Karnaugh Maps, Part I

CS 350: Computer Organization & Assembler Language Programming

[2/9: Redrew some maps, cleaned up some row/column bugs, and added the cubical map; 2/12: Circle on p.1.]

A. Why?

- Simplification of logical expressions can improve their readability and implementability.
- Karnaugh maps are one way to simplify logical expressions.

B. Outcomes

After this lecture, you should

- Know how to simplify boolean expressions using Karnaugh maps.

C. Using Truth Tables to Simplify Boolean Expressions

- Simplifying boolean expressions makes them easier to read and to implement.
- One way to simplify expressions uses factoring, excluded middle, and domination:
    - For all E and X, we have $E\ X + E\ \overline{X} = E\ (X + \overline{X}) = E\ 1 = E$
- We can also use idempotency to take an expression and repeat it so that we can use it two ways.
- Example: Below, we duplicate $X\,\overline{Y}$.

  $\overline{X}\ \overline{Y} + X\,\overline{Y} + X\,Y$
  $= (\overline{X}\ \overline{Y} + X\,\overline{Y}\,) + (X\,\overline{Y} + X\,Y)$
  $= (\overline{X} + X)\,\overline{Y} + X\,(\overline{Y} + Y)$
  $= \overline{Y} + X$

- In terms of truth tables, we can apply a particular factoring iff the terms involved have 1's in the truth table column. E.g., for $\overline{X}\ \overline{Y} + X\,\overline{Y} = \overline{Y}$, we need 1's in the $\overline{X}\ \overline{Y}$ and $X\,\overline{Y}$ rows.

- Take the truth table for $Z = \overline{X}\,\overline{Y} + X\,\overline{Y} + X\,Y$ — we can check for $X\,\overline{Y} + X\,Y$ by looking for 1s in those two rows.

 © James Sasaki, 2014

- Check for $\overline{X}\ \overline{Y} + X\ \overline{Y}$ by looking for 1s in those two rows.

X	Y	Z
0	0	1
0	1	0
1	0	1
1	1	1

- If you have a DNF expression, then each DNF term corresponds to one or more 1's in the truth table column for the overall expression.
- For example, in the table below with $\overline{X}\ \overline{Y} + X\ \overline{Y} + XY$, each term contributes a single 1. The expressions X and $\overline{Y}$ both correspond to two 1's; if we OR together X and $\overline{Y}$, two of the 1's are in the same row, so we get only three 1's, not four. Note $X + \overline{X}\ \overline{Y}$ also has three 1's: two from X and one from $\overline{X}\ \overline{Y}$.

X	Y	$X+\overline{Y}$	$X+\ \overline{X}\ \overline{Y}$	$\overline{X}\ \overline{Y}+X\ \overline{Y}+XY$
0	0	1	1	1
0	1	0	0	0
1	0	1	1	1
1	1	1	1	1

- **Definition**: If $P_0 + P_1 + \dots + P_n$ implies Z, then each P_i is an **implicant** of Z.
- If we have an equality, not just an implication, then we have a representation of Z. With our example, if $Z = \overline{X}\ \overline{Y} + X\ \overline{Y} + XY$, then $X + \overline{Y}$ and $X + \overline{X}\ \ \overline{Y}$ are also representations of Z.
- **Definition**: An implicant of Z is a **prime implicant** if it cannot be replaced by a simpler implicant without changing Z. In our example, $\overline{Y}$ is prime; $X + \overline{X}\ \overline{Y}$ isn't.
- The problem of simplifying an expression E corresponds to

 © James Sasaki, 2014

1. Finding enough implicants to get an equality $P_0 + P_1 + \dots + P_n = E$
2. Minimizing the number of implicants, and
3. Getting all the implicants to be prime.

