Computational Physics: Solving the Poisson Equation in 1D & 2D

Introduction

The poisson equation is a famous PDE that is used to derive solutions to various physical problems. One such problem is deriving the charge potential and electric field of a system, given some initial and boundary conditions and a source charge distribution. All of the units of the variables of the problems presented below are dimensionless.

1D Problem

A simple problem that can be solved with the poisson equation is finding the electric potential on a 1 dimensional rod that is insulated on each end. The definition of this problem is:

$$-\frac{d^2\phi(x)}{dx^2} = S(x)\ ,\ \ \phi(0) = \phi(1) = 0\ ,\ \ S(x) = 12x^2$$

with $\phi(x)$ being a real valued function representing the electric potential and $S(x)$ being the source charge distribution. The solution of this problem is provided from the course notes and is $\phi(x) = x(1-x^3)$ and the energy of the system should asymptotically converge to the value $E = -0.64286$. We can see the numerical solution in the figures below, along with the analytical solution provided.

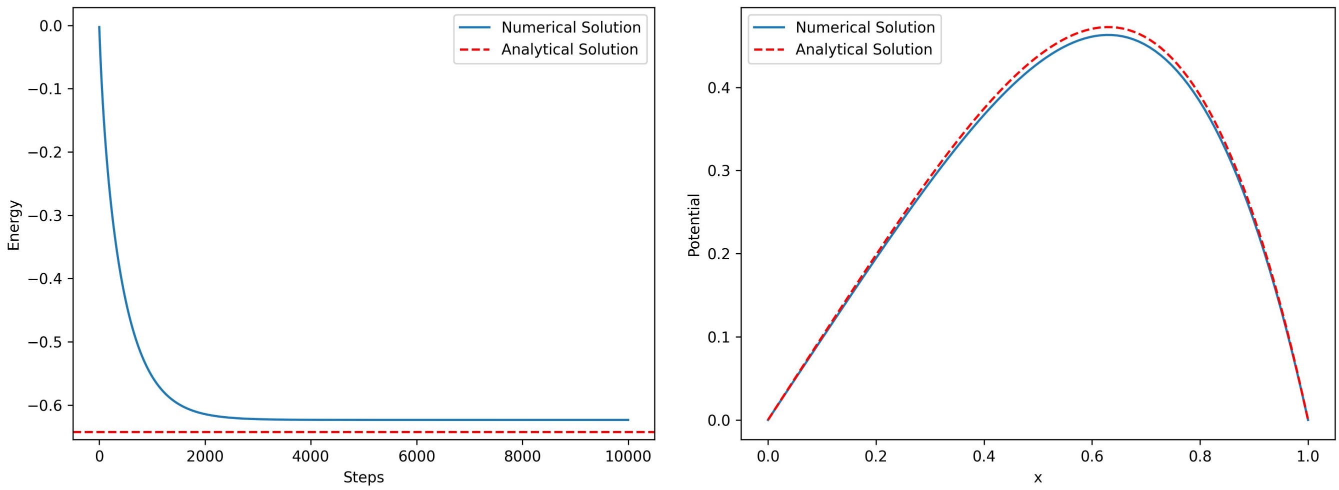

Figure 1 - Left: Energy of the system as a function of the timesteps. Right: Electric Potential as a function of x. The red dashed lines corresponds to the analytical result provided in the notes.

