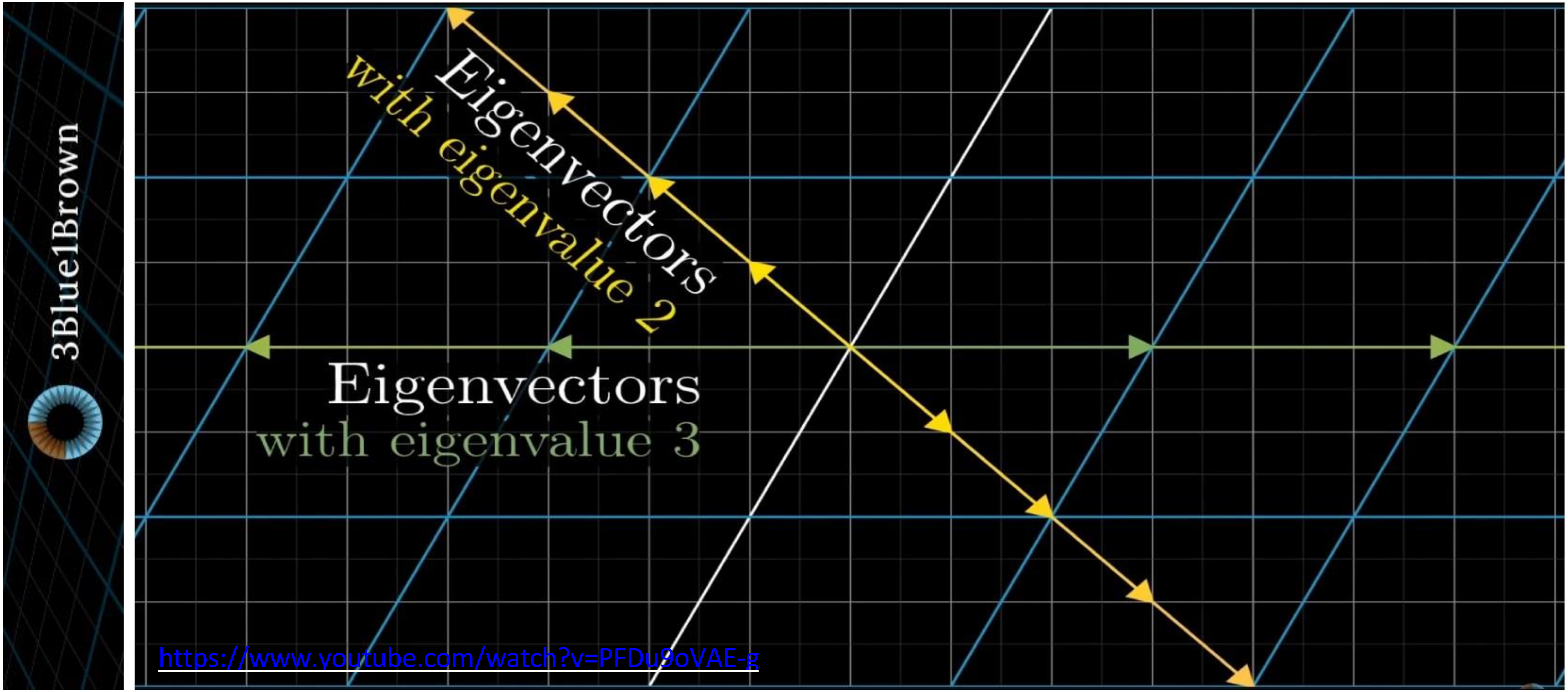
The range for the linear transformation $T: R^2 \to R^2$ defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ y - x \end{bmatrix} \text{ is } R^2.$$

$$R(T) = \text{span}\left\{ T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \neq R^2$$