- For our example, $X + \overline{Y}$ is a simplest representation
  - It has fewer terms than $\overline{X}\ \overline{Y} + X\,\overline{Y} + XY$
  - It has all-prime implicants; $X + \overline{X}\ \overline{Y}$ does not.
- We especially want to avoid redundant implicants (one that's implied by another)
  - In $X + \overline{Y} + \overline{X}\ \overline{Y}$, the $\overline{X}\ \overline{Y}$ is redundant
  - In general, $E_1\ E_2 + E_1 = E_1$. ("**Absorption**" law)

D. Karnaugh Maps

- If an implicant corresponds to a particular pattern of 1's, then a prime implicant is one that can't be extended. In the simplest case, we're be looking for an individual 1 that can't be paired with another 1 for factoring. (We'll also see more complicated cases.)
- **Karnaugh maps** ("car-no" maps or just **K-maps**) are a way to write the truth table for an expression in a format that makes it easier to find implicants for the expression and to recognize when the implicants are prime. The table below shows how adjacent rows of 1's can be combined to form a larger implicant. **It's very important to note that the rows are in the order 00, 01, 11, 10**, not 00, 01, 10, 11. I.e., the DNF terms are in the order $\overline{X}\ \overline{Y}$, $\overline{X}\ Y$, $X\ Y$, and $X\,\overline{Y}$.
- By reordering the rows of the table, the bit pattern for each row differs in exactly one bit from the bit pattern of its adjacent neighbors. In boolean expression nomenclature, each DNF term differs from its adjacent neighbors in that one variable is negated. E.g., $X\,Y$ is adjacent to the $\overline{X}\ Y$ above it. $X\,Y$ is also adjacent to the $X\,\overline{Y}$ below it. Since adjacent terms differ only in that one variable is negated, this means you can factor out that variable if you OR together the two terms.
- E.g., the last two rows correspond to $X\,Y + X\,\overline{Y}$; we can drop the Y and $\overline{Y}$ and get just X.

© James Sasaki, 2014

	X	**Y**	***DNF Term***	
	0	0	$\overline{X}\,\overline{Y}$	$\overline{X}\,\overline{Y} + \overline{X}\,Y = \overline{X}$
$\overline{X}\,Y + X\,Y = Y$	0	1	$\overline{X}\,Y$	
	1	1	$X\,Y$	$X\,\overline{Y} + X\,Y = X$
	1	0	$X\,\overline{Y}$	

- Conceptually, the table wraps around top to bottom. This way, $\overline{X}\,\overline{Y}$ is adjacent to $X\,\overline{Y}$, and $\overline{X}\,\overline{Y} + X\,\overline{Y}$ simplifies to just $\overline{Y}$. Note that the inner pair of $\overline{X}\,Y$ and $X\,Y$ simplifies to Y, so the inner pair is the opposite of the wraparound pair.

	X	**Y**	***DNF Term***
	0	0	$\overline{X}\,\overline{Y}$
$\overline{X}\,Y + X\,Y = Y$	0	1	$\overline{X}\,Y$
	1	1	$X\,Y$
$\overline{X}\,\overline{Y} + X\,\overline{Y} = \overline{Y}$	1	0	$X\,\overline{Y}$
	0	0	$\overline{X}\,\overline{Y}$

- In a 2-variable Karnaugh map, there are three possible kinds of implicants: Single squares, which correspond to a 2-variable conjunction; pairs, which correspond to a single variable, and the 4-high column of all 1's, which corresponds to T (true, the 0-variable non-false expression).
- In a Karnaugh map, we indicate implicants by drawing a rectangle around the 1's for the implicant.
- Let's do a Karnaugh map for $Z = \overline{X}\,\overline{Y} + X\,\overline{Y} + X\,Y$…
  - We can circle/box the 1s for the pair $X\,Y + X\,\overline{Y} = X$

 © James Sasaki, 2014

- To circle the wraparound pair $\overline{X}\ \overline{Y} + X\ \overline{Y} = \overline{Y}$, we'll use two half circles/ boxes.
- We've circled all the 1's (and none of the 0's), so we're done. The simplified expression is $X + \overline{Y}$.

