concepts.tex

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1 Vorticity equation

is given by

$$\rho \frac{D\omega}{Dt} = \rho(\omega \cdot \nabla)\mathbf{V} + \mu \nabla^2 \omega \tag{1}$$

You get it by taking the curl of every term in NS (study group exercise). *Note: no pressure term, since P is from potential and $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\Phi = 0$*

2 Enstrophy

Defined as

$$Z \equiv \int_{Vol} \frac{1}{2}\omega^2 dr \tag{2}$$

compare this to **kinetic energy**

$$KE \equiv \int_{Vol} \frac{1}{2}u^2 dr. \tag{3}$$

They fill about the same purpose, but enstrophy is for turbulent flow.

Kinetic energy is distributed around injection scale, enstrophy is distributed around *Kolmogorov scale.*

$$\frac{D}{Dt}KE = -2vZ \tag{4}$$

so high enstrophy gives fast decay of kinetic energy.

3 Energy cascade

Larger eddies break up to smaller and smaller eddies and then to heat. Large eddies carry most of the energy. They interact with the mean flow and extract energy from it. Small eddies, *assumed isotropic*, dissipate energy to heat.

Most important cascade mechanism is Vortex stretching and tilting.

Note: in 2D, KE tends to end up in large-scale structures. for example storms on jupiter.

At large scales we have

$$\begin{aligned} Re &= \frac{UL}{v} \\ \varepsilon \approx \frac{U^3}{L} &= \frac{v^3 Re^3}{L^4} \end{aligned} \tag{5}$$

Now, assuming the system is in quasi-equilibrium during energy cascade, we have

$$\begin{aligned} \varepsilon_{\text{large}} &\approx \varepsilon_{\text{small}} \\ \Rightarrow \frac{v^3 Re^3}{L^4} &\approx \frac{v^3}{L_K^4} \end{aligned} \tag{6}$$

4 Kolmogorov

Kolmogorov's hypothesis of local isotropy: At sufficiently high Re, the small scale turbulent motions are statistically isotropic.

First similarity hypothesis: In every turbulent flow, at sufficiently high Re, the description of the small scale motions have a universal form that is uniquely determined by the energy dissipation rate and viscosity.

Second similarity hypothesis: In every turbulent flow, at sufficiently high Re, the description of the intermediate scale motions have a universal form that is uniquely determined by the energy dissipation rate only.

From 1st similarity hypothesis, we get the Kolmogorov micro scales

$$\begin{aligned} \text{Time: } T_K &= \left(\frac{\nu}{\varepsilon}\right)^{1/2} \\ \text{Velocity: } U_K &= (\varepsilon\nu)^{1/4} \\ \text{Length: } L_K &= \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \end{aligned} \tag{7}$$

5 RANS

Reynolds Averaged Navier Stokes.

Notation- time, spatial, ensemble averages:

$$\begin{aligned} \langle f\rangle(\mathbf{r}) &= \lim_{T\leftarrow\infty}\left[\frac{1}{T}\int_t^{t+T} f(\bar{\mathbf{r}},t)\,\mathrm{d}t\right] \\ \langle f\rangle(t) &= \lim_{V\leftarrow\infty}\left[\frac{1}{V}\iiint f(\bar{r},t)\,\mathrm{d}V\right] \\ \langle f\rangle(\mathbf{r},t) &= \lim_{N\leftarrow\infty}\left[\frac{1}{N}\sum_{n=1}^{\infty} f_n(\mathbf{r},t)\,\mathrm{d}t\right] \end{aligned} \tag{8}$$

Reynolds decomposition

$$
\begin{aligned}
u_i &= U_i + u_i' \\
p &= P + p' \\
\tau_{ij} &= T_{ij} + \tau_{ij}'
\end{aligned}
\tag{9}
$$

where u, p, τ are instantaneous values, U, P, T are mean values, and u', p', τ' are fluctuation values.