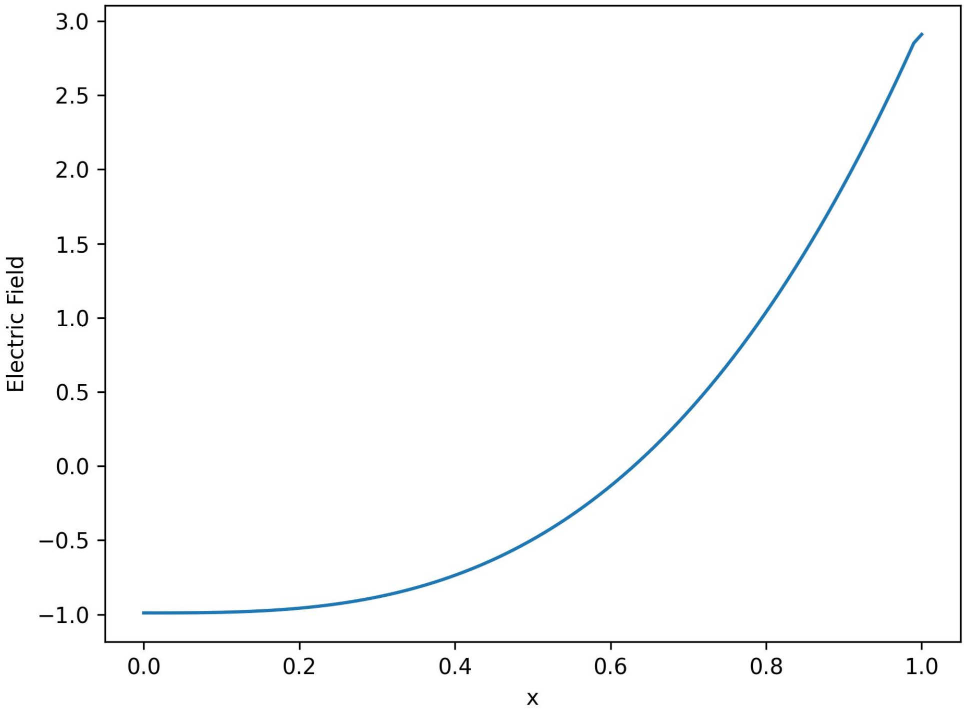

Figure 2 - The electric Field produced by the final charge distribution.

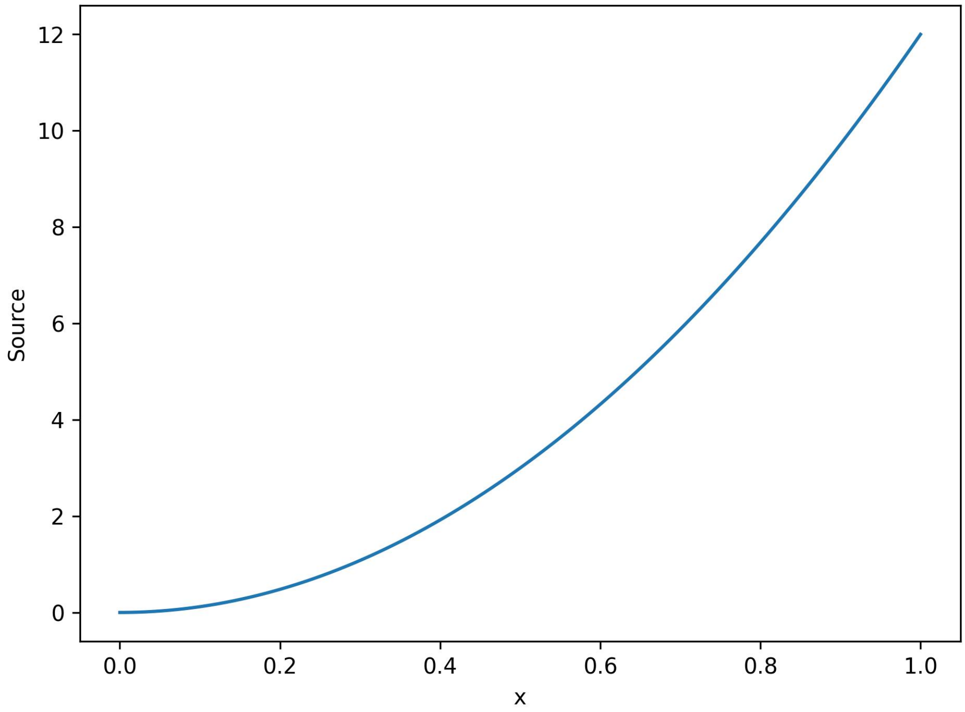

Figure 4 - The initial charge distribution on the rod.

The electric potential & field that are shown represent their respective quantities after 10000 repetitions. As we can surmise the numerical solution is producing satisfactory results.

2D Problem

The poisson equation presented in the previous section is a special case of the generalized equation:

$$\nabla^2 \phi(x,\ y) = S(x,\ y)$$

where ∇^2 is the laplacian. We can define a source distribution:

$$S(x,\ y) = sin(x^2 + y^2)$$

which we can see in . The boundary conditions are, for a cartesian domain Ω where $\Omega \subset \mathrm{R}^2$, $\phi(x,\ y) = 0$ on $\partial\,\Omega$, where $\partial\,\Omega$ is the boundary of Ω. The resulting numerical solution is presented in the figures below.