X	**Y**	**Z**
0	0	1
0	1	0
1	1	1
1	0	1

- An implicant is prime if it can't be extended to be a larger implicant. E.g., the map above for $X + \overline{Y}$ is preferred over the maps below for $\boldsymbol{X}\ \boldsymbol{Y} + \overline{Y}$ and $X + \overline{\boldsymbol{X}}\ \overline{\boldsymbol{Y}}$ because neither $\boldsymbol{X}\ \boldsymbol{Y}$ nor $\overline{\boldsymbol{X}}\ \overline{\boldsymbol{Y}}$ is prime.

X	**Y**	**Z**
0	0	1
0	1	0
1	1	1
1	0	1

X	**Y**	**Z**
0	0	1
0	1	0
1	1	1
1	0	1

- An implicant is redundant if it only circles 1's that are already circled. E.g., below, $X\ Y$ is redundant because it's already included in X.

X	Y	Z
0	0	1
0	1	0
1	1	1
1	0	1

- A 2-variable Karnaugh map can also be drawn as a square: Below are maps for the expressions $X + \overline{X}\ \overline{Y}$ and $X + \overline{Y}$. (Again, $X + \overline{Y}$ is preferred because both its implicants are prime.)

X \ Y	0	1
0	1	0
1	1	1

X \ Y	0	1
0	1	0
1	1	1

E. 3-Variable Karnaugh Maps

- A 3-variable Karnaugh map is basically two 2-variable maps joined together. So one possibility is to look for pairs within each column.
- In this example, the left column is a map for $X\,Y$ when $Z = 0$; the right column is a map for $X\,Y$ when $Z = 1$. We've circled
  - $\overline{X}\ \overline{Z} = \overline{X}\ \overline{Y}\ \overline{Z} + \overline{X}\ Y\ \overline{Z}$,
  - $Y\overline{Z} = \overline{X}\ Y\overline{Z} + X\,Y\overline{Z}$, and
  - $X\,Z = X\,Y\,Z + X\,\overline{Y}\ Z$

XY \ Z	0	1
00	1	0
01	1	0
11	1	1
10	0	1

- (In a 3-variable map, a pair of 1's corresponds to factoring one 1 variable, so we reduce a pair of 3-variable terms to one 2-variable term.)

- It's also possible to look for pairs going left-right. These pairs correspond to ones where we factor out a term of $\overline{Z} + Z$. We do have such a pair: $X\,Y = X\,Y\,\overline{Z} + X\,Y\,Z$.
    - **Note**: It would be redundant to circle both the $X\,Z$ and $X\,Y$ pairs, so we have two simplest expressions: $(\overline{X}\,\overline{Z} + Y\,\overline{Z} + X\,Z)$ and $(\overline{X}\,\overline{Z} + X\,Y + X\,Z)$.

XY \ Z	0	1
00	1	0
01	1	0
11	1	1
10	0	1

- In a 3-variable map, we can also try to factor out 2 variables at the same time, as in
    - $X\,Y\,\overline{Z} + X\,\overline{Y}\,\overline{Z} + X\,Y\,Z + X\,\overline{Y}\,Z = X$
- If you like, you can also see this as first factoring out two pairs and then combining the pairs:
    - $X\,Y\,\overline{Z} + X\,\overline{Y}\,\overline{Z} + X\,Y\,Z + X\,\overline{Y}\,Z = (X\,Y\,\overline{Z} + X\,\overline{Y}\,\overline{Z}) + (X\,Y\,Z + X\,\overline{Y}\,Z) = X\,\overline{Z} + X\,Z = X$.
- In a map, these 4 terms correspond to a 2×2 square (two pairs joined left-to-right). The maps below show the implicants X, $\overline{X}$, and Y.