Eigenvalues and Eigenvectors

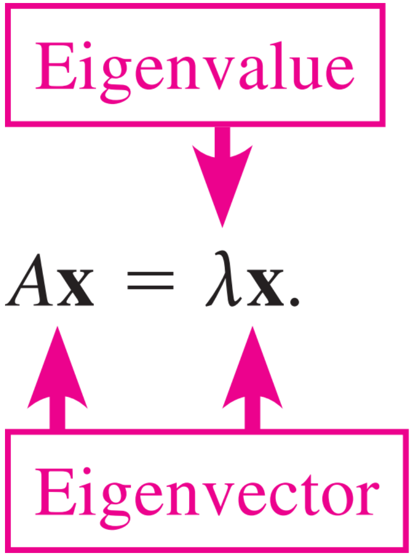


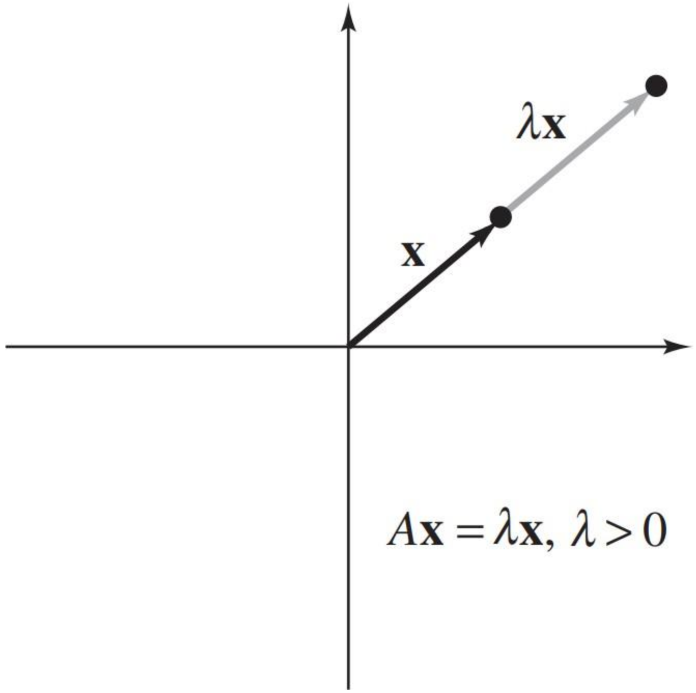
https://www.youtube.com/watch?v=PFDu9oVAE-g

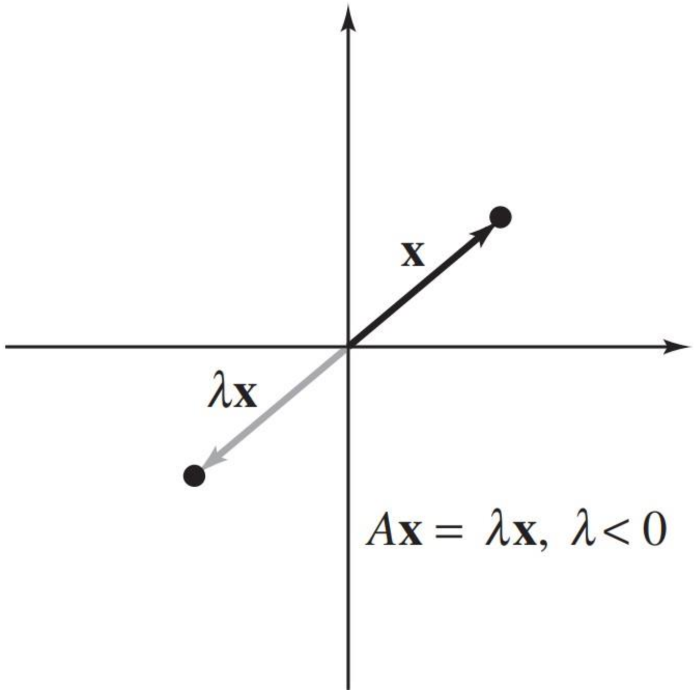
Eigenvalues and Eigenvectors

Definitions of Eigenvalue and Eigenvector

Let A be an $n \times n$ matrix. The scalar λ is an eigenvalue of A when there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is an eigenvector of A corresponding to λ.







Eigenvalues and Eigenvectors

Finding Eigenvalues and Eigenvectors

A be an $n \times n$ matrix, and nonzero vector $\mathbf{x} \in R^n$.

$$A\mathbf{x} = \lambda\mathbf{x}$$
$$\lambda\mathbf{x} - A\mathbf{x} = \mathbf{0}$$
$$(\lambda I_n - A)\mathbf{x} = \mathbf{0}$$

Eigenvalue

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Eigenvector

1. An eigenvalue of A is a scalar λ such that $\det(\lambda I_n - A) = 0$.
2. The eigenvectors of A corresponding to λ are the nonzero solutions of $(\lambda I_n - A)\mathbf{x} = \mathbf{0}$.

The **sum** of the eigenvalues is called the *trace*.

The **product** of the eigenvalues is the *determinant* of the matrix.

Eigenvalues and Eigenvectors

Finding Eigenvalues and Eigenvectors

A be an $n \times n$ matrix.

