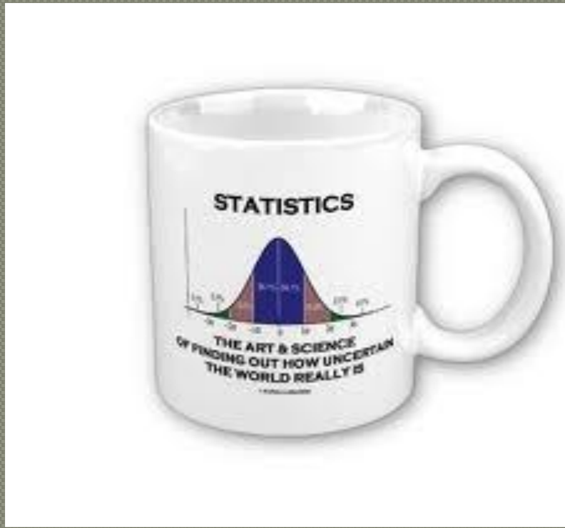
Probability Theory

Math 232



- Contents

1. Basic Probability Concepts
2. Random Variables.
3. Some Probability Distributions
4. Moments and Moment generating function.

Lecture 1

A random experiment:

is an experiment whose outcome is not known until it is observed.

Sample space:

is the set of all possible outcomes of an experiment. It is denoted by *S*

Example:

A coin is tossed three times and the sequence of heads / tails is recorded

S ={HHH, HHT, HTH, HTT, THH, THT,TTH,TTT}

The event

An event is a subset of the sample space.

Probability of an event

Is a measure of chance of occurrence of the event.

It is a number between 0 and 1

If the experiment has **N** equally likely outcomes, then the probability of the event A is denoted by P(A) and is defined by:

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Total number of outcomes in } S} = \frac{n(A)}{N}$$

Random Variables

Random Variable: is a variable that takes on numerical values determined by the outcome of a random experiment.

A random variable is a function from the sample space **S** to the real numbers .

- Random Variables are generally denoted with capital letters such as X, Y, or Z.

Types of a random variable

1- Discrete Random Variables

2-Contionus Random Variables

Example: *A Simple Random Variable*

- Toss a coin twice, and let the random variable X be the number of tails appearing.
- **Questions:**
  - What are the possible values of X?
- **Answer:**

- Possible values: 0, 1, 2

Discrete Random Variables

A random variable is discrete if its set of possible outcomes is countable.
(like the numbers 1, 2, 3, … .
Or 1,2,3,…,n)

Discrete Random Variable (Example)

- The number of children in a family.
- The number of defective items .
- The number of student in class.
- The number of calls in hour.

Probability Distributions for Discrete Random Variables

Probability distribution is a distribution of the probabilities associated with each of the values of a random variable.

For discrete random variable X with the values $x_1, x_2, x_3,$ We denoted the probability function of x by $f(x)$

$$p_k(x) = P(X = x_k),\ k = 1, 2, 3, \ldots$$

$$f(x) = P(X = x)$$

Any probability distribution (Probability mass function)of a discrete random variable X must satisfy:

1. $0 \le f(x) \le 1$

2. $\sum_x f(x) = 1$

Example

Find the probability distribution of the pair of dice sum of the numbers when a are tossed

---

Solution.

$$n(S) = 6^2 = 36$$

$$S = \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(2,6),\ldots,(6,1),\ldots,(6,6)\}$$

Let X be a random variable whose values

x are the possible totals.

Then x take the values 2, 3, 4, . . . , 12

Die 1	Die 2					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

The probability distribution of the random variable X is given by

$f(x) = P[X = x]; \quad x = 2, 3, \ldots, 12.$

Therefore

X	2	3	4	5	6	7	8	9	10	11	12
f(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Example: *Probability Distribution of a Random Variable*

- The random variable *X* is the number of tails in two tosses of a coin.
- **Questions:**
  - What are the probabilities of the possible outcomes?
  - What is the probability distribution of *X*?
- **Answer** : Possible outcomes:

Each has probability ¼ so the probability distribution is:

X = Number of tails	0	1	2
Probability	1/4	1/2	1/4

Non-overlapping "Or" Rule→$P(X=1)=1/2$

Example: *Probability Distribution of a Random Variable*

- We have the probability distribution of the random variable *X* for number of tails in two tosses of a coin.