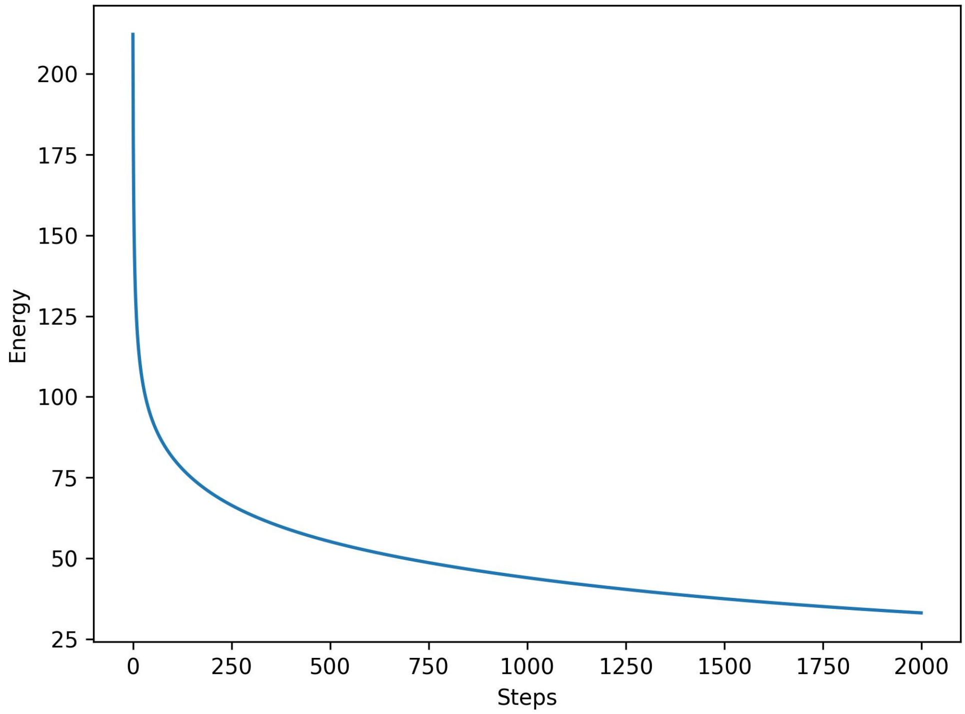

Figure 5 - Energy of the system as a function of the timesteps. It assymptotically converges to the value 33.0973.

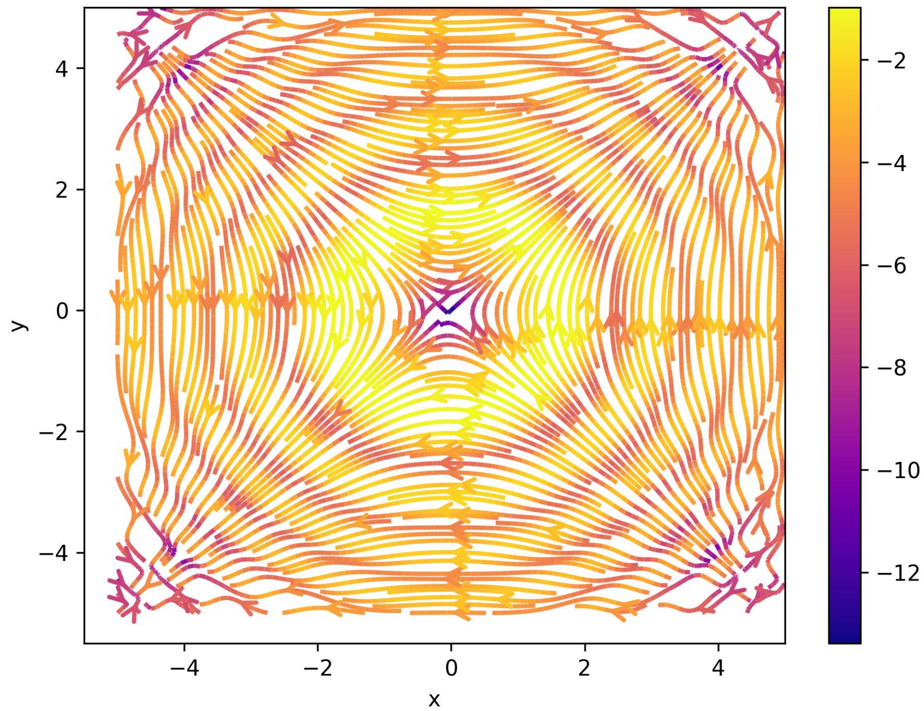

Figure 6 - Stream lines of the electric Field produced by the final charge distribution.

