Tautologies; Normal Forms

CS 350: Computer Organization & Assembler Language Programming

A. Why?

- Tautologies describe properties that are always true and transformations that are always valid.
- Disjunctive Normal Form (DNF) is a particular standard way to represent a logical expression.
- DNF corresponds to a sum of the rows of 1's for the truth table of an expression.

B. Outcomes

At the end of today, you should:

- Know what tautologies, contradictions, and contingencies are and how to recognize them.
- Know what disjunctive normal form (DNF) is and how it's related to truth tables.

C. Laws of Boolean Logic

- In addition to comparing the truth tables of expressions, we can also manipulate them syntactically, substituting equals for equals.
- There exist many alternative ways to write laws for Boolean logic; here are some.
  - **Associativity and Commutativity**
    - *AND* and *OR* are associative; *XOR*, *NAND*, *NOR*, *IMPL*, *IFF* are not associative.
    - *AND*, *OR*, *XOR*, *NAND*, *NOR*, *IFF* are commutative, *IMPL* is not commutative.
  - **Distributivity**
    - $X\ AND\ (Y\ OR\ Z) = (X\ AND\ Y)\ OR\ (X\ AND\ Z)$

    - $X\ OR\ (Y\ AND\ Z) = (X\ OR\ Y)\ AND\ (X\ OR\ Z)$
  - **DeMorgan's Laws**
    - $P\ NAND\ Q = NOT\ (P\ AND\ Q) = (NOT\ P)\ OR\ (NOT\ Q)$
    - $P\ NOR\ Q = NOT\ (P\ OR\ Q) = (NOT\ P)\ AND\ (NOT\ Q)$
  - **Contradiction**: $X\ AND\ NOT\ X = 0$
  - **Excluded Middle**: $X\ OR\ NOT\ X = 1$
  - **Double Negation** (Pierce's Law): $NOT\ NOT\ X = X$
  - **Identity**: $X\ AND\ 1 = X$; also, $X\ OR\ 0 = X$
  - **Domination**: $X\ OR\ 1 = 1$; also, $X\ AND\ 0 = 0$
  - **Idempotency**: $X\ OR\ X = X\ AND\ X = X$
  - **Definition of** *IMPL*: $X\ IMPL\ Y = NOT\ X\ OR\ Y$
  - **Definition of** *IFF*: $X\ IFF\ Y = (X\ IMPL\ Y)\ AND\ (Y\ IMPL\ X)$

D. Disjunctive Normal Form

- A **normal form** is a standard way of writing something.
- In **DNF (Disjunctive Normal Form)**, a logical expression is written as as a disjunction (*OR*) of terms.
  - Each term is the conjunction (*AND*) of 1 or more "atoms" (a.k.a. "literals").
  - An **atom** (= "**literal**") is a variable or *NOT* variable.
  - **Example**: $\overline{X}\,Y + X\,Z$ is in DNF.
  - **Example**: $X\,Z + Y\,Z$ is in DNF.
  - **Example**: $(X+Y)\,Z$ is not in DNF (even though it's equivalent to $X\,Z + Y\,Z$).
- Using Boolean algebra (especially DeMorgan's laws and distribution) you can always simplify a complex expression into one in DNF.
  - Typically use DeMorgan's laws and distribution to reduce the number of parentheses in the expression, then use rules like contradiction, identity, idempotency, etc., to get rid of unnecessary terms.

- **Example**:

$$
\begin{array}{ll}
\neg\,(X\,\overline{Y} + Y) & \\
= \neg(X\,\overline{Y})\ \overline{Y} & \text{by DeMorgan's law } (\mathit{NOT} \text{ of } \mathit{OR}) \\
= (\overline{X}+Y)\ \overline{Y} & \text{by DeMorgan's law } (\mathit{NOT} \text{ of } \mathit{AND}) \text{ and double negation} \\
= \overline{X}\,\overline{Y} + Y\,\overline{Y} & \text{by Distributivity } (\mathit{AND} \text{ over } \mathit{OR}) \\
= \overline{X}\,\overline{Y} + 0 & \text{by Contradiction} \\
= \overline{X}\,\overline{Y} & \text{by Domination}
\end{array}
$$