XY \ Z	0	1
00	0	0
01	0	0
11	1	1
10	1	1

XY \ Z	0	1
00	1	1
01	1	1
11	0	0
10	0	0

XY \ Z	0	1
00	0	0
01	1	1
11	1	1
10	0	0

 © James Sasaki, 2014

- As with the 2-variable maps, we can wrap around from top to bottom, within each column separately, or both at the same time. Doing both gives us a wrap-around square for the implicant $\overline{Y}$.

		Z 0	1
	00	1	1
XY	01	0	0
	11	0	0
	10	1	1
	00	1	1

		Z 0	1
	00	1	1
XY	01	0	0
	11	0	0
	10	1	1

- We can also look for 1×4 rectangles (two pairs combined end-to-end) for the implicants $\overline{Z}$ and Z:

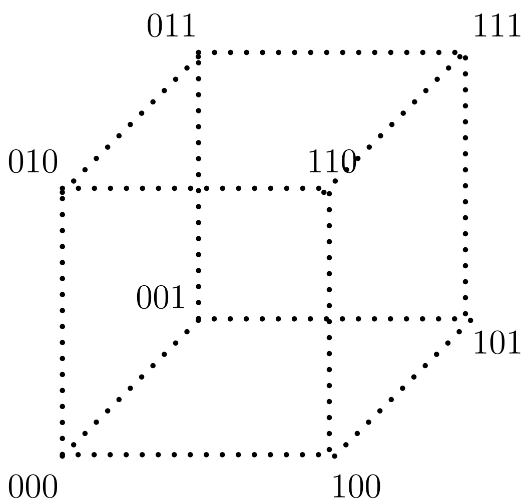
		Z 0	1
	00	1	0
XY	01	1	0
	11	1	0
	10	1	0

		Z 0	1
	00	0	1
XY	01	0	1
	11	0	1
	10	0	1

- An equally valid way to draw a 3-variable map is the 2×4 transpose of the 4 $\times$ 2 table: It has one variable on the side and two variables on the top; each row forms a 2-variable map. Below, the 1×4 rectangle is for X; the left-right wrap-around square is $\overline{Z}$.

		Y Z			
		00	01	11	10
X	0	1	0	0	1
	1	1	1	1	1

- A 3-variable map can also be drawn as a $2 \times 2 \times 2$ cube, with $\overline{X}$ and X along one dimension, $\overline{Y}$ and Y along a second, and $\overline{Z}$ and Z along the third. To factor out one variable, we look for a pair of adjacent 1's; to factor out two variables, we look for a 2×2 slice of the cube.



3-D Karnaugh Map For 3 Variables

- In a 3-variable map, the implicants are of sizes 1, 2, 4, and 8 (all 8 subterms gives the expression T (true)). **Implicants always have sizes that are powers of 2**.

Karnaugh Maps, Part I

CS 350: Computer Organization & Assembler Language Programming

A. Why?

- Simplification of logical expressions can improve their readability and implementability.
- Karnaugh maps are one way to simplify logical expressions.

B. Outcomes

After this activity, you should be able to

- Translate between 2-variable DNF terms and their Karnaugh map implicants.
- Determine whether an implicant in a 2-variable Karnaugh map is prime.

C. Questions

1. (a) Draw a Karnaugh map for the variables X and Y. Fill in with 0's and 1's (and circle the 1(s)) for the expression $X\,\overline{Y}$. (b) Repeat, for $X\,Y$. (c) … for $\overline{X}$. (d) … for $\overline{X}$. (e) … for Y. (f) … for $\overline{Y}$.

2. (a) Study the circled 1 in the Karnaugh map below. Does they form an implicant? If so, what is the DNF term for the implicant? (b) Repeat. (c) Repeat. (d) Repeat.