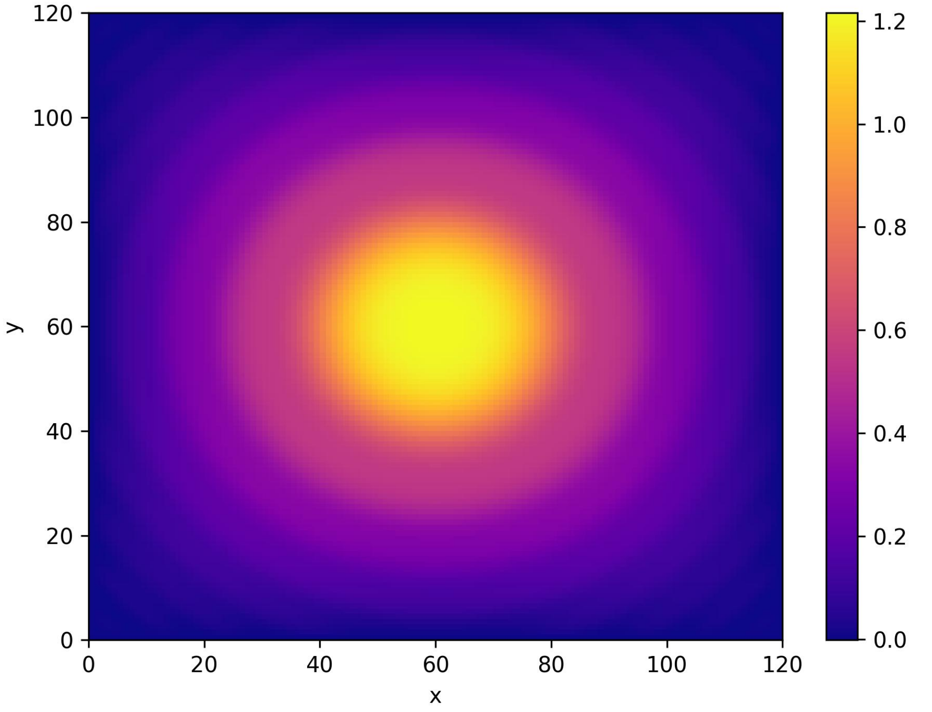

Figure 7 - The electric potential after 10000 timesteps.

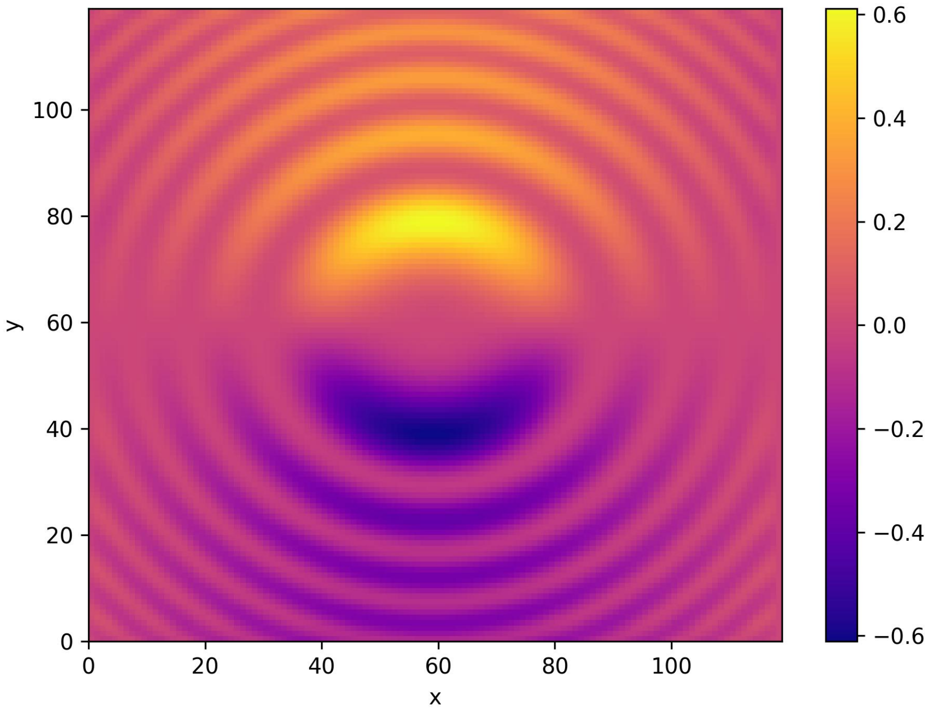

Figure 8 - The x component of the electric field after 10000 timesteps.