E. Full DNF and Truth Tables

- A DNF expression is in **Full DNF** if each term includes all the variables of the expression.
  - **Example**: $X\,Y\,Z + X\,\overline{Y}\,Z + X\,Y\,Z + \overline{X}\,Y\,Z$ is in full DNF.
  - **Example**: $X\,Z + Y\,Z$ is in DNF but not full DNF
- You can convert an expression from DNF to full DNF by inserting $(V + \overline{V})$ for missing variables V and expanding by distributing AND over OR
  - **Example**: $X\,Z + Y\,Z$
    $= X\,(Y + \overline{Y})\,Z + (X + \overline{X})\,Y\,Z$
    $= X\,Y\,Z + X\,\overline{Y}\,Z + X\,Y\,Z + \overline{X}\,Y\,Z$
    $= X\,Y\,Z + X\,\overline{Y}\,Z + \overline{X}\,Y\,Z$
- Full DNF format is useful because a full DNF term corresponds to exactly one row of a truth table.
- **Example**: $Z = \overline{X}\ \overline{Y} + X\,\overline{Y} + X\,Y$ is in full DNF (there are two variables, X and Y), so each term specifies exactly one row where the truth table for Z should be 1.

X	***Y***	***DNF Term***	***Z***
0	*0*	$\overline{X}\,\overline{Y}$	*1*
0	*1*	$\overline{X}\ Y$	*0*
1	*0*	$X\,\overline{Y}$	*1*
1	*1*	$X\,Y$	*1*

- For 3 variables, there are 8 rows, with each row indexed by a 3-variable term.
- **Example**: The table below defines $W = \overline{X}\ \overline{Y}\ \overline{Z} + \overline{X}\ Y Z + X\,\overline{Y}\ Z$.

X	*Y*	*Z*	*DNF Term*	*W*
0	0	0	$\overline{X}\ \overline{Y}\ \overline{Z}$	1
0	0	1	$\overline{X}\,\overline{Y}\,Z$	0
0	1	0	$\overline{X}\ Y\overline{Z}$	0
0	1	1	$\overline{X}\ Y Z$	1
1	0	0	$X\,\overline{Y}\,\overline{Z}$	0
1	0	1	$X\,\overline{Y}\ Z$	1
1	1	0	$X\,Y\overline{Z}$	0
1	1	1	$X\,Y Z$	0

- To convert a full DNF expression to a truth table, take the column for the expression and place a 1 in each row corresponding to a full DNF term; place a 0 in each leftover row.
- In the other direction, to convert a truth table to a full DNF term, find each row that contains a 1 and add the DNF term for that row.
  - (If the column is nothing but 0, the expression is F (false).
- Length of Full DNF Expressions
  - How many terms can a full DNF expression with k variables have?
  - Converting from arbitrary boolean expression to equivalent full DNF representation can cause “exponential blowup”.
  - **Problem to solve** in next class: Given a boolean expression, how can we find a shortest equivalent expression?

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A. Why?

- Tautologies describe properties that are always true and transformations that are always valid.
- Disjunctive Normal Form (DNF) is a particular standard way to represent a logical expression.

B. Outcomes

After this activity, you should be able to

- Recognize tautologies, contradictions, and contingencies via their truth tables.
- Translate between an expression's disjunctive normal form (DNF) and its truth table.

C. Questions

1. Write a truth table to show that *((X IMPL Y) AND (Y IMPL Z)) IMPL (X IMPL Z)* is a tautology.
2. Write a truth table to test one of DeMorgan's laws on three variables: *NOT(X OR Y OR Z) = NOT X AND NOT Y AND NOT Z.* Is this a tautology, contradiction, or contingency?
3. Write a truth table for *((X IMPL Y) IMPL Z) IFF (X IMPL (Y IMPL Z))*; is this a tautology, a contradiction, or a contingency?
4. Given an expression, the number of 1's in its truth table column is =, >, or < than the number of terms in its full DNF representation?
5. (a) Translate $\neg(X + Y\,Z)$ into minimal DNF (non-full DNF with a minimal set of terms). (b) Give the equivalent full DNF representation. (c) Use the full DNF representation to write a truth table for the expression.
6. Repeat the previous problem on $(X + \overline{Y})\,(Y + Z)$.

Solution

1. The table below shows that $((X\ IMPL\ Y)\ AND\ (Y\ IMPL\ Z))\ IMPL\ (X\ IMPL\ Z)$ is a tautology.

X	***Y***	***Z***	***((X***	***IMPL***	***Y)***	***AND***	***(Y***	***IMPL***	***Z))***	***IMPL***	***(X***	***IMPL***	***Z)***
0	0	0	0	1	0	1	0	1	0	***1***	0	1	0
0	0	1	0	1	0	1	0	1	1	***1***	0	1	1
0	1	0	0	1	1	0	1	0	0	***1***	0	1	0
0	1	1	0	1	1	1	1	1	1	***1***	0	1	1
1	0	0	1	0	0	0	0	1	0	***1***	1	0	0
1	0	1	1	0	0	0	0	1	1	***1***	1	1	1
1	1	0	1	1	1	0	1	0	0	***1***	1	0	0
1	1	1	1	1	1	1	1	1	1	***1***	1	1	1

2. The table below shows that $NOT\,(X\ OR\ Y\ OR\ Z) = NOT\ X\ AND\ NOT\ Y\ AND\ NOT\ Z$ is a tautology.

X	***Y***	***Z***	***(NOT***	***((X OR Y)***	***OR Z))***	***=***	***((NOT X***	***AND***	***NOT Y)***	***AND***	***NOT Z)***
0	0	0	1	0	0	***1***	1	1	1	1	1
0	0	1	0	0	1	***1***	1	1	1	0	0
0	1	0	0	1	1	***1***	1	0	0	0	1
0	1	1	0	1	1	***1***	1	0	0	0	0
1	0	0	0	1	1	***1***	0	0	1	0	1
1	0	1	0	1	1	***1***	0	0	1	0	0
1	1	0	0	1	1	***1***	0	0	0	0	1
1	1	1	0	1	1	***1***	0	0	0	0	0

3. Since the truth table for $((X\ IMPL\ Y)\ IMPL\ Z)\ IFF\ (X\ IMPL\ (Y\ IMPL\ Z))$ has a mix of 0's and 1's, it's a contingency.

X	***Y***	***Z***	***((X IMPL Y)***	***IMPL***	***Z)***	***=***	***(X***	***IMPL***	***(Y IMPL Z))***
0	0	0	1	0	0	***0***	0	1	1
0	0	1	1	1	1	***1***	0	1	1
0	1	0	1	0	0	***0***	0	1	0
0	1	1	1	1	1	***1***	0	1	1
1	0	0	0	1	0	***1***	1	1	1
1	0	1	0	1	1	***1***	1	1	1
1	1	0	1	0	0	***1***	1	0	0
1	1	1	1	1	1	***1***	1	1	1

4. The number of 1's in the truth table column for an expression equals the number of terms in the full DNF representation.

5a. $\neg(X + Y\,Z) = \overline{X}\ \neg(Y\,Z) = \overline{X}\,(\overline{Y} + \overline{Z}) = \overline{X}\ \overline{Y} + \overline{X}\ \overline{Z}$.

5b. $\overline{X}\ \overline{Y} + \overline{X}\ \overline{Z} = \overline{X}\ \overline{Y}\,(Z + \overline{Z}) + \overline{X}\,(Y + \overline{Y})\,\overline{Z}$
$= \overline{X}\ \overline{Y}\ Z + \overline{X}\ \overline{Y}\ \overline{Z} + \overline{X}\,Y\,\overline{Z} + \overline{X}\ \overline{Y}\ \overline{Z}$
$= \overline{X}\ \overline{Y}\ \overline{Z} + \overline{X}\ \overline{Y}\ Z + \overline{X}\,Y\,\overline{Z}$.

5c. The table below shows the full DNF expression from part (b) and also the original expression $\neg(X + Y\,Z)$. (You weren't asked for this last part.)

X	Y	Z	$\overline{X}\ \overline{Y}\ \overline{Z} + \overline{X}\ \overline{Y}\ Z + \overline{X}\,Y\,\overline{Z}$	NOT	(X	+	$Y\,Z$)
0	0	0	1	1	0	0	0
0	0	1	1	1	0	0	0
0	1	0	1	1	0	0	0
0	1	1	0	0	0	1	1
1	0	0	0	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	0	0	1	1	0
1	1	1	0	0	1	1	1

6a. $(X+\overline{Y})\,(Y+Z) = X\,Y + X\,Z + \overline{Y}\,Y + \overline{Y}\,Z = X\,Y + X\,Z + \overline{Y}\,Z.$

6b. $X\,Y + X\,Z + \overline{Y}\,Z = X\,Y\,(\overline{Z}+Z) + X\,(\overline{Y}+Y)\,Z + (\overline{X}+X)\,\overline{Y}\,Z$

$= X\,Y\,\overline{Z} + X\,Y\,Z + X\,\overline{Y}\,Z + X\,Y\,Z + \overline{X}\,\overline{Y}\,Z + X\,\overline{Y}\,Z$

$= \overline{X}\,\overline{Y}\,Z + X\,\overline{Y}\,Z + X\,Y\,\overline{Z} + X\,Y\,Z$

6c. The table:

X	Y	Z	$\overline{X}\,\overline{Y}\,Z + X\,\overline{Y}\,Z + X\,Y\,\overline{Z} + X\,Y\,Z$	$(X+\overline{Y})$	AND	$(Y+Z)$
0	0	0	0	1	0	0
0	0	1	1	1	1	1
0	1	0	0	0	0	1
0	1	1	0	0	0	1
1	0	0	0	1	0	0
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