X	**Y**	**Z**
0	0	0
0	1	0
1	1	1
1	0	0

(a)

X	**Y**	**Z**
0	0	0
0	1	0
1	1	1
1	0	1

(b)

X	**Y**	**Z**
0	0	1
0	1	1
1	1	1
1	0	0

(c)

X	**Y**	**Z**
0	0	1
0	1	0
1	1	0
1	0	1

(d)

© James Sasaki, 2014

3. (a) This Karnaugh map is incomplete because not all the 1s are circled. Try to cover the uncovered 1(s) by extending the given circle; if that's not possible, add a new circle. Give the DNF expression for the resulting map. (b) Repeat. (c) Repeat. (d) Repeat.

X	***Y***	***Z***
0	0	0
0	1	1
1	1	$\boxed{1}$
1	0	0

(a)

X	***Y***	***Z***
0	0	0
0	1	1
1	1	0
1	0	$\boxed{1}$

(b)

X	***Y***	***Z***
0	0	0
0	1	$\boxed{1}$
1	1	$\boxed{1}$
1	0	1

(c)

X	***Y***	***Z***
0	0	$\boxed{1}$
0	1	0
1	1	0
1	0	1

(d)

4. Simplify $(X\,Y + X\,\overline{Y} + Y)$ using Karnaugh maps: Draw a truth table for it, then draw a Karnaugh map for the expression (place the 0's and 1's, then circle prime implicants to exactly cover the 1's), then give the expression specified by the implicants.

5. Simplify $(X + \overline{Y})(\overline{X} + \overline{Y})$ using Karnaugh maps.

Solution

Problem 1:

X	Y	Z
0	0	0
0	1	0
1	1	0
1	0	1

(a): $X\ Y$

X	Y	Z
0	0	1
0	1	0
1	1	0
1	0	0

(b): $\overline{X}\ \overline{Y}$

X	Y	Z
0	0	0
0	1	0
1	1	1
1	0	1

(c): X

X	Y	Z
0	0	1
0	1	1
1	1	0
1	0	0

(d): $\overline{X}$

X	Y	Z
0	0	0
0	1	1
1	1	1
1	0	0

(e): Y

X	Y	Z
0	0	1
0	1	0
1	1	0
1	0	1

(f): $\overline{Y}$

2. (a) Yes, $X\ Y$. (b) Yes, X. (c) No. (An easy way to see this is that implicants always have sizes that are powers of 2.) (d) Yes, $\overline{Y}$.

3. (a) Yes, we can extend (the term $X\,Y$) to Y. (b) We can't extend, but we can introduce a new implicant, $\overline{X}\ \ Y$. (c) We can't extend, but we can introduce X. (d) We can extend (the term $\overline{X}\ \ \overline{Y}$) to $\overline{Y}$.

X	***Y***	***Z***
0	0	0
0	1	1
1	1	1
1	0	0

(a)

X	***Y***	***Z***
0	0	0
0	1	1
1	1	0
1	0	1

(b)

X	***Y***	***Z***
0	0	0
0	1	1
1	1	1
1	0	1

(c)

X	***Y***	***Z***
0	0	1
0	1	0
1	1	0
1	0	1

(d)

4. Below, the truth table for $(X\,Y + \overline{X}\ \overline{Y} + \overline{X})$ is to the left; the Karnaugh map is to the right. The circled implicants correspond to $\overline{X} + Y$.

X	***Y***	***Z***
0	0	1
0	1	1
1	0	0
1	1	1

X	***Y***	***Z***
0	0	1
0	1	1
1	1	1
1	0	0

5. We can expand $(X+\overline{Y})(\overline{X}+\overline{Y}) = X\,\overline{X} + X\,\overline{Y} + \overline{X}\ \overline{Y} + \overline{Y}\ \overline{Y} = X\,\overline{Y} + \overline{X}\ \overline{Y} + \overline{Y}$. Again, the truth table for the expression is to the left; the Karnaugh map is to the right. The circled implicant is $\overline{Y}$.

X	***Y***	***Z***
0	0	1
0	1	0
1	0	1
1	1	0

X	***Y***	***Z***
0	0	1
0	1	0
1	1	0
1	0	1

 © James Sasaki, 2014