Assignment VIII ISE 5414

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5.1.2 Let Min(t) be the Poisson process representing the minor defects over the length t. Let Maj(t) be the Poisson process representing the major defects over the length t.

Now we want to calculate, $Pr[Min(t) + Maj(t) = k]$, where $X(t) = Min(t) + Maj(t)$ which represents number of defects, either major or minor, in the cable length of t.

$Pr[Min(t) + Maj(t) = k]$ can be written as a convolution sum of $Min(t)$ and $Maj(t)$.

$$Pr[Min(t) + Maj(t) = k] = \sum_{m=0}^{t} Pr[Min(t) = m]Pr[Maj(t) = k - m]$$

$$Pr[Min(t) + Maj(t) = k] = \sum_{m=0}^{t} \frac{(\alpha t)^m e^{-\alpha t}}{m!} \times \frac{(\beta t)^{k-m} e^{-\beta t}}{(k-m)!}$$

Multiplying and dividing by k! and taking all the "m" independent terms outside summation's scope.

$$Pr[Min(t) + Maj(t) = k] = \frac{e^{-(\alpha+\beta)t}}{k!} \sum_{m=0}^{t} \frac{k!}{(k-m)!m!} (\alpha t)^m (\beta t)^{k-m}$$

This summation is actually a binomial theorem which leads to $(\alpha t + \beta t)^k$

$$Pr[Min(t) + Maj(t) = k] = \frac{e^{-(\alpha+\beta)t}}{k!} (\alpha t + \beta t)^k$$

$$Pr[X(t) = k] = \frac{e^{-(\alpha+\beta)t}[(\alpha + \beta)t]^k}{k!}$$

Thus, proving that X(t) is a Poisson process of rate $(\alpha + \beta)$

5.1.7 Let k be the number of shocks that the system would have received till time t. Then,

$$Pr[N(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Also, we know that,

Pr[System survives each shock] $= \alpha$

Pr[System survives k shocks] $= \alpha^k$

Pr[System is surviving at time t]

$$= \sum_k Pr[N(shocks) = k \cap N(t) = k]$$

Pr[System is surviving at time t]

$$= \sum_k Pr[N(shocks) = k | N(t) = k] Pr[N(t) = k]$$

Therefore, Pr[System is surviving at time t]

$$= \sum_k \alpha^k \times \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

5.1.9 We know that the Number of passengers arriving at the bus stop at time T, where T is a random variable is a Poisson process and can be written as follows,

$$Pr[X(T) = k | T = t] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$Pr[X(T) = k | T = t] = \frac{(2t)^k e^{-2t}}{k!}$$

Therefore,

$$E[X(T) | T = t] = 2$$

$$E[X(T)^2 | T = t] = Var[X(T) | T = t] + E[X(T) | T = t]^2 = 2 + 2^2 = 6$$

$$Pr[X(t) = k] = \int_{-\infty}^{\infty} Pr[X(T) = k | T = t] f_T(t) dt$$

We know that,

$$f_T(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore,

$$Pr[X(t) = k] = \int_0^1 Pr[X(T) = k | T = t] dt$$

$$Pr[X(t) = k] = \int_0^1 \frac{(2t)^k e^{-2t}}{k!} dt$$

$$Pr[X(t) = k] = \frac{\Gamma(k+1) - \Gamma(k+1, 2)}{2k!} dt$$

5.1.12 We know that,

$$Pr[X'(t) = j] = \int_0^\infty \frac{(\theta t)^k e^{-\theta t}}{k!} f(\theta) d\theta$$

We are given,
$f(\theta) = e^{-\theta}$ for $\theta > 0$

$$Pr[X'(t) = j] = \int_0^\infty \frac{(\theta t)^k e^{-\theta t}}{k!} e^{-\theta} d\theta$$

Solving this integral, we obtain,

$$Pr[X'(t) = j] = \left(\frac{t}{1+t}\right)^j \left(\frac{1}{1+t}\right)$$

For j = 0, 1, ...

5.2.1 We write the binomial distribution of $Pr[X(n,p) = 0]$ as,

$$Pr[X(n,p) = 0] = \binom{n}{0} p^0 (1-p)^{n-0}$$

$$Pr[X(n,p) = 0] = (1-p)^n$$

We are given that, $np = \lambda$ i.e. $p = \lambda / n$

$$Pr[X(n,p) = 0] = \left(1 - \frac{\lambda}{n}\right)^n$$

Now, taking limit of n tending towards infinity on both sides,

$$\lim_{n\to\infty} Pr[X(n,p)=0] = \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n$$

$$\lim_{n\to\infty} Pr[X(n,p)=0] = e^{-\lambda}$$

Hence proved!

In the second part,

$$\frac{Pr[X(n,p)=k+1]}{Pr[X(n,p)=k]} = \frac{\binom{n}{k+1}p^{k+1}(1-p)^{n-k-1}}{\binom{n}{k}p^{k}(1-p)^{n-k}}$$

On applying limit on both side with n tending towards infinity, we can get the desired output.

$$\lim_{n\to\infty}\frac{Pr[X(n,p)=k+1]}{Pr[X(n,p)=k]} = \frac{\lambda}{k+1}$$