1) λ : a scalar (could be zero)

$\det(\lambda I_n - A) = |\lambda I_n - A| = 0$ (characteristic equation of A)

2) $\mathbf{x}$: a nonzero vector in R^n

The eigenvector $\mathbf{x}$ associted with the eigenvalue λ is a solusion of the homogeneous linear syetem $[\lambda I_n - A]\mathbf{x} = 0$.

3) V_λ: the eigenspace of A corresponding to λ in R^n

$V_\lambda = \text{span}\{\mathbf{x}\} = \{r\mathbf{x}\,, r \in R\}$.

Eigenvalues and Eigenvectors

Example 1:

Find the eigenvalues and corresponding eigenvectors and eigenspace of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Example 1: $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

$\boxed{1}$ Find the eigenvalues

$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$$

$$|\lambda I_2 - A| = 0$$

$$\begin{vmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda+5) - (-1)(12) = 0$$

$$\lambda^2 + 5\lambda - 2\lambda - 10 + 12 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

eigenvalues

$$(\lambda+2)(\lambda+1) = 0 \Rightarrow \lambda_1 = -2,\ \lambda_2 = -1$$

$\boxed{2}$ Find the eigenvector for $\lambda_1 = -2$

$$[\lambda I_2 - A]X = 0$$

$$-2I_2 = -2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

$$[-2I_2 - A]X = 0$$

$$\begin{bmatrix} -2-2 & 0-(-12) \\ 0-1 & -2-(-5) \end{bmatrix} X = 0$$

$$\left[\begin{array}{cc|c} -4 & 12 & 0 \\ -1 & 3 & 0 \end{array}\right] \xrightarrow{R_1 = \frac{-1}{4}R_1} \left[\begin{array}{cc|c} 1 & -3 & 0 \\ -1 & 3 & 0 \end{array}\right] \xrightarrow{R_2 = R_1 + R_2} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let $x_2 = r$, $r \in \mathbb{R}$

$$x_1 - 3x_2 = 0$$

$$x_1 - 3r = 0 \Rightarrow x_1 = 3r$$

$$\therefore X = \begin{bmatrix} 3r \\ r \end{bmatrix},\ r \in \mathbb{R}$$

$$X = r\begin{bmatrix} 3 \\ 1 \end{bmatrix},\ r \in \mathbb{R}$$

$\therefore X_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = -2$

Eigenvalues and Eigenvectors

Example 1: $A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$.

For $\lambda_2 = -1$

$$[\lambda I_2 - A]X = 0 \Longrightarrow [-I_2 - A]X = 0$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$$

$$\begin{pmatrix} -1-2 & 0-(-12) \\ 0-1 & -1-(-5) \end{pmatrix} X = 0 \Rightarrow \begin{pmatrix} -3 & 12 \\ -1 & 4 \end{pmatrix} X = 0, \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{array}{cc} x_1 & x_2 \end{array}$$

$$\left(\begin{array}{cc|c} -3 & 12 & 0 \\ -1 & 4 & 0 \end{array}\right) \xrightarrow{R_1 = -\frac{1}{3}R_1} \left(\begin{array}{cc|c} 1 & -4 & 0 \\ -1 & 4 & 0 \end{array}\right) \xrightarrow{R_2 = R_1 + R_2}$$

$$\left(\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $x_2 = s$, $s \in R$

$x_1 - 4x_2 = 0$

$x_1 - 4s = 0 \Longrightarrow x_1 = 4s$

$$\therefore X = \begin{pmatrix} 4s \\ s \end{pmatrix} = s\begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad s \in R$$

$\therefore X_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvector for $\lambda_2 = -1$

$\boxed{3}$ The eigenspace for $\lambda_1 = -2$

$$V_{-2} = \text{span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\} = \left\{r\begin{bmatrix} 3 \\ 1 \end{bmatrix}, r \in R\right\}$$

The eigenspace for $\lambda_2 = -1$

$$V_{-1} = \text{span}\left\{\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right\} = \left\{s\begin{bmatrix} 4 \\ 1 \end{bmatrix}, s \in R\right\}$$

Eigenvalues and Eigenvectors

Example 2:

Find the eigenvalues and corresponding eigenvectors and eigenspace of

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Example 2: $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

$\boxed{1}$ Find the eigenvalues

$$|\lambda I_3 - A| = 0 \Longrightarrow \begin{vmatrix} \lambda-1 & 0-2 & 0-0 \\ 0-2 & \lambda-1 & 0-0 \\ 0-0 & 0-0 & \lambda-(-3) \end{vmatrix} = 0$$

$$\begin{array}{c} \begin{vmatrix} \lambda-1 & -2 & 0 \\ -2 & \lambda-1 & 0 \\ 0 & 0 & \lambda+3 \end{vmatrix} = 0 \\ \begin{array}{ccc} + & - & + \end{array} \end{array}$$

$$(\lambda+3)\big((\lambda-1)(\lambda-1)-(-2)(-2)\big) = 0$$

$$(\lambda+3)(\lambda^2-\lambda-\lambda+1-4) = 0$$

$$(\lambda+3)(\lambda^2-2\lambda-3) = 0$$

$$\begin{array}{cc} \lambda & -3 \\ \lambda & 1 \end{array}$$

$$(\lambda+3)(\lambda-3)(\lambda+1) = 0$$

$\lambda_1 = -3$ $\lambda_2 = 3$ $\lambda_3 = -1$ the eigenvalues.

$\boxed{2}$ Find the eigenvectors:

For $\lambda_1 = -3$

$$[\lambda I_3 - A]X = 0, \quad X \in \mathbb{R}^3 \to \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[-3I_3 - A]X = 0$$

$$\left(\begin{array}{ccc|c} -3-1 & 0-2 & 0-0 & 0 \\ 0-2 & -3-1 & 0-0 & 0 \\ 0-0 & 0-0 & -3-(-3) & 0 \end{array}\right) \Longrightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} -4 & -2 & 0 & 0 \\ -2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$$\xrightarrow{R_1 = \frac{-1}{4}R_1} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ -2 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = 2R_1 + R_2} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = \frac{-1}{3}R_2}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = \frac{-1}{2}R_2 + R_1} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Let $x_3 = r$, $r \in \mathbb{R}$

$x_2 = 0$, $x_1 = 0$

$$\therefore X = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} = r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}$$

$X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_1 = -3$

Eigenvalues and Eigenvectors

Example 2: $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$

For $\lambda_2 = 3$

$[3I_3 - A]X = 0$

$$\left(\begin{array}{ccc|c} 3-1 & 0-2 & 0-0 & 0 \\ 0-2 & 3-1 & 0-0 & 0 \\ 0-0 & 0-0 & 3-(-3) & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right) \xrightarrow{R_1 = \frac{1}{2}R_1}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right) \xrightarrow{R_2 = 2R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = \frac{1}{6}R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$x_3 = 0$

$x_1 - x_2 = 0$

$x_1 - s = 0$

Let $x_2 = s$, $s \in R$

$x_3 = 0$

$x_1 = s$

$$\therefore X = \begin{bmatrix} s \\ s \\ 0 \end{bmatrix}, \ s \in R \ = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ s \in R$$

$X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 3$.

For $\lambda_3 = -1$

$[-I_3 - A]X = 0$

$$\left(\begin{array}{ccc|c} -1-1 & 0-2 & 0-0 & 0 \\ 0-2 & -1-1 & 0-0 & 0 \\ 0-0 & 0-0 & -1-(-3) & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -2 & -2 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \xrightarrow{R_1 = -\frac{1}{2}R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \xrightarrow{R_2 = 2R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = \frac{1}{2}R_2} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$x_3 = 0$

$x_1 + x_2 = 0$

$x_1 + t = 0$

$x_1 = -t$

Let $x_2 = t$, $t \in R$

$x_3 = 0$

$x_1 = -t$

$$\therefore X = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ t \in R$$

$X_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for $\lambda_3 = -1$

Eigenvalues and Eigenvectors

Example 2: $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

$\boxed{3}$ Find the eigenspace:

* For $\lambda_1 = -3$

$$V_{-3} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, r \in R \right\}$$

* For $\lambda_2 = 3$

$$V_3 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, s \in R \right\}$$

* For $\lambda_3 = -1$

$$V_{-1} = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, t \in R \right\}$$

Eigenvalues and Eigenvectors

Note:

If the matrix A is an $n \times n$ triangular (diagonal)matrixthen its eigenvalues are the entries on its main diagonal.

Example: Finding the eigenvalues of

$$\textbf{a. } A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$$

$$\textbf{b. } A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Note:

If the matrix A is an $n \times n$ triangular (diagonal)matrixthen its eigenvalues are the entries on its main diagonal.

Example: Finding the eigenvalues of

a. $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}$ $\therefore \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$

b. $A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ $\therefore \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = -4, \lambda_5 = 3$

Eigenvalues and Eigenvectors

Example 3:

Find the eigenvalues and corresponding eigenvectors and eigenspace of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Example 3:

Find the eigenvalues and corresponding eigenvectors and eigenspace of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

$\boxed{1}$ $\because$ A is a traingular matrix

$\therefore$ The eigenvalues are : $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 3$

$\boxed{2}$?

$\boxed{3}$?

Eigenvalues and Eigenvectors

Example 4:

Find the eigenvalues and corresponding eigenvectors and eigenspace of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Example 4:

Find the eigenvalues and corresponding eigenvectors and eigenspace of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$\underbrace{\lambda_1 = 2 \quad \lambda_2 = 2 \quad \lambda_3 = 2}$$

$\boxed{1}$ $\boxed{\lambda = 2}$

$\downarrow$

$\boxed{2}$ X

$\downarrow$

$\boxed{3}$ $V_2 = \text{span}\{X\}$

Eigenvalues and Eigenvectors

Example 5: Choose The Correct Answer

An eigenvalue for the matrix $A = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$ is

$a)\ \ 0$

$b)\ \ 1$

$c) - 2$

$d) - 6$

Eigenvalues and Eigenvectors

Example 5: Choose The Correct Answer

An eigenvalue for the matrix $A = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$ is

$a)\ \ 0$

$b)\ \ 1$

$\boxed{c) - 2}$

$d) - 6$

Eigenvalues and Eigenvectors

Example 6: Choose The Correct Answer

An eigenvalue for the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

$a)$ 0

$b)$ 1

$c)$ 2

$d)$ 3

Eigenvalues and Eigenvectors

Example 6: Choose The Correct Answer

An eigenvalue for the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

$a)$ 0

$b)$ 1

$\boxed{c)\ \ 2}$

$d)$ 3

Eigenvalues and Eigenvectors

Example 7: Choose The Correct Answer

An eigenvector for value of $\lambda = -2$ for matrix $\begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}$ is

$a)\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$b)\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c)\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$d)\ \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenvalues and Eigenvectors

Example 7: Choose The Correct Answer

An eigenvector for value of $\lambda = -2$ for matrix $\begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}$ is

$a)\ \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$b)\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$c)\ \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{d)\ \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

$[-2I_2 - A]X = 0$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -r \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\left(\begin{array}{cc|c} -5 & -5 & 0 \\ -3 & -3 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right)$$

$$\begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \end{array}$$

let $x_2 = r$, $r \in R$

$x_1 + x_2 = 0$

$x_1 + r = 0 \rightarrow x_1 = -r$

Eigenvalues and Eigenvectors

Example 8: True / False

The matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ have the same eigenvalues.

Eigenvalues and Eigenvectors

Example 8: True / False

The matrices $\overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}$ and $\overset{B}{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$ have the same eigenvalues.

$$|\lambda I_2 - A| = 0$$

$$\begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = 0$$

$$(\lambda - a)(\lambda - d) - (cb) = 0$$

$$|\lambda I_2 - B| = 0$$

$$\begin{vmatrix} \lambda - d & b \\ c & \lambda - a \end{vmatrix} = 0$$

$$(\lambda - d)(\lambda - a) - (cb) = 0$$

Eigenvalues and Eigenvectors

Example 9: True / False

$\lambda = 2$ is an eigenvalue for $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$.

Eigenvalues and Eigenvectors

Example 9: True / False

$\lambda = 2$ is an eigenvalue for $A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$.

$\lambda_1 = 1$ $\lambda_2 = 2$

Diagonalization

Diagonal Matrix :

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -5 \end{pmatrix}_{3\times 3}$$

$$D^2 = D \cdot D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -5 \end{pmatrix} = \begin{pmatrix} \overset{2^2}{4} & 0 & 0 \\ 0 & \overset{4^2}{16} & 0 \\ 0 & 0 & \underset{(-5)^2}{25} \end{pmatrix}$$

$$D^K = \begin{pmatrix} 2^K & 0 & 0 \\ 0 & 4^K & 0 \\ 0 & 0 & (-5)^K \end{pmatrix} \quad , K \in Z^+$$

Diagonalization

Definition of a Diagonalizable Matrix

An $n \times n$ matrix A is diagonalizable when A is similar to a diagonal matrix. That is, A is diagonalizable when there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

Then $B = P^{-1}AP$ and $A = P^{-1}BP$

$$A^2 = (P^{-1}BP)^2 = P^{-1}BP \cdot P^{-1}BP = P^{-1}B\,(PP^{-1})BP = P^{-1}B^2P$$

Thus, $A^k = (P^{-1}BP)^k = P^{-1}B^kP$

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable because

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has the property that

$$B = P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Diagonalization

Similar Matrices Have the Same Eigenvalues

If A and B are similar $n \times n$ matrices, then they have the same eigenvalues.
There exists an invertible matrix P such that $B = P^{-1}AP$

$$
\begin{aligned}
|\lambda I - B| &= |\lambda I - P^{-1}AP| \\
&= |P^{-1}\lambda IP - P^{-1}AP| \\
&= |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| \\
&= |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|.
\end{aligned}
$$

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

is diagonalizable because

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has the property that

$$B = P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

This means that A and B have the same characteristic polynomial.

Diagonalization

Condition for Diagonalization

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. Moreover, if

$$B = P^{-1}AP$$

with B a diagonal matrix, then the diagonal entries of B are the eigenvalues of A and the column vectors of P are the corresponding eigenvectors.

$$\underset{\text{eigenvalues of } A}{\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{A}{\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}} \underset{\text{eigenvectors of } A}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

If A has n distinct eigenvalues, then A is diagonalizable.

Diagonalization

Steps for Diagonalizing a Square Matrix

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ for A (if possible) with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.
2. Let P be the $n \times n$ matrix whose columns consist of these eigenvectors. That is, $P = [\mathbf{p}_1\ \mathbf{p}_2 \ldots \mathbf{p}_n]$.
3. The diagonal matrix $B = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ on its main diagonal. Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main diagonal of B.

Diagonalization

Example 1:

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

a) Show that $\lambda = 2$ and $\lambda = 3$ are eigenvalues of A.

b) Find a matrix P such that $P^{-1}AP = D, D$ is a diagonal matrix.

Diagonalization

Example 1: Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

a) Find the eigenvalues:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$$|\lambda I_2 - A| = 0 \longrightarrow \begin{vmatrix} \lambda-1 & -1 \\ 2 & \lambda-4 \end{vmatrix} = 0$$

$$(\lambda-1)(\lambda-4) - (-2) = 0$$

$$\lambda^2 - 4\lambda - \lambda + 4 + 2 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda-3)(\lambda-2) = 0$$

$$\lambda_1 = 3, \quad \lambda_2 = 2 \qquad \text{eigenvalues}$$

b) Find the eigenvectors:

For $\lambda_1 = 3$:

$$[\lambda I_2 - A]X = 0$$

$$[3 I_2 - A]X = 0 \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\overbrace{\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}}^{3I_2} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 = \frac{1}{2}R_1}$$

$$\left[\begin{array}{cc|c} 1 & \frac{-1}{2} & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 = -2R_1 + R_2} \begin{array}{ccc} x_1 & x_2 & \\ \end{array} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $x_2 = r$, $r \in R$

$$x_1 - \tfrac{1}{2}x_2 = 0$$

$$x_1 - \tfrac{1}{2}r = 0 \rightarrow x_1 = \tfrac{1}{2}r$$

$$\therefore X = \begin{bmatrix} \frac{1}{2}r \\ r \end{bmatrix} = r\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}, r \in R$$

$\therefore X_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda_1 = 3$

Diagonalization

Example 1: Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

For $\lambda_2 = 2$:

$[\lambda I_2 - A]X = 0$

$[2I_2 - A]X = 0 \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\overbrace{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}^{2I_2} - \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$

$$\begin{array}{cc} x_1 & x_2 \\ \end{array}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \end{array}\right) \xrightarrow{R_2 = -2R_1 + R_2}$$

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $x_2 = t$, $t \in R$

$x_1 - x_2 = 0$

$x_1 - t = 0 \rightarrow x_1 = t$

$\therefore X = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $t \in R$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda_2 = 2$.

$\lambda_1 = 3$, $\lambda_2 = 2$

$X_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{vmatrix} \frac{1}{2} & 1 \\ 1 & 1 \end{vmatrix} = (\frac{1}{2})(1) - (1)(1) = -\frac{1}{2} \neq 0$

$\therefore X_1, X_2$ are linearly independent .

$$\therefore P = \begin{array}{c} \begin{array}{cc} X_1 & X_2 \end{array} \\ \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

$$\therefore D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

λ_1 λ_2

$D = P^{-1}AP$

Diagonalization

Example 2:

Let $A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

a) Show that $\lambda = 0, -1, -2$ are eigenvalues of A.

b) Find an eigenvector corresponding to $\lambda = 0$.

c) Show that the matrix A is diagonalizable.

Diagonalization

Example 2: Let $A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

a) Find the eigenvalues:

$$|\lambda I_3 - A| = 0$$

$$\overset{\lambda I_3}{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}} - \overset{A}{\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}}$$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} \lambda+1 & 0 & 1 \\ 0 & \lambda+1 & -1 \\ 1 & 0 & \lambda+1 \end{vmatrix} = 0$$

$$(\lambda+1)(\lambda+1)(\lambda+1) + (-(\lambda+1)) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda+1) - (\lambda+1) = 0$$

$$(\lambda+1)[(\lambda+1)(\lambda+1) - 1] = 0$$

$$(\lambda+1)[\lambda^2+\lambda+\lambda+1-1] = 0$$

$$(\lambda+1)[\lambda^2+2\lambda] = 0$$

$$(\lambda+1)[\lambda(\lambda+2)] = 0 \Longrightarrow \lambda(\lambda+1)(\lambda+2) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = -1, \quad \lambda_3 = -2$$

b) Find the eigenvector for $\lambda_1 = 0$

$$[\lambda I_3 - A]X = 0$$

$$[0 - A]X = 0 \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in R^3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3 = -R_1 + R_3} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$x_1 + x_3 = 0$
$x_1 + r = 0$
$x_1 = -r$

$x_2 - x_3 = 0$
$x_2 - r = 0$
$x_2 = r$

let $x_3 = r$, $r \in R$

$x_2 = r$, $x_1 = -r$

$$\therefore X = \begin{bmatrix} -r \\ r \\ r \end{bmatrix} = r\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad r \in R$$

$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is the eigenvector for $\lambda_1 = 0$

c)

$\because$ A has 3 distinct eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -2$

$\therefore$ A is a diagonalizable matrix.

Diagonalization

Example 3:

Show that the matrix A is diagonalizable

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

Diagonalization

Example 3:

1 Find the eigenvalues:

$$|\lambda I_3 - A| = 0 \qquad \begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix} = 0$$

$$(\lambda-1)\big((\lambda-3)(\lambda+1)-1\big) - \big(-(\lambda+1)-(-3)\big) + \big(1-(3)(\lambda-3)\big) = 0$$

$$(\lambda-1)(\lambda^2-2\lambda-3-1) + (\lambda+1) - 3 + 1 - 3\lambda + 9 = 0$$

$$(\lambda-1)(\lambda^2-2\lambda-4) + (-2\lambda) + 8 = 0$$

$$\lambda^3 - 2\lambda^2 - 4\lambda - \lambda^2 + \cancel{2\lambda} + 4 \cancel{-2\lambda} + 8 = 0$$

$$\lambda^3 - 3\lambda^2 - 4\lambda + 12 = 0$$

$$\therefore \lambda_1 = -2, \quad \lambda_2 = 3, \quad \lambda_3 = 2$$

eigenvalues

2 Find the eigenvectors:

For $\lambda_1 = -2$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}$$

$$[\lambda I_3 - A]X = 0$$

$$\left(\begin{array}{ccc|c} -3 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 3 & -1 & -1 & 0 \end{array}\right) \xrightarrow{R_1 = -\frac{1}{3}R_1} \left(\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -1 & -1 & -1 & 0 \\ 3 & -1 & -1 & 0 \end{array}\right) \xrightarrow[R_3 = -3R_1 + R_3]{R_2 = R_1 + R_2}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & -\frac{4}{3} & -\frac{4}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = -\frac{3}{4}R_2} \left(\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = \frac{1}{3}R_2 + R_1}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_3 = r$, $r \in R$

$x_1 = 0$ $\qquad x_1 = 0$

$x_2 + x_3 = 0$

$x_2 + r = 0$ $\qquad x_2 = -r$

$$\therefore X = \begin{bmatrix} 0 \\ -r \\ r \end{bmatrix} = r\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad r \in R$$

$$\therefore X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ is an eigenvector for } \lambda_1 = -2$$

Diagonalization

Example 3:

For $\lambda_2 = 3$

$$[\lambda I_3 - A]X = 0$$

$$[3I_3 - A]X = 0$$

$$\begin{array}{cc} 3I_3 & A \\ \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} - & \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix} \end{array}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right) \xrightarrow{R_1 = \frac{1}{2}R_1} \left(\begin{array}{ccc|c} \boxed{1} & \frac{1}{2} & \frac{1}{2} & 0 \\ -1 & 0 & -1 & 0 \\ 3 & -1 & 4 & 0 \end{array}\right) \xrightarrow[R_3 = -3R_1 + R_3]{R_2 = R_1 + R_2}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{5}{2} & \frac{5}{2} & 0 \end{array}\right) \xrightarrow[R_3 = \frac{2}{5}R_3]{R_2 = 2R_2} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow[R_3 = R_2 + R_3]{R_1 = -\frac{1}{2}R_2 + R_1}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_3 = t$, $t \in R$

$x_1 + x_3 = 0$

$x_1 + t = 0$ $\quad \therefore x_1 = -t$

$x_2 - x_3 = 0$

$x_2 - t = 0$ $\quad \therefore x_2 = t$

$$\therefore X = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad t \in R$$

$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_2 = 3$.

for $\lambda_3 = 2$

$$[\lambda I_3 - A]X = 0$$

$$[2I_3 - A]X = 0$$

$$\begin{array}{cc} 2I_3 & A \\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - & \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix} \end{array}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 3 & -1 & 3 & 0 \end{array}\right) \xrightarrow[R_3 = -3R_1 + R_3]{R_2 = R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 = -\frac{1}{4}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = -R_2 + R_1}$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_3 = s$, $s \in R$

$x_1 + x_3 = 0$

$x_1 + s = 0$ $\quad \therefore x_1 = -s$

$x_2 = 0$ $\quad \therefore x_2 = 0$

$$\therefore X = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad s \in R$$

$\therefore X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_3 = 2$

Diagonalization

Example 3:

$\therefore \lambda_1 = -2, \quad \lambda_2 = 3, \quad \lambda_3 = 2$ eigenvalues

$X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ eigenvectors

$$\because \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} 0 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} \end{array} = -1 - 1(-2) = -1 + 2 = 1 \neq 0$$

$\therefore X_1, X_2, X_3$ are linearly independent.

$\therefore D = P^{-1}AP$, where.

$$P = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example 4:

Show that the matrix A is diagonalizable

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & -2 & 0 \\ 7 & -4 & 2 \end{bmatrix}$$

Then find a matrix P such that $P^{-1}AP$ is diagonal.

Diagonalization

Example 5:

Show that the matrix A is not diagonalizable

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 7 & -4 & 2 \end{bmatrix}$$

Diagonalization

Example 5:

$\therefore\ \lambda_1 = -2\ ,\ \lambda_2 = 3\ ,\ \lambda_3 = 2$ eigenvalues

$X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},\ X_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\ X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ eigenvectors

$$\because \begin{array}{ccc} + & - & + \end{array}$$
$$\begin{vmatrix} 0 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = -1 - 1(-2) = -1 + 2 = 1 \neq 0$$

$\therefore\ X_1, X_2, X_3$ are linearly independent.

$\therefore\ D = P^{-1}AP$, where.

$$P = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},\quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & \frac{-4}{7} & \frac{1}{7} & 0 \\ 0 & \boxed{1} & \frac{-1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 = \frac{4}{7}R_2 + R_1} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & \frac{-1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$x_1 = 0$

$x_2 - \frac{1}{4}x_3 = 0$

$x_2 - \frac{1}{4}t = 0$

Let $x_3 = t\ ,\ t \in \mathbb{R}$

$x_1 = 0$

$x_2 = \frac{1}{4}t$

$$\therefore X = \begin{bmatrix} 0 \\ \frac{1}{4}t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ \frac{1}{4} \\ 1 \end{bmatrix}$$

$X_1 = \begin{bmatrix} 0 \\ \frac{1}{4} \\ 1 \end{bmatrix} = X_2$ are eigenvector for $\lambda_1 = \lambda_2 = 1$

Diagonalization

Example 5:

For $\lambda_3 = 2$

$$\overset{2I_3}{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}} - \begin{pmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 7 & -4 & 2 \end{pmatrix}$$

$$[\lambda I_3 - A]X = 0$$

$$[2I_3 - A]X = 0$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 \\ -7 & 4 & 0 & 0 \end{array}\right) \xrightarrow[R_3 = 7R_1 + R_3]{R_2 = 6R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right) \xrightarrow{R_3 = -4R_2 + R_3}$$

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

let $x_3 = r$, $r \in R$

$x_1 = 0$ $\quad \therefore x_1 = 0$

$x_2 = 0$ $\quad \therefore x_2 = 0$

$$\therefore X = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} = r\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} , \; r \in R$$

$\therefore X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda_3 = 2$

$\therefore \lambda_1 = 1 \; , \; \lambda_2 = 1 \; , \; \lambda_3 = 2$ (eigenvalues)

$\therefore X_1 = \begin{bmatrix} 0 \\ 1/4 \\ 1 \end{bmatrix} , \; X_2 = \begin{bmatrix} 0 \\ 1/4 \\ 1 \end{bmatrix} , \; X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (eigenvectors)

$$\begin{vmatrix} 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$\therefore X_1 , X_2 , X_3$ are linearly dependent

$\therefore$ A is NOT diagonalizable.