Important average properties:

$$
\begin{aligned}
\langle u_i' \rangle = 0, \langle U_i \rangle = U_i \\
\langle U_i u_j' \rangle = 0, \langle u_i' u_j' \rangle \neq 0
\end{aligned}
\tag{10}
$$

5.1 Averaging process

starting from NS

$$
\rho \frac{\partial u_i}{\partial t} + \rho \left(u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} + f_i \tag{11}
$$

and taking average (time average?) we get

$$
\begin{aligned}
\langle LHS \rangle &= \rho \frac{\partial U_i}{\partial t} + \rho U_j \frac{\partial U_i}{\partial x_j} + \rho \frac{\partial}{\partial x_j} \langle u_i' u_j' \rangle \\
\langle RHS \rangle &= -\frac{\partial P}{\partial x_i} + \mu \frac{\partial}{\partial x_j} \frac{\partial U_i}{\partial x_j} + \langle f_i \rangle.
\end{aligned}
\tag{12}
$$

So then , the averga of the momentum equations become

$$
\begin{aligned}
\rho \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right) &= -\frac{\partial P}{\partial x_i} + \frac{\partial T_{ij}}{\partial x_j} + \frac{\partial R_{ij}}{\partial x_j} + \langle f_i \rangle \\
&= \frac{\partial}{\partial x_j} \left(-p \delta_{ij} + T_{ij} + R_{ij} \right) + \langle f_i \rangle
\end{aligned}
\tag{13}
$$

with

$$
T_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \tag{14}
$$

$$
R_{ij} \equiv -\langle \rho u_i' u_j' \rangle. \tag{15}
$$

Note here that the Reynolds stress R_{ij} depends on the fluctuating velocities for which we have no governing equations. therefore, **The system is not closed.**

5.2 Closure using Boussinesq hypothesis

Assume simple relationship between Reynolds stress and velocity gradients:

$$
R_{ij} \equiv -\langle \rho u_i' u_j' \rangle = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \rho k \delta_{ij} \tag{16}
$$

with

$$k \equiv \frac{1}{2}\langle u_i' u_j' \rangle. \tag{17}$$

We add the last term to agree with pure definitions which give

$$R_{ii} = -2\rho k. \tag{18}$$

6 Standard k-ε model

Important assumptions

- turbulent fluctuations are locally isotropic
- production and dissipation are locally equal (not true near walls)

Model k-equation:

$$\frac{\partial k}{\partial t} = U_i \frac{\partial k}{\partial x_j} = \frac{\mu_t}{\rho} S^2 - \varepsilon + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_y}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] \tag{19}$$

Model ε equation

$$\frac{\partial \varepsilon}{\partial t} + U_i \frac{\partial \varepsilon}{\partial x_j} = C_{\varepsilon 1}\frac{\mu_t}{\rho}\frac{\varepsilon}{k} S^2 - C_{\varepsilon 2}\frac{\varepsilon^2}{k} + \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] \tag{20}$$

Defining **k-production**

$$P_k \equiv \frac{R_{ij}}{\rho}\frac{\partial U_i}{\partial x_j} = \frac{\mu_t}{\rho} S^2 \tag{21}$$

k-diffusion

$$D_k \equiv \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right] \tag{22}$$

Dissipation

$$\varepsilon \equiv \frac{\mu}{\rho}\left\langle \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right\rangle \tag{23}$$

ε-diffusion

$$D_\varepsilon \equiv \frac{\partial}{\partial x_j}\left[\frac{1}{\rho}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right] \tag{24}$$

gives the **easy to memorize model:**

$$\boxed{\frac{Dk}{Dt} = P_k - \varepsilon + D_k} \tag{25}$$

$$\boxed{\frac{D\varepsilon}{Dt} = \frac{\varepsilon}{k}\left(C_{\varepsilon 1} P_k - C_{\varepsilon 2}\varepsilon\right) + D_\varepsilon} \tag{26}$$

Note: ε/k to match the units to k-eq then just constants.