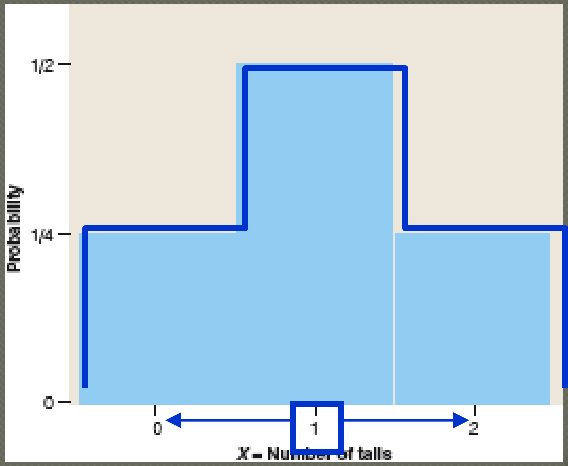
X = Number of tails	0	1	2
Probability	1/4	1/2	1/4

- **Question:** How do we display and summarize *X*?
- **Answer**: Use **probability histogram.**

Summarize: (center) mean=median=1

(spread) Typical distance from 1 is a bit less than 1.

(shape) unimodal, symmetric



Example: *Probability Distribution*

- **Background:** A coin is tossed 3 times and the random variable *X* is number of tails tossed.
- **Questions:** What are the possible outcomes, values of X, and probabilities? How do we find probability that *X* =1? *X* =2?
- **Answer:**
  - Interim Table:

Rule to combine probabilities

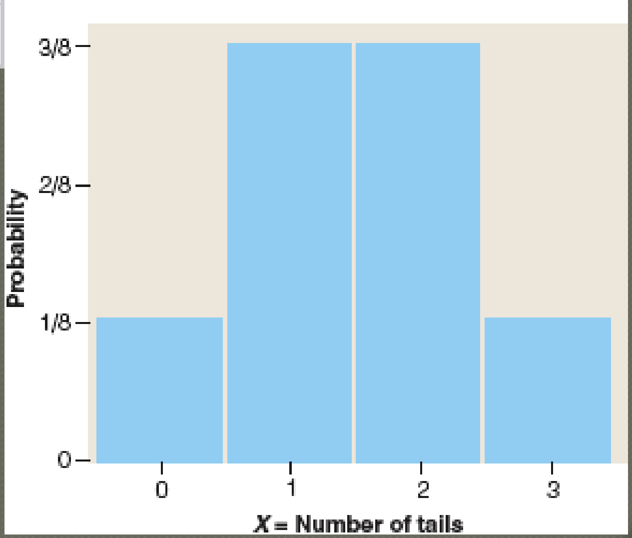
Outcome	X=no.of tails	Probability
HHH	0	1/8
HHT	1	1/8
HTH	1	1/8
THH	1	1/8
HTT	2	1/8
THT	2	1/8
TTH	2	1/8
TTT	3	1/8

Example: *Probability Distribution and Histogram*

- **Background**: X is number of tails in 3 coin tosses.
- **Question:** What are the probability distribution of X and probability histogram?
- **Answer**: Use the interim table to determine probabilities.

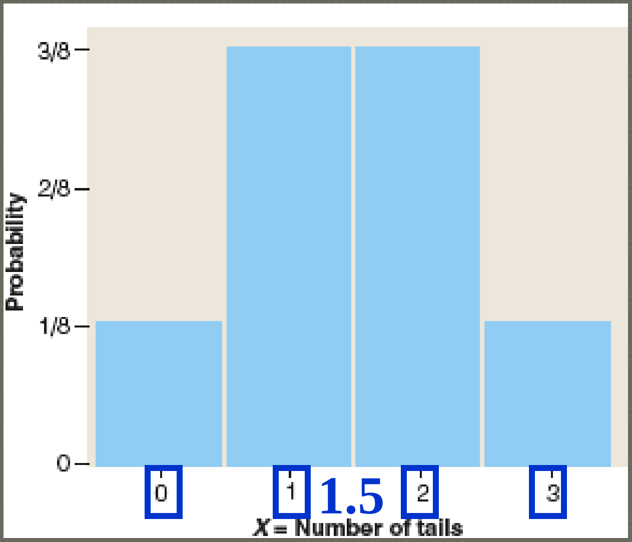
X = Number of tails	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

Use the probability distribution to sketch the histogram.