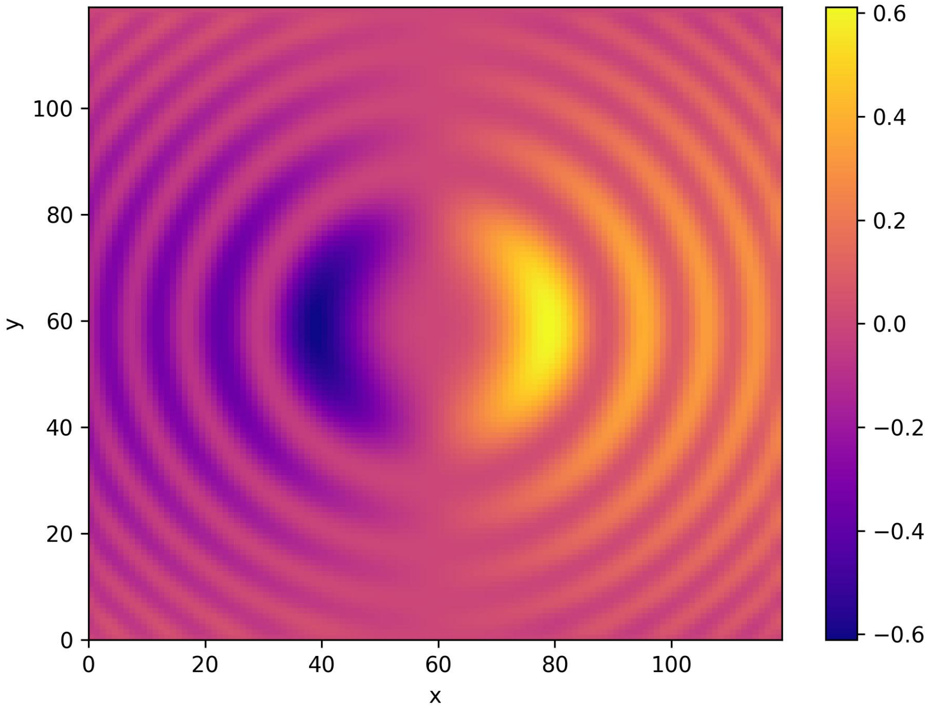

Figure 9- The x component of the electric field after 10000 timesteps.

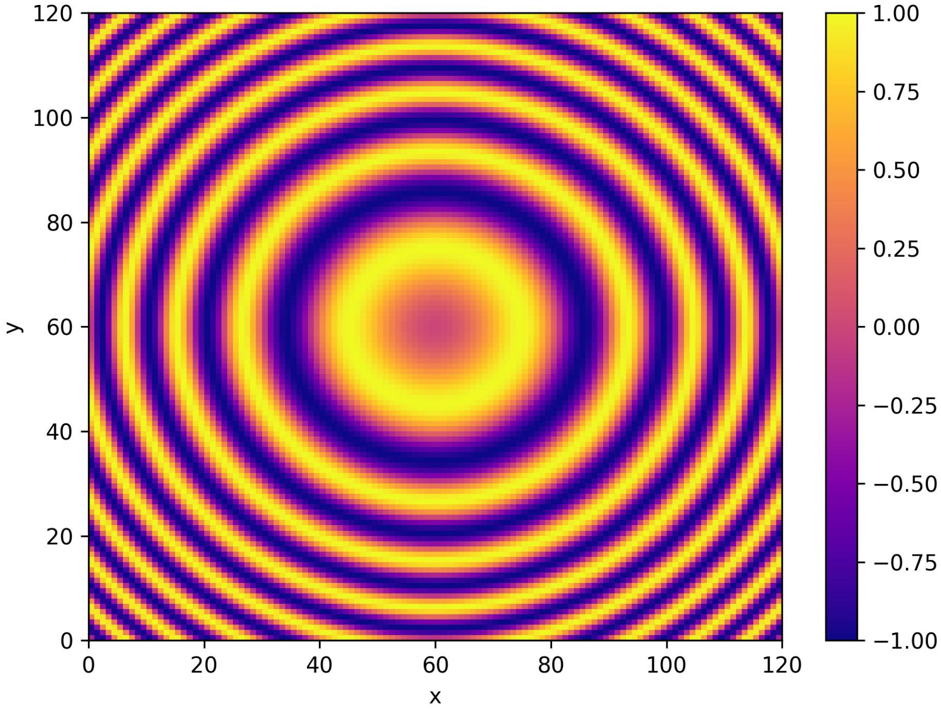

Figure 10 - The source charge distribution.