Example: *Summaries from Probability Histogram*

- **Background:** Histogram for number of tails in 3 coin tosses.



- **Question:** What does it show?
- **Response:**
  Histogram has
  - Shape: symmetric, unimodal
  - Center: median = mean = 1.5

The cumulative distribution function F(x)

Let X be a discrete random variable takes the values $x_1, x_2, x_3, \cdots$ with probabilities $p_1, p_2, p_3, \cdots$ then F(x) given by

$$F(x) = P(X \le x) = \sum_{t \le x} f(t).$$

1- $F(x) \ge 0$

2- $\lim_{x \to \infty} F(x) = 1$

3- $\lim_{x \to -\infty} F(x) = 0$

4- If a and b two real numbers and a < b then

$$P(a < X \le b) = F(b) - F(a).$$

Example:

Find the cumulative distribution of the number of heads when a coin is tossed four times.

Solution

$2^4 = 16$ points in the sample space, $x = 0,1,2,3,4$

The probability distribution of the random variable X,

$f(x) = P[X = x]$, $x = 0,1,2,3,4$.

x	0	1	2	3	4
f(x)	1/16	4/16	6/16	4/16	1/16

$F(0)=P(x \leq 0)= f(0)=1/16$

$F(1)=P(x \leq 1)= f(0)+f(1)= 1/16 + 4/16 = 5/16$

$F(2)=P(x \leq 2)= f(0)+f(1)+f(2)= 1/16 + 4/16 + 6/16 = 11/16$

$F(3)=P(x \leq 3)= f(0)+f(1)+f(2)+f(3)= 1/16 + 4/16 + 6/16 + 4/16 = 15/16$

$F(4)=P(x \leq 4)= f(0)+f(1)+f(2)+f(3)+f(4)= 1/16 + 4/16 + 6/16 + 4/16+ 1/16=1$

$$F(x)=\begin{cases} 0 & \text{for} \quad x<0 \\ \dfrac{1}{16} & \text{for} \quad 0 \le x < 1 \\ \dfrac{5}{16} & \text{for} \quad 1 \le x < 2 \\ \dfrac{11}{16} & \text{for} \quad 2 \le x < 3 \\ \dfrac{15}{16} & \text{for} \quad 3 \le x < 4 \\ 1 & \text{for} \quad 4 \le x \end{cases}$$

Example: The probability distribution of Y (the number of blood typing) was

y	1	2	3	4
$p(y)$	.4	.3	.2	.1

We first determine $F(y)$ for each value in the set $\{1, 2, 3, 4\}$ of possible values:

$$F(1) = P(Y \le 1) = P(Y = 1) = p(1) = .4$$
$$F(2) = P(Y \le 2) = P(Y = 1 \text{ or } 2) = p(1) + p(2) = .7$$
$$F(3) = P(Y \le 3) = P(Y = 1 \text{ or } 2 \text{ or } 3) = p(1) + p(2) + p(3) = .9$$
$$F(4) = P(Y \le 4) = P(Y = 1 \text{ or } 2 \text{ or } 3 \text{ or } 4) = 1$$

Now for any other number y, $F(y)$ will equal the value of F at the closest possible value of Y to the left of y. For example, $F(2.7) = P(Y \le 2.7) = P(Y \le 2) = .7$, and $F(3.999) = F(3) = .9$. The cdf is thus

$$F(y) = \begin{cases} 0 & \text{if } y < 1 \\ .4 & \text{if } 1 \le y < 2 \\ .7 & \text{if } 2 \le y < 3 \\ .9 & \text{if } 3 \le y < 4 \\ 1 & \text{if } 4 \le y \end{cases}$$

Examples

Given that the cumulative distribution function of T, the number of years to maturity for a randomly selected bond, is

$$F(t) = \begin{cases} 0, & t < 1, \\ \frac{1}{4}, & 1 \le t < 3, \\ \frac{1}{2}, & 3 \le t < 5, \\ \frac{3}{4}, & 5 \le t < 7, \\ 1, & t \ge 7, \end{cases}$$

find

(a) $P(T = 5)$;

(b) $P(T > 3)$;

(c) $P(1.4 < T < 6)$;

2- Determine the value c so that each of the following function can serve as a probability distribution of the discrete random variable X:

$$f(x) = c(x^2 + 4), for \;\; x = 0,1,2,